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# NON-RECTILINEAR WAVEGUIDES: ANALYTICAL AND NUMERICAL RESULTS BASED ON THE GREEN'S FUNCTION

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# Introduction

An optical waveguide is a dielectric structure which guides and confines an optical signal along a desired path. Probably, the best known example is the optical fiber, where the light signal is confined in a cylindrical structure. Optical waveguides are largely used in long distance communications, integrated optics and many other applications.

At first, the study of optical waveguides was mostly confined to rectilinear structures (used for some medical application and in long distance communications); later, the growing interest in optical integrated circuits stimulated the study of waveguides with different geometries. In fact, electromagnetic wave propagation along perturbed waveguides is still continuing to be widely investigated because of its importance in the design of optical devices, such as couplers, tapers, gratings, bendings imperfections of structures and so on.

In an optical waveguide, the central region (the *core*) is surrounded by a layer with lower index of refraction called *cladding*. A protective *jacket* covers the cladding. The difference between the indices of refraction of core and cladding makes possible to guide an optical signal and to confine its energy in proximity of the core.

There are two relevant ways of modeling wave propagation in optical waveguides. In *closed waveguides* one considers a tubular neighbourhood of the core and impose Dirichlet, Neumann or Robin conditions on its boundary. The use of these boundary conditions is efficient but somewhat artificial, since it creates spurious waves reflected by the boundary of the cladding. In this thesis we will study *open waveguides*, i.e. we will assume that the cladding (or the jacket) is extended to infinity. This choice provides a more accurate model to study the energy radiated outside the core (see [SL] and [Ma]).

Thinking of an optical signal as a superposition of waves of different frequency (the *modes*), it is observed that in a rectilinear waveguide most of the energy provided by the source propagates as a finite number of such waves (the *guided modes*). The guided modes are mostly confined in the core; they decay exponentially transversally to the waveguide's axis and propagate along that axis without any significant loss of energy. The rest of the energy (the *radiating energy*) is made of *radiation* and *evanescent* modes, according to their different behaviour along the waveguide's axis (see Section

2.4 for further details). The electromagnetic field can be represented as a discrete sum of guided modes and a continuous sum of radiation and evanescent modes.

In this thesis we present an analytical approach to the study of electromagnetic wave propagation in 2-D optical waveguides. The first part of the thesis deals with rectilinear waveguides, the second one with the non-rectilinear case. These two parts are closely connected one another: the results obtained for rectilinear waveguides play a crucial role for developing a mathematical framework for studying wave propagation in perturbed ones, as we are going to explain shortly.

We shall consider the time harmonic wave propagation in a 2-D optical waveguide. As a model equation, we will use the following *Helmholtz equation* (or *reduced wave equation*):

$$\text{(H)} \qquad \Delta u(x,z) + k^2 n(x,z)^2 u(x,z) = f(x,z),$$

with $(x,z) \in \mathbb{R}^2$, where $n(x,z)$ is the index of refraction of the waveguide, $k$ is the wavenumber and $f$ is a function representing a source. The axis of the waveguide is assumed to be the $z$ axis, while $x$ denotes the transversal coordinate.

In Chapters 2 and 3, we find a resolution formula for (H) and state a condition which guarantees the uniqueness of the solutions of (H). The non-rectilinear case is analyzed in Chapters 4 and 5 where we study small perturbations of rectilinear waveguides from the theoretical and numerical points of view, respectively.

Our approach to rectilinear waveguides is strictly connected to the ones proposed in [MS] and [AC1]. In Chapter 2 we derive a resolution formula for (H) in the case in which the function $n$ is of the form

$$\text{(RI)} \qquad n := n_0(x) = \begin{cases} n_{co}(x), & |x| \leq h, \\ n_{cl}, & |x| > h, \end{cases}$$

where $n_{co}$ is a bounded function and $2h$ is the width of the core. Such a choice of $n$ corresponds to an index of refraction depending only on the transversal coordinate and, thus, (H) describes the electromagnetic wave propagation in a rectilinear open waveguide.

The solution that we find was already obtained in [MS] by using a different technique when $n_{co}$ is an even function decreasing along the positive $x$ axis. In this thesis we adopt the approach developed in [AC1] for the case of a cylindrical optical fiber. In particular, we separate the variables and use the theory of Titchmarsh on eigenfunction expansion (see Section 2.2). This approach leads to a resolution formula for (H) with general assumptions on $n_{co}(x)$. The obtained Green's function consists of a finite sum, representing the guided modes, and of a continuous superposition of radiation and evanescent modes which represents the energy radiated outside the waveguide.

Since this approach is based on a rigorous transform theory, the superposition of guided, radiation and evanescent modes is complete.

Chapter 2 also contains some asymptotic results which will be instrumental for the analysis carried out in Chapters 4 and 5.

The study of rectilinear waveguides continues in Chapter 3, where we deal with the problem of the uniqueness of solutions. Here, our main result is Theorem 3.1, where we propose an outgoing radiation condition in integral form.

Historically, the uniqueness of solutions for Helmholtz equation has been widely investigated for problems in which energy cannot be trapped along any direction (see Section 3.1 for some reference). In such cases, the well known Sommerfeld radiation condition (or its successive generalizations) is used to restore the uniqueness of solutions.

In infinite waveguides, Sommerfeld radiation condition is useless because of the presence of guided modes. From a more physical point of view, we can say that the concept of energy radiating towards infinity suggested by Sommerfeld must be replaced or integrated by a new concept of waves carrying power to infinity.

The problem of uniqueness for waveguides has been studied mostly in the Russian literature (see [NS], [No], [KNH] and references therein). For closed waveguides, the Sveshnikov condition has been used (see [Sv]). For open waveguides (which is the case considered here) Reichardt condition has been introduced (see [Rei]). This condition guarantees the uniqueness of guided modes, but it does not apply to radiating energy.

The radiation condition that we give in Chapter 3 guarantees uniqueness of guided and radiated energy and is stated in a form which recalls a versions of Sommerfeld radiation condition due to Rellich (see [Rel]). Furthermore, when no guided mode is supported by the waveguide, our condition coincides that one.

In Chapters 4 and 5 we deal with the study of non-rectilinear waveguides. In Chapter 4 we present a mathematical framework which allows us to study the problem of wave propagation in perturbed waveguides. In particular, we are able to prove the existence of a solution for small perturbations of 2-D rectilinear waveguides.

The method used here is based on the results obtained for the rectilinear case. In fact, most of Chapter 4 consists on the study of the integral operator (denoted by $L_0^{-1}$) whose kernel is the Green's function obtained in Chapter 2. We shall prove the boundness of the operator $L_0^{-1}$ in certain weighted Sobolev spaces and then, by applying a standard fixed point argument, we are able to prove the existence of a solution.

The results obtained in Chapter 4 are applied to cases of physical interest in Chapter 5 by showing several numerical results. In particular, we will focus our attention on the so called *grating assisted* and *out-of-plane couplers*, where imperfections of the waveguide are used to couple a guided mode with other guided and radiating modes, respectively.

The thesis is completed by Chapter 1, where we collect some physical motivations to our work, and by two technical Appendices.

# Acknowledgements

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Most of the topics studied in this thesis have been motivated by discussions I had with Prof. Fadil Santosa. I am indebted to him for his numerous enlightening suggestions and the stimulating problems he has proposed to me. I wish to thank him also for his warm hospitality in Minneapolis.

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# Chapter 1

# Physical Framework

## 1.1 Waveguides

Optical waveguides are thin glass strands designed for light transmission and they fill a role of primary importance in optoelectronics and long distance communications. An optical waveguide is used to transfer light signals along a precise direction, in the sense that the electromagnetic field propagates in that direction and is (mostly) confined in a bounded region of the plane transversal to the direction of propagation. Such a phenomena can happen only by using materials with different index of refraction, as we are going to explain shortly.

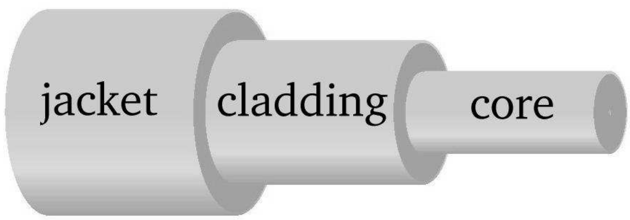


Figure 1.1: An optical fiber.

Optical waveguides can be classified according to their geometry (planar or rectangular waveguides, optical fibers), refractive index distribution (step or gradient index) and material (glass, polymer, semiconductor).

For example, an optical fiber is made of a central region called *core*, through which most of the light signal propagates. The core is surrounded by a *cladding* layer which has a slight lower index of refraction. A plastic coating, called *jacket*, covers the cladding to protect the glass surface.

Optical waveguides work thanks to the difference of the indexes of refraction of core and cladding. A first and simple explanation of what happens can be given by doing a ray analysis of the problem and using the Snell's law. Consider a ray propagating inside the core which hits the interface between core and cladding. It is easy to prove that if such a ray has an angle of incidence smaller than a critical one, then it is reflected back to the core (*total internal reflection*).

A more accurate description can be done when we pass to an electromagnetic analysis of the problem (which will be our point of view in this thesis), i.e. thinking of a light signal as a superposition of waves (*modes*) of different frequency.

Optical waveguides can have various transmission modes which exhibit different behaviours. Most of the electromagnetic radiation propagates without loss as a finite set of *guided modes* along the fiber axis. The electromagnetic field intensity of the guided modes in the cladding decays exponentially transversally to the waveguide's axis. Guided modes are the ones which carry the data we want to transmit. The remaining part of the traveling light is not trapped inside the waveguide and it is called *radiating* energy. The radiating energy can be seen as a superposition of *radiation* and *evanescent modes*, according to their different behaviours along the direction of propagation (see Section 2.4).

Communication using optical waveguides has lots of attractive features. Optical fibers have a very large bandwidth, usually from $10^{13}$ to $10^{16}$ Hz, which implies an information-carrying capacity far superior from the best copper cable systems. Wave propagation in optical waveguides exhibits a very low attenuation or transmission loss, which makes possible the transmission of data over long distances without using intermediate signal repeaters. Among other important features of optical fibers we cite their very small size and weight (the diameter is usually the same as the one of an human hair), their electrical isolation (which makes possible their use in an electrically noisy environment) and the signal security (thanks to the low radiating energy it is hard to recover information in a non-invasive way).

There is an vaste bibliography on optical waveguides. We will refer mostly to [Ma] and [SL]; other fundamental references are [Ya], [LL1], [Ha] and [Ta].

## 1.2 From Maxwell's equations to Helmholtz equation

The study of electromagnetic wave propagation is modelled by Maxwell's equations. In this section we describe how, starting from them, we can derive Helmholtz equation, which gives a good approximation of wave propagation in optical waveguides.

In a linear and isotropic media, the time-harmonic Maxwell equations can be written in the following form:

$$\nabla \times \mathbf{E} = -i\mu\omega\mathbf{H}, \quad \nabla \times \mathbf{H} = \mathbf{J} + i\omega\varepsilon\mathbf{E},$$
$$\nabla \cdot (\varepsilon\mathbf{E}) = \rho, \qquad \nabla \cdot (\mu\mathbf{H}) = 0.$$

Here, $\mathbf{E}$ is the *electric field*, $\mathbf{H}$ is the *magnetic field*, $\rho$ is the *charge density*, $\mathbf{J}$ is the *current density*, $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, $\nabla \times \mathbf{v} = \mathrm{curl}(\mathbf{v})$, $\nabla \cdot \mathbf{v} = \mathrm{div}(\mathbf{v})$, $\varepsilon$ is the dielectric permittivity, $\mu$ is the magnetic permeability and $\omega$ is the angular frequency. We will assume that the media are non-magnetic, thus $\mu$ equals the magnetic permeability of vacuum, i.e. $\mu = \mu_0$.

We will rewrite these equations by using a terminology more familiar in optics. We set $n = \sqrt{\mu_0\varepsilon}$, $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ and $k = \omega\sqrt{\mu_0\varepsilon_0}$, which are, respectively, the position-dependent *index of refraction*, the *free space impedance* and the *wavenumber*; Maxwell's equations become

$$\nabla \times \mathbf{E} = -i\eta_0 k\mathbf{H}, \qquad \nabla \times \mathbf{H} = \mathbf{J} + i\eta_0^{-1}kn^2\mathbf{E}, \tag{1.1a}$$
$$\nabla \cdot (n^2\mathbf{E}) = \rho/\varepsilon_0, \qquad \nabla \cdot \mathbf{H} = 0. \tag{1.1b}$$

By taking the curl of the first equation in (1.1a), we obtain

$$\nabla \times (\nabla \times \mathbf{E}) = -i\eta_0 k\nabla \times \mathbf{H};$$

thus, by using the second equation in (1.1a),

$$\nabla \times (\nabla \times \mathbf{E}) - k^2n^2\mathbf{E} + i\eta_0 k\mathbf{J} = 0. \tag{1.2}$$

From the first equation in (1.1b) and by applying the vector identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \Delta\mathbf{E},$$

(here $\Delta = \nabla \cdot \nabla =$ laplacian), (1.2) can be written as

$$\Delta\mathbf{E} + k^2n^2\mathbf{E} = -\nabla\left[\mathbf{E} \cdot \nabla \ln n^2\right] + \mathbf{F}, \tag{1.3}$$

where

$$\mathbf{F} = \nabla\left(\frac{\rho}{\varepsilon_0 n^2}\right) + i\eta_0 k\mathbf{J}.$$

We notice that $\mathbf{F}$ has to do only with the charges and currents present, so it will be the *source* which generates the fields $\mathbf{E}$ and $\mathbf{H}$.

If the medium is homogeneous (i.e. $n$ constant), then the first term on the right-hand side of (1.3) disappears, and we arrive at the familiar time-harmonic vector wave equation with a source,

$$\Delta\mathbf{E} + k^2n^2\mathbf{E} = \mathbf{F}. \tag{1.4}$$

This equation has the advantage that, written in Cartesian coordinates, it allows for the *decoupling* of the components of the electric field, that is, each of the components of the vector $\mathbf{E}$ will satisfy the scalar wave equation.

Equation (1.4) still holds, but only approximately, if $n$ varies in space, provided its variation is very slow along the distance of one light wavelength. A justification of this can be found in [Ma], Section 1.3. Since the change in the index of refraction between the core and the cladding of a typical optical fiber is very small, it turns out that we can still apply (1.4) in this case. This enables us to use the so called *weakly guiding approximation*, as we are going to show in the following.

We will describe the weakly guiding approximation following [SL], Chapters 30 and 32. Consider a Cartesian coordinate system $(x, y, z)$, so that the $z-$axis is the axis of symmetry of the fiber. Thus, the index of refraction will not depend on $z$, i.e $n = n(x, y)$. We denote by $\mathbf{E}_t$ the *transverse* component of $\mathbf{E}$ (i.e. the component perpendicular to the $z-$axis) and by $\nabla_t$ the *transverse gradient* $\nabla_t = (\partial_x, \partial_y, 0)$. By using these notations, (1.3) becomes

$$\Delta \mathbf{E} + k^2 n^2 \mathbf{E} = -\nabla \left[ \mathbf{E}_t \cdot \nabla_t \ln n^2 \right] + \mathbf{F}. \tag{1.5}$$

Let $n_*$ be the maximum of the index of refraction of the fiber, $n_{cl}$ be the index of refraction of the cladding and set

$$\delta = \frac{1}{2}\left(1 - \frac{n_{cl}^2}{n_*^2}\right).$$

An optical fiber is called *weakly guiding* if $n_*$ does not differ much from $n_{cl}$, or equivalently, $\delta \ll 1$. Let $\varphi(x, y)$ be a function such that $0 \le \varphi(x, y) \le 1$, $\varphi(x, y) = 1$ in the cladding and

$$n^2(x, y) = n_*^2\{1 - 2\Delta\varphi(x, y)\}.$$

Since

$$\nabla_t \ln n^2 = \nabla_t \ln(1 - 2\Delta\varphi) = 2\Delta\nabla_t\varphi + 2\Delta^2\nabla_t\varphi^2 + \dots,$$

then, up to zeroth order approximation, $\nabla_t \ln n^2 = 0$, and equation (1.5) becomes equation (1.4).

In Section 32-2 of [SL], it is also shown that the $z-$component of $\mathbf{E}$ is of order $O(\sqrt{\delta})$. Thus, the electric field of weakly guiding fibers is essentially transverse, and its $x$ and $y$ components satisfy approximately the Helmholtz equation

$$\Delta u + k^2 n(x)^2 u = f, \quad x \in \mathbb{R}^3.$$

This equation will be our model for the electromagnetic field propagation in an optical fiber.

In this thesis we will study mostly the case of a 2-D optical waveguide, i.e. when the index of refraction $n$ depends only on the $x$ and $z$-coordinates. In such a case, we can assume that the electric and magnetic fields do not vary along the $y$-axis and thus we obtain

$$\Delta u + k^2 n(x,z)^2 u = f, \quad x \in \mathbb{R}^2.$$

## 1.3 Mode coupling

It is well known that when a single guided mode is excited in a rectilinear optical waveguide, it propagates undisturbed along the guide's axis and no other mode appears. This fact happens only if the dielectric guide is free of imperfections and its diameter remain constant throughout its length. Otherwise, the fiber can not support an individual mode and the propagation constant becomes complex. However, for small imperfections, the propagation constant can be considered the same as the case of a "perfect" rectilinear waveguide.

Real-life waveguides are never perfect, since they might contain imperfections due to inhomogeneities or changes in the core's width and shape. When a pure guided mode is excited inside a guide with imperfections, a sort of resonation takes place and the other allowed modes are excited. This effect causes a signal distortion, since every guided mode propagate at its own characteristic velocity, and a loss in the signal power, due to the transfer of power to radiation, evanescent and the other guided modes. This phenomenon is called *mode coupling*.

Mode coupling is not always an effect to avoid. For example, imperfections can be used to design dielectric antennas and in many other applications, see [Ma], [SL], [Erd] and [HDL]. At the same time, it has important applications in design of multimode fibers, see [Ma] and [Pk].

Coupled mode theory is particularly useful when a slight perturbation has a large effect on the distribution of modal power, i.e. when a resonance condition causes a large exchange of power between the modes of the unperturbed fiber.

As already mentioned, the electromagnetic field of the perturbed fiber at position $z$ can be described by a superposition of guided and radiating modes of the unperturbed fiber. Clearly, an individual mode of the set does not satisfy Maxwell (or Helmholtz) equation for the perturbed fiber, and hence the perturbed field must generally be distributed between all modes of the set. This distribution varies with position along the fiber and is described by a set of *coupled mode equations*, which determine the amplitude of every mode.

## 1.4 Coupling theory

There are several ways of describing mode coupling. In the following, we give the main ideas for describing the coupling between guided modes, following the theory developed in [SL].

We assume that the guided part of the field can be represented as

$$u^g(x,z) = \sum_{j\in\{s,a\}} \sum_{m=1}^{M_j} \left[b^+_{j,m}(z) + b^-_{j,m}(z)\right] v_j(x),$$

where $z$ is the direction of propagation, $x$ is the coordinate transversal to the fiber's axis and $v_j$ satisfies

$$v'' + (\lambda - q)v = 0,$$

$b^+_{j,m}$ and $b^-_{j,m}$ contain the amplitude and phase-dependence of the forward and backward propagating guided modes and $j = s, a$ denotes, respectively, the symmetric and antisymmetric modes (in Chapter 2 we will derive a rigorous framework which allows us to represent the modes in such a way).

As shown in [SL], this representation leads to the following set of coupled mode equations:

$$\frac{db^+_{j,m}}{dz} - i\beta^j_m b^+_{j,m} = i \sum_{j\in\{s,a\}} \sum_{l=1}^{M_j} D^j_{ml}[b^+_{j,l} + b^-_{j,l}],$$

for each forward-propagating mode, and

$$\frac{db^-_{j,m}}{dz} + i\beta^j_m b^-_{j,m} = -i \sum_{j\in\{s,a\}} \sum_{l=1}^{M_j} D^j_{ml}[b^+_{j,l} + b^-_{j,l}],$$

for the backing-propagating ones, where

$$D^j_{ml}(z) = \frac{k}{2n_{co}\|v(\cdot,\lambda^j_m)\|_2^2} \int_{-\infty}^{+\infty} (n^2 - n_*^2) v_j(x,\lambda^j_m) v_j(x,\lambda^j_l)\, dx. \tag{1.6}$$

When only a small fraction of the total power of the perturbed fiber is transferred between modes, the coupled mode equations can be solved iteratively (see [SL]).

Under certain resonance conditions, a small perturbation of the waveguide leads to a considerable exchange of power between different modes, usually referred as *strong power transfer*. If we suppose that the strong power transfer happens mostly between only two guided modes, then it is possible to formulate a simplest version of the above coupling equations, because the coupling involving the other modes can be neglected.

In order to derive the simplified equations, we further suppose that only coupling between two forward propagating modes happens and we assume that they have propagation constants $\beta_1$ and $\beta_2$ and modal amplitudes $b_1(z)$ and $b_2(z)$, respectively. Thus, the coupled mode equations reduce to

$$\frac{db_1}{dz} - i(\beta_1 + D_{11})b_1 = iD_{12}b_2,$$
$$\frac{db_2}{dz} - i(\beta_2 + D_{22})b_2 = iD_{21}b_1,$$

where the coupling coefficients $D_{ij}$ are defined by (1.6). Other analogous cases are described in [SL] and [Ma].

We will return on the above equations in Section 5.5, where we describe how to obtain the modal amplitudes $b_1$ and $b_2$ by using a different approach, which is a consequence of the results obtained in this thesis.

## 1.5 Grating couplers

Grating couplers are optical devices where, generally speaking, there is a perturbation of the index of refraction of the core (and/or of the cladding) in order to couple a guided mode with other guided modes or also radiating ones. Thus, they are usually studied by using the coupled theory described above.

An important kind of grating couplers are the *grating-assisted direction couplers* (see [Ma]), where two or more waveguides are close to each other and the coupling between modes is created by using a perturbation in the index of refraction of the core (and/or of the cladding). In this setting, coupling between guided modes is the most interesting phenomena to be studied. In Chapter 5 we will discuss this problem by showing some numerical examples.

Besides coupling between guided modes, perturbations in the index of refraction along the fiber lead also to a different behaviour of the radiating part of the electromagnetic field. As already mentioned, this can be used to excite a guided mode by using radiated power, as we will explain shortly.

In this case the grating is outside the waveguide, i.e. only the index of refraction of the cladding is perturbed and the coupling between guided and radiating modes has to be studied. This kind of couplers are used to excite a guided mode supported by the fiber together with an incident beam coming from the region outside the fiber, see [Ma], [SL], [Hu], [UFSN], [HLL] and [WJN]. We will give some numerical results in Section 5.4.

# Chapter 2

# Rectilinear Waveguides

## 2.1 Introduction

As shown in Section 1.2, electromagnetic wave propagation in 2-D rectilinear optical waveguides can be described by the Helmholtz equation

$$\Delta u + k^2 n(x)^2 u = f, \quad \text{in } \mathbb{R}^2. \tag{2.1}$$

In this chapter we will study (2.1) in the case that $n$ is of the form

$$n(x) = \begin{cases} n_b, & \text{if } x \geq h, \\ n_{co}(x), & \text{if } -h \leq x \leq h, \\ n_a, & \text{if } x \leq -h. \end{cases} \tag{2.2}$$

If not stated otherwise, the function $n(x)$ will be supposed to be positive and to belong to $L^\infty(\mathbb{R})$. An example of such a function is represented in Fig. 2.1.

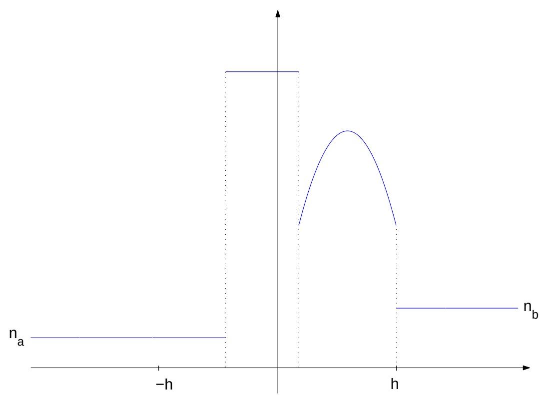


Figure 2.1: A possible choice of $n(x)$.

A particular and relevant case of (2.2) was studied in [MS]. In [MS] the authors found a resolution formula for (2.1), with

$$n := n_0(x) = \begin{cases} n_{cl}, & \text{if } |x| \geq h, \\ n_{co}(x), & \text{if } |x| < h, \end{cases} \tag{2.3}$$

where $n_{co}(x)$ is supposed to be an even function, decreasing for $x \geq 0$.

Both in (2.2) and (2.3), the cladding is assumed to be extended to infinity and to have constant index of refraction ($n_a$ and $n_b$ in one case, $n_{cl}$ in the other one).

In Sections 2.3 and 2.4 we will generalize the results of [MS]. As in [MS], we will separate the variables and study an associated eigenvalue problem; however, the resolution formula will be obtained by using a different technique. In our work, the theory of Titchmarsh on eigenvalue problems (see [CL] and [Ti]) will be of fundamental importance. For this reason, we will recall the principal results of this theory in Section 2.2.

