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A note on the generalized Rayleigh quotient for non-self-adjoint linear stability operators

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As a generalization of the classical Rayleigh–Ritz technique for self-adjoint systems, a variational method that appears capable of providing rapid estimates of the dispersion relation for complicated, non-self-adjoint linear stability operators is outlined. The method is illustrated by examining the linear stability of Taylor–Dean flow against general three-dimensional disturbances. Relatively simple trial functions provide surprisingly accurate estimates for the dispersion relation.

The basic equations governing the linear stability of a steady state in a large class of physical systems can be described by the following eigenvalue problem:

$$L\psi=\sigma M\psi \quad \text{with} \quad B\psi=0, \quad \psi\in\mathscr{H}, \tag{1}$$

where $\mathscr{H}$ is a Hilbert space containing an m-component vector perturbation function ψ and endowed with an appropriately defined scalar product. The symbols $L=L(\alpha,\beta)$ and $M=M(\alpha,\beta)$ denote linear partial differential operators that, in general, depend on the base flow, while β and α are vector parameters characterizing the base state and the perturbations, respectively, and B denotes a linear operator defining the associated linear boundary conditions satisfied by the perturbations.

To the eigenvalue problem (1), there corresponds a dispersion relation of the form

$$\sigma=f(\alpha,\beta). \tag{2}$$

In general, the eigenvalues σ's are complex, with real and imaginary parts equal, respectively, to the growth rates and frequencies of the perturbation eigenmodes. Derivation of the relation (2) is the major objective of linear stability analysis, which provides sufficient conditions for the base state to be unstable. One determines the marginal stability curve, the locus in β-α parameter space along which $\mathrm{Re}(\sigma)=0$, and this curve provides the minimum critical base-state parameters β_c for onset of instability and the corresponding parameters α_c's that represent the neutral mode.

We define L^* and M^* to be adjoints of operators L and M with respect to the suitably chosen scalar product (,) and let ϕ be the adjoint eigenfunctions satisfying

$$L^*\phi=\sigma^*M^*\phi \quad \text{with} \quad B^*\phi=0, \quad \phi\in\mathscr{H}, \tag{3}$$

where $B^*\phi$ denotes the associated boundary conditions.

If both operators L and M are self-adjoint, then σ is real and the classical Rayleigh quotient provides the variational formula

$$\sigma= min\frac{(\psi,L\psi)}{(\psi,M\psi)}, \quad \psi\in\mathscr{H}. \tag{4}$$

This provides a powerful method for estimating the extremal eigenvalue, since first-order errors in the eigenfunction will give second-order errors in the eigenvalue. By choosing a trial function ψ with one or two variational parameters, one can often obtain a rapid estimate. However, the Rayleigh–Ritz method provides by means of (4) a technique for systematically approximating the eigenvalue spectrum based on the selection of an appropriate trial function ψ, expressed in terms of basis functions that are members of a complete set and that satisfy the boundary conditions. The success of the variational method depends, of course, on how well the trial functions represent the form of the actual solution. For a more detailed discussion, many texts are available. See, e.g., Weinberger.[1]

When the operators L and M are non-self-adjoint, a weaker form of the variational formulation, succinctly stated by Morse and Feshbach,[2] still holds. This principle, apparently not well recognized by applied mathematicians (Weinberger,[1] pp. 149–150), has not been widely applied in the stability literature. The most common approach so far has been to treat the boundary-value problem by Galerkin or shooting methods, which in terms of effort is far from trivial.

In what follows, we will discuss the variational method for eigenvalues of non-self-adjoint operators based on the generalized Rayleigh quotient, then examine the linear stability of Taylor–Dean flow by this approach and show that the marginal stability curve agrees well with that of Mutazabi *et al.*[3] obtained by more detailed calculation. Before proceeding, we note that, if all the eigenvalues of non-self-adjoint operators L and M are known to be real (corresponding to the so-called principle of exchange of stabilities in the hydrodynamic-stability literature), one may be able to show that these operators are self-adjoint with respect to a modified scalar product. This can be achieved if, as sufficient condition, there exists an invertible bounded operator A such that the similarity transforms $A^{-1}LA$ and $A^{-1}MA$ are self-adjoint. Even though the choice of A is not always obvious, this generalization has important applications and has been recently utilized by Galdi and Straughan[4] to provide necessary and sufficient conditions for the finite-amplitude stability of a class of hydrodynamic-stability problems.

