Methods of analyzing planar optical waveguides

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We present a new approximate solution of the scalar-wave equation for planar optical waveguides with arbitrary refractive-index profiles. Test calculations are done for an index profile with a known solution. The comparison demonstrates the accuracy of our method. The method may also be applied to circularly symmetric optical fibers.

The analysis of optical waveguides with an arbitrary index distribution is of considerable importance in integrated optics. For index profiles for which exact solutions are not known, we must resort to numerical or approximate methods. For waveguides that support a large number of guided modes, the WKB approximation[1] may be used, but it is not applicable close to a turning point or for single-mode waveguides.

We present here an approximate method that gives accurate results even for single-mode waveguides and is applicable close to and at the turning points. Solutions similar to those given here have been used[2] to describe formal series solutions and uniform asymptotic expansions of solutions of ordinary differential equations. Use of first-order perturbation theory along with our method greatly improves the accuracy of the calculated propagation constant.

Even though the method is applicable to circularly symmetric optical fibers with arbitrary index profiles, we consider here an asymmetric graded planar waveguide. As we demonstrate, our approximation agrees extremely well with the exact solution.

Consider the scalar-wave equation

$$\frac{d^2\psi}{dx^2} + \kappa^2(x)\psi(x) = 0, \tag{1}$$

where $\kappa^2 = k^2n^2(x) - \beta^2$, k is the free-space wave number, $n(x)$ is the x-dependent refractive index, and β is the propagation constant.

We consider the following index profile:

$$\begin{aligned} n^2(x) &= n_2^2 + (n_1^2 - n_2^2)\exp(-x/d) \quad \text{for } x > 0 \\ &= n_c^2 \quad \text{for } x < 0 \end{aligned}, \tag{2}$$

where n_1, n_2, and n_c are the refractive indices shown in Fig. 1. Using Eqs. (1) and (2), we get

$$\frac{d^2\psi}{dX^2} + V^2[\exp(-X) - b]\psi = 0 \quad \text{for } X > 0, \tag{3a}$$

$$\frac{d^2\psi}{dX^2} - V^2(b + B)\psi = 0 \quad \text{for } X < 0, \tag{3b}$$

where

$$V^2 = k^2d^2(n_1^2 - n_2^2), \quad X = x/d, \quad B = \frac{n_2^2 - n_c^2}{n_1^2 - n_2^2}, \tag{3c}$$

$$b = \frac{n_e^2 - n_2^2}{n_1^2 - n_2^2}, \quad n_e = \beta/k. \tag{3d}$$

The exact solution of Eqs. (3) is given by[3]

$$\psi(X) = \begin{cases} \dfrac{J_\nu[2V\exp(X/2)]}{J_\nu(2V)} & \text{for } X > 0 \\ \exp[V(b+B)^{1/2}X] & \text{for } X < 0 \end{cases}, \tag{4a}$$

where

$$\nu = 2V\sqrt{b}. \tag{4b}$$

In writing Eqs. (4) we have used the fact that for a guided mode $\psi(X) \to 0$ as $X \to \pm\infty$, and the multiplying constants have been chosen so that $\psi(0) = 1$. The continuity of $\psi'(X)$ at $X = 0$ will yield the following eigenvalue equation for the normalized propagation constant b:

$$\frac{J_\nu'(2V)}{J_\nu(2V)} = -\left(\frac{n_e^2 - n_c^2}{n_1^2 - n_2^2}\right)^{1/2}. \tag{5}$$

