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Non-self-adjoint boundary-value problem with discontinuous density function

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We determine spectrum and principal functions of the non-self-adjoint differential operator corresponding to 1-D non-self-adjoint Schrödinger equation with discontinuous density function, provide some sufficient conditions guaranteeing finiteness of eigenvalues and spectral singularities, and introduce the convergence properties of principal functions. Copyright © 2009 John Wiley & Sons, Ltd.

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1. Introduction and preliminaries

In this paper, we are concerned with the non-self-adjoint boundary-value problem (BVP) corresponding to

$$-y''+q(x)y=k^2\rho(x)y, \quad x\in\mathbb{R}_+=[0,\infty) \tag{1}$$

and

$$y(0)=0, \tag{2}$$

where k is a complex parameter and ρ is the discontinuous density function given by

$$\rho(x)=\begin{cases}\gamma^2 & \text{if } 0\leqslant x\leqslant a\\ 1 & \text{if } a<x<\infty\end{cases} \quad \text{and} \quad \gamma>1. \tag{3}$$

The complex-valued function q is assumed to satisfy the condition

$$\int_0^\infty x|q(x)|\,dx<\infty, \tag{4}$$

unless otherwise stated.

In the recent years, Sturm–Liouville problems (SLPs) with discontinuous coefficients took a prominent attention since they arise naturally in diverse fields of application. For instance, SLPs with quasi-periodic boundary conditions have been treated in [1, 2] to investigate the propagation of harmonic waves in elastic composites with periodic structure. SLPs with discontinuous coefficients also appear in vibration problems in geophysics (see e.g. [3–5]). Upper and lower bounds for the eigenvalues of a SLP with discontinuous coefficients arising in heat conduction in a layered composite have been determined by Horgan and Nasser in [6].

For $\gamma=1$, i.e. $\rho(x)\equiv 1$, spectral analysis of the non-self-adjoint BVP (1)–(2) is intensively investigated in [7]. In the case $0<\gamma\neq 1$, the inverse problem corresponding to BVP (1)–(2) is handled in [4, 8], in which q is assumed to be real-valued function. Note that q is allowed to take complex values in this study. In addition, the existence of discontinuous coefficient (3) in Equation (1) makes the

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investigation of the problem quite challenging as it strongly influences the structure of Jost solution and hence the treatment of the problem. For instance, while the Jost solution is given by

$$f(x,k)=e^{ikx}+\int_x^\infty K(x,t)e^{ikt}\,dt$$

in continuous case (see [7, Appendix II, Theorem 2]), the Jost solution in discontinuous case (see [8, Theorem 1]) is obtained as

$$e(x,k)=e_0(x,k)+\int_{\mu^+(x)}^\infty H(x,t)e^{ikt}\,dt, \tag{5}$$

where

$$e_0(x,k)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)e^{i\mu^+(x)}+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)e^{i\mu^-(x)}$$

and

$$\mu^\pm(x)=\pm x\sqrt{\rho(x)}+a(1\mp\sqrt{\rho(x)}).$$

It is obvious that

$$e(x,k)=f(x,k)\quad \text{for } a<x<\infty.$$

It is worth mentioning that the non-self-adjoint singular BVP

$$-y''+q(x)y=\lambda\rho(x)y,\quad x\in\mathbb{R}_+=[0,\infty), \tag{6}$$

with

$$y'(0)-\lambda\sum_{n=1}^m \alpha_n y(a_n)+\lambda\int_0^\infty G(x)y(x)\,dx=0 \tag{7}$$

is handled by [9] in which the adjoint problem is constructed and the discrete and continuous spectrums are determined. However, convergence properties of the principal vectors corresponding to eigenvalues and spectral singularities are not discussed in [9].

Let $L_2(0,\infty,\rho(x))$ denote the Hilbert space of complex-valued functions having the following property:

$$\int_0^\infty \rho(x)|f(x)|^2\,dx<\infty.$$

Hereafter, we shall denote by L_ρ the differential operator generated by the BVP (1)–(2) in $L_2(0,\infty,\rho(x))$.

The purpose of this paper is two fold: First, we investigate the quantitative properties of spectrum of the operator L_ρ and second, we introduce the convergence properties of the principal vectors. The treatment of the problem can be outlined as follows:

- Construction of the Jost solution and the solutions $S(x,k)$ and $C(x,k)$ satisfying

$$\left.\begin{array}{ll} S(0,k)=0, & S'(0,k)=1 \\ C(0,k)=1, & C'(0,k)=0 \end{array}\right\}. \tag{8}$$

- Formulation of the resolvent operator.
- Expression of the sets of eigenvalues and spectral singularities in terms of singularities of the kernel of resolvent operator.
- Use of boundary uniqueness theorems to show that

$$\sup_{x\in\mathbb{R}_+}\{\exp(\varepsilon x^\delta)|q(x)|\}<\infty\quad \text{for some } \varepsilon>0 \text{ and } \tfrac{1}{2}\leqslant\delta\leqslant 1 \tag{9}$$

  is the condition guaranteeing finiteness of eigenvalues, spectral singularities, and their multiplicities,
- Determination of the principal vectors corresponding to eigenvalues and spectral singularities and investigation of their convergence properties.

