A Boundary Value Problem of Ordinary Self-Adjoint Differential Equations with Singularities.

By Marion Cameron Gray.

Introduction. The problem of solving the ordinary homogeneous self-adjoint differential equation of the second order with singularities, subject to given boundary conditions, has been discussed by Hilb.* He considers the non-singular equation

$$(d/dx)(xdu/dx) + [-g(x) + \lambda h(x)/x]\, u = 0 \qquad (\epsilon \leq x \leq 1)$$

with the boundary conditions $u(\epsilon) = u(1) = 0$, and introduces a singularity by allowing ϵ to become zero. The functions g and h are continuous, and h is positive in $(\epsilon, 1)$. Hilb finds a Green's function for the equation, and the corresponding integral equation; then, in passing to the limiting case $\epsilon = 0$, he applies the Hilbert † theory of quadratic forms in infinitely many variables.

Weyl ‡ has discussed the more general equation

$$(d/dx)(p(x)du/dx) - q(x)u + \lambda u = 0 \qquad (0 \leq x < \infty)$$

with homogeneous boundary conditions at zero and infinity. The function q is continuous, and p is continuous and positive in the interval $(0, \infty)$, so that the only singularity is at infinity. Weyl also introduces a Green's function, and applies to the resulting integral equation his theory of singular integral equations § which is based on Hilbert's work. He distinguishes two types of

* E. Hilb, "Über Integraldarstellungen willkürlicher Funktionen," *Mathematische Annalen*, Vol. 66, pp. 1-66. The work of Wirtinger, *Mathematische Annalen*, Vol. 48, pp. 387-389, may also be cited.

† D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig (1912).

‡ H. Weyl, "Über gewöhnliche lineare Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen," *Göttinger Nachrichten*, Mathematische-Physische Klasse, 1909, pp. 37-63, also "Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen," *Mathematische Annalen*, Vol. 68, pp. 220-269.

§ H. Weyl, "Singuläre Integralgleichungen," *Mathematische Annalen*, Vol. 66, pp. 273-324.

singularity, the limit circle type, which leads to solutions for complex values of λ, and the limit point type, which leads to solutions only for real values of λ.

In a later paper * Weyl extends his method to the equation

$$(d/dx)(p(x)du/dx) - q(x)u + \lambda k(x)u = 0 \qquad (-\infty < x < \infty)$$

where the function k is continuous, and there are singularities at both ends of the interval $(-\infty, \infty)$, first for the case in which k is positive, and then for the polar case in which k may change sign.

In the present paper we consider the equation

$$(d/dx)(p(x)du/dx) + q(x)u + \lambda u = 0 \qquad (0 \leq x \leq \pi)$$

with boundary conditions $u(0) = u(\pi) = 0$. We assume that the function q is continuous, and that the function p is continuous with a continuous first derivative, and vanishes a finite number of times in the interval. We introduce a complete normed orthogonal system of functions, $\phi_m(x)$, satisfying the boundary conditions, and use it to pass from the differential equation to a system of infinitely many equations in an infinite number of unknowns.† We consider the case in which this system of equations has solutions only for real values of λ, Weyl's limit point type, and we apply the results of the Carleman ‡ theory of integral equations of Class I, for the special case of quadratic forms. From the solutions of the system of infinitely many equations we derive solutions of the given differential equation and discuss their properties, referring in this connexion also to the Hellinger § theory of limited quadratic forms (§§ 1 and 2).

In § 3 we introduce a theory of *limited differential systems*, by means of which we derive (§ 4) an expansion theorem for an arbitrary function $f(x)$, subject to the conditions that f be continuous, satisfy the boundary conditions, and have a first derivative which is integrable and of integrable square in $(0, \pi)$. The corresponding expansion theorems of Weyl and Hilb require further the existence of the function

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* H. Weyl, "Über gewöhnliche lineare Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen," *Göttinger Nachrichten*, Mathematische-Physische Klasse, 1910, pp. 442-467. For further references to the literature of the subject we may refer to those given by Weyl in the introduction to this paper.

† Anna Pell Wheeler, "Linear Ordinary Self-Adjoint Differential Equations of the Second Order," *American Journal of Mathematics*, Vol. 49 (1927), p. 310.

‡ T. Carleman, *Sur les équations intégrales singulières à noyau reél et symétrique.*

§ E. Hellinger, "Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen," *Journal für Mathematik*, Vol. 136, pp. 211-271.

$$(d/dx)(p(x)\,df/dx),$$

though we may note that Plancherel * has obtained an expansion theorem for the Hilb problem in which he requires only that f be continuous and of bounded variation in $(0,1)$, vanish at each end of the interval, and be such that the integral

$$\int_0^1 |f(t)/t|\,dt$$

converges.

1. *Properties of the Solutions of the System of Infinitely Many Equations Arising from the Differential Equation.* We consider solutions of the linear homogeneous self-adjoint differential equation of the second order

$$(1) \qquad (d/dx)(p(x)\,du/dx) + q(x)u + \lambda u = 0$$

which are of integrable square in the interval $(0,\pi)$. The function q is continuous in $(0,\pi)$, the function p is continuous with a continuous first derivative, and vanishes a finite number of times in the interval. The boundary conditions are assumed to be

$$(2) \qquad u(0) = u(\pi) = 0,$$

if $p(0) \neq 0$ and $p(\pi) \neq 0$, but if for example $p(0) = 0$ the conditions are

$$(2') \qquad \lim_{x\to 0} p(x)u(x) = 0 \text{ and } u(\pi) = 0.$$

An auxiliary system of normed orthogonal functions satisfying the boundary conditions (2) is

$$(3) \qquad \phi_m(x) = (2/\pi)^{1/2} \sin mx,$$

and this system is also complete.

To obtain from the system (1) and (2) [or (2′)] a system of infinitely many equations in an infinite number of unknowns we multiply equation (1) by $\phi_m(x)$, and integrate between 0 and π,†

$$\int_0^\pi (pu')'\phi_m + \int_0^\pi qu\phi_m + \lambda \int_0^\pi u\phi_m = 0.$$

* M. Plancherel, "Integraldarstellungen willkürlicher Funktionen," *Mathematische Annalen*, Vol. 67, pp. 519-534.

† When there is no ambiguity we shall omit the variable of integration; for definite integrals in $(0, \pi)$ this variable is always x. We use primes throughout to denote derivatives with respect to x.

Now

$$\int_0^\pi (pu')'\phi_m = \int_0^\pi (p\phi_m')'u,$$

since ϕ_m satisfies the boundary conditions (2), while u satisfies the conditions (2) or (2'). Also

$$\int_0^\pi (p\phi_m')'u = \sum_n \int_0^\pi (p\phi_m')'\phi_n \int_0^\pi u\phi_n$$

by the known properties of the complete system (3). A solution u of the system (1) and (2) [or (2')] therefore satisfies the system of equations

$$\sum_{n=1}^{\infty} \int_0^\pi (p\phi_m')'\phi_n \int_0^\pi u\phi_n + \sum_{n=1}^{\infty} \int_0^\pi q\phi_m\phi_n \int_0^\pi u\phi_n + \lambda \int_0^\pi u\phi_m = 0,$$

which may be written

$$\sum_{n=1}^{\infty} p_{mn}x_n + \sum_{n=1}^{\infty} q_{mn}x_n + \lambda x_m = 0, \tag{4}$$

where

$$x_m = \int_0^\pi u\phi_m,$$

$$p_{mn} = \int_0^\pi (p\phi_m')'\phi_n,$$

and

$$q_{mn} = \int_0^\pi q\phi_m\phi_n.$$

The matrix $P = (p_{mn})$ is symmetric, the rows are of finite norm, that is Σp_{mn}^2 converges for every value of m, but P is not limited. The matrix $Q = (q_{mn})$ is symmetric and limited.

The matrix $P + Q$ is of the type for which Carleman has given an existence theorem.* In this paper we consider only the case in which $P + Q$ is of Class I,† that is, the system of equations (4) has no solution of finite norm, not identically zero, for complex values of λ.

The system of equations (4) may, however, have solutions of finite norm, not identically zero, for real values of λ, the characteristic values. Let these values be λ_a, and the corresponding characteristic solutions l_{am}. These characteristic values of λ form the *point spectrum.*

Let $\Delta f(\lambda) = f(\lambda_2) - f(\lambda_1)$ denote the increment of the continuous func-

* Carleman, *l. c.*, ch. 1.