Since the solution of Eq. (1) can be stated in terms of

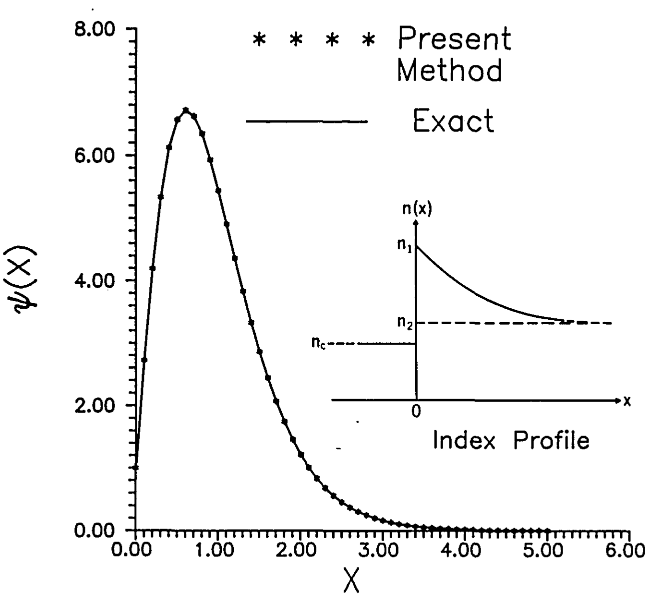

Fig. 1. Electromagnetic field as a function of X for $V = 4$. The index profile is shown in the inset.

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Airy functions[2] when $\kappa^2(x)$ is a linear function of x, we look for a solution of the form

$$\psi(x) = \begin{cases} F(x)Ai[\xi(x)] \\ G(x)Bi[\xi(x)] \end{cases}, \tag{6}$$

where $\xi(x)$, $F(x)$, and $G(x)$ are to be determined.

Using Eq. (6) in Eq. (1) and neglecting the term proportional to $F''(x)$, we get

$$\xi(x)[\xi'(x)]^2 + \kappa^2(x) = 0, \tag{7a}$$

$$2F'(x)Ai'(\xi)\xi' + F(x)A'(\xi)\xi'' = 0. \tag{7b}$$

Thus

$$F(x) = \frac{\text{const.}}{[\xi'(x)]^{1/2}}, \quad \xi(x) = \left\{\frac{3}{2}\int_{x_t}^{x} [-\kappa^2(x)]^{1/2}\mathrm{d}x\right\}^{2/3}, \tag{8}$$

where x_t is the turning point, i.e., $\kappa(x_t) = 0$. We will get a similar expression for $G(x)$. We thus find the following approximate solution of Eq. (1):

$$\psi(x) \cong [C_1 Ai(\xi) + C_2 Bi(\xi)](\xi')^{-1/2}. \tag{9}$$

The solution of Eq. (3a) using the present method [see relation (9)] is thus written as

$$\psi(X) = \left(\frac{\xi_0'}{\xi'}\right)^{1/2} \frac{Ai(\xi)}{Ai(\xi_0)} \quad \text{for } X > 0, \tag{10}$$

where $\xi(X)$ is given by [see Eq. (8)]

$$\begin{aligned} \xi &= -\left\{3V[\exp(-X) - b]^{1/2} - b^{1/2}\tan^{-1} \times \left[\frac{\exp(-X)}{b} - 1\right]^{1/2}\right\}^{2/3} \quad \text{for } X < X_t, \\ &= \left\{3V - [b - \exp(-X)]^{1/2} + \frac{b^{1/2}}{2} \times \ln\frac{b^{1/2} + [b - \exp(-X)]^{1/2}}{b^{1/2} - [b - \exp(-X)]^{1/2}}\right\}^{2/3} \quad \text{for } X > X_t. \end{aligned} \tag{11}$$

X_t is the turning point, i.e., $b = \exp(-X_t)$, and the subscript 0 indicates the value at $X = 0$.

The field given by Eq. (10) is our proposed solution to the stated problem. It satisfies the boundary condition $\psi(\pm\infty) \to 0$ at infinity and the continuity condition $\psi(0) = 1$ at $X = 0$. Our Eq. (8), on which the solution relies, indicates that truly arbitrary index profiles require numerical integration. This is not surprising and should be expected when one deals with an arbitrary profile. There are, of course, special cases for which Eq. (8) is integrable in closed form.

