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Unbounded upper and lower solutions method for Sturm–Liouville boundary value problem on infinite intervals[☆]

Hairong Lian[a,b,*], Peiguang Wang[c], Weigao Ge[b]

[a] School of Information Engineering, China University of Geoscience, Beijing 100083, PR China
[b] Department of Mathematics, Beijing Institute of Technology, Beijing 100081, PR China
[c] College of Electronic and Information Engineering, Hebei University, Baoding 071002, PR China

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ABSTRACT

Unbounded upper and lower solutions theories are established for the Sturm–Liouville boundary value problem of a second order ordinary differential equation on infinite intervals. By using such techniques and the Schäuder fixed point theorem, the existence of solutions as well as the positive ones is obtained. Nagumo conditions play an important role in the nonlinear term involved in the first-order derivatives.

1. Introduction

Boundary value problems (BVPs) on infinite intervals, arising from the study of radially symmetric solutions of the nonlinear elliptic equation [1,4], have received much attention in recent years. There have been many existence results for some boundary value problems of differential equations on the half line, see [1–4,6–9,12] and the references therein.

It is well know that the upper and lower solution method is a powerful tool to prove the existence of solutions to a differential equation subject to certain boundary value problem, see [2–6,10–12] and the reference therein. In many cases, the upper and lower solutions are defined on compact intervals, so they are bounded. When applying this method to discuss the infinite intervals problem, the solutions are limited to the bounded case. see [2,3,5,6,10,11].

In [3], Agarwal and O'Regan discussed the Sturm–Liouville boundary value problem of the second-order differential equation

$$\begin{cases} \dfrac{1}{p(t)}\left(p(t)y'(t)\right)' = q(t)f\left(t, y(t)\right), & t \in (0, +\infty), \\ -a_0y(0) + b_0 \lim\limits_{t\to 0^+} p(t)y'(t) = c_0, \quad \text{or} \quad b_0 \lim\limits_{t\to 0^+} p(t)y'(t) = 0, \\ y(t) \text{ bounded on } [0, +\infty), \quad \text{or} \quad \lim\limits_{t\to +\infty} y(t) = 0, \end{cases}$$

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* Corresponding author at: School of Information Engineering, China University of Geoscience, Beijing 100083, PR China.
E-mail address: lianhr@126.com (H. Lian).

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where $a_0 > 0, b_0 \geqslant 0, C_0 \geqslant 0$. General existence criteria were obtained to guarantee the existence of bounded solutions. The methods used therein were based on diagonalization arguments and existence results of appropriate boundary value problems on finite intervals.

In [6], Eloe, Kaufmann and Tisdell studied the boundary value problem for the ordinary differential equation

$$\begin{cases} x''(t) - a(t)x(t) + f(t, x(t)) = 0, & t \in (0, +\infty), \\ x(0) = x_0, \quad x(t) \text{ bounded on } [0, +\infty). \end{cases}$$

By employing the degree theory and upper and lower solutions on compact domains, the authors obtained the existence of at least triple solutions on the sequential arguments.

In [12], Yan, Agarwal and O'Regan established a upper and lower solution theory for the boundary value problem

$$\begin{cases} y''(t) + \Phi(t)f\left(t, y(t), y'(t)\right) = 0, & t \in (0, +\infty), \\ ay(0) - by'(0) = y_0 \geqslant 0, & \lim\limits_{t\to+\infty} y'(t) = k > 0, \end{cases}$$

where $a > 0, b > 0$. By using the upper and lower solutions method, the authors presented sufficient conditions for the existence of unbounded positive solutions.

We cannot but ask if or not we can establish the existence of unbounded upper and lower solutions for infinite intervals problems, especially with full boundary conditions.

Motivated by the papers mentioned above, in this paper, we aim to establish the general unbounded upper and lower solution theory. Consider the Sturm–Liouville boundary value problem of the second order differential equation on the half line

$$\begin{cases} u''(t) + \phi(t)f\left(t, u(t), u'(t)\right) = 0, & t \in (0, +\infty), \\ u'(0) - au''(0) = B, \quad u''(+\infty) = C, \end{cases} \tag{1}$$

where $\phi : (0, +\infty) \to (0, +\infty), f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ are continuous, $a > 0, B, C \in \mathbb{R}$. Here we note that under weaker conditions than those in [12], the existence of positive solutions are also discussed.

