An Introduction to Clifford Algebra

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Definitions and Notations

- $\mathbf{e_1}, \mathbf{e_2} \in \mathbb{R}^2$ are said to be **orthonormal unit vectors** if

$$|\mathbf{e_1}| = |\mathbf{e_2}| = 1, \quad \langle \mathbf{e_1}, \mathbf{e_2} \rangle = 0,$$

  where $|\cdot|$ is the norm and $\langle \cdot, \cdot \rangle$ is the inner product. They form a basis of $\mathbb{R}^2$, and geometrically this means that $\mathbf{e_1}$ and $\mathbf{e_2}$ have length 1 and are perpendicular to each other.

- For $\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2}$ and $\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2}$, the **dot product** is defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

- For $\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2}$ and $\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2}$, the **bivector** is defined by

$$\mathbf{a} \wedge \mathbf{b} = (a_1 b_2 + a_2 b_1) e_{12}$$

Bivectors on $\mathbb{R}^2$

Given a vector space $\mathbb{R}^2$, the addition and scalar multiplication of the vectors are well-defined. Then a natural question to ask is: can we define a vector multiplication on $\mathbb{R}^2$? Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, we want the multiplication preserves the norm, i.e.,

$$|\mathbf{ab}| = |\mathbf{a}||\mathbf{b}|,$$

and it is distributive and associative.

Let $\mathbf{e_1}, \mathbf{e_2}$ be the orthonormal unit vectors on $\mathbb{R}^2$. The norm of a vector

$$\mathbf{r} = r_1\mathbf{e_1} + r_2\mathbf{e_2}$$

is defined by

$$|\mathbf{r}| = \sqrt{r_1^2 + r_2^2}.$$

If $\mathbf{r}$ is multiplied by itself, we require that $\mathbf{r}^2 = |\mathbf{r}|^2$, that is, we want

$$\begin{aligned}(r_1\mathbf{e_1} + r_2\mathbf{e_2})^2 &= (r_1\mathbf{e_1} + r_2\mathbf{e_2})(r_1\mathbf{e_1} + r_2\mathbf{e_2}) \\ &= r_1^2\mathbf{e_1}^2 + r_1 r_2(\mathbf{e_1 e_2} + \mathbf{e_2 e_1}) + r_2^2\mathbf{e_2^2} \\ &= r_1^2 + r_2^2.\end{aligned}$$

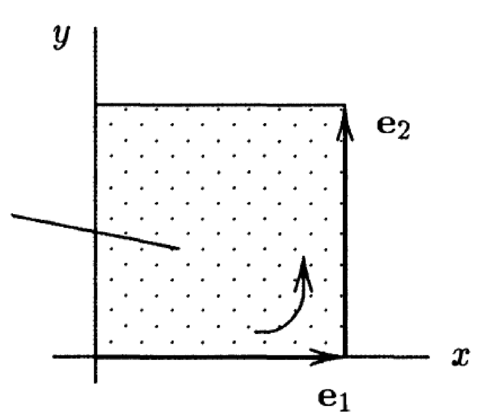
This equality is satisfied if the orthonormal unit vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ satisfy the following properties:

$$\mathbf{e_1}^2 = \mathbf{e_2}^2 = 1, \quad \mathbf{e_1 e_2} = -\mathbf{e_2 e_1}.$$

Consequently, we have

$$(\mathbf{e_1 e_2})^2 = \mathbf{e_1 e_2 e_1 e_2} = \mathbf{e_1}(-\mathbf{e_1 e_2})\mathbf{e_2} = -1,$$

which follows that $\mathbf{e_1 e_2}$ is neither a scalar nor a vector of $\mathbb{R}^2$. This product is called a **bivector**, representing the oriented plane area of the square with sides $\mathbf{e_1}$ and $\mathbf{e_2}$. Denote $\mathbf{e_1 e_2} = \mathbf{e_{12}}$, it is illustrated by the following figure from [1]



We defined the **Clifford product** of two vectors $\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2}$ and $\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2}$ to be

$$\mathbf{ab} = a_1 b_2 + a_2 b_2 + (a_1 b_2 - a_2 b_1)\mathbf{e_{12}} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}.$$

Reflections

Given two vectors $\mathbf{a}$ and $\mathbf{r}$ in $\mathbb{R}^2$, $\mathbf{r}$ has a parallel component to $\mathbf{a}$ denoted as $\mathbf{r}_\parallel$ that is given by the dot product of $\mathbf{r}$ and $\mathbf{a}$ multiplied by the vector $\mathbf{a}^{-1} = \frac{\mathbf{a}}{|\mathbf{a}|^2}$, that is

$$\mathbf{r}_\parallel = (\mathbf{r} \cdot \mathbf{a})\frac{\mathbf{a}}{|\mathbf{a}|^2} = (\mathbf{r} \cdot \mathbf{a})\mathbf{a}^{-1}$$

