

A (gentle) Introduction to Sturm-Liouville Problems

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Sturm-Liouville Theory

- ① Mathematicians have studied Sturm-Liouville problems for over 200 years.
- ② Highly developed theory and remains an active area of interest.
- ③ Major results:
 - a. A complete “spectral theory” for problems where $[a, b]$ is finite and the potential $q(x)$ is nice.
 - b. A satisfying but complex theory for “singular” problems.
- ④ Value: Solutions to separable PDEs, major advances in spectral theory, various applications in physics.

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A Sturm-Liouville Problem

A *Sturm-Liouville problem* (SLBVP) on $[a, b] \subset \mathbb{R}$ is a second order ODE with boundary conditions at a and b :

$$\begin{cases} -u''(x) + q(x)u(x) = \lambda u(x) & x \in [a, b] \\ u(a) \cos(\alpha) + u'(a) \sin(\alpha) = 0 \\ u(b) \cos(\beta) + u'(b) \sin(\beta) = 0 \end{cases}$$

- ① We assume that $p(x), q(x)$ are real.
- ② Very general equation: every second order differential operator can be put into a Sturm-Liouville form.
- ③ The parameter λ suggests that our focus will be a kind of eigenvalue problem. The theory of this type of eigenproblem is called **Sturm-Liouville theory**.
- ④ A form of the time-independent Schrödinger equation for energy level λ .

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The Nonsingular SL Problem

Introduce the operator notation

$$L(u) := -u''(x) + q(x)u(x).$$

The **nonsingular SL problem** is

$$\begin{cases} Lu = \lambda u(x) & x \in [a, b] \\ u(a) \cos(\alpha) + u'(a) \sin(\alpha) = 0 \\ u(b) \cos(\beta) + u'(b) \sin(\beta) = 0 \end{cases}$$

where $q(x) \geq 0$ and is continuously differentiable on $[a, b]$.

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An Eigenvalue Problem

We seek a complex number λ and a function u which satisfies the boundary conditions and

$$Lu = \lambda u.$$

The pair (λ, u) is called an **eigenpair**, λ an **eigenvalue**, and u an **eigenvector** (or eigenfunction) of the regular Sturm-Liouville problem.

The Plan

With the spectral theorem for the matrix A in mind, we will seek to develop a spectral theory for the operator L in our SLBVP. A satisfying spectral theory should:

- ① Describe an expansion in eigenvectors for the Hilbert space on which L acts.
- ② Find a way to enact the transformation L as multiplication and addition in an appropriate basis.

An analogy

The main reason for considering the eigenvalue problem in matrix algebra is the result:

Theorem (Spectral Theorem for Square Matrices)

Let A be an $n \times n$ matrix on a finite dimensional, complex inner product space V . If $(Ax, y) = (x, Ay)$ where (\cdot, \cdot) is the inner product on V then

- ① V has an orthonormal basis $\{v_1, v_2, \dots, v_n\}$ of eigenvectors corresponding to distinct eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.
- ② The matrix A can be enacted as $\sum_{i=1}^n \lambda_i P_i$ where λ_i are distinct eigenvalues of A and P_i are orthogonal projections.

The Spectral Theorem for the Nonsingular Sturm-Liouville Problem

Theorem

- ① The operator L is self-adjoint on $L^2(a, b)$: $(Lu, v) = (u, Lv)$.
- ② The λ for which the non-singular SLBVP has a solution is a discrete subset $\{\lambda_k\}$ of \mathbb{R} .
- ③ The set $\{\phi_k\}$ of solutions corresponding to $\{\lambda_k\}$ forms an orthogonal set in $L^2(a, b)$.
- ④ Every f in $L^2(a, b)$ can be written as $f = \sum_{k=1}^{\infty} c_k \phi_k$ for some choice of coefficients c_k .

