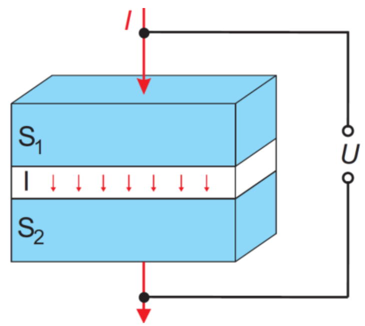
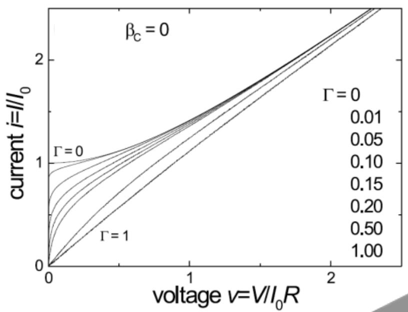
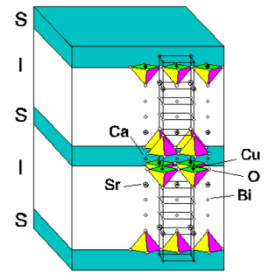
EBERHARD KARLS UNIVERSITÄT TÜBINGEN

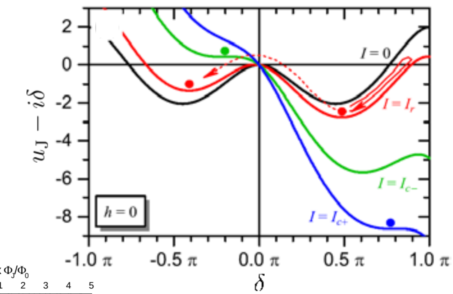
FACULTY OF SCIENCE

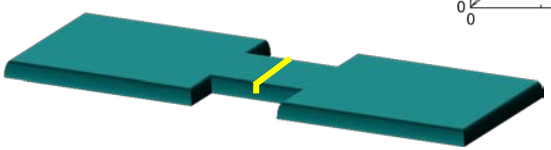
Basic Properties of Josephson Junctions

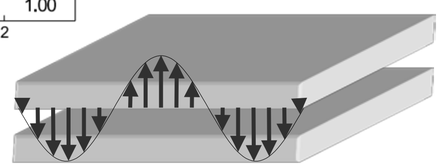


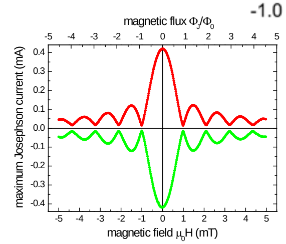












Dieter Koelle

Physikalisches Institut & Center for Quantum Science (CQ)
in the Center for Light-Matter Interaction, Sensors & Analytics (LISA$^+$)

NANOSCALE COHERENT HYBRID DEVICES FOR SUPERCONDUCTING QUANTUM TECHNOLOGIES

School on Quantum Materials,
8-10 October 2018, Braga (Portugal)

Motivation

Josephson junction (JJ) = two weakly coupled superconductors („weak link“)

quantum mechanical coupling of the superconductor wave functions

⇨ insights into basic properties of superconductors

e.g. phase-sensitive experiments on the order parameter symmetry of unconventional superconductors, based on inteference effects

⇨ JJ is the key element in superconducting electronics

large variety of devices for many applications, e.g.

- voltage standards,
- SQUID magnetometers,
- radiation detectors,
- qubits,
- ultrafast processors,
- ...

Outline

I. Macroscopic Wave Function

II. Josephson Relations & Consequences

III. Josephson Junction in a Magnetic Field

IV. Resistively & Capacitively Shunted Junction (RCSJ) model

V. Fluctuations in Josephson Junctions

VI. Classification of JJs – Ground States: 0- π-, φ-Junctions

Books:

A. Barone & G. Paterno, *Physics & Applications of the Josephson Effect*, J. Wiley & Sons (1982)

K.K. Likharev, *Dynamics of Josephson Junctions and Circuits*, Gordon & Breach (1986)

T.P. Orlando, K.A. Delin, *Foundations of Applied Superconductivity*, Addison-Wesley (1991)

W. Buckel, R. Kleiner, *Superconductivity*, Wiley-VCH, 3rd ed. (2016)

Reviews:

K.K. Likharev, *Superconducting weak links*, Rev. Mod. Phys. **51**, 101 (1979)

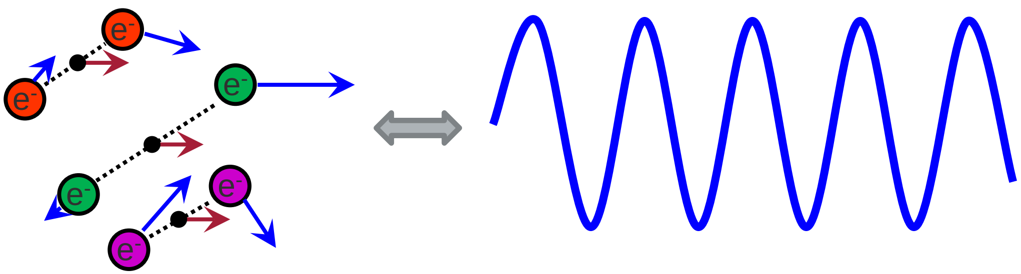
A.A. Golubov, M.Yu. Kupriyanov, E. Il'ichev, *The current phase relation in Josephson junctions*, Rev. Mod. Phys. **76**, 411 (2004)

A.I. Buzdin, *Proximity effects in superconductor-ferromagnet heterostructures*, Rev. Mod. Phys. **77**, 935 (2005)

Macroscopic Wave Function

a single (macroscopic) wave function
describes the state of all Cooper pairs in a superconductor
→ highly correlated, coherent many-particle quantum state

superconducting charge carriers: Cooper pairs **(mobile electrons correlated in momentum *k*-space)**



macroscopic wave function

$$\boxed{\Psi = \Psi_0 \cdot e^{i\varphi}}$$

phase φ
determined by carrier velocity $\boldsymbol{v}_\mathrm{s}$
& vector potential $\boldsymbol{A}$
(connected to magnetic field via relation for magnetic induction (flux density) $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$

amplitude $\Psi_0 = \sqrt{n_\mathrm{s}}$
n_s: Cooper pair density

$$\boxed{\hbar \boldsymbol{\nabla} \varphi = m_\mathrm{s} \boldsymbol{v}_\mathrm{s} + q_\mathrm{s} \boldsymbol{A}}$$

Cooper pair charge $q_\mathrm{s} = 2e$
and mass $m_\mathrm{s} = 2m_e$

Macroscopic Wave Function

a single (macroscopic) wave function
describes the state of all Cooper pairs in a superconductor
→ highly correlated, coherent many-particle quantum state

Cooper pairs
highly correlated
motion

supercurrent density: $$\boxed{\boldsymbol{j}_\mathrm{s} = q_\mathrm{s} n_\mathrm{s} \boldsymbol{v}_\mathrm{s} = \frac{q_\mathrm{s} n_\mathrm{s}}{m_\mathrm{s}} \left(\hbar \boldsymbol{\nabla} \varphi - q_\mathrm{s} \boldsymbol{A} \right)}$$

Macroscopic Wave Function

with the definition of the **gauge-invariant phase gradient**

$$\boldsymbol{\nabla}\phi \equiv \boldsymbol{\nabla}\varphi - \frac{q_\mathrm{s}}{\hbar}\boldsymbol{A} \quad \text{or} \quad \boxed{\boldsymbol{\nabla}\phi \equiv \boldsymbol{\nabla}\varphi - \frac{2\pi}{\Phi_0}\boldsymbol{A}}$$

with $q_s = 2e$ and magnetic flux quantum $\Phi_0 \equiv \frac{h}{2e}$

$$\boldsymbol{j}_\mathrm{s} = \frac{q_\mathrm{s}\hbar}{m_s} n_\mathrm{s} \boldsymbol{\nabla}\phi \quad \text{i.e.} \quad \boxed{\boldsymbol{j}_\mathrm{s} \propto n_\mathrm{s}\boldsymbol{\nabla}\phi}$$

integration of $\boldsymbol{\nabla}\phi \equiv \boldsymbol{\nabla}\varphi - \frac{2\pi}{\Phi_0}\boldsymbol{A}$ $\Rightarrow$ **gauge-invariant phase**

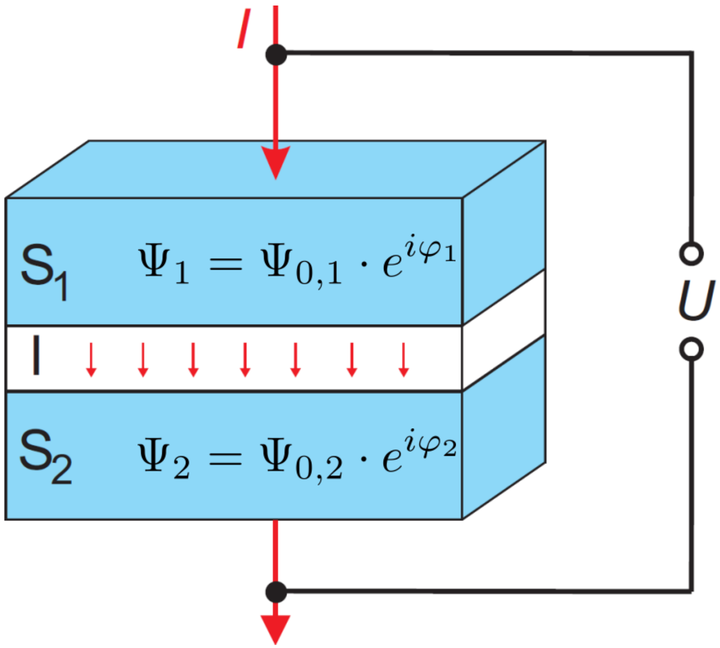
$$\boxed{\phi(\boldsymbol{r}) = \varphi(\boldsymbol{r}) - \frac{2\pi}{\Phi_0}\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{A}\,\mathrm{d}\boldsymbol{r}}$$

Weakly Coupled Superconductors

consider now two superconductors S_1, S_2 with macroscopic wave functions

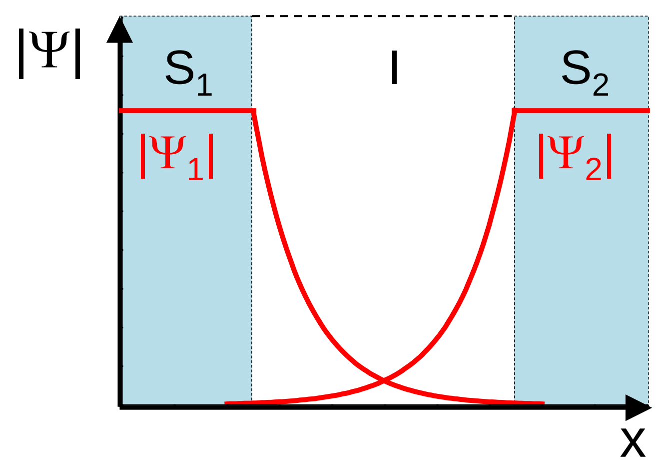
$$\Psi_i = \Psi_{0,i} \cdot e^{i\varphi_i} \qquad (i = 1, 2)$$

What is the relation between the wave functions Ψ_i (phases φ_i) if the two superconductors are coupled via a weak link?
(e.g. via insulating (I) tunnel barrier in a SIS junction)



finite coupling ⟺ overlap of the wave functions Ψ_i

⇨ supercurrent through weak link (across barrier)



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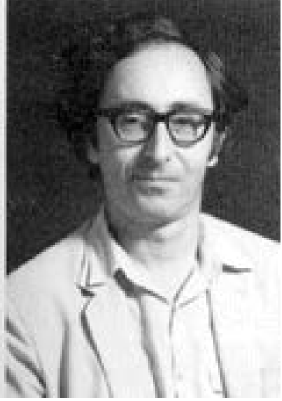
Josephson Relations

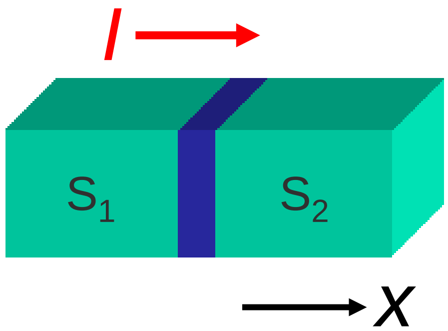


B. D. Josephson
Nobel prize physics 1973

connect phase of the wave functions
to current *I* and voltage *U* across weak link

derivation by solving Schrödinger Eq. for two coupled quantum mechanical systems → Feynman

Alternative: following general arguments by Landau & Lifschitz

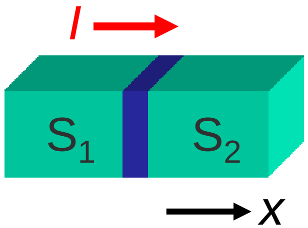


assume

- barrier in (y,z) plane
- constant current density in (y,z)
- constant phase gradient and n_s in the S_1, S_2 electrodes

B.D. Josephson, *Possible new effects in superconductive tunneling*, Phys. Lett. **1**, 251 (1962)
R.P. Feynman, R.B. Leighton, M. Sands, *Feynman Lectures on Physics* ***3***, Addison-Wesley (1965)
L.D. Landau, E.M. Lifschitz, *Lehrbuch der Theoretischen Physik* ***IX***, Akademie-Verlag (1980)
W. Buckel, R. Kleiner, *Superconductivity*, Wiley-VCH, 3rd ed. (2016)

Josephson Relations

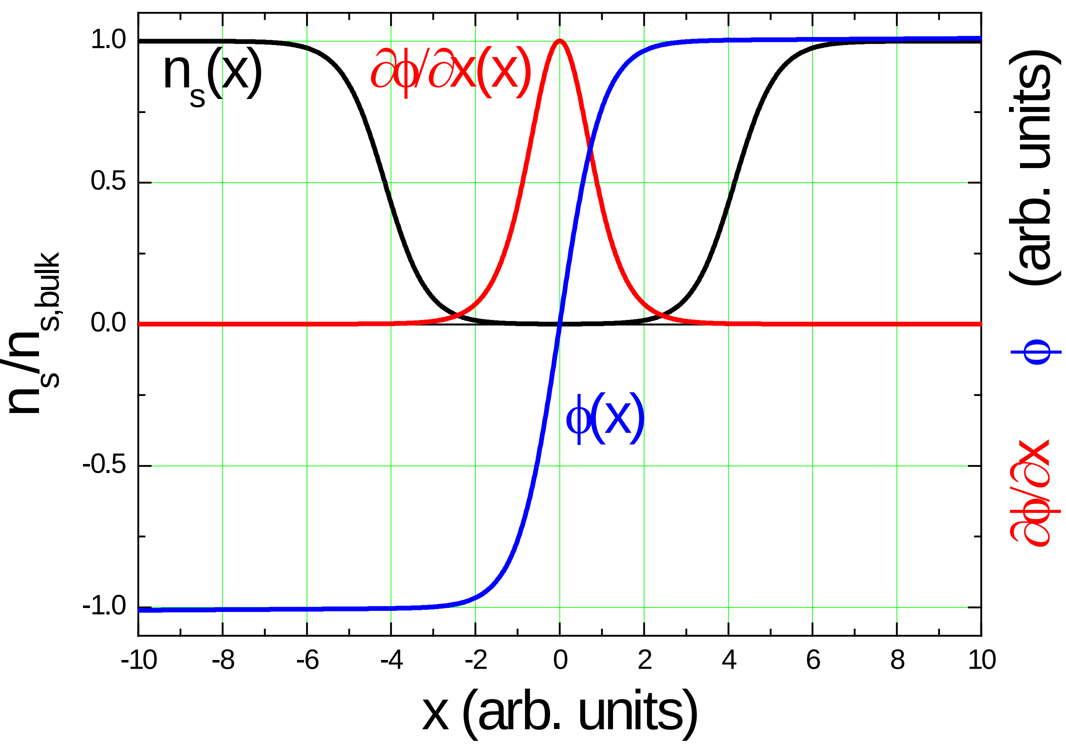


for current I along x and cross section A_J=const

$$j_s(x) = I/A_J = const$$

i.e., because of $\boldsymbol{j}_s \propto n_s \nabla\phi$ ⇨ $n_s(x) \cdot \frac{\partial\phi}{\partial x}(x) = const$

⇨ **change in n_s at weak link has to be compensated by change in $\partial\phi/\partial x$**

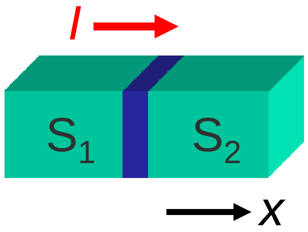


suppressed n_s at barrier
→ dip in $n_s(x)$

⇨ peak in $\partial\phi/\partial x$

⇨ $\phi(x)$ makes a step

Josephson Relations



weak link is characterized by a **phase difference**

$$\boxed{\delta \equiv \phi_2 - \phi_1 = \varphi_2 - \varphi_1 - \frac{2\pi}{\Phi_0}\int_1^2 A_x \mathrm{d}x}$$

integral across barrier

⇨ analogous to $\boldsymbol{j}_\mathrm{s} \propto \boldsymbol{\nabla}\phi$ in the bulk superconductor,
j_s across the weak link is a function of the phase difference

$$\boldsymbol{j}_\mathrm{s} = \boldsymbol{j}_\mathrm{s}(\delta)$$

Josephson Relations

Question: what is the functional dependence of j_s (δ) ?