† Carleman, *l. c.*, p. 187.

tion $f(\lambda)$ for the interval $\Delta = (\lambda_1, \lambda_2)$ of the real λ axis, and consider the system of equations

$$\sum_n p_{mn} \Delta\rho_n(\lambda) + \sum_n q_{mn} \Delta\rho_n(\lambda) + \int_\Delta \lambda d\rho_n(\lambda) = 0, \tag{5}$$

which may have solutions of finite norm, and not constant, for certain intervals Δ of the real λ axis. These intervals Δ form the *continuous spectrum,* and the corresponding characteristic differential solutions $\rho_m(\lambda)$ are continuous and of bounded variation in λ. They are determined up to an additive constant, which we fix by the condition $\rho_m(0) = 0$ for every value of m. The sequence $\{\rho_m(\lambda)\}$ defines a continuous monotonic non-decreasing function of bounded variation in λ

$$\begin{aligned} \rho(\lambda) &= \sum_m [\rho_m(\lambda)]^2 && \lambda > 0 \\ &= 0 && \lambda = 0 \\ &= -\sum_m [\rho_n(\lambda)]^2 && \lambda < 0 \end{aligned} \tag{6}$$

The function $\rho(\lambda)$ is called the basis function of the solutions $\rho_m(\lambda)$.

The term *spectrum* is used to denote the values of λ which belong to the point spectrum, the limiting points of the point spectrum, and the continuous spectrum, and we know by the theory of forms of Class I that such a spectrum exists, that is, there are solutions of finite norm either of the linear system of equations (4), or of the differential system (5) or of both. Further, since the matrix P is not limited, the spectrum may extend over any part of the real λ axis.

We may note that there may be more than one sequence of characteristic differential solutions satisfying all the conditions imposed, but for the present we shall assume that the continuous spectrum is simple, and return later (§ 5) to the more general case.

The sequences $\{l_{\alpha m}\}$ and $\{\rho_m(\lambda)\}$ form an orthogonal system, satisfying the relations

$$\begin{aligned} \sum_m l_{\alpha m} l_{\beta m} = e_{\alpha\beta} &= 0 && \alpha \neq \beta \\ &= 1 && \alpha = \beta, \end{aligned} \tag{7}$$

$$\sum_m \Delta_1\rho_m(\lambda)\, \Delta_2\rho_m(\lambda) = \Delta_{12}\rho(\lambda), \tag{8}$$

where Δ_1 and Δ_2 are any two intervals of the real λ axis in which there is a continuous spectrum, and Δ_{12} the interval common to the two intervals Δ_1 and Δ_2. Finally the two types of solutions are orthogonal

$$(9) \qquad \sum_m l_{am}\,\Delta\rho_m(\lambda) = 0$$

for every α, and every value of λ.*

If the system of equations

$$(10) \qquad \sum_n \,(p_{mn} + q_{mn})x_n = 0$$

has no solution of finite norm other than zero, the solutions l_{am} and $\rho_m(\lambda)$ form a *complete* system, satisfying the relation

$$(11) \qquad \sum_a l_{am}l_{an} + \int_{-\infty}^{\infty} \frac{d\rho_m(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)}\dagger = e_{mn}.$$

The proof is easily obtained from the following result which is a special case of a theorem given by Carleman.‡

Let the matrix of the system of equations (4) be of Class I, with characteristic numbers λ_a and corresponding characteristic solutions l_{am}. Let the continuous spectrum be simple with characteristic differential forms $\rho_m(\lambda)$ and basis function $\rho(\lambda)$. Write

$$Y_a = \sum_n l_{an}y_n, \qquad X_a = \sum_n l_{an}x_n$$

$$\Delta y(\lambda) = \sum_n \Delta\rho_n(\lambda)y_n, \qquad \Delta x(\lambda) = \sum_n \Delta\rho_n(\lambda)x_n,$$

where $\{x_n\}$ and $\{y_n\}$ are sequences of finite norm, and $\{x_n\}$ is such that $\sum x_m[\sum(p_{mn} + q_{mn})^2]^{1/2}$ converges. Then

$$(12) \qquad -\sum_n \,(p_{mn} + q_{mn})y_n = \sum_a \lambda_a Y_a l_{am} + \int_{-\infty}^{\infty} \frac{\lambda\, d\rho_m(\lambda)\, dy(\lambda)}{d\rho(\lambda)},$$

$$(13) \qquad -\sum_{m,n} (p_{mn} + q_{mn})x_m y_n = \sum_a \lambda_a X_a Y_a + \int_{-\infty}^{\infty} \frac{\lambda\, dx(\lambda)\, dy(\lambda)}{d\rho(\lambda)}$$

The series and integrals involved are absolutely convergent.

These expansions show at once that there is no sequence of finite norm, not identically zero, orthogonal to all the sequences $\{l_{am}\}$ and $\{\rho_m(\lambda)\}$. For any two sequences $\{x_m\}$ and $\{y_n\}$ of finite norm we obtain expansions

$$(14) \qquad y_m = \sum_a l_{am} \sum_n y_n l_{an} + \int_{-\infty}^{\infty} \frac{d\rho_m(\lambda)\; d\sum_n y_n\rho_n(\lambda)}{d\rho(\lambda)}$$

* For proofs of these statements about the spectrum we may refer to Hellinger, *l. c.*, pp. 240 et seq., and Carleman, *l. c.*, ch. 3.

† Hellinger, *l. c.*, p. 237, defines this integral in a finite interval, and the definition is extended to an infinite interval by Weyl, *l. c.*, *Mathematische Annalen*, Vol. 66, p. 288.

‡ Carleman, *l. c.*, pp. 101-102.

$$(15)\qquad \sum_m x_m y_m = \sum_a \sum_n l_{am} x_m \sum_n l_{an} y_n + \int_{-\infty}^{\infty} \frac{d \sum_m x_m \rho_m(\lambda)\, d \sum_n y_n \rho_n(\lambda)}{d\rho(\lambda)}^*$$

and the relation (11) follows immediately.

2. *The Solutions of the Differential Equation.* From the solutions l_{am} and $\rho_m(\lambda)$ of the last section we can now derive solutions of the differential equation (1). Consider the sequence $\{\rho_m(\lambda)\}$ which is a solution of finite norm for the system of equations (5). By the Riesz-Fischer theorem

$$(16)\qquad \rho_m(\lambda) = \int_0^\pi u(x,\lambda)\,\phi_m(x)\,dx$$

for a function $u(x, \lambda)$ which is of integrable square in x in $(0, \pi)$. Further the sequence $\{\Sigma\, p_{mn}\rho_n(\lambda)\}$ is of finite norm. Now

$$\begin{aligned}\sum_n p_{mn}\rho_n(\lambda) &= \sum_n \int_0^\pi (p\phi_m')'\phi_n \int_0^\pi u\phi_n \\ &= \int_0^\pi (p\phi_m')'u \\ &= -(2/\pi)^{1/2} m^2 \int_0^\pi pu \sin mx + (2/\pi)^{1/2} m \int_0^\pi p'u \cos mx \\ &= (2/\pi)^{1/2} m \left[\int_0^\pi p'u \cos mx - m \int_0^\pi pu \sin mx\right]\end{aligned}$$

The function u is such that this is of finite norm, and therefore the factor in the bracket is of finite norm. But the sequence $\{\int_0^\pi p'u \cos mx\}$ is itself of finite norm, therefore the sequence $\{m \int_0^\pi pu \sin mx\}$ is likewise. It follows that the function pu has a derivative almost everywhere which is integrable and of integrable square in the interval $(0, \pi)$, and, since p has a derivative, u has a derivative almost everywhere except perhaps for the zeros of the function p. We have also $u(0, \lambda) = u(\pi, \lambda) = 0$, unless $p(0) = 0$ or $p(\pi) = 0$, and in all cases $\lim_{x\to 0} p(x)u(x, \lambda) = \lim_{x\to\pi} p(x)u(x, \lambda) = 0$.

Further, the expression

$$m \int_0^\pi p'u \cos mx - m^2 \int_0^\pi pu \sin mx$$

* Carleman, *l. c.*, p. 103.

is of finite norm, and

$$-m^2\int_0^\pi pu\sin mx = -m\int_0^\pi (pu)'\cos mx.$$

Hence

$$m\int_0^\pi [p'u-(pu)']\cos mx$$

is of finite norm, and therefore the function $p'u-(pu)'$ has a derivative almost everywhere which is integrable and of integrable square in the interval $(0,\pi)$. Except at the zeros of the function p this function may be written pu', so that at all points for which $p(x)\neq 0$ the functions u' and $(pu')'$ exist almost everywhere. Further we have

$$\int_0^\pi [u(x,\lambda)]^2\,dx = \sum_m \Big[\int_0^\pi u\phi_m\Big]^2 = \rho(\lambda),$$

and therefore $\int_0^\pi [u(x,\lambda)]^2\,dx$ converges to a continuous function of λ.