1.1. *Solutions of Equation* (1)

Next, we present some preliminary results from [7] and [10] which will be used in our further work.

1.1.1. *The solutions $S(x,k)$ and $C(x,k)$.*

We denote by $S(x,k)$ and $C(x,k)$ the solutions of Equation (1) satisfying (8). $S(x,k)$ and $C(x,k)$ have the following representations:

$$S(x,k)=\frac{\sin k\gamma x}{k\gamma}+\int_0^x A(x,t)\frac{\sin k\gamma t}{k\gamma}\,dt \tag{10}$$

and

$$C(x,k)=\cos k\gamma x+\int_0^x B(x,t)\cos k\gamma t\,\mathrm{d}t \tag{11}$$

for $x\in[0,a]$, where the kernels $A(x,t)$ and $B(x,t)$ have continuous derivatives with respect to x and t and satisfy

$$\frac{\partial^2}{\partial x^2}A(x,t)-\frac{\partial^2}{\partial t^2}A(x,t)=q(x)A(x,t),$$

$$\frac{\partial^2}{\partial x^2}B(x,t)-\frac{\partial^2}{\partial t^2}B(x,t)=q(x)B(x,t),$$

and

$$A'_x(x,x)=\tfrac{1}{2}q(x),\quad A'_t(x,t)|_{t=0}=0,$$

$$B'_x(x,x)=\tfrac{1}{2}q(x),\quad B'_t(x,t)|_{t=0}=0.$$

Therefore, $S(x,k)$ and $C(x,k)$ are entire functions of k and satisfy the following estimates:

$$\begin{aligned} S(x,k)&=\frac{\sin k\gamma x}{k\gamma}\left(1+O\left(\frac{1}{k}\right)\right) \quad \text{as } |k|\to\infty\\ C(x,k)&=\cos k\gamma x\left(1+O\left(\frac{1}{k}\right)\right) \quad \text{as } |k|\to\infty \end{aligned} \tag{12}$$

for $x\in[0,a]$ (see [7, Sections 25–26] and [11, Chapter VI]).

1.1.2. Jost solution $F(x,k)$. As mentioned before, (5) is the Jost solution of Equation (1). However, in this paper we will consider slightly different Jost solution for convenience.

On the interval (a,∞), Equation (1) becomes

$$-y''+q(x)y=k^2y,\quad a<x<\infty, \tag{13}$$

which has the solution $f(x,k)$ satisfying

$$\lim_{x\to\infty} f(x,k)\exp(-ikx)=1,$$

for $k\in\overline{\mathbb{C}}_+:=\{k:k\in\mathbb{C},\mathrm{Im}k\geqslant 0\}$. In [7], $f(x,k)$ is called the Jost solution of (13). Under the condition (4), $f(x,k)$ has the following representation:

$$f(x,k)=\exp(ikx)+\int_x^\infty K(x,t)\exp(ikt)\,\mathrm{d}t\quad \text{for } x\in(a,\infty), \tag{14}$$

where the kernel $K(x,t)$ is expressed in terms of q, and is continuously differentiable with respect to its arguments. On the other hand, $K(x,t)$ satisfies

$$K(x,x)=\frac{1}{2}\sigma(x),$$

$$|K(x,t)t|\leqslant\frac{1}{2}\sigma\left(\frac{x+t}{2}\right)\exp\left(\sigma_1(x)-\sigma_1\left(\frac{x+t}{2}\right)\right), \tag{15}$$

$$|K_x(x,t)|\leqslant\frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right|+C\sigma\left(\frac{x+t}{2}\right), \tag{16}$$

where $C>0$ is constant and

$$\sigma(x)=\int_x^\infty|q(t)|\,\mathrm{d}t\quad\text{and}\quad\sigma_1(x)=\int_x^\infty\sigma(t)\,\mathrm{d}t.$$

Hence, $f(x,k)$ is analytic with respect to k in $\mathbb{C}_+:=\{k:k\in C,\mathrm{Im}k>0\}$ and continuous on the real axis. Moreover, for $k\in\overline{\mathbb{C}}_+$ and $x\in(a,\infty)$ the following estimates hold:

$$f(x,k)=\exp(ikx)\left[1+O\left(\frac{1}{k}\right)\right]\quad\text{as } |k|\to\infty, \tag{17}$$

$$f'_x(x,k)=ik\exp(ikx)\left[1+O\left(\frac{1}{k}\right)\right]\quad\text{as } |k|\to\infty. \tag{18}$$