In Section 2.3 we will apply Titchmarsh theory to our problem, in the case in which $n$ is given by (2.2). As a special case, in Section 2.4 we derive the resolution formula for (2.1) in the case $n$ of the form (2.3), already obtained in [MS]. When $n \equiv 1$, (2.1) becomes the well-known free-space Helmholtz equation and, as proven in Section 2.5, our resolution formula coincides with the usual one.

In Section 2.6 we will prove some asymptotic lemmas that are crucial for the continuation of this thesis.

In Section 2.7 we will study the far-field of the solution obtained in Section 2.4.

The chapter ends with Section 2.8, where we analyze the behaviour of the singularity of the Green's function derived in Section 2.4.

## 2.2 The Titchmarsh theory of eigenvalue problems

In this section we recall the main results of Titchmarsh theory on eigenfunction expansions, which will be useful for the construction of a resolution formula for (2.1). In particular, we will adopt the same notations as in [CL] and recall Theorems 5.1 and 5.2 from [CL].

Let $B$ be the operator

$$Bv := -(pv')' + qv$$

defined in $-\infty < x < \infty$. Here, $p$ and $q$ are supposed to be an absolutely continuous and a locally integrable function, respectively. We say that $B$ is in the *limit circle* case at $+\infty$ if, for some complex number $l_0$, every solution of

$$Bv = l_0 v$$

belongs to $L^2(0,\infty)$. Otherwise $B$ is said to be in the *limit point* case at $+\infty$. As shown in Theorem 2.1, Chapter 9 in [CL], the above classification depends only on the operator $B$ and not on the particular $l_0$ chosen, in the sense that if every solution of $Bv = l_0 v$ belongs to $L^2(0,\infty)$, then the same holds for the solutions of $Bv = lv$, for every complex number $l$.
Analogously we can define the limit point and circle cases at $-\infty$.

Let $\eta_1$ and $\eta_2$ be solutions of $Bv = lv$ satisfying the initial conditions

$$\begin{aligned} \eta_1(0,l) &= 1, & \eta_2(0,l) &= 0, \\ p(0)\eta_1'(0,l) &= 0, & p(0)\eta_2'(0,l) &= 1, \end{aligned} \tag{2.4}$$

respectively. We define the limit point at $+\infty$ as the quantity $m_{+\infty}(l)$ such that

$$\eta_1(x,l) + m_{+\infty}(l)\eta_2(x,l) \in L^2(0,+\infty),$$

for $l \in \mathbb{C}$, with $\operatorname{Im} l > 0$. Similarly, the limit point at $-\infty$ is $m_{-\infty}(l)$ and it is such that

$$\eta_1(x,l) + m_{-\infty}(l)\eta_2(x,l) \in L^2(-\infty,0),$$

for every $l \in \mathbb{C}$, with $\operatorname{Im} l > 0$.

Let $I = [a,b]$ be any bounded interval containing zero, and consider the equation $Bv = lv$ on $I$, together with the boundary conditions

$$\begin{aligned} v(a)\cos\alpha + p(a)v'(a)\sin\alpha = 0, \\ v(b)\cos\beta + p(b)v'(b)\sin\beta = 0, \end{aligned}$$

where $0 \leq \alpha, \beta < \pi$.

Since we are considering a finite interval, standard Sturm-Liouville theory allows us to write a Parseval identity in terms of a complete orthonormal set of eigenfunctions $\{h_n^I\}_{n\in\mathbb{N}}$ corresponding to countably many real eigenvalues $\{\lambda_n^I\}_{n\in\mathbb{N}}$:

$$\int_I |f(x)|^2 dx = \sum_{n=1}^{+\infty} \Big| \int_I f(x)\, \overline{h_n^I(x)}\, dx \Big|^2, \quad f \in L^2(I). \tag{2.5}$$

Since $\eta_1(x,l)$ and $\eta_2(x,l)$ are a basis for the solutions of $Bv = lv$, we can write

$$h_n^I(x) = r_{n,1}^I \eta_1(x,\lambda_n^I) + r_{n,2}^I \eta_2(x,\lambda_n^I),$$

with $r_{n,1}^I, r_{n,2}^I \in \mathbb{C}$. Therefore, by pluggin this formula into the Parseval identity (2.5), we have

$$\int_I |f(x)|^2 dx = \sum_{j,k=1}^{2} \int_{-\infty}^{+\infty} \overline{g_j^I(\lambda)} g_k^I(\lambda) d\rho_{j,k}^I(\lambda).$$

with

$$g_j^I(\lambda) = \int_I f(x)\eta_j(x,\lambda)dx.$$

Notice that each element of the *spectral matrix* $\rho^I = (\rho^I_{jk})$ turns out to be a piecewise constant function with jumps at the points $\lambda = \lambda^I_n$ given by

$$\rho^I_{jk}(\lambda^I_n + 0) - \rho^I_{jk}(\lambda^I_n - 0) = r^I_{n,j}\,\overline{r^I_{n,k}}.$$

We notice that the spectral matrix $\rho^I$ is Hermitian, non-decreasing (i.e. $\rho^I(\lambda) - \rho^I(\mu)$ is positive semidefinite if $\lambda > \mu$) and that the total variation of $\rho^I_{jk}$ is finite over every bounded interval (see [CL] for further details).

We are interested in what happens when the interval $I$ becomes the whole real line. Fundamental results on this topic are stated in the next theorem (see [CL] for a proof).

**Theorem 2.1.** *Let $B$ be in the limit-point case at $-\infty$ and $\infty$. It exists a nondecreasing Hermitian matrix $\rho = (\rho_{jk})$ whose elements are of bounded variation on every bounded $\lambda$ interval, and which is essentially unique in the sense that*

$$\rho^I_{jk}(\lambda) - \rho^I_{jk}(\mu) \to \rho_{jk}(\lambda) - \rho_{jk}(\mu) \quad \text{as } I \to (-\infty, \infty)$$

*at points of continuity $\lambda,\ \mu$ of $\rho_{jk}$. Furthermore*

$$\rho_{jk}(\lambda) - \rho_{jk}(\mu) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_\mu^\lambda \operatorname{Im} \mathcal{M}_{jk}(\nu + i\varepsilon)\, d\nu, \tag{2.6}$$

*where*

$$\begin{aligned} &\mathcal{M}_{11}(l) = [m_{-\infty}(l) - m_\infty(l)]^{-1}, \\ &\mathcal{M}_{12}(l) = \mathcal{M}_{21}(l) = \frac{1}{2}[m_{-\infty}(l) + m_\infty(l)][m_{-\infty}(l) - m_\infty(l)]^{-1}, \\ &\mathcal{M}_{22}(l) = m_{-\infty}(l) m_\infty(l)[m_{-\infty}(l) - m_\infty(l)]^{-1}. \end{aligned} \tag{2.7}$$

*Moreover, the following Parseval identity and inversion transform formula hold for any $f \in L^2(\mathbb{R})$:*

$$\int_{-\infty}^{\infty} |f(x)|^2 dt = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \overline{g_j(\lambda)} g_k(\lambda) d\rho_{jk}(\lambda),$$

$$f(x) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \overline{\eta_j(x,\lambda)} g_k(\lambda) d\rho_{jk}(\lambda),$$

*where*

$$g_j(\lambda) = \int_{-\infty}^{\infty} f(x) \eta_j(x,\lambda) dx,$$

*with $j = 1, 2$.*

## 2.3 Green's function: the general case

In this section we study the Helmholtz equation (2.1) in the case in which $n$ is given by (2.2). Thanks to the symmetry of the problem, we look for solutions of the homogeneous equation associated to (2.1) in the form

$$u(x,z) = v(x,\beta)e^{ik\beta z};$$

$v(x,\beta)$ satisfies the associated eigenvalue problem for $v$:

$$v'' + [\lambda - q(x)]v = 0, \quad \text{in } \mathbb{R}, \tag{2.8}$$

where

$$n_* = \max_{\mathbb{R}} n, \quad \lambda = k^2(n_*^2 - \beta^2), \quad q(x) = k^2[n_*^2 - n(x)^2]. \tag{2.9}$$

For convenience we set

$$q(x) = \begin{cases} q_b, & \text{if } x > h, \\ q_{co}(x), & \text{if } -h \leq x \leq h, \\ q_a, & \text{if } x < -h, \end{cases} \tag{2.10}$$

where the definition of $q_a, q_b > q_{co}(x)$ is evident from (2.9).

As shown in Section 2.2, we need to find two solutions of (2.8) which satisfy (2.4). Following the mathematical framework used in [MS], we prefer to impose different initial conditions on the solutions of (2.8); in particular we look for two solutions $v_s(x,\lambda)$ and $v_a(x,\lambda)$ such that

$$\begin{array}{ll} v_s(0,\lambda) = 1 & v_a(0,\lambda) = 0, \\ v_s'(0,\lambda) = 0 & v_a'(0,\lambda) = \sqrt{\lambda}, \end{array} \tag{2.11}$$

and find

$$v_j(x,\lambda) = \begin{cases} \phi_j(h,\lambda)\cos Q_b(x-h) + \frac{\phi_j'(h,\lambda)}{Q_b}\sin Q_b(x-h), & \text{if } x \geq h, \\ \phi_j(x,\lambda), & \text{if } -h \leq x \leq h, \\ \phi_j(-h,\lambda)\cos Q_a(x+h) + \frac{\phi_j'(-h,\lambda)}{Q_a}\sin Q_a(x+h), & \text{if } x \leq -h, \end{cases} \tag{2.12}$$

for $j = s, a$, with

$$Q_b = \sqrt{\lambda - q_b}, \quad Q_a = \sqrt{\lambda - q_a}. \tag{2.13}$$

Now, we find the quantities we need in order to apply Titchmarsh theory. It is easy to verify (see [CL], Corollary 1 pag. 231) that, under our assumptions on $n$, (2.8) is in the limit-point case at $+\infty$ and $-\infty$.

Our first step will be to find the two limit-points $m_{+\infty}(\lambda)$ and $m_{-\infty}(\lambda)$. We study (2.8) separately in the two intervals $(-\infty, 0]$ and $[0, +\infty)$. Consider $[0, +\infty)$ first and suppose $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda > 0$. If $x \in (h, +\infty)$, the function $\exp\left[iQ_b(x-h)\right]$ belongs to $L^2(h, +\infty)$ and is a solution of (2.8). The limit point at infinity $m_{+\infty}(\lambda)$ must be such that $v_s(x, \lambda) + m_{+\infty}(\lambda) v_a(x, \lambda)$ is proportional to $\exp\left[iQ_b(x-h)\right]$, with $x \in (h, +\infty)$.

From (2.12) it follows that

$$m_{+\infty}(\lambda) = -\frac{iQ_b\phi_s(h,\lambda) - \phi_s'(h,\lambda)}{iQ_b\phi_a(h,\lambda) - \phi_a'(h,\lambda)}. \tag{2.14}$$

By analogous computations and by requiring that $v_s(x, \lambda) + m_{-\infty}(\lambda) v_a(x, \lambda)$ is proportional to $\exp\left[-iQ_a(x+h)\right]$ in $(-\infty, -h)$, we find

$$m_{-\infty}(\lambda) = -\frac{iQ_a\phi_s(-h,\lambda) + \phi_s'(-h,\lambda)}{iQ_a\phi_a(-h,\lambda) + \phi_a'(-h,\lambda)}. \tag{2.15}$$

In the next step, we find the quantities $\mathcal{M}_{11}$, $\mathcal{M}_{12}$ and $\mathcal{M}_{22}$, which are defined by (2.7). We use the following notations:

$$\begin{aligned} \alpha(\lambda) &= -iQ_b\phi_s(h,\lambda) + \phi_s'(h,\lambda), & \beta(\lambda) &= iQ_b\phi_a(h,\lambda) - \phi_a'(h,\lambda), \\ \gamma(\lambda) &= -iQ_a\phi_s(-h,\lambda) - \phi_s'(-h,\lambda), & \delta(\lambda) &= iQ_a\phi_a(-h,\lambda) + \phi_a'(-h,\lambda), \end{aligned} \tag{2.16}$$

and we have

$$m_{+\infty}(\lambda) = \frac{\alpha(\lambda)}{\beta(\lambda)}, \quad m_{-\infty}(\lambda) = \frac{\gamma(\lambda)}{\delta(\lambda)}, \tag{2.17}$$

and

$$\begin{aligned} \mathcal{M}_{11}(\lambda) &= \frac{\beta(\lambda)\delta(\lambda)}{\gamma(\lambda)\beta(\lambda) - \alpha(\lambda)\delta(\lambda)}, \\ 2\mathcal{M}_{12}(\lambda) &= \frac{\gamma(\lambda)\beta(\lambda) - \alpha(\lambda)\delta(\lambda)}{\gamma(\lambda)\beta(\lambda) - \alpha(\lambda)\delta(\lambda)}, \\ \mathcal{M}_{22}(\lambda) &= \frac{\alpha(\lambda)\gamma(\lambda)}{\gamma(\lambda)\beta(\lambda) - \alpha(\lambda)\delta(\lambda)}. \end{aligned} \tag{2.18}$$

Since the functions $v_j(x, \lambda)$ defined by (2.12) are real for $\lambda$ real, from Theorem 2.1 we have the following inversion formula

$$f(x) = \int_{-\infty}^{+\infty} \sum_{j,j' \in \{s,a\}} v_j(x,\lambda) F^{j'}(\lambda) d\rho_{jj'}(\lambda), \tag{2.19}$$

for any $f \in L^2(\mathbb{R})$, with

$$F^j(\lambda) = \int_{-\infty}^{+\infty} f(x) v_j(x,\lambda) dx, \tag{2.20}$$

and where we set $\rho_{ss}(\lambda) := \rho_{11}(\lambda)$, $\rho_{sa}(\lambda) := \rho_{12}(\lambda)$ and $\rho_{aa}(\lambda) := \rho_{22}(\lambda)$.

In the last part of this section we follow the proof of Theorem 3.1 in [MS] to derive the expression of a Green's function for (2.1).

By multiplying (2.1) by $v_j(x,\lambda)$, integrating in $x$ over $\mathbb{R}$ and after two integrations by parts we find

$$U^j_{zz}(z,\lambda) + (k^2n_*^2 - \lambda)U^j(z,\lambda) = F^j(z,\lambda), \quad j \in \{s,a\},$$

where $U^j$ and $F^j$ are, respectively, the coefficients of the Titchmarsh expansion of $u$ and $f$, given by the formula (2.20).

The solution of the above equation, which is outgoing for $\lambda < k^2n_*^2$ or decays for $\lambda > k^2n_*^2$ as $|z| \to +\infty$ is given by

$$U^j(z,\lambda) = \int_{-\infty}^{+\infty} \frac{e^{i|z-\zeta|\sqrt{k^2n_*^2-\lambda}}}{2i\sqrt{k^2n_*^2-\lambda}} F^j(\lambda,\zeta)d\zeta, \quad j \in \{s,a\}.$$

Thus, by using (2.19) for $f$ and $u$, we obtain

$$u(x,z) = \int_{\mathbb{R}^2} G(x,z;\xi,\zeta)f(\xi,\zeta)d\xi d\zeta,$$

where

$$G(x,z;\xi,\zeta) = \sum_{j,j' \in \{s,a\}} \int_{-\infty}^{+\infty} \frac{e^{i|z-\zeta|\sqrt{k^2n_*^2-\lambda}}}{2i\sqrt{k^2n_*^2-\lambda}} v_j(x,\lambda)v_{j'}(x,\lambda)d\rho_{jj'}(\lambda).$$

## 2.4 Green's function: the case $q$ axially symmetric

We will develop in detail the case $q(x) = q(-x)$. This case was studied in [MS], together with the assumption $q(x)$ increasing for $x \geq 0$. We will drop this condition and obtain analogous results by applying Titchmarsh theory described in Section 2.2.

We recall that

$$q(x) = k^2[n_*^2 - n(x)^2],$$

where now $n$ and $n_*$ will be given by (2.3) and (2.9). As in [MS], we require that $n_* > n_{cl}$, otherwise no guided modes are allowed, but we don't have to assume that $n$ is decreasing for $x > 0$. By using the same notations as in [MS], we put

$$d^2 = k^2(n_*^2 - n_{cl}^2), \quad Q = \sqrt{\lambda - d^2}. \tag{2.21}$$

Now we can write (2.12), (2.14), (2.15) and (2.18) in this case. We notice that here $Q = Q_a = Q_b$ and that $\phi_s$ and $\phi_a$ are even and odd functions respectively. Hence,

$$\begin{aligned} \phi_s(-h,\lambda) &= \phi_s(h,\lambda), \quad \phi_s'(-h,\lambda) = -\phi_s'(h,\lambda), \\ \phi_a(-h,\lambda) &= -\phi_a(h,\lambda), \quad \phi_a'(-h,\lambda) = \phi_a'(h,\lambda), \end{aligned}$$

and then

$$
v_j(x,\lambda)=\begin{cases}\phi_j(h,\lambda)\cos Q(x-h)+\frac{\phi_j'(h,\lambda)}{Q}\sin Q(x-h), & \text{if } x>h,\\ \phi_j(x,\lambda), & \text{if } |x|\le h,\\ \phi_j(-h,\lambda)\cos Q(x+h)+\frac{\phi_j'(-h,\lambda)}{Q}\sin Q(x+h), & \text{if } x<-h,\end{cases} \tag{2.22}
$$

for $j=s,a$. Furthermore,

$$
m_{-\infty}(\lambda)=-m_{+\infty}(\lambda)=\frac{iQ\phi_s(h,\lambda)-\phi_s'(h,\lambda)}{iQ\phi_a(h,\lambda)-\phi_a'(h,\lambda)}, \tag{2.23}
$$

and

$$
\mathcal{M}_1(\lambda):=\mathcal{M}_{11}(\lambda)=\frac{1}{2}\cdot\frac{iQ\phi_a(h,\lambda)-\phi_a'(h,\lambda)}{iQ\phi_s(h,\lambda)-\phi_s'(h,\lambda)},
$$

$$
\mathcal{M}_{12}(\lambda)=0, \tag{2.24}
$$

$$
\mathcal{M}_2(\lambda):=\mathcal{M}_{22}(\lambda)=-\frac{1}{2}\cdot\frac{iQ\phi_s(h,\lambda)-\phi_s'(h,\lambda)}{iQ\phi_a(h,\lambda)-\phi_a'(h,\lambda)}.
$$

**Remark 2.2.** We observe that, if $n(x)\equiv 0$, we obtain the same results as in [CL], Example 1, pag. 252-253.

**Lemma 2.3.** *Let $v_j(x,\lambda_m^j)$ and $v_l(x,\mu)$, $j,l\in\{s,a\}$, be solutions of* (2.8) *given by* (2.22)*, with $\mu>0$ and where $\lambda_j^m\in(0,d^2)$ is a root of*

$$
\phi_j'(h,\lambda)+\sqrt{d^2-\lambda}\,\phi_j(h,\lambda)=0. \tag{2.25}
$$

*Then,*

$$
\int_{-\infty}^{\infty} v_j(x,\lambda_m^j)v_l(x,\lambda)dx=0. \tag{2.26}
$$

*Proof.* Consider the equations

$$
\begin{aligned}
v''(x,\lambda)+[\lambda-q(x)]v(x,\lambda)&=0,\\
v''(x,\mu)+[\mu-q(x)]v(x,\mu)&=0,
\end{aligned}
$$

for $\lambda\neq\mu$. We multiply them by $v(x,\mu)$ and $v(x,\lambda)$, respectively, and integrate in $x$ over the interval $(a,b)$. An integration by parts and a subtraction gives:

$$
\int_a^b v(x,\lambda)v(x,\mu)dx=\frac{1}{\lambda-\mu}\left[v(x,\lambda)v'(x,\mu)-v'(x,\lambda)v(x,\mu)\right]_a^b.
$$

The conclusion follows by observing that, if $\lambda=\lambda_m^j$, $v_j(x,\lambda_m^j)$ and its first derivative vanish exponentially as $|x|\to+\infty$, while $v_l(x,\mu)$ and $v_l'(x,\mu)$ grow at most polynomially as $|x|\to+\infty$. □

**Theorem 2.4.** *Let* $f \in L^2(\mathbb{R})$ *and let* $v_j(x,\lambda),\ j=s,a$, *be defined by* (2.22) *and*

$$F_j(\lambda) = \int_{-\infty}^{\infty} f(x) v_j(x,\lambda)dx, \quad j \in \{s,a\}. \tag{2.27}$$

*Then, the inversion formula,*

$$f(x) = \sum_{j\in\{s,a\}} \int_0^{+\infty} F_j(\lambda) v_j(x,\lambda) d\rho_j(\lambda), \tag{2.28}$$

*and the Parseval identity,*

$$\int_{-\infty}^{\infty} f(x)^2 dx = \sum_{j\in\{s,a\}} \int_0^{+\infty} F_j(\lambda)^2 d\rho_j(\lambda), \tag{2.29}$$

*hold, with*

$$\langle d\rho_j, \eta\rangle = \sum_{m=1}^{M_j} r_j^m \eta(\lambda_j^m) + \frac{1}{2\pi}\int_{d^2}^{\infty} \frac{\sqrt{\lambda - d^2}}{(\lambda - d^2)\phi_j(h,\lambda)^2 + \phi_j'(h,\lambda)^2}\eta(\lambda)d\lambda, \tag{2.30}$$

*for all* $\eta \in C_0^\infty(\mathbb{R})$, *where*

$$r_j^m = \left[\int_{-\infty}^{\infty} v_j(x,\lambda_j^m)^2 dx\right]^{-1}$$

$$= \frac{\sqrt{d^2 - \lambda_j^m}}{\sqrt{d^2 - \lambda_j^m}\int_{-h}^{h} \phi_j(x,\lambda_j^m)^2 dx + \phi_j(h,\lambda_j^m)^2}. \tag{2.31}$$

*Proof.* We will give only a sketch of the proof, because similar arguments were used in [AC1].

Suppose $\lambda \in (0, d^2]$. In this interval $\mathcal{M}_1(\lambda)$ and $\mathcal{M}_2(\lambda)$ are real-valued functions of $\lambda$. We note that they may be also infinite, because their denominators can vanish. In particular, this happens for the values of $\lambda$ verifying (2.25), with $j = s$ or $j = a$. As shown in [MS], these eigenvalues are finite in number. As a consequence, $\rho_j(\lambda)$ has a jump in $\lambda_j^m$, which will be denoted by $r_j^k$.

In order to find $\rho_j$ for $\lambda > d^2$ we need to find $\operatorname{Im}\mathcal{M}_1$ and $\operatorname{Im}\mathcal{M}_2$. From (2.24) and from

$$\operatorname{Im}\left(\frac{iA - B}{iC - D}\right) = \frac{BC - AD}{C^2 + D^2},$$

it follows that

$$\operatorname{Im}\mathcal{M}_j(\lambda) = \frac{1}{2}\cdot\frac{\sqrt{\lambda - d^2}}{(\lambda - d^2)\phi_j(h,\lambda)^2 + \phi_j'(h,\lambda)^2}.$$

With the above formulas we have proven that (2.30) holds and then, from Theorem 2.1, (2.28) and (2.29) follow.

Now, we compute the jumps $r_j^m$ by using the Parseval identity (2.29). From Lemma 2.3 it follows that $v_j(x, \lambda_m^j)$ is orthogonal to every other solution of (2.8). Since $v_j(x, \lambda_m^j)$ belongs to $L^2(\mathbb{R})$, we can write the Parseval identity for it and obtain the first line of (2.31).

We notice that, for $\lambda = \lambda_j^m$, $v(x, \lambda_m^j) = \phi_j(h, \lambda_j^m) e^{-\sqrt{d^2 - \lambda_m^j}(x-h)}$; thus

$$\int_h^{+\infty} v_j(x, \lambda_m^j)^2 dx = \frac{\phi(h, \lambda_m^j)^2}{Q},$$

and then we obtain the second line of (2.31) from the first one. □

**Remark 2.5.** *Classification of the solutions*

The eigenvalue problem (2.8) leads to three different types of solutions of (2.1) of the form $u_\beta(x, z) = v(x, \lambda) e^{ik\beta z}$, with $\lambda = k^2(n_*^2 - \beta^2)$.

- *Guided modes*: $0 < \lambda < d^2$. It exists a finite number of eigenvalues $\lambda_m^j$, $m = 1, \ldots, M_j$, $j \in \{s, a\}$, which satisfies (2.25). In this case, the solutions $v_j(x, \lambda_m^j)$ decay exponentially for $|x| > h$:

$$v_j(x, \lambda_m^j) = \begin{cases} \phi_j(h, \lambda_m^j) e^{-\sqrt{d^2 - \lambda_m^j}(x-h)}, & x > h, \\ \phi_j(x, \lambda_m^j), & |x| \leq h, \\ \phi_j(-h, \lambda_m^j) e^{\sqrt{d^2 - \lambda_m^j}(x+h)}, & x < -h. \end{cases}$$

  In the $z$ direction, $u_\beta$ is bounded and oscillatory, because $\beta$ is real.

- *Radiation modes*: $d^2 < \lambda < k^2 n_*^2$. In this case, $u_\beta$ is bounded and oscillatory both in the $x$ and $z$ directions.

- *Evanescent modes*: $\lambda > k^2 n_*^2$. The functions $v_j$ are bounded and oscillatory. In this case $\beta$ becomes imaginary and hence $u_\beta$ decays exponentially in one direction along the $z$-axis and increase exponentially in the other one.

