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A dynamic IS-LM model with delayed taxation revenues

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Abstract

Some recent contributions to Economic Dynamics have shown a new interest for delay differential equations. In line with these approaches, we re-proposed the problem of the existence of a finite lag between the accrual and the payment of taxes in a framework where never this type of lag has been considered: the well known IS-LM model. The qualitative study of the system of functional (delay) differential equations shows that the finite lag may give rise to a wide variety of dynamic behaviours. Specifically, varying the length of the lag and applying the "stability switch criteria", we prove that the equilibrium point may lose or gain its local stability, so that a sequence of alternated stability/instability regions can be observed if some conditions hold. An important scenario arising from the analysis is the existence of limit cycles generated by sub-critical and supercritical Hopf bifurcations. As numerical simulations confirm, if multiple cycles exist, the so called "crater bifurcation" can also be detected. Economic considerations about a stylized policy analysis stand by qualitative and numerical results in the paper.

1. Introduction

In recent years Economic Dynamics seem to devote new interest to delay differential equations. This is because some economic phenomena cannot be described exhaustively with pure (linear or non-linear) differential equations. As an historical overview may show, after the pioneering work on the time-to-build investments by Kalecki [18], subsequently elaborated upon further by Frisch and Holme [9] and James and Belz [16], only some few contributions appeared from the fifties to the eighties. [1] But, starting from the nineties, an increasing number of works began to make use of mixed differential-difference equations to perform dynamic economic models (e.g. [35,15,17,1,29]). Nevertheless it cannot be forgotten that, although the presence of lagged variables is stressed in many other papers, there are some recent contributions which deliberately avoid the formal difficulties of functional (delay) differential equations. Additional assumptions (sometime ad hoc) on the lag's nature are used to change the dynamical system into a more tractable system of ordinary differential equations (ODEs), e.g. [29,22]. These assumptions may concern the length of the lag or the

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[1] Among others here we recall the papers by Furuno [10], where a two sector growth model is considered, and by Burmeister–Turnovsky [4] on adaptive expectations in continuous time models. Further references can be founded in [11].

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distribution of different lags over different agents. In the former case, economically or analytically justified a lag to be small, the function embodying the lag is approximated by a linear Taylor expansion. In the latter case, justified a random distribution over the economic agents, the finite lag is changed into an exponentially distributed lag. In both cases, the original system of mixed differential-difference equations is transformed into a system of ODEs.

Here we are not interested to argue if these last approaches to Economic Dynamics are or are not admissible. We would rather stress that there are many economic situations where a finite time delay cannot be ignored or avoided without altering the real dynamic behaviour of the system. It is in this perspective that here we re-proposed the problem of the existence of a (finite) lag between the accrual and the payment of taxes in a framework where usually no lag of this type is considered: the well known dynamic IS-LM model.

Originally the question about collection lags in the tax system was raised, and its economic implications stressed, by Tanzi [32,33] for less developed countries. His insight was very simple: when there are lags in tax collection, inflation erodes the real value of the government's revenues. In the wake of Tanzi the role of lags has been extensively studied focalising on typical arguments of public finance (e.g. [5,6]).

This paper aims to investigate a more general theme: the impact of delayed tax revenues on the fiscal policy outcomes. Specifically, our approach deals with the dynamical effects that such lags (only partially controllable by policy makers) may have on the stabilisation policies.

From the analytical point of view, like the standard ODEs IS-LM model our system of functional (delay) differential equation has a unique equilibrium point. The qualitative analysis shows that, varying the length of the lag and then applying the "stability switch criteria", this critical point loses or gains its local stability. Furthermore, if some conditions hold, a sequence of intervals, where regions of stability/instability are alternated, may exist. This implies that, given the tax structure, policy makers might have to face serious difficulties to stabilise the economic system, by using the usual fiscal policy instruments.

Another important scenario arising from the qualitative analysis is the existence of limit cycles generated by sub-critical and supercritical Hopf bifurcations. As numerical simulations will show, if multiple limit cycles exist, the so called "crater bifurcation" can also be detected. [2]

The paper is organised as follows. Section 2 briefly explains the reasons of a finite lag in tax revenues and suggests a practical way to capture its broad qualitative features in a macro-model. Section 3 contains a description of the model and a qualitative study of its dynamic behaviour. Section 4 is devoted to the numerical simulations of a specified version of the model. Finally, Section 5 concludes.

2. There are lags in tax collection, but which type of lag is suitable for a macro-model?

Any economic system has collection lags in its tax system. The real world suggests that the lengths of the lags are not different for different agents and are not distributed in a random manner over all the population. For the same tax all the agents face the same lag, if it exists. The lag is institutionally fixed and any agent know about it, as, for example, it happens in the case of value-added tax (VAT) payments, which are made at fixed dates.

In a model aimed to depict economic system dynamics, even if continuous time is considered, its tax revenues (T) come, at a given time t, from the weighted sum of a finite sequence of differently lagged incomes (Y), that may be differently taxed.