Also, $\mathbf{r}$ has a perpendicular component to $\mathbf{a}$ denoted as $\mathbf{r}_\perp$ that is given by

$$\mathbf{r}_\perp = \mathbf{r} - \mathbf{r}_\parallel = \mathbf{r} - (\mathbf{r} \cdot \mathbf{a})\mathbf{a}^{-1} = (\mathbf{ra} - \mathbf{r} \cdot \mathbf{a})\mathbf{a}^{-1} = (\mathbf{r} \wedge \mathbf{a})\mathbf{a}^{-1}$$

Thus the reflection of $\mathbf{r}$ denoted as $\mathbf{r}'$ can be obtained by decomposing $\mathbf{r} = \mathbf{r}_\parallel + \mathbf{r}_\perp$ and sending it to $\mathbf{r}' = \mathbf{r}_\parallel - \mathbf{r}_\perp$. Note that since $\mathbf{r}_\perp$ is a bivector,

$$\mathbf{r}_\perp = (\mathbf{r} \wedge \mathbf{a})\mathbf{a}^{-1} = -\mathbf{a}^{-1}(\mathbf{r} \wedge \mathbf{a}) = \mathbf{a}^{-1}(\mathbf{a} \wedge \mathbf{r}) = -(\mathbf{a} \wedge \mathbf{r})\mathbf{a}^{-1}$$

Then we can find two direct formulas for $\mathbf{r}'$ as

$$\begin{aligned}\mathbf{r}' &= \mathbf{r}_\parallel + \mathbf{r}_\perp \\ &= (\mathbf{r} \cdot \mathbf{a})\mathbf{a}^{-1} - (\mathbf{r} \wedge \mathbf{a})\mathbf{a}^{-1} \\ &= (\mathbf{r} \cdot \mathbf{a} - \mathbf{r} \wedge \mathbf{a})\mathbf{a}^{-1} \\ &= (\mathbf{a} \cdot \mathbf{r} + \mathbf{a} \wedge \mathbf{r})\mathbf{a}^{-1} \\ &= \mathbf{ara}^{-1}\end{aligned}$$

$$\begin{aligned}\mathbf{r}' &= (\mathbf{r} \cdot \mathbf{a} - \mathbf{r} \wedge \mathbf{a})\mathbf{a}^{-1} \\ &= (2\mathbf{r} \cdot \mathbf{a} - \mathbf{ra})\mathbf{a}^{-1} \\ &= 2\frac{\mathbf{a} \cdot \mathbf{r}}{\mathbf{a}^2}\mathbf{a} - \mathbf{r}\end{aligned}$$

With the commutative properties of Clifford products,

$$\mathbf{ar}_\parallel\mathbf{a}^{-1} = \mathbf{r}_\parallel\mathbf{aa}^{-1} = \mathbf{r}_\parallel$$

and

$$\mathbf{ar}_\perp\mathbf{a}^{-1} = -\mathbf{r}_\perp\mathbf{aa}^{-1} = -\mathbf{r}_\perp$$

which yields to formula $\mathbf{r}' = \mathbf{ara}^{-1}$.

Reflections and Rotation in 3D

In the Euclidean space $\mathbb{R}^3$ the vectors $\mathbf{r}$ and $\mathrm{ar}^{-1} = 2(\mathbf{a} \cdot \mathbf{r})\mathbf{a}^{-1} - \mathbf{r}$ are symmetric with respect to the axis $\mathbf{a}$. The opposite of $\mathbf{ar}^{-1}$, the vector

$$-\mathbf{ara}^{-1} = \mathbf{r} - 2\frac{\mathbf{a} \cdot \mathbf{r}}{\mathbf{a}^2}\mathbf{a}$$

is obtained by reflecting $\mathbf{r}$ across the mirror perpendicular to $\mathbf{a}$ [reflection across the plane $\mathbf{ar}_{123}$]

Consider a vector $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and the bivector $\mathbf{ae}_{123} = a_1\mathbf{e}_{23} + a_2\mathbf{e}_{31} + a_3\mathbf{e}_{12}$ dual to $\mathbf{a}$. The vector $\mathbf{a}$ has positive square

$$\mathbf{a}^2 = |\mathbf{a}|^2, \quad \text{where} \quad |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2},$$

but the bivector $\mathbf{ae}_{123}$ has negative square

$$(\mathbf{ae}_{123})^2 = -|\mathbf{a}|^2$$

It follows that

$$\exp(\mathbf{ae}_{123}) = \cos\alpha + \mathbf{e}_{123}\frac{\mathbf{a}}{\alpha}\sin\alpha$$

where $\alpha = |\mathbf{a}|$. A spatial rotation of the vector $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ around the axis $\mathbf{a}$ by the angle α is given by

$$\mathbf{r} \to a\mathbf{r}a^{-1}, \quad a = \exp\left(\frac{1}{2}\mathbf{ae}_{123}\right).$$