The general variational principle for non-self-adjoint

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problems can be stated as follows. The eigenvalue is given by the generalized Rayleigh quotient

$$\sigma = ext\, Q, \quad \text{where } Q = \frac{(\phi, L\psi)}{(\phi, M\psi)}, \quad \text{with } \psi,\phi \in \mathscr{H}, \tag{5}$$

where *ext* denotes a stationary point and where the first variation of Q, given by

$$\delta Q = \frac{[\delta\phi,(L-\sigma M)\psi] + [(L^* - \sigma^* M^*)\phi, \delta\psi],}{(\phi, M\psi)}, \tag{6}$$

vanishes. We note that the second variation is given by

$$\delta^2 Q = \frac{2[\delta\phi,(L-\sigma M)\delta\psi]}{(\phi, M\psi)}. \tag{7}$$

Whenever L and M are self-adjoint and positive-definite, the second variation is positive and, hence, (5) reduces to the Rayleigh quotient (4) and $ext \equiv min$. However, for non-self-adjoint operators, no definite statement can be made regarding the second variation. Still, one can obtain approximate eigenvalues using the principle of stationarity alone. As in the case of self-adjoint operators, we can choose the trial functions ϕ and ψ to be one- or two-parameter functions having a suitably simple form and satisfying the boundary conditions, in order to obtain a rapid estimate of the extremal eigenvalue. For a systematic estimate of the eigenvalue spectrum, we can choose the trial functions, in a manner analogous to the Rayleigh–Ritz method, as

$$\psi = \sum a_n s_n \quad \text{and} \quad \phi = \sum a_n^* h_n, \tag{8}$$

where s_n and h_n are members of complete sets that satisfy the boundary conditions of (1) and (3), respectively. The constants a_n^* and a_n are, in general, to be determined by the stationarity principle. When B^* and B are identical h_n and s_n can be chosen to be the identical without loss of generality.

We substitute these expressions for ψ and ϕ in (5) and then apply the stationarity condition implied by (6):

$$\frac{\partial\sigma}{\partial a_n} = 0, \quad \frac{\partial\sigma}{\partial a_n^*} = 0, \tag{9}$$

which yields the following algebraic equation for σ:

$$\det(L_{ij} - \sigma M_{ij}) = 0, \tag{10}$$

where

$$L_{ij} = (h_i, L s_j) \quad \text{and} \quad M_{ij} = (h_i, M s_j) \tag{11}$$

and the determinant is understood to be $Nm \times Nm$, where N is the number of terms in (8) and m is the dimension of ψ.

As in the Rayleigh–Ritz method, the number of eigenvalues so obtained is given by the number of members chosen from the complete set. Solving (10), we can obtain the dispersion relation (2). Thus this variational approach can provide an approximate dispersion relation, with relatively little effort, in hydrodynamic-stability problems, where the operators L and M are frequently variable-coefficient, multidimensional partial differential operators. The method is best illustrated by the simple example of hydrodynamic stability presented next.

Let us consider the linear stability problem of Taylor–Dean flow in the thin gap approximation, which has been recently analyzed by Mutazabi *et al.*[3] for general three-dimensional disturbances by means of a numerical shooting method. Employing their notation, the linear stability operator (1) is given here by

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} (D^2-q^2)^2 + ipTa[V(D^2-q^2)] & -2q^2 TaV \\ TaDV & (D^2-q^2) - ipTaV \end{bmatrix}, \tag{12}$$

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} (D^2-q^2) & 0 \\ 0 & 1 \end{bmatrix}, \tag{13}$$

where $D = d/dx$ and ψ are given in terms of the radial and azimuthal velocity perturbations u,v as

$$\psi = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix}. \tag{14}$$

The boundary conditions (1) are

$$u = Du = v = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1. \tag{15}$$

One relevant β parameter is the Taylor number Ta defined as

$$Ta = \frac{\Omega R d}{\nu} \left(\frac{d}{R} \right)^{1/2}, \tag{16}$$

where R is the radius of the inner cylinder, d is the gap between the two cylinders, and Ω is the rotation rate of the inner cylinder. When nondimensionalized by ΩR, the steady azimuthal base flow is given by

$$V(x) = 3(1+\mu)x^2 - 2(2+\mu)x + 1, \tag{17}$$

where $x = (r-R)/d$ and μ, the second β parameter, is the ratio of rotation rates of outer and inner cylinders. The parameters α correspond here to q and p, the respective axial and azimuthal wave numbers of the perturbation, and the dispersion relation obviously reduces in this case to

$$\sigma = f(\mu, Ta, p, q). \tag{18}$$

We choose trial functions ψ and ϕ of the form

$$\psi = \begin{bmatrix} \sum a_n f_n(x) \\ \sum b_n g_n(x) \end{bmatrix} \quad \text{and} \quad \phi = \begin{bmatrix} \sum a_n^* f_n(x) \\ \sum b_n^* q_n(x) \end{bmatrix}, \tag{19}$$

where f_n and g_n, given by

$$f_n = x^{2+n}(1-x)^2, \tag{20}$$

$$g_n = x^{1+n}(1-x), \tag{21}$$

satisfy the boundary conditions (15). We substitute these expressions for ψ and ϕ and (13) for L and M into (5) and