⇨ from simple considerations:

- phases ϕ_i in the electrodes are defined modulo 2π
  (phase change of $2\pi n$ (n: integer) does not change Ψ_i)

  ⇨ j_s = 2π-periodic function of δ

$$j_s = \sum_n j_{0n} \sin n\delta + \sum_n \tilde{j}_{0n} \cos n\delta \qquad (n = 1, 2, \ldots)$$

- time reversal symmetry: $j_s(\delta) = -j_s(-\delta)$
  (both, currrents and phases ($\sim\omega t$) change sign upon time reversal)

  ⇨ excludes cosine terms

- rapid convergence of sin-series (e.g. for conventional SIS junctions)

$$\boxed{j_s = j_0 \, \sin\delta}$$

1. Josephson relation (current-phase relation = CPR)

Josephson Relations

Question: what is the evolution of δ in time ?

take time derivative of gauge-invariant phase difference $\delta = \varphi_2 - \varphi_1 - \frac{2\pi}{\Phi_0}\int_1^2 \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l}$

$$\frac{\partial \delta}{\partial t} = \frac{\partial \varphi_2}{\partial t} - \frac{\partial \varphi_1}{\partial t} - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l}$$

with energy-phase relation $-\hbar\frac{\partial \varphi}{\partial t} = \frac{\mu_0\lambda_\mathrm{L}^2}{2n_\mathrm{s}}\boldsymbol{j}_\mathrm{s}^2 + q_\mathrm{s}\tilde{\phi}$ (derived from Schrödinger equation for n_s=const in the electrodes)

with London penetration depth $\lambda_\mathrm{L} = \left(\frac{m_\mathrm{s}}{\mu_0 q_\mathrm{s}^2 n_\mathrm{s}}\right)^{1/2}$

and electrochemical potential $\tilde{\phi}$ related to electric field $\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\tilde{\phi}$

$$\frac{\partial \delta}{\partial t} = -\frac{1}{\hbar}\left(\frac{\mu_0\lambda_\mathrm{L}^2}{2n_\mathrm{s}}\underbrace{\left[\boldsymbol{j}_\mathrm{s}^2(2) - \boldsymbol{j}_\mathrm{s}^2(1)\right]}_{\text{=0 (current continuity)}} + q_\mathrm{s}\left[\tilde{\varphi}_2 - \tilde{\varphi}_1\right]\right) - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l}$$

$$\frac{\partial \delta}{\partial t} = \frac{2\pi}{\Phi_0}\int_1^2 \underbrace{\left(-\boldsymbol{\nabla}\tilde{\varphi} - \frac{\partial \boldsymbol{A}}{\partial t}\right)}_{\text{= electric field } \boldsymbol{E}}\cdot \mathrm{d}\boldsymbol{l} \quad \Rightarrow \quad \frac{\partial \delta}{\partial t} \equiv \boxed{\dot{\delta} = \frac{2\pi}{\Phi_0}U}$$

voltage across junction

2. Josephson relation (voltage-phase relation)

T.P. Orlando, K.A. Delin, *Foundations of Applied Superconductivity*, Addison-Wesley (1991)

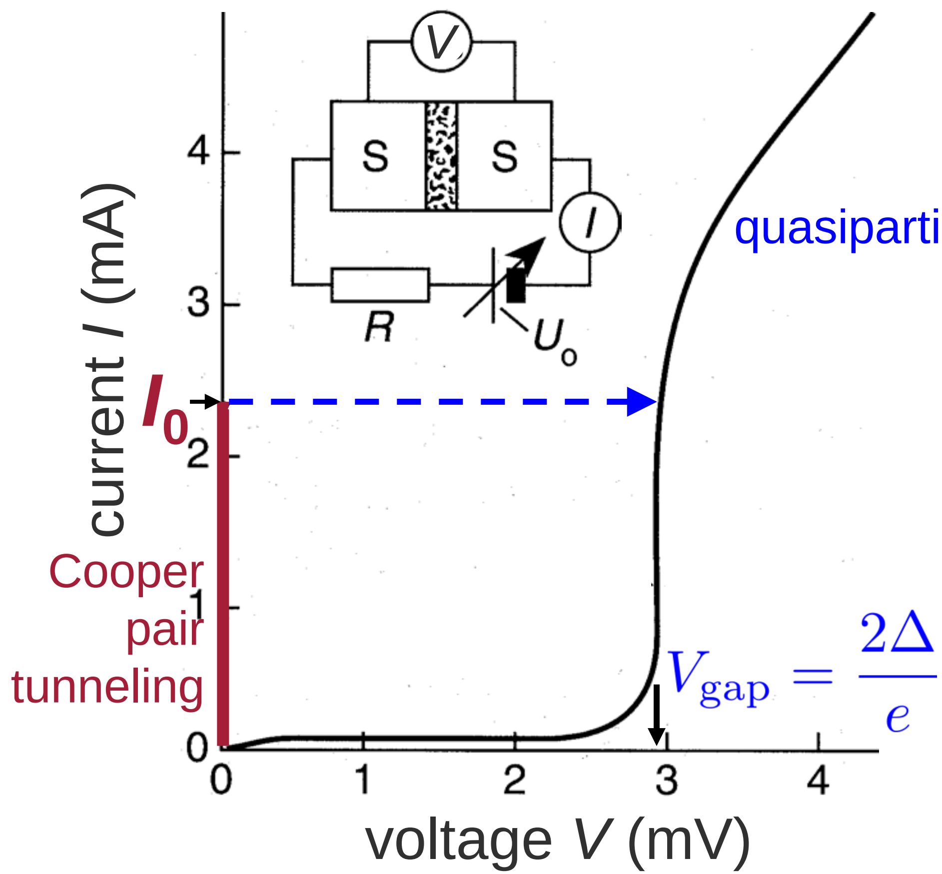
Consequences of Josephson Relations

1. zero voltage state (static case) $U = 0 \Rightarrow \delta = const = \delta_0$

⇨ $j_s = j_0 \sin \delta_0$ ⇨ maximum supercurrent density across junction:

$\sin \delta_0 = 1 \Rightarrow \boxed{j_{s,max} = j_0}$ critical current density

⇨ $j_0 \ll j_{c,deparing}$ ⇨ „weak link“



Δ: energy gap in the quasiparticle density of states

Consequences of Josephson Relations

2. finite voltage state (dynamic case) $U \neq 0$

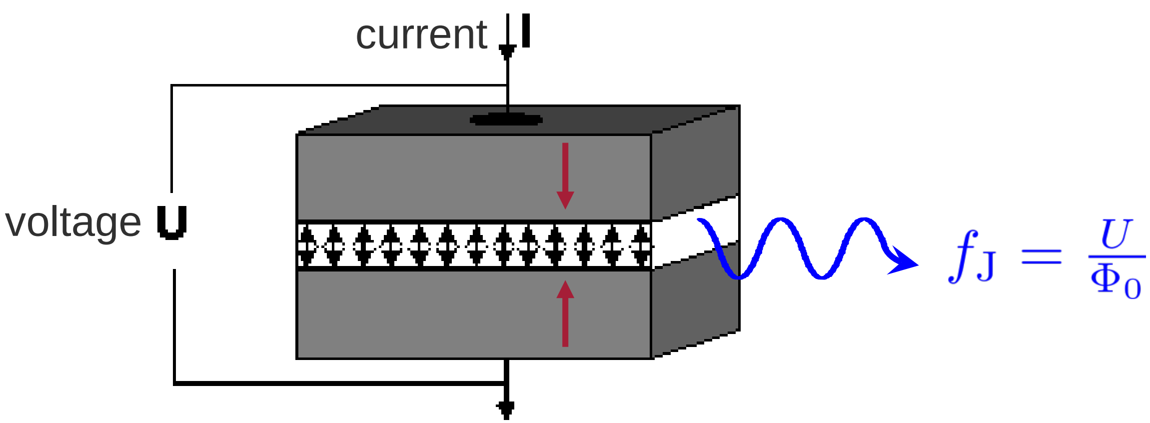
for $U = const$ integrate 2nd Josephson Eq. $\delta(t) = \delta_0 + 2\pi \frac{U}{\Phi_0} t$

insert into 1st Josephson Eq. $j_s = j_0 \sin\{\delta_0 + 2\pi f_J t\}$

⇨ Cooper pair current across junction oscillating with the **Josephson frequency**

$$\boxed{f_J \equiv \frac{U}{\Phi_0} \approx 483.6 \frac{\text{GHz}}{\text{mV}} \cdot U}$$

⇨ **quantum interference of the wave functions across the barrier**



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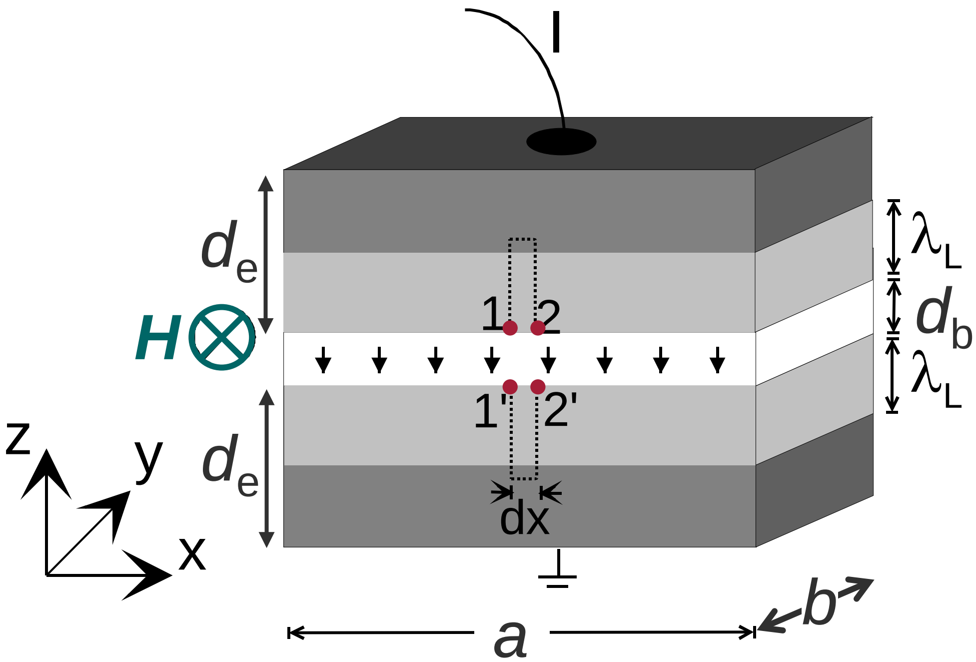
Josephson Junction in a Magnetic Field



field ***H*** applied in the JJ plane

(rectangular barrier)

magnetic flux density ***B*** penetrates into electrodes

→ effective magnetic thickness:

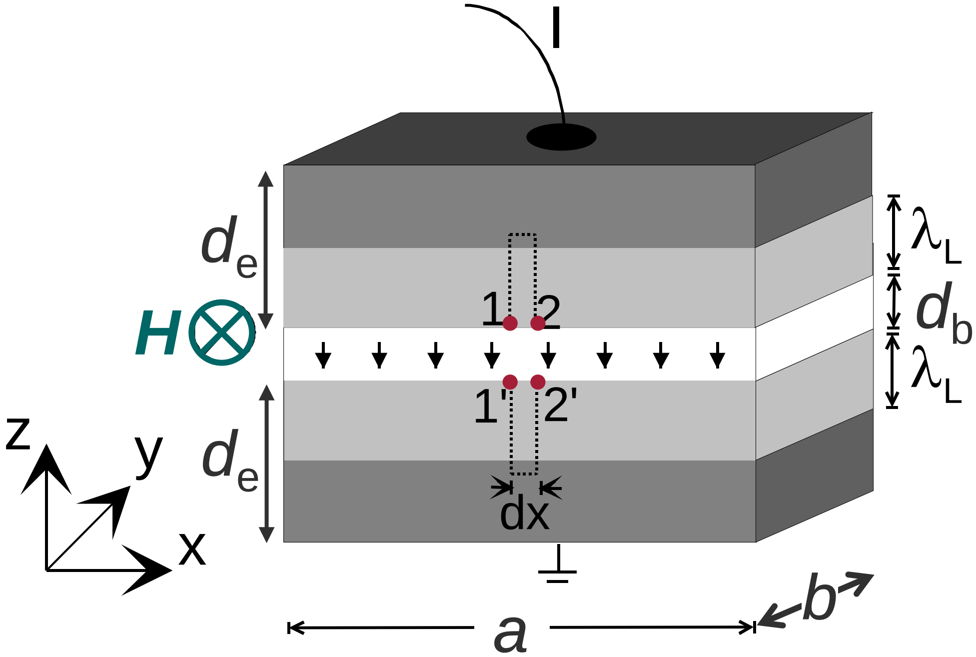
$$d_{\text{eff}} \approx d_{\text{b}} + 2\lambda_{\text{L}}$$

for identical electrode materials with thickness $d_{\text{e}} \gtrsim 2\lambda_{\text{L}}$

for different electrode materials with thickness $d_{e,i}$:

$$d_{\text{eff}} \equiv d_{\text{b}} + \lambda_{\text{L},1} \tanh \frac{d_{\text{e},1}}{\lambda_{\text{L},1}} + \lambda_{\text{L},2} \tanh \frac{d_{\text{e},2}}{\lambda_{\text{L},2}}$$

Josephson Junction in a Magnetic Field



relation between *B* and δ ?