The continuity of $\psi'(X)$ at $X = 0$ will yield the eigenvalue equation

$$\frac{V(B + b)^{1/2}}{\xi_0'} = \frac{Ai'(\xi_0)}{Ai(\xi_0)} - \frac{1}{2}\frac{\xi''}{\xi_0'^2}. \tag{12}$$

If follows from Eqs. (3a), (7a), and (10) that $\psi(X)$ given by Eq. (10) is an exact solution of the differential equation,

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}X^2} + V^2[\exp(-X) - b]\psi + \left(\frac{1}{2}\frac{\xi'''}{\xi'} - \frac{3}{4}\frac{\xi''^2}{\xi'^2}\right)\psi = 0. \tag{13}$$

This solution will be a good approximation to the solution of Eq. (3a) provided that the last term in Eq. (13) is small. This is so in practical cases, for which ξ''' and ξ'' are small, and this will be corroborated by means of the present example. Comparing Eqs. (13) and (3a) and considering the last term in Eq. (13) as a perturbation, we get a first-order correction δb to the normalized propagation constant,[4]

$$\delta b \cong \frac{\displaystyle\int_0^\infty \left(\frac{3}{4}\frac{\xi''^2}{\xi'^2} - \frac{1}{2}\frac{\xi''^2}{\xi'}\right)\psi^2 \mathrm{d}X}{\displaystyle V^2\int_{-\infty}^{\infty} \psi^2(X)\mathrm{d}X}. \tag{14}$$

We have used[5,6] $n_2 = 2.177$, $n_c = 1.0$, and $n_1^2 - n_2^2 = 0.187$ in our calculations. The value of d has been varied to obtain results as a function of V. The calculated values of b are given in Table 1 with the exact values [Eq. (5)] and those calculated by the WKB method. The agreement is good compared with the WKB method for all values of V. The correction given by relation (14) gives extremely accurate eigenvalues. The error is only ~0.03% even at $V = 1.5$, i.e., near cutoff. The error Δb in b is related to error $\Delta\beta$ in β by the relation [see Eq. (3d)]

$$\frac{\Delta b}{b} \cong \frac{\Delta\beta}{\beta}\frac{n_1}{n_1 - n_2}\frac{1}{b}. \tag{15}$$

Thus an error of 0.03% in b in this case will correspond to an error of ~0.00002% in β. In Fig. 1 we plotted ψ versus X for $V = 4$ as calculated by Eq. (10) (the present method) and Eq. (4) (the exact values). The figure shows no discernable difference between the two curves, even at the turning point.

Because our approximate solution [relations (9) and (10)] uses the Airy functions, it might be confused with the traditional WKB solution that uses Airy functions in the connection formulas to handle the turning points. On the contrary, we give here a single closed-form solution that holds for all x, including at and in the vicinity of the turning points, and it is amenable to a desktop computer, on which the Airy functions are easily evaluated. If the refractive index is truly arbitrary, numerical integration over the nonuniform re-

Table 1. Normalized Propagation Constants for Different Values of V

V	Exact [Eq. (5)]	Present Method [Eq. (12)]	Present Method with Correction [relation (14)]	WKB
1.5	0.035007	0.036088	0.035017	0.037833
2.0	0.104954	0.105896	0.105028	0.108613
2.5	0.171442	0.172159	0.171520	0.175311
3.0	0.299188	0.229736	0.229260	0.233076
3.5	0.278650	0.279081	0.278717	0.282486
4.0	0.321179	0.321520	0.321235	0.324927
5.0	0.390292	0.390522	0.390366	0.393845
6.0	0.444075	0.444241	0.444110	0.447436
7.0	0.487244	0.487369	0.487272	0.490429
8.0	0.522776	0.522874	0.522800	0.525803

gion is always required; in this sense, our solution is typical [see Eq. (8)]. Even without the first-order correction of relation (14), our method gives excellent agreement with the exact value of b (see Table 1). Although the use of relation (14) requires additional computational complexity, we know of no other method that yields accuracy in β to 0.00002% at $V = 1.5$. Our solution for b, using the first-order correction of relation (14), is nearly as accurate at $V = 1.5$ as it is at $V = 5.0$.

Our method gives the electromagnetic field (directly in terms of Airy functions) as well as the eigenfunctions. It is simpler than alternative numerical methods, especially if the accuracy of Eq. (12) is adequate. Variational methods require complicated numerical integrations, and finite-element methods are difficult to program. Expansion methods that use orthogonal functions usually require the eigenfunctions of a matrix, which usually cannot be obtained on a desktop computer unless only a few terms are used, in which case the accuracy is jeopardized. If the refractive index is arbitrary, the numerical integration complicates our analysis but no more so than for any other method.

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