Proof: L is Self-Adjoint

- 1 In the finite dimensional case, $(Ax, y) = (x, Ay)$ for all x is sufficient to guarantee that all the eigenvalues of A are real.
- 2 This turns out to also be the case for the operator L .
- 3 Notation:

$$[fg](x) = f(x)\bar{g}'(x) - f'(x)\bar{g}(x).$$

Proof: All Real Eigenvalues

A similar integration by parts argument shows that

$$\int_{x_1}^{x_2} (\bar{g}Lf - f\bar{L}g)dt = [fg](x_2) - [fg](x_1)$$

holding on every interval $[x_1, x_2]$ and for any f, g for which Lf, Lg are defined.

Let λ be an eigenvalue of L corresponding to eigenfunction f . Then

$$\begin{aligned} \bar{f}Lf - f\bar{L}f &= (\lambda - \bar{\lambda})|f|^2 \\ (\lambda - \bar{\lambda})\|f\|_2^2 &= [ff](b) - [ff](a) = 0 \end{aligned}$$

Now $f \neq 0$ in L^2 so that $(\lambda - \bar{\lambda}) = 0$ and hence

$$\Im(\lambda) = \frac{(\lambda - \bar{\lambda})}{2i} = 0.$$

Proof: L is Self-Adjoint

For self-adjointness

$$\begin{aligned} (Lu, v) &= \int_a^b (-u''(x) + q(x)u(x))\bar{v}(x) dx \\ &= \int_a^b q(x)\bar{v}(x)u(x) dx - \int_a^b u''(x)\bar{v}(x) dx \\ &= \int_a^b q(x)\bar{v}(x)u(x) dx + \int_a^b u'(x)\bar{v}'(x) dx + u'(x)\bar{v}(x) \Big|_a^b \\ &= \int_a^b q(x)\bar{v}(x)u(x) dx - \int_a^b u(x)\bar{v}''(x) dx + u(x)\bar{v}(x) \Big|_a^b \\ &= \int_a^b u(x)\overline{-v''(x) + q(x)v(x)} dx \\ &= \int_a^b u(x)\bar{Lv} dx \\ &= (u, Lv) \end{aligned}$$

Proof: Orthogonal Eigenfunctions

Now suppose that there f and g are eigenfunctions corresponding to distinct eigenvalues λ and γ . Using

$$\int_{x_1}^{x_2} (\bar{g}Lf - f\bar{L}g)dt = [fg](x_2) - [fg](x_1)$$

we can write

$$(\lambda - \gamma) \int_a^b f\bar{g}dx = (\lambda - \gamma)(f, g) = [fg](b) - [fg](a) = 0.$$

Thus any two eigenfunctions are orthogonal.

Proof: Green's Function

A Green's function for the operator L is a function satisfying the following properties.

- 1 Symmetry: $G(x, y) = G(y, x)$
- 2 $G(x, y)$ is continuous on $[a, b] \times [a, b]$. $G(x, y)$ is twice continuously differentiable away from the set $D = \{(x, x) | x \in [a, b]\}$.
- 3 At $x = y$, $\frac{\partial x}{\partial G}$ has a jump discontinuity. The jump satisfies

$$\frac{\partial x}{\partial G}(y^+, y) - \frac{\partial x}{\partial G}(y^-, y) = \lim_{\delta \downarrow 0} \left[\frac{\partial x}{\partial G}(y + \delta, y) - \frac{\partial x}{\partial G}(y - \delta, y) \right] = 1$$

- 4 On $[a, b] \times [a, b] \setminus D$, $G(x, y)$ satisfies the differential equation

$$L_x G(x, \xi) = G_{xx}(x, y) + q(x)G(x, \xi)$$

Proof: A Lemma

Lemma

If $G(x, y)$ is a Green's function for the SLBVP, $\mu \in \mathbb{R}$ and ψ a to the SLBVP, the operator

$$T(u(x)) = \int_a^b [G(x, y) - \mu \psi(x) \bar{\psi}(y)] u(y) dy$$

has either $-\|T\|$ or $\|T\|$ as an eigenvalue.