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

$$\delta = \varphi_2 - \varphi_1 - \frac{2\pi}{\Phi_0} \int_1^2 \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$$

from $\boldsymbol{j}_\mathrm{s} = \frac{q_\mathrm{s} n_\mathrm{s}}{m_\mathrm{s}} (\hbar \boldsymbol{\nabla}\varphi - q_\mathrm{s} \boldsymbol{A})$

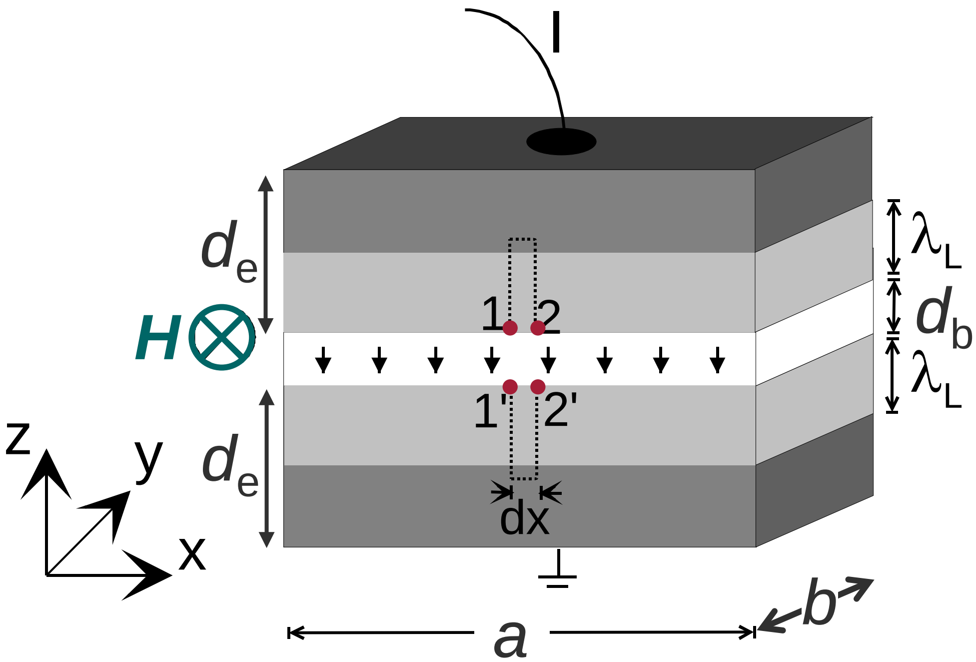
$$\boldsymbol{\nabla}\varphi = \frac{2\pi}{\Phi_0} \left(\mu_0 \lambda_\mathrm{L}^2 \boldsymbol{j}_\mathrm{s} + \boldsymbol{A} \right)$$

integrate $\boldsymbol{\nabla}\varphi$ along dotted lines in the two electrodes

path 2→1: $$\varphi(1) - \varphi(2) = \frac{2\pi}{\Phi_0} \mu_0 \lambda_\mathrm{L}^2 \int_2^1 \boldsymbol{j}_\mathrm{s}\, \mathrm{d}\boldsymbol{l} + \frac{2\pi}{\Phi_0} \int_2^1 \boldsymbol{A}\, \mathrm{d}\boldsymbol{l}$$

path 1´→2´: $$\varphi(2') - \varphi(1') = \frac{2\pi}{\Phi_0} \mu_0 \lambda_\mathrm{L}^2 \int_{1'}^{2'} \boldsymbol{j}_\mathrm{s}\, \mathrm{d}\boldsymbol{l} + \frac{2\pi}{\Phi_0} \int_{1'}^{2'} \boldsymbol{A}\, \mathrm{d}\boldsymbol{l}$$

Josephson Junction in a Magnetic Field



sum up both Eqs. and add
on both sides integrals across barrier

$$\frac{2\pi}{\Phi_0}\int_1^{1'} \boldsymbol{A}\,\mathrm{d}\boldsymbol{l} + \frac{2\pi}{\Phi_0}\int_{2'}^{2} \boldsymbol{A}\,\mathrm{d}\boldsymbol{l}$$

flux through integration path

$$\underbrace{\varphi(2')-\varphi(2)-\frac{2\pi}{\Phi_0}\int_2^{2'} \boldsymbol{A}\,\mathrm{d}\boldsymbol{l}}_{\delta(x+\mathrm{d}x)} - \Bigg(\underbrace{\varphi(1')-\varphi(1)-\frac{2\pi}{\Phi_0}\int_1^{1'} \boldsymbol{A}\,\mathrm{d}\boldsymbol{l}}_{\delta(x)}\Bigg) = \frac{2\pi}{\Phi_0}\underbrace{\oint \boldsymbol{A}\,\mathrm{d}\boldsymbol{l}}_{\Phi = Bd_{\mathrm{eff}}\mathrm{d}x} + \frac{2\pi}{\Phi_0}\mu_0\lambda_L^2\underbrace{\left(\int_2^1 \boldsymbol{j}_s\,\mathrm{d}\boldsymbol{l} + \int_{1'}^{2'} \boldsymbol{j}_s\,\mathrm{d}\boldsymbol{l}\right)}_{\approx 0}$$

for thick enough electrodes
in the Meissner state

magnetic field induces a gradient of δ along the JJ

$$\frac{\partial\delta}{\partial x} = \frac{2\pi}{\Phi_0}Bd_{\mathrm{eff}}$$

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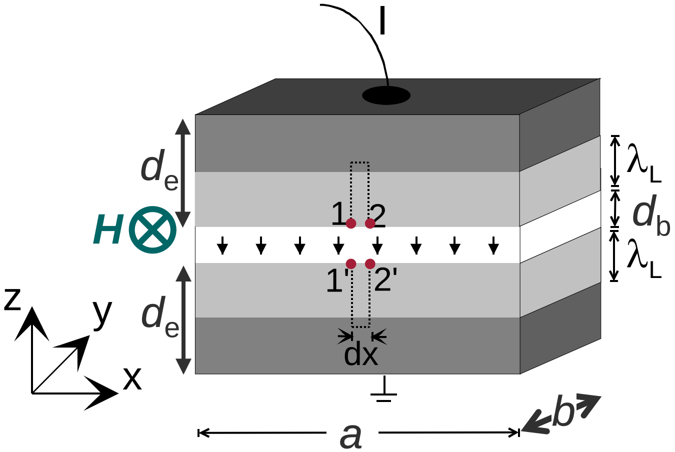
Josephson Junction in a Magnetic Field



General case: applied field ***H*** can be screened by supercurrents flowing across the JJ

with Ampere's law $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$

for our geometry $\boldsymbol{B} = B_y \hat{\boldsymbol{e}}_y$: $\dfrac{\partial B_y(x)}{\partial x} = \mu_0 j_z(x)$

combined with $\dfrac{\partial \delta}{\partial x} = \dfrac{2\pi}{\Phi_0} B_y d_{\text{eff}}$ $\Rightarrow$ $\dfrac{\partial^2 \delta}{\partial x^2} = \dfrac{2\pi}{\Phi_0} d_{\text{eff}} \dfrac{\partial B_y}{\partial x} = \dfrac{1}{\lambda_J^2} \dfrac{j_z(x)}{j_0} = \dfrac{1}{\lambda_J^2} \sin \delta(x)$

Ferrel-Prange Eq.

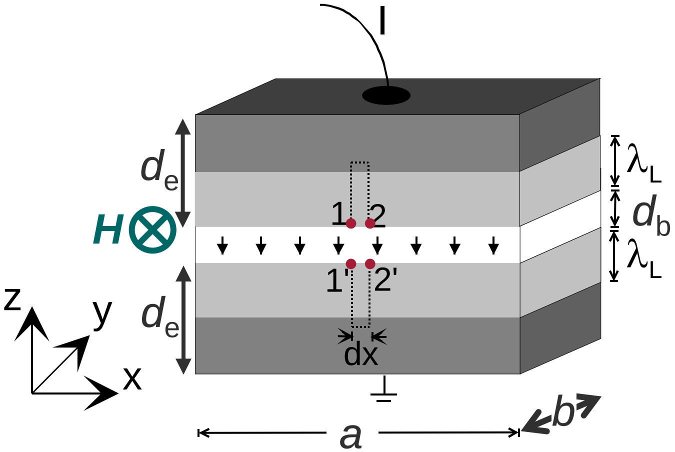
with the Josephson length $\lambda_J \equiv \left(\dfrac{\Phi_0}{2\pi \mu_0 d_{\text{eff}} j_0} \right)^{1/2}$

for small applied fields: $\dfrac{\partial^2 \delta}{\partial x^2} = \dfrac{1}{\lambda_J^2} \delta(x)$ $\Rightarrow$ $\delta(x) = \delta(0) e^{-\frac{x}{\lambda_J}}$

$\left(\dfrac{\partial \delta}{\partial x} \ll \dfrac{1}{\lambda_J} \right)$ $\Rightarrow$ $B_y(x) = B_y(0) e^{-\frac{x}{\lambda_J}}$

λ_J is the characteristic length over which a JJ can screen external magnetic fields
(similar to λ_L in a bulk superconductor)

Short JJ in a Magnetic Field



„short junction“ limit:
size of JJ along direction $\perp$***H***

$$\boxed{a \lesssim 4\lambda_J}$$

magnetic field penetrates JJ homogeneously along the barrier

$$B(x) = const$$

magnetic flux in the JJ:

$$\boxed{\Phi_J = B\, d_{\rm eff}\, a}$$

integration of $\frac{\partial \delta}{\partial x} = \frac{2\pi}{\Phi_0} B d_{\rm eff}$ along x

$$\delta(x) = \delta_0 + \frac{2\pi}{\Phi_0} B d_{\rm eff} x$$

δ(x) grows linearly along barrier
(slope determined by *B*)

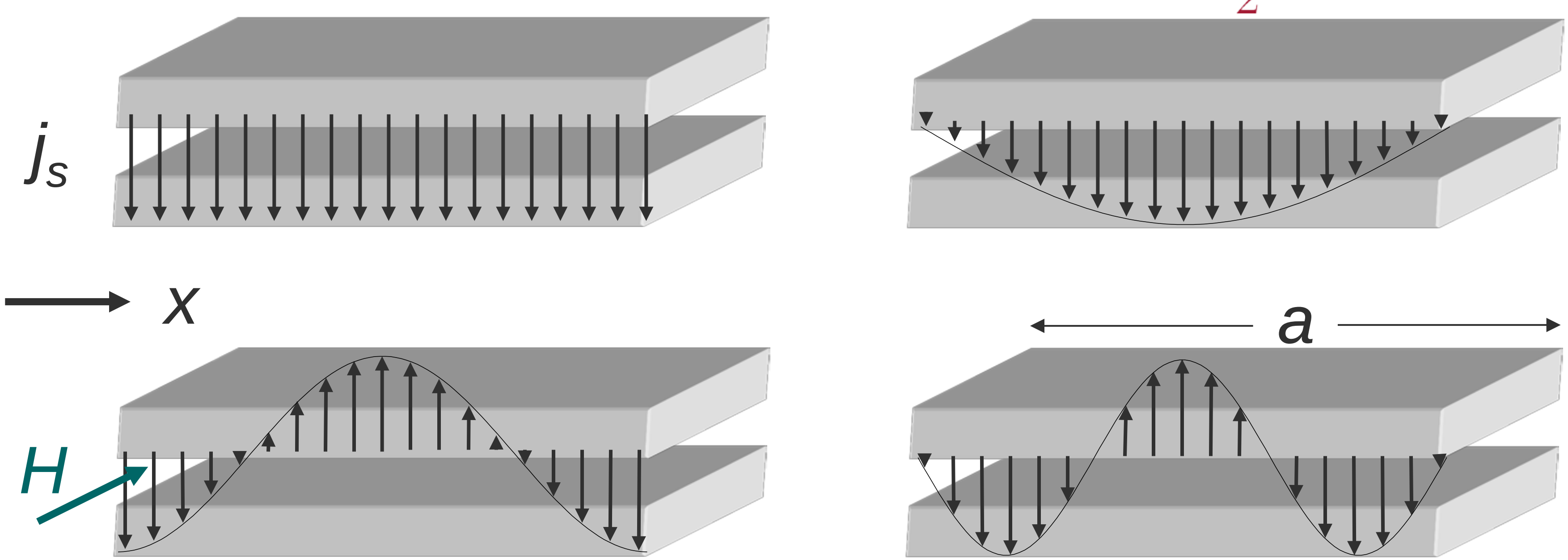
Short JJ in a Magnetic Field

$\delta(x) = \delta_0 + \frac{2\pi}{\Phi_0} B d_{\mathrm{eff}} x$ inserted into 1st Josephson relation:

$$j_{\mathrm{s}}(x) = j_0 \sin\left\{\delta(0) + \frac{2\pi}{\Phi_0} B\, d_{\mathrm{eff}} \cdot x\right\}$$

⇨ **$j_s(x)$ oscillates with a wavelength** $\propto \frac{1}{\Phi_{\mathrm{J}}} \propto \frac{1}{B}$

$\Phi_{\mathrm{J}} = 0$ $\qquad$ $\Phi_{\mathrm{J}} = \frac{1}{2}\Phi_0$



$\Phi_{\mathrm{J}} = \Phi_0$ $\qquad$ $\Phi_{\mathrm{J}} = \frac{3}{2}\Phi_0$

Short JJ in a Magnetic Field

total supercurrent I_s through the JJ → integrate j_s(x) over JJ area $A_\mathrm{J} = ab$

$$I_\mathrm{s}(\Phi_\mathrm{J},\delta_0) = \int_0^b \mathrm{d}y \int_0^a \mathrm{d}x j_0 \sin\delta(x) = -j_0 \cdot b \cdot \left.\frac{\cos\left(\delta_0 + \frac{2\pi}{\Phi_0} B d_\mathrm{eff} x\right)}{\left(\frac{2\pi}{\Phi_0} B d_\mathrm{eff}\right)}\right|_0^a$$

$\Phi_\mathrm{J} = B\, d_\mathrm{eff}\, a$

$$= \underbrace{j_0 \cdot b \cdot a}_{I_0} \cdot \frac{\sin \pi \frac{\Phi_J}{\Phi_0}}{\pi \frac{\Phi_J}{\Phi_0}} \cdot \sin(\delta_0 + \pi \frac{\Phi_J}{\Phi_0})$$

for given I_s, Φ_J → δ_0 adjusts accordingly

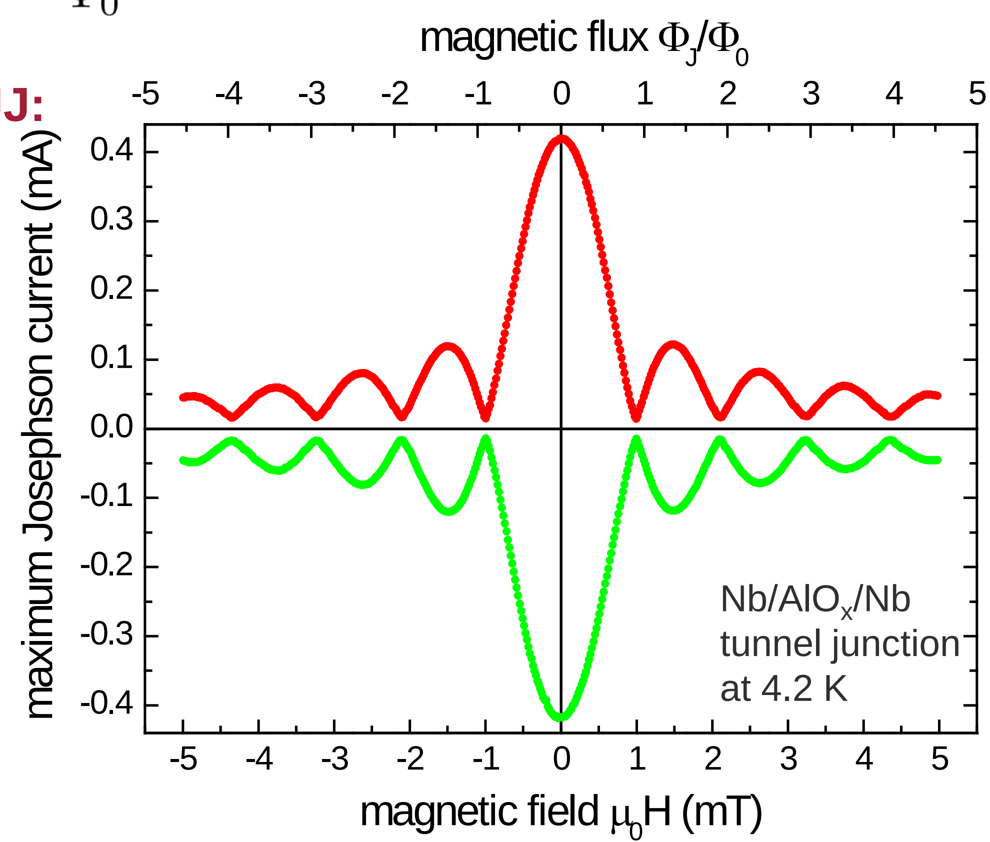
maximum supercurrent I_c through the JJ:

→ $\sin(\delta_0 + \pi \frac{\Phi_\mathrm{J}}{\Phi_0}) \pm 1$

$$\Rightarrow \boxed{I_c(\Phi_\mathrm{J}) = I_0 \cdot \left| \frac{\sin \pi \frac{\Phi_\mathrm{J}}{\Phi_0}}{\pi \frac{\Phi_\mathrm{J}}{\Phi_0}} \right|}$$

Fraunhofer pattern

(analogous to diffraction at single slit in optics)



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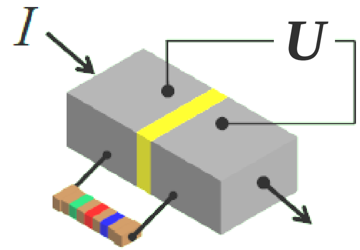
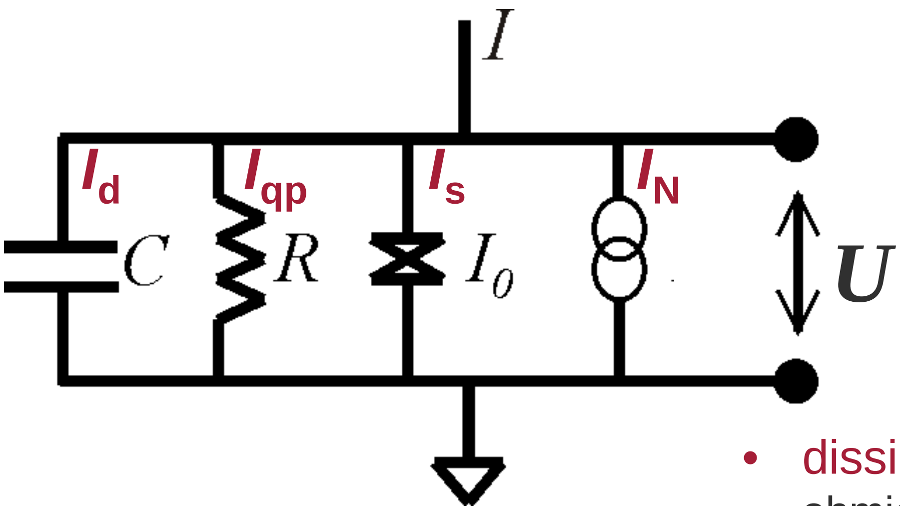
RCSJ Model & Washboard Potential

resistively and capacitively shunted junction (RCSJ)