The sense of the rotation is clockwise when regarded from the arrow-head of $\mathbf{a}$. The axis of two consecutive rotations around the axes $\mathbf{a}$ and $\mathbf{b}$ is given by the Rodrigues formula

$$\mathbf{c}' = \frac{\mathbf{a}' + \mathbf{b}' + \mathbf{a}' \times \mathbf{b}'}{1 - \mathbf{a}' \cdot \mathbf{b}'} \quad \text{where} \quad \mathbf{a}' = \frac{\mathbf{a}}{\alpha}\tan\frac{\alpha}{2}$$

This result is obtained by dividing both sides of the formula

$$\exp\left(\frac{1}{2}\mathbf{ce}_{123}\right) = \exp\left(\frac{1}{2}\mathbf{be}_{123}\right)\exp\left(\frac{1}{2}\mathbf{ae}_{123}\right)$$

by their scalar parts and then by inspecting the bivector parts.

Linear Space of Bivectors In $\mathbb{R}^3$

Let $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3} \in \mathbb{R}^3$ be orthonormal unit vectors, so they form a basis of $\mathbb{R}^3$. The bivectors

$$\mathbf{e_1} \wedge \mathbf{e_2}, \mathbf{e_1} \wedge \mathbf{e_3}, \mathbf{e_2} \wedge \mathbf{e_3}$$

form a basis for the linear space of bivectors, denoted by $\bigwedge^2 \mathbb{R}^3$. The inner product on Euclidean space $\mathbb{R}^3$ can be extended to a symmetric bilinear product (an inner product) on $\bigwedge^2 \mathbb{R}^3$, given by

$$\langle \mathbf{x_1} \wedge \mathbf{x_2}, \mathbf{y_1} \wedge \mathbf{y_2} \rangle = \begin{vmatrix} \mathbf{x_1} \cdot \mathbf{y_1} & \mathbf{x_1} \cdot \mathbf{y_2} \\ \mathbf{x_2} \cdot \mathbf{y_1} & \mathbf{x_2} \cdot \mathbf{y_2} \end{vmatrix}.$$

In particular, we have

$$\langle \mathbf{a} \cdot \mathbf{b} \rangle = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2.$$

The Hodge Dual

The Hodge dual sending a vector $\mathbf{a} \in \mathbb{R}^3$ to a bivector $\star\mathbf{a} \in \bigwedge^2 \mathbb{R}^3$, defined by

$$\mathbf{b} \wedge \star\mathbf{a} = (\mathbf{b} \cdot \mathbf{a})\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \text{ for all } \mathbf{b} \in \mathbb{R}^3$$

The Hodge dual depends not only on the metric but also on the choice of orientation - it is customary to use a right-handed and orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ Thus, we have assigned to each vector

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \in \mathbb{R}^3$$

a bivector

$$\mathbf{A} = \star\mathbf{a} = a_1\mathbf{e}_2 \wedge \mathbf{e}_3 + a_2\mathbf{e}_3 \wedge \mathbf{e}_1 + a_3\mathbf{e}_1 \wedge \mathbf{e}_2 \in \bigwedge^2 \mathbb{R}^3.$$

Using the induced metric on the bivector space $\Lambda^2\mathbb{R}^3$ we can extend the Hodge dual to a mapping sending a bivector $\mathbf{A} \in \Lambda^2\mathbb{R}^3$ to a vector $\star\mathbf{A} \in \mathbb{R}^3$, defined by

$$\mathbf{B} \wedge \star\mathbf{A} = \langle \mathbf{B}, \mathbf{A} \rangle \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \text{ for all } \mathbf{B} \in \bigwedge^2 \mathbb{R}^3.$$

Using duality, the relation between the cross product and the exterior product can be written as

$$\begin{aligned}\mathbf{a} \wedge \mathbf{b} &= \star(\mathbf{a} \times \mathbf{b}), \\ \mathbf{a} \times \mathbf{b} &= \star(\mathbf{a} \wedge \mathbf{b}).\end{aligned}$$

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References

[1] Pertti Lounesto. *Clifford Algebras and Spinors, Second edition.* The Press Syndicate of the University of Cambridge, 2001.