Now Weyl * has defined a *differential solution* $u(x,\lambda)$ of equation (1) as a real continuous function of x and of λ, for which the integral $\int_0^\pi [u(x,\lambda)]^2\,dx$ exists as a continuous function of λ, and which satisfies the boundary conditions (2) or (2′) and the equation

$$(pu')' + qu + \int_0^\lambda \lambda\,du\,\dagger = 0. \tag{17}$$

By the properties we have already found of the function $u(x,\lambda)$ defined by equation (16), we need only show that it satisfies equation (17) in order to prove that it is a differential solution of equation (1). Return to the system of equations (5) satisfied by $\rho_m(\lambda)$, and write it in the form

$$\sum_n \int_0^\pi (p\phi_m')'\,\phi_n\Delta\rho_n(\lambda) + \sum_n\int_0^\pi q\,\phi_m\phi_n\,\Delta\rho_n(\lambda) + \int_\Delta \lambda d\rho_m(\lambda) = 0,$$

Substitute for $\Delta\rho_m(\lambda)$ from equation (16)

$$\int_0^\pi (p\phi_m')'\,\Delta u(x,\lambda) + \int_0^\pi q\phi_m\Delta u(x,\lambda) + \int_0^\pi \phi_m \int_\Delta \lambda\,du(x,\lambda) = 0.$$

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* Weyl. *l. c., Mathematische Annalen,* Vol. 68, p. 239.

† In the Stieltjes and Hellinger integrals used throughout the variable of integration is always λ.

Take Δ to be the interval $(0, \lambda)$, then $\Delta u(x, \lambda) = u(x, \lambda)$. Further

$$\int_0^\pi (p\phi_m')'u(x,\lambda) = \int_0^\pi [p'u - (pu)']'\phi_m$$

by the boundary properties of the functions $u(x, \lambda)$ and $\phi_m(x)$, and therefore finally

$$(18) \qquad \int_0^\pi \phi_m\{[p'u - (pu)']' + qu + \int_0^\lambda \lambda du(x,\lambda)\} = 0.$$

At all points for which $p(x) \neq 0$ this may be written

$$(pu')' + qu + \int_0^\lambda \lambda du(x,\lambda) = 0,$$

so that the function $u(x, \lambda)$ is a differential solution of equation (1), except perhaps at the zeros of the function p.

If we consider the solutions l_{am} of the system of equations (4) we can show in a similar manner that the functions $u_a(x)$ defined by

$$(19) \qquad l_{am} = \int_0^\pi u_a(x)\,\phi_m(x)\,dx$$

are solutions of equation (1) except perhaps at the zeros of the function p.* The functions $u_a(x)$ are such that the sequence $\{m\int_0^\pi pu_a\phi_m\}$ is of finite norm, so that the function pu_a has a derivative almost everywhere which is integrable and of integrable square in the interval $(0, \pi)$.

The solutions $u(x, \lambda)$ and $u_a(x)$ have orthogonal properties similar to those of the systems $\rho_m(\lambda)$ and l_{am}. They are derived immediately from the corresponding relations (7), (8) and (9), and have the form

$$(20) \qquad \int_0^\pi u_a(x)\,u_\beta(x)\,dx = e_{a\beta},$$

$$(21) \qquad \int_0^\pi \Delta_1 u(x,\lambda)\,\Delta_2 u(x,\lambda)\,dx = \Delta_{12}\rho(\lambda)$$

for any two intervals Δ_1 and Δ_2 of the real λ axis. and

$$(22) \qquad \int_0^\pi u_a(x)\,\Delta u(x,\lambda)\,dx = 0$$

for every value of α, and every interval Δ.

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* These solutions have been obtained for the non-singular differential equation $u'' + qu + \lambda u = 0$ by Anna Pell Wheeler, *l. c.*, *American Journal of Mathematics*, Vol. 49 (1927), p. 312.

3. *Limited Differential Systems.* We now introduce the conception of limited differential systems, and prove some properties of these systems analogous to the known properties of limited matrices. These are used later in obtaining an expansion theorem for an arbitrary function $f(x)$, subject to suitably imposed conditions, in terms of the solutions $u_a(x)$ and $u(x,\lambda)$ of § 2. The functions $\rho_m(\lambda)$ and $\rho(\lambda)$ used throughout this section are the characteristic differential solutions and corresponding basis function of § 1.

Definition I. Let $f(\lambda)$ be a continuous function of λ in the infinite interval $(-\infty, \infty)$, and let $u(\lambda)$ be continuous and of bounded variation in that interval. Then the function $\int_\Delta f(\lambda)\,du(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$ if the Hellinger integral

$$\int_{-\infty}^{\infty} [f(\lambda)]^2 \, \frac{[du(\lambda)]^2}{d\rho(\lambda)}$$

converges.* If $\int_\Delta f(\lambda)\,du(\lambda)$ is integrable $H(\rho)$, and if, further, $\Delta g(\lambda)$ is any function integrable $H(\rho)$, then the integral

$$\int_{-\infty}^{\infty} f(\lambda) \, \frac{du(\lambda)\,dg(\lambda)}{d\rho(\lambda)}$$

is absolutely convergent.

Definition II. The system $\int_\Delta f(\lambda)\,du_m(\lambda)$ is a *differential system* in $(-\infty, \infty)$ if $\int_\Delta f(\lambda)\,du_m(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$ for every value of m, and if, further, the series $\sum [u_m(\lambda)]^2$ converges to a continuous function $u(\lambda)$.

Definition III. The differential system $\int_\Delta f(\lambda)\,du_m(\lambda)$ is *limited* in $(-\infty, \infty)$ if the sequence

$$\left\{ \int_{-\infty}^{\infty} f(\lambda) \, \frac{du_m(\lambda)\,dg(\lambda)}{d\rho(\lambda)} \right\}$$

is of finite norm and norm $< M \int_{-\infty}^{\infty} [f(\lambda)]^2 \, \frac{[dg(\lambda)]^2}{d\rho(\lambda)}$, for every function $\Delta g(\lambda)$ integrable $H(\rho)$.
A similar definition holds for a finite interval.

THEOREM I. *The system $\Delta\rho_m(\lambda)$ is a limited differential system.*

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* This definition is similar to that given by A. J. Pell, "Linear Equations with unsymmetric Systems of Coefficients," *Transactions of the American Mathematical Society*, Vol. 20, p. 26.

By the statements of § 1 we see immediately that $\Delta\rho_m(\lambda)$ is a differential system, and it remains to prove that it is limited. The orthogonal relation (8) is equivalent to the relation

$$(23)\qquad \int_{-\infty}^{\infty}\frac{du(\lambda)\,dv(\lambda)}{d\rho(\lambda)}=\sum_m\int_{-\infty}^{\infty}\frac{du(\lambda)\,d\rho_m(\lambda)}{d\rho(\lambda)}\int_{-\infty}^{\infty}\frac{dv(\lambda)\,d\rho_m(\lambda)}{d\rho(\lambda)}$$

for any two functions $u(\lambda)$ and $v(\lambda)$ integrable $H(\rho)$ in $(-\infty,\infty)$.* Then, for the differential system in question,

$$\sum_m\left\{\int_{-\infty}^{\infty}\frac{d\rho_m(\lambda)\,dg(\lambda)}{d\rho(\lambda)}\right\}^2=\int_{-\infty}^{\infty}\frac{[dg(\lambda)]^2}{d\rho(\lambda)}$$

for every $\Delta g(\lambda)$ integrable $H(\rho)$, therefore the condition of Definition III is satisfied, and the differential system is limited.

Definition IV. The matrix B whose coefficients are

$$(24)\qquad b_{mn}=\int_{-\infty}^{\infty}f(\lambda)\,\frac{du_m(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)}$$

is the matrix corresponding to the differential system $\int_\Delta f(\lambda)\,du_m(\lambda)$.