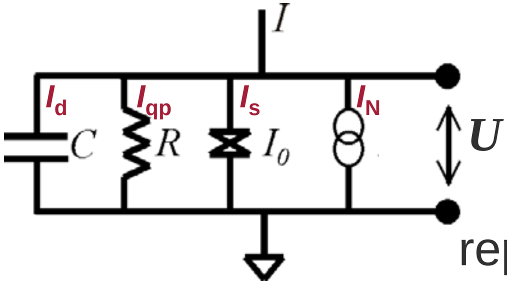
- simple model to describe dynamics of JJs
  → current voltage characteristics (IVC)



- Josephson current $$I_\mathrm{s} = I_0 \sin\delta$$
- displacement current
  across junction capacitance *C* $$I_\mathrm{d} = C\dot{U}$$
- dissipative (quasiparticle) current
  ohmic current through shunt resistor *R* $$I_\mathrm{qp} = U/R$$
- current noise source
  thermal noise of shunt resistor *R*
  at temperature *T*, with
  spectral density $S_I(f) = \frac{4k_BT}{R}$ $$I_\mathrm{N}(t)$$

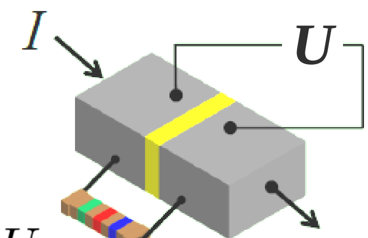
W.C. Stewart, *Current-voltage characteristics of Josephson junctions*, Appl. Phys. Lett. **12**, 277 (1968)
D.E. McCumber, *Effect of ac impedance on dc voltage-current characteristics of Josephson junctions*, J. Appl. Phys. **39**, 3113 (1968)

RCSJ Model & Washboard Potential



from Kirchoff's law:

$$I + I_N(t) = I_0 \sin\delta + \frac{U}{R} + C\dot{U}$$



replace $U \rightarrow \dot{\delta}$ via 2nd Josephson relation $\dot{\delta} = \frac{2\pi}{\Phi_0} U$

⇨ Eq. of motion for δ :

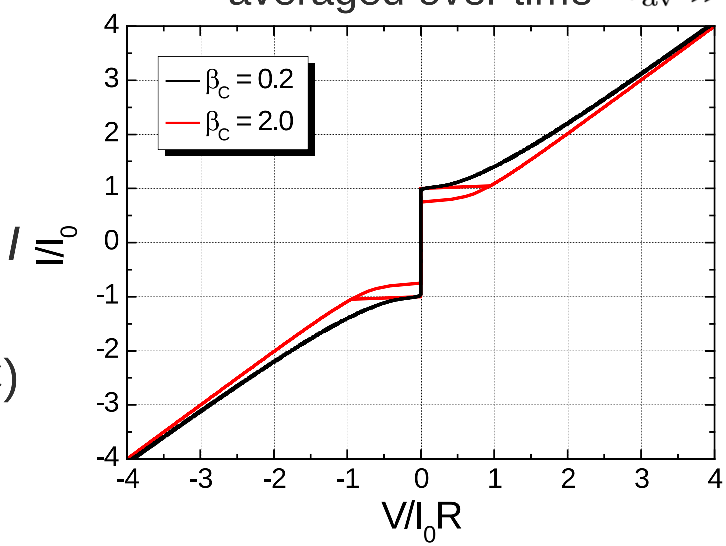
$$\boxed{I + I_N(t) = I_0 \sin\delta + \frac{\Phi_0}{2\pi R}\,\dot{\delta} + \frac{\Phi_0 C}{2\pi}\,\ddot{\delta}}$$

finite voltage contains high-frequency Josephson oscillations → experimentally detected voltage: $V \equiv \langle U \rangle = \frac{1}{t_{av}} \int_0^{tav} \mathrm{d}t\, U(t)$

averaged over time $t_{av} \gg \frac{1}{f_J}$

solution of Eq. of motion (numerical simulations)

$\delta(t) \Rightarrow \dot{\delta}(t) \propto U(t) \Rightarrow V$ for given bias current *I*



⇨ current-voltage characteristics (IVC)

RCSJ Model & Washboard Potential

rearrange Eq. of motion for δ (for simplicity we set T=0, i.e. I_N=0)

$$\frac{\Phi_0}{2\pi} C\ddot{\delta} + \frac{\Phi_0}{2\pi}\frac{1}{R}\dot{\delta} = -I_0 \sin\delta + I \equiv -\frac{2\pi}{\Phi_0}\frac{\partial U_\mathrm{J}}{\partial \delta}$$

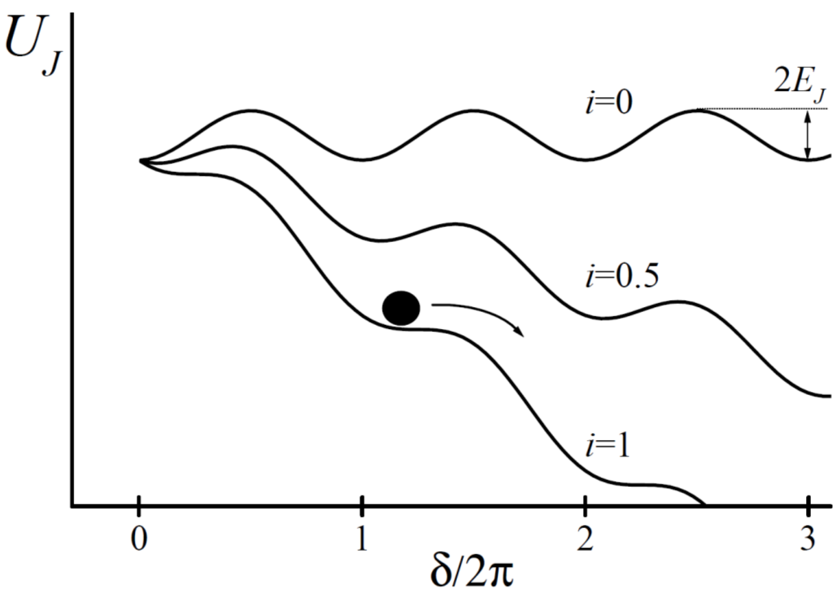
with the **tilted washboard potential** $U_\mathrm{J} \equiv E_\mathrm{J}\{1 - \cos\delta - i\,\delta\}$

Josephson coupling energy $E_\mathrm{J} \equiv \dfrac{I_0 \Phi_0}{2\pi}$ normalized bias current $i \equiv \dfrac{I}{I_0}$

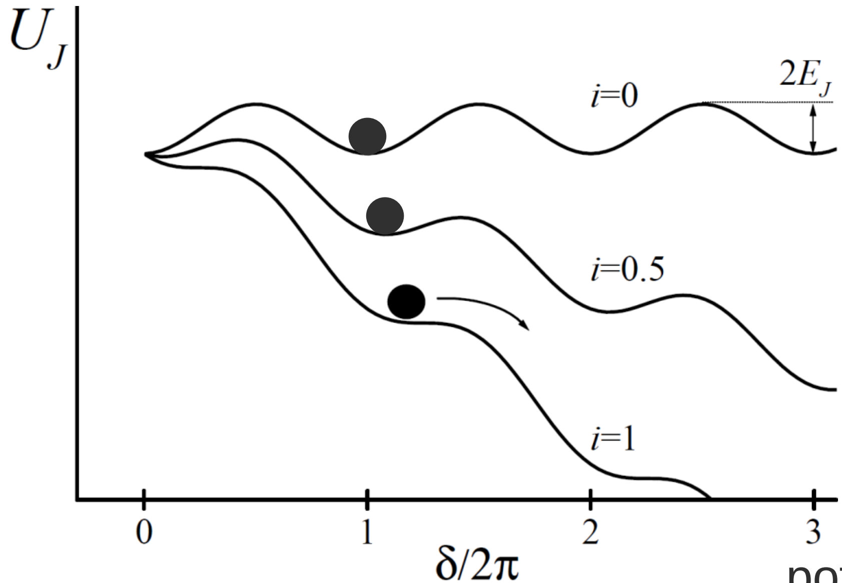
⇨ analogous system: point-like particle in the tilted washboard potential

$$m\ddot{x} + \xi\dot{x} = -\frac{\partial\{W(x) - F_\mathrm{d} x\}}{\partial x}$$

		⇔		
mass	m		C	capacitance
friction coeff.	ξ		$1/R$	conductance
force	F_d		I	current
velocity	$\dot{x}$		$\dot{\delta}\,\dfrac{\Phi_0}{2\pi} = U$	voltage



RCSJ Model & Washboard Potential



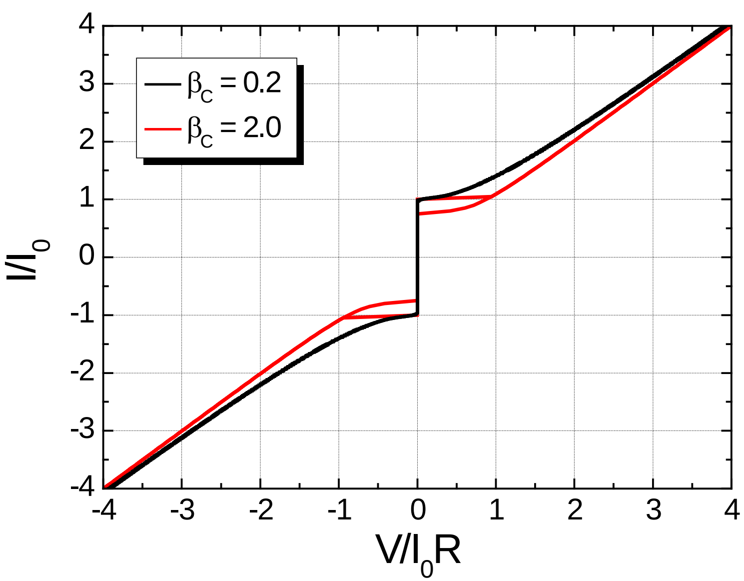
static case:
„particle" is trapped in potential minimum

$$\langle\dot{\delta}\rangle \propto V = 0$$

dynamic case:
„particle" rolls down the tilted potential

$$\langle\dot{\delta}\rangle \propto V \neq 0$$

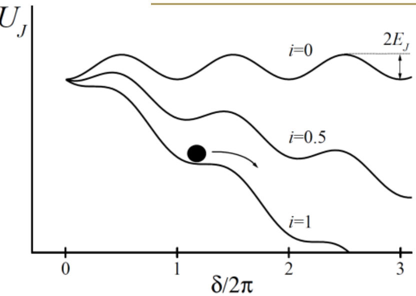
potential minima disappear at $i = 1 \Leftrightarrow I = I_0$
i.e. when critical current I_0 is reached



for large tilt $F_\mathrm{d} \gg \dfrac{\partial W(x)}{\partial x} \Rightarrow \dot{x} = \dfrac{F_\mathrm{d}}{\xi}$

i.e. for $I \gg I_0 \Rightarrow V = I\,R$

Effect of Damping in the RCSJ Model



normalized Eq. of motion

$$\beta_C \ddot{\delta} + \dot{\delta} + \sin\,\delta = i + i_N$$

with Stewart-McCumber parameter

$$\beta_C \equiv \frac{2\pi}{\Phi_0} I_0 R^2 C$$

$$i \equiv \frac{I}{I_0}\ ,\ i_{\rm N} \equiv \frac{I_{\rm N}}{I_0}$$

characteristic voltage (I_0R product) $V_{\rm c} \equiv I_0 R$

characteristic frequency

$$\omega_{\rm c} \equiv \frac{2\pi}{\Phi_0} I_0 R$$

normalized time $\tau \equiv t\omega_{\rm c}$

decrasing *I* from *I*>I_0:

strong damping: friction term $\dot{\delta}$ dominates, i.e. $\beta_{\rm C} \ll 1$

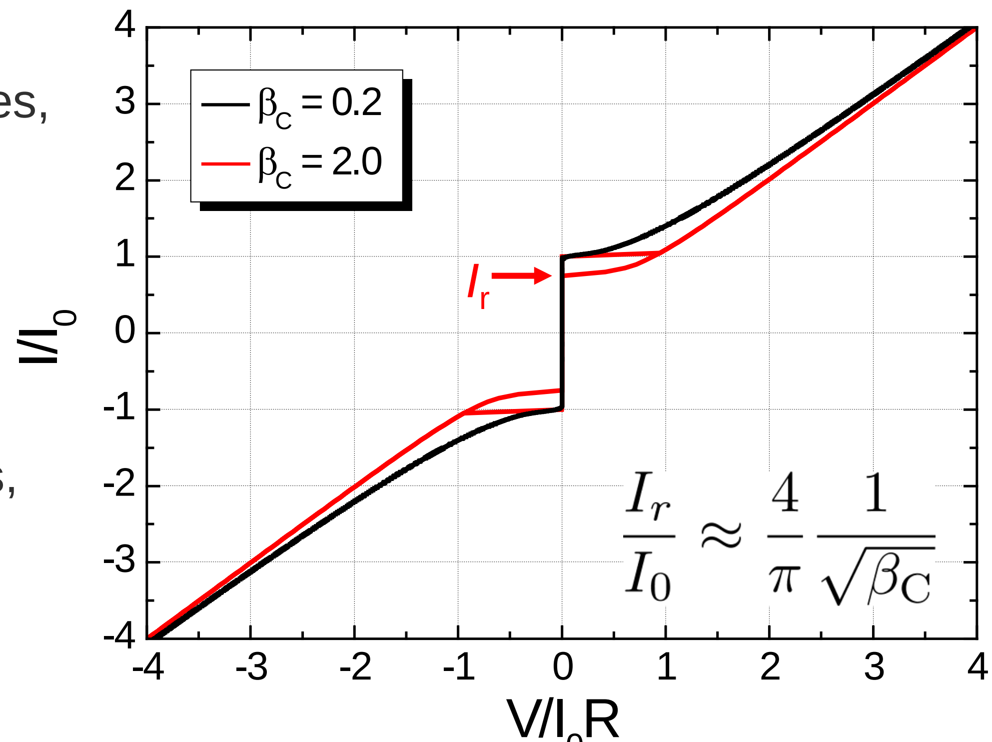
particle gets retrapped at *I*=I_0

non-hysteretic IVC

weak damping: inertial term $\ddot{\delta}$ dominates, i.e. $\beta_{\rm C} \gg 1$