By making use of continuity of the functions f and f'_x on the closed interval $[0,a]$, one can obtain the Jost solution $F(x,k)$ of Equation (1), where

$$F(x,k)=\begin{cases} H(k)S(x,k)+J(k)C(x,k) & \text{for } x\in[0,a], \\ \exp(ikx)+\displaystyle\int_x^\infty K(x,t)\exp(ikt)\,dt & \text{for } x\in(a,\infty), \end{cases} \tag{19}$$

and

$$\begin{aligned} H(k) &= C(a,k)f'(a,k)-C'(a,k)f(a,k), \\ J(k) &= f(a,k)S'(a,k)-f'(a,k)S(a,k), \end{aligned} \tag{20}$$

for $k\in\overline{\mathbb{C}}_+$. Note that

$$F(0,k)=J(k) \tag{21}$$

Thus, from (12), (17), and (19) the solution F can be represented asymptotically as follows:

$$F(x,k)=\begin{cases} \exp(ika)\left[\cos k\gamma(x-a)+\dfrac{i}{\gamma}\sin k\gamma(x-a)\right]\left[1+O\left(\dfrac{1}{k}\right)\right], & 0\leqslant x\leqslant a, \\ \exp(ikx)\left[1+O\left(\dfrac{1}{k}\right)\right], & a<x<\infty, \end{cases} \tag{22}$$

for $k\in\overline{\mathbb{C}}_+$ as $|k|\to\infty$ (see [10, Equation 5]).

Observe that the Jost solution given by (5) also satisfies the estimate (22).

1.1.3. The solution $F_1(x,k)$. We now give another solution $F_1(x,k)$ of Equation (1) on the half-axis $[0,\infty)$.

Denote by $f_1(x,k)$ the solution of Equation (13) satisfying

$$\lim_{x\to\infty} f_1(x,k)\exp(ikx)=1,$$

for $x\in(a,\infty)$ and $k\in\overline{\mathbb{C}}_+:=\{k:k\in\mathbb{C},\ \operatorname{Im}k\geqslant 0\}$. It is shown in [7] that

$$f_1(x,k)=\exp(-ikx)\left[1+O\left(\frac{1}{k}\right)\right] \quad \text{as } |k|\to\infty.$$

Therefore, the solution $F_1(x,k)$ can be expressed asymptotically as follows:

$$F_1(x,k)=\begin{cases} \exp(-ika)\left[\cos k\gamma(x-a)-\dfrac{i}{\gamma}\sin k\gamma(x-a)\right]\left[1+O\left(\dfrac{1}{k}\right)\right], & 0\leqslant x\leqslant a, \\ \exp(-ikx)\left[1+O\left(\dfrac{1}{k}\right)\right], & a<x<\infty, \end{cases}$$

for $k\in\overline{\mathbb{C}}_+$ as $|k|\to\infty$.

2. The resolvent and the spectrum

In this section we construct the resolvent of the BVP (1)–(2), determine the sets of eigenvalues and spectral singularities, and discuss their quantitative properties.

The resolvent set $\mathscr{R}(L_\rho)$ is given by

$$\mathscr{R}(L_\rho)=\{\lambda:\lambda=k^2, \operatorname{Im}k>0 \text{ and } J(k)\neq 0\}.$$

If $\lambda\in\mathscr{R}(L_\rho)$, then the resolvent $R_\lambda(L_\rho)$ of the operator L_ρ exists. That is, there exists a solution $y(x,k)$ of the BVP

$$-y''+[q(x)-k^2\rho(x)]y=\rho\psi, \quad x\in\mathbb{R}_+=[0,\infty),$$

$$y(0)=0,$$

which belongs to $L_2(0,\infty,\rho(x))$. The solution $y(x,k)$ has the following representation:

$$y(x,k)=R_\lambda(L_\rho)\psi(x)=\int_0^\infty G(x,t;k)\rho(t)\psi(t)\,dt \quad \text{for } \lambda\in\mathscr{R}(L_\rho), \tag{23}$$

where $\psi \in L_2(0,\infty,\rho(x))$ and $G(x,t;k)$ is Green's Function given by

$$G(x,t;k):=\begin{cases} \dfrac{F_1(0,k)F(x,k)F(t,k)}{2ikJ(k)}-\dfrac{F(x,k)F_1(t,k)}{2ik} & \text{for } 0\leqslant t\leqslant x,\\[2ex] \dfrac{F_1(0,k)F(x,k)F(t,k)}{2ikJ(k)}-\dfrac{F(t,k)F_1(x,k)}{2ik} & \text{for } x<t<\infty, \end{cases} \tag{24}$$

in which (21) is taken into account.