By using Theorem 2.4, we can write a resolution formula of (2.1) as the superposition of guided, radiation and evanescent modes. Next theorem has been proven in [MS].

**Theorem 2.6.** *Let $f \in C_0(\mathbb{R}^2)$ and, for $(x, z), (\xi, \zeta) \in \mathbb{R}^2$, let*

$$G(x, z; \xi, \zeta) = \sum_{j \in \{s,a\}} \int_0^{+\infty} \frac{e^{i|z-\zeta|\sqrt{k^2 n_*^2 - \lambda}}}{2i\sqrt{k^2 n_*^2 - \lambda}} v_j(x, \lambda) v_j(\xi, \lambda) d\rho_j(\lambda), \tag{2.32}$$

*where* $v_j(x,\lambda)$ *and* $d\rho_j(\lambda)$ *are given by* (2.22) *and* (2.30), *respectively.*

*Then the function*

$$u(x,z) = \int_{\mathbb{R}^2} G(x,z;\xi,\zeta) f(\xi,\zeta) d\xi d\zeta, \quad (x,z) \in \mathbb{R}^2, \tag{2.33}$$

*is of class* $C^1(\mathbb{R}^2)$ *and satisfies* (2.1) *in the sense of distributions.*

Unlike the analogous result obtained in the previous section for a generic index of refraction, we are able to perform an accurate study of (2.33). In fact, we note that (2.33) can be split up into three components

$$u = u^g + u^r + u^e,$$

where

$$u^l = \int_{\mathbb{R}^2} G^l f, \quad l = g, r, e,$$

and

$$G^g(x,z;\xi,\zeta) = \sum_{j\in\{s,a\}} \sum_{m=1}^{M_j} \frac{e^{i|z-\zeta|\sqrt{k^2 n_*^2 - \lambda_m^j}}}{2i\sqrt{k^2 n_*^2 - \lambda_m^j}} v_j(x,\lambda_m^j) v_j(\xi,\lambda_m^j) r_m^j, \tag{2.34a}$$

$$G^r(x,z;\xi,\zeta) = \frac{1}{2\pi} \sum_{j\in\{s,a\}} \int_{d^2}^{k^2 n_*^2} \frac{e^{i|z-\zeta|\sqrt{k^2 n_*^2 - \lambda}}}{2i\sqrt{k^2 n_*^2 - \lambda}} v_j(x,\lambda) v_j(\xi,\lambda) \sigma_j(\lambda) d\lambda, \tag{2.34b}$$

$$G^e(x,z;\xi,\zeta) = -\frac{1}{2\pi} \sum_{j\in\{s,a\}} \int_{k^2 n_*^2}^{+\infty} \frac{e^{-|z-\zeta|\sqrt{\lambda - k^2 n_*^2}}}{2\sqrt{\lambda - k^2 n_*^2}} v_j(x,\lambda) v_j(\xi,\lambda) \sigma_j(\lambda) d\lambda, \tag{2.34c}$$

with

$$\sigma_j(\lambda) = \frac{\sqrt{\lambda - d^2}}{(\lambda - d^2)\phi_j(h,\lambda)^2 + \phi_j'(h,\lambda)^2}. \tag{2.35}$$

$G^g$ represents the guided part of the Green's function, which describes the guided modes, i.e. the modes propagating mainly inside the core; $G^r$ and $G^e$ are the parts of the Green's function corresponding to the radiation and evanescent modes, respectively.

The radiation and evanescent components altogether form the radiating part $G^{rad}$ of $G$:

$$G^{rad} = G^r + G^e = \frac{1}{2\pi} \sum_{j\in\{s,a\}} \int_{d^2}^{+\infty} \frac{e^{i|z-\zeta|\sqrt{k^2 n_*^2 - \lambda}}}{2i\sqrt{k^2 n_*^2 - \lambda}} v_j(x,\lambda) v_j(\xi,\lambda) \sigma_j(\lambda) d\lambda. \tag{2.36}$$

## 2.5 The free-space Green's function

When $n \equiv 1$, the Helmholtz equation (2.1) becomes

$$\Delta u + k^2 u = f, \tag{2.37}$$

which is the well-known Helmholtz equation in the free-space. The Green's function of (2.37) which satisfies the Sommerfeld radiation condition at infinity is

$$G_0(x, z; \xi, \zeta) = -\frac{i}{4} H_0^{(1)}(kr), \tag{2.38}$$

where $r = \sqrt{(x-\xi)^2 + (z-\zeta)^2}$ and $H_0^{(1)}$ denotes the zeroth order Hankel equation of the first kind.

On the other hand, as done in Section 2.4 for the Helmholtz equation (2.1), we can apply the Titchmarsh theory and obtain a Green's formula for the free-space Helmholtz equation (2.37) analogous to (2.32):

$$G_0^T(x, z; \xi, \zeta) = \frac{1}{2\pi} \int_0^{+\infty} \frac{e^{i|z-\zeta|\sqrt{k^2-\lambda}}}{2i\sqrt{k^2-\lambda}} \frac{\cos(\sqrt{\lambda}(x-\xi))}{\sqrt{\lambda}} d\lambda, \tag{2.39}$$

which can also be obtained from (2.32) by taking the limit as $n_0 \to n_{cl}$ and by setting $n_{cl} = 1$.

In the rest of this section, we prove that (2.38) and (2.39) are actually the same function.

We recall (see [AS]) that $H_0^{(1)}(y) = J_0(y) + iY_0(y)$, $y \in \mathbb{C}$, where $J_0$ and $Y_0$ are the zeroth order Bessel functions of first and second kind, respectively. Finally, we report here formulas (9.1.18) and (9.1.22) from [AS]:

$$J_0(y) = \frac{1}{\pi} \int_0^{\pi} \cos(y \cos t) dt, \tag{2.40a}$$

$$Y_0(y) = \frac{1}{\pi} \int_0^{\pi} \sin(y \sin t) dt - \frac{2}{\pi} \int_0^{+\infty} e^{-y \sinh t} dt, \quad |\arg y| < \frac{\pi}{2}. \tag{2.40b}$$

We write $x$ and $z$ by using the polar coordinates

$$x = R \sin\vartheta, \quad z = R\cos\vartheta,$$

and set $\xi, \zeta = 0$. Furthermore, thanks to the symmetry of $G_0$ and $G_0^T$ with respect to the coordinate axes, we can assume (without loss of generality) that $\vartheta \in [0, \pi/2]$.

**Lemma 2.7.** *The following identity holds for every $\vartheta \in [0, \pi/2]$ and $R \geq 0$:*

$$\int_0^k \frac{\cos(R\cos\vartheta\sqrt{k^2-\mu^2})}{\sqrt{k^2-\mu^2}} \cos(\mu R \sin\vartheta) d\mu = \frac{\pi}{2} J_0(kR). \tag{2.41}$$

*Proof.* We set $\mu = k\sin t$ and change the coordinates in the integral in (2.41):

$$\int_0^k \frac{\cos(R\cos\vartheta\sqrt{k^2-\mu^2})}{\sqrt{k^2-\mu^2}}\cos(\mu R\sin\vartheta)d\mu = \int_0^{\frac{\pi}{2}} \cos(kR\cos\vartheta\cos t)\cos(kR\sin\vartheta\sin t)dt$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(kR\cos(\vartheta+t))dt + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(kR\cos(\vartheta-t))dt,$$

where the second equality follows from the trigonometric addition formulas. Again, by changing the variables in the above integrals, it is easy to find

$$\int_0^k \frac{\cos(R\cos\vartheta\sqrt{k^2-\mu^2})}{\sqrt{k^2-\mu^2}}\cos(\mu R\sin\vartheta)d\mu = \frac{1}{2}\int_{-\frac{\pi}{2}+\vartheta}^{\frac{\pi}{2}+\vartheta} \cos(kR\cos t)dt.$$

Since $\cos(kR\cos t)$ is periodic in $t$ with period $\pi$, we finally obtain

$$\int_0^k \frac{\cos(R\cos\vartheta\sqrt{k^2-\mu^2})}{\sqrt{k^2-\mu^2}}\cos(\mu R\sin\vartheta)d\mu = \frac{1}{2}\int_0^{\pi} \cos(kR\cos t)dt,$$

which, together with (2.40a), implies (2.41). □

**Lemma 2.8.** *The following identity holds for every $\vartheta \in [0, \pi/2]$ and $R > 0$:*

$$(2.42)\quad \int_0^k \frac{\sin(R\cos\vartheta\sqrt{k^2-\mu^2})}{\sqrt{k^2-\mu^2}}\cos(\mu R\sin\vartheta)d\mu$$
$$- \int_k^{+\infty} \frac{e^{-R\cos\vartheta\sqrt{\mu^2-k^2}}}{\sqrt{\mu^2-k^2}}\cos(\mu R\sin\vartheta)d\mu = \frac{\pi}{2}Y_0(kR).$$

*Proof.* In the second integral in (2.42) we change the variables by setting $\mu = k\cosh t$ and we use the trigonometric addition formulas:

$$\int_k^{+\infty} \frac{e^{-R\cos\vartheta\sqrt{\mu^2-k^2}}}{\sqrt{\mu^2-k^2}}\cos(\mu R\sin\vartheta)d\mu = \operatorname{Re}\int_0^{+\infty} e^{-kR\sinh(t+i\vartheta)}dt$$
$$= \operatorname{Re}\int_{i\vartheta}^{i\vartheta+\infty} e^{-kR\sinh\tau}d\tau.$$

The last integral in the equations above can be computed by using the Residues Theorem:

$$\int_{i\vartheta}^{i\vartheta+\infty} e^{-kR\sinh\tau}d\tau = \int_0^{+\infty} e^{-kR\sinh t}dt - i\int_0^{\vartheta} e^{-ikR\sin t}dt,$$

and thus we have

$$\int_k^{+\infty} \frac{e^{-R\cos\vartheta\sqrt{\mu^2-k^2}}}{\sqrt{\mu^2-k^2}}\cos(\mu R\sin\vartheta)d\mu = \int_0^{+\infty} e^{-kR\sinh t}dt - \int_0^{\vartheta} \sin(kR\sin t)dt. \tag{2.43}$$

Now, we consider the first integral in (2.42). We change the variables by setting $\mu = k\cos t$ and, again, use the trigonometric addition formulas:

$$\begin{aligned}\int_0^k &\frac{\sin(R\cos\vartheta\sqrt{k^2-\mu^2})}{\sqrt{k^2-\mu^2}}\cos(\mu R\sin\vartheta)d\mu \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(kR\sin(t+\vartheta))dt + \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(kR\sin(t-\vartheta))dt \\ &= \frac{1}{2}\int_{\vartheta}^{\frac{\pi}{2}+\vartheta} \sin(kR\sin t)dt + \frac{1}{2}\int_{-\vartheta}^{\frac{\pi}{2}-\vartheta} \sin(kR\sin t)dt.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2}\int_{\vartheta}^{\frac{\pi}{2}+\vartheta} + \frac{1}{2}\int_{-\vartheta}^{\frac{\pi}{2}-\vartheta} \sin(kR\sin t)dt &= \frac{1}{2}\int_{-\vartheta}^{\vartheta} + \int_{\vartheta}^{\frac{\pi}{2}-\vartheta} + \frac{1}{2}\int_{\frac{\pi}{2}-\vartheta}^{\frac{\pi}{2}+\vartheta} \sin(kR\sin t)dt \\ = \int_{\vartheta}^{\frac{\pi}{2}-\vartheta} + \int_{\frac{\pi}{2}-\vartheta}^{\frac{\pi}{2}} \sin(kR\sin t)dt &= \int_{\vartheta}^{\frac{\pi}{2}} \sin(kR\sin t)dt,\end{aligned}$$

and from (2.43), it follows that the left side of (2.42) can be written as

$$\int_0^{\frac{\pi}{2}} \sin(kR\sin t)dt - \int_0^{+\infty} e^{-kR\sinh t}dt.$$

Since $\sin(kR\sin t)$ is symmetric with respect to the axis $t = \pi/2$, from (2.40b) we get (2.42). □

**Corollary 2.9.** $G_0 \equiv G_0^T$.

*Proof.* We change the variables in (2.39) by setting $\mu = \sqrt{\lambda}$:

$$\begin{aligned}G_0^T = &-\frac{i}{2\pi}\int_0^k \frac{\cos(z\sqrt{k^2-\mu^2})\cos(\mu x)}{\sqrt{k^2-\mu^2}}d\mu \\ &+ \frac{1}{2\pi}\int_0^k \frac{\sin(|z|\sqrt{k^2-\mu^2})\cos(\mu x)}{\sqrt{k^2-\mu^2}}d\mu - \frac{1}{2\pi}\int_k^{+\infty} \frac{e^{-|z|\sqrt{\mu^2-k^2}}}{\sqrt{\mu^2-k^2}}\cos(\mu x)d\mu.\end{aligned}$$

Thus, the conclusion follows by representing $x$ and $z$ in polar coordinates and from Lemmas 2.7 and 2.8. □

## 2.6 Asymptotic Lemmas

In this section we will prove some result which will be useful in the rest of this thesis. In particular, next lemma will play a crucial role.

**Lemma 2.10.** *Let $q \in L^1_{loc}(\mathbb{R})$, $\omega \in \mathbb{R}$ and $r > 0$. Let $v_\pm(x,\omega)$ be the respective solutions of*

$$\begin{cases} v'' + [\omega^2 - q(x)]v = 0, \quad \text{in } (-r,r), \\ v(0,\omega) = 1, \\ v'(0,\omega) = \pm i\omega. \end{cases} \tag{2.44}$$

*Then $v_\pm$ are analytic in $\omega$ for $x \in [-r,r]$ and the following asymptotic expansions hold uniformly for $x \in [-r,r]$ as $\omega \to \infty$:*

$$\begin{aligned} v_\pm(x,\omega) &= e^{\pm i\omega x}\left\{1 + \mathcal{O}\left(1/\omega\right)\right\}, \\ v'_\pm(x,\omega) &= \pm i\omega e^{\pm i\omega x}\left\{1 + \mathcal{O}\left(1/\omega\right)\right\}. \end{aligned} \tag{2.45}$$

*Proof.* We prove the Lemma only for $v = v_+$, i.e. we study the Cauchy problem (2.44), where in the second initial condition we choose the sign "+". As is well-known, $v$ satisfies (2.44) if and only if it is a solution of the integral equation

$$v(x,\omega) = e^{i\omega x} + \int_0^x q(s)\frac{\sin\omega(x-s)}{\omega}v(s,\omega)ds, \quad x \in [-r,r], \tag{2.46}$$

that we rewrite as

$$v(x) - T_\omega v(x) = e^{i\omega x}, \quad x \in [-r,r], \tag{2.47}$$

where

$$T_\omega v(x) = \int_{-r}^{r} \tau(x,s;\omega)v(s)ds,$$

with

$$\tau(x,s;\omega) = \frac{q(s)\sin\omega(x-s)}{\omega}\chi_{[0,x]}(s).$$

We consider the Banach space $C^0([-r,r])$ equipped by the usual norm

$$\|v\|_{C^0} = \sup_{x\in[-r,r]} |v(x,\omega)|.$$

Since $q \in L^1_{loc}(\mathbb{R})$, it holds that

$$|T_\omega v(x)| \le \|v\|_{C^0}\int_{-r}^{r} |\tau(x,s;\omega)|ds \le \|v\|_{C^0}\int_{-r}^{r} |x-s|q(s)ds,$$

and hence

$$|T_\omega v(x)| \le 2r\|q\|_1\|v\|_{C^0}, \tag{2.48}$$

for every $x \in [-r, r]$ and $\omega \in \mathbb{R}$.

Similarly, for every $x_1, x_2 \in [-r, r]$ and $\omega \in \mathbb{R}$ we have:

$$|T_\omega v(x_1) - T_\omega v(x_2)| \le \left[ 2r \left| \int_{x_1}^{x_2} q(s)ds \right| + |x_1 - x_2| \right] \|v\|_{C^0}. \tag{2.49}$$

Estimates (2.48) and (2.49) show that $T_\omega$ maps bounded sequences in $C^0([-r, r])$ into equibounded and equicontinuous sequences in $C^0([-r, r])$. By Arzelà's theorem, we infer that $T_\omega$ is a compact operator from $C^0([-r, r])$ into itself. Thus, from Fredholm's alternative theorem, (2.47) has the unique solution

$$v(x, \omega) = \sum_{n=0}^{\infty} T_\omega^n(e^{i\omega x}), \quad x \in [-r, r]. \tag{2.50}$$

In fact, $v$ is solution of the homogeneous equation $v - T_\omega v = 0$ if and only if $v$ satisfies

$$\begin{cases} v'' + [\omega^2 - q(x)]v = 0 & \text{in } (-r, r), \\ v(0, \omega) = v'(0, \omega) = 0, \end{cases}$$

i.e. if and only if $v \equiv 0$ in $[-r, r]$.

Since (2.48) and (2.49) hold uniformly with respect to $\omega \in \mathbb{R}$, the convergence of the series in (2.50) is uniform and hence $v(x, \omega)$ is analytic for $\omega \in \mathbb{R}$.

The asymptotic expansions (2.45) follow from (2.46) and from the formula

$$v'(x, \omega) = i\omega e^{i\omega x} + \int_0^x q(s)\cos\omega(x - s)v(s, \omega)d\omega,$$

which is obtained by differentiating (2.46) with respect to $x$. □

**Remark 2.11.** With minor modifications of the proof, Lemma 2.10 can be proven for $\omega \in S \subset \mathbb{C}$, where $S = (-\infty, \infty) \times [-t, t]$, with $t \ge 0$.

**Lemma 2.12.** *Let $q \in L^1_{loc}(\mathbb{R})$. The functions $\phi_s(x, \lambda)$ and $\phi_a(x, \lambda)$, $x \in [-h, h]$, defined in* (2.22)*, are analytic in $\lambda$ and $\sqrt{\lambda}$ respectively, with $\lambda \ge 0$.*

*Moreover, the following asymptotic expansions hold uniformly in $x \in [-h, h]$ as $\lambda \to \infty$:*

(2.51)

$$\phi_s(x, \lambda) = \cos(\sqrt{\lambda}x)\left\{1 + \mathcal{O}\left(\tfrac{1}{\sqrt{\lambda}}\right)\right\}, \quad \phi_s'(x, \lambda) = -\sqrt{\lambda}\sin(\sqrt{\lambda}x)\left\{1 + \mathcal{O}\left(\tfrac{1}{\sqrt{\lambda}}\right)\right\},$$

$$\phi_a(x, \lambda) = \sin(\sqrt{\lambda}x)\left\{1 + \mathcal{O}\left(\tfrac{1}{\sqrt{\lambda}}\right)\right\}, \quad \phi_a'(x, \lambda) = \sqrt{\lambda}\cos(\sqrt{\lambda}x)\left\{1 + \mathcal{O}\left(\tfrac{1}{\sqrt{\lambda}}\right)\right\}.$$

*Proof.* We set $\lambda = \omega^2$ and observe that $\phi_s$ and $\phi_a$ are, respectively, the real and imaginary part of $v_+$ in Lemma 2.10. Hence, we can deduce that $\phi_j$, $j \in \{s, a\}$, are analytic in $\omega$ and (2.51) follows.

We observe that

$$\frac{\sin \omega(x-s)}{\omega} \quad \text{and} \quad \operatorname{Re} e^{i\omega x}$$

are analytic in $\omega^2$. Then, from (2.46) and (2.50), we have that $\phi_s$ is analytic in $\omega^2$, i.e. in $\lambda$, for $\lambda \geq 0$. $\square$

**Lemma 2.13.** *Let $q \in L^1_{loc}(\mathbb{R})$ and $\lambda_0 = \min(\lambda_1^s, \lambda_1^a)$, where $\lambda_1^s$ and $\lambda_1^a$ are defined in Remark 2.5. Let $\phi_j$, $j \in \{s, a\}$, be defined by* (2.22)*. Then, the following estimates hold for $x \in [-h, h]$ and $\lambda \geq \lambda_0$:*

$$|\phi_j(x, \lambda)| \leq \Phi_*, \quad |\phi_j'(x, \lambda)| \leq \Phi_* \sqrt{\lambda}, \tag{2.52}$$

*where*

$$\Phi_* := \exp\left\{ \frac{1}{2\sqrt{\lambda_0}} \int_{-h}^{h} |q(t)| dt \right\}. \tag{2.53}$$

*Proof.* We consider the function

$$\psi_j(x, \lambda) = \phi_j'(x, \lambda)^2 + \lambda \phi_j(x, \lambda)^2,$$

and notice that

$$\psi_j'(x, \lambda) = 2q(x)\phi_j(x, \lambda)\phi_j'(x, \lambda),$$

as it follows from (2.8). By using Young inequality, we get that $\psi_j$ satisfies

$$\begin{cases} \psi_j'(x, \lambda) \leq \dfrac{|q(x)|}{\sqrt{\lambda}} \psi_j(x, \lambda), \\ \psi_j(0, \lambda) = \lambda. \end{cases}$$

Therefore, by integrating the above inequality, we obtain that

$$\psi_j(x, \lambda) \leq \lambda \exp\left\{ \frac{1}{\sqrt{\lambda}} \int_{-h}^{h} |q(t)| dt \right\} \leq \lambda \Phi_*^2,$$

which implies (2.52). $\square$

**Corollary 2.14.** *Let $\sigma_j(\lambda)$, $j \in \{s, a\}$, be the quantities defined in* (2.35)*. Then it holds that*

$$\sigma_j(\lambda) = \frac{1}{\sqrt{\lambda - d^2}} + \mathcal{O}\left(\frac{1}{\lambda}\right), \text{ as } \lambda \to \infty. \tag{2.54}$$

*Proof.* By multiplying

$$\phi_j''(x,\lambda) + [\lambda - q(x)]\phi_j(x,\lambda) = 0, \quad x \in [-h,h],$$

by $\phi_j'(x,\lambda)$ and integrating in $x$ over $(0,h)$, we find

$$\phi_j'(h,\lambda)^2 - \phi_j'(0,\lambda)^2 + (\lambda - d^2)[\phi_j(h,\lambda)^2 - \phi_j(0,\lambda)^2] = 2\int_0^h [q(x) - d^2]\phi_j(x,\lambda)\phi_j'(x,\lambda)dx.$$

Thus, by using (2.11), we obtain

$$\left|\phi_s'(h,\lambda)^2 + (\lambda - d^2)\phi_s(h,\lambda)^2 - (\lambda - d^2)\right| \le 2k^2(n_*^2 + n_{cl}^2)\int_0^h |\phi_s(x,\lambda)\phi_s'(x,\lambda)|dx,$$

and

$$\left|\phi_a'(h,\lambda)^2 + (\lambda - d^2)\phi_a(h,\lambda)^2 - \lambda\right| \le 2k^2(n_*^2 + n_{cl}^2)\int_0^h |\phi_a(x,\lambda)\phi_a'(x,\lambda)|dx,$$

The asymptotic formula (2.54) follows from the two equations above, (2.35) and the bounds (2.52) for $\phi_j(x,\lambda)$ and $\phi_j'(x,\lambda)$. □

**Lemma 2.15.** *Let $\sigma_j(\lambda)$, $j \in \{s,a\}$, be the quantities defined in* (2.35). *The following formulas hold for $\lambda \to d^2$:*

$$\sigma_j(\lambda) = \begin{cases} \dfrac{\sqrt{\lambda - d^2}}{\phi_j'(h,d^2)^2} + \mathcal{O}\left(\lambda - d^2\right), & \textit{if } \phi_j'(h,d^2) \neq 0, \\[2ex] \dfrac{1}{\phi_j(h,d^2)^2\sqrt{\lambda - d^2}} + \mathcal{O}\left(\sqrt{\lambda - d^2}\right) & \textit{otherwise.} \end{cases} \tag{2.55}$$

*Proof.* From Lemma 2.12 we know that $\phi_s$ and $\phi_a$ are analytic in $\lambda$ and $\sqrt{\lambda}$ respectively, with $\lambda \ge 0$. Thus, in a neighbour of $\lambda = d^2$, we write

$$\phi(h,\lambda) = \sum_{m=0}^{\infty}(\lambda - d^2)^m a_m, \quad \phi'(h,\lambda) = \sum_{m=0}^{\infty}(\lambda - d^2)^m b_m,$$

$$\phi(h,\lambda)^2 = \sum_{m=0}^{\infty}(\lambda - d^2)^m \alpha_m, \quad \phi'(h,\lambda)^2 = \sum_{m=0}^{\infty}(\lambda - d^2)^m \beta_m,$$

where we omitted the dependence on $j$ to avoid too heavy notations.