This paper is organized as follows. In Section 2, some definitions and lemmas are presented. We establish an upper and lower solution theory for BVP (1) in Section 3. Sufficient conditions are given for the existence of solutions and positive solutions. An explicit example is given to illustrate our main results in the last section.

2. Preliminaries

In this section, we present some definitions and lemmas which are essential in the proof of our main results.

Definition 2.1. A function $\alpha \in C^1[0, +\infty) \cap C^2(0, +\infty)$ is called a lower solution of BVP (1) if

$$\begin{cases} \alpha''(t) + \phi(t)f\left(t, \alpha(t), \alpha'(t)\right) \geqslant 0, & t \in (0, +\infty), \\ \alpha(0) - a\alpha'(0) \leqslant B, \quad \alpha'(+\infty) < C. \end{cases} \tag{2}$$

Similarly, a function $\beta \in C^1[0, +\infty) \cap C^2(0, +\infty)$ is called an upper solution of BVP (1) if

$$\begin{cases} \beta''(t) + \phi(t)f\left(t, \beta(t), \beta'(t)\right) \leqslant 0, & t \in (0, +\infty), \\ \beta'(0) - a\beta''(0) \geqslant B, \quad \beta''(+\infty) > C. \end{cases} \tag{3}$$

Definition 2.2. Given a pair of functions $\alpha, \beta \in C^1[0, +\infty)$ satisfying $\alpha(t) \leqslant \beta(t), t \in [0, +\infty)$. A function $f[0, +\infty) \times R^2 \to R$ is said to satisfy the Nagumo condition with respect to the pair of functions α, β, if there exists a nonnegative function $\psi \in C[0, +\infty)$ and a positive one $h \in C[0, +\infty)$ such that

$$|f(t, x, y, z)| \leqslant \psi(t)h\left(|z|\right)$$

for all $0 \leqslant t < +\infty, \alpha(t) \leqslant x \leqslant \beta(t), y \in \mathbb{R}$ and

$$\int_0^{+\infty} \psi(s)\phi(s)\mathrm{d}s < +\infty, \qquad \int^{+\infty} \frac{s}{h(s)}\mathrm{d}s = +\infty.$$

Consider the space X defined by

$$X = \left\{x \in C^1[0, +\infty), \lim_{t\to+\infty} x'(t) \text{ exist}\right\}, \tag{4}$$

with the norm $\|x\| = \max\{\|x\|_1, \|x'\|_\infty\}$, where $\|x\|_1 = \sup_{t\in[0,+\infty)} \left|\frac{x(t)}{1+t}\right|$, $\|x'\|_\infty = \sup_{t\in[0,+\infty)} |x'(t)|$. By the standard arguments, we can prove that $(X, \|\cdot\|)$ is a Banach space.

Lemma 2.1. *If $e \in L^1[0,+\infty)$, then the Sturm–Liouville BVP of the second order linear differential equation*

$$\begin{cases} u''(t) + e(t) = 0, & t \in (0,+\infty), \\ u(0) - au'(0) = B, & u'(+\infty) = C, \end{cases} \tag{5}$$

has a unique solution in X. Moreover this solution can be expressed as

$$u(t) = aC + B + Ct + \int_0^{+\infty} G(t,s)e(s)\mathrm{d}s, \tag{6}$$

where

$$G(t,s) = \begin{cases} a+s, & 0 \leqslant s \leqslant t < +\infty, \\ a+t, & 0 \leqslant t \leqslant s < +\infty. \end{cases}$$

The proof is standard. So we omit it here.

Theorem 2.2 ([12]). *Let $M \subset X$, then M is relatively compact if the following conditions hold:*

(a) *all functions from M are uniformly bounded in X;*
(b) *the functions from $\{y : y = \frac{x}{1+t}, x \in M\}$ and $\{z : z(t) = x'(t), x \in M\}$ are equicontinuous on any compact interval of $[0,+\infty)$;*
(c) *the functions from $\{y : y = \frac{x}{1+t}, x \in M\}$ and $\{z : z(t) = x'(t), x \in M\}$ are equiconvergent at infinity, that is, for any $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that*

$$|y(t) - y(+\infty)| < \epsilon, \qquad |z(t) - z(+\infty)| < \epsilon,$$

for all $t > T$, and $x \in M$.

3. Main results

In this section, we will present the existence criteria for the existence of solutions and positive ones of BVP (1). We first cite the conditions (H_1) and (H_2) here.