Formally we might write:

$$T(t) = \sum_{i=0}^{n} \epsilon_i \tau_i Y(t - \theta_i),$$

where τ_i is the tax rate such that $0 < \tau_i < 1$, $(i = 0, \ldots, n)$, $\sum_{i=0}^{n} \epsilon_i = 1$ the sum of all the income tax shares and θ_i a finite time delay such that $\theta_i > 0$, $\forall i \neq 0$ and $\theta_i = 0$ if $i = 0$.

Although our formal definition of tax revenues is what the real world suggests, a theoretical macro-model needs a more simple expression to make behind mathematics tractable. Therefore, without losing of generality, here we propose a practical compromise able to capture only the broad qualitative feature of time lags in tax collection. We shall assume that, at each time t, tax revenues have only two complementary components: the first based on the current income, the second on a past income, and both subjected to the same tax rate. Formally we shall set:

$$T(t) = (1 - \epsilon)Y(t) + \epsilon Y(t - \theta), \tag{1}$$

where θ is a fixed mean time lag of income the current tax revenue is based on, and τ a common average tax rate.

[2] Details on this argument can be found in [19].

Let us note that also multiple lags might be considered. In this case the model's behind mathematics becomes more sophisticated.

What it is really important to stress here is that, if a dynamic macro-model includes tax collection lags and it is formulated in continuous time, then the resulting mathematical system has functional (delay) differential equations, exactly as it happens when a growth model takes a time-to-build technology into account. [3]

3. The model

3.1. Assumptions

Let us consider a fixed-price disequilibrium intermediate run IS-LM model augmented by a government budget constraint, in the tradition of the well known Schinasi's works [24,25].

In the case of a pure money financing deficit, the resulting system is the following: [4]

$$\begin{cases} Y'(t) = \alpha[I(Y(t), r(t)) + g - S(Y^{\mathrm{D}}(t)) - T(t)], \\ r'(t) = \beta[L(Y(t), r(t)) - M(t)], \\ M'(t) = g - T(t), \end{cases} \tag{2}$$

where α and β are positive constants.

It may be useful to recall that: the first equation represents the traditional disequilibrium adjustment in the product market with I, r, g, S, T and $Y^{\mathrm{D}}(t) = Y(t) - T(t)$ representing investment, the interest rate, (constant) government expenditure, tax collections and disposable income, respectively; the second equation represents the disequilibrium dynamic adjustment in the money market for which M represents the real money supply (prices are fixed at unity) and L represents the liquidity preference function; the third equation is the government's budget constraint. It suggests that the government must issue (retire) debt only in the form of money in order to finance deficits (surpluses).

The following are assumed about derivatives: $I_Y > 0$, $I_r < 0$, $0 < S'(Y^{\mathrm{D}}(t)) < 1$, $L_Y > 0$, $L_r < 0$. Further it is postulated that a portion of current tax revenues come from a past income as in (1), where the tax rate is such that $0 < \tau < 1$, $\tau > S'(Y^{\mathrm{D}}(t))$.

3.2. The qualitative analysis

Given $I_r < 0$, let us suppose that:

$$\lim_{r\to 0^+} I\left(\frac{g}{\tau}, r\right) \geqslant S\left(g\frac{1-\tau}{\tau}\right) \quad \text{and} \quad \lim_{r\to +\infty} I\left(\frac{g}{\tau}, r\right) \leqslant S\left(g\frac{1-\tau}{\tau}\right).$$

As usual, to define the steady-state solutions we set (the delay) $\theta = 0$, and, under the previous conditions, we have a unique critical point (Y^*, r^*, M^*) such that

$$Y^* = \frac{g}{\tau},$$

$$r^* \text{ solution of } I\left(\frac{g}{\tau}, r\right) = S\left(g\frac{1-\tau}{\tau}\right),$$

$$M^* = L\left(\frac{g}{\tau}, r^*\right).$$

3.3. Linear stability analysis

The linearization of the system in a neighbourhood of the fixed point (Y^*, r^*, M^*) yields:

$$\begin{pmatrix} Y(t) \\ r(t) \\ M(t) \end{pmatrix}' = A \begin{pmatrix} Y(t) - Y^* \\ r(t) - r^* \\ M(t) - M^* \end{pmatrix} + B \begin{pmatrix} Y(t-\theta) - Y^* \\ r(t-\theta) - r^* \\ M(t-\theta) - M^* \end{pmatrix}, \tag{3}$$

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3 See, for example, the recent paper by Asea and Zak [1] and Szydlowski [27,28], and Szydlowski and Krawiec [30,31].