From equation (14) of § 1 we obtain the result that *for every sequence* $\{y_m\}$ *of finite norm there exists a sequence* $\{c_a\}$ *of finite norm, and a function* $\Delta g(\lambda)$ *integrable* $H(\rho)$ *in* $(-\infty,\infty)$ *such that*

$$(25)\qquad y_m=\sum_a l_{am}c_a+\int_{-\infty}^{\infty}\frac{d\rho_m(\lambda)\,dg(\lambda)}{d\rho(\lambda)}.$$

For functions $\Delta g(\lambda)$ integrable $H(\rho)$ in $(-\infty,\infty)$ we have a similar result which we prove as the theorem

THEOREM II. *For every function* $\Delta g(\lambda)$ *integrable* $H(\rho)$ *in* $(-\infty,\infty)$ *there exists a sequence* $\{y_n\}$ *of finite norm such that*

$$\Delta g(\lambda)=\sum_n\Delta\rho_n(\lambda)\,y_n.$$

The proof follows immediately from an expansion given by Hellinger,†

$$g(\lambda)=\sum_n\rho_n(\lambda)\int_{-\infty}^{\infty}\frac{dg(\lambda)\,d\rho_n(\lambda)}{d\rho(\lambda)}$$

* Weyl, *l. c.*, *Mathematische Annalen*, Vol. 66, p. 289.

† Hellinger, *l. c.*, p. 248. The proof here given is for a finite interval, but it may be extended to include the present case.

which holds for any function $\Delta g(\lambda)$ integrable $H(\rho)$ in $(-\infty, \infty)$. The sequence

$$y_n = \int_{-\infty}^{\infty} \frac{dg(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)}$$

is of finite norm by equation (23), and therefore the theorem is established.

We now show that the properties of the differential system may be derived from those of its corresponding matrix by means of the following theorem

THEOROM III. *If the differential system* $\int_\Delta f(\lambda)\, du_m(\lambda)$ *is limited the corresponding matrix is limited, and conversely.*

We consider the sequence $\{\sum_n b_{mn} y_n\}$ and prove that it is of finite norm for every sequence $\{y_n\}$ of finite norm. Expand the sequence $\{y_n\}$ in the form (25), then

$$\sum_n b_{mn} y_n = \sum_n \int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)} \left[\sum_a l_{an} c_a + \int_{-\infty}^{\infty} \frac{dg(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)} \right]$$

and therefore, by equations (9) and (23)

$$\sum_n b_{mn} y_n = \int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda)\, dg(\lambda)}{d\rho(\lambda)}$$

The function $\Delta g(\lambda)$ is integrable $H(\rho)$, therefore by the definition of a limited differential system the sequence $\{\sum b_{mn} y_n\}$ is of finite norm for every sequence $\{y_n\}$ of finite norm, which proves that the matrix B is limited.*

We now assume that the matrix B is limited and prove that the differential system is also limited. Let $\Delta g(\lambda)$ be any function integrable $H(\rho)$ in $(-\infty, \infty)$. By equation (23) we have

$$\int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda)\, dg(\lambda)}{d\rho(\lambda)}$$
$$= \sum_n \int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)} \int_{-\infty}^{\infty} \frac{dg(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)}$$

and this integral exists for every function $\Delta g(\lambda)$. Hence

$$\int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda)\, dg(\lambda)}{d\rho(\lambda)} = \sum_n b_{mn} y_n$$

* Hellinger and Toeplitz, "Grundlagen für eine Theorie der unendlichen Matrizen," *Mathematische Annalen*, Vol. 69, pp. 321-324.

where the sequence $\{y_n\}$ is of finite norm. The sequence $\{\sum b_{mn}y_n\}$ is of finite norm $\leqq M$, since B is limited, and it follows that the differential system is also limited.

THEOREM IV. *If the differential system* $\int_\Delta f(\lambda)\,du_m(\lambda)$ *is limited, the function* $\int_\Delta f(\lambda)\,d\sum x_m u_m(\lambda)$ *is integrable* $H(\rho)$ *in* $(-\infty, \infty)$ *for every sequence* $\{x_m\}$ *of finite norm.*

The differential system is limited and therefore the corresponding matrix B is limited. Also, for any sequence $\{x_m\}$ of finite norm, the function $\int_\Delta f(\lambda)\,d\sum_{m=1}^{N} x_m u_m(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$, where N has any finite value. By Theorem II there exists a sequence $\{c_p{}^N\}$ of finite norm such that

$$\int_\Delta f(\lambda)\,d\sum_{m=1}^{N} x_m u_m(\lambda) = \Delta \sum_p c_p{}^N \rho_p(\lambda)$$

where

$$c_p{}^N = \int_{-\infty}^{\infty} f(\lambda)\,\frac{d\sum_{m=1}^{N} x_m u_m(\lambda)\,d\rho_p(\lambda)}{d\rho(\lambda)}\,.$$

By the definition of the matrix B this gives

$$\begin{aligned}\sum_p (c_p{}^N)^2 &= \sum_p \Big(\sum_{m=1}^{N} x_m b_{mp}\Big)^2\\ &\leq M\sum_{m=1}^{N} x_m{}^2\\ &\leq M\sum_{m=1}^{\infty} x_m{}^2,\end{aligned}$$

since B is limited, and hence the series $\sum_p c_p{}^N \rho_p(\lambda)$ converges uniformly for every finite value of N. Further

$$\lim_{N\to\infty} c_p{}^N = c_p = \sum_{m=1}^{\infty} x_m b_{mp}$$

and this sequence is of finite norm since B is limited. It follows that

$$\begin{aligned}\lim_{N\to\infty}\int_\Delta f(\lambda)\,d\sum_{m=1}^{N} x_m u_m(\lambda) &= \lim_{N\to\infty}\sum_p c_p{}^N \rho_p(\lambda)\\ &= \sum_p c_p\rho_p(\lambda)\end{aligned}$$

and this is sufficient to show that the function $\int_\Delta f(\lambda)\, d\sum_m x_m u_m(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$.

By the aid of this theorem we now prove the following important theorem

THEOREM V. *If* $\int_\Delta f(\lambda) du_m(\lambda)$ *is a limited differential system in* $(-\infty, \infty)$

$$\sum_m x_m \int_{-\infty}^{\infty} f(\lambda) \frac{du_m(\lambda)dg(\lambda)}{d\rho(\lambda)} = \int_{-\infty}^{\infty} f(\lambda) \frac{d\sum_m x_m u_m(\lambda) dg(\lambda)}{d\rho(\lambda)} \tag{26}$$

for every sequence $\{x_m\}$ *of finite norm, and every function* $\Delta g(\lambda)$ *integrable* $H(\rho)$ *in* $(-\infty, \infty)$.

Write

$$U(x, \lambda) = \sum_m x_m u_m(\lambda)$$

and consider a finite interval $(-l, l)$ of the real λ axis. Divide this interval into N parts, $\Delta_1 \cdots \Delta_N$, and let λ_i be any point of the interval Δ_i. Then for any sequence $\{x_m\}$ of finite norm, and any function $\Delta g(\lambda)$ integrable $H(\rho)$

$$\sum_{i=1}^{N} f(\lambda_i) \cdot \frac{\Delta_i U(x,\lambda)\Delta_i g(\lambda)}{\Delta_i \rho(\lambda)} = \sum_m x_m \sum_{i=1}^{N} f(\lambda_i) \frac{\Delta_i u_m(\lambda)\Delta_i g(\lambda)}{\Delta_i\rho(\lambda)} \tag{27}$$

and this is a limited linear form by Theorem IV. The coefficients

$$\sum_{i=1}^{N} f(\lambda_i) \frac{\Delta_i u_m(\lambda)\Delta_i g(\lambda)}{\Delta_i\rho(\lambda)}$$

form a sequence of finite norm, and

$$\lim_{N\to\infty} \sum_{i=1}^{N} f(\lambda_i) \frac{\Delta_i u_m(\lambda)\Delta_i g(\lambda)}{\Delta_i\rho(\lambda)} = \int_{-l}^{l} f(\lambda) \frac{du_m(\lambda)dg(\lambda)}{d\rho(\lambda)}$$

exists for every value of m. Then for any fixed system of values of the sequence $\{x_m\}$ we may take the limit of the right hand member of equation (27) term by term, and thus obtain the equation

$$\int_{-l}^{l} f(\lambda) \frac{dU(x,\lambda)dg(\lambda)}{d\rho(\lambda)} = \sum_m x_m \int_{-l}^{l} f(\lambda) \frac{du_m(\lambda)dg(\lambda)}{d\rho(\lambda)} \tag{28}$$

for any finite interval $(-l, l)$. Further, by Theorem IV, the integral

$$\lim_{l\to\infty} \int_{-l}^{l} f(\lambda) \frac{dU(x,\lambda)dg(\lambda)}{d\rho(\lambda)}$$

converges, and by Definition III the sequence

$$\left\{ \int_{-l}^{l} f(\lambda) \frac{du_m(\lambda) dg(\lambda)}{d\rho(\lambda)} \right\}$$

is of finite norm and norm $< M \int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[dg(\lambda)]^2}{d\rho(\lambda)}$.