From the definition of eigenvalues, (17) and (24), we have

$$\sigma_d(L_\rho)=\{\lambda=k^2:\operatorname{Im}k>0 \text{ and } J(k)=0\}, \tag{25}$$

where $\sigma_d(L_\rho)$ denotes the set of all eigenvalues of the operator L_ρ.

In [9], the adjoint problem of non-self-adjoint singular BVP (6)–(7) is constructed and the continuous spectrum is determined. Similar to the one in [9, Theorem 5], the continuous spectrum $\sigma_c(L_\rho)$ of the operator L_ρ can be obtained as

$$\sigma_c(L_\rho)=[0,\infty).$$

Spectral singularities are defined to be the poles of the kernel of the resolvent operator, which are embedded in continuous spectrum [12]. Thus, the set of spectral singularities can be expressed as

$$\sigma_{ss}(L_\rho)=\{\lambda=k^2:\operatorname{Im}k=0,\ k\neq 0 \text{ and } J(k)=0\}. \tag{26}$$

Hereafter, we will discuss quantitative properties of eigenvalues and spectral singularities of the operator L_ρ. In order to do so, we should discuss quantitative properties of the set of zeros of $J(k)$ in the upper half-plane $\overline{\mathbb{C}}_+$.

Lemma 1

Suppose (4) holds. Then

(i) the set of eigenvalues is bounded, is no more than countable, and its limits points can lie only in the positive real axis,
(ii) the set of spectral singularities is compact and its linear Lebesgue measure is zero.

Proof

It follows from (14)–(15) that the function J is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$. On the other hand, by (21)–(22) we have the following estimate:

$$\begin{aligned} J(k) &= \exp(ika)\left[\cos k\gamma a-\frac{i}{\gamma}\sin k\gamma a\right]\left[1+O\left(\frac{1}{k}\right)\right],\\ &=\left[\frac{1}{2}e^{ika(1+\gamma)}\left(1-\frac{1}{\gamma}\right)+\frac{1}{2}e^{-ika(-1+\gamma)}\left(1+\frac{1}{\gamma}\right)\right]\left[1+O\left(\frac{1}{k}\right)\right], \end{aligned} \tag{27}$$

for $k=\xi+i\tau\in\overline{\mathbb{C}}_+$ as $\tau\to\infty$. Since $\gamma>1$, the term $\exp(-ika(\gamma-1))$ in (27) explodes while the first term approaches zero as $\tau\to\infty$. That is, J cannot tend to zero for sufficiently large $k\in\overline{\mathbb{C}}_+$. Thus, boundedness of the sets $\sigma_d(L_\rho)$ and $\sigma_{ss}(L_\rho)$ follows from (25) and (26). Furthermore, from the analyticity of the function J in $\mathbb{C}_+$, the set $\sigma_d(L_\rho)$ has at most countable number of elements and its limit points can lie only in positive semi-axis. From boundary uniqueness theorem of analytic functions [13], we deduce that the set $\sigma_{ss}(L_\rho)$ is closed and that its linear Lebesgue measure is zero. □

Henceforth, we will call the multiplicity of a zero of the function J in $\overline{\mathbb{C}}_+$ as the multiplicity of the corresponding eigenvalue and spectral singularity of the BVP (1) and (2).

In the following, we put an additional restriction on q to guarantee the finiteness of $\sigma_d(L_\rho)$ and $\sigma_{ss}(L_\rho)$.

Theorem 1

If for every $\varepsilon>0$

$$\sup_{x\in[0,\infty)}\{e^{\varepsilon x}|q(x)|\}<\infty \tag{28}$$

holds, then the operator L_ρ has finite number of eigenvalues and spectral singularities and each of them is of a finite multiplicity.

Proof

It follows from (15) that

$$|K(a,t)|\leqslant C\exp\left(-\frac{\varepsilon}{2}\right)t.$$

That is, the function J has an analytic continuation from the real axis to the lower half-plane $\operatorname{Im}k>-\varepsilon/2$. Thus, the sets $\sigma_d(L_\rho)$ and $\sigma_{ss}(L_\rho)$ have no limit points on positive real line. As they are bounded, they have finite number of elements. Finally, analyticity of J in $\operatorname{Im}k>-\varepsilon/2$ implies that all zeros of J in $\overline{\mathbb{C}}_+$ has finite multiplicity. □

Let us consider the condition

$$\sup_{x\in\mathbb{R}_+}\{\exp(\varepsilon x^{\delta})|q(x)|\}<\infty \quad \text{for some } \varepsilon>0 \text{ and } \tfrac{1}{2}\leqslant\delta<1. \tag{29}$$

Remark 1

The condition (29) is weaker than the condition (28), and under the condition (29) the function J does not have any analytic continuation from the real axis to the lower half-plane. Therefore, finiteness of eigenvalues cannot be proved in a similar way given in the preceding theorem.