4 We have to notice that Schinasi assumed an instantaneous adjustment in the money market to simplify his analysis. A qualitative of the three-dimensional system was carried out by Sasakura [23].

where

$$A = \begin{pmatrix} \alpha(I_y - S_y) + \alpha(S_y - 1)(1-\epsilon)\tau & \alpha I_r & 0 \\ \beta L_y & \beta L_r & -\beta \\ -(1-\epsilon)\tau & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} \alpha\tau(S_y - 1)\epsilon & 0 & 0 \\ 0 & 0 & 0 \\ -\tau\epsilon & 0 & 0 \end{pmatrix},$$

$$I_y = \frac{\partial I}{\partial Y}(Y^*, r^*); \quad I_r = \frac{\partial I}{\partial r}(Y^*, r^*); \quad S_y = \frac{\mathrm{d}S}{\mathrm{d}Y}(Y^* - g); \quad L_y = \frac{\partial L}{\partial Y}(Y^*, r^*); \quad L_r = \frac{\partial L}{\partial r}(Y^*, r^*).$$

The characteristic equation of system (3) is $\det(\lambda I - A - B\mathrm{e}^{-\theta\lambda}) = 0$, which leads to

$$D(\lambda, \theta) := Q(\lambda)\mathrm{e}^{-\theta\lambda} + P(\lambda) = 0, \tag{4}$$

where $Q(\lambda) = q_2\lambda^2 + q_1\lambda + q_0$, $P(\lambda) = \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0$ and

$$\begin{aligned} q_2 &= \alpha\tau(1 - S_y)\epsilon \geqslant 0, & p_2 &= \alpha[S_y - I_y + (1-\epsilon)\tau(1 - S_y)] - \beta L_r, \\ q_1 &= -L_r\alpha\beta\tau(1 - S_y)\epsilon \geqslant 0, & p_1 &= \alpha\beta[L_r(I_y - S_y + (1-\epsilon)\tau(1 - S_y)) - I_r L_y], \\ q_0 &= -\alpha\beta\tau I_r\epsilon \geqslant 0, & p_0 &= -\alpha\beta\tau I_r(1-\epsilon) \geqslant 0. \end{aligned}$$

To investigate the local stability of the equilibrium point, we begin by considering, as usual, the case without delay ($\theta = 0$). In this case the characteristic polynomial is

$$D(\lambda, 0) = Q(\lambda) + P(\lambda)$$

hence, according to the Hurwitz criterion, the fixed point is stable if and only if

$$\begin{cases} p_2 + q_2 > 0, \\ (p_1 + q_1)(p_2 + q_2) > p_0 + q_0. \end{cases}$$

3.4. Stability with positive delay

When $\theta > 0$, standard results on stability of systems of delay differential equations postulate that a fixed point is asymptotically stable if and only if all roots of Eq. (4) have negative real part, while instability implies the existence (at least one) of roots with positive real part. It is well known that Eq. (4) is a transcedental equation which has an infinite number of complex roots and the possible ones with positive real part are finite in number.

Since the degree of the polynomial Q is lower than that of the polynomial P, the sum of multiplicities of roots with positive real part can change, as θ varies in $]0, +\infty[$, if a root appears on or crosses the imaginary axis (see [8]).

We want to obtain the values θ^* such that the fixed point (Y^*, r^*, M^*) changes from local asymptotic stability to instability or vice versa. To determine such stability switches we need the imaginary solutions of Eq. (4). Let $\lambda = \pm\mathrm{i}\omega$ be these solutions. As $D(0, \theta) = q_0 \neq 0$, then $\omega \neq 0$, i.e. the imaginary axis cannot be crossed by real values. Without loss of generality, we assume $\omega > 0$. We look for roots $\lambda = \mathrm{i}\omega$ of (4), because, being real the coefficients of Eq. (4), also $\lambda = -\mathrm{i}\omega$ is a root of (4).

We suppose that

$$\forall\omega \in \mathbb{R}, \quad P(\mathrm{i}\omega) + Q(\mathrm{i}\omega) \neq 0.$$

This implies that P and Q have no common imaginary roots. The previous conditions is equivalent to

$$(p_1 + q_1)(p_2 + q_2) \neq p_0 + q_0.$$

Therefore, if $\mathrm{i}\omega$ is root of D, then $Q(\mathrm{i}\omega) \neq 0$, otherwise $Q(\mathrm{i}\omega) = P(\mathrm{i}\omega) = 0$. By using the standard Euler formula, from $Q(\mathrm{i}\omega)\mathrm{e}^{-\mathrm{i}\theta\omega} + P(\mathrm{i}\omega) = 0$, we have

$$\begin{aligned} \sin\omega\theta &= \mathrm{Im}\left(\frac{P(\mathrm{i}\omega)}{Q(\mathrm{i}\omega)}\right) = \omega\frac{\omega^4 q_2 - \omega^2(q_0 - q_1p_2 + q_2p_1) + q_0p_1 - p_0q_1}{\omega^4q_2^2 + \omega^2(q_1^2 - 2q_0q_2) + q_0^2}, \\ \cos\omega\theta &= -\mathrm{Re}\left(\frac{P(\mathrm{i}\omega)}{Q(\mathrm{i}\omega)}\right) = \frac{\omega^4(q_1 - q_2p_2) + \omega^2(q_0p_2 - q_1p_1 + p_0q_0) + p_0q_0}{\omega^4q_2^2 + \omega^2(q_1^2 - 2q_0q_2) + q_0^2}. \end{aligned} \tag{5}$$

A necessary condition to have ω as a solution of (5) is that ω must be a root of the following equation:

$$F(\omega) := |P(\mathrm{i}\omega)|^2 - |Q(\mathrm{i}\omega)|^2 = \omega^6 + (p_2^2 - q_2^2 - 2p_1)\omega^4 + (2q_0q_2 - 2p_0p_2 - q_1^2 + p_1^2)\omega^2 + p_0^2 - q_0^2 = 0. \tag{6}$$

Let $z = \omega^2$. Then

$$F(z) = z^3 + a_F z^2 + b_F z + c_F = 0, \tag{7}$$

where $a_F = p_2^2 - q_2^2 - 2p_1$, $b_F = 2q_0q_2 - 2p_0p_2 - q_1^2 + p_1^2$, $c_F = p_0^2 - q_0^2$.