Proof: An Equivalent Problem

Given such a Green's function for L , we can construct the integral operator

$$(Tf)(x) = \int_a^b G(x, y) f(y) dy$$

Integrate by parts and use the boundary conditions to get:

$$L(Tu)(x) = u(x) \text{ and } T(Lu)(x) = u(x).$$

Apply T to the eigenvalue equation $Lu = \lambda u$. Obtain the equivalent problem:

$$Tu = \mu u$$

where $\mu = \frac{-1}{\lambda}$ and WLOG $\lambda \neq 0$.

Proof: Building an Orthonormal Set

By the lemma with $\mu = 0$, T has an eigenvector, so normalize it and call it ψ_0 . Define a new operator T_1 by

$$(T_1 u)(x) = \int_a^b [G(x, y) - \mu_0 \psi_0(x) \bar{\psi}_0(y)] u(y) dy.$$

Now T_1 satisfies the conditions of the lemma so that it has either $\|T_1\|$ or $-\|T_1\|$ as an eigenvalue, say μ_1 . Normalize its eigenvector and call it ψ_1 . Then

$$\begin{aligned} (T_1 u, \psi_0) &= \mu_1 (\psi_1, \psi_0) \\ &= (Tu, \psi_0) - \mu_0 (u, \psi_0) \cdot (\psi_0, \psi_0) \\ &= (u, T\psi_0) - \mu_0 (u, \psi_0) \\ &= (u, \mu \psi_0) - (u, \mu \psi_0) \\ &= 0. \end{aligned}$$

Thus ψ_0 and ψ_1 are an orthonormal set and

$$|\mu_1| = \|T\psi_1\| \leq \|T\| = |\mu_0|.$$

Building an Orthonormal Set

We continue in this way, writing next

$$(T_2 u)(x) = \int_a^b [G(x, y) - \mu_0 \psi_0(x) \bar{\psi}_0(y) - \mu_1 \psi_1(x) \bar{\psi}_1(y)] dy.$$

Use a similar argument with inner products and operator norms to obtain a vector ψ_2 orthogonal to ψ_1 and ψ_0 , with an eigenvalue μ_2 satisfying

$$|\mu_0| \geq |\mu_1| \geq |\mu_2|.$$

We can continue this process provided that no $\mu_n = 0$. But this won't happen for any $n \in \mathbb{N}$ since otherwise

$$0 = L(T_n u) = L(Tu) - \sum_{k=0}^{n-1} \mu_k(u, \psi_k) L\psi_k = u - \sum_{k=0}^{n-1} \mu_k(u, \psi_k) L\psi_k$$

which means that

$$u = \sum_{k=0}^{n-1} \mu_k(u, \psi_k) L\psi_k = \sum_{k=0}^{n-1} (u, \psi_k) LT\psi_k = \sum_{k=0}^{n-1} (u, \psi_k) \psi_k$$

holds for all $u \in C^2[a, b]$.

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Example 1

Let $b > 0$, $f(x) \in L^2(0, b)$ and consider the heat problem

$$\begin{cases} u_t - u_{xx} = 0 \text{ on } [0, b] \times [0, \infty) \\ u(0, t) = u(b, t) = 0 \text{ for } t \in [0, \infty) \\ u(x, 0) = f(x) \text{ for } x \in [0, b] \end{cases}.$$

This problem models the distribution of heat in rod of length b with initial temperature $f(x)$ and insulated ends.

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Proof: Completeness (Sketch)

Now that we have an infinite sequence of solutions ψ_k , we want to establish that this sequence is complete. That is, we want to show that the sequence spans the set of $L^2[a, b]$ functions. Here is an outline of that proof:

- 1 Use the structure of T_n and Bessel's inequality to show that $\|Tu - \sum_{k=0}^{\infty} \mu_k(u, \psi_k) \psi_k\| \rightarrow 0$ uniformly for $u \in C[a, b]$.
- 2 Set $u = Lf$ for $f \in C^2[a, b]$ satisfying the boundary conditions. Notice that

$$\mu_k(u, \psi_k) = (u, \mu_k \psi_k) = (u, T\psi_k) = (Tu, \psi_k) = (f, \psi_k)$$

- 3 Use the equality $Tu(x) = \sum_{k=0}^{\infty} \mu_k(u, \psi_k) \psi_k$ to prove that

$$f(x) = \sum_{k=0}^{\infty} (f, \psi_k) \psi_k$$

- 4 Use standard approximation arguments to prove that the formula holds for any $f \in L^2[a, b]$ (regardless of the boundary conditions).