particle gets retrapped at *I*=I_r<I_0

hysteretic IVC

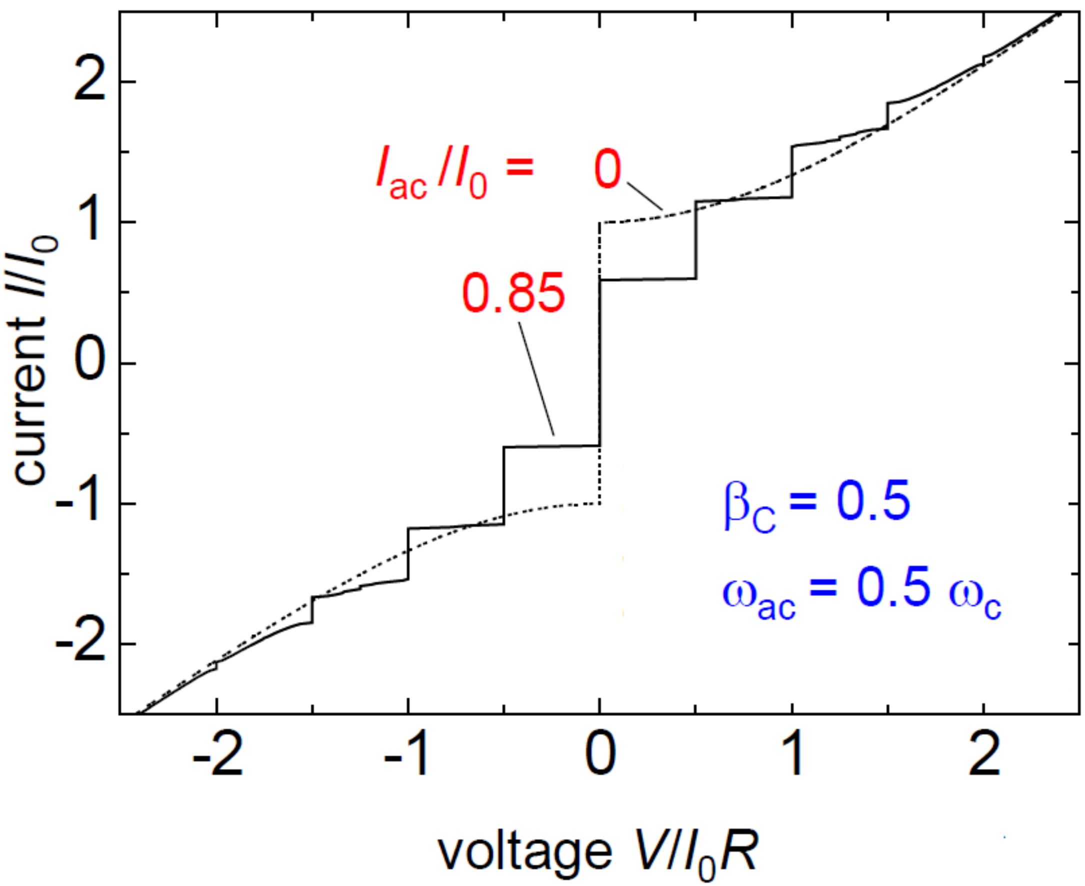


$$\frac{I_r}{I_0} \approx \frac{4}{\pi}\frac{1}{\sqrt{\beta_{\rm C}}}$$

Microwave Absorption: Shapiro Steps

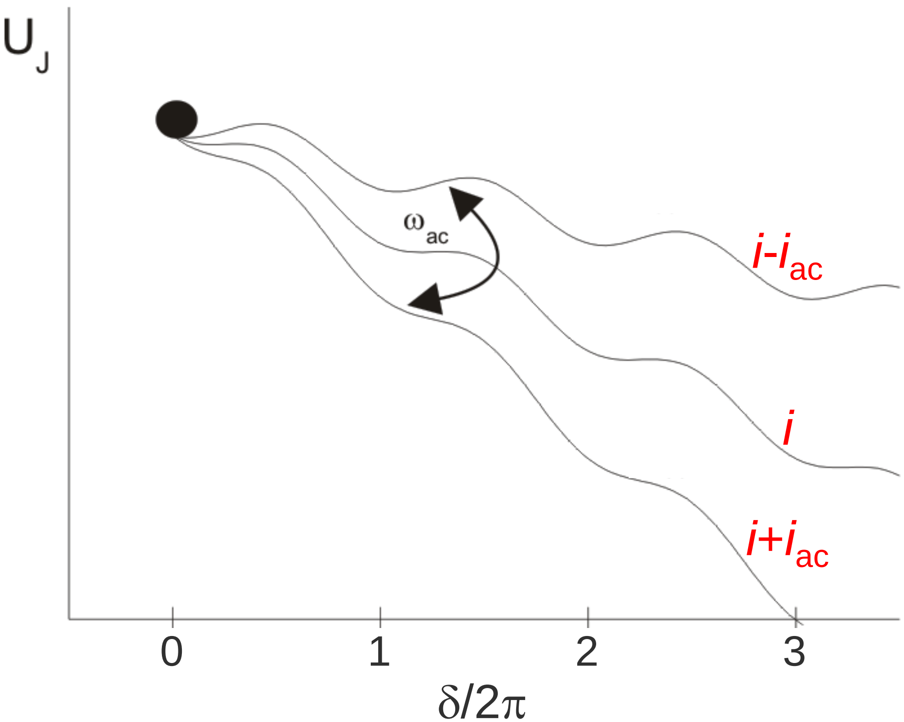
apply alternating current in addition to dc current $I_{\mathrm{tot}} = I + I_{\mathrm{ac}} \sin \omega_{\mathrm{ac}} t$

⇨ **regimes of constant voltage V_n in the IVC = Shapiro steps**



Microwave Absorption: Shapiro Steps

illustrative interpretation with particle in tilted washboard potential



motion of the „particle" synchronizes with the external drive

change of δ by $2\pi\, n$ $\quad (n = 1, 2, \dots)$

per excitation period $T_{\mathrm{ac}} = 1/f_{\mathrm{ac}}$

$f_{\mathrm{ac}} = 2\pi\omega_{\mathrm{ac}}$

velocity $\dot{\delta}_n = \dfrac{2\pi n}{T_{\mathrm{ac}}} = 2\pi n f_{\mathrm{ac}}$

stable within some intervall of applied dc current *I*

steps of constant voltage V_n on IVC at $V_n = \dfrac{\Phi_0}{2\pi}\dot{\delta}_n = n\Phi_0 f_{\mathrm{ac}}$

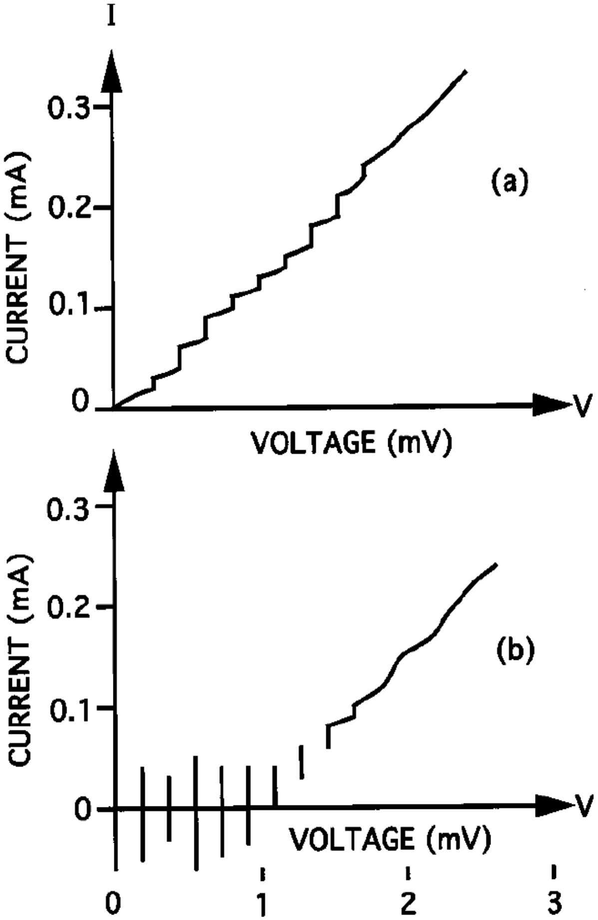
equidistant Shapiro steps with separation

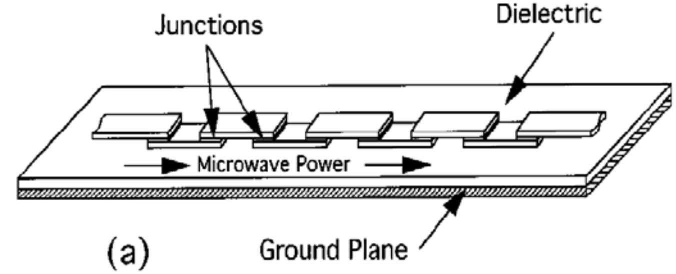
$$\Delta V = V_{n+1} - V_n = \Phi_0 f_{\mathrm{ac}} \approx \frac{1\,\mathrm{mV}}{483.6\,\mathrm{GHz}} \cdot f_{\mathrm{ac}}$$

Josephson Normal: Voltage Standard

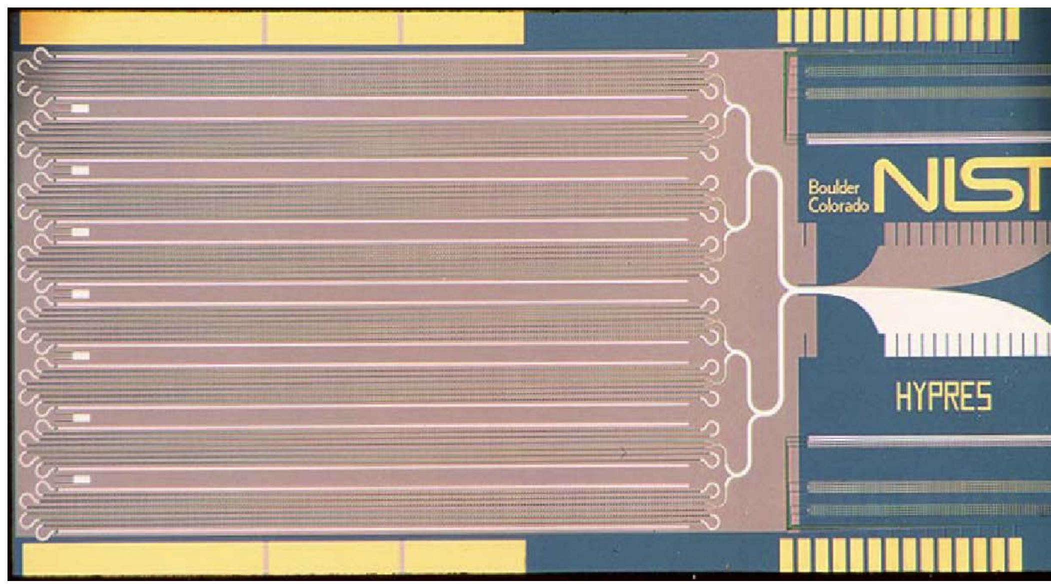
irradiation of microwaves on Josephson junctions
(typically: f = 70 GHz $\rightarrow n \cdot 0.144$ mV)

$$V_n = n\Phi_0 f_{\mathrm{ac}}$$





20.000 Josephson junctions



reproducible voltages with relative uncertainty
< 1 : 10^{10}, corresponds to 1 nV at 10 V)

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Fluctuations in Josephson Junctions

- **Thermal noise**

What is the effect of finite temperature *T* ?

for a JJ described within the RCSJ model:

$$\frac{\Phi_0}{2\pi} C\ddot{\delta} + \frac{\Phi_0}{2\pi}\frac{1}{R}\dot{\delta} = -I_0 \sin\delta + I + I_\mathrm{N} \equiv -\frac{2\pi}{\Phi_0}\frac{\partial U_\mathrm{J}}{\partial \delta}$$

noise current acts as a stochastic force → Langevin Eq.

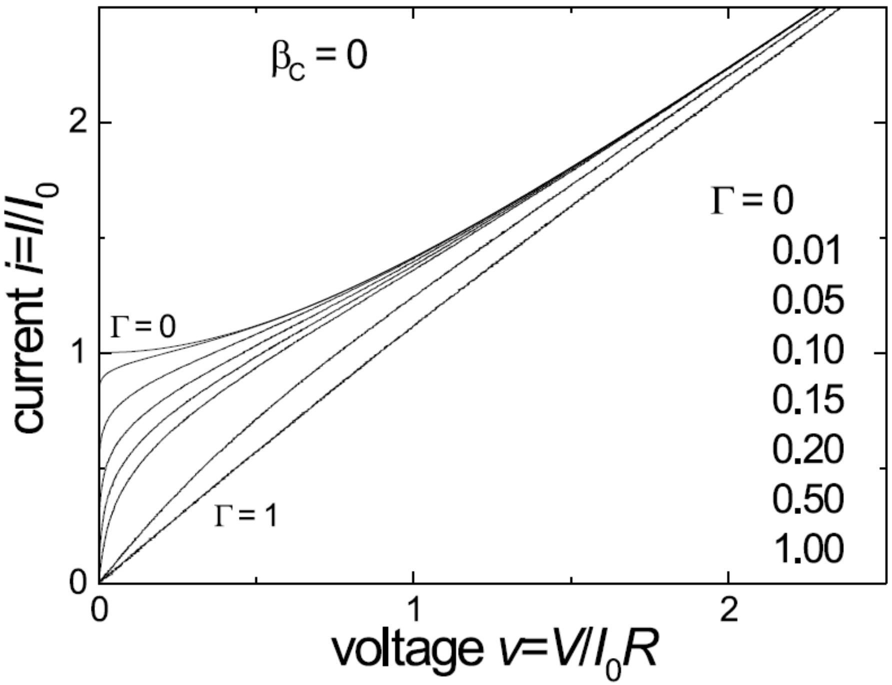
induces fluctuating tilt of the washboard potential

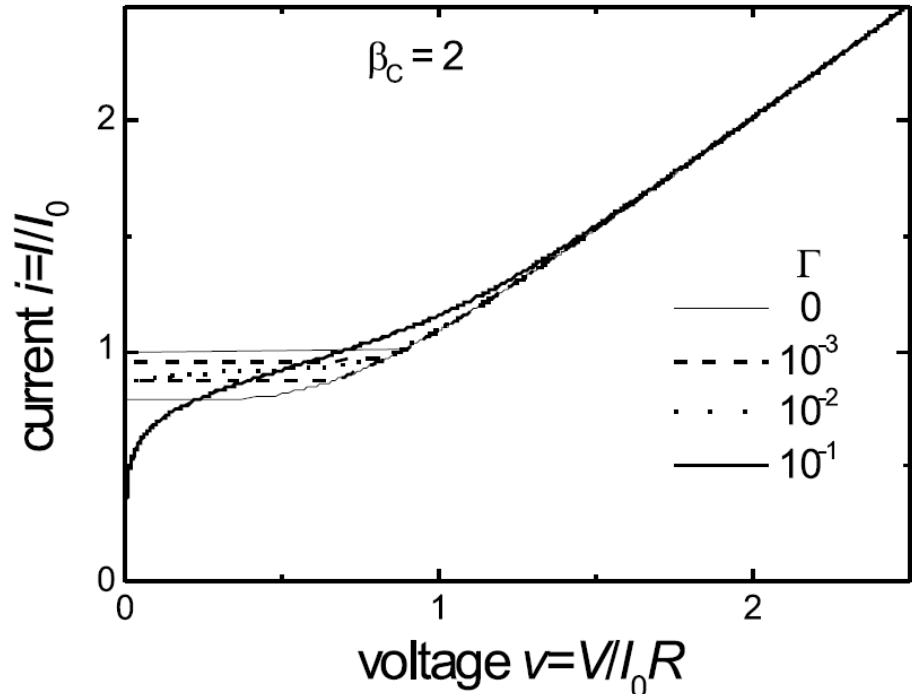
$$U_\mathrm{J} \equiv E_\mathrm{J}\{1 - \cos\delta - (i + i_\mathrm{N})\,\delta\} \qquad i_\mathrm{N} \equiv I_\mathrm{N}/I_0$$

Fluctuations in Josephson Junctions

for $I \lesssim I_0$ fluctuations can lead to $I + I_{\mathrm{N}}(t) > I_0$ ⇨ voltage pulses $U(t)$ with V>0

⇨ thermal smearing (noise rounding) of IVCs

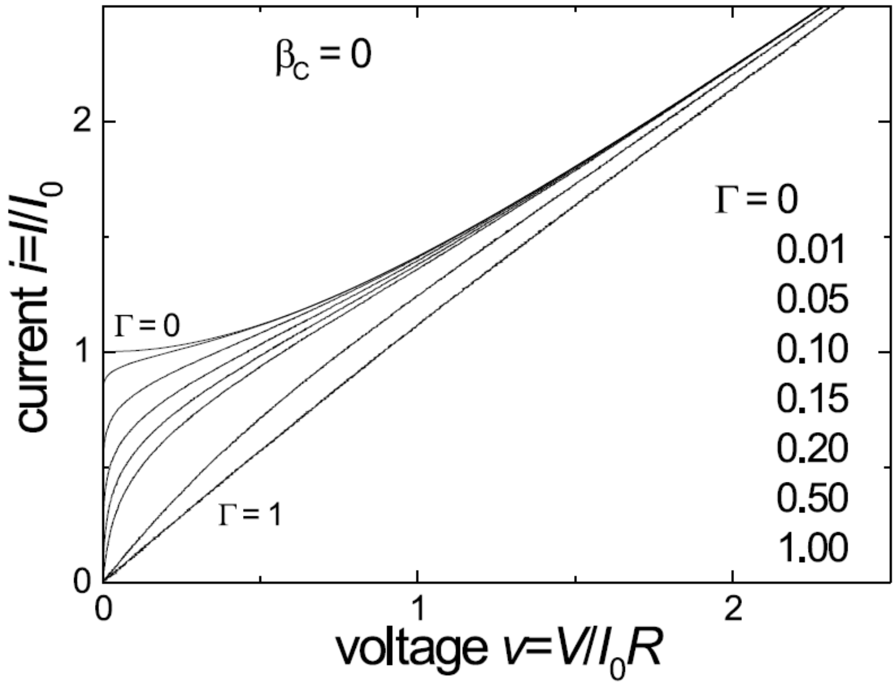




quantified by thermal noise parameter $\Gamma \equiv \dfrac{k_{\mathrm{B}}T}{E_{\mathrm{J}}}$

E_{J}: Josephson coupling energy = amplitude of washboard potential

Fluctuations in Josephson Junctions



$$\Gamma = \frac{2\pi k_\mathrm{B} T}{I_0 \Phi_0} = \frac{2\pi k_\mathrm{B} T/\Phi_0}{I_0} = \frac{I_\mathrm{th}}{I_0}$$

thermal fluctuations „destroy" Josephson coupling

regime of small thermal fluctuations:

$$\Gamma \ll 1$$

corresponds to $I_0 \gg I_\mathrm{th} = \frac{2\pi}{\Phi_0} k_\mathrm{B} T \propto T$

for T= 4.2 K: I_{th} ~ 0.18 µA
for T = 77 K: I_{th} ~ 2.3 µA

⇨ significant suppression of „measurable I_c already at Γ=10^{-2} !