(H_1): (1) BVP (1) has a pair of upper and lower solutions $\beta, \alpha \in X$ with

$$\alpha(t) \leqslant \beta(t), \quad t \in [0,+\infty).$$

(2) $f \in C\left([0,+\infty) \times \mathbb{R}^2, \mathbb{R}\right)$ satisfies the Nagumo condition with respect to α and β.

(H_2): $\phi \in L^1[0,+\infty)$ and there exists $\alpha > 1$ such that

$$\sup_{0 \leqslant t < +\infty} (1+t)^{\alpha}\phi(t)\psi(t) < +\infty.$$

Remark 3.1. Condition (H_2) holds when we have (H_1). And the function $\psi(t)$ is in the Nagumo condition.

Theorem 3.1. *Suppose the conditions (H_1) and (H_2) hold. Then BVP (1) has at least one solution $u \in C^1[0,+\infty) \cap C^2(0,+\infty)$ such that*

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \quad t \in [0,+\infty).$$

Proof. Let $\delta > 0$ and choose

$$\eta \geqslant \max\left\{ \sup_{t \in [\delta,+\infty)} \frac{\beta(t) - \alpha(0)}{t}, \sup_{t \in [\delta,+\infty)} \frac{\beta(0) - \alpha(t)}{t} \right\}$$

and $R > C$ such that

$$\int_{\eta}^{R} \frac{s}{h(s)}\mathrm{d}s \geqslant M\left(\sup_{t\in[0,+\infty)} \frac{\beta(t)}{(1+t)^{\alpha}} - \inf_{t\in[0,+\infty)} \frac{\alpha(t)}{(1+t)^{\alpha}} + \frac{\alpha}{\alpha-1} \cdot \sup_{t\in[0,+\infty)} \frac{\beta(t)}{1+t} \right),$$

where $M = \sup_{0 \leqslant t < +\infty}(1+t)^{\alpha}\phi(t)\psi(t)$.

Consider the boundary value problem

$$\begin{cases} u''(t) + \phi(t)f^*\left(t, u(t), u'(t)\right) = 0, & t \in (0,+\infty), \\ u(0) - au'(0) = B, & u''(+\infty) = C, \end{cases} \tag{7}$$

where

$$f^*(t,x,y,z)=\begin{cases}F_R\left(t,\alpha(t),y\right)+\dfrac{x-\alpha(t)}{1+|x-\alpha(t)|}, & x<\alpha(t),\\ F_R(t,x,y), & \alpha(t)\leqslant y\leqslant\beta(t),\\ F_R\left(t,\beta(t),y\right)-\dfrac{x-\beta(t)}{1+|x-\beta(t)|}, & x>\beta(t),\end{cases}$$

and

$$f_R(t,x,y,z)=\begin{cases}f\left(t,x,-R\right), & y<-R,\\ f(t,x,y), & -R\leqslant y\leqslant R\\ f\left(t,x,R\right), & y>R,\end{cases}$$

Step 1. BVP (7) has at least one solution u.

To this end, define the operator $T:X\to X$ by

$$(Tu)(t)=aC+B+Ct+\int_0^{+\infty}G(t,s)\phi(s)f^*\left(s,u(s),u'(s)\right)\mathrm{d}s.$$

By Lemma 2.1, we can see that the fixed points of T coincide with the solutions of BVP (7). So it is enough to prove that T has at least one fixed point.

We claim that $T:X\to X$ is completely continuous.

(1) $T:X\to X$ is well defined. For any $u\in X$, it holds

$$\int_0^{+\infty}\phi(s)f^*\left(s,u(s),u'(s)\right)\mathrm{d}s\leqslant\int_0^{+\infty}\phi(s)\left(H_0\psi(s)+1\right)\mathrm{d}s<+\infty,$$

where $H_0=\max_{0\leqslant s\leqslant\|u'\|_\infty}h(s)$. By Lebesgue dominated convergent theorem, we have

$$\begin{aligned}\lim_{t\to+\infty}\frac{(Tu)(t)}{1+t}&=\lim_{t\to+\infty}\left(\frac{aC+B+Ct}{1+t}+\int_0^{+\infty}\frac{G(t,s)}{1+t}\phi(s)f^*\left(s,u(s),u'(s)\right)\mathrm{d}s\right)\\&=C+\int_0^{+\infty}\phi(s)f^*\left(s,u(s),u'(s)\right)\mathrm{d}s<+\infty,\\ \lim_{t\to+\infty}(Tu)'(t)&=\lim_{t\to+\infty}\left(C+\int_t^{+\infty}\phi(s)f^*\left(s,u(s),u'(s)\right)\mathrm{d}s\right)\\&=C<+\infty,\end{aligned}$$

so $Tu\in X$.