We classify zeros of the function J in the closed-upper half-plane $\overline{\mathbb{C}}_+$ as follows:

$$N^+=\{k\in\mathbb{C}_+ : J(k)=0\}, \tag{30}$$

$$N=\{k\in\mathbb{R}-\{0\} : J(k)=0\}. \tag{31}$$

Denote by N_0^+ and N_0 the sets of limit points of N^+ and N, respectively. In addition, N_∞ represents the set of zeros of $J(k)$ with infinite multiplicity. From the uniqueness theorem of analytic functions (see [14]), we obtain that

$$N^+\cap N_\infty=\emptyset,\quad N_0^+\subset N,\quad N_0\subset N,\quad N_\infty\subset N.$$

From continuity of all derivatives of J up to the real axis, we observe that

$$N_0^+\subset N_\infty \quad\text{and}\quad N_0\subset N_\infty. \tag{32}$$

To prove the next result we will employ the following theorem [15]:

Theorem 2

Assume that the function g is analytic in $\mathbb{C}_+$, all of its derivatives are continuous up to the real axis, and there exists a $T>0$ such that

$$|g^{(r)}(\lambda)|\leqslant M_r \quad \text{for } r=0,1,2,\ldots,\lambda\in\mathbb{C}_+,\ |\lambda|<T$$

and

$$\int_{-\infty}^{-T}\frac{\ln|g(x)|}{1+x^2}\,dx<\infty,\quad \int_T^{\infty}\frac{\ln|g(x)|}{1+x^2}\,dx<\infty \tag{33}$$

hold. If the set P with linear Lebesgue measure zero is the set of all zeros of the function g with infinite multiplicity and if

$$\int_0^h \ln E(s)\,d\mu(P_s)=-\infty,$$

then $g(\lambda)\equiv 0$, where $E(s)=\inf\{M_r s^r/r! : r=0,1,2,\ldots\}$, $\mu(P_s)$ is the Lebesgue measure of the s-neighborhood of P, and h is an arbitrary positive constant.

Theorem 3

If (29) holds, then

$$N_\infty=\emptyset. \tag{34}$$

Proof

From (20) we find

$$J(k)=\sum_{i=1}^{2}(-1)^{i-1}J_i(k)J_{i+2}(k), \tag{35}$$

where

$$J_1(k)=1+\int_a^{\infty}K(a,t)e^{ik(t-a)}\,dt,$$

$$J_2(k)=ik-\frac{K(a,a)}{2}+\int_a^{\infty}K_x(a,t)e^{ik(t-a)}\,dt,$$

$$J_3(k)=\cos k\gamma a e^{ika}+\frac{A(a,a)}{2}\zeta(a,k)+\int_0^a A_x(a,t)\zeta(t,k)e^{ik(a-t)}\,dt,$$

$$J_4(k)=\zeta(a,k)+\int_0^a A(a,t)\zeta(t,k)e^{ik(a-t)}\,dt,$$

$$\zeta(t,k)=\frac{\sin k\gamma t}{k\gamma}e^{ikt}=\frac{1}{2\gamma}\int_{(1-\gamma)t}^{(1+\gamma)t}e^{ikr}\,dr.$$

It follows from (10), (11), and (14)–(16) that J is analytic in $\mathbb{C}_+$ and all of its derivatives are continuous up to the real axis. As the set N is bounded, there exists a sufficiently large $T>0$ such that the function J satisfies the inequalities in (33). On the other hand, from (15)–(16) and (29) we have

$$\left|\frac{d^m}{dk^m}J_i(k)\right| \leqslant c_1\int_a^\infty (2(1+\gamma)t)^m \exp\left(-\frac{\varepsilon}{2}\left(\frac{t}{2}\right)^\delta\right)dt, \quad i=1,2, \tag{36}$$

where $k\in\mathbb{C}_+$, $|k|<T$, and $m=0,1,2,\ldots$. Taking continuity of the functions A and A_x into account we obtain

$$\left|\frac{d^m}{dk^m}J_i(k)\right| \leqslant c_2(2a(1+\gamma))^m, \quad i=3,4, \tag{37}$$

where $k\in\mathbb{C}_+$, $|k|<T$, and $m=0,1,2,\ldots$. Therefore, from (35), (36), and (37) we arrive at

$$\begin{aligned}\left|\frac{d^n}{dk^n}J(k)\right| &\leqslant \sum_{i=1}^{2}\sum_{s=0}^{n}\binom{n}{s}\left|\frac{d^{n-s}}{dk^{n-s}}J_i(k)\right|\left|\frac{d^s}{dk^s}J_{i+2}(k)\right| \\ &\leqslant C2^n\sum_{s=0}^{n}\binom{n}{s}\int_a^\infty (a(1+\gamma))^s(t(1+\gamma))^{n-s}\exp\left(-\frac{\varepsilon}{2}\left(\frac{t}{2}\right)^\delta\right)dt \\ &\leqslant C2^{2n}(1+\gamma)^n\int_0^\infty t^n\exp\left(-\frac{\varepsilon}{2}\left(\frac{t}{2}\right)^\delta\right)dt\end{aligned} \tag{38}$$