Fluctuations in Josephson Junctions

• Low-frequency excess noise: 1/*f* noise

description of tunnel junctions by parameter fluctuations rather than Langevin force

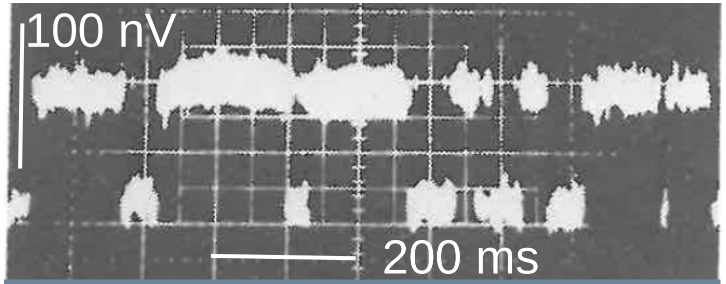
origin:

fluctuations in I_0 due to trapping and release of electrons at defects in the tunnel barrier (change barrier height, and hence I_0, (also R))

single trap:

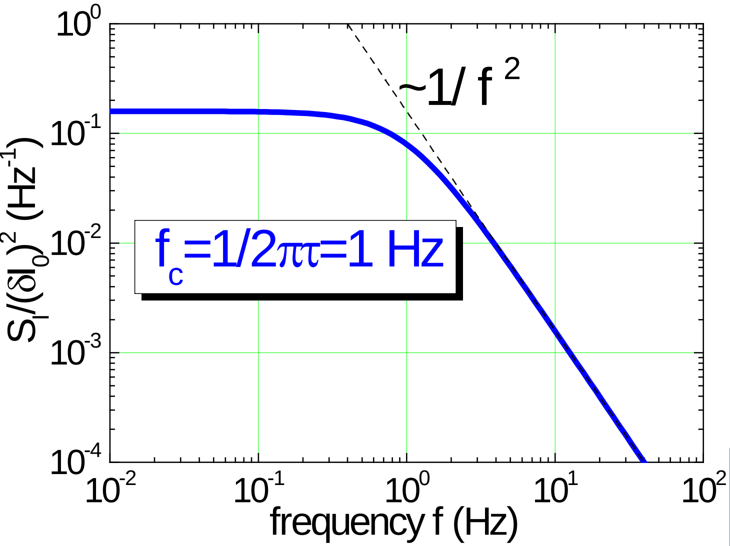
random switching of I_0 between two values with difference δI_0 and effective lifetime τ

⇒ random telegraph signal (RTS)



R. Gross, B. Mayer, Physica C **180**, 235 (1991)

$V(t)$ of grain boundary junction at I=1.2 I_0



with Lorentzian spectral density

$$S_I(f) = \frac{(\delta I_0)^2 \cdot \tau}{1 + (2\pi\tau \cdot f)^2}$$

with $\tau^{-1} \equiv \tau_1^{-1} + \tau_2^{-1}$ for mean life times τ_1=τ_2 in the two states

C.T. Rogers & R.A. Burman, *Composition of 1/f noise in metal-insulator-metal tunnel junctions*, Phys. Rev. Lett. **53**, 1272 (1984)

Fluctuations in Josephson Junctions

for thermally activated trapping processes

$$\tau = \tau_0 \, \exp\left(\frac{E}{k_\mathrm{B}T}\right)$$

with τ_0=const, and activation energy *E*

e.g. τ_0=0.1 s, and *E*=1.8 meV
for Nb-AlO_x-Nb tunnel JJs

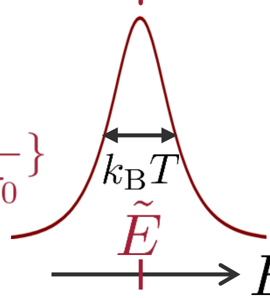
B. Savo, F.C. Wellstood, J. Clarke, Appl. Phys. Lett. **50**, 1757 (1987)

superposition of several (or many) traps

$$S(f) \propto \int dE \, \underset{\uparrow}{D}(E) \underbrace{\left[\frac{\tau_0 \, e^{\frac{E}{k_\mathrm{B}T}}}{1 + (2\pi f \tau_0)^2 \, e^{\frac{2E}{k_\mathrm{B}T}}}\right]}$$

distribution function

peak at $\tilde{E} \equiv k_\mathrm{B}T \ln\{\frac{1}{2\pi f \tau_0}\}$



for given *T*, only traps contribute with

$$\tilde{E} - k_\mathrm{B}T \lesssim E \lesssim \tilde{E} + k_\mathrm{B}T$$

for broad distribution $D(E)$ with respect to $k_\mathrm{B}T$:

take $D(\tilde{E})$ out of the integral

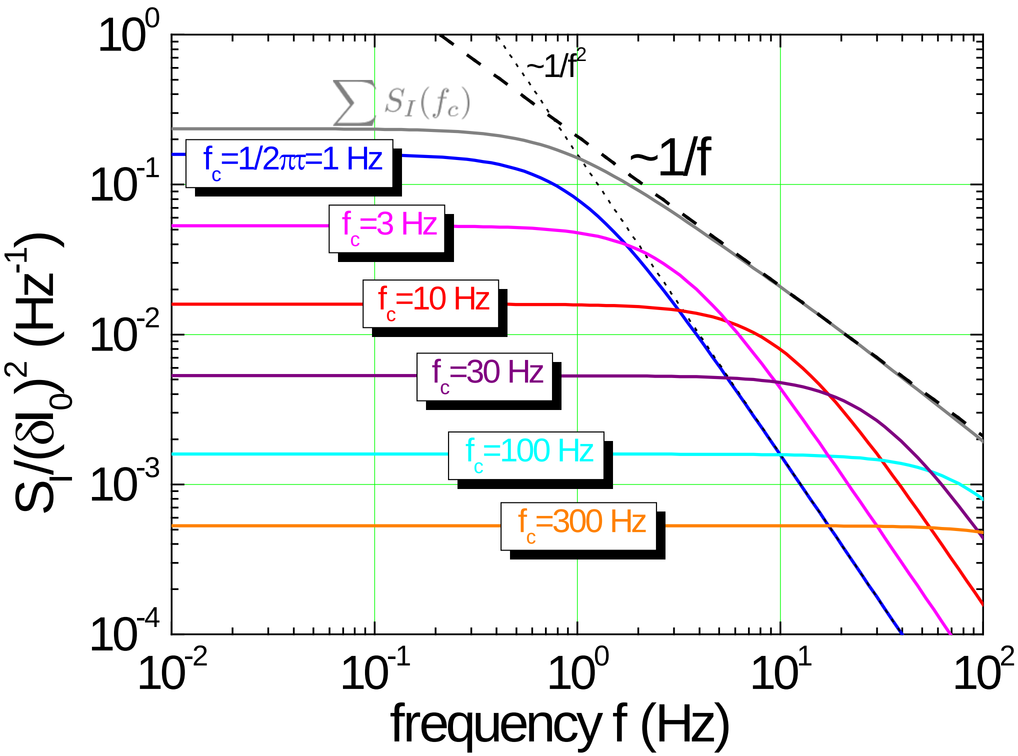
$$\Rightarrow \quad \boxed{S(f) \propto k_\mathrm{B}T \, D(\tilde{E}) \, \frac{1}{f}}$$

P. Dutta, P.M. Horn, *Low-frequency fluctuations in solids: 1/f noise*, Rev. Mod. Phys. **53**, 497 (1981); M.B. Weissman, *1/f noise and other slow, nonexponential kinetics in condensed matter*, Rev.Mod.Phys. **60**, 537 (1988)

Fluctuations in Josephson Junctions

superposition of several (or many) traps

the superposition of only few traps already yields $S(f) \propto \dfrac{1}{f}$



P. Dutta, P.M. Horn, *Low-frequency fluctuations in solids: 1/f noise*, Rev. Mod. Phys. **53**, 497 (1981); M.B. Weissman, *1/f noise and other slow, nonexponential kinetics in condensed matter*, Rev.Mod.Phys. **60**, 537 (1988)

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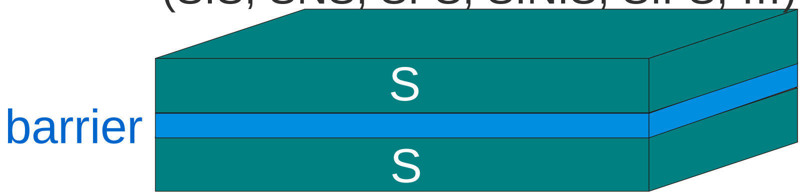
Types of Josephson Junctions

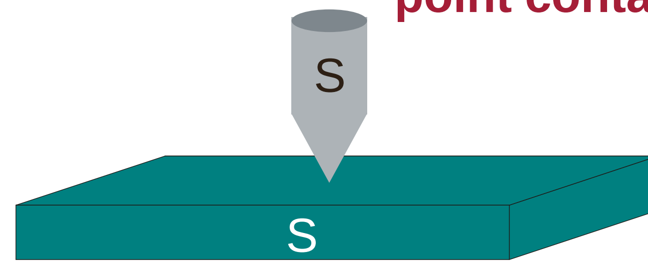
planar sandwich-type JJ

(SIS, SNS, SFS, SINIS, SIFS, ...)

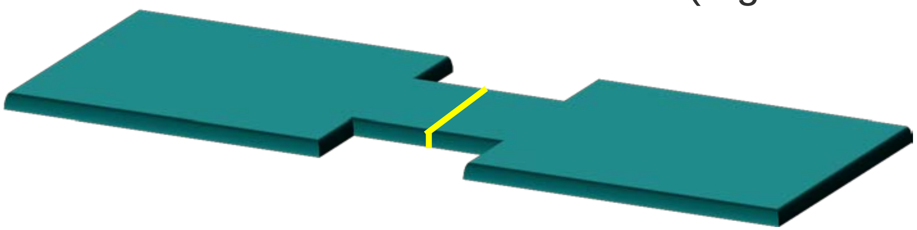


insulator (I), normal conductor (N), ferromagnet (F)

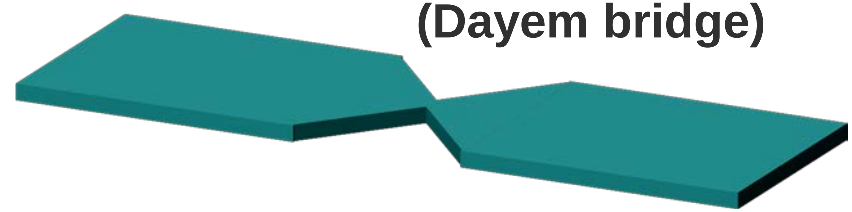
point contact



irradiation-induced barrier JJ (e.g. ion beam)

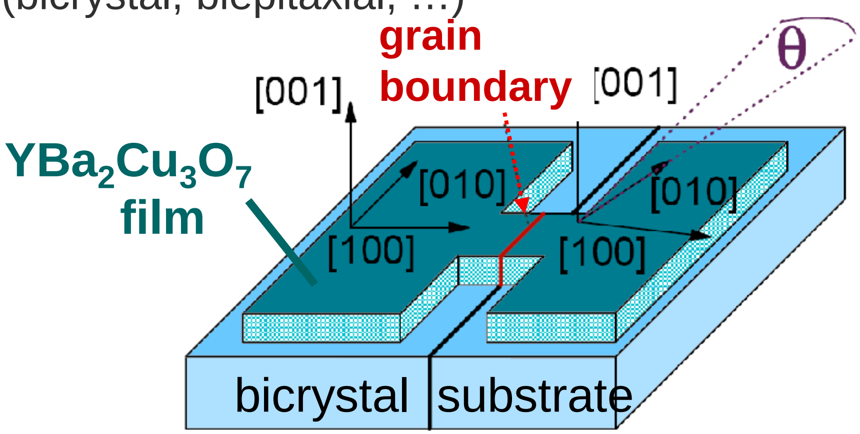


constriction junction (Dayem bridge)



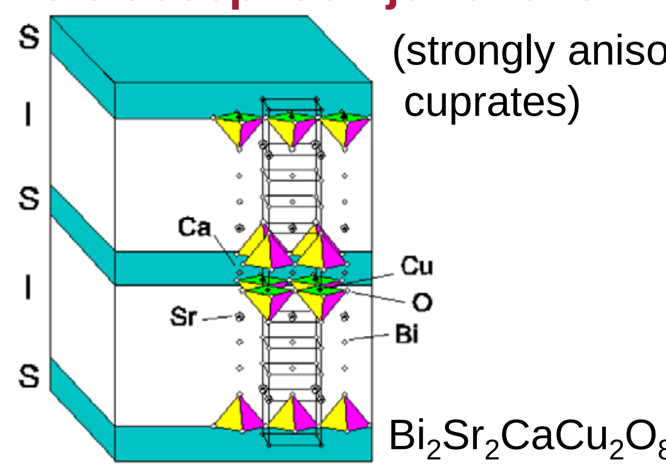
grain boundary junction (cuprates)

(bicrystal, biepitaxial, ...)



intrinsic Josephson junctions

(strongly anisotropic cuprates)

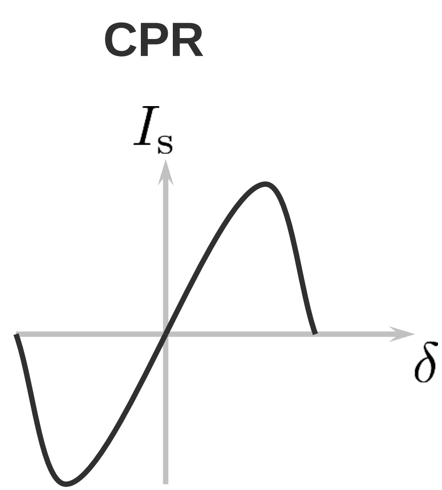


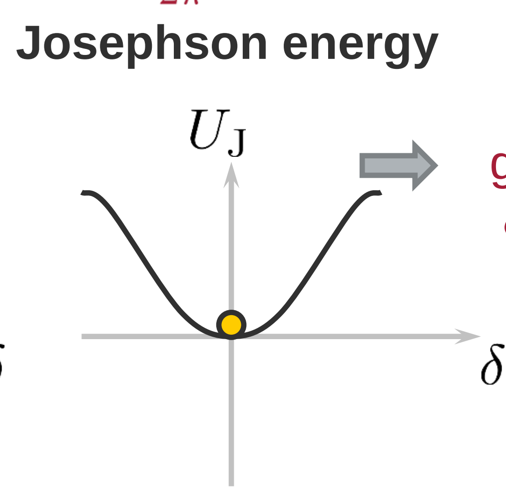
JJs with Different Ground States

The **Josephson energy $U_J(\delta)$** can be derived for any CPR, i.e. $I_s(\delta)$:
Increase current I_s in time t → change of $\delta(t)$ → finite voltage U

$$U_J(\delta) = \int_{t_0}^{t} I_s(\delta(\tilde t))\underset{\uparrow\atop =\frac{\Phi_0}{2\pi}\dot\delta(\tilde t)}{U}\,\mathrm{d}\tilde t = \frac{\Phi_0}{2\pi}\int_{t_0}^{t} I_s(\tilde\delta(\tilde t))\,\underbrace{\dot\delta(\tilde t)\,\mathrm{d}\tilde t}_{=\mathrm{d}\tilde\delta} = \frac{\Phi_0}{2\pi}\int_{\delta_0}^{\delta} I_s(\tilde\delta)\,\mathrm{d}\tilde\delta$$

for $I_s = I_0 \sin\delta$ ⇨ $U_J(\delta) = \underset{\uparrow\atop =\frac{I_0\Phi_0}{2\pi}}{E_J}(1-\cos\delta)$