Given the reduced form of Eq. (7), we set $k = -\frac{a_F}{3}$, thus its discriminant will be

$$F_D = \frac{1}{4}[F(k)]^2 + \frac{1}{9}[F'(k)]^3. \tag{8}$$

We remind the reader that a cubic equation has three distinct real roots if $F_D < 0$. If $F_D > 0$ there exists one real root and two complex conjugate roots. If $F_D = 0$, then Eq. (7) has a root of multiplicity two or three.

Lemma 1. *Let $\epsilon > \frac{1}{2}$ be the share of tax revenues linked with the current income. Then the following cases can be discerned*:

(a) *If $a_F \geqslant 0$ or $b_F \leqslant 0$, then Eq. (7) has only one real positive root.*
(b) *If $a_F < 0$ and $b_F > 0$ and $F_D > 0$, then Eq. (7) has only one real positive root.*
(c) *If $a_F < 0$ and $b_F > 0$ and $F_D \leqslant 0$, then Eq. (7) has three real positive roots that are distinct if $F_D \neq 0$.*

Proof. Let us note that $\epsilon > \frac{1}{2}$ implies $c_F = p_0^2 - q_0^2 = \alpha^2\beta^2\tau^2 I_r^2(1-2\epsilon) < 0$. Therefore, in virtue of Descartes' rule, the three enunciated cases follow immediately. □

By similar arguments the following two Lemmas can be proved.

Lemma 2. *If $\epsilon < \frac{1}{2}$, then*:

(a) *$a_F \geqslant 0$ and $b_F \geqslant 0$ imply that (7) has no real positive roots.*
(b) *$a_F < 0$ or $b_F < 0$ and $F_D > 0$ imply that (7) has no real positive roots.*
(c) *$a_F < 0$ and $b_F < 0$ and $F_D \leqslant 0$ imply that (7) has two real positive roots which are distinct if $F_D \neq 0$.*

Lemma 3. *If $\epsilon = \frac{1}{2}$, then*:

(a) *$a_F \geqslant 0$ and $b_F \geqslant 0$ imply that (7) has no real positive roots.*
(b) *$b_F < 0$ imply that (7) has only one real positive root.*
(c) *$a_F < 0$ and $b_F = 0$ imply that (7) has only one real positive root.*
(d) *$a_F < 0$ and $b_F > 0$ imply that (7) has two real positive roots.*

Fig. 1 depicts the solutions generated by Eq. (7) with all the possible values of ϵ.

Assuming that ω^* is a solution of (6), let $\phi \in [0, 2\pi]$ a solution of

$$\begin{aligned}
\sin\phi &= \omega^* \frac{(\omega^*)^4 q_2 - (\omega^*)^2(q_0 - q_1p_2 + q_2p_1) + q_0p_1 - p_0q_1}{(\omega^*)^4 q_2^2 + (\omega^*)^2(q_1^2 - 2q_0q_2) + q_0^2}, \\
\cos\phi &= \frac{(\omega^*)^4(q_1 - q_2p_2) + (\omega^*)^2(q_0p_2 - q_1p_1 + p_0q_0) + p_0q_0}{(\omega^*)^4 q_2^2 + (\omega^*)^2(q_1^2 - 2q_0q_2) + q_0^2}.
\end{aligned} \tag{9}$$

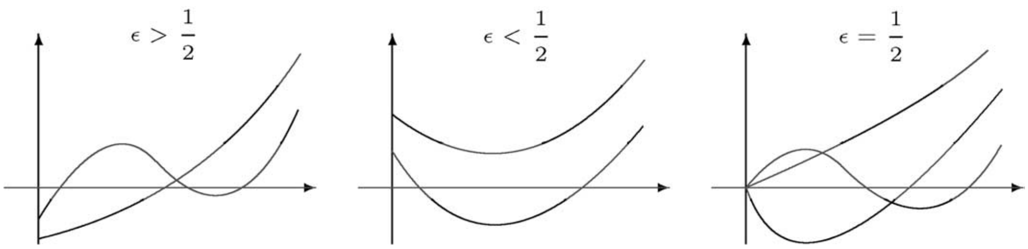

Fig. 1. Solutions generated by Eq. (7) with different values of ϵ.