We notice that $\alpha_0 = a_0^2$, $\alpha_1 = 2a_0a_1$ and the same for $b_0$ and $b_1$. From (2.35) we have

$$\begin{aligned} \sigma_j(\lambda)^{-1} &= \sqrt{\lambda - d^2}\phi_j(h,\lambda)^2 + \frac{1}{\sqrt{\lambda - d^2}}\phi_j'(h,\lambda)^2 \\ &= \frac{\beta_0}{\sqrt{\lambda - d^2}} + \sum_{m=0}^{\infty}(\lambda - d^2)^{m+\frac{1}{2}}(\alpha_m + \beta_{m+1}). \end{aligned} \tag{2.56}$$

If $\beta_0 \neq 0$, since $\beta_0 = b_0^2$, (2.55) follows. If $\beta_0 = 0$ we have that the leading term in (2.56) is $\alpha_0 + \beta_1$. We notice that $\alpha_0 \neq 0$, otherwise $\phi(x, d^2) \equiv 0$ for all $x \in \mathbb{R}$. We know that $\beta_1 = 2b_0 b_1 = 0$, because $b_0 = 0$. Then $\alpha_0 + \beta_1 = \alpha_0$ and (2.55) follows. $\square$

## 2.7 Far field

The far field region is the zone of the plane where the angular field distribution is essentially independent of the distance from the source. In this section we derive the far field of (2.33). The first thing we do is to write the far field of the Green's function (2.32). Then, the one of (2.33) will follow easily.

A brief description of the asymptotic methods used in this section can be found in Appendix A.

The far field will be written in terms of the polar coordinates

$$\begin{cases} x = R\sin\vartheta, \\ z = R\cos\vartheta, \end{cases}$$

where $R = \sqrt{x^2+z^2} > 0$ and $\vartheta \in [-\pi, \pi]$.

As already mentioned in Section 2.4 (see formulas (2.34a) and (2.36)) we split the expression in (2.32) up in two parts: $G^g$, corresponding to the guided modes, and $G^{rad}$, corresponding to the radiating ones (which include radiation and evanescent modes).

We begin with a formula for $G^g$; we notice that, for $R \to +\infty$, $G^g$ decays exponentially in all directions, except for the ones along the fiber's axis ($\vartheta = 0, \pi$).

For the sake of simplicity, in all the lemmas and theorems of this section, we will carry out the computations when $\vartheta$ belongs to $[0, \pi/2]$. The remaining cases can be derived similarly.

**Lemma 2.16.** *For fixed $\vartheta \in [0, \pi/2]$ and $R$ sufficiently large, the far field of $G^g$ is given by*

$$G^g = \sum_{m=1}^{M_s} \frac{r_m^s e^{iR\sqrt{k^2n_*^2-\lambda_m^s}}}{2i\sqrt{k^2n_*^2-\lambda_m^s}} v_s(\xi, \lambda_m^s) e^{-i\zeta\sqrt{k^2n_*^2-\lambda_m^s}}, \quad \textit{if } \vartheta = 0, \tag{2.57a}$$

$$\begin{aligned} G^g = \sum_{j\in\{s,a\}} \sum_{m=1}^{M_j} & \frac{r_m^j e^{h\sqrt{d^2-\lambda_m^j}}}{2i\sqrt{k^2n_*^2-\lambda_m^j}} \phi_j(h, \lambda_m^j) \times \\ & \times e^{R\left[i\sqrt{k^2n_*^2-\lambda_m^j}\cos\vartheta - \sqrt{d^2-\lambda_m^j}\sin\vartheta\right]} v_j(\xi, \lambda_m^j) e^{-i\zeta\sqrt{k^2n_*^2-\lambda_m^j}}, \\ & \qquad \textit{if } 0 < \vartheta < \frac{\pi}{2}, \end{aligned} \tag{2.57b}$$

*and*

$$(2.57c)\quad G^g = \sum_{j\in\{s,a\}} \sum_{m=1}^{M_j} \frac{r_m^j e^{h\sqrt{d^2-\lambda_m^j}}}{2i\sqrt{k^2n_*^2-\lambda_m^j}} \phi_j(h,\lambda_m^j) e^{-R\sqrt{d^2-\lambda_m^j}} v_j(\xi,\lambda_m^j) e^{i|\zeta|\sqrt{k^2n_*^2-\lambda_m^j}},$$

$$\text{if } \vartheta = \frac{\pi}{2}.$$

*Proof.* Formulas (2.57) are obtained by a mere change of variables and a rearrangement of the coefficients in (2.34a). The case $\vartheta = 0$ is trivial. Otherwise, thanks to (2.25) and (2.22) we can write $G^g$ as

$$G^g = \sum_{j\in\{s,a\}} \sum_{m=1}^{M_j} \frac{e^{i|z-\zeta|\sqrt{k^2n_*^2-\lambda_m^j}}}{2i\sqrt{k^2n_*^2-\lambda_m^j}} \phi_j(h,\lambda_m^j) e^{-\sqrt{d^2-\lambda_m^j}|x-h|} v_j(\xi,\lambda_m^j) r_m^j,\ x > h.$$

By substituting the polar coordinates, (2.57b) and (2.57c) follow. □

Now, we derive the far-field pattern for $G^{rad}$. As it is shown in the next theorem, the radiating part $G^{rad}$ of the Green's function vanishes in all directions for $R \to \infty$.

In the following, it will be useful to write $G^{rad}$ in a slightly different way. Suppose $|x| \geq h$. From (2.22), we can write $v_j(x,\lambda)$ as combination of the two exponentials $e^{iQx}$ and $e^{-iQx}$, where $Q$ is given by (2.21). We have

$$(2.58)\qquad v_j(R\sin\vartheta,\lambda) = \alpha_j(\lambda)e^{iQR\sin\vartheta} + \overline{\alpha_j(\lambda)}e^{-iQR\sin\vartheta}, \quad R\sin\vartheta \geq h,$$

where

$$(2.59)\qquad \alpha_j(\lambda) = \frac{e^{-ihQ}}{2}\left[\phi_j(h,\lambda) + \frac{\phi_j'(h,\lambda)}{iQ}\right].$$

By using (2.58) and (2.59) we can write $G^{rad}$ in terms of polar coordinates as

$$(2.60)\quad G^{rad} =$$

$$\frac{1}{2\pi}\sum_{j\in\{s,a\}} \int_{d^2}^{\infty} \frac{e^{i|R\cos\vartheta-\zeta|\sqrt{k^2n_*^2-\lambda}}}{2i\sqrt{k^2n_*^2-\lambda}} \left[\alpha_j(\lambda)e^{iQR\sin\vartheta} + \overline{\alpha_j(\lambda)}e^{-iQR\sin\vartheta}\right] v_j(\xi,\lambda)\sigma_j(\lambda)d\lambda.$$

**Theorem 2.17.** *Let $(\xi,\zeta) \in \mathbb{R}^2$ be fixed. For a fixed $\vartheta \in (0,\pi/2]$, the following asymptotic expansion holds as $R \to +\infty$:*

(2.61)

$$G^{rad} = \frac{e^{i\left(Rkn_{cl}-\frac{3}{4}\pi\right)}}{\sqrt{R}}\sqrt{\frac{kn_{cl}}{2\pi}}\sin\vartheta \sum_{j\in\{s,a\}} \alpha_j(k^2n_*^2-\mu_0(\vartheta)^2) g_j(\xi,\zeta,\mu_0(\vartheta)) + \mathcal{O}\left(\frac{1}{R}\right),$$

*where $\alpha_j(\lambda)$ are given by (2.59) and*

$$(2.62)\qquad \mu_0(\vartheta) = kn_{cl}\cos\vartheta,$$

(2.63) $$g_j(\xi,\zeta,\mu)=v_j(\xi,k^2n_*^2-\mu^2)\sigma_j(k^2n_*^2-\mu^2)e^{-i\zeta\mu}.$$

*In the case $\vartheta=0$, we have for $R\to+\infty$:*

(2.64)
$$G^{rad}=\begin{cases}\dfrac{1}{2\pi}v_s(\xi,k^2n_*^2)\sigma_s(k^2n_*^2)\left(\dfrac{1}{R}-\dfrac{1}{|R-\zeta|}\right)+\mathcal{O}\left(\dfrac{1}{R^2}\right), & \text{if } \lim\limits_{\lambda\to d^2}\sigma_s(\lambda)=0,\\[2ex] \dfrac{e^{i(Rkn_{cl}-\frac{\pi}{4})}}{2\sqrt{R}}\cdot\dfrac{v_s(\xi,d^2)e^{-i\zeta kn_{cl}}}{i\phi_s(h,d^2)^2}\left(\dfrac{1}{2\pi kn_{cl}}\right)^{\frac12}+\mathcal{O}\left(\dfrac{1}{R}\right) & \text{otherwise.}\end{cases}$$

*Proof.* Case 1: $\vartheta\in(0,\pi/2)$. For each fixed $\vartheta$, it is possible to find $R_0(\vartheta)$ such that $x=R\sin\vartheta>h$ and $z=R\cos\vartheta>\zeta$, for all $R>R_0(\vartheta)$, and hence $|R\cos\vartheta-\zeta|=R\cos\vartheta-\zeta$, for all $R>R_0(\vartheta)$.

Firstly we calculate the contribution of $G^r$. In (2.60), we consider the integral over the interval $(d^2,k^2n_*^2)$ and change the variables by setting $\mu=\sqrt{k^2n_*^2-\lambda}$. We have:

(2.65) $$G^r=-\frac{i}{2\pi}\sum_{j\in\{s,a\}}\int_0^{kn_{cl}}e^{iR(\mu\cos\vartheta+\sqrt{k^2n_{cl}^2-\mu^2}\sin\vartheta)}\alpha_j(k^2n_*^2-\mu^2)g_j(\xi,\zeta,\mu)d\mu$$
$$+\int_0^{kn_{cl}}e^{iR(\mu\cos\vartheta-\sqrt{k^2n_{cl}^2-\mu^2}\sin\vartheta)}\overline{\alpha_j(k^2n_*^2-\mu^2)}g_j(\xi,\zeta,\mu)d\mu,$$

where $\alpha_j$ and $g_j$, $j\in\{s,a\}$, are given by (2.59) and (2.63) respectively.

Since the exponents of the two exponentials in (2.65) are purely imaginary, we can apply the method of the stationary phase (see Appendix A, formula (A.3)). The exponential in the first integral in (2.65) has a single turning point for $\mu=\mu_0(\vartheta)=kn_{cl}\cos\vartheta$, and gives a contribution of $R^{-\frac12}+\mathcal{O}(1/R)$, as $R\to\infty$. The one in the second integral has no turning points; hence, an integration by parts easily shows that the second integral is an $\mathcal{O}(1/R)$ as $R\to+\infty$.

As a result of these observations and of formula (A.3), we then find:

(2.66) $$G^r=\frac{e^{i\left(Rkn_{cl}-\frac34\pi\right)}}{\sqrt{R}}\sqrt{\frac{kn_{cl}}{2\pi}}\sin\vartheta\sum_{j\in\{s,a\}}\alpha_j(k^2n_*^2-\mu_0(\vartheta)^2)g_j(\xi,\zeta,\mu_0)+\mathcal{O}\left(\frac{1}{R}\right).$$

Now we study the behaviour of $G^e$. In (2.60), we consider the integral for $\lambda\in(k^2n_*^2,+\infty)$ and change the variables by setting $\mu=\sqrt{\lambda-k^2n_*^2}$. As we did for (2.65), we obtain
$$G^e=-\frac{1}{2\pi}\sum_{j\in\{s,a\}}I_j+\overline{I}_j,$$
where

(2.67) $$I_j=\int_0^\infty g_j^e(\xi,\mu)e^{-|R\cos\vartheta-\zeta|\mu+iR\sqrt{\mu^2+k^2n_{cl}^2}\sin\vartheta}d\mu$$

and

$$g_j^e(\xi,\mu) = \alpha_j(\mu^2 + k^2 n_*^2) v_j(\xi, \mu^2 + k^2 n_*^2)\sigma_j(\mu^2 + k^2 n_*^2).$$

We will study the behaviour of $I_j$ as $R \to \infty$. We can assume to have chosen $R_0(\vartheta)$ big enough such that $|R\cos\vartheta - \zeta| \neq 0$, for all $R > R_0(\vartheta)$.

Since $\xi$ is fixed, from Lemma 2.12 we know that $v_j(\xi,\lambda)$ and $\alpha_j(\lambda)$ are bounded for $\lambda \geq k^2 n_*^2$. From Lemma 2.15 it follows that also $\sigma_j$ is bounded. Therefore, for fixed $\xi$, we have that $g_j^e$ is bounded by a constant $C$ and then

$$|I_j| \leq C \int_0^\infty e^{-|R\cos\vartheta - \zeta|\mu} d\mu = \mathcal{O}\left(\frac{1}{R}\right),$$

as can be shown by an integration by parts.

Therefore, for fixed $\xi, \zeta$ and $\vartheta \in (0, \pi/2)$, we have that

$$G^e = \mathcal{O}\left(\frac{1}{R}\right). \tag{2.68}$$

From (2.66) and (2.68) formula (2.66) follows at once, which concludes the proof of this case.

Case 2: $\vartheta = \pi/2$. This case is different from the previous one because also $G^e$ behaves as $\mathcal{O}(1/\sqrt{R})$ and hence comes into play.

Again, we write $G^r$ as in (2.65) and note that $\cos\vartheta = 0$. The only turning point is for $\mu = 0$ for both integrals in (2.65). In this case, by a slightly different formula (see (A.4) in Appendix A), we obtain

$$G^r = -\frac{i}{2}\left(\frac{kn_{cl}}{2\pi R}\right)^{\frac{1}{2}} \sum_{j\in\{s,a\}} v_j(\xi, k^2 n_*^2)\sigma_j(k^2 n_*^2) 2\,\mathrm{Re}\left[\alpha_j(k^2 n_*^2) e^{i(Rkn_{cl} - \frac{\pi}{4})}\right] + \mathcal{O}\left(\frac{1}{R}\right).$$

We proceed in a similar way for $G^e$ and find

$$G^e = -\frac{1}{2}\left(\frac{kn_{cl}}{2\pi R}\right)^{\frac{1}{2}} \sum_{j\in\{s,a\}} v_j(\xi, k^2 n_*^2)\sigma_j(k^2 n_*^2) 2\,\mathrm{Re}\left[\alpha_j(k^2 n_*^2) e^{i(Rkn_{cl} + \frac{\pi}{4})}\right] + \mathcal{O}\left(\frac{1}{R}\right).$$

We sum these two formulas and get

$$G^{rad} = \frac{e^{i\left(Rkn_{cl} - \frac{3}{4}\pi\right)}}{\sqrt{R}} \sqrt{\frac{kn_{cl}}{2\pi}} \sum_{j\in\{s,a\}} v_j(\xi, k^2 n_*^2)\sigma_j(k^2 n_*^2)\alpha_j(k^2 n_*^2) + \mathcal{O}\left(\frac{1}{R}\right).$$

We notice that the last formula is just (2.61) when $\sin\vartheta = 1$.

Case 3: $\vartheta = 0$. We notice that in this case $x = 0$, which implies

$$v_j(x,\lambda) = \begin{cases} 1 & \text{if } j = s, \\ 0 & \text{if } j = a. \end{cases}$$

Therefore, in the expression of $G^r$ and $G^e$, only the term $j = s$ remain. From (2.34b) and (2.34c) we have, for $R$ big enough,

$$G^r(R,0,\xi,\zeta) = -\frac{i}{2\pi}\int_0^{kn_{cl}} e^{iR\mu}e^{-i\mu\zeta}v_s(\xi,k^2n_*^2-\mu^2)\sigma_s(k^2n_*^2-\mu^2)d\mu, \tag{2.69}$$

and

$$G^e(R,0,\xi,\zeta) = -\frac{1}{2\pi}\int_0^{+\infty} e^{-|R-\zeta|\mu}v_s(\xi,k^2n_*^2+\mu^2)\sigma_s(k^2n_*^2+\mu^2)d\mu,$$

where we changed the variables as in (2.65) and (2.67), respectively.

The contribution of $G^e$ can be easily computed by using integration by parts:

$$G^e = -\frac{1}{2\pi|R-\zeta|}v_s(\xi,k^2n_*^2)\rho_s(k^2n_*^2) + \mathcal{O}\left(\frac{1}{R^2}\right).$$

For $G^r$ we have to distinguish two different cases. When the denominator of $\sigma_s$ does not vanish at $d^2$, then

$$\lim_{\lambda\to d^2}\sigma_s(\lambda) = 0,$$

and we can integrate by parts to obtain:

$$\begin{aligned} G^r &= -\frac{1}{2\pi}\int_0^{kn_{cl}} \frac{de^{iR\mu}}{d\mu}\frac{v_s(\xi,k^2n_*^2-\mu^2)\sigma_s(k^2n_*^2-\mu^2)}{R}d\mu \\ &= \frac{v_s(\xi,k^2n_*^2)\sigma_s(k^2n_*^2)}{2\pi R} + \mathcal{O}(\frac{1}{R^2}), \quad R\to\infty. \end{aligned}$$

By summing up the contributions of $G^r$ and $G^e$, then we have the first case of (2.64).

If the denominator of $\sigma_s$ vanishes at $d^2$, from Lemma 2.15 we have that $\sigma_s(\lambda) = \mathcal{O}\big((\lambda-d^2)^{-\frac{1}{2}}\big)$, as $\lambda\to d^2$. The behaviour of $G^r$ (given by (2.34b)) is influenced by that of the integrand near both ends of the path of integration. We split the integral in (2.34b) up into two integrals, the first over the interval $(d^2, d^2+k^2n_{cl}^2)$, the second over $d^2+k^2n_{cl}^2, k^2n_*^2)$. By two appropriate change of variables ($\lambda = \beta^2+d^2$ in the former integral and $\lambda = k^2n_*^2-\mu^2$ in the latter), we write $2\pi G^r = H_1 + H_2$, where

$$H_1 = \int_0^{\frac{kn_{cl}}{2}} \frac{e^{iR\sqrt{k^2n_{cl}^2-\beta^2}}}{i\sqrt{k^2n_{cl}^2-\beta^2}}v_s(\xi,\beta^2+d^2)e^{-i\zeta\sqrt{k^2n_{cl}^2-\beta^2}}\sigma_s(\beta^2+d^2)\beta d\beta,$$

and

$$H_2 = -i\int_0^{\frac{\sqrt{3}kn_{cl}}{2}} e^{iR\mu}e^{-i\mu\zeta}v_s(\xi,k^2n_*^2-\mu^2)\sigma_s(k^2n_*^2-\mu^2)d\mu.$$

In $H_1$ there is only one turning point, for $\beta = 0$, while for $\beta = kn_{cl}$ there will be a contribution of $\mathcal{O}(1/R)$. By applying the method of the stationary phase and using (2.55), we get

$$G^r = \frac{e^{i(Rkn_{cl}-\frac{\pi}{4})}}{2\pi\sqrt{R}} \cdot \frac{v_s(\xi, d^2)e^{-i\zeta kn_{cl}}}{i\phi_s(h, d^2)^2} \left(\frac{\pi}{2kn_{cl}}\right)^{\frac{1}{2}}. \tag{2.70}$$

In $H_2$ there are no turning points and hence it behaves as $\mathcal{O}(1/R)$ and then, from (2.70), (2.64) follows. This completes the proof of Case 3. $\square$

From the far field of the Green's function, we can easily derive the far field of the solution of (H), in the case the source $f$ has compact support.

We will use the following notations:

$$F^j(\lambda, \zeta) = \int_{-\infty}^{+\infty} f(\xi, \zeta)v_j(\xi, \lambda)d\xi, \quad \hat{F}^j(\lambda, \mu) = \int_{-\infty}^{+\infty} F^j(\lambda, \zeta)e^{-i\zeta\mu}d\zeta. \tag{2.71}$$

**Corollary 2.18.** *Let $u$ be the solution of* (H) *defined in* (2.33). *The far field pattern of $u$ is given by the following asymptotic formulas, for fixed $\vartheta$ and $R \to +\infty$:*

*(i) if $\vartheta \in (0, \pi/2)$,*

(2.72a)

$$u = \sum_{j\in\{s,a\}} \sum_{m=1}^{M_j} \frac{r_m^j e^{h\sqrt{d^2-\lambda_m^j}}}{2i\beta_m^j} \phi_j(h, \lambda_m^j) e^{R\left(i\beta_m^j \cos\vartheta - \sqrt{d^2-\lambda_m^j}\sin\vartheta\right)} \hat{F}^j\left(\lambda_m^j, \beta_m^j\right)$$
$$+ \frac{e^{i\left(Rkn_{cl}-\frac{3}{4}\pi\right)}}{\sqrt{R}} \sqrt{\frac{kn_{cl}}{2\pi}} \sin\vartheta\ \sigma_j(\lambda_0(\vartheta))\alpha_j(k^2n_*^2 - \mu_0(\vartheta)^2)\hat{F}^j(\lambda_0(\vartheta), \mu_0(\vartheta))$$
$$+ \mathcal{O}\left(\frac{1}{R}\right),$$

*where $\alpha_j$ are given by* (2.59) *and*

$$\mu_0(\vartheta) = kn_{cl}\cos\vartheta, \quad \lambda_0(\vartheta) = k^2(n_*^2 - n_{cl}^2\cos^2\vartheta), \quad \beta_m^j = \sqrt{k^2n_*^2 - \lambda_m^j};$$

*(ii) if $\vartheta = \pi/2$,*

(2.72b)

$$u = \sum_{j\in\{s,a\}} \sum_{m=1}^{M_j} \frac{r_m^j e^{h\sqrt{d^2-\lambda_m^j}}}{2i\beta_m^j} \phi_j(h, \lambda_m^j) e^{-R\sqrt{d^2-\lambda_m^j}} \frac{\hat{F}^j(\lambda_m^j, \beta_m^j) + \hat{F}^j(\lambda_m^j, -\beta_m^j)}{2}$$
$$+ \frac{e^{i\left(Rkn_{cl}-\frac{3}{4}\pi\right)}}{\sqrt{R}} \sqrt{\frac{kn_{cl}}{2\pi}} \sigma_j(k^2n_*^2)\alpha_j(k^2n_*^2)\hat{F}^j(k^2n_*^2, 0) + \mathcal{O}\left(\frac{1}{R}\right);$$

*(iii) if* $\vartheta = 0$,

$$u = \sum_{m=1}^{M_s} \frac{r_m^s e^{iR\sqrt{k^2n_*^2-\lambda_m^s}}}{2i\sqrt{k^2n_*^2 - \lambda_m^s}} \hat{F}^j(\lambda_m^s, \beta_m^s) + \mathcal{O}\left(\frac{1}{R^2}\right), \tag{2.72c}$$

*for* $\lim_{\lambda \to d^2} \sigma_s(\lambda) = 0$, *and*

$$\begin{aligned} u = \sum_{m=1}^{M_s} \frac{r_m^s e^{iR\sqrt{k^2n_*^2-\lambda_m^s}}}{2i\sqrt{k^2n_*^2 - \lambda_m^s}} \hat{F}^j(\lambda_m^s, \beta_m^s) \\ + \frac{e^{i(Rkn_{cl} - \frac{\pi}{4})}}{\sqrt{R}} \cdot \frac{F^s(d^2, kn_{cl})}{2i\phi_s(h, d^2)^2}\left(\frac{1}{2\pi kn_{cl}}\right)^{\frac{1}{2}} + \mathcal{O}\left(\frac{1}{R}\right), \end{aligned} \tag{2.72d}$$

*otherwise.*

## 2.8 The singularity of the Green's function

In this section we continue the study of the features of the Green's function. In particular, we will estimate the behaviour of the Green's function in the singularity.

**Lemma 2.19.** *Let* $(\xi, \zeta)$ *be a fixed point in* $\mathbb{R}^2$ *and* $p = (x - \xi, z - \zeta)$. *Then*

$$\left| G(x, z; \xi, \zeta) - \ln|p| \right| \le C_s, \quad \text{as } |p| \to 0, \tag{2.73}$$

*where* $C_s$ *does not depend on* $x, z, \xi, \zeta$.

*Proof.* In Lemma 4.4 we will prove that the guided and radiating part of the Green's function are bounded. Thus, the singularity of the Green's function arise only from the evanescent part (2.34c).

We will prove (2.73) only when $(\xi, \zeta)$ is in the interior of the core, the other cases can be treated in a similar way. Since we are interested in the behaviour of the Green's function for $|p| \to 0$, we can suppose $|p| \le 1$.

In (2.34c), we split the path of integration in two intervals: $(k^2n_*^2, k^2n_*^2 + \Lambda^2)$ and $(k^2n_*^2 + \Lambda^2, +\infty)$, where $\Lambda > 0$ is fixed.