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Example 1

Guess that there is a solution to the problem of the form

$$u(x, t) = v(t)w(x).$$

Substitute:

$$\frac{v'(t)}{v(t)} = \frac{w''(x)}{w(x)}.$$

Such a relation can hold only when both of these terms are constants, that is

$$-\lambda = \frac{v'(t)}{v(t)} = \frac{w''(x)}{w(x)}.$$

This leads to the pair of ordinary differential equations

$$w'' = -\lambda w$$

and

$$\dot{v} = -\lambda v.$$

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Example 1

With the boundary condition $w(0) = w(b) = 0$ and Calculus IV methods all solutions of the w problem are of the form

$$w(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x)$$

and all solutions of the v problem are of the form:

$$v(t) = Ce^{-\lambda t}$$

Example 1

For each $n \in \mathbb{N}$

$$u_n(x, t) = D_n e^{-\lambda t} \left(\sqrt{\frac{2}{b}} \sin\left(\frac{n\pi}{b}x\right) \right)$$

satisfies the heat equation with the condition $u_n(0, t) = u_n(L, t) = 0$. Need to construct:

$$u(x, 0) = f(x).$$

Example 1

From the left-hand boundary condition $w(0) = 0$ we see that

$$w(x) = B \sin(\sqrt{\lambda}x)$$

Now from the right-hand boundary condition $w(b) = 0$

$$\lambda_n = \frac{n^2 \pi^2}{b^2} \text{ for } n \in \mathbb{N}.$$

So we have determined all the eigenvalues for L with these boundary conditions. Moreover it is easy to see that to each λ_n there corresponds an eigenfunction

$$w_n(x) = \sqrt{\frac{2}{b}} \sin\left(\frac{n\pi}{b}x\right).$$

An Example

Write:

$$u_n(x, 0) = D_n w_n(x).$$

- The w problem is a regular Sturm-Liouville problem. So $\{w_n\}$ is a complete orthonormal system.
- Thus there must be a choice of D_n so that

$$f(x) = \sum_{n=1}^{\infty} D_n w_n.$$

- In fact, the correct choice of D_n is given by

$$D_n = \frac{2}{b} \int_0^b f(x) w_n(x) dx.$$

- Conclude:

$$u(x, t) = \sum_{n=1}^{\infty} D_n e^{-\lambda_n t} w_n(x).$$

Singularity in Sturm-Liouville Problems

There are two notions of singularity in Sturm-Liouville problems. The first kind of singularity occurs when the potential $q(x)$ has singularity. The second is when the interval of the problem is allowed to be infinite. The simple theory of the nonsingular problem has been well understood since the time of Sturm and Liouville. Weyl in his dissertation of 1910 laid out a rigorous framework for analyzing singular problems. The theory of the singular problems however remains an area of active inquiry.

Spectrum

Not all half-line problems have eigenvalues. Thus, it will be impossible to phrase our new spectral theorem in terms of eigenvalues and eigenvectors. We need a more general set of terminology.

Definition

The **spectrum** of an operator L is the set of $\lambda \in \mathbb{C}$ so that $(L - \lambda I)$ does not have a bounded inverse.

Clearly, the spectrum includes any eigenvalues an operator might have but is far more general. Note that for the nonsingular problem the spectrum is precisely the set of eigenvalues.

One of the major difficulties in the theory of singular problems is that their spectra are extremely complex, in general having a discrete part and a positive measure part.