Let us note that a solution ω^* of (6) is also a solution of the characteristic equation (4) if and only if

$$\theta^* = \frac{\phi(\omega^*) + n2\pi}{\omega^*} \quad n \in \mathbb{N}. \tag{10}$$

Therefore (see [3]) a pair of pure imaginary roots $\lambda = \pm i\omega$ cross the imaginary axis from left to right if $\frac{\mathrm{dRe}\lambda(\theta^*)}{\mathrm{d}\theta} > 0$. On the contrary, if that derivative is negative the crossing of imaginary axis is from right to left. It can be proved easily (see [3]) that the sign of $\frac{\mathrm{dRe}\lambda(\theta^*)}{\mathrm{d}\theta}$ is equal to the sign of $\frac{\mathrm{d}F(\omega^*)}{\mathrm{d}\omega}$ which, in its turn, equal to the sign of

$$3\omega^4 + 2a_F\omega^2 + b_F. \tag{11}$$

Theorem 1. *Suppose that the fixed point (Y^*, r^*, M^*) is locally asymptotically stable without time delay, then the following taxonomy holds:*

- *$F(z) = 0$ has no solutions, then no stability switches exists.*
- *$F(z) = 0$ has one or two solutions, then there exists only one stability switch.*
- *$F(z) = 0$ has three solutions, then there exists at least a stability switch.*

Proof. If Eq. (7) has no solutions the proof is trivial.

Suppose that (7) has a unique solution. Let ω^* be the unique positive root of Eq. (6) and ϕ^* the corresponding solution of (9). Since the polynomial (6) is an increasing function in a neighbourhood of ω^*, the derivative of $F(\omega)$ at ω^* is positive. This means that $\lambda = \pm i\omega^*$ crosses the imaginary axis from left to right. Hence a stability switch occurs at $\theta^* = \phi^*/\omega^*$.

Suppose now that (7) has two solutions. Let $\omega_1^* < \omega_2^*$ be the two positive roots of Eq. (6). It is trivial to prove that the polynomial (7) is a decreasing function in a neighbourhood of ω_1^* and an increasing function in a neighbourhood of ω_2^*. Therefore we have a crossing of imaginary axis from left to right in correspondence of ω_2^* and a crossing from right to left in correspondence of ω_1^*. Hence there exists one stability switch.

Suppose that (7) has three solutions. Let $\omega_1^* < \omega_2^* < \omega_3^*$ the three positive roots of the Eq. (6). It is trivial to prove that the polynomial (7) is an increasing function in a neighbourhood of ω_1^* and ω_3^* and is a decreasing function in a neighbourhood of ω_2^*. Therefore we have a crossing of imaginary axis from left to right in correspondence of ω_1^* and ω_3^* and a crossing from right to left in correspondence of ω_2^*. Hence there exists at least one stability switch. □

Similarly we can prove the following theorem.

Theorem 2. *Suppose that the fixed point (Y^*, r^*, M^*) is locally asymptotically unstable without time delay, then the following taxonomy holds:*

- *$F(z) = 0$ has no solutions or a unique solution then no stability switches exists.*
- *$F(z) = 0$ has two or three solutions then a lot of stability switches may exist.*

Let us note that, looking at the distribution of the points

$$\theta^*_{i,m_i} = \frac{\phi(\omega_i^*) + m_i 2\pi}{\omega_i^*} \quad \forall m_i \in \mathbb{N},$$

where i is the number of positive roots of Eq. (7), we can obtain the time delay region of stability or instability. Indeed, crossing (increasing θ) a point such that the corresponding expression (11) is positive, the roots of characteristic equation (4) with positive real part increase by two. Crossing (increasing θ) a point such that the corresponding expression (11) is negative, the roots of (4) with positive real part reduce by two. Therefore, the computation of roots with (negative) positive real part allows us to discern stability and instability regions.

We saw that, if some conditions hold, complex conjugate eigenvalues cross the imaginary axis with non-zero speed if the sign of expression (11) differs from zero. Therefore, by using the length of the lag as a bifurcation parameter, we can set the following theorem (see [13]):

Theorem 3. *If θ^* is a stability switch and $F_D \neq 0$ then an Hopf bifurcation occur at θ^*.*

Proof. If $F_D \neq 0$, F has no double roots hence his derivative does not vanish at ω^*. □

In the neighbourhood of θ^* the period of the orbit is close to $\frac{2\pi}{\omega^*}$.

4. The specific dynamical system and its numerical simulation

To perform our numerical simulation, we consider the following specific functions:

- Investment
$$I(Y(t), r(t)) = A\frac{[Y(t)]^a}{[r(t)]^b},$$
where $a, b > 0$, $A > 0$.
- Saving
$$S(Y^D) = sY^D(t),$$
where $0 < s < 1$.
- Liquidity preference
$$L(Y(t), r(t)) = L_1(Y(t)) + L_2(r(t)) = \gamma Y(t) + \frac{\lambda}{r(t) - \hat{r}},$$
where $\gamma, \lambda > 0$ and $\hat{r} > 0$ is a fixed very small rate of interest generating the liquidity trap as $r(t)$ falls to the level $\hat{r} > 0$ (i.e. $L_2(r(t)) \to +\infty$ as $r(t) \to \hat{r}$).