From Lemma 2.52 $v_j(x, \lambda)$, $j \in \{s, a\}$, are bounded for $(x, \lambda) \in [-h, h] \times [\lambda_0, +\infty)$ and we denote by $\Phi_*$ their maximum. Then

$$\begin{aligned} \left| \sum_{j \in \{s,a\}} \int_{k^2n_*^2}^{k^2n_*^2+\Lambda^2} \frac{e^{-|z-\zeta|\sqrt{\lambda - k^2n_*^2}}}{2\sqrt{\lambda - k^2n_*^2}} v_j(x, \lambda) v_j(\xi, \lambda) \sigma_j(\lambda) d\lambda \right| \\ \le \frac{\Phi_*^2}{2} \sum_{j \in \{s,a\}} \int_{k^2n_*^2}^{k^2n_*^2+\Lambda^2} \frac{\sigma_j(\lambda)}{\sqrt{\lambda - k^2n_*^2}} d\lambda = \frac{\Phi_*^2}{2} C_1(\Lambda). \end{aligned} \tag{2.74}$$

In estimating the integral over $(k^2n_*^2+\Lambda^2,+\infty)$, we need to use (2.75) and (2.76) below: it holds that

$$\sigma_j(\lambda)=\frac{1}{\sqrt{\lambda-k^2n_*^2}}+\mathcal{O}\left((\lambda-k^2n_*^2)^{-1}\right),\quad \text{as } \lambda\to+\infty, \tag{2.75}$$

$$\begin{aligned}\sum_{j\in\{s,a\}} v_j(x,\lambda)v_j(\xi,\lambda) &= \cos(x\sqrt{\lambda})\cos(\xi\sqrt{\lambda})\left(1+\mathcal{O}(1/\sqrt{\lambda})\right)\\ &\qquad + \sin(x\sqrt{\lambda})\sin(\xi\sqrt{\lambda})\left(1+\mathcal{O}(1/\sqrt{\lambda})\right)\\ &= \cos(\sqrt{\lambda}|x-\xi|)+\mathcal{O}(1/\sqrt{\lambda}),\quad \text{as } \lambda\to+\infty,\end{aligned} \tag{2.76}$$

uniformly for $x,\xi\in[-h,h]$. The above formulas are straightforward consequences of Corollary 2.14 and Lemma 2.12, respectively. Thus, for $|x|\le h$,

$$\Bigg|\sum_{j\in\{s,a\}}\int_{k^2n_*^2+\Lambda^2}^{+\infty}\frac{e^{-|z-\zeta|\sqrt{\lambda-k^2n_*^2}}}{2\sqrt{\lambda-k^2n_*^2}}v_j(x,\lambda)v_j(\xi,\lambda)\sigma_j(\lambda)d\lambda - \int_{k^2n_*^2+\Lambda^2}^{+\infty}\frac{e^{-|z-\zeta|\sqrt{\lambda-k^2n_*^2}}}{2(\lambda-k^2n_*^2)}\cos(\sqrt{\lambda}|x-\xi|)d\lambda\Bigg|\le C_2(\Lambda).$$

It only remains to study the integral

$$\begin{aligned}I&=\int_{k^2n_*^2+\Lambda^2}^{+\infty}\frac{e^{-|z-\zeta|\sqrt{\lambda-k^2n_*^2}}}{2(\lambda-k^2n_*^2)}\cos(\sqrt{\lambda}|x-\xi|)d\lambda\\ &=\mathrm{Re}\int_{k^2n_*^2+\Lambda^2}^{+\infty}\frac{e^{-|z-\zeta|\sqrt{\lambda-k^2n_*^2}+i|x-\xi|\sqrt{\lambda}}}{2(\lambda-k^2n_*^2)}d\lambda.\end{aligned}$$

We set

$$I'=\mathrm{Re}\int_{k^2n_*^2+\Lambda^2}^{+\infty}\frac{e^{-\sqrt{\lambda-k^2n_*^2}(|z-\zeta|-i|x-\xi|)}}{2(\lambda-k^2n_*^2)}d\lambda$$

and prove that $|I-I'|\le k^2n_*^2(2\Lambda)^{-1}$. In fact, from

$$e^{i|x-\xi|\sqrt{\lambda}}-e^{i|x-\xi|\sqrt{\lambda-k^2n_*^2}}=i\int_{|x-\xi|\sqrt{\lambda-k^2n_*^2}}^{|x-\xi|\sqrt{\lambda}}e^{is}ds,$$

it follows that

$$\begin{aligned}\left|e^{-|z-\zeta|\sqrt{\lambda-k^2n_*^2}+i|x-\xi|\sqrt{\lambda}}-e^{(-|z-\zeta|+i|x-\xi|)\sqrt{\lambda-k^2n_*^2}}\right|&=\left|ie^{-|z-\zeta|\sqrt{\lambda-k^2n_*^2}}\int_{|x-\xi|\sqrt{\lambda-k^2n_*^2}}^{|x-\xi|\sqrt{\lambda}}e^{is}ds\right|\\ &\le|x-\xi|(\sqrt{\lambda}-\sqrt{\lambda-k^2n_*^2}),\end{aligned}$$

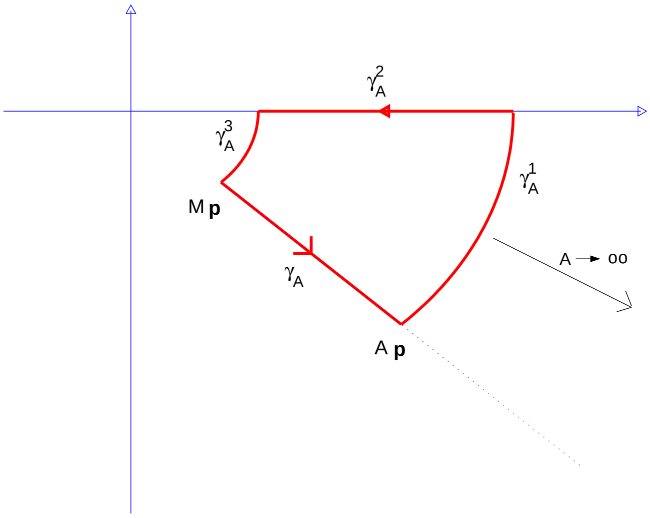


Figure 2.2: The path of integration $\gamma_A$.

and hence, if $|p| \leq 1$,

$$\left| e^{-|z-\zeta|\sqrt{\lambda-k^2n_*^2}+i|x-\xi|\sqrt{\lambda}} - e^{(-|z-\zeta|+i|x-\xi|)\sqrt{\lambda-k^2n_*^2}} \right| \leq \frac{k^2n_*^2}{2\sqrt{\lambda-k^2n_*^2}}.$$

Therefore we have

$$|I - I'| \leq \int_{k^2n_*^2+\Lambda^2}^{+\infty} \frac{k^2n_*^2}{4(\lambda-k^2n_*^2)^{\frac{3}{2}}} d\lambda,$$

and hence

$$|I - I'| \leq \frac{k^2n_*^2}{2\Lambda}. \tag{2.77}$$

Now, we estimate the integral $I'$. Let $p' = |z-\zeta| - i|x-\xi|$ and change the variable of integration inside $I'$:

$$I' = \mathrm{Re} \int_\gamma \frac{e^{-\mu}}{\mu} d\mu,$$

where $\mu = p'\sqrt{\lambda - k^2n_*^2}$ and $\gamma$ is the half-line starting from $\Lambda p'$ and going toward the direction of $p'$.

In order to calculate

$$\int_\gamma \frac{e^{-\mu}}{\mu} d\mu,$$

we truncate the path of integration at a point $Ap'$, $A > \Lambda$, and then we do the limit for $A \to +\infty$, as shown in Fig.2.8.

By the Residue's theorem we have that

$$\int_{\gamma_A} = -\int_{\gamma_A^1} - \int_{\gamma_A^2} - \int_{\gamma_A^3} \frac{e^{-\mu}}{\mu} d\mu.$$

We change the variable of integration in the integral over $\gamma_A^3$ by putting $\mu = \Lambda|p'|e^{i\vartheta}$, with $0 \leq \vartheta \leq \arg p'$. Since $0 \leq \arg p' \leq \pi/2$ we have

$$\left| -\int_{\gamma_A^3} \frac{e^{-\mu}}{\mu} d\mu \right| = \left| \int_{\arg p'}^{0} ie^{-\Lambda|p|e^{i\vartheta}} d\vartheta \right| \leq \frac{\pi}{2}. \tag{2.78a}$$

In the same way, by setting $\mu = A|p'|e^{i\vartheta}$ in the integral over $\gamma_A^2$, we find

$$\left| \int_{\gamma_A^1} \frac{e^{-\mu}}{\mu} d\mu \right| \leq \frac{\pi}{2}. \tag{2.78b}$$

By using (2.78a) and (2.78b) and taking the limit for $A \to \infty$ we have

$$\left| \int_{\gamma} \frac{e^{-\mu}}{\mu} d\mu - \int_{+\infty}^{\Lambda|p'|} \frac{e^{-\mu}}{\mu} d\mu \right| \leq \pi.$$

Since

$$\begin{aligned} \int_{\Lambda|p'|}^{+\infty} \frac{e^{-\mu}}{\mu} d\mu &= [e^{-\mu} \ln \mu]_{\Lambda|p'|}^{+\infty} + \int_{\Lambda|p'|}^{+\infty} \ln \mu e^{-\mu} d\mu \\ &= -\ln|p'| + \mathcal{O}(1), \quad \text{as } |p'| \to 0, \end{aligned}$$

we have

$$\left| \int_{\gamma} \frac{e^{-\mu}}{\mu} d\mu - \ln|p'| \right| \leq \mathcal{O}(1), \quad \text{as } |p'| \to 0,$$

and then, since $|p'| = |p|$, we obtain (2.73). □

# Chapter 3

# An outgoing radiation condition

## 3.1 The problem of uniqueness for the Helmholtz equation

It is well known that the Dirichlet Problem

$$\tag{3.1} \begin{cases} \Delta u + k^2 u = f & \text{outside a close and bounded surface } \Sigma \subset \mathbb{R}^N \\ u = U & \text{on } \Sigma \end{cases}$$

has not an unique solution. If $k = 0$ (Poisson's equation), in order to obtain the uniqueness, it is required that the solution vanishes at infinity. If $k \neq 0$, this is no more sufficient. In fact, there are two different solutions of (3.1) which vanish at infinity, representing the outward and inward radiation. Hence, an additional (or different) condition is required. The first and simpler condition we can add is

$$\tag{3.2} \lim_{r\to\infty} r^{\frac{N-1}{2}} \left( \frac{\partial u}{\partial r} - iku \right) = 0,$$

uniformly, which is the so-called *Sommerfeld's radiation condition.* Here, $\frac{\partial u}{\partial r}$ denotes the radial derivative of $u$.

The physical meaning of this condition is that there are no sources of energy at infinity. Moreover, it assures that, far from the surface $\Sigma$, $u$ behaves as a wave generated by a point source.

Stated as in (3.2) and together with the assumption that $u$ vanishes at infinity, this condition is due to Sommerfeld, see [So1] and [So2] (see also [Mag1] and [Mag2]).

The vanishing assumption on $u$ was dropped by Rellich (see [Rel]), who also proved that (3.2) can be replaced by the weaker condition

$$\tag{3.3} \lim_{R\to\infty} \int\limits_{\partial B_R} \left| \frac{\partial u}{\partial r} - iku \right|^2 d\sigma = 0 \ ,$$

where $B_R$ is the ball centered in the origin with radius $R$. In the same paper, Rellich proved also that the radiation condition can be written in the integral form

$$\int_{R^N} \left| \frac{\partial u}{\partial r} - iku \right|^2 dx < +\infty. \tag{3.4}$$

Condition (3.4) can be considered the starting point for the results in this chapter, as we are going to explain shortly. Before describing our results, we cite some generalizations of the work of Rellich.

When $n$ is a function which is identically 1 outside a compact set, (3.3) still guarantees the uniqueness of

$$\Delta u + k^2 n(x)^2 u = f, \quad x \in \mathbb{R}^N, \tag{3.5}$$

see [Mi1] and [Sc] and references therein.

Several authors (see [Rel],[Mi2],[Mo],[RS],[Zh],[PV]) studied the case in which $n$ is no more constant at infinity, but has an angular dependency like $n(\mathbf{x}) \to n_\infty(\mathbf{x}/|\mathbf{x}|)$ and it approaches to the limit with a certain behaviour.

Among these papers, we want to mention the results in [Zh] and [PV], where the authors proved the uniqueness of (3.5) by means of the limiting absorbtion method and by introducing a Sommerfeld radiation condition of the form

$$\lim_{R\to+\infty} \frac{1}{R} \int_{B_R} \left| \frac{\partial u}{\partial r} - ikn_\infty u \right|^2 dx = 0.$$

In this chapter we will study the uniqueness of the solutions of the Helmholtz equation when $n$ is as in (RI). We observe that such a function $n$ is not considered in the works cited before. In fact, it is easy to show that a pure guided mode supported by the Helmholtz equation does not satisfy the Sommerfeld radiation conditions above. A function $n$ similar to (RI) was considered by Jäger and Saitō in [JS1]-[JS3] but, again, the authors added assumptions on $n$ in order to avoid the presence of guided modes in the wave propagation.

As far as we know, the only works dealing with uniqueness, in an optical waveguide setting, have appeared in the Russian literature (see [Rei],[No],[NS],[KNH] and references therein). However, the *Reichardt condition* studied there in only deals with guided modes and does not apply to the radiating energy.

The main result of this chapter is Theorem 3.1, where we present a new radiation condition that guarantees the uniqueness of solution of (H) with $n$ given by (RI). We observe that, if we suppose that no guided modes are present (this is possible by choosing special parameters in the function $n$), our radiation condition reduces to (3.4). In this setting, our results provide a different proof of special cases studied in [Rel] and [JS1]-[JS3].

The key ingredients of our proof are essentially 3: (i) if (H) possesses two solutions satisfying our radiation condition, then their difference $w$ must belong to the space $H^2(\mathbb{R}^2)$; (ii) as a consequence of (i), the Fourier transform of $w$ in the $z$ direction (parallel to the fiber's axis) is square integrable for almost all $x \in \mathbb{R}$ and satisfies an ordinary differential equation in $x$; (iii) the only square integrable solution of such an equation is identically zero.

The chapter is organized as follows. In Section 3.2 we introduce some notations. The main results are proved in Section 3.3. In Section 3.4 we give a sufficient condition for the technical assumption (3.11) which is needed to prove Theorem 3.1.

## 3.2 Preliminaries

We consider the Helmholtz equation (H), where $n$ is given by (RI). From the analysis done in Chapter 2, we know that it exists a finite number of solutions of

$$\Delta u + k^2 n(x)^2 u = 0, \quad (x, z) \in \mathbb{R}^2, \tag{3.6}$$

of the form

$$u(x, z) = v_j(x, \lambda_m^j) e^{\pm i\beta_m^j z}, \quad m = 1, \ldots, M_j,\ j \in \{s, a\},$$

with $\lambda_m^j = k^2[n_*^2 - (\beta_m^j)^2]$; $v_j(x, \lambda)$ have the properties:

(V1) $v_j(x, \lambda_m^j) \in L^2(\mathbb{R}) \cap C^1(\mathbb{R})$;

(V2) $v_j(x, \lambda_m^j)$ vanishes exponentially as $|x| \to \infty$;

(V3) $v_j$ satisfies (2.8) .

To simplify notations, we denote by $\gamma_l,\ l = 1, \ldots, M,\ M = M_s + M_a$, the values $\lambda_m^j,\ m = 1, \ldots, M_j,\ j = s, a$, and by $\gamma_*$ their maximum.

We set

$$e(x, \gamma_l) = \frac{v_j(x, \gamma_l)}{\|v_j(\cdot, \gamma_l)\|_2}, \tag{3.7}$$

where we choose $v_j,\ j \in \{s, a\}$, according to the $\lambda_m^j$ corresponding to $\gamma_l$, and define

$$u_l(x, z) = e(x, \gamma_l) U(z, \gamma_l), \tag{3.8}$$

with

$$U(z, \gamma_l) = \int_{-\infty}^{\infty} u(\xi, z) e(\xi, \gamma_l) d\xi, \quad l = 1, \ldots, M, \tag{3.9}$$

and

$$u_0(x,z) = u(x,z) - \sum_{l=1}^{M} u_l(x,z). \tag{3.10}$$

Here and in the following, we will assume that the solution $u$ of (3.5) satisfies the following technical assumption:

$$\lim_{|x|\to+\infty} u(x,z)e^{-|x|\sqrt{d^2-\gamma_*}} = 0, \qquad \lim_{|x|\to+\infty} u_x(x,z)e^{-|x|\sqrt{d^2-\gamma_*}} = 0. \tag{3.11}$$

In such a way, since $e(x,\gamma_l)$ vanishes exponentially as $|x| \to +\infty$, the integral defining $U(z,\gamma_l)$ converges.

We conclude this section by introducing some notation. We set

$$\begin{aligned} B_R &= \left\{(x,z) \in \mathbb{R}^2 : x^2 + z^2 \leq R^2\right\}, \\ Q_R &= \left\{(x,z) \in \mathbb{R}^2 : |x| \leq R \text{ and } |z| \leq R\right\}, \end{aligned}$$

and, for $l = 0, 1, \dots, M$,

$$\Omega_R^l = \begin{cases} B_R, & \text{if } l = 0, \\ Q_R, & \text{otherwise.} \end{cases} \tag{3.12}$$

Finally, $\nu$ will denote the outward normal to $\partial\Omega_R^l$.

## 3.3 Uniqueness theorem

**Theorem 3.1.** *There exists at most one weak solution $u$ of* (3.5) *which satisfies* (3.11) *and such that*

$$\sum_{l=0}^{M} \int_0^\infty \int_{\partial\Omega_\rho^l} \left|\frac{\partial u_l}{\partial \nu} - i\beta_l u_l\right|^2 d\sigma_l d\rho \leq C^2. \tag{3.13}$$

*Here $u_l$, $l = 0, 1, \dots, M$, are given by* (3.8) *and* (3.10).

For the proof of this theorem, we preliminary prove Lemma 3.2 and Theorem 3.3.

**Lemma 3.2.** *Let $\beta \in \mathbb{R}$ and $w$ be a weak solution of* (3.6). *Then*

$$\int_{\partial\Omega} \left|\frac{\partial w}{\partial \nu} - i\beta w\right|^2 d\sigma = \int_{\partial\Omega} \left(\left|\frac{\partial w}{\partial \nu}\right|^2 + \beta^2|w|^2\right) d\sigma, \tag{3.14}$$

*for every $\Omega \subset \mathbb{R}^2$ bounded and sufficiently smooth.*

*Proof.* Since $w$ is a weak solution of (3.6), we have that

$$-\int_{\mathbb{R}^2} \nabla w \cdot \nabla \varphi dx + k^2 \int_{\mathbb{R}^2} n(x)^2 w \varphi dx = 0, \tag{3.15}$$

for every $\varphi \in H_0^1(\mathbb{R}^2)$. By using a standard approximation argument, we can set $\varphi = \mathcal{X}_\Omega \bar{w}$ and obtain that

$$\int_{\partial\Omega} \bar{w} \frac{\partial w}{\partial \nu} d\sigma - \int_{\Omega} |\nabla w|^2 dx + k^2 \int_{\Omega} n(x)^2 |w|^2 dx = 0,$$

where we used $\nabla \mathcal{X}_\Omega = -\nu \mathcal{X}_{\partial\Omega}$ (in the sense of distributions).

By taking the imaginary part of the above equation we find

$$\mathrm{Im} \int_{\partial\Omega} \bar{w} \frac{\partial w}{\partial \nu} d\sigma = 0,$$

which implies (3.14). □

**Theorem 3.3.** *Let $w$ be a weak solution of* (3.6) *satisfying* (3.13) *and* (3.11). *Then*

$$\sum_{l=0}^{M} \int_0^{+\infty} d\rho \int_{\partial\Omega_\rho^l} \left[ \left| \frac{\partial w_l}{\partial \nu} \right|^2 + \beta_l^2 |w_l|^2 \right] d\sigma_l \le C^2, \tag{3.16}$$

*and, in particular,*

$$\int_{\mathbb{R}^2} |w_l|^2 dx dz \le C^2, \tag{3.17}$$

*for every $l = 0, 1, 2, \ldots, M$.*

*Proof.* By Lemma 3.2, it is enough to prove that each $w_l$, $l = 0, 1, \ldots, M$, satisfies (3.6). Then, (3.16) and (3.17) follow from (3.14) and (3.13).

Suppose $l \ge 1$. Since $w$ is a weak solution of (3.6), we set $\varphi(x, z) = e(x, \gamma_l)\eta(z)$ in (3.15), with $\eta \in C_c^1(\mathbb{R})$:

$$-\int_{\mathbb{R}^2} w_x e' \eta dx dz - \int_{\mathbb{R}^2} w_z e \eta' dx dz + k^2 \int_{\mathbb{R}^2} n(x)^2 w e \eta dx dz = 0;$$

integration by parts gives

$$\int_{\mathbb{R}^2} w e'' \eta dx dz - \int_{\mathbb{R}^2} w_z e \eta' dx dz + k^2 \int_{\mathbb{R}^2} n(x)^2 w e \eta dx dz = 0.$$

Since $e(x, \gamma_l)$ satisfies (2.8), we obtain

$$-\int_{\mathbb{R}^2} w_z e \eta' dx dz + (k^2 n_*^2 - \gamma_l) \int_{\mathbb{R}^2} w e \eta dx dz = 0,$$

and thus, from (3.9),

$$-\int_{\mathbb{R}} W_z(z,\gamma_l)\eta'(z)dz + (k^2n_*^2-\gamma_l)\int_{\mathbb{R}} W(z,\gamma_l)\eta(z)dz = 0, \tag{3.18}$$

for every $\eta \in C_c^1(\mathbb{R})$.

Now, we prove that $w_l(x,z)$, $l=1,\dots,M$, is a weak solution of (3.6): by setting $\varphi(x,z)=\psi(x)\eta(z)$, with $\psi,\eta\in C_c^1(\mathbb{R})$, and integrating by parts, we have

$$\begin{aligned}
-\int_{\mathbb{R}^2} \nabla w_l\cdot\nabla\varphi dx &+ k^2\int_{\mathbb{R}^2} n(x)^2 w_l\varphi dx \\
&= -\int_{\mathbb{R}^2} We'\psi'\eta dxdz - \int_{\mathbb{R}^2} W_z e\psi\eta' dxdz + k^2\int_{\mathbb{R}^2} n(x)^2 We\psi\eta dxdz \\
&= \int_{\mathbb{R}^2} We''\psi\eta dxdz - \int_{\mathbb{R}^2} W_z e\psi\eta' dxdz + k^2\int_{\mathbb{R}^2} n(x)^2 We\psi\eta dxdz = 0,
\end{aligned}$$

which follows from (2.8) and (3.18).

Since $w$ and $w_l$, $l=1,\dots,M$, satisfy (3.6), the same holds for $w_0$. Thus, as already mentioned, we can apply Lemma 3.2 to each $w_l$, $l=0,1,\dots,M$, and obtain

$$\sum_{l=0}^{M}\int_{\partial\Omega_\rho^l}\Big|\frac{\partial w_l}{\partial\nu}-i\beta_l w_l\Big|^2 d\sigma_l = \sum_{l=0}^{M}\int_{\partial\Omega_\rho^l}\left(\Big|\frac{\partial w_l}{\partial\nu}\Big|^2+\beta_l^2|w_l|^2\right)d\sigma_l,$$

for every $\rho>0$, and then, since $w$ satisfies (3.13), we get (3.16) and (3.17). □

Now, we are ready to prove the Theorem 3.1.

*Proof of Theorem 3.1.* Let $u_1$ and $u_2$ be two solutions of (3.5) which satisfy the assumptions of the theorem. We set $w=u_1-u_2$ and observe that $w$ satisfies (3.6) and (3.13), as follows from the triangle inequality ((3.13) defines a seminorm).

From Theorem 3.3 we have that $w\in L^2(\mathbb{R}^2)$ and, by using Lemmas 4.1 and 4.3, we get $w\in H^2(\mathbb{R}^2)$. Therefore, $w(x,\cdot)\in L^2(\mathbb{R})$ for almost every $x\in\mathbb{R}$, and the same holds for $w_x(x,\cdot)$ and $w_{xx}(x,\cdot)$. Hence, we can transform (3.6) by using the Fourier transform in the $z$-coordinate

$$\hat{w}(x,t)=\int_{-\infty}^{+\infty} w(x,z)e^{-izt}dz, \quad \text{for a.e. } x\in\mathbb{R},$$

and obtain:

$$\hat{w}_{xx}(x,t)+[k^2n(x)^2-t^2]\hat{w}(x,t)=0, \quad \text{a.e. } x\in\mathbb{R}. \tag{3.19}$$

From Fubini-Tonelli's theorem, the integrals

$$\int_{\mathbb{R}^2}|\hat{w}(x,t)|^2dxdt, \quad \int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}|\hat{w}(x,t)|^2dx \ \text{ and } \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}|\hat{w}(x,t)|^2dt$$

have the same value, finite or infinite.

Since $w(x,\cdot)$ belongs to $L^2(\mathbb{R})$ for almost every $x \in \mathbb{R}$, the same holds for $\hat{w}(x,\cdot)$ and, furthermore, we have

$$\int_{-\infty}^{+\infty} |\hat{w}(x,t)|^2 dt = 2\pi \int_{-\infty}^{+\infty} |w(x,z)|^2 dz \quad \text{a.e. } x \in \mathbb{R}.$$

By integrating the above equation and using Fubini-Tonelli's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} |\hat{w}(x,t)|^2 dx dt &= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} |\hat{w}(x,t)|^2 dt = 2\pi \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} |w(x,z)|^2 dz \\ &= 2\pi \int_{\mathbb{R}^2} |w(x,z)|^2 dx dz < +\infty.\end{aligned}$$

Therefore $\hat{w}(\cdot,t) \in L^2(\mathbb{R})$ for almost every $t \in \mathbb{R}$.