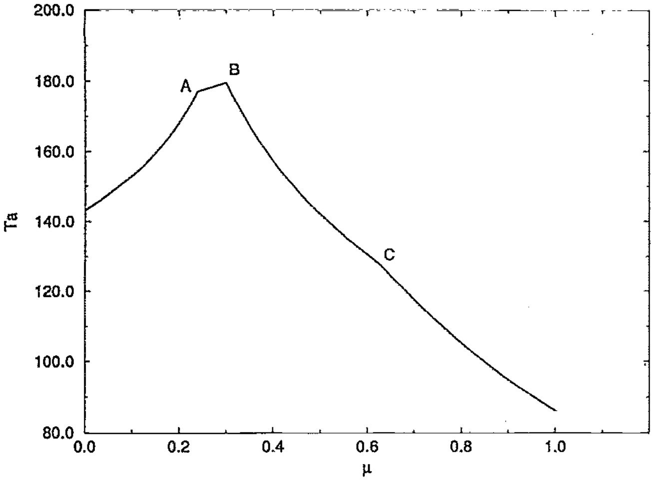

FIG. 1. Critical Taylor number Ta versus rotation ratio μ at marginal stability, corresponding to various branches of the solution.

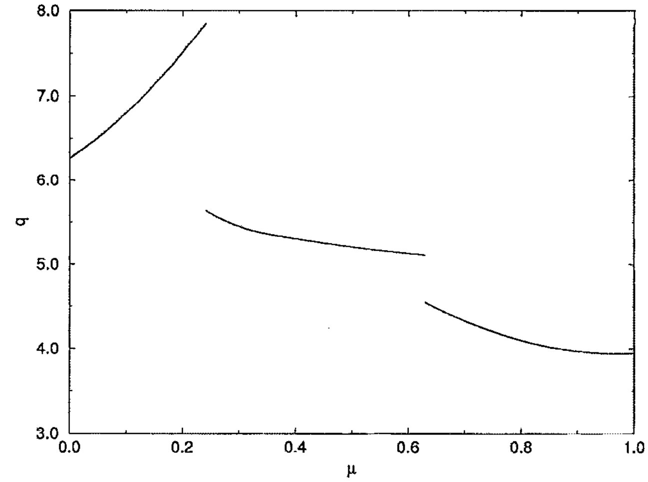

FIG. 2. Critical axial wave number q of the marginally stable mode versus rotation ratio μ corresponding to various branches of the solution.

apply the stationarity principle by differentiating with respect to the a_n, b_n, a_n^*, and b_n^*'s, to obtain the following equation for σ:

$$\det(\widetilde{L}_{ij}-\sigma\widetilde{M}_{ij})=0, \tag{22}$$

where

$$\widetilde{L}_{ij}=\begin{bmatrix}(f_i,L_{11}f_j) & (f_i,L_{12}g_j)\\ (g_i,L_{21}f_j) & (g_i,L_{22}g_j)\end{bmatrix}, \tag{23}$$

$$\widetilde{M}_{ij}=\begin{bmatrix}(f_i,M_{11}f_j) & (f_i,M_{12}g_j)\\ (g_i,M_{21}f_j) & (g_i,M_{22}g_j)\end{bmatrix}. \tag{24}$$

Then (22) provides us with the corresponding approximation to the dispersion relation (18). For the computational results presented here, we have chosen a two-term approximation, with $n=0$ and 1 in (20) and (21).

We solve Eq. (22) for the marginal stability curve to obtain the critical Ta for onset of instability as a function of rotation ratio μ and the corresponding values p, q, $\text{Im}(\sigma)$. The results of the computation are summarized in Fig. 1. We find for values of rotation ratio $\mu<0.24$ that the most unstable mode is axisymmetric and nonoscillatory, with $p=0$ and $\text{Im}(\sigma)=0$. For $0.24<\mu<0.31$, the most unstable modes are axisymmetric with $p=0$, but oscillatory, with $11.12<\text{Im}(\sigma)<17.91$. For $0.31<\mu<0.63$, the most unstable modes are nonaxisymmetric and oscillatory with $0.4<p<0.55$ and $11.12<\text{Im}(\sigma)<20.91$, whereas for $\mu>0.63$, they are always nonoscillatory and axisymmetric. These different branches correspond to different physical states and at their points of intersections (A,B,C) different physical states may coexist. Thus the critical points (A,B,C) are points of codimension-2, as noted by Mutazabi *et al.*[3]

In Fig. 2 the critical wave number q is plotted as a function of μ and is seen to exhibit discontinuities at the critical points A and C. As judged from a comparison with their Figs. 6 and 7, these computed results are within about 7% of those obtained by Mutazabi *et al.*[3] by means of more elaborate calculations, over the entire range of parameters, and they display the same qualitative features.

We conclude, therefore, that the generalized Rayleigh quotient has the potential to provide easy and reliable estimates of the dispersion relation for non-self-adjoint stability problems and, hence, may merit much wider application.

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