Taking into account these last three functions and given the equation (2), the specific dynamical system becomes:

$$\begin{cases} Y'(t) = \alpha\left\{A\frac{[Y(t)]^a}{[r(t)]^b} + g - s[(1-(1-\epsilon)\tau)Y(t) - \epsilon\tau Y(t-\theta)] - \tau[(1-\epsilon)Y(t) + \epsilon Y(t-\theta)]\right\}, \\ r'(t) = \beta\left[\gamma Y(t) + \frac{\lambda}{r(t)-\hat{r}} - M(t)\right], \\ M'(t) = g - \tau[(1-\epsilon)Y(t) + \epsilon Y(t-\theta)]. \end{cases} \tag{12}$$

4.1. Numerical results

Though we left out the case where no stability switch exists, the set of parameters linked with the system (12) may give rise to a lot of fairly different dynamic behaviours. [5] Nevertheless all the innumerable possible cases can be grouped. So that we have distinguished the case where, without time delay, the equilibrium point would be stable, from the one where, without time delay, the equilibrium point would be unstable. Furthermore, we have considered the cases where the set of parameters makes the equilibrium point a centre of closed orbits.

Before starting our numerical investigation, we would remind the reader that, given the income tax rate, the level of Y^* depends on the level of the public expenditure g. Therefore, government decisions about public expenditure are reflected on the system's dynamics.

We examine two case studies. In the first case we consider a fixed point locally asymptotically stable without time delay (see Theorem 1). In the other case we consider an unstable fixed point without time delay (see Theorem 2).

4.1.1. From stability to instability

When fixed parameters (see Table 1) are such that the fixed point is stable without time delay, statements of Theorem 1 hold. In this case two different situations can be examined. The former assumes a low taxation rate ($\tau = 10\%$), the latter a high taxation rate ($\tau = 40\%$). Furthermore in each situation, we have distinguished three different shares ϵ of delay tax revenues, i.e. 90%, 50%, 35%.

For each of the two selected tax rates, we have three shapes of the bifurcation curve in the parameter plane (θ, g), in accordance with the share of delay tax revenues (see Figs. 2 and 3). By comparing the three different cases of the two

[5] Numerical simulations have been performed using the package DDE-BIFTOOL v. 2.00 by Engelborghs and Roose [7].

Table 1
Fixed parameters

	A	a	b	γ	λ	$\hat{r}$	α	s	β
Stable	0.03	1.03	0.60	0.050	1	0.001	1	0.08	1
Unstable	0.38	1.05	0.83	0.077	1	0.003	0.96	0.30	1

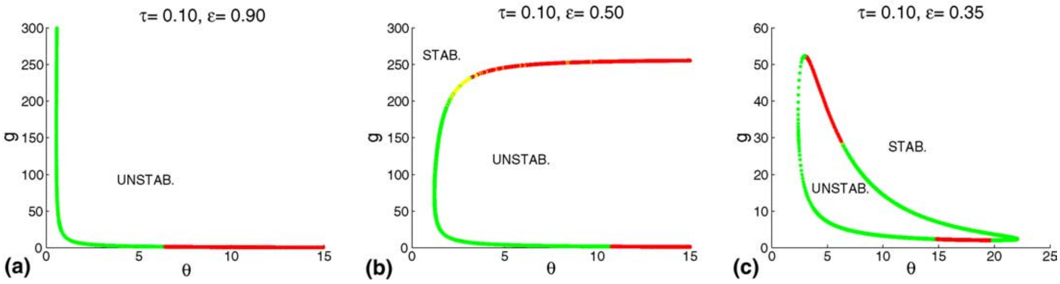

Fig. 2. Bifurcation curves with fixed τ and different values of ϵ.

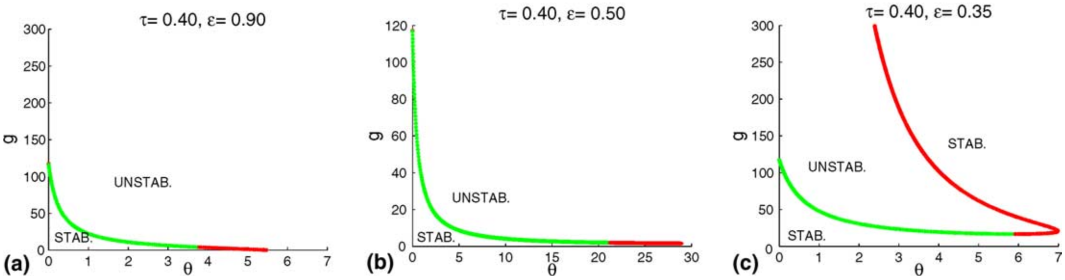

Fig. 3. Bifurcation curves with fixed τ and different values of ϵ.

figures, it is easy to check that the area of instability becomes less wide when the fraction of lagged tax revenues is reduced. Although this is an obvious result, from the point of view of policy makers, there are serious stabilisation problems if the tax rate and the share of lagged tax revenues give the bifurcation curve like an iperbola (here Figs. 2a, 3a and b). Even though we take into account the different scale on the g axis, cases of this sort need deep changes on the tax structure. Policy makers should reduce the lag θ and/or the share of postponed tax revenues ϵ to pursue stability conditions. In fact, given the income target and the tax structure, there are only few cases where a simple change of the tax rate may generate stable equilibria. Among the different situations here considered, when at the initial stage, the tax rate is $\tau = 40\%$ and the share of lagged tax rate is $\epsilon = 50\%$ or 35% (see Fig. 3b and c), a cut in the tax rate (e.g. at 10%) is able to fold up the bifurcation curve making the system stable, if the government expenditure (i.e. the income target) is reasonably high (see Figs. 2b and c). Money supply adjustments are implicit in these choices.