The right hand member of equation (28) is therefore uniformly convergent, and again we may take the limit term by term to obtain equation (26).

We now apply the preceding theory to prove the following theorem which is required in the next section:

THEOREM VI. *If the matrix K is limited, the differential system $\Delta \sum_n k_{mn}\rho_n(\lambda)$ is limited in $(-\infty, \infty)$, and the differential system $\int_\Delta^n (1/\lambda)\, d \sum_n k_{mn}\rho_n(\lambda)$ is limited in $(-\infty, \infty)$ when we exclude an interval $(-\delta, \delta)$ containing the origin.*

By Theorem I the differential system $\Delta\rho_m(\lambda)$ is limited. Further the series $\sum k_{mn}^2$ converges for every value of m since the matrix K is limited, and therefore, by Theorem V,

$$\int_{-\infty}^{\infty} \frac{d \sum_n k_{mn}\rho_n(\lambda) dg(\lambda)}{d\rho(\lambda)} = \sum_n k_{mn} \int_{-\infty}^{\infty} \frac{d\rho_n(\lambda) dg(\lambda)}{d\rho(\lambda)} \tag{29}$$

for every value of m, and every function $\Delta g(\lambda)$ integrable $H(\rho)$. Write

$$y_n = \int_{-\infty}^{\infty} \frac{dg(\lambda) d\rho_n(\lambda)}{d\rho(\lambda)} ,$$

then, by Theorem II such a sequence $\{y_n\}$ exists and is of finite norm for every function $\Delta g(\lambda)$ integrable $H(\rho)$. Further, the sequence $\{\sum k_{mn}y_n\}$ on the right hand side of equation (29) is of finite norm, and it follows that the integral on the left hand side of equation (29) is also of finite norm for every function $\Delta g(\lambda)$ integrable $H(\rho)$, and therefore finally the differential system is limited.

We now consider the differential system $\int_\Delta (1/\lambda)\, d \sum_n k_{mn}\rho_n(\lambda)$ and prove that it is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞), that is, that the sequence

$$\left\{ \int_{-\infty}^{-\delta} (1/\lambda)\, d \sum_n k_{mn}\rho_n(\lambda) dg(\lambda)/d\rho(\lambda) \right. \\ \left. + \int_{\varepsilon}^{\infty} (1/\lambda)\, d \sum k_{mn}\rho_n(\lambda) dg(\lambda)/d\rho(\lambda) \right\} \tag{30}$$

is of finite norm for every function $\Delta g(\lambda)$ integrable $H(\rho)$. Now if $\Delta g(\lambda)$ is integrable $H(\rho)$ in $(-\infty, \infty)$ the function $\int_\Delta (1/\lambda)\, dg(\lambda)$ is also integrable $H(\rho)$ in $(-\infty, -\delta)$ and (δ, ∞). Hence if we write

$$\Delta h(\lambda) = \int_\Delta (1/\lambda)\, dg(\lambda)$$

the expression (30) becomes

$$\left\{ \int_{-\infty}^{-\delta} \frac{d \sum_n k_{mn}\rho_n(\lambda)\, dh(\lambda)}{d\rho(\lambda)} + \int_\delta^\infty \frac{d \sum_n k_{mn}\rho_n(\lambda)\, dh(\lambda)}{d\rho(\lambda)} \right\}$$

which is of finite norm, since we have just proved that the differential system $\Delta \sum k_{mn}\rho_n(\lambda)$ is limited, and the theorem is proved.

A similar method may be used to show that the differential system $\int_\Delta (1/|\lambda|^{1/2})\, d\sum_n k_{mn}\rho_n(\lambda)$ is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞).

4. *Expansion Theorems.* We first derive an expansion theorem for $\int_0^\pi f(x)g(x)dx$ in terms of the solutions $u_a(x)$ and $u(x, \lambda)$ of § 2, where f and g are any two functions of integrable square in $(0, \pi)$. We start from the known expansion in terms of the complete normed orthogonal system, $\phi_n(x)$, of equation (3),

$$\int_0^\pi f(x)g(x)dx = \sum_m \int_0^\pi f\phi_m \int_0^\pi g\phi_m.$$

Now the sequences $\left\{ \int_0^\pi f\phi_m \right\}$ and $\left\{ \int_0^\pi g\phi_m \right\}$ are of finite norm, therefore by equation (15)

$$\int_0^\pi fg = \sum_a \sum_m l_{am} \int_0^\pi f\phi_m \sum_n l_{an} \int_0^\pi g\phi_n$$
$$+ \int_{-\infty}^\infty \frac{d \sum_m \rho_m(\lambda) \int_0^\pi f\phi_m \; d\sum_n \rho_n(\lambda) \int_0^\pi g\phi_n}{d\rho(\lambda)}$$

We now substitute for l_{am} and $\rho_m(\lambda)$ from equations (19) and (16) respectively. This last expansion then becomes

$$\int_0^\pi f(x)g(x)dx = \sum_a \int_0^\pi f(x)u_a(x)dx \int_0^\pi g(x)u_a(x)dx$$
(31)
$$+ \int_{-\infty}^\infty \frac{d \int_0^\pi f(x)u(x,\lambda)dx \; d\int_0^\pi g(x)u(x,\lambda)dx}{d\rho(\lambda)},$$

and this is the desired form.

To find a similar expansion for an arbitrary function $f(x)$, subject to suitably imposed conditions we prove some properties of the differential systems that occur in the expansion.

LEMMA I. *The differential system*

$$\int_{\Delta} (1/\lambda) \cdot d \int_0^{\pi} (pu')'\phi_m$$

is limited in $(-\infty, \infty)$ *if we exclude an interval* $(-\delta, \delta)$ *containing the origin.*

We return to the system of equations (5), and let $\Delta = (\delta, \lambda)$ be an interval of the positive real λ axis. $[\lambda > \delta > 0]$. Then, by equation (18)

$$\int_0^{\pi} (pu')'\phi_m + \int_0^{\pi} qu\phi_m + \int_{\delta}^{\lambda} \lambda d\rho_m(\lambda) = 0. \tag{32}$$

We invert the last integral by means of a theorem of Hellinger,* namely, that if $F(\lambda) = \int_a^{\lambda} u(\lambda) df(\lambda)$ where $u(\lambda)$ is positive and $f(\lambda)$ is of bounded variation in (a, λ), then

$$f(\lambda) - f(a) = \int_a^{\lambda} \frac{dF(\lambda)}{u(\lambda)} .$$

If we write

$$-F(\lambda) = \int_0^{\pi} (pu')'\phi_m + \int_0^{\pi} qu\phi_m$$

we have

$$\rho_m(\lambda) = \int_{\delta}^{\lambda} \frac{dF(\lambda)}{\lambda} ,$$

or

$$\int_{\Delta} (1/\lambda)\, d \int_0^{\pi} (pu')'\phi_m + \int_{\Delta} (1/\lambda)\, d \int_0^{\pi} qu\phi_m + \Delta\rho_m(\lambda) = 0. \tag{33}$$

This theorem of Hellinger is also applicable if the function $u(\lambda)$ is constantly negative in the interval of integration, hence equation (33) holds for any interval Δ of the real λ axis, outside the interval $(-\delta, \delta)$.

The differential system $\Delta\rho_m(\lambda)$ is limited, by Theorem I. Also

$$\int_{\Delta} (1/\lambda)\, d \int_0^{\pi} qu\phi_m = \int_{\Delta} (1/\lambda)\, d \sum_n q_{mn}\rho_n(\lambda)$$

where Q is the limited matrix defined in § 1, and therefore, by Theorem VI, the differential system $\int_{\Delta} (1/\lambda)\, d \int_0^{\pi} qu\phi_m$ is limited in the intervals

* Hellinger, *l. c.*, p. 236.

$(-\infty, -\delta)$ and (δ, ∞). It follows immediately from equation (33) that the differential system $\int_\Delta (1/\lambda)\, d \int_0^\pi (pu')'\phi_m$ is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞).