From (3.19), it follows that

$$\hat{w}(x,t) = a(t)\cos\sqrt{\lambda - d^2}(x-h) + b(t)\sin\sqrt{\lambda - d^2}(x-h), \quad \text{a.e. } x > h,$$

where $\lambda = k^2 n_*^2 - t^2$ and $d^2 = k^2(n_*^2 - n_{cl}^2)$. Since

$$\int_{-\infty}^{+\infty} |\hat{w}(x,t)|^2 dx \geq \int_{h}^{+\infty} |\hat{w}(x,t)|^2 dx,$$

we obtain that $\hat{w}(x,t)$ can be not identically zero only for some values $0 < \lambda_j^m \leq d^2$, and furthermore, in that case

$$\hat{w}(x,t) = a(t) v_s(x, \lambda_s^m) + b(t) v_a(x, \lambda_a^m).$$

Hence we should have

$$w(x,z) = A Z_s(z) v_s(x, \lambda_s^m) + B Z_a(z) v_a(x, \lambda_a^m),$$

where $Z_j(z) = e^{\pm\sqrt{k^2 n_*^2 - \lambda_j^m} z}$, since $w$ is a solution of (3.6). Thus, $w \notin L^2(\mathbb{R}^2)$, because $w(x,\cdot) \notin L^2(\mathbb{R})$. Therefore we found a contradiction due to having supposed $w \not\equiv 0$.

□

## 3.4 Remarks

In the following lemma, we give a sufficient condition for (3.11), which is needed to prove Theorem (3.1). In particular, we shall show that (3.11) follows by assuming that $w$ belongs to the weighted Lebesgue space $L^2(\mu)$, i.e. $w$ is such that

$$\int_{\mathbb{R}^2} |w(x,z)|^2 \mu(x,z) dx dz < +\infty,$$

where $\mu$ is a weight function of the form

$$\mu(x,z) = \frac{C_1^2}{(C_2^2 + x^2 + z^2)^{2\gamma}}, \tag{3.20}$$

with $\gamma > 0$. Such a weighted Lebesgue space will play a crucial role in Chapter 4 where, in Corollary 4.7, we will prove that the solution found in (2.33) belongs to $L^2(\mu)$, for $\gamma > 1/2$.

**Lemma 3.4.** *Let $w \in L^2(\mu)$ be a solution of* (3.6)*. Then $w$ satisfies* (3.11)*.*

*Proof.* Since $w$ is a solution of (3.6), from from Lemmas 4.1 and 4.3 we have that $w \in H^2(\mu)$. Thus , it easily follows that the function $\Psi = w\mu^{\frac{1}{2}}$ belongs to $H^2(\mathbb{R}^2)$. From the Sobolev Imbedding Theorem (see Theorem 4.12 in [AF]) we have that $\Psi \in L^\infty(\mathbb{R}^2)$, i.e. $w$ may grow at infinity at most as $(x^2+z^2)^{\frac{\gamma}{2}}$, which implies the first condition in (3.11).

Now, we are going to obtain an analogous result for the gradient of $w$, which will imply the second condition in (3.11). From

$$\nabla\mu^{\frac{1}{2}} = -\frac{2\gamma\mu^{\frac{1}{2}}}{C_2^2 + x^2 + z^2}\,(x,z), \quad \Delta\mu^{\frac{1}{2}} = \frac{4\gamma(\gamma+1)(x^2+z^2)\mu^{\frac{1}{2}}}{(C_2^2 + x^2 + z^2)^2} - \frac{4\gamma\mu^{\frac{1}{2}}}{C_2^2 + x^2 + z^2},$$

and by using (3.6), we have that $\Psi$ satisfies the following equation

$$\begin{aligned}\Delta\Psi + \frac{4\gamma}{C_2^2 + x^2 + z^2}\,(x,z)\cdot\nabla\Psi + \\ \left[k^2 n(x)^2 + \frac{4\gamma}{C_2^2 + x^2 + z^2} + \frac{4\gamma(\gamma-1)(x^2+z^2)}{(C_2^2 + x^2 + z^2)^2}\right]\Psi = 0.\end{aligned} \tag{3.21}$$

Since $\Psi \in H^2(\mathbb{R}^2)$, from Theorem 8.10 in [GT] we have

$$\|\Psi\|_{W^{3,2}(H'_+)} \le C'\|\Psi\|_{W^{2,2}(H_+)},$$

where we set $H_+ = \{(x,z) \in \mathbb{R}^2 | x \ge h\}$ and $H'_+ = \{(x,z) \in \mathbb{R}^2 | x \ge h'\}$, with $h' > h$, and $C'$ is a constant independent from $\psi$. Again, we apply the Sobolev Imbedding Theorem and get that $|\nabla\Psi|$ is bounded and thus the second condition in (3.11) holds for $w$ as $x \to +\infty$. In an analogous way, one proves that the same result holds as $x \to -\infty$ and the proof is complete. □

# Chapter 4

# Non-Rectilinear Waveguides. Mathematical Framework

## 4.1 Framework description

When a rectilinear waveguide has some imperfection (hopefully small) or when the waveguide slightly bends from the rectilinear position, we cannot assume that its index of refraction $n$ depends only on the transversal coordinate $x$, see Figure 4.1. From the mathematical point of view, in this case, we shall study the Helmholtz equation

$$\Delta u + k^2 n_\varepsilon(x, z)^2 u = f, \quad \text{in } \mathbb{R}^2, \tag{4.1}$$

where $n_\varepsilon(x, z)$ is a perturbation of the function $n_0(x)$ defined in (2.3), representing a "perfect" rectilinear configuration.

We denote by $L_0$ and $L_\varepsilon$ the Helmholtz operators corresponding to $n_0(x)$ and $n_\varepsilon(x, z)$ respectively:

$$L_0 = \Delta + k^2 n_0(x)^2, \quad L_\varepsilon = \Delta + k^2 n_\varepsilon(x, z)^2. \tag{4.2}$$

As a first approximation, our problem can be described by studying $L_0 u = f$. If we are interested in a better approximation, we shall study the equation $L_\varepsilon u = f$.

In Chapter 2, we found a resolution formula for

$$L_0 u = f,$$

i.e. we were able to write explicitly (in terms of a Green's function) the operator $L_0^{-1}$ and then a solution of (H). Now, we want to use $L_0^{-1}$ to write higher order approximations of solutions of (4.1), i.e. of

$$L_\varepsilon u = f. \tag{4.3}$$

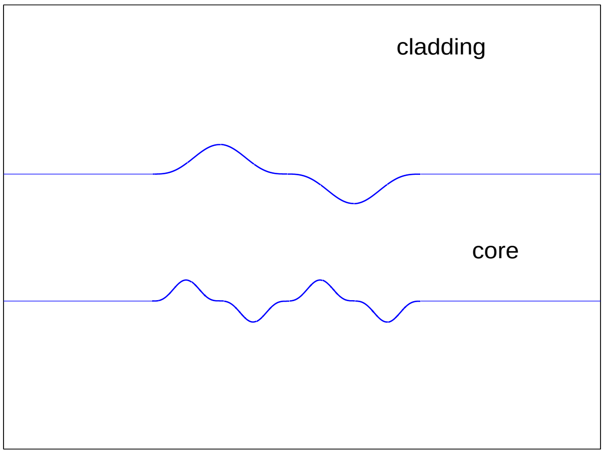


Figure 4.1: A waveguide with imperfections. We cannot assume anymore that the index of refraction $n$ depends only on the transversal coordinate $x$.

In order to do this, we formally represent $u$ and $L_\varepsilon$ by their Neumann series

$$u = u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots,$$
$$L_\varepsilon = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \dots,$$

where $\varepsilon > 0$ is supposed to be small. By using the above formulas and equating the asymptotic terms of the same order, we can solve (4.3) by iteration:

$$\begin{aligned} L_0 u_0 &= f, \\ L_0 u_1 &= -L_1 u_0, \\ &\vdots \\ L_0 u_j &= -\sum_{r=0}^{j-1} L_{j-r} u_r, \\ &\vdots \end{aligned} \tag{4.4}$$

Every step of (4.4) can be solved by using the formula already derived for $L_0^{-1}$.

The above method gives us an algorithm to compute $u_j$ for $j = 0, 1, \dots$. Then the asymptotic expansion of the solution $u$ in terms of $\varepsilon$ can be calculated up to the desired order.

The existence of a solution of (4.3) will be proven in Theorem 4.9 by using a standard fixed point argument: since (4.3) is equivalent to

$$L_0 u = f + (L_0 - L_\varepsilon) u,$$

then we have

$$u = L_0^{-1} f + \varepsilon L_0^{-1} \left( \frac{L_0 - L_\varepsilon}{\varepsilon} \right) u.$$

Our goal is to find suitable function spaces on which $L_0^{-1}$ and $\frac{L_0 - L_\varepsilon}{\varepsilon}$ are continuous; then, by choosing $\varepsilon$ sufficiently small, the existence of a solution will follow by the contraction mapping theorem.

Crucial to our construction are the estimates contained in Sections 4.3, in particular the ones in Theorem 4.7.

Section 4.2 contains some estimates of the solution of the Helmholtz equation in $\mathbb{R}^N$, $N \geq 2$, that we need in Theorem 4.9.

## 4.2 Regularity results

In this section we study the global regularity of weak solutions of the Helmholtz equation. Since our results hold in $\mathbb{R}^N$, $N \geq 2$, it will be useful to denote a point in $\mathbb{R}^N$ by $x$, i.e. $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$.

Let $\mu : \mathbb{R}^N \to \mathbb{R}$ be a positive function. We will denote by $L^2(\mu)$ the weighted space consisting of all the complex valued measurable functions $u(x)$, $x \in \mathbb{R}^N$, such that

$$\mu^{\frac{1}{2}} u \in L^2(\mathbb{R}^N),$$

equipped with the natural norm

$$\|u\|^2_{L^2(\mu)} = \int_{\mathbb{R}^N} |u(x)|^2 \mu(x) dx.$$

In a similar way we define the weighted Sobolev spaces $H^1(\mu)$ and $H^2(\mu)$. The norms in $H^1(\mu)$ and $H^2(\mu)$ are given respectively by:

$$\|u\|^2_{H^1(\mu)} = \int_{\mathbb{R}^N} |u(x)|^2 \mu(x) dx + \int_{\mathbb{R}^N} |\nabla u(x)|^2 \mu(x) dx,$$

and

$$\|u\|^2_{H^2(\mu)} = \int_{\mathbb{R}^N} |u(x)|^2 \mu(x) dx + \int_{\mathbb{R}^N} |\nabla u(x)|^2 \mu(x) dx + \int_{\mathbb{R}^N} |\nabla^2 u(x)|^2 \mu(x) dx.$$

The proofs in this section hold true whenever the (non-negative) weight $\mu$ has the following properties:

$$\begin{gathered} \mu \in C^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \\ |\nabla \mu| \leq C_1 \mu, \quad |\nabla^2 \mu| \leq C_2 \mu, \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{4.5}$$

where $C_1$ and $C_2$ are positive constants.

By taking into account the symmetries of the problem, when $N = 2$, a good choice of $\mu$ is given by

$$\mu(x) = \mu_1(x_1)\mu_2(x_2). \tag{4.6}$$

We will suppose $f \in L^2(\mu^{-1})$. Since $\mu$ is bounded, it is clear that $f \in L^2(\mu)$ too.

**Lemma 4.1.** *Let $u \in H^1_{loc}(\mathbb{R}^N)$ be a weak solution of*

$$\Delta u + k^2 n(x)^2 u = f, \quad x \in \mathbb{R}^N, \tag{4.7}$$

*with $n \in L^\infty(\mathbb{R}^N)$. Let $\mu$ satisfy the assumptions in (4.5). Then*

$$\int\limits_{\mathbb{R}^N} |\nabla u|^2 \mu dx \leq \frac{1}{2} \int\limits_{\mathbb{R}^N} |f|^2 \mu dx + \left(2C_2 + k^2 n_*^2 + \frac{1}{2}\right) \int\limits_{\mathbb{R}^N} |u|^2 \mu dx, \tag{4.8}$$

*where $n_* = \|n\|_{L^\infty(\mathbb{R}^N)}$.*

*Proof.* Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be such that

$$\eta(0) = 1, \quad 0 \leq \eta \leq 1, \quad |\nabla \eta| \leq 1, \quad |\nabla^2 \eta| \leq 1, \tag{4.9}$$

and consider the function defined by

$$\mu_m(x) = \mu(x)\eta\left(\frac{x}{m}\right). \tag{4.10}$$

Then $\mu_m(x)$ increases with $m$ and converges to $\mu(x)$ as $m \to +\infty$; furthermore

$$\begin{aligned} |\nabla \mu_m(x)| &\leq |\nabla \mu(x)| + \frac{1}{m}\mu(x) \leq \left(C_1 + \frac{1}{m}\right)\mu(x), \\ |\nabla^2 \mu_m(x)| &\leq |\nabla^2 \mu(x)| + \frac{2}{m}|\nabla \mu(x)| + \frac{1}{m^2}\mu(x) \leq \left(C_2 + \frac{2C_1}{m} + \frac{1}{m^2}\right)\mu(x), \end{aligned} \tag{4.11}$$

for every $x \in \mathbb{R}^N$.

Since $u$ is a weak solution of (4.7), we have that

$$\int\limits_{\mathbb{R}^N} \nabla u \cdot \nabla \phi dx - k^2 \int\limits_{\mathbb{R}^N} n(x)^2 u \phi dx = -\int\limits_{\mathbb{R}^N} f \phi dx,$$

for every $\phi \in H^1_{\text{loc}}(\mathbb{R}^N)$. We choose $\phi = \bar{u}\mu_m$ and obtain by Theorem 6.16 in [LL]:

$$\int\limits_{\mathbb{R}^N} |\nabla u|^2 \mu_m dx = -\int\limits_{\mathbb{R}^N} \bar{u} \nabla u \cdot \nabla \mu_m dx + k^2 \int\limits_{\mathbb{R}^N} n(x)^2 |u|^2 \mu_m dx - \int\limits_{\mathbb{R}^N} f \bar{u} \mu_m dx. \tag{4.12}$$

Integration by parts gives:

$$\operatorname{Re} \int\limits_{\mathbb{R}^N} \bar{u} \nabla u \cdot \nabla \mu_m dx = -\frac{1}{2} \int\limits_{\mathbb{R}^N} |u|^2 \Delta \mu_m dx;$$

hence, by considering the real part of (4.12), we obtain:

$$\int_{\mathbb{R}^N} |\nabla u|^2 \mu_m dx \leq 2\left(C_2 + \frac{2C_1}{m} + \frac{1}{m^2}\right) \int_{\mathbb{R}^N} |u|^2 \mu dx + k^2 n_*^2 \int_{\mathbb{R}^N} |u|^2 \mu_m dx + \left(\int_{\mathbb{R}^N} |f|^2 \mu_m dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |u|^2 \mu_m dx\right)^{\frac{1}{2}};$$

here we have used (4.11) and Hölder inequality.

Young inequality and the fact that $\mu_m \leq \mu$ then yield:

$$\int_{\mathbb{R}^N} |\nabla u|^2 \mu_m dx \leq \left[2\left(C_2 + \frac{2C_1}{m} + \frac{1}{m^2}\right) + k^2 n_*^2 + \frac{1}{2}\right] \int_{\mathbb{R}^N} |u|^2 \mu dx + \frac{1}{2}\int_{\mathbb{R}^N} |f|^2 \mu dx.$$

The conclusion then follows by the monotone convergence theorem. □

**Lemma 4.2.** *The following identity holds for every* $u \in H^2_{loc}(\mathbb{R}^N)$ *and every* $\phi \in C^2_0(\mathbb{R}^N)$*:*

$$\int_{\mathbb{R}^N} |\Delta u|^2 \phi dx + \int_{\mathbb{R}^N} |\nabla u|^2 \Delta\phi dx = \int_{\mathbb{R}^N} |\nabla^2 u|^2 \phi dx + \mathrm{Re} \int_{\mathbb{R}^N} (\nabla^2 \phi \nabla u, \nabla u) dx. \tag{4.13}$$

*Proof.* It is obvious that, without loss of generality, we can assume that $u \in C^3(\mathbb{R}^N)$; a standard approximation argument will then lead to the conclusion.

For $u \in C^3(\mathbb{R}^N)$, (4.13) follows by integrating over $\mathbb{R}^N$ the differential identity

$$\begin{aligned}\phi \sum_{i,j=1}^N u_{ii}\bar{u}_{jj} - \phi \sum_{i,j=1}^N u_{ij}\bar{u}_{ij} + \sum_{i,j=1}^N u_i\bar{u}_i\phi_{jj} - \mathrm{Re}\sum_{i,j=1}^N u_i\bar{u}_j\phi_{ij} \\ = \mathrm{Re}\left\{\sum_{i,j=1}^N [(\phi\bar{u}_j u_{ii})_j + (u_i\bar{u}_i\phi_j)_j - (\phi\bar{u}_j u_{ij})_i - (u_j\bar{u}_i\phi_j)_i]\right\},\end{aligned}$$

and by divergence theorem. □

**Lemma 4.3.** *Let* $u \in H^1(\mu)$ *be a weak solution of* (4.7)*. Then*

$$\int_{\mathbb{R}^N} |\nabla^2 u|^2 \mu dx \leq 2\int_{\mathbb{R}^N} |f|^2 \mu dx + 2k^4 n_*^4 \int_{\mathbb{R}^N} |u|^2 \mu dx + 4C_2 \int_{\mathbb{R}^N} |\nabla u|^2 \mu dx. \tag{4.14}$$

*where* $C_2$ *is the constant in* (4.11)*.*

*Proof.* From well-known interior regularity results on elliptic equations (see Theorem 8.8 in [GT]), we have that if $u \in H^1_{\text{loc}}(\mathbb{R}^N)$ is a weak solution of (4.7), then $u \in H^2_{\text{loc}}(\mathbb{R}^N)$. Then we can apply Lemma 4.2 to $u$ by choosing $\phi = \mu_m$:

$$\int_{\mathbb{R}^N} |\Delta u|^2 \mu_m dx + \int_{\mathbb{R}^N} |\nabla u|^2 \Delta\mu_m dx = \int_{\mathbb{R}^N} |\nabla^2 u|^2 \mu_m dx + \mathrm{Re}\int_{\mathbb{R}^N} (\nabla^2 \mu_m \nabla u, \nabla u) dx.$$

From (4.7), (4.11) and the above formula, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla^2 u|^2 \mu_m dx &= \int_{\mathbb{R}^N} |f - k^2 n(x)^2 u|^2 \mu_m dx \\
&\quad + \int_{\mathbb{R}^N} |\nabla u|^2 \Delta \mu_m dx - 2\operatorname{Re} \int_{\mathbb{R}^N} (\nabla^2 \mu_m \nabla u, \nabla u) dx \\
&\le 2 \int_{\mathbb{R}^N} |f|^2 \mu_m dx + 2k^4 n_*^4 \int_{\mathbb{R}^N} |u|^2 \mu_m dx \\
&\quad + 2\left(C_2 + \frac{2C_1}{m} + \frac{1}{m^2}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \mu dx + 2 \int_{\mathbb{R}^N} |\nabla^2 \mu_m| |\nabla u|^2 dx \\
&\le 2 \int_{\mathbb{R}^N} |f|^2 \mu_m dx + 2k^4 n_*^4 \int_{\mathbb{R}^N} |u|^2 \mu_m dx \\
&\quad + 4\left(C_2 + \frac{2C_1}{m} + \frac{1}{m^2}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \mu dx.
\end{aligned}
$$

Since $\mu_m \le \mu$, the conclusion of the proof follows by taking the limit as $m \to \infty$. □

## 4.3 Existence of a solution

In this section we will consider the case $N = 2$. For the sake of simplicity, we will assume that $\mu$ is given by (4.6). Analogous results hold for every $\mu$ satisfying (4.5) and such that

$$\mu(x,z) \le \mu_1(x)\mu_2(z), \tag{4.15}$$

with $\mu_j \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}),\ j = 1, 2$.

Before starting with the estimates on the solution $u$, we prove a preliminary result on the boundness of the guided and radiated part of the Green's function.

**Lemma 4.4.** *Let $G^g$ and $G^r$ be the functions defined in* (2.34a) *and* (2.34b)*, respectively. Then*

$$|G^g(x,z;\xi,\zeta)| \le \Phi_*^2 \sum_{j\in\{s,a\}} \sum_{m=1}^{M_j} \frac{r_m^j}{2\sqrt{k^2 n_*^2 - \lambda_m^j}}, \tag{4.16a}$$

*and*

$$|G^r(x,z;\xi,\zeta)| \le \max\left\{\frac{1}{\pi},\ \frac{1}{4\sqrt{\pi}} \sum_{j\in\{s,a\}} \Phi_* \Upsilon_j,\ \frac{1}{2\pi} \sum_{j\in\{s,a\}} \Phi_*^2 \Upsilon_j^2 \right\}. \tag{4.16b}$$

*Here,*

$$\Upsilon_j = \left( \int_{d^2}^{k^2 n_*^2} \frac{\sigma_j(\lambda)}{2\sqrt{k^2 n_*^2 - \lambda}} d\lambda \right)^{\frac{1}{2}}, \quad j \in \{s, a\},$$

*where, as in Lemma 2.13, $\Phi_*$ denotes the maximum of $|\phi_j(x,\lambda)|$ for $x \in [-h, h]$ and $\lambda \in \{\lambda_j^m\}_{m=1,\dots,M_j} \cup [d^2, +\infty)$, with $j \in \{s, a\}$.*

*Proof.* Since $G^g$ is a finite sum, from Remark 2.5 and Lemma 2.13, it is easy to deduce (4.16a).

In the study of $G^r$ we have to distinguish three different cases, according to whether $(x, z)$ and $(\xi, \zeta)$ belong to the core or not. Furthermore, we observe that $\Upsilon_j < +\infty$ as follows form Lemma 2.15.

Case 1: $x, \xi \in [-h, h]$. From Lemma 2.13 we have that $v_j(x, \lambda)$ are bounded by $\Phi_*$. From (2.34b) we have

$$|G^r| \leq \frac{1}{2\pi} \sum_{j\in\{s,a\}} \Phi_*^2 \Upsilon_j^2.$$

Case 2: $|x|, |\xi| \geq h$. We can use the explicit formula for $v_j$ (see (2.22)) and obtain

$$|v_j(x, \lambda)| \leq \sqrt{\phi_j(h, \lambda)^2 + Q^{-2} \phi_j'(h, \lambda)^2} = [Q\, \sigma_j(\lambda)]^{-\frac{1}{2}}.$$

Therefore we have

$$|G^r(x, z; \xi, \zeta)| \leq \frac{1}{2\pi} \sum_{j\in\{s,a\}} \int_{d^2}^{k^2 n_*^2} \frac{d\lambda}{2\sqrt{\lambda - d^2}\sqrt{k^2 n_*^2 - \lambda}} = \frac{1}{2},$$

and hence (4.16b) follows.

Case 3: $|x| \leq h$ and $|\xi| \geq h$. We estimate $|v_j(x, \lambda)|$ by $\Phi_*$ and $v_j(\xi, \lambda)$ by $[Q\sigma_j(\lambda)]^{-\frac{1}{2}}$, and write:

$$\begin{aligned} |G^r(x, z; \xi, \zeta)| &\leq \frac{1}{2\pi} \Phi_* \sum_{j\in\{s,a\}} \int_{d^2}^{k^2 n_*^2} \frac{1}{2\sqrt{k^2 n_*^2 - \lambda}} \left[ \frac{\sigma_j(\lambda)}{Q} \right]^{\frac{1}{2}} d\lambda \\ &\leq \frac{1}{4\pi} \Phi_* \sum_{j\in\{s,a\}} \Upsilon_j \left( \int_{d^2}^{k^2 n_*^2} \frac{d\lambda}{\sqrt{\lambda - d^2}\sqrt{k^2 n_*^2 - \lambda}} \right)^{\frac{1}{2}}. \end{aligned}$$

Again (4.16b) follows. □

In the next lemma we prove some estimate that will be useful in Theorem 4.7.

**Lemma 4.5.** *For $j, l \in \{s, a\}$ and $\lambda, \eta \geq k^2 n_*^2$, let*

$$p_{j,l}(\lambda, \eta) = \int_{-\infty}^{\infty} v_j(x, \lambda) v_l(x, \eta) \mu_1(x) dx, \tag{4.17}$$

*and*

$$q(\lambda, \eta) = \int_{-\infty}^{\infty} (e_{\lambda,\eta} \star \mu_2)(z) \mu_2(z) dz, \tag{4.18}$$

*where $e_{\lambda,\eta}(z) = e^{-|z|(\sqrt{\lambda - k^2 n_*^2} + \sqrt{\eta - k^2 n_*^2})}$.*

*Then $p_{j,l}(\lambda, \eta) = 0$ for $j \neq l$,*

$$p_{j,j}(\lambda, \eta)^2 \sigma_j(\lambda) \sigma_j(\eta) \leq P_j(\lambda) P_j(\eta), \tag{4.19}$$

*with*

$$P_j(\lambda) \leq 2\Phi_*^2 \sigma_j(\lambda) \int_0^h \mu_1(x) dx + \frac{2}{\sqrt{\lambda - d^2}} \int_h^{+\infty} \mu_1(x) dx,$$

*and*

$$q(\lambda, \eta) \leq \min \left( \|\mu_2\|_1^2, \frac{\|\mu_2\|_2^2}{\sqrt[4]{\lambda - k^2 n_*^2} \sqrt[4]{\eta - k^2 n_*^2}} \right). \tag{4.20}$$

*Proof.* Since $v_s$ and $v_a$ are, respectively, even and odd functions, then $p_{j,l}(\lambda, \eta) = 0$ for $j \neq l$.