for $n=1,2,\ldots$. We obtain by (27) and (38) that

$$\left|J(k)-\frac{1}{2}e^{-ika(-1+\gamma)}\left(1+\frac{1}{\gamma}\right)\right|<\infty$$

and

$$\left|\frac{d^n}{dk^n}J(k)\right| \leqslant M_n \quad \text{for } n=1,2,\ldots \text{ and } k\in\mathbb{C}_+, |k|<T,$$

where

$$M_n = C2^{2n}(1+\gamma)^n\int_0^\infty t^n\exp\left(-\frac{\varepsilon}{2}\left(\frac{t}{2}\right)^\delta\right)dt.$$

As J is not equally zero, from Theorem 2 we find

$$\int_0^h \ln E(s)\, d\mu(N_\infty, s) > -\infty. \tag{39}$$

For M_n we have the following estimate:

$$M_n \leqslant Dd^n n^{n(1-\delta)/\delta} n!, \tag{40}$$

where D and d are constants depending on C, ε, and δ. Thus,

$$\begin{aligned}E(s) &\leqslant D\inf\{Dd^n n^{n(1-\delta)/\delta} n! : n=0,1,2,\ldots\}, \\ &\leqslant D\exp\left\{-\frac{1-\delta}{\delta}e^{-1/(1-\delta)}b^{-\delta/(1-\delta)}s^{-\delta/(1-\delta)}\right\}.\end{aligned} \tag{41}$$

It follows from $\delta/(1-\delta)\geqslant 1$, (39), and (41) that

$$\int_0^h s^{-\delta/(1-\delta)}\, d\mu(N_\infty, s) < \infty,$$

which is possible only if $d\mu(N_\infty, s)=0$. That is, $N_\infty=\emptyset$. The proof is complete. □

Theorem 4

If (29) holds, then the operator L_ρ has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.

Proof

From Lemma 1, (32), and (34) we conclude that the sets $\sigma_d(L_\rho)=\{\lambda=k^2 : k\in N^+\}$ and $\sigma_{ss}(L_\rho)=\{\lambda=k^2 : k\in N\}$ are countable and bounded and they have no limit points. Thus, the sets $\sigma_d(L_\rho)$ and $\sigma_{ss}(L_\rho)$ have finite number of elements. Moreover, it follows from (34) that eigenvalues and spectral singularities are of finite multiplicity. □

It is natural to ask what happens when $0<\delta<\frac{1}{2}$. The next theorem shows that the set of eigenvalues of L_ρ may not have finite number of elements if the number δ is allowed to be less than $\frac{1}{2}$.

Theorem 5
If

$$\sup_{x\in[0,\infty)}\{e^{\varepsilon x^\delta}|q(x)|\}<\infty,\quad 0<\delta<\frac{1}{2}$$

holds for every $\varepsilon>0$, then

$$\sum_n(\ell_n^+)^{(1-2\delta)/(1-\delta)}<\infty,$$

where $\{\ell_n^+\}$ is the sequence of lengths of all finite complementary intervals of N_∞.

Proof
By making use of Carleson's theorem [16], the proof can be done by similar to that of [17, Theorem 4.6]. □

Combining Theorems 1 and 4, we arrive at the next result.

Corollary 1
Given the differential operator corresponding to the BVP (1)–(2). Equation (9) is the condition that guarantees the finiteness of eigenvalues, spectral singularities, and their multiplicities.

3. The principal vectors

In this section, we determine the principal vectors of the operator L_ρ corresponding to its eigenvalues and spectral singularities. We start with the following definition.

Definition 1
Let $\lambda=\lambda_0$ be an eigenvalue of L_ρ. If the vectors $y^{(0)}(x,\lambda_0)$, $y^{(1)}(x,\lambda_0),\ldots,y^{(s)}(x,\lambda_0)$ satisfy the equations

$$\left[-\frac{d^2}{dx^2}+q(x)-\lambda_0\rho(x)\right]y^{(0)}(x,\lambda_0)=0$$

$$\left[-\frac{d^2}{dx^2}+q(x)-\lambda_0\rho(x)\right]y^{(n)}(x,\lambda_0)-\rho(x)y^{(n-1)}(x,\lambda_0)=0$$

for $n=0,1,\ldots,s$, then the vector $y^{(0)}(x,\lambda_0)$ is said to be the eigenvector corresponding to the eigenvalue $\lambda=\lambda_0$ of L_ρ. The vectors $y^{(1)}(x,\lambda_0),\ldots,y^{(s)}(x,\lambda_0)$ are called associated vectors corresponding to the eigenvalue $\lambda=\lambda_0$. The eigenvectors and associated vectors corresponding to $\lambda=\lambda_0$ are called the principal vectors of the eigenvalue $\lambda=\lambda_0$. The principal vectors corresponding to the spectral singularities are defined similarly.