The Half-Line SL Problem

Consider the following singular problem:

$$\begin{cases} Lu := -u'' + q(x)u = \lambda u, & x \in [0, \infty) \\ u(0) \cos(\alpha) + u'(0) \sin(\alpha) = 0 \\ u(x) \in L^2(0, \infty) \end{cases}$$

- 1 This problem is called the **half-line Sturm-Liouville problem** and is the most straightforward singular problem to analyze.
- 2 Note that the $L^2(0, \infty)$ condition can be thought of as a boundary condition at ∞ .
- 3 Our goal is to state a spectral theory for this problem.

A Spectral Theorem for the Half-Line Problem

Theorem

Assume $q(x) \geq -kx^2$ for k a positive constant. Let $\psi(x, \lambda)$ be the solution to the problem

$$\begin{cases} Lu = \lambda u \\ u(0) = 0 \\ u'(0) = 1 \end{cases}$$

Then there exists a monotone non-decreasing function ρ on $(-\infty, \infty)$ so that the expansion

$$f(x) = \int_0^\infty (f, \psi(\cdot, \lambda)) \psi(x, \lambda) d\rho(\lambda)$$

holds for every $f \in L^2(0, \infty)$.

Think of ρ as the measure which weights parts of the spectrum of L in such a way as to resolve the function f . The formula agrees with the nonsingular problem expansion formula if we set ρ equal to a step function with jumps at the eigenvalues.

Example 2

Consider the problem

$$\begin{cases} -u'' = \lambda u \\ u(0) = 0 \\ u \in L^2(0, \infty) \end{cases}.$$

Plan: regard as a limiting problem of finite interval problems.
That is, consider the sequence of problems

$$\begin{cases} -u'' = \lambda u \\ u(0) = 0 \quad u(b) = 0 \end{cases}$$

and attempt to “take a limit” as $b \rightarrow \infty$.

Example 2

Let $f(t)$ be a locally L^2 function which vanishes for $t > c$ where $0 < c < b$. By the theory for regular problem:

$$f(x) = \sum_{k=0}^{\infty} (f, \psi_k) \psi_k.$$

Set

$$\psi(x, \lambda) = \sin(\lambda x)$$

and let $\rho_b(\lambda)$ be the nondecreasing step function which jumps by $\frac{2}{b}$ when λ passes through $\frac{k\pi}{b}$. Then we have

$$f(x) = \int_0^{\infty} (f, \psi(\cdot, \lambda)) \psi(x, \lambda) d\rho_b(\lambda).$$

Example 2

- For a fixed b we are back to solving the problem

$$-\frac{d^2}{dx^2} w = \lambda w \quad w(0) = w(b)$$

which we treated in detail in example one.

- Orthonormal system

$$\psi_k(x) = \sqrt{\frac{2}{b}} \sin\left(\frac{k\pi x}{b}\right)$$

and corresponding eigenvalues

$$\lambda_n = \frac{n^2 \pi^2}{b^2} \text{ for } n \in \mathbb{N}.$$

Example 2

Now recklessly taking limits as $b \rightarrow \infty$ conclude

$$\rho_b(\lambda) \rightarrow \frac{2\lambda}{\pi}$$

and hence that

$$f(x) = \frac{2}{\pi} \int_0^{\infty} (f, \sin(\lambda x)) \sin(\lambda x) d\lambda.$$

Comments on Example 2

The transformation

$$\mathcal{F}f(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin(\xi t) dt$$

is the Fourier sine transform. For odd functions $f(t)$ this transform has an inverse transformation that is also a sine transform that is,

$$f(x) = \mathcal{F}\mathcal{F}f(x).$$

This is precisely the formula

$$f(x) = \frac{2}{\pi} \int_0^\infty (f, \sin(\lambda x)) \sin(\lambda x) d\lambda.$$

References

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Comments on Singular Theory

- ① Example 2 furnishes an example of the truly bizarre behavior of singular problems: since $\rho_b(\lambda) \rightarrow \frac{2\lambda}{\pi}$, every point in $[0, \infty)$ is in the spectrum of L but no point in the spectrum is an eigenvalue!
- ② The example suggests how we might prove the spectral theory in the singular case. Need to find a framework for taking limits in the singular problem. This is Weyl's major contribution to the singular theory.