(2) $T:X\to X$ is continuous. For any convergent sequence $u_n\to u$ in X, there exists $r_1>0$ such that $\sup_{n\in N}\|u_n\|\leqslant r_1$, where $u_0=u$. Similarly, we have

$$\int_0^{+\infty}\phi(s)\left|f^*\left(s,u_n(s),u_n'(s)\right)-f^*\left(s,u(s),u'(s)\right)\right|\mathrm{d}s\leqslant 2\int_0^{+\infty}\phi(s)\left(H_{r_1}\psi(s)+1\right)\mathrm{d}s<+\infty,$$

where $H_r=\max_{0\leqslant s\leqslant r}h(s)$. And then

$$\begin{aligned}\|Tu_n-Tu\|&=\max\{\|Tu_n-Tu\|_1,\|(Tu_n)'-(Tu)'\|_\infty\}\\&\leqslant\int_0^{+\infty}\max\left\{\sup_{0\leqslant t<+\infty}\left|\frac{G(t,s)}{1+t}\right|,1\right\}\phi(s)\left|f^*\left(s,u_n(s),u_n'(s)\right)-f^*\left(s,u(s),u'(s)\right)\right|\mathrm{d}s\\&\leqslant\int_0^{+\infty}\phi(s)\left|f^*\left(s,u_n(s),u_n'(s)\right)-f^*\left(s,u(s),u'(s)\right)\right|\mathrm{d}s\\&\to 0,\quad\text{as } n\to+\infty,\end{aligned}$$

so $T:X\to X$ is continuous.

(3) $T:X\to X$ is compact. Let B be any bounded subset of X, then there exists $r>0$ such that for any $u\in B$, it holds $\|u\|\leqslant r$. Then $\forall u\in B$, one has

$$\begin{aligned}\|Tu\|&=\max\{\|Tu\|_0,\|(Tu)'\|_\infty\}\\&\leqslant\int_0^{+\infty}\phi(s)\left|f^*\left(s,u(s),u'(s)\right)\right|\mathrm{d}s\\&\leqslant\int_0^{+\infty}\phi(s)\left(H_r\psi(s)+1\right)\mathrm{d}s<+\infty,\end{aligned}$$

so TB is uniformly bounded. Meanwhile, for any $T>0$, if $t_1,t_2\in[0,T]$, we have

$$\left|\frac{Tu(t_1)}{1+t_1}-\frac{Tu(t_2)}{1+t_2}\right|=\left|\int_0^{+\infty}\left(\frac{G(t_1,s)}{1+t_1}-\frac{G(t_2,s)}{1+t_2}\right)\phi(s)f^*\left(s,u(s),u'(s)\right)\mathrm{d}s\right|$$

$$\begin{aligned} &\leqslant \int_0^{+\infty} \left| \frac{G(t_1, s)}{1+t_1} - \frac{G(t_2, s)}{1+t_2} \right| \phi(s) \left(H_r \psi(s) + 1\right) \mathrm{d}s \\ &\to 0, \quad \text{as } t_1 \to t_2, \end{aligned}$$

and

$$\begin{aligned} \left|(Tu)'(t_1) - (Tu)'(t_2)\right| &= \left| \int_{t_1}^{t_2} \phi(s) f^*\left(s, u(s), u'(s)\right) \mathrm{d}s \right| \\ &\leqslant \int_{t_1}^{t_2} \phi(s) \left(H_r \psi(s) + 1\right) \mathrm{d}s \\ &\to 0, \quad \text{as } t_1 \to t_2, \end{aligned}$$

that is, TB is equicontinuous. From Theorem 2.2, we can see that if TB is equiconvergent at infinity, then TB is relatively compact. In fact,