Henceforth, we assume that the condition (28) holds. Let $k_1,k_2,\ldots,k_p$ denote the zeros of the function $J(k)=F(0,k)$ in $\mathbb{C}_+$ (whose squares are the eigenvalues of L_ρ) with multiplicities $m_1,m_2,\ldots,m_p$, respectively. Similarly, let $k_{p+1},k_{p+2},\ldots,k_q$ be zeros of $J(k)$ in $\mathbb{R}$ with multiplicities $m_{p+1},m_{p+2},\ldots,m_q$, respectively. It is obvious that

$$\frac{d^n}{dk^n}\{W[F,S]\}_{k=k_j}=\left\{\frac{d^n}{dk^n}J(k)\right\}_{k=k_j}=0 \tag{42}$$

for $n=0,1,\ldots,m_j-1$ and $j=1,2,\ldots,q$.

Theorem 6
The formula

$$\left\{\frac{d^n}{dk^n}S(x,k)\right\}_{k=k_j}=\sum_{s=0}^{n}\binom{n}{s}\beta_{n-s}\left\{\frac{d^s}{dk^s}F(x,k)\right\}_{k=k_j} \tag{43}$$

holds for $x\in[0,\infty)$, $n=0,1,\ldots,m_j-1$, and $j=1,2,\ldots,q$.

Proof
We shall use the mathematical induction. It is evident from (42) that

$$S(x,k_j)=\beta_0F(x,k_j)$$

for $j=1,2,\ldots,q$. Suppose that (43) holds for an arbitrary integer n_0 such that $1\leqslant n_0\leqslant m_j-2$. Differentiating the equation

$$\left[-\frac{d^2}{dx^2}+q(x)-k^2\rho(x)\right]y(x,k)=0 \tag{44}$$

with respect to k and substituting $k=k_j$, we have

$$\left[-\frac{d^2}{dx^2}+q(x)-k_j^2\rho(x)\right]y^{(n_0+1)}(x,k_j)=2\rho(x)[k_j(n_0+1)y^{(n_0)}(x,k_j)+n_0(n_0+1)y^{(n_0-1)}(x,k_j)], \tag{45}$$

where

$$y^{(n)}(x,k_j)=\left\{\frac{d^n}{dk^n}y(x,k)\right\}_{k=k_j}.$$

As $S(x,k)$ and $F(x,k)$ solve Equation (44), we obtain by (43)–(45) that

$$\left[-\frac{d^2}{dx^2}+q(x)-k^2\rho(x)\right]g^{(n_0)}(x,k_j)=0 \quad \text{for } j=1,2,\ldots,q,$$

where

$$g^{(n_0)}(x,k_j)=\left\{\frac{d^{n_0+1}}{dk^{n_0+1}}S(x,k)\right\}_{k=k_j}-\sum_{s=1}^{n_0+1}\binom{n_0+1}{s}\beta_{n_0+1-s}\left\{\frac{d^s}{dk^s}F(x,k)\right\}_{k=k_j}. \tag{46}$$

On the other hand, (42) and (46) imply that

$$W[g^{(n_0)}(x,k),F(x,k)]_{k=k_j}=\frac{d^{n_0+1}}{dk^{n_0+1}}W[S(x,k),F(x,k)]_{k=k_j}=0,$$

and therefore,

$$g^{(n_0)}(x,k_j)=\beta_{n_0+1}F(x,k_j)$$

for $j=1,2,\ldots,q$. The proof is complete. □

Now we introduce the vectors

$$\Phi^{(n)}(x,\lambda_j)=\begin{cases}\dfrac{1}{n!}\left\{\dfrac{d^n}{dk^n}S(x,k)\right\}_{k=k_j} & \text{for } x\in[0,a]\\ \displaystyle\sum_{s=0}^{n}B_{n,s}\left\{\frac{d^s}{dk^s}F(x,k)\right\}_{k=k_j} & \text{for } x\in(a,\infty)\end{cases} \tag{47}$$

for $\lambda_j=k_j^2$, we have

$$B_{n,s}=\frac{1}{n!}\binom{n}{s}\beta_{n-s},$$

where $n=0,1,\ldots,m_j-1$ and $j=1,2,\ldots,q$. It follows from the definition that the vectors $\Phi^{(n)}(x,\lambda_j)$, for $n=0,1,\ldots,m_j-1$ and $j=1,2,\ldots,p$ and the vectors $\Phi^{(n)}(x,\lambda_j)$, for $n=0,1,\ldots,m_j-1$ and $j=p+1,p+2,\ldots,q$, are principal vectors corresponding to the eigenvalues and the spectral singularities of L_ρ, respectively.