By Hölder inequality, $p_{j,j}(\lambda, \eta)^2 \leq p_{j,j}(\lambda, \lambda) p_{j,j}(\eta, \eta)$; also, by the formula (2.22), we obtain

$$\begin{aligned} p_{j,j}(\lambda, \lambda) &= 2 \int_0^h |\phi_j(x, \lambda)|^2 \mu_1(x) dx + 2 \int_h^{+\infty} |v_j(x, \lambda)|^2 \mu_1(x) dx \\ &\leq 2\Phi_*^2 \int_0^h \mu_1(x) dx + \frac{2}{\sqrt{\lambda - d^2} \sigma_j(\lambda)} \int_h^{+\infty} \mu_1(x) dx, \end{aligned}$$

and hence (4.19).

Now we have to estimate $q(\lambda, \eta)$. Firstly we observe that

$$|q(\lambda, \eta)| \leq \int_{-\infty}^{\infty} \mu_2(z) dz \int_{-\infty}^{\infty} \mu_2(\zeta) d\zeta = \|\mu_2\|_1^2,$$

which proves part of (4.20). Furthermore, from Young's inequality (see Theorem 4.2 in [LL]) and the arithmetic-geometric mean inequality, we have

$$\begin{aligned} |q(\lambda, \eta)| \leq \|e_{\lambda,\eta}\|_1 \|\mu_2\|_2^2 &= \frac{2\|\mu_2\|_2^2}{\sqrt{\lambda - k^2 n_*^2} + \sqrt{\eta - k^2 n_*^2}} \\ &\leq \frac{\|\mu_2\|_2^2}{\sqrt[4]{\lambda - k^2 n_*^2} \sqrt[4]{\eta - k^2 n_*^2}}. \end{aligned}$$

□

**Theorem 4.6.** *Let $G$ be the Green's function* (2.32)*. Then*

$$\|G\|_{L^2(\mu\times\mu)} < +\infty. \tag{4.21}$$

*Proof.* We write $G = G^g + G^r + G^e$, as in (2.34a)-(2.34c), and use Minkowski inequality:

$$\|G\|_{L^2(\mu\times\mu)} \leq \|G^g\|_{L^2(\mu\times\mu)} + \|G^r\|_{L^2(\mu\times\mu)} + \|G^e\|_{L^2(\mu\times\mu)}. \tag{4.22}$$

From Lemma 4.4 and (4.5) it follows that

$$\|G^g\|_{L^2(\mu\times\mu)}, \|G^r\|_{L^2(\mu\times\mu)} < +\infty. \tag{4.23}$$

It remains to prove that $\|G^e\|_{L^2(\mu\times\mu)} < +\infty$. From (4.6) we have

$$\begin{aligned}
\|G^e\|^2_{L^2(\mu\times\mu)} &= \int\limits_{\mathbb{R}^2\times\mathbb{R}^2} |G^e(x,z;\xi,\zeta)|^2 \mu_1(x)\mu_2(z)\mu_1(\xi)\mu_2(\zeta)\,dx\,dz\,d\xi\,d\zeta \\
&= \int\limits_{\mathbb{R}^2\times\mathbb{R}^2} G^e(x,z;\xi,\zeta)\overline{G^e(x,z;\xi,\zeta)}\mu_1(x)\mu_2(z)\mu_1(\xi)\mu_2(\zeta)\,dx\,dz\,d\xi\,d\zeta;
\end{aligned}$$

hence, thanks to Lemma 4.5, the definition (2.34c) of $G^e$ and Fubini's theorem, we obtain:

$$\begin{aligned}
\|G^e\|^2_{L^2(\mu\times\mu)} &\leq \frac{1}{4\pi^2} \sum_{j\in\{s,a\}} \int\limits_{k^2n_*^2}^{\infty} \int\limits_{k^2n_*^2}^{\infty} q(\lambda,\eta) p_{j,l}(\lambda,\eta)^2 \frac{\sigma_j(\lambda)d\lambda}{2\sqrt{\lambda - k^2n_*^2}} \cdot \frac{\sigma_l(\eta)d\eta}{2\sqrt{\eta - k^2n_*^2}} \\
&\leq \frac{1}{4\pi^2} \sum_{j\in\{s,a\}} \int\limits_{k^2n_*^2}^{\infty} \int\limits_{k^2n_*^2}^{\infty} q(\lambda,\eta) \frac{P_j(\lambda)d\lambda}{2\sqrt{\lambda - k^2n_*^2}} \cdot \frac{P_j(\eta)d\eta}{2\sqrt{\eta - k^2n_*^2}}.
\end{aligned}$$

The conclusion follows from Lemmas 4.5 and 2.15. □

**Corollary 4.7.** *Let $u$ be the solution* (2.33) *of* (H)*, then*

$$\|u\|_{H^2(\mu)} \leq C\|f\|_{L^2(\mu^{-1})}, \tag{4.24}$$

*where*

$$C^2 = \frac{5}{2} + 2C_2 + \left[\frac{3}{2} + 4C_2 + 8C_2^2 + (1+4C_2)k^2n_*^2 + 2k^4n_*^4\right] \|G\|^2_{L^2(\mu\times\mu)}. \tag{4.25}$$

*Proof.* From (2.33) and by using Hölder inequality, it follows that

$$\begin{aligned}
\|u\|^2_{L^2(\mu)} &= \int\limits_{\mathbb{R}^2} \left| \int\limits_{\mathbb{R}^2} G(x,z;\xi,\zeta) f(\xi,\zeta)\,d\xi\,d\zeta \right|^2 \mu(x,z)\,dx\,dz \\
&\leq \|f\|^2_{L^2(\mu^{-1})} \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} |G(x,z;\xi,\zeta)|^2 \mu(\xi,\zeta)\mu(x,z)\,d\xi\,d\zeta\,dx\,dz \\
&= \|f\|^2_{L^2(\mu^{-1})} \|G\|^2_{L^2(\mu\times\mu)},
\end{aligned}$$

and then we obtain (4.24) from Lemmas 4.1 and 4.3. □

**Remark 4.8.** Bounds for the constant $C$ can be computed numerically. Below, we present a table showing such bounds in various instances. Further examples will be given in Chapter 5.

| $k$ | $n_*$ | $n_{cl}$ | $h$ | $C$ |
|---|---|---|---|---|
| $2\pi$ | 1.45 | 1.0 | 1.4 | 754.64 |
| 10 | 2 | 1 | 0.2 | 196.69 |

Now, we are ready to prove the result that guarantees the existence of a solution of $L_\varepsilon u = f$. It will be useful to assume in general that $L_\varepsilon$ is of the form

$$L_\varepsilon = \sum_{i,j=1}^{2} a_{ij}^\varepsilon \partial_{ij} + \sum_{i=1}^{2} b_i^\varepsilon \partial_i + c^\varepsilon. \tag{4.26}$$

We suppose that

$$a_{ij}^\varepsilon = \delta_{ij} + \varepsilon \tilde{a}_{ij}^\varepsilon,\ i,j = 1,2; \quad b_i = \varepsilon \tilde{b}_i^\varepsilon,\ i = 1,2; \quad c^\varepsilon = k^2 n_0(x)^2 + \varepsilon \tilde{c}^\varepsilon,$$

and that there exists a constant $K$, independent of $\varepsilon$ such that

$$\left[\sum_{i,j=1}^{2} (\tilde{a}_{ij}^\varepsilon)^2\right]^{\frac{1}{2}}, \ \left[\sum_{i=1}^{2} (\tilde{b}_i^\varepsilon)^2\right]^{\frac{1}{2}}, \ |\tilde{c}^\varepsilon| \le K\mu \quad \text{in } \mathbb{R}^2. \tag{4.27}$$

**Theorem 4.9.** *Let $f \in L^2(\mu^{-1})$. Then there exists a positive number $\varepsilon_0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, equation $L_\varepsilon u = f$ admits a (weak) solution $u^\varepsilon \in H^2(\mu)$.*

*Proof.* We write

$$L_\varepsilon = L_0 + \varepsilon \tilde{L}_\varepsilon; \tag{4.28}$$

clearly, the coefficients of $\tilde{L}_\varepsilon$ are $\tilde{a}_{ij}^\varepsilon$, $\tilde{b}_i^\varepsilon$ and $\tilde{c}^\varepsilon$. We can write (4.28) as

$$u + \varepsilon L_0^{-1} \tilde{L}_\varepsilon u = L_0^{-1} f;$$

$L_0^{-1} f$ is nothing else than the solution of (H) defined in (2.33).

We shall prove that $L_0^{-1} \tilde{L}_\varepsilon$ maps $H^2(\mu)$ continuously into itself. In fact, for $u \in H^2(\mu)$, we easily have:

$$\begin{aligned} \|\tilde{L}_\varepsilon u\|^2_{L^2(\mu^{-1})} &\le \int_{\mathbb{R}^2} \left[\sum_{i,j=1}^{2} (\tilde{a}_{ij}^\varepsilon)^2 \sum_{i,j=1}^{2} |u_{ij}|^2 + \sum_{i=1}^{2} (\tilde{b}_i^\varepsilon)^2 \sum_{i=1}^{2} |u_i|^2 + (\tilde{c}^\varepsilon)^2 |u|^2\right] \mu^{-1} dx dz \\ &\le K^2 \|u\|^2_{H^2(\mu)}. \end{aligned}$$

Moreover, Corollary 4.7 implies that

$$\|L_0^{-1} f\|_{H^2(\mu)} \le C \|f\|_{L^2(\mu^{-1})},$$

and hence

$$\|L_0^{-1}\tilde{L}_\varepsilon u\|_{H^2(\mu)} \leq CK\|u\|_{H^2(\mu)}.$$

Therefore, we choose $\varepsilon_0 = (CK)^{-1}$ so that, for $\varepsilon \in (0, \varepsilon_0)$, the operator $\varepsilon L_0^{-1}\tilde{L}_\varepsilon$ is a contraction and hence our conclusion follows from Picard's fixed point theorem. □

**Remark 4.10.** In the rest of this thesis we will suppose that the solution $u^\varepsilon$ can be approximated by the iterative method described in (4.4). One thing missing here is to prove that the resulting series converges. Related to this issue we cite [NR1]-[NR3] where the authors proved the convergence of such a series for a different problem which has many points in common with ours. This will be the object of future work.

# Chapter 5

# Non-Rectilinear Waveguides. Numerical examples.

## 5.1 The numerical scheme

In this chapter we shall show how our results in Chapters 2 and 4 can be used to derive some numerical simulations of the physical phenomena described in Sections 1.3 and 1.5.

As in Chapter 4, we study the Helmholtz equation

$$\Delta u + k^2 n_\varepsilon(x,z)^2 u = f, \quad (x,z) \in \mathbb{R}^2,$$

where $n_\varepsilon$ is a small perturbation of a function $n_0 := n_0(x)$, depending only on the $x$-coordinate, which models the index of refraction of a rectilinear waveguide.

In particular, we are interested in perturbations which can be described by a geometric transformation of the plane, in a sense that we are going to explain shortly. In fact, the expression of $L_\varepsilon$ given in (4.2) is hardly exploitable by using (4.4), because it may be difficult to write $n_\varepsilon(x,z)$ in terms of its Neumann series.

Our approach consists in finding a suitable change of coordinates such that $n_\varepsilon$ can be rewritten in a more handy way. The best we can hope is that, after having changed the coordinates, $n_\varepsilon$ depends only on the "new" transversal coordinate or it can be represented in Neumann series with the zeroth order term depending on the transversal coordinate. In such cases, the iterative method (4.4) can be applied.

We will change the coordinates by using an invertible map $\Gamma$ which, in most of the

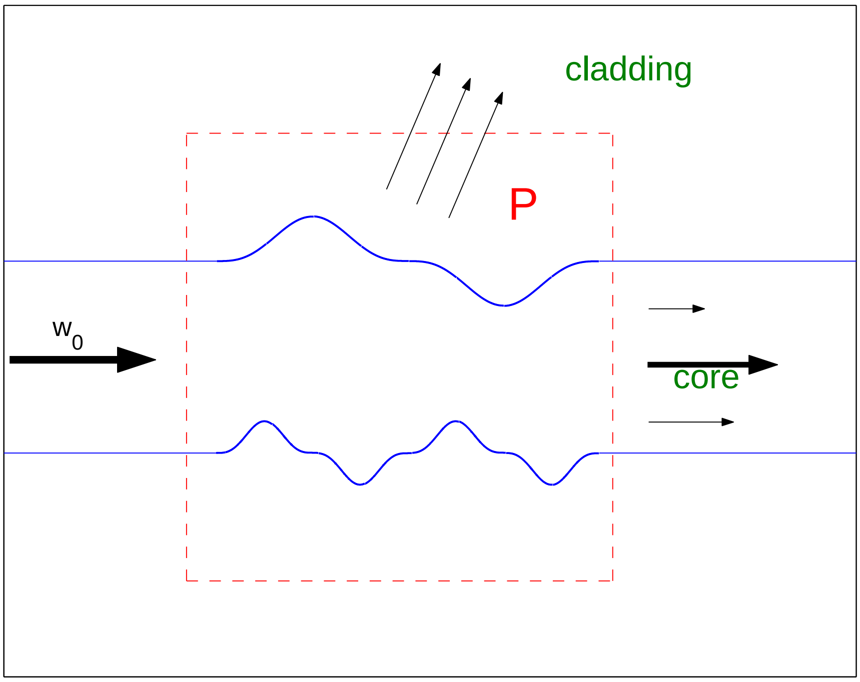


Figure 5.1: The presence of imperfections in a rectilinear waveguide affect the wave propagation of a pure guided mode: the radiating and other guided modes are excited.

studied cases, will be of the form[1]

$$\Gamma^\varepsilon(t,s) = \begin{cases} x = t + \varepsilon T(t)S(s), \\ z = s, \end{cases} \tag{5.1}$$

where $T \in C_0^2(\mathbb{R})$ and $S \in C_0^2(\mathbb{R})$ describe the "profile" of the perturbation, in a sense that we are going to explain shortly. In our simulations we have chosen $T$ to be a piecewise polynomial function for numerical reasons. We will explain which roles are played by $T$ and $S$ later.

We consider $\Gamma^\varepsilon$ is as in (5.1) and change the coordinates; the operator $L_\varepsilon$ in (4.2) becomes (see Appendix B)

$$\begin{aligned} L^\varepsilon w := {} & \frac{1+\varepsilon^2\psi_s^2}{(1+\varepsilon\psi_t)^2} w_{tt} - 2\frac{\varepsilon\psi_s}{1+\varepsilon\psi_t} w_{ts} + w_{ss} \\ & - \frac{1}{(1+\varepsilon\psi_t)^3}\left[\varepsilon(1+\varepsilon^2\psi_s^2)\psi_{tt} - 2\varepsilon^2(1+\varepsilon\psi_t)\psi_s\psi_{st} + \varepsilon(1+\varepsilon\psi_t)^2\psi_{ss}\right] w_t \\ & + k^2 n_0 \varepsilon(t)^2 w = \tilde{f}(t,s), \quad (t,s) \in \mathbb{R}^2, \end{aligned} \tag{5.2}$$

[1] Here and in the rest of the chapter, we use the following notation: by a subscript, as in $L_\varepsilon u$, we denote functions of the variables $(x,z)$, whereas a superscript, as in $L^\varepsilon w$, indicates (the corresponding) functions of the variables $(t,s)$.

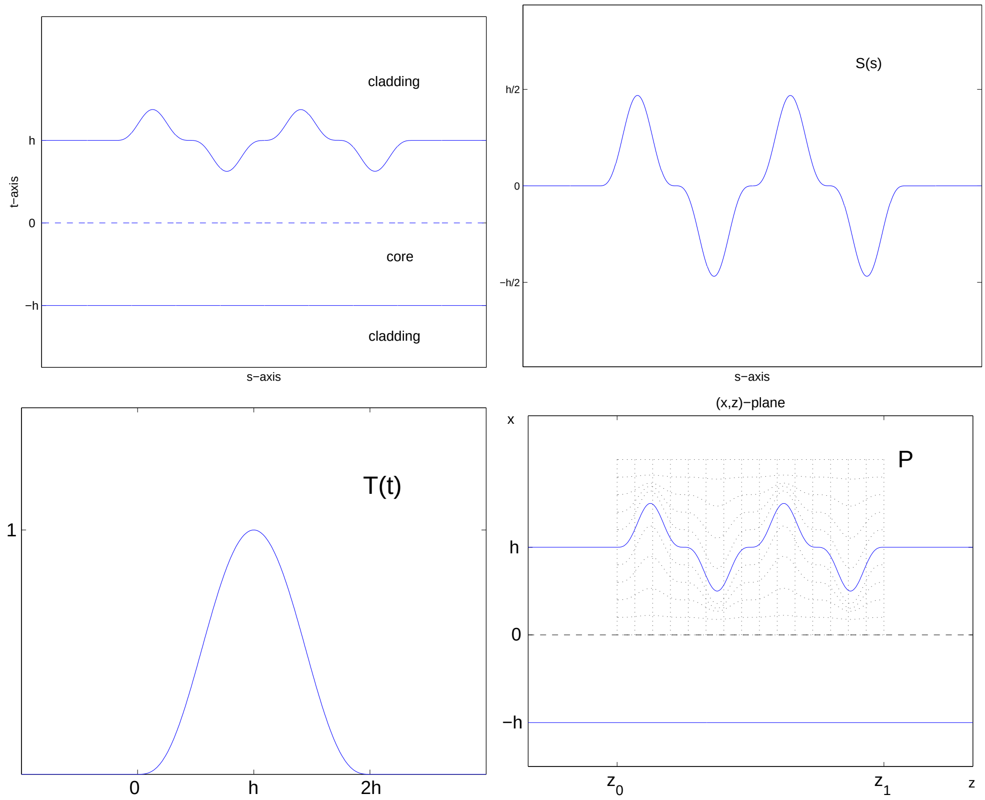


Figure 5.2: In the first figure we show an example of non-rectilinear waveguide and in the second and third ones the consequent choice of the functions $S$ and $T$, respectively. In the last figure, the effect of $\Gamma$ on a rectangular grid in the $(t,s)$-plane is represented.

with $\tilde{f}(t,s) = f(x,z)$, $w(t,s) = u(x,z)$ and where we put

$$\psi(t,s) = T(t)S(s). \tag{5.3}$$

We stress the fact that $\Gamma^\varepsilon$ is chosen in such a way that the new refraction coefficient in (5.2) is $n_0(t)$, where $n_0$ is the function defined in (2.3) that models a rectilinear waveguide.

The smoothness assumption on $S$ and $T$ makes the coefficients of $L^\varepsilon$ continuous. By choosing $S$ compactly supported we suppose that the waveguide is rectilinear outside a bounded region of the plane, that is $L^\varepsilon = L^0$ outside a compact set. The function $T(t)$ is introduced in order to make $\Gamma$ be a smooth and invertible transformation of the plane.

For instance, in the example shown in Fig. 5.2, a good choice of $T$ is the one represented in Fig. 5.2. In Fig. 5.2 we visualize how $\Gamma$ transforms the plane, by representing in the $(x,z)$-plane the image of a rectangular grid in the $(t,s)$-plane.

The parameter $\varepsilon$ controls the amplitude of the perturbation (note that when $\varepsilon = 0$

there is no perturbation at all).

We will always suppose that the "perturbed zone" $P$ is a compact set in $\mathbb{R}^2$, as follows from the assumptions on $T$ and $S$. This choice is motivated by two main reasons. First, it reduces the computations in the numerical simulations in comparison with those needed for a perturbation extended to infinity. Furthermore, we are sure that we can apply Theorem 4.9, because the coefficients of $\tilde{L}$ are smooth and with compact support.

As shown in Chapter 4, we write $w$, solution of $L^\varepsilon w = \tilde{f}$, by using the Neumann series

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots$$

and then we compute the $w_j$'s by using the following algorithm:

$$L_0 w_0 = \tilde{f},\ L_0 w_1 = -L_1 w_0, \dots,\ L_0 w_j = -\sum_{r=0}^{j-1} L_{j-r} w_r, \dots$$

as in (4.4). As already mentioned in Remark 4.10, we did not prove the convergence of the Neumann series, which will be the object of future work. In this thesis we will assume that such a series exists and converges.

In the simulations presented in this chapter, we will suppose $w_0$ to be a guided mode of a rectilinear waveguide, without perturbations. In other words we are taking a special choice of $f$. Thus, $w_0$ propagates undisturbed if no imperfections are present. If the waveguide is perturbed, radiating energy together with the remaining guided modes (if any) supported by the fiber appear. The occurrence of these phenomena will be made clear by pictures presented in this chapter.

In all our simulations we compute $w_1$: its computation is made easier by the fact that we know the explicit expression of $w_0$. The computation of $w_2, w_3, \dots$ would require a larger numerical effort. However, since we know that $\varepsilon$ cannot be taken larger than $\varepsilon_0 = (CK)^{-1}$, the contribution of $\varepsilon^2 w_2$, $\varepsilon^3 w_3, \dots$ would be, generally, rather small.

In Sections 5.2-5.5, we will present and compare pictures of $w_1$ and $w_0 + \varepsilon w_1$ in three instances. In Section 5.3 we will present an example showing the totality of the coupling effects (i.e. the whole $w_1$ will be considered). In Section 5.4 we will focus our attention on the coupling between $w_0$ and the radiating energy and thus we will consider only the radiating part of $w_1$. Section 5.5 concerns the coupling between guided modes and hence only the guided part of $w_1$ will be considered.

In Section 5.2, we compute possible values of $\varepsilon_0$.

## 5.2 Computing $\varepsilon_0$

From Theorem 4.9, we know that a solution of $L^\varepsilon u = f$ exists whenever $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 = (CK)^{-1}$, with $C$ and $K$ given by (4.25) and (4.27), respectively. We notice that, while $C$ depends only on the weight function $\mu$ chosen, $K$ depends also on the perturbation.

In this section we describe how we obtain an estimate for $\varepsilon_0$. We will calculate $\varepsilon_0$ for several choices of the weight function $\mu$. In particular, we will consider functions $\mu$ of the form:

$$\mu(x,z) = 16\left(4 + \frac{|x-x_0|^2 + |z-z_0|^2}{B^2}\right)^{-m}, \tag{5.4}$$

and

$$\mu(x,z) = \mu_{a_1}(|x-x_0|)\mu_{a_2}(|z-z_0|), \tag{5.5}$$

where

$$\mu_{a_l}(t) = \begin{cases} 1, & t \geq 0 \\ [1+(t-a_l)^2]^{-m}, & t > a_l, \end{cases}$$

with $l = 1, 2$, $m > 0$ and where $P_0 := (x_0, z_0)$ denotes the center of the perturbed zone $P$ and $a_l$, $l = 1, 2$ is half of the side of P parallel to the $x$ and $z$ directions, respectively.

Formula (4.25) says that $C$ depends on the parameters $x_0, z_0, k, m$ and the function $n$. We notice that if $\mu$ is as in (5.4), the functions $\mu_1$ and $\mu_2$ in Section 4.3 are replaced by the function $\tilde{\mu}(|x - x_0|)$ and $\tilde{\mu}(|z - z_0|)$, respectively, defined by

$$\tilde{\mu}(r) = 4\left(4 + \frac{|r|^2}{B^2}\right)^{-\frac{m}{2}};$$

in fact, in such a case, (4.15) holds.

Now, we show how we can derive an estimate for $K$ from (4.27), in the case in which $L^\varepsilon$ is as in (5.2). $K$ will depend on the $L^\infty$ norms of the coefficients $\tilde{a}_{ij}$, $\tilde{b}_i$ and on the $L^\infty$ norm of $\mu^{-1}$ on $P$.

We set $J = (1 + \varepsilon\psi_t)^{-1}$; the coefficients of $\tilde{L}^\varepsilon$ are

$$\begin{aligned}
\tilde{a}^\varepsilon_{11} &= -\left[2\psi_t + \varepsilon(\psi_t^2 - \psi_s^2)\right] J^2, \\
2\tilde{a}^\varepsilon_{12} &= -\psi_s J, \quad \tilde{a}^\varepsilon_{22} = 0, \\
\tilde{b}^\varepsilon_1 &= -(1 + \varepsilon^2\psi_s^2)J^3 + 2\varepsilon\psi_s\psi_{st}J^2 - \psi_{ss}J, \\
\tilde{b}^\varepsilon_2 &= 0, \quad \tilde{c}^\varepsilon = 0.
\end{aligned}$$

We assume that $\varepsilon \leq \varepsilon_*$ and $|T| \leq 1$ and use the following notations:

$$T'_* = \max_{t\in\mathbb{R}} |T'(t)|, \quad S_* = \max_{s\in\mathbb{R}} |S(s)|, \quad S''_* = \max_{s\in\mathbb{R}} |S''(s)|, \quad S'_* = \max_{s\in\mathbb{R}} |S'(s)|;$$

furthermore, we notice that

$$J \leq J_* := \frac{1}{1 - \varepsilon_* |T'_* S_*|}.$$

Thus,

$$\begin{aligned}
&|\tilde{a}^\varepsilon_{11}| \leq \left\{2S_* T'_* + \varepsilon_* [(S_* T'_*)^2 + (S'_*)^2]\right\} J_*^2, \\
&2|\tilde{a}^\varepsilon_{12}| \leq S'_* J_*, \\
&|\tilde{b}^\varepsilon_1| \leq [1 + (\varepsilon_* S'_*)^2] J_*^3 + 2\varepsilon_* T'_* (S'_*)^2 J_*^2 + S''_* J_*.
\end{aligned}$$

By standard arguments, we can infer that these bounds are controlled by only four parameters: $S''_*$ (the curvature of the perturbation), $T'_*$, $\varepsilon_*$ and the size of $P$. By (4.27) $K$ is computed as a function $K(\varepsilon_*)$ of $\varepsilon_*$; then $\varepsilon_0$ is given by

$$\varepsilon_0 = \min\left(\frac{1}{CK(\varepsilon_*)}, \varepsilon_*\right).$$

In each of the following sections, a table will show how $\varepsilon_0$ changes by choosing a different weight function $\mu$. In particular we will calculate numerically an estimate for $\varepsilon_0$ for the weights given by (5.4) and (5.5) for several choices of the parameters.