4.1.2. From instability to stability

When fixed parameters (see Table 1) are such that make the fixed point unstable without time delay, then statements of Theorem 1 hold. In this case, it can be detected that, given the tax rate τ, the time delay θ is able to generate some regions of stability, whose number is increasing as the public expenditure increases. As the Fig. 4 shows, the eigenvalue with initial positive higher real part eigenvalue becomes repeatedly negative as g increases and this allows the system to reach stable regions. Our numerical simulations confirmed that this system's behaviour is preserved for different value of the tax rate. Specifically, the higher the tax rate, is the higher values of public expenditure will have to be.

To set out this scenario, here we selected two different tax rates (10% and 20%) each together with two different shares ϵ (i.e. 75% and 50%) of delay tax revenues (see Figs. 5 and 6). In the plane (θ, g), all the bifurcation curves are like hyperbolas displayed in such a way as to separate unstable from stable regions. The alternation between unstable and stable areas implies that, given the lag and/or the share of postponed tax revenues, policy makers can stabilise the system combining reductions of the tax rate with right increases of public expenditure. As our model provides a pure money financing deficit, money supply expansion is also needed in this situation.

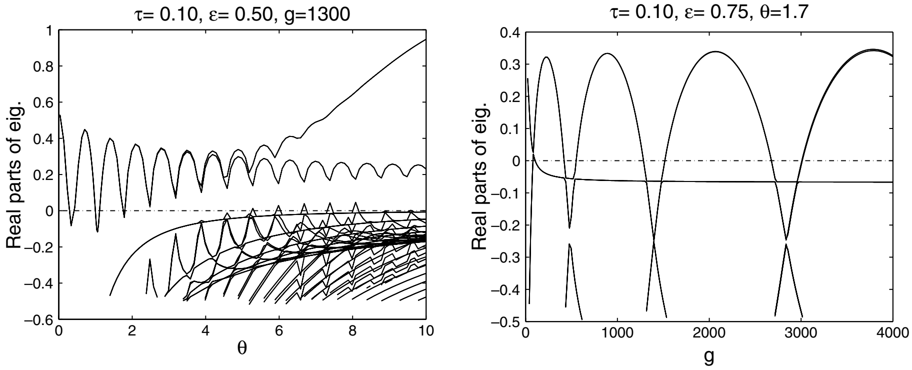

Fig. 4. Real parts of eigenvalues.

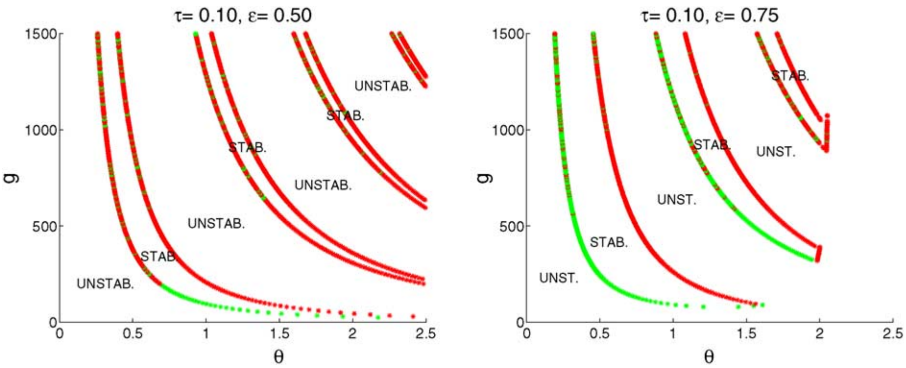

Fig. 5. Bifurcation curves with fixed τ and different values of ϵ.

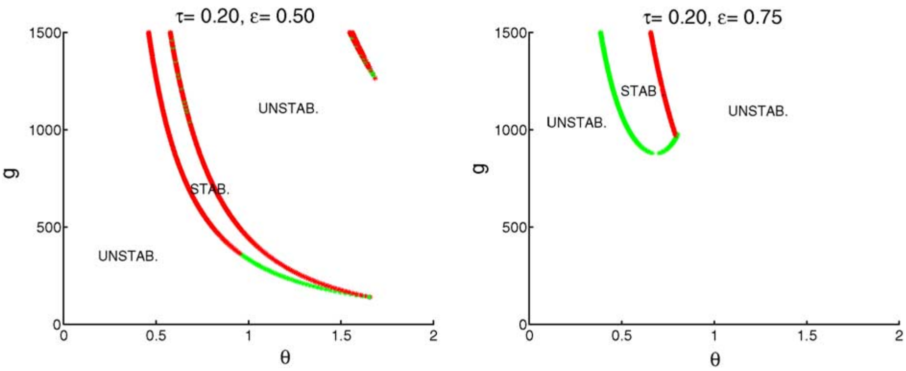

Fig. 6. Bifurcation curves with fixed τ and different values of ϵ.