⇨ ground state phase:
δ=0 (for I=0)

⇨ **„0-JJ“**

JJs with Different Ground States

CPR: $I_\mathrm{s} = I_0 \sin(\delta)$ → ground state: $\delta = 0$

CPR: $I_\mathrm{s} = I_0 \sin(\delta - \pi)$ → ground state: $\delta = \pi$

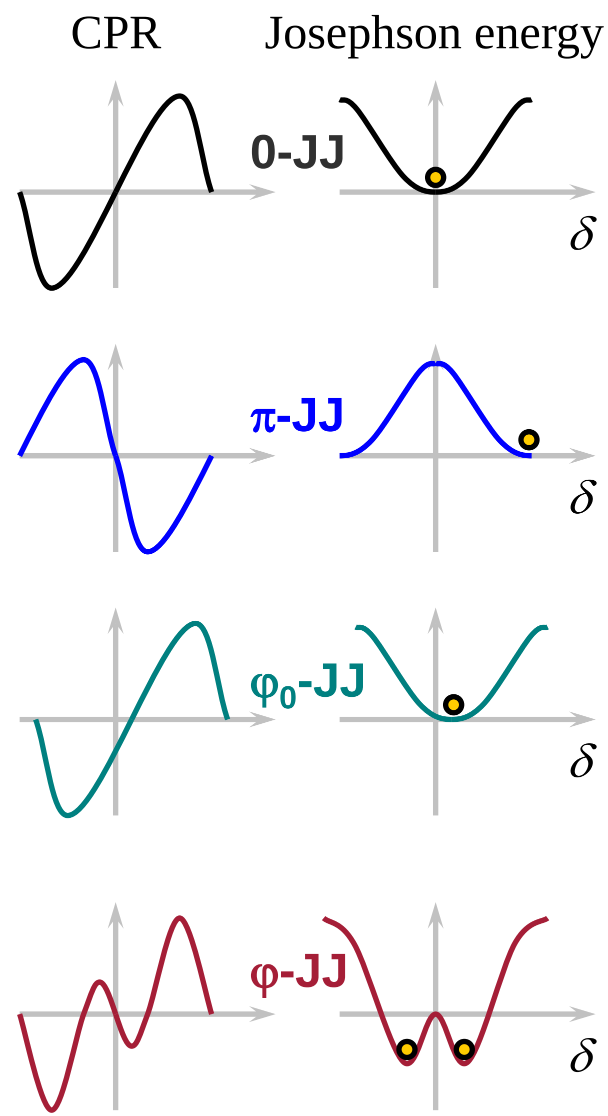
Experiment: V.V. Ryazanov *et al.*, Phys. Rev. Lett. **86**, 2427 (2001)

CPR: $I_\mathrm{s} = I_0 \sin(\delta - \varphi_0)$ → ground state: $\delta = \varphi_0$
(with $0 < \varphi_0 < \pi$)

Experiment: D.B. Szombati *et al.*, Nat. Phys. **12**, 568 (2016)

CPR: $I_\mathrm{s} = I_{0,1} \sin(\delta) + I_{0,2} \sin(2\delta)$
→ ground state: $\delta = \pm\varphi$

(with $0 < \varphi < \pi$ and $\varphi = \arccos(-\frac{I_{0,1}}{2I_{0,2}})$ for $-I_{0,2} > \frac{I_{0,1}}{2}$)



Theory: Yu.S. Barash *et al.*, Phys. Rev. B **52**, 665 (1995)
Y. Tanaka , S. Kashiwaya, Phys. Rev. B **53**, R11957 (1996)
A. Buzdin, A.E. Koshelev, Phys. Rev. B **67**, 220504(R) (2003)

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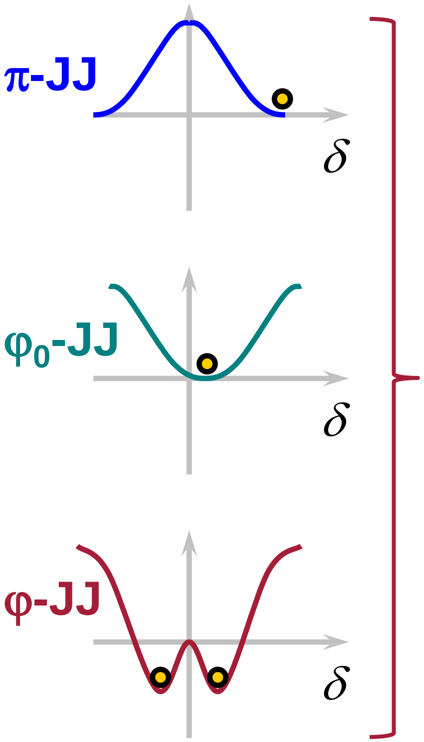
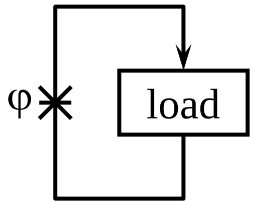
JJs with Different Ground States

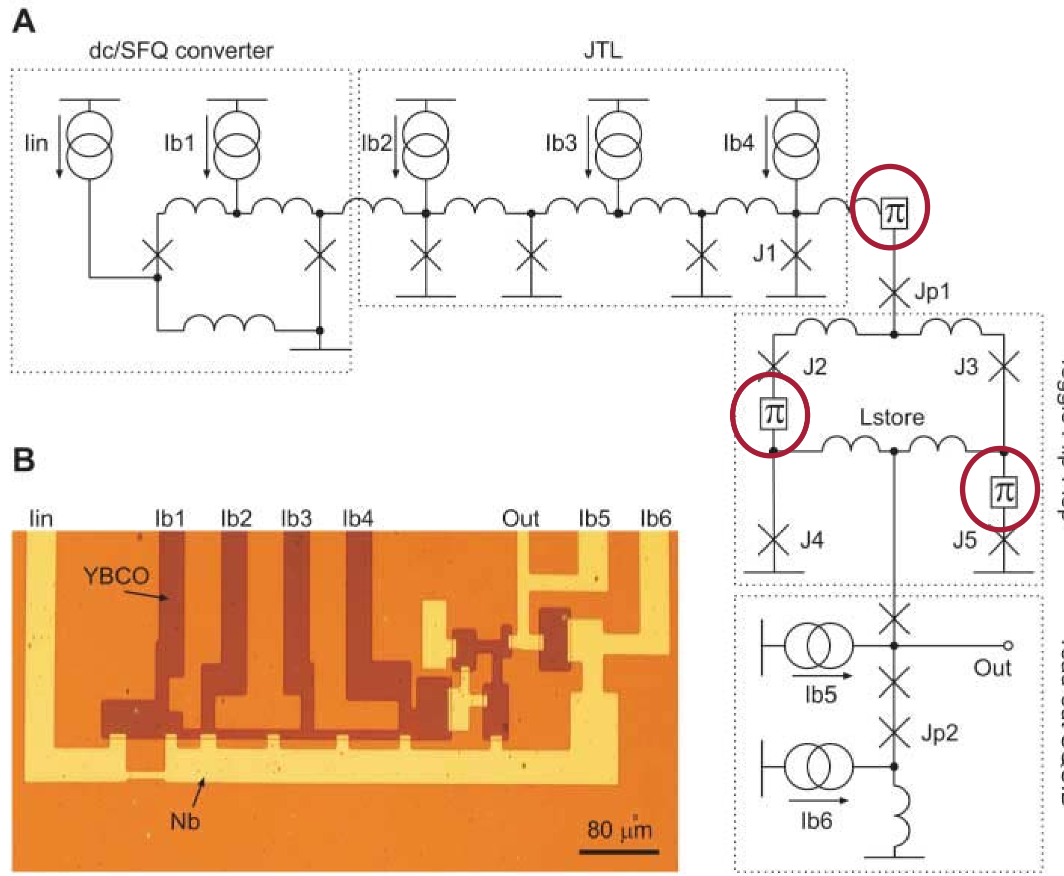


phase battery
phase shifter



Self-biased RSFQ flip-flop:

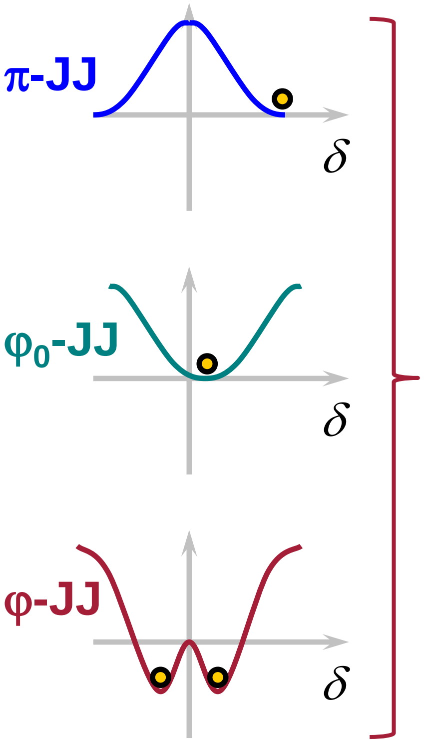
integrated π-rings based on YBCO-Nb *s*-/*d*-wave JJs



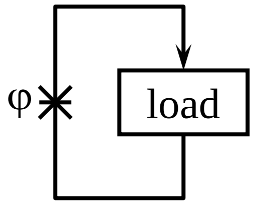
T. Ortlepp *et al.*, Science **312**, 1495 (2006)

RSFQ: rapid single flux quantum

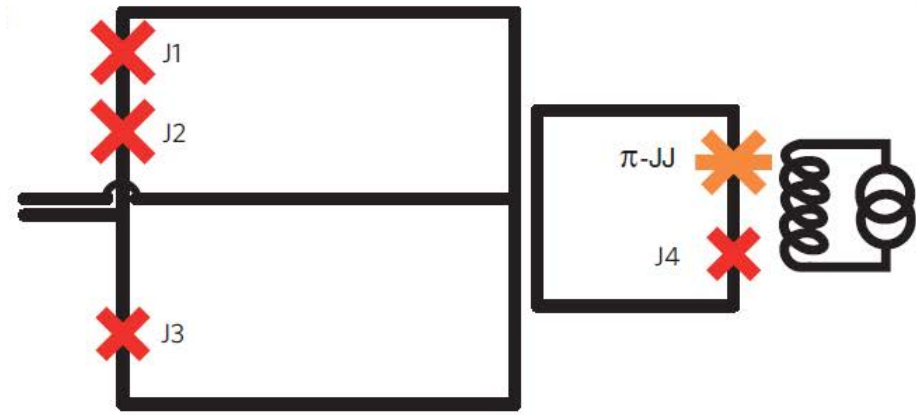
JJs with Different Ground States

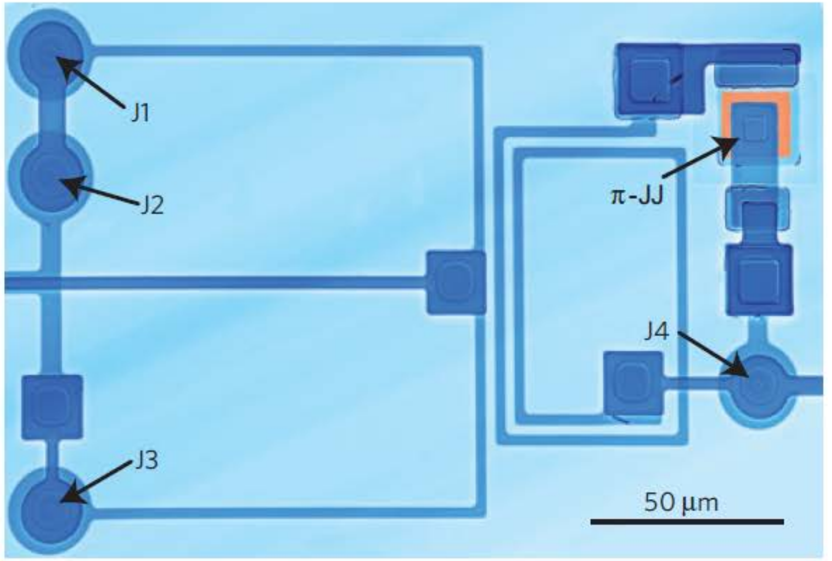


phase battery
phase shifter



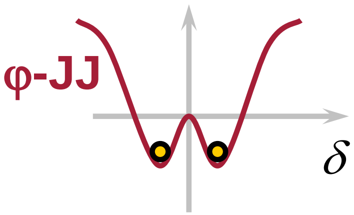
Phase qubit with SFS π-JJ:
based on Nb-CuNi-Nb JJs





Feofanov *et al.*, Nature Physics **6**, 595 (2010)

φ-JJ: Tunable Bistable System

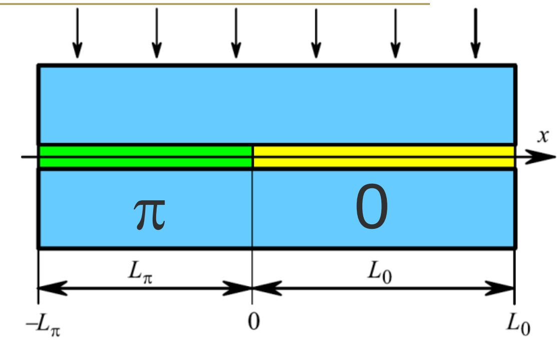


based on periodic 0-π JJs

A. Buzdin, A.E. Koshelev, PRB **67** (2003)

⇨ simplest case: **0-π JJ**

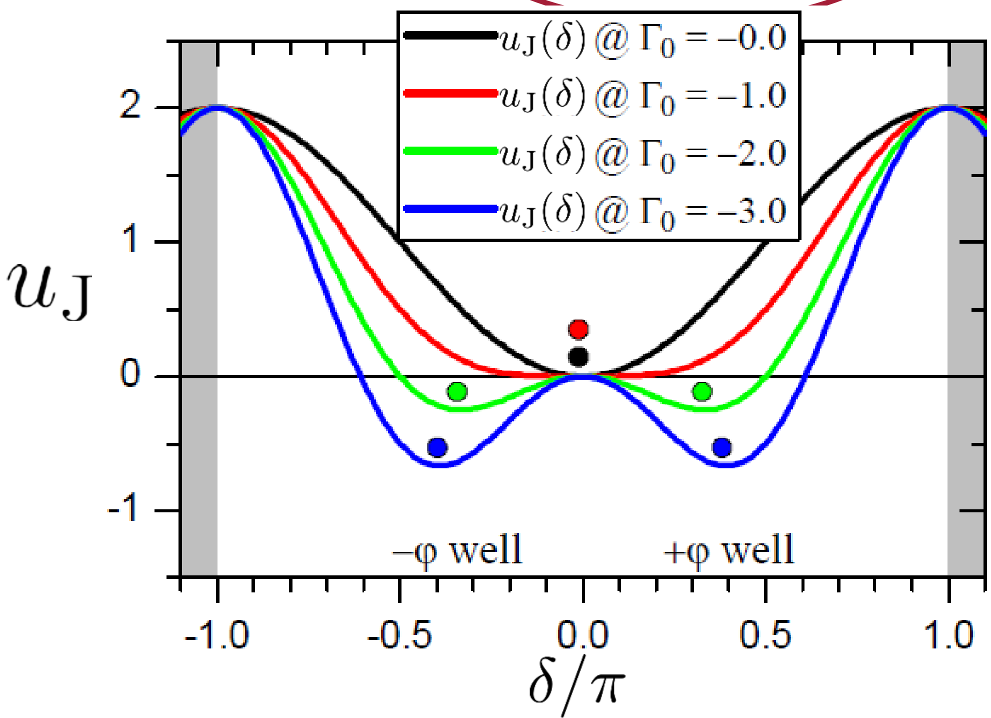
E. Goldobin *et al*., Phys. Rev. Lett. **107**, 227001 (2011)



$$u_J \equiv \frac{U_J(\delta)}{E_J} = 1 - \cos(\delta) + \frac{\Gamma_0}{4}\{1-\cos(2\delta)\} + \Gamma_h h \sin(\delta)$$