Theorem 7

$$\Phi^{(n)}(x,\lambda_j)\in L_2(0,\infty,\rho(x)) \quad \text{for } n=0,1,\ldots,m_j-1 \text{ and } j=1,2,\ldots,p, \tag{48}$$

$$\Phi^{(n)}(x,\lambda_j)\notin L_2(0,\infty,\rho(x)) \quad \text{for } n=0,1,\ldots,m_j-1 \text{ and } j=p+1,p+2,\ldots,q. \tag{49}$$

Proof

From (15) and (28) we obtain

$$K(x,t)\leqslant c\exp\left(-\frac{\varepsilon}{2}(x+t)\right),$$

and therefore,

$$\frac{\mathrm{d}^n}{\mathrm{d}k^n}F(x,k)\leqslant D\left[x^n\exp(-x\mathrm{Im}k)+\int_x^\infty t^n\exp\left(-\frac{\varepsilon}{2}x-t\left(\frac{\varepsilon}{2}+\mathrm{Im}k\right)\right)\mathrm{d}t\right] \tag{50}$$

for $x>a$, where D is a constant. Since $\mathrm{Im}\,k_j>0$ for the eigenvalues $\lambda_j=k_j^2$, $j=1,2,\dots,p$, (48) follows from (47) and (50). For the spectral singularities $\lambda_j=k_j^2$, $j=p+1,p+2,\dots,q$ we know that $\mathrm{Im}\,k_j=0$; hence, (47) implies (49). □

Define the following Hilbert spaces:

$$H_m=\{f:\|f\|_m<\infty\}\quad\text{and}\quad H_{-m}=\{f:\|f\|_{-m}<\infty\}$$

for $m=1,2,\dots$ with norms

$$\|f\|_m^2=\int_0^\infty \rho(x)(1+x)^{2m}|f(x)|^2\,\mathrm{d}x$$

and

$$\|f\|_{-m}^2=\int_0^\infty \rho(x)(1+x)^{-2m}|f(x)|^2\,\mathrm{d}x,$$

respectively. It is evident that

$$H_0=L_2(0,\infty,\rho(x))$$

and

$$H_{m+1}\subsetneqq H_m\subsetneqq L_2(0,\infty,\rho(x))\subsetneqq H_{-m}\subsetneqq H_{-(m+1)}$$

As a consequence of the preceding theorem we have the following:

Corollary 2

$$\Phi^{(n)}(x,\lambda_j)\in H_{-m_0}\quad\text{for } n=0,1,\dots,m_j-1 \text{ and } j=p+1,p+2,\dots,q,$$

where

$$m_0=\max\{m_{p+1},m_{p+2},\dots,m_q\}.$$

4. Summary and conclusions

In this study we have discussed spectral properties of a Sturm–Liouville problem (SLP) with discontinuous density function. This type of problems has been the topic of many research papers in the existing literature. The reference list is by no means complete; but we can refer to [2–4, 6, 8–10]. In particular, Gasymov [4] considered the SLP (1)–(2) with a real-valued potential q to show that it has only a finite number of simple eigenvalues and that the half-axis constitutes its absolutely continuous spectrum. Afterwards, in [8] Guseinov and Pashaev handled the same BVP to investigate the inverse scattering problem. In [10], Darwish brought a non-self-adjoint SLP with discontinuous density function under investigation and gave two examples to draw the attention to the difficulty in the study of distribution and finiteness of eigenvalues.

The BVP (1)–(2) considered in this paper includes a potential function q, which is assumed to be complex valued. Hence, being a non-self-adjoint problem, the BVP (1)–(2) treated in this study requires a completely different approach. This is not only because of the existence of spectral singularities embedded in the continuous spectrum but also because of the different structure of the Jost solutions. The notable achievements of this paper can be summarized as the proposition of sufficient conditions guaranteeing the finiteness of eigenvalues and spectral singularities (Theorem 4) and the investigation of convergence properties principal vectors corresponding to eigenvalues and spectral singularities (Theorems 7 and Corollary 2). By using the substitution $\gamma=1$ in Equation (1), the problem (1)–(2) can be reduced to the one treated by Naimark in [7, Appendix II, p. 292]. Considering this and the non-self-adjointness of the problem (1)–(2), the results obtained in this study can be regarded as the extension and generalization of the respective ones in [4, 8], and [7, Appendix II, p. 292]. Furthermore, this study prepares a groundwork for the applications since its results can be found significant for the study of inverse scattering problem.

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