$$\begin{aligned} \left| \frac{Tu(t)}{1+t} - \lim_{t\to+\infty} \frac{Tu(t)}{1+t} \right| &= \left| \frac{aC + B + Ct}{1+t} - C + \int_0^{+\infty} \left(\frac{G(t,s)}{1+t} - 1 \right) \phi(s) f^*\left(s, u(s), u'(s)\right) \mathrm{d}s \right| \\ &\leqslant \left| \frac{l(t)}{1+t} - C \right| + \int_0^{+\infty} \left| \frac{G(t,s)}{1+t} - 1 \right| \phi(s) \left(H_r \psi(s) + 1\right) \mathrm{d}s \\ &\to 0, \quad \text{as } t \to +\infty, \end{aligned}$$

$$\begin{aligned} \left|(Tu)'(t) - C\right| &= \left| \int_t^{+\infty} \phi(s) f^*\left(s, u(s), u'(s)\right) \mathrm{d}s \right| \\ &\leqslant \int_t^{+\infty} \phi(s) \left(H_r \psi(s) + 1\right) \mathrm{d}s \\ &\to 0, \quad \text{as } t \to +\infty, \end{aligned}$$

Then we can obtain that $T : X \to X$ is completely continuous.

By the Schäuder fixed point theorem, we can easily obtain that T has at least one fixed point $u \in X$.

Step 2. The function u satisfying $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, $t \in [0, +\infty)$.

Otherwise, if $u(t) \leqslant \beta(t)$, $t \in [0, +\infty)$ does not hold, then,

$$\sup_{0\leqslant t<+\infty} (u(t) - \beta(t)) > 0.$$

Because $u'(+\infty) - \beta'(+\infty) < 0$, so there are two cases.

Case 1. $\lim_{t\to 0^+} (u(t) - \beta(t)) = \sup_{0\leqslant t<+\infty} (u(t) - \beta(t)) > 0$.

Easily, it holds $u'(0^+) - \beta'(0^+) \leqslant 0$. While by the boundary condition, we have

$$u(0) - \beta(0) \leqslant a(u'(0) - \beta'(0)) \leqslant 0,$$

which is a contradiction.

Case 2. There exists $t^* \in (0, +\infty)$ such that

$$u(t^*) - \beta(t^*) = \sup_{0\leqslant t<+\infty} (u(t) - \beta(t)) > 0.$$

So we have $u'(t^*) - \beta'(t^*) = 0$, $u''(t^*) - \beta''(t^*) \leqslant 0$. Unfortunately,

$$\begin{aligned} u''(t^*) - \beta''(t^*) &\geqslant \phi(t^*) \left(f\left(t^*, \beta(t^*), \beta'(t^*)\right) - f^*\left(t^*, u(t^*), u'(t^*)\right) \right) \\ &= \phi(t^*) \frac{u(t^*) - \beta(t^*)}{1 + |u(t^*) - \beta(t^*)|} \\ &> 0. \end{aligned}$$

Which is also a contradiction.

Consequently, $u(t) \leqslant \beta(t)$ holds for all $t \in [0, +\infty)$. Similarly, we can show that $\alpha(t) \leqslant u(t)$ for all $t \in [0, +\infty)$.

Step 3. The function u is a solution of BVP (1).

In fact, we show that $|u'(t)| \leqslant R$, $t \in [0, +\infty)$ from the following three cases.

Case 1. $|u'(t)| > \eta$, $\forall t \in [0, +\infty)$.

Without loss of generality, we suppose $u'(t) > \eta$, $t \in [0, +\infty)$. While for any $T \geqslant \delta$,

$$\frac{\beta(T) - \alpha(0)}{T} \geqslant \frac{u(T) - u(0)}{T} = \frac{1}{T} \int_0^T u'(s) \mathrm{d}s > \eta \geqslant \frac{\beta(T) - \alpha(0)}{T},$$

which is a contradiction. so there must exist $t_0 \in [0, +\infty)$ such that $|u'(t_0)| \leqslant \eta$.

Case 2. $|u'(t)| \leqslant \eta$, $\forall t \in [0, +\infty)$.

Just take $R = \eta$ in the beginning and we can complete the proof.

Case 3. There exists $[t_1, t_2] \subset [0, +\infty)$ such that $|u'(t_1)| = \eta$, $|u'(t)| > \eta$, $t \in (t_1, t_2]$ or $|u'(t_2)| = \eta$, $|u'(t)| > \eta$, $t \in [t_1, t_2)$.