For the characteristic solutions $u_a(x)$ belonging to the characteristic numbers λ_a we have the corresponding result that *the matrix*

$$(1/\lambda_a) \int_0^\pi (pu_a')'\phi_m$$

*is limited.**

From this result we wish to show that the differential system $\int_\Delta (m/|\lambda|^{1/2})\, d \int_0^\pi pu\phi_m$ is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞), but we first require the following Lemma:

LEMMA II. *The differential system*

$$\int_\Delta (1/m\,|\lambda|^{1/2})\, d \int_0^\pi (pu')'\phi_m \tag{34}$$

is limited in the intervals $(-\infty, -\delta)$ *and* (δ, ∞).

We prove this by introducing corresponding matrices [Definition IV, § 3]. Write

$$v_m(\lambda) = \int_0^\pi (pu')'\phi_m,$$

then the matrix A,

$$a_{mn} = \int_{-\infty}^{\infty} \frac{1}{\lambda} \frac{dv_m(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)}$$

is the limited matrix corresponding to the limited differential system $\int_\Delta (1/\lambda)\, dv_m(\lambda)$ of Lemma I (Theorem III). The matrix B defined by the relation

$$b_{mn} = \int_{-\infty}^{\infty} \frac{1}{m\,|\lambda|^{1/2}} \frac{dv_m(\lambda)\, d\rho_n(\lambda)}{d\rho(\lambda)}$$

exists and is of finite norm for every value of m, and we wish to prove that it is limited. Consider the iterated matrix C

* The results for the characteristic solutions which we state in this section have all been proved for the non-singular equation $u'' + qu + \lambda u = 0$, by Anna Pell Wheeler, *l. c.*, *American Journal of Mathematics*, Vol. 49 (1927), pp. 317-318.

$$c_{mn} = \sum_p b_{mp} b_{np}$$
$$= \sum_p \int_{-\infty}^{\infty} \frac{1}{m\,|\,\lambda\,|^{1/2}} \frac{dv_m(\lambda)\,d\rho_p(\lambda)}{d\rho(\lambda)} \int_{-\infty}^{\infty} \frac{1}{n\,|\,\lambda\,|^{1/2}} \frac{dv_n(\lambda)\,d\rho_p(\lambda)}{d\rho(\lambda)} .$$

By applying equation (23) we find

$$c_{mn} = \int_{-\infty}^{\infty} \frac{dv_m(\lambda)\,dv_n(\lambda)}{mn\,|\,\lambda\,|\,d\rho(\lambda)} ,$$

and by a rearrangement of terms this becomes

$$c_{mn} = \int_{-\infty}^{\infty} \frac{\dfrac{dv_m(\lambda)}{|\,\lambda\,|}\;\dfrac{dv_n(\lambda)}{n^2}\;\dfrac{n}{m}}{d\rho(\lambda)} .$$

Now consider the matrix D,

$$d_{mn} = \int_{-\infty}^{\infty} \frac{\dfrac{dv_m(\lambda)}{|\,\lambda\,|}\;\dfrac{dv_n(\lambda)}{n^2}}{d\rho(\lambda)}$$
$$= \sum_p \int_{-\infty}^{\infty} \frac{dv_m(\lambda)\,d\rho_p(\lambda)}{|\,\lambda\,|\,d\rho(\lambda)} \int_{-\infty}^{\infty} \frac{dv_n(\lambda)\;d\rho_p(\lambda)}{n^2\,d\rho(\lambda)} .$$

By Lemma I the differential system $\int_\Delta (1/|\,\lambda\,|)\; dv_m(\lambda)$ is limited, and

$$v_n(\lambda)/n^2 = (1/n^2) \int_0^\pi (pu')'\phi_m$$
$$= \int_0^\pi pu \sin nx - (1/n) \int_0^\pi p'u \cos nx$$
$$= \sum_p r_{np}\rho_p(\lambda),$$

where R is the limited matrix

$$r_{np} = \int_0^\pi p(x) \sin nx \sin px - (1/n) \int_0^\pi p'(x) \cos nx \sin px.$$

Hence by Theorem VI $\Delta v_n(\lambda)/n^2$ is a limited differential system. It follows that D is the product of two limited matrices and is therefore limited. Further the matrix C is symmetric, and has been expressed in the form

$$c_{mn} = d_{mn} \cdot (n/m)$$

where D is a limited matrix. Hence C is a limited matrix,* and therefore finally B is a limited matrix, which proves the Lemma.

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* Cf. A. J. Pell, *l. c.*, *Transactions of the American Mathematical Society*, Vol. 20, p. 35.

We now deduce the required result, namely,

LEMMA III. *The differential system*

$$(m/|\lambda|^{1/2})\, d\int_0^\pi pu\,\phi_m$$

is limited in the intervals $(-\infty, -\delta)$ *and* (δ, ∞).

By the preceding Lemma the differential system (34) is limited. Also

$$\int_\Delta (1/m\,|\lambda|^{1/2})\, d\int_0^\pi (pu')'\phi_m = \int_\Delta (m/|\lambda|^{1/2})\, d\int_0^\pi pu\sin mx$$
$$-\int_\Delta (1/|\lambda|^{1/2})\, d\int_0^\pi p'u\cos mx$$

where $\int_\Delta (1/|\lambda|^{1/2})\, d\int_0^\pi p'u\cos mx$ is a limited differential system since it may be written in the form $\int_\Delta (1/|\lambda|^{1/2})\, d\sum_n t_{mn}\rho_n(\lambda)$ where T is a limited matrix (Theorem VI). We conclude immediately that the differential system $\int_\Delta (1/|\lambda|^{1/2})\, m\, d\int_0^\pi pu\,\phi_m$ is limited in the intervals $(-\infty, -\delta)$ and (δ, ∞).

The corresponding result for the point spectrum is that *the matrix*

$$(m/|\lambda_a|^{1/2})\int_0^\pi pu_a\phi_m$$

is limited.

We now determine the conditions which must be imposed on an arbitrary function $f(x)$ in order that the function $\int_\Delta |\lambda|^{1/2}\, d\int_0^\pi f(x)\,u(x,\lambda)$ be integrable $H(\rho)$ in $(-\infty, -\delta)$ and (δ, ∞). Return to the equation (33) and apply the theorem given by Carleman *, that if $f(\lambda)$ and $w(\lambda)$ are continuous functions in the interval (δ, λ) and $\alpha(\lambda)$ a function of bounded variation in that interval, then

$$\int_\delta^\lambda f(\lambda)\, d\int_\delta^\lambda \omega(\lambda)\,d\alpha(\lambda) = \int_\delta^\lambda f(\lambda)\omega(\lambda)\,d\alpha(\lambda).$$

For the positive interval (δ, λ) this gives

* Carleman, *l. c.*, p. 11.

$$-\int_\delta^\lambda (\lambda)^{1/2}\, d\rho_m(\lambda) = \int_\delta^\lambda (\lambda)^{1/2}\, d\int_\delta^\lambda (1/\lambda)\, d\int_0^\pi (pu')'\phi_m$$
$$+\int_\delta^\lambda (\lambda)^{1/2}\, d\int_\delta^\lambda (1/\lambda)\, d\int_0^\pi qu\phi_m$$
$$=\int_\delta^\lambda (1/(\lambda)^{1/2})\, d\int_0^\pi (pu')'\phi_m + \int_\delta^\lambda (1/(\lambda)^{1/2})\, d\int_0^\pi qu\phi_m.$$

For the negative interval $(-\delta, -\lambda)$ we obtain similarly

$$-\int_\delta^\lambda |\lambda|^{1/2}\, d\rho_m(\lambda) = \int_\delta^\lambda (\lambda/|\lambda|)\,(1/|\lambda|^{1/2})\, d\int_0^\pi (pu')'\phi_m$$
$$+\int_\delta^\lambda (\lambda/|\lambda|)\,(1/|\lambda|^{1/2})\, d\int_0^\pi qu\phi_m,$$

so that for any interval Δ outside $(-\delta, \delta)$ we have

$$\int_\Delta [1/m(|\lambda|)^{1/2}]\, d\int_0^\pi (pu')'\phi_m + \int_\Delta [1/m(|\lambda|)^{1/2}]\, d\int_0^\pi qu\phi_m$$
$$+\int_\Delta (\lambda/m\,|\lambda|)\,|\lambda|^{1/2}\, d\rho_m(\lambda) = 0.$$

Now let $f(x)$ be a function of x such that the sequence $\{m\int_0^\pi f\phi_m\}$ is of finite norm, multiply this last equation by $m\int_0^\pi f\phi_m$ and sum for m

$$\sum_m m\int_0^\pi f\phi_m \int_\Delta [1/m(|\lambda|)^{1/2}]\, d\int_0^\pi (pu')'\phi_m$$
$$+\sum_m m\int_0^\pi f\phi_m \int_\Delta [1/m(|\lambda|)^{1/2}]\, d\int_0^\pi qu\phi_m$$
$$+\sum_m m\int_0^\pi f\phi_m \int_\Delta (\lambda/|\lambda|)\,(|\lambda|^{1/2}/m)\, d\rho_m(\lambda) = 0.$$

In each of the first two terms of this equation we have the product of a limited differential system and a sequence of finite norm, and they therefore represent functions which are integrable $H(\rho)$ in the intervals $(-\infty, -\delta)$ and (δ, ∞) (Theorem IV). It follows that *the function*

$$(\lambda/|\lambda|)\,|\lambda|^{1/2}\, d\int_0^\pi f(x)u(x,\lambda)\,dx$$

is integrable $H(\rho)$ in $(-\infty, -\delta)$ and (δ, ∞) for every function $f(x)$ such that the sequence $\{m\int_0^\pi f\phi_m\}$ is of finite norm.