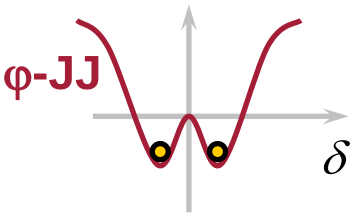
Γ_h : asymmetry parameter

$$\Gamma_0 \equiv \frac{2I_{0,2}}{I_{0,1}}$$



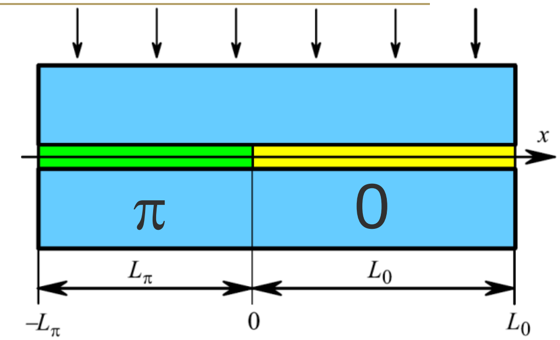
bistabe/two-level system for $-\Gamma_0>1$

φ-JJ: Tunable Bistable System



simplest case: **0-π JJ**

E. Goldobin *et al.*, Phys. Rev. Lett. **107**, 227001 (2011)



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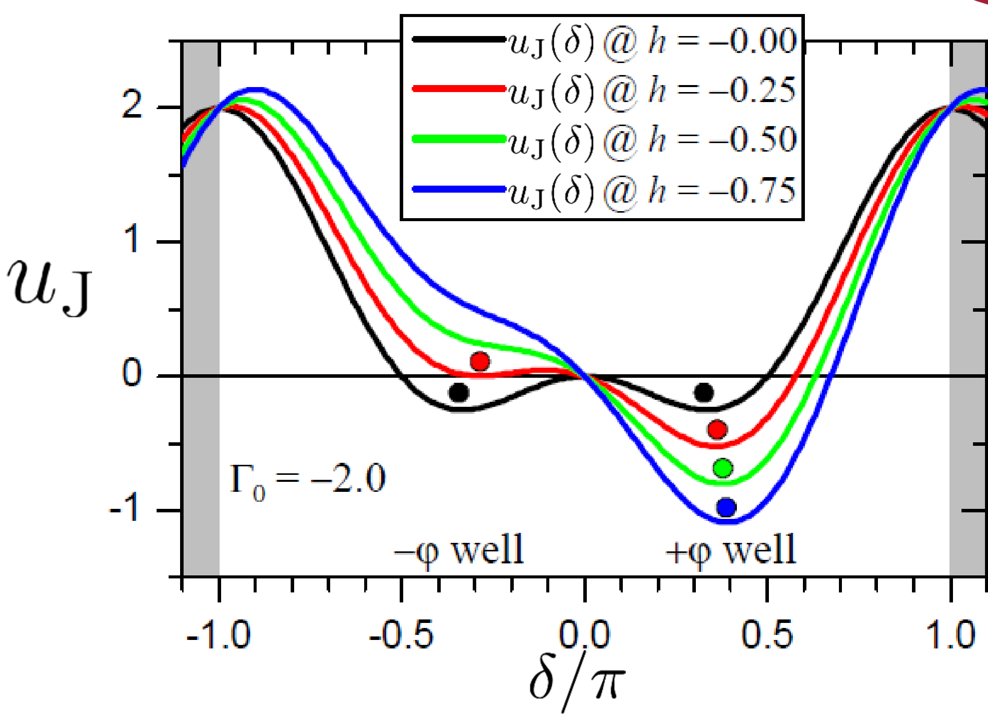
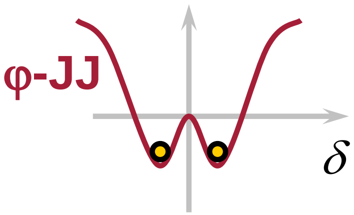


tunable by magnetic field

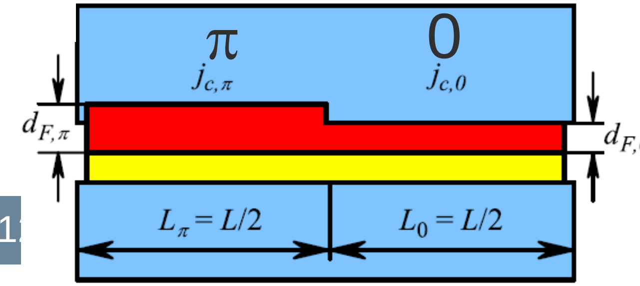
lifts degeneracy of double-well potential

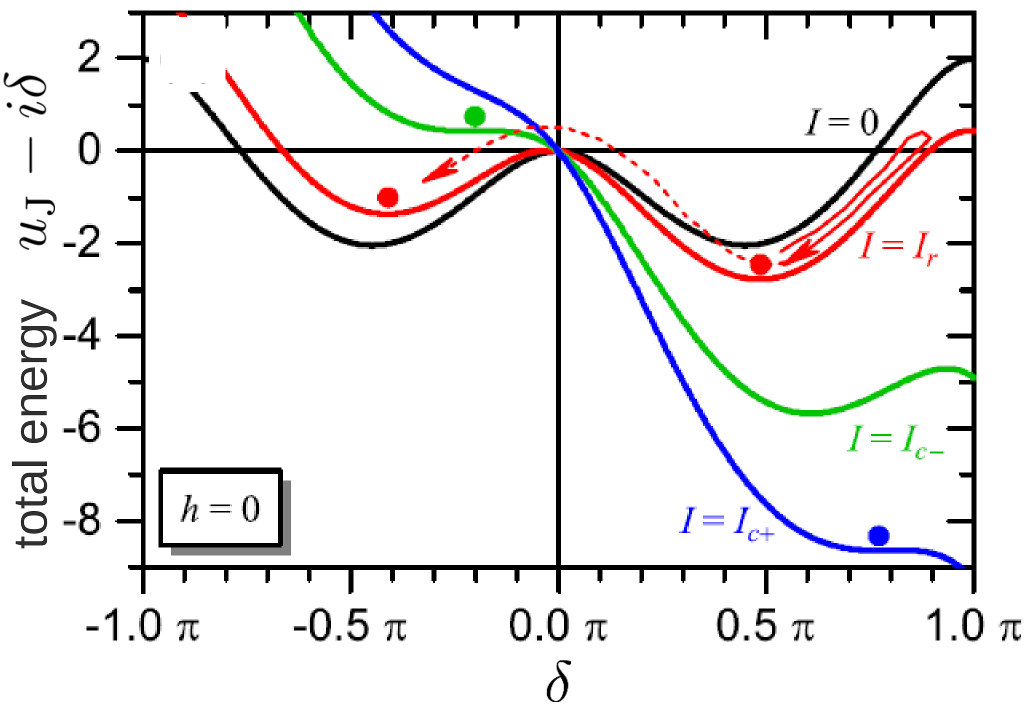
φ-JJ: Two Critical Currents

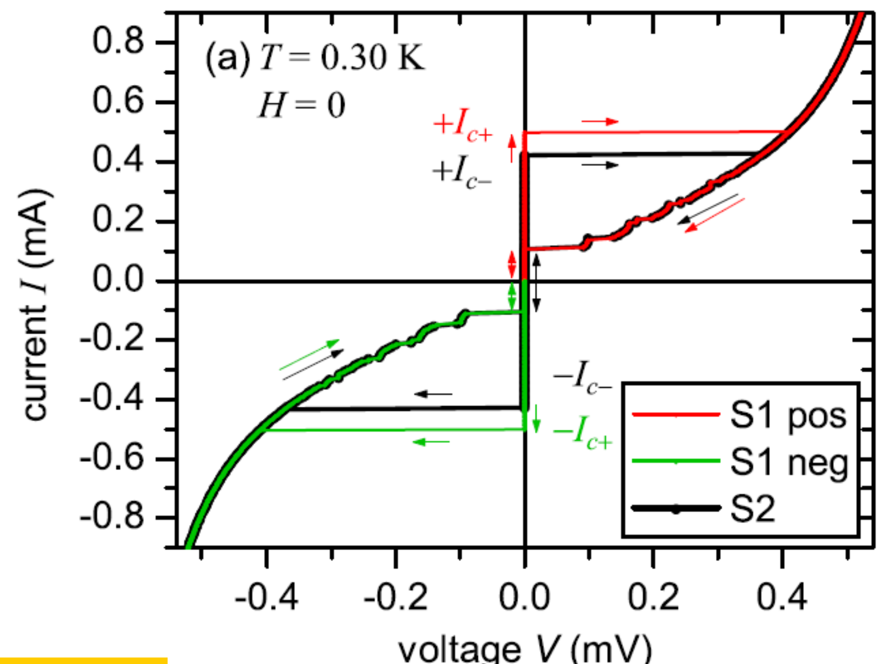


experimental realization:
Nb-AlO$_x$-Cu$_{0.4}$Ni$_{0.6}$-Nb SIFS JJ
with step in the F layer

H. Sickinger *et al.*, Phys. Rev. Lett. **109**, 107002 (201



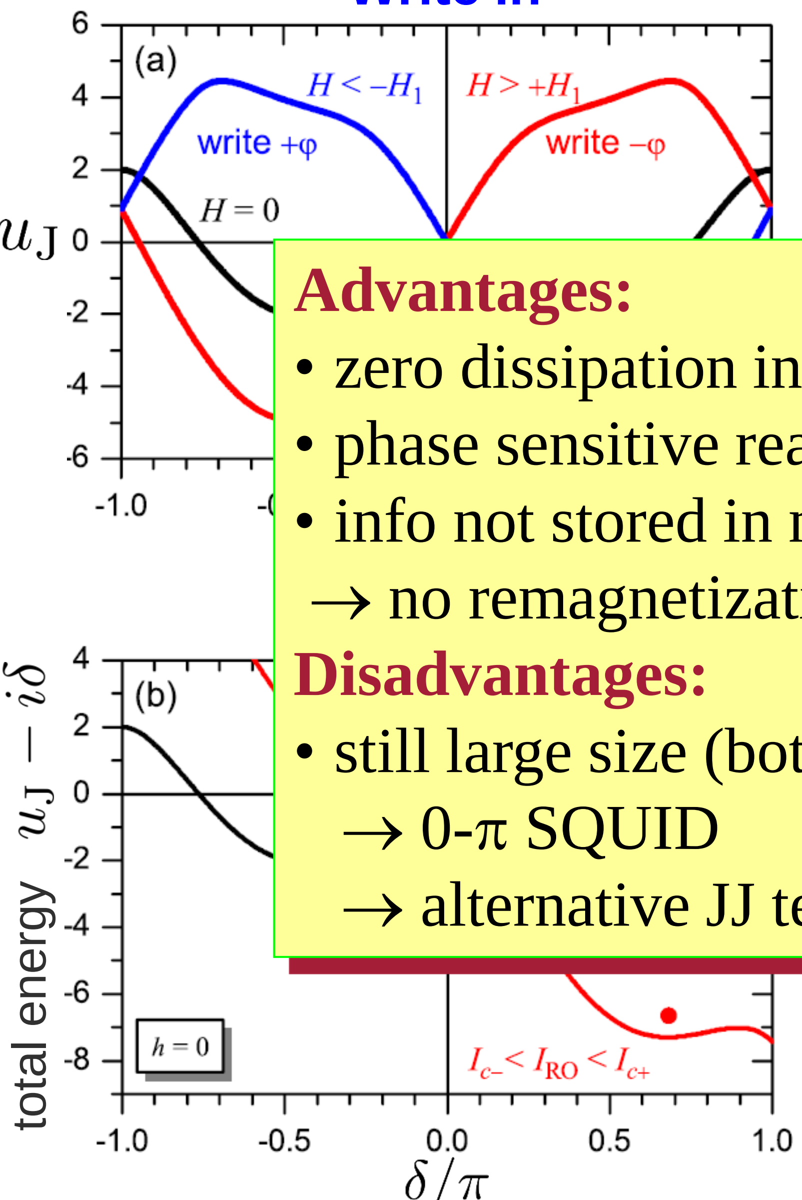


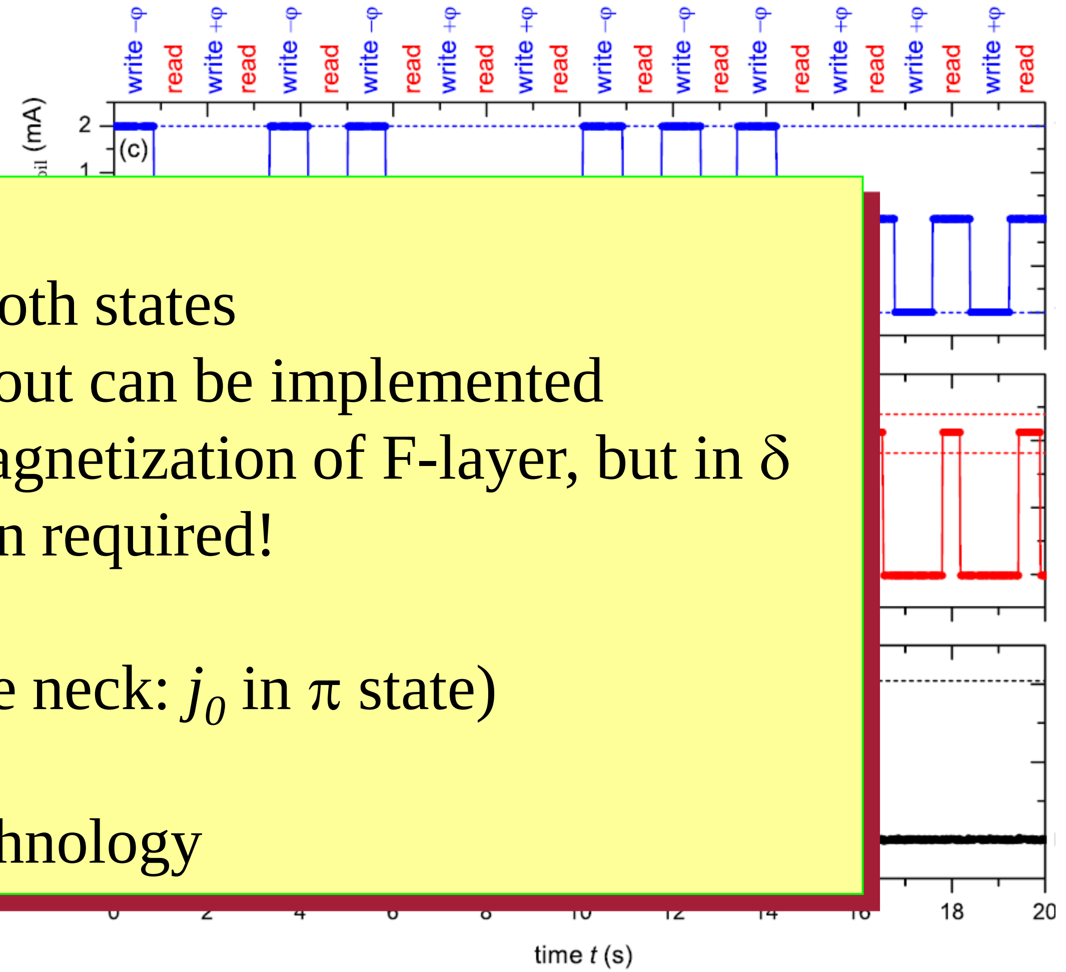


state detector!

φ-JJ Application: φ bit (memory cell)

Write in





Advantages:

- zero dissipation in both states
- phase sensitive readout can be implemented
- info not stored in magnetization of F-layer, but in δ
  → no remagnetization required!

Disadvantages:

- still large size (bottle neck: j_0 in π state)
  → 0-π SQUID
  → alternative JJ technology

E. Goldobin et al., APL **102**, 242602 (2013)

Summary

Macroscopic Wave Function → coherent state of Cooper pairs

⇩

Weak coupling of two condensates

→Josephson Relations & Consequences (static & dynamic cases)

- Static case:

Josephson Junction in a Magnetic Field → $I_c(H)$ Fraunhofer pattern for short JJs

- Dynamic case:

Resistively & Capacitively Shunted Junction (RCSJ) model
→ I-V-characteristics (particle in the tilted washboard potential)

Fluctuations in Josephson Junctions → thermal noise & I_c fluctuations important for device applications , e.g. SQUIDS

Classification of JJs – Ground States: 0- π-, φ-Junctions
→ new applications: phase batteries, memory devices, qubits,..