Without loss of generality, we suppose that $u'(t_1) = \eta$, $u'(t) > \eta$, $t \in (t_1, t_2]$. Obviously,

$$\begin{aligned}
\int_{u'(t_1)}^{u'(t_2)} \frac{s}{h(s)} \mathrm{d}s &= \int_{t_1}^{t_2} \frac{u'(s)}{h\left(u'(s)\right)} u''(s) \mathrm{d}s \\
&= \int_{t_1}^{t_2} \frac{-\phi(s) f\left(s, u(s), u'(s)\right) u''(s)}{h\left(u'(s)\right)} \mathrm{d}s \\
&\leqslant \int_{t_1}^{t_2} u'(s)\phi(s)\psi(s)\mathrm{d}s \leqslant M \int_{t_1}^{t_2} \frac{u'(s)}{(1+s)^{\alpha}} \mathrm{d}s \\
&= M\left(\int_{t_1}^{t_2} \left(\frac{u(s)}{(1+s)^{\alpha}}\right)' \mathrm{d}s + \int_{t_1}^{t_2} \frac{\alpha u(s)}{(1+s)^{1+\alpha}} \mathrm{d}s\right) \\
&\leqslant M\left(\sup_{t\in[0,+\infty)} \frac{\beta(t)}{(1+t)^{\alpha}} - \inf_{t\in[0,+\infty)} \frac{\alpha(t)}{(1+t)^{\alpha}} + \sup_{t\in[0,+\infty)} \frac{\beta(t)}{1+t} \int_0^{+\infty} \frac{\alpha}{(1+t)^{\alpha}} \mathrm{d}s\right) \\
&\leqslant \int_{\eta}^{R} \frac{s}{h(s)} \mathrm{d}s,
\end{aligned}$$

which concludes that $u'(t_2) \leqslant R$. For t_1 and t_2 are arbitrary, we obtain that if $u'(t) \geqslant \eta$, then $u'(t) \leqslant R$, $t \in [0, +\infty)$.

Similarly, we can also obtain that if $u'(t_1) = -\eta$, $u'(t) < -\eta$, $t \in (t_1, t_2]$, then $u'(t) > -R$, $t \in [0, +\infty)$. So,

$$u''(t) = -f^*\left(t, u(t), u'(t)\right) = -f\left(t, u(t), u'(t)\right)$$

that is, u is a solution of BVP (1). □

If $f : [0, +\infty)^4 \to [0, +\infty)$, we can establish a criteria for the existence of positive solutions.

Theorem 3.2. *Let $f : [0, +\infty)^3 \to [0, +\infty)$ be continuous and $\phi \in L^1[0, +\infty)$. Suppose the following conditions hold.*

(H_3) BVP (1) *has a pair of positive upper and lower solutions $\beta, \alpha \in X$ satisfying*

$$\alpha(t) \leqslant \beta(t), \quad t \in [0, +\infty).$$

(H_4) *For any $r > 0$, there exists φ_r such that for $\alpha(t) \leqslant x \leqslant \beta(t)$, $0 \leqslant y \leqslant r$, we have*

$$f(t, x, y) \leqslant \varphi_r(t), \quad t \in [0, +\infty),$$

and

$$\int_0^{+\infty} \phi(s)\psi(s)\mathrm{d}s < +\infty.$$

Then BVP (1) *with $B, C \geqslant 0$ has at least one solution such that*

$$\alpha(t) \leqslant u(t) \leqslant \beta(t) \quad t \in [0, +\infty).$$

Proof. Choose $R = \frac{1}{a}\left(B + \beta(0)\right)$ and consider the boundary value problem (7) except f_R substituting by

$$f_R(t, x, y) = \begin{cases} f\left(t, x, 0\right), & y < 0, \\ f(t, x, y), & 0 \leqslant y \leqslant R \\ f\left(t, x, R\right), & y > R. \end{cases}$$

Similarly, we can obtain that (7) has at least one solution u satisfying $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, $t \in [0, +\infty)$. Because

$$u''(t) = -\phi(t) f^*\left(t, u(t), u'(t)\right) \leqslant 0$$

and $u'(+\infty) = C \geqslant 0$, we have

$$0 \leqslant u'(t) \leqslant u'(0) = \frac{1}{a}\left(B + u(0)\right) \leqslant R.$$

Consequently, the solution u is a positive solution of (1). □

Remark 3.2. The conditions in Theorem 3.3 are weaker than those of Theorem 3.1 presented in [12].