For the point spectrum we have the similar result that *the sequence* $\{|\lambda_a|^{1/2}\int_0^\pi f(x)u_a(x)\}$ *is of finite norm for every function* $f(x)$ *such that the sequence* $\{m\int_0^\pi f\phi_m\}$ *is of finite norm.*

An expansion theorem for the function $p(x)f(x)$, where f is any function satisfying the above condition, may now be obtained, starting from the known expansion

$$p(x)f(x)=\sum_m \phi_m(x)\int_0^\pi p(x)f(x)\phi_m(x)\,dx.$$

We divide the real λ axis into three intervals $(-\infty,-\delta)$, $(-\delta,\delta)$ and (δ,∞), and denote the characteristic values of λ belonging to these intervals by λ_{a_1}, λ_{a_2} and λ_{a_3}, respectively. Then by the expansion (31)

$$\begin{aligned}
p(x)f(x)=&\sum_m\phi_m(x)\sum_{a_1}\int_0^\pi p\phi_m u_{a_1}\int_0^\pi fu_{a_1}\\
&\qquad+\sum_m\phi_m(x)\int_{-\infty}^{-\delta}\frac{d\int_0^\pi p\phi_m u\,d\int_0^\pi fu}{d\rho(\lambda)}\\
&+\sum_m\phi_m(x)\sum_{a_2}\int_0^\pi p\phi_m u_{a_2}\int_0^\pi fu_{a_2}\\
&\qquad+\sum_m\phi_m(x)\int_{-\delta}^{\delta}\frac{d\int_0^\pi p\phi_m u\,d\int_0^\pi fu}{d\rho(\lambda)}\\
&+\sum_m\phi_m(x)\sum_{a_3}\int_0^\pi p\phi_m u_{a_3}\int_0^\pi fu_{a_3}\\
&\qquad+\sum_m\phi_m(x)\int_{\delta}^{\infty}\frac{d\int_0^\pi p\phi_m u\,d\int_0^\pi fu}{d\rho(\lambda)}\\
=&A_1+A_2+A_3.
\end{aligned}$$

Consider first the positive interval (δ,∞), the terms in the expansion belonging to this interval being denoted by A_3. Then

$$\begin{aligned}
A_3=&\sum_m\frac{\phi_m(x)}{m}\sum_{a_3}\frac{m\int_0^\pi pu_{a_3}\phi_m}{(\lambda_{a_3})^{1/2}}(\lambda_{a_3})^{1/2}\int_0^\pi fu_{a_3}\\
&+\sum_m\frac{\phi_m(x)}{m}\int_\delta^\infty\frac{\dfrac{m\,d\int_0^\pi pu\phi_m}{(\lambda)^{1/2}}(\lambda)^{1/2}\,d\int_0^\pi fu}{d\rho(\lambda)}
\end{aligned}$$

and, by the properties of the differential system $\int_\Delta (1/\lambda^{1/2})\, m\, d\int_0^\pi pu\phi_m$ and of the matrix $(1/(\lambda_{a_3})^{1/2})\, m\int_0^\pi pu_{a_3}\phi_m$ (Lemma III and Theorem V) we see that this may be written

$$(35)\qquad A_3 = \sum_{a_3} p(x)u_{a_3}(x)\int_0^\pi fu_{a_3} + \int_\delta^\infty \frac{p(x)\,du(x,\lambda)\; d\int_0^\pi f(x)u(x,\lambda)\,dx}{d\rho(\lambda)}.$$

For the negative interval $(-\infty, -\delta)$ we find the corresponding expansion

$$(36)\qquad A_1 = \sum_{a_1} p(x)u_{a_1}(x)\int_0^\pi fu_{a_1} + \int_{-\infty}^{-\delta} \frac{p(x)\,du(x,\lambda)\; d\int_0^\pi f(x)u(x,\lambda)\,dx}{d\rho(\lambda)}$$

by the same method.

We now consider the interval $(-\delta, \delta)$ in which the method of proof has to be modified since the differential system $\int_\Delta (1/|\lambda|^{1/2})\, md\int_0^\pi pu\phi_m$ is not limited in this interval. The proof is obtained from the following Lemmas.

Lemma IV. *The differential system* $m\int_0^\pi pu\phi_m$ *is limited in the interval* $(-\delta, \delta)$.

We return to equation (18) satisfied by $u(x,\lambda)$, and write it in the form

$$\Delta\int_0^\pi (p\phi_m')'u(x,\lambda) + \Delta\int_0^\pi q\phi_m u(x,\lambda) + \int_0^\pi \phi_m \int_\Delta \lambda du(x,\lambda) = 0.$$

In this we substitute the value of $\phi_m(x)$, and obtain the equation

$$-m\,\Delta\int_0^\pi pu\sin mx + \Delta\int_0^\pi p'u\cos mx + (1/m)\,\Delta\int_0^\pi qu\sin mx + (1/m)\int_0^\pi \sin mx\int_\Delta \lambda\, du(x,\lambda) = 0.$$

The differential systems $\Delta \int_0^\pi p'u \cos mx$ and $(1/m) \Delta \int_0^\pi qu \sin mx$ are limited in $(-\delta, \delta)$. Also

$$(1/m) \int_0^\pi \sin mx \int_\Delta \lambda \, du(x, \lambda) = (1/m) \int_\Delta \lambda \, d\rho_m(\lambda)$$

and this differential system is limited in $(-\delta, \delta)$. It follows immediately that the differential system $m \Delta \int_0^\pi pu \sin mx$ is limited in $(-\delta, \delta)$.

LEMMA V. *The function* $\Delta \int_0^\pi f(x) u(x, \lambda) dx$ *is integrable* $H(\rho)$ *in the interval* $(-\delta, \delta)$ *for every function* $f(x)$ *which is of integrable square in* $(0, \pi)$. For

$$\Delta \int_0^\pi f(x) u(x, \lambda) = \Delta \sum_m x_m \rho_m (\lambda),$$

where $\{x_m\}$ is a sequence of finite norm, and $\Delta\rho_m(\lambda)$ is a limited differential system. The truth of the Lemma follows immediately by Theorem IV.

For the characteristic values λ_{α_2} we have similarly that *the matrix* $m \int_0^\pi u_{\alpha_2}\phi_m$ *is limited,* and that *the sequence* $\left[\int_0^\pi f u_{\alpha_2}\right]$ *is of finite norm for every function* $f(x)$ *which is of integrable square in* $(0, \pi)$.

In this interval we have then

$$A_2 = \sum_m \frac{\phi_m(x)}{m} \sum_{\alpha_2} m \int_0^\pi p u_{\alpha_2}\phi_m \int_0^\pi f u_{\alpha_2}$$
$$+ \sum_m \frac{\phi_m(x)}{m} \int_{-\delta}^{\delta} \frac{md \int_0^\pi pu\phi_m \, d \int_0^\pi fu}{d\rho(\lambda)}$$

from which, by the properties we have just proved,

$$A_2 = \sum_{\alpha_2} p(x) u_{\alpha_2}(x) \int_0^\pi f u_{\alpha_2} + \int_{-\delta}^{\delta} \frac{p(x) du(x, \lambda) \, d \int_0^\pi f(x) u(x, \lambda) \, dx}{d\rho(\lambda)} \tag{37}$$

By adding the results (35), (36) and (37) we see that the complete expansion may be expressed as the theorem.