4.1.3. Cyclical behaviours

Whatever the qualitative system's behaviour may be without lag, when θ are close to anyone of its bifurcation values θ^*, then our IS-LM model exhibits closed orbits. These are generated by sub-critical and supercritical Hops bifurcations, so that we may have unstable and stable limit cycles. If we refer to Figs. 2 and 3, moving θ from a stable to an unstable region and chosen g in such a way that it is possible to intercept the bifurcation curve, then, crossing this

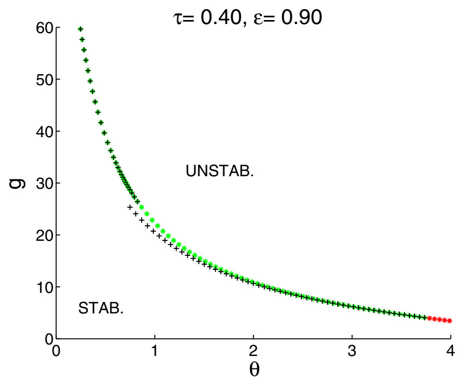

Fig. 7. Crater bifurcation diagram.

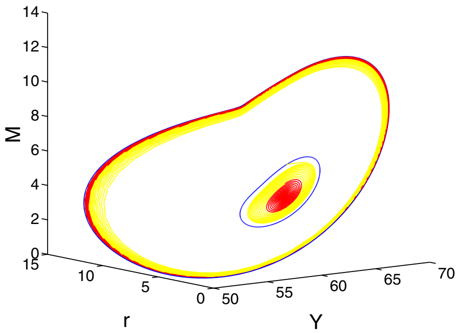

Fig. 8. Limit cycles.

curve through a green [6] zone, each $\theta \leqslant \theta^*$ in a neighbourhood of θ^* gives rise to a sub-critical Hopf bifurcation. So that closer to θ^* (at least one) closed orbits exist and trajectories starting at initial value in a neighbourhood of these orbits are repelled from them. Consequently, initial points outside the orbits' basin of attraction spiral away from the orbits; initial points inside the orbits' basin of attraction are attracted by the appropriate fixed point. In this case, from the economic point of view (cf. [2]), as a closed orbit defines an attractive region, the sub-critical Hopf bifurcation is related to the notion of "corridor stability", [7] in as much as long as an initial point is located inside this attractive region, it will stay in the corridor defined by the orbit and will eventually converge towards the equilibrium point.

When, given an appropriate g, an increasing θ allows us to pass from an unstable to a stable region, then crossing the bifurcation curve through a green zone, each $\theta > \theta^*$ makes unstable the fixed point and attracting orbits surrounding this point appear. This case of supercritical Hopf bifurcation is seen in economics as a stylized business cycle dynamics.

With reference to the red zone, outcomes of Hopf bifurcations are reversed. Passing from stability to instability supercritical bifurcations emerge; while bifurcations are sub-critical if we pass from an unstable to a stable region.

Our numerical simulation shows that there are only few cases where the nature of bifurcations is ambiguous. [8] In contrast, there are quite a lot of cases where the system exhibits multiple limit cycles. If there are two, as it happens for θ strictly enclosed in a neighbourhood of 1 with $\tau = 40\%$ and $\epsilon = 90\%$ (cf. the Fig. 7), the inner limit cycle is unstable, the outer one is stable. Situations like these are known as "crater bifurcation" [20, p. 67]. Fig. 8 depicts a crater obtained with the parameters τ and ϵ above said.

Likewise the sub-critical Hopf bifurcation, the economic interpretation of the crater is related to the concept of corridor stability (cf. [26,12]). The crater's peculiarity is that the equilibrium does not have to be unstable in order to gen-

[6] For interpretation of color in Figs. 2, 3, 5–8, the reader is referred to the web version of this article.
[7] The concept of corridor stability in economics was developed by Leijonhufvud [21] and refined by Howitt [14].
[8] Areas where ambiguity subsists are depicted as yellow zones in the bifurcation curves.

erate a business cycle dynamics. In our model inside a crater's basin of attraction an appropriate fiscal policy allows the government to choose between cyclical oscillations of economic activity, if these are not too large and violent, and a gradual convergence towards the equilibrium.

5. Conclusions

While the static IS-LM model is still seen as one of the main model useful for macro-economics' teaching purposes, its importance as a tool of stylized policy analysis has declined in the last decades. [9] On the contrary the dynamic version of this model has preserved its theoretical relevance and has been rarely considered for policy analysis. Its main purpose has been the explanation of endogenous cycles in the business activity. This paper is an attempt to model the persistence of endogenous fluctuations characterizing the macro-economic system together with a stylized policy analysis. Fluctuations in our model deals with the collection lags linked with the tax system. Consequently, as we have shown in the previous section, stabilisation policies by means of the traditional fiscal rules are not always efficacious. Often, as our numerical simulation suggests, appropriate structural change in the tax system is needed to make the fiscal policy useful for stability issues. If the model were carefully fitted to observed data and then capable of generating realistic values, surely it would be able to shed new light on the stability effects that fiscal policy may have on the real system's dynamics.

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