4. Example

In this section, we provide an explicit example to illustrate our main results.
Consider the boundary value problem

$$\begin{cases} x''(t) - \mathrm{e}^{-\gamma t}\arctan(x'(t))\left(2(t-x(t)) + (1-x'(t))^2\right) = 0, \quad t \in (0,+\infty), \\ x(0) = 0, \quad x'(+\infty) = 0. \end{cases} \tag{8}$$

Conclusion: BVP (8) has at least one solution.

Proof. Set

$$\phi(t) = \mathrm{e}^{-\gamma t}, \qquad f(t,u,v) = \arctan v\left(2(t-u) + (1-v)^2\right).$$

It is easy to prove that $\alpha(t) = -t$, $\beta(t) = t$ are a pair of upper and lower solutions of (8). Moreover, $\alpha, \beta \in X$, $\alpha(t) < \beta(t)$, $t \in [0,+\infty)$.

Meanwhile, when $0 \le t < +\infty$, $-t \le u \le t$, $-1 \le v \le 1$, it holds

$$\begin{aligned} |f(t,u,v)| &= \left|\arctan v\left(2(t-u) + (1-v)^2\right)\right| \\ &\le 4(1+t)|v|. \end{aligned}$$

If we choose $h(s) = s$, $\psi(t) = 4(1+t)$, then

$$\begin{aligned} &|f(t,u,v)| \le \psi(t)h(|v|), \\ &\int^{+\infty} \frac{s}{h(s)} = \int^{+\infty} \mathrm{d}s = +\infty, \\ &\int_0^{+\infty} \phi(s)\psi(s)\mathrm{d}s = 4\int_0^{+\infty} \mathrm{e}^{-\gamma s}(1+s)\mathrm{d}s < +\infty. \end{aligned}$$

that is, f satisfies the Nagumo condition with respect to $-t$ and t. And for any fixed constant $\alpha > 1$,

$$\begin{aligned} \sup_{0\le t<+\infty} (1+t)^\alpha \phi(t)\psi(t) &= 4 \sup_{0\le t<+\infty} (1+t)^{\alpha+1}\mathrm{e}^{-\gamma t} \\ &\le 4\max\left\{\left(\frac{\alpha+1}{\gamma}\right)^{\alpha+1}, 1\right\} < +\infty, \end{aligned}$$

So by Theorem 3.1, we have that (8) has at least one solution. □

References

[1] R.P. Agarwal, D. O'Regan, Infinite Interval Problems for Differential, Difference and Integral Equations, Kluwer Academic Publisher, 2001.
[2] R.P. Agarwal, D. O'Regan, Nonlinear boundary value problems on the semi-infinite interval: An upper and lower solution approach, Mathematika 49 (2002) 129–140.
[3] R.P. Agarwal, D. O'Regan, Infinite interval problems modeling phenomena which arise in the theory of plasma and elesrical potential theory, Stud. Appl. Math. 111 (2003) 339–358.
[4] S.W. Chen, Y. Zhang, Singular boundary value problems on a half-line, J. Math. Anal. Appl. 195 (1995) 449–468.
[5] J. Ehme, P.W. Eloe, J. Henderson, Upper and lower solution methods for fully nonlinear boundary value problems, J. Differential Equations 180 (2002) 51–64.
[6] P.W. Eloe, E.R. Kaufmann, C.C. Tisdell, Multiple solutions of a boundary value problem on an unbounded domain, Dynam. Systems Appl. 15 (1) (2006) 53–63.
[7] J.M. Gomes, J.M. Sanchez, A variational approach to some boundary value problems in the half-line, Z. Angew. Math. Phys. 56 (2005) 192–209.
[8] G.Sh. Guseinov, I. Yaslan, Boundary value problems for second order nonlinear differential equations on infinite intervals, J. Math. Anal. Appl. 290 (2004) 620–638.
[9] Y.S. Liu, Boundary value problems for second order differential equations on infinite intervals, Appl. Math. Comput. 135 (2002) 211–216.
[10] H.B. Thomopson, Second order ordinary differential equations with fully nonlinear two point boundary conditions, Pacific J. Math. 172 (1996) 255–276.
[11] R.A. Khan, J.R.L. Webb, Existence of at least three solutions of a second-order three-point boundary value problem, Nonlinear Anal. 64 (2006) 1356–1366.
[12] B.Q. Yan, D. O'Regan, R.P. Agarwal, Unbounded solutions for singular boundary value problems on the semi-infinite interval: Upper and lower solutions and multiplicity, J. Comput. Appl. Math. 197 (2006) 365–386.