THEOREM VII. *Let $f(x)$ be a function of x such that the sequence $\left[m \int_0^\pi f(x) \sin mx \right]$ is of finite norm, and let $u_\alpha(x)$ and $u(x,\lambda)$ be the functions defined by equations (19) and (17), respectively. Then the function $p(x)f(x)$ may be expanded in the interval $0 \leq x \leq \pi$ as the uniformly convergent series and integral*

$$(38) \qquad p(x)\,f(x) = \sum_\alpha p(x)u_\alpha(x) \int_0^\pi f u_\alpha + \int_{-\infty}^{\infty} \frac{p(x)\,du(x,\lambda)\; d\int_0^\pi f(x)u(x,\lambda)\,dx}{d\rho(\lambda)}$$

where the summation and integration extend over every part of the real λ axis in which there is a spectrum.

5. *Extension of the preceding theory to a continuous spectrum which is not simple.* We now state briefly the generalizations of the preceding theory which are required when we assume the existence of a number (at most denumerable) of continuous spectra. Let $\rho_m{}^{(\alpha)}(\lambda)$ be the characteristic differential forms with corresponding basis functions $\rho^{(\alpha)}(\lambda)\cdot$ $(\alpha = 1, 2, \cdots)$. The orthogonality relations now are

$$\sum_m l_{\alpha m} l_{\beta m} = e_{\alpha\beta},$$

$$\begin{aligned} \sum_m \Delta_1\rho_m{}^{(\alpha)}(\lambda)\,\Delta_2\rho_m{}^{(\beta)}(\lambda) &= 0 \qquad && \alpha \neq \beta \\ &= \Delta_{12}\rho^{(\alpha)}(\lambda) && \alpha = \beta, \end{aligned}$$

$$\sum_m l_{\alpha m}\rho_m{}^{(\beta)}(\lambda) = 0.$$

For the matrix expansions (12) and (13) we find the more general forms

$$-\sum_n (p_{mn} + q_{mn})y_n = \sum_\alpha \lambda_\alpha Y_\alpha l_{\alpha m} + \sum_\alpha \int_{-\infty}^{\infty} \lambda \,\frac{dy^{(\alpha)}(\lambda)\; d\rho_m{}^{(\alpha)}(\lambda)}{d\rho^{(\alpha)}(\lambda)}$$

$$-\sum_{m,n} (p_{mn} + q_{mn})x_m y_n = \sum_\alpha \lambda_\alpha X_\alpha Y_\alpha + \sum_\alpha \int_{-\infty}^{\infty} \lambda \,\frac{dx^{(\alpha)}(\lambda)\; dy^{(\alpha)}(\lambda)}{d\rho^{(\alpha)}(\lambda)}$$

where X and Y have the same values as before, while

$$y^{(\alpha)}(\lambda) = \sum_m \rho_m{}^{(\alpha)}(\lambda) y_m, \qquad x^{(\alpha)}(\lambda) = \sum_m \rho_m{}^{(\alpha)}(\lambda) x_m.$$

From these expansions it follows that the systems l_{am} and $\rho_m^{(a)}(\lambda)$ are complete, and therefore

$$\sum_a l_{am} l_{an} + \sum_a \int_{-\infty}^{\infty} \frac{d\rho_m^{(a)}(\lambda)\, d\rho_n^{(a)}(\lambda)}{d\rho^{(a)}(\lambda)} = e_{mn}.$$

We now have a sequence of differential solutions $u^{(a)}(x, \lambda)$, defined by

$$\rho_m^{(a)}(\lambda) = \int_0^{\pi} u^{(a)}(x, \lambda)\phi_m(x)\, dx \tag{39}$$

satisfying equation (18) and the other conditions imposed on the solutions $u(x, \lambda)$ of § 2. The orthogonality relations in this case are

$$\int_0^{\pi} u_\alpha(x) u_\beta(x)\, dx = e_{\alpha\beta},$$

$$\begin{aligned} \int_0^{\pi} \Delta_1 u^{(\alpha)}(x, \lambda)\, \Delta_2 u^{(\beta)}(x, \lambda)\, dx &= 0 & \alpha \neq \beta \\ &= \Delta_{12}\rho^{(\alpha)}(\lambda) & \alpha = \beta, \end{aligned}$$

$$\int_0^{\pi} u_\alpha(x) \Delta u^{(\beta)}(x, \lambda) = 0.$$

The definitions of § 3 are modified as follows. We say that the sequence $\left\{ \int_\Delta f(\lambda)\, du^{(a)}(\lambda) \right\}$ is *integrable* $H(\rho^{(a)})$ if the integral

$$\int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[du^{(a)}(\lambda)]^2}{d\rho^{(a)}(\lambda)}$$

exists for every value of α, and if further the series

$$\sum_a \int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[du^{(a)}(\lambda)]^2}{d\rho^{(a)}(\lambda)}$$

converges.

The system of functions $\int_\Delta f(\lambda)\, du_m^{(a)}(\lambda)$ is a *limited differential system* if the functions $\int_\Delta f(\lambda)\, du_m^{(a)}(\lambda)$ are integrable $H(\rho^{(a)})$ for every value of m, and if further the sequence

$$\left\{ \int_{-\infty}^{\infty} f(\lambda) \frac{du_m^{(a)}(\lambda)\, dg^{(a)}(\lambda)}{d\rho^{(a)}(\lambda)} \right\}$$

is of finite norm and norm

$$\leq M \sum_{(a)} \int_{-\infty}^{\infty} [f(\lambda)]^2 \frac{[dg^{(a)}(\lambda)]^2}{d\rho^{(a)}(\lambda)}$$

for every value of α, and every $\Delta g^{(a)}(\lambda)$ integrable $H(\rho^{(a)})$.

The *corresponding matrix B* is defined by

$$b_{mn} = \sum_a \int_{-\infty}^{\infty} f(\lambda) \; \frac{du_m^{(a)}(\lambda)\, d\rho_n^{(a)}(\lambda)}{d\rho^{(a)}(\lambda)} \quad .$$

With these definitions the theorems of § 3 are still true, and the expansions of § 4 may be summed up as the theorem.

THEOREM VIII. *Let $u_a(x)$ and $u^{(a)}(x, \lambda)$ be the functions defined by equations (19) and (39) respectively. Then for any two functions $f(x)$ and $g(x)$ of integrable square in the interval $(0, \pi)$*

$$\int_0^\pi f(x) g(x)\, dx = \sum_a \int_0^\pi f(x) u_a(x)\, dx \int_0^\pi g(x) u_a(x)\, dx$$
$$+ \sum_a \int_{-\infty}^{\infty} \frac{d\int_0^\pi f(x) u^{(a)}(x,\lambda)\, dx \; d\int_0^\pi g(x) u^{(a)}(x,\lambda)\, dx}{d\rho^{(a)}(\lambda)} \quad .$$

Further, if $f(x)$ is a continuous function of x satisfying the boundary conditions (2) or (2′) and having a first derivative which is integrable and of integrable square in the interval $(0, \pi)$, then

$$p(x) f(x) = \sum_a p(x) u_a(x) \int_0^\pi f(x) u_a(x)\, dx$$
$$+ \sum_a \int_{-\infty}^{\infty} \frac{p(x)\, du^{(a)}(x,\lambda)\; d\int_0^\pi f(x) u^{(a)}(x,\lambda)\, dx}{d\rho^{(a)}(\lambda)} \quad ,$$

where the series and integrals involved are uniformly and absolutely convergent.

Note. Instead of equation (1) we might consider the more general equation

(1′) $$(d/dx)[p(x)\, du/dx] + q(x) u + \lambda k(x) u = 0,$$

where k is continuous and positive in the interval $(0, \pi)$, the boundary conditions being given by equation (2) or (2′). If we assume that the function

$$\frac{1}{(k)^{1/2}} \frac{d}{dx} \left(p(x) \; \frac{d}{dx} \; \frac{1}{(k)^{1/2}} \right)$$

exists and is continuous, equation (1′) may be reduced to the form of the original equation (1)*. The solutions of equation (1′) are therefore found from the solutions of the corresponding equation (1).

* Cf. Hilbert, *l. c.*, p. 51.

In this case we find the general expansion

$$p(x)f(x) = \sum_a p(x)u_a(x)\int_0^\pi kfu_a + \sum_a \int_{-\infty}^{\infty} \frac{p(x)\,du^{(a)}(x,\lambda)\; d\int_0^\pi kfu^{(a)}}{d\rho^{(a)}(\lambda)}$$

where the series and integrals converge uniformly and absolutely for every function $f(x)$ satisfying the conditions previously imposed, namely, that f be continuous, satisfy the boundary conditions, and have a first derivative which is integrable and of integrable square in the interval $(0, \pi)$.