**Remark 5.1.** While the function $S$ is determined by the perturbation, the function $T$ is just a technical device. It is desirable to choose $T$ such that $T'_*$ is as small as possible.

## 5.3 A preliminary example

In this section we consider a waveguide as in Figure 5.3 and show the effect of a perturbation of the interface between core and cladding.

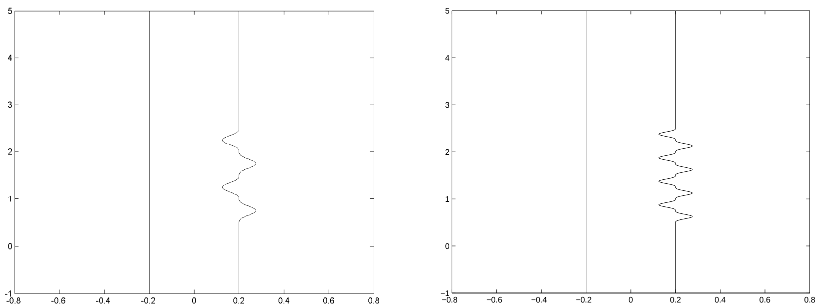


Figure 5.3: The profile of the waveguide in the two studied examples. Only one side of the interface between core and cladding is perturbed.

The examples in this section have only an introductory scope; indeed, the choice of the parameters is purely academic with the aim of obtaining pictures as clear as

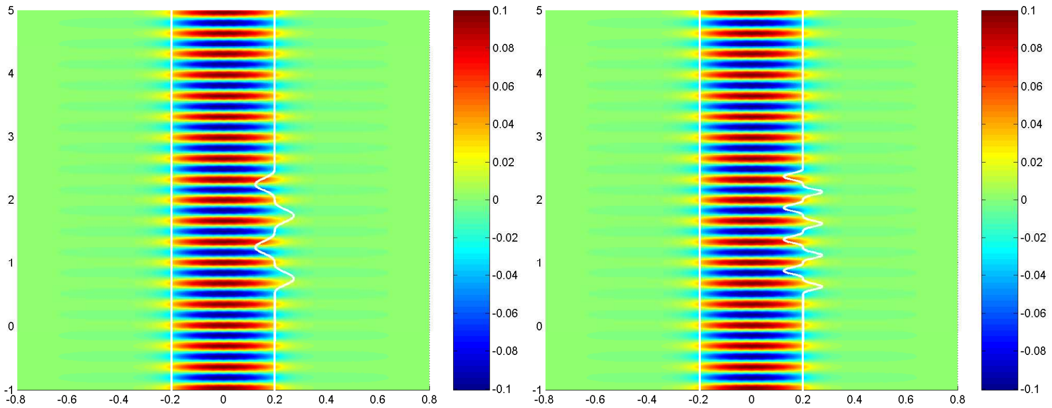

Figure 5.4: Real part of $w_0$, i.e. real part of the guided mode corresponding to $\lambda_1^s$.

possible. Later, in Sections 5.4 and 5.5, we will consider *real life* parameters and applications.

A rectilinear waveguide with parameters as in Table 5.1 support three guided modes, for $\lambda_1^s = 36.78$, $\lambda_2^s = 284.31$ and $\lambda_1^a = 141.96$.

| $n_{co}$ | $n_{cl}$ | $k$ | $h = R_{co}$ | $d^2$ |
|---|---|---|---|---|
| 2.0 | 1.0 | 10.0 | 0.2 | 300.0 |

Table 5.1: The parameters of the waveguide

We suppose that $w_0$ is a pure forward propagating guided mode corresponding to the first eigenvalue $\lambda_1^s$, see Fig. 5.4.

As described in Section 5.1, we operate a change of coordinates based on the functions $S$ and $T$. Details on the choice of such functions will be given in Sections 5.4 and 5.5.

In Table 5.2 we write approximate bounds on $\varepsilon_0$ for the first perturbation considered.

| $\mu$ as in (5.4) | | | |
|---|---|---|---|
| $(x_0, z_0)$ | $\varepsilon_0$ | C | K |
| $O$ | 9.77e-5 | 196.69 | 52.06 |
| $P_0$ | 3.75e-4 | 196.69 | 13.54 |
| $P_0$ | 5.32e-4 | 138.77 | 13.54 |
| $P_0$ | 2.35e-4 | 314.57 | 13.54 |
| $P_0$ | 4.89e-4 | 518.75 | 3.93 |
| $P_0$ | 8.86e-5 | 37.84 | 298.07 |

| $\mu$ as in (5.5) | | | |
|---|---|---|---|
| $(x_0, z_0)$ | $\varepsilon_0$ | C | K |
| $P_0$ | 1.73e.4 | 584.23 | 9.89 |
| $P_0$ | 9.90e-5 | 1.02e+3 | 9.89 |
| $P_0$ | 2.12e-4 | 477.63 | 9.89 |
| $P_0$ | 1.43e-4 | 708.38 | 9.89 |

Table 5.2: Estimates of $\varepsilon_0$ corresponding to the weight function $\mu$ as in (5.4) and (5.5), respectively.

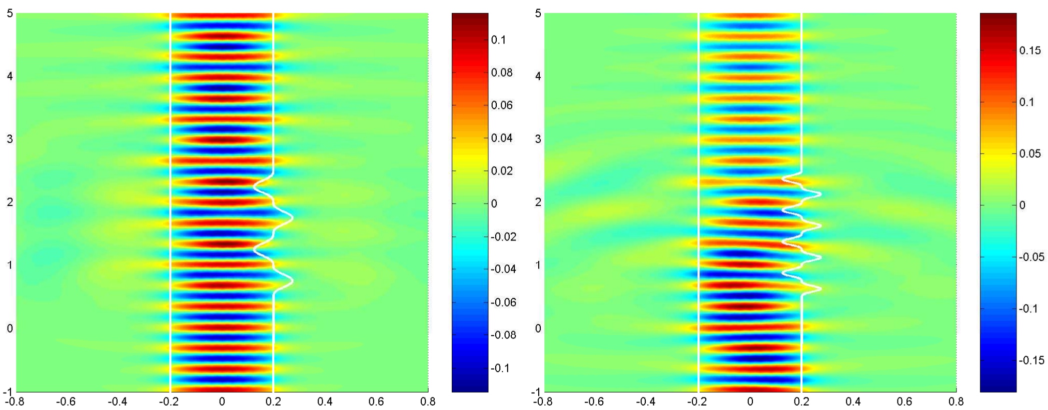

Figure 5.5: Real part of the near field of $w_0 + \varepsilon w_1$.

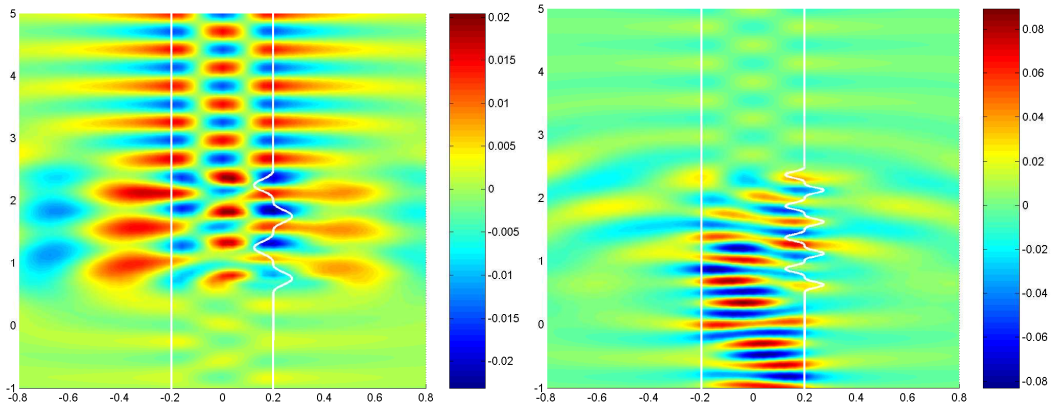

Figure 5.6: Real part of the near field of $w_1$.

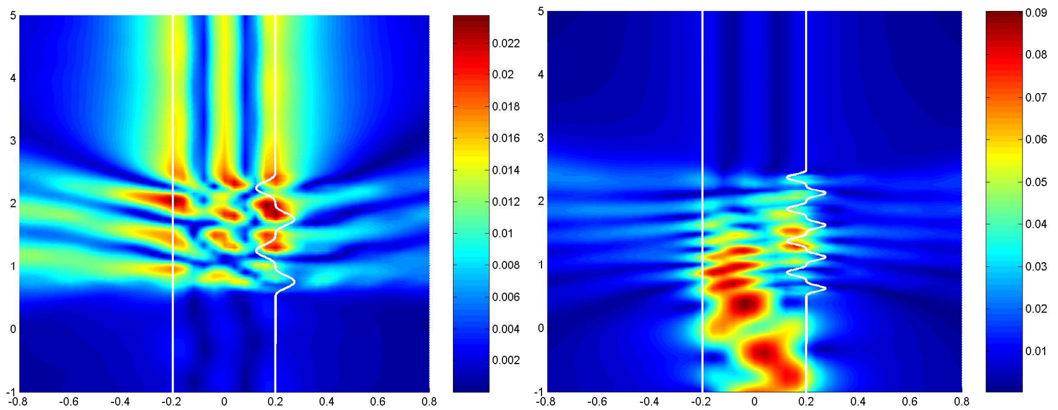

Figure 5.7: Modulus of $w_1$.

In Figure 5.5 the resulting effect of the perturbation on the wave propagation is represented. Here, the first two terms of the Neumann series are shown. Details on $w_1$ are given in Figures 5.6 and 5.7 where we show its real part and modulus, respectively. In Figure 5.8 the real part of the guided component is represented; Figures 5.9 and 5.10 show the effect of the perturbation on the coupling between $w_0$ and the other guided modes in the two different perturbations considered, i.e. in Fig. 5.8 we show the sum of the three guided modes appearing in $w_1$, while in Figures 5.9 and 5.10 each of such modes is represented separately. Finally, in Figure 5.11 the radiating component of $w_1$ show the effect of the perturbation on the energy radiating outside the waveguide.

We notice that different kinds of perturbation may have a different result on the wave propagation. In both the cases studied here, there is a coupling effect between $w_0$ (a pure guided mode) and the other modes supported by the waveguide (guided

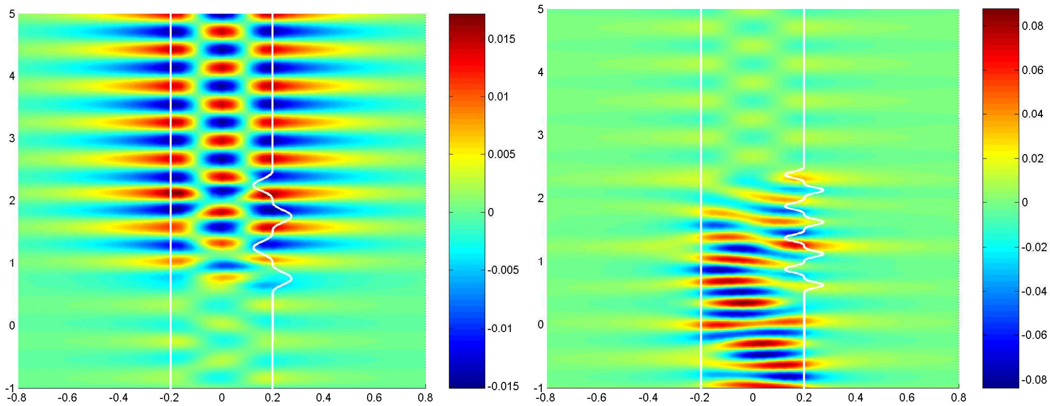


Figure 5.8: Real part of the guided component of $w_1$.

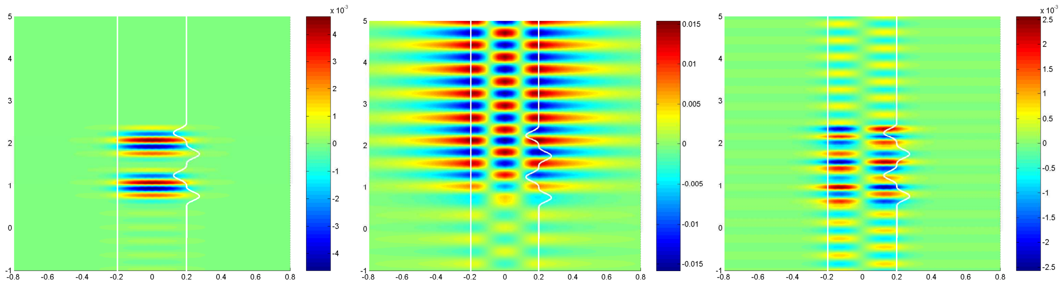

Figure 5.9: Details of the three components of the guided part of $w_1$ corresponding to $\lambda_1^s, \lambda_2^s$ and $\lambda_1^a$, respectively, in the first example considered. The real part is shown.

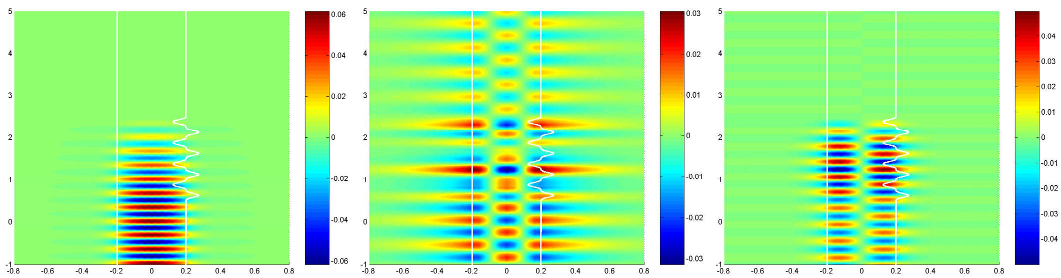

Figure 5.10: Details of the three components of the guided part of $w_1$ corresponding to $\lambda_1^s, \lambda_2^s$ and $\lambda_1^a$, respectively, in the second example considered. The real part is shown.

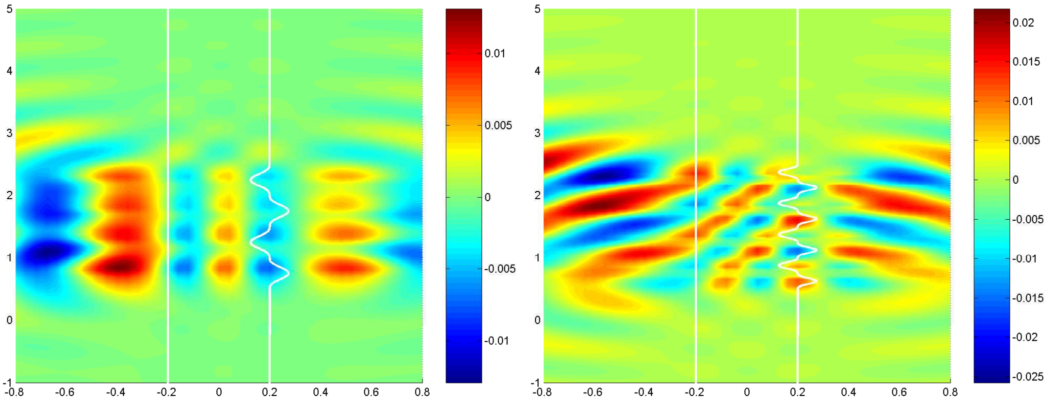


Figure 5.11: Real part of the radiating component of $w_1$.

and radiating ones).On the other hand, as it is clear from Figures 5.8, 5.9 and 5.10, the first perturbation (see Figure 5.9 and the first picture in Figure 5.8) generates mostly a new forward-propagating guided mode corresponding to $\lambda_2^s$; in the second perturbation considered (see Figure 5.10 and the second picture in Figure 5.8), the most remarkable effect is that the resulting guided part of $w_1$ is backward-propagating. From Figure 5.7, we notice that the first perturbation generates a considerable part of energy which escapes the waveguide, while in the second perturbation most of the energy of $w_1$ is confined inside the core of the waveguide, i.e. the effect of the perturbation on the radiating modes is minor.

## 5.4 Out-of-plane waveguide couplers

In this section, we will focus our attention on the study of the (radiating) energy that escapes the fiber as a consequence of a perturbation. We have in mind the applications described in Section 1.5.

We consider a 2-D waveguide in all its components: a central zone (the core), a finite cladding and then an infinite jacket, see Figure 5.12.

The index of refraction is supposed to be piecewise constant. In particular $n_{co}$, $n_{cl}$ and $n_{ja}$ will denote the index of refraction of the core, cladding and jacket, respectively.

As in Section 5.3, there is no symmetry in the perturbation, but, thanks to the fact that the the index of refraction is piecewise constant, it is possible to find a transformation $\Gamma$ such that $n_\varepsilon(x, z)$ is mapped into $n_0(t)$, where $n_0(t)$ represents the index of refraction in the new coordinates $(t, s)$. We choose $S(s)$ and $T(t)$ as in Fig. 5.13:

$$S(s) = A\left[1 - \left(\frac{s - s_0}{\omega}\right)^2\right]^3 \sin(2\pi a_0 s)\chi_{(s_0-\omega, s_0+\omega)}(s), \tag{5.6}$$

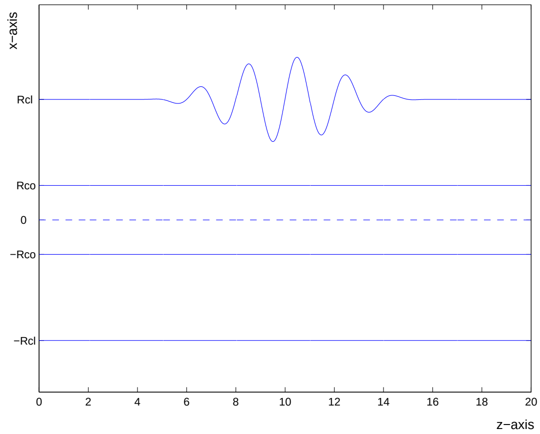


Figure 5.12: The perturbed waveguide. In this example, we are considering a waveguide made of a core, a cladding and an infinite jacket.

and

$$T(t) = \left[1 - \left(\frac{t - R_{cl}}{\rho}\right)^2\right]^3 \chi_{(R_{cl}-\rho, R_{cl}+\rho)}(t), \tag{5.7}$$

with $\rho = R_{cl} - R_{co}$. This choice of $S$ and $T$ amounts to a perturbation of the interface between cladding and jacket as in Figure 5.12.

In Tables 5.3 and 5.4 we computed estimates of $\varepsilon_0$ as described in Section 5.2.

In the next two subsections, we present results on the near-field and on the far-field.

### 5.4.1 Near-field

We study what happens near the perturbed zone of the waveguide represented in Figure 5.12. We will consider perturbations represented by the function in (5.6) corresponding to the values $a_0 = 0.5$ and $a_0 = 1.0$ and show the numerical results in Figures 5.16-5.19 and 5.20-5.22, respectively.

In Table 5.5, we report the values of the relevant parameters of the waveguides. With these parameters, the fiber supports several guided modes; the first one corresponds to the values $\lambda_1^s = 3.3016$ and $\beta_1^s = 8.9276$.

As already mentioned, we are interested in what happens to the wave propagation when a pure guided mode is propagating in the waveguide. Thus, we suppose that $w_0$ is the first forward propagating guided mode supported by the rectilinear fiber (see Figure 5.14):

$$w_0(t, s) = v_s(t, \lambda_1^s) e^{i\beta_1^s s},$$

with $\beta_1^s = \sqrt{k^2 n_*^2 - \lambda_1^s}$.

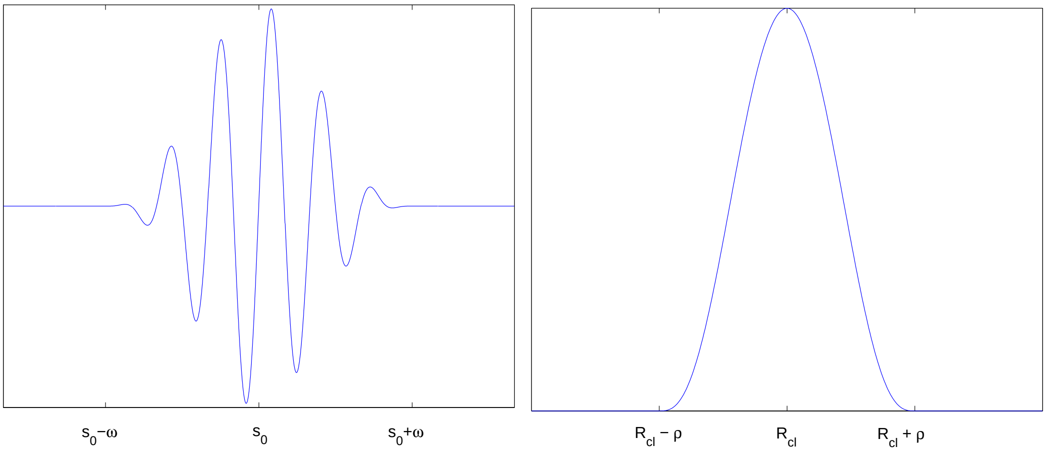


Figure 5.13: The figures show the choice we made for the functions $S$ and $T$, respectively. Such a choice corresponds to a perturbed waveguide as in Figure 5.12.

| $\mu$ as in (5.4) | | | | | | |
|---|---|---|---|---|---|---|
| $(x_0, z_0)$ | $B$ | $m$ | $a_0$ | $\varepsilon_0$ | C | K |
| $O$ | 1 | 2 | 0.5 | 1.77e-7 | 7.54e+2 | 7.46e+3 |
| $O$ | 1 | 2 | 1.0 | 6.01e-8 | 7.54e+2 | 2.21e+4 |
| $P_0$ | 1 | 2 | 0.5 | 7.62e-6 | 7.54e+2 | 1.74e+2 |
| $P_0$ | 1 | 2 | 1.0 | 7.62e-6 | 7.54e+2 | 1.74e+2 |
| $P_0$ | 0.5 | 2 | 0.5 | 5.21e-6 | 1.11e+3 | 1.73e+2 |
| $P_0$ | 0.5 | 2 | 1.0 | 5.21e-6 | 1.11e+3 | 1.73e+2 |
| $P_0$ | 2 | 2 | 0.5 | 7.52e-6 | 7.66e+2 | 1.73e+2 |
| $P_0$ | 2 | 2 | 1.0 | 7.52e-6 | 7.66e+2 | 1.73e+2 |
| $P_0$ | 1 | 1.2 | 0.5 | **8.56e-5** | 1.29e+3 | 9.03 |
| $P_0$ | 1 | 1.2 | 1.0 | **5.50e-5** | 1.29e+3 | 14.05 |
| $P_0$ | 1 | 4 | 0.5 | 1.24e-8 | 2.87e+2 | 2.80e+5 |
| $P_0$ | 1 | 4 | 1.0 | 1.24e-8 | 2.87e+2 | 2.80e+5 |

Table 5.3: Bounds for $\varepsilon_0$ when $\mu$ is as in (5.4).

| $\mu$ as in (5.5) | | | | | |
|---|---|---|---|---|---|
| $(x_0, z_0)$ | $m$ | $a_0$ | $\varepsilon_0$ | C | K |
| $P_0$ | 1 | 0.5 | **3.95e-5** | 4.29e+3 | 5.91 |
| $P_0$ | 1 | 1.0 | **1.12e-5** | 4.29e+3 | 20.48 |
| $P_0$ | 0.4 | 0.5 | 3.91e-5 | 4.32e+3 | 5.91 |
| $P_0$ | 0.4 | 1.0 | 1.12e-5 | 4.32e+3 | 20.48 |
| $P_0$ | 2 | 0.5 | 3.60e-5 | 4.69e+3 | 5.91 |
| $P_0$ | 2 | 1.0 | 1.04e-5 | 4.69e+3 | 20.48 |
| $P_0$ | 2/3 | 0.5 | 3.96e-5 | 4.26e+3 | 5.91 |
| $P_0$ | 2/3 | 1.0 | 1.14e-5 | 4.26e+3 | 20.48 |

Table 5.4: Bounds for $\varepsilon_0$ when $\mu$ is as in (5.5).

| $n_{co}$ | $n_{cl}$ | $n_{ja}$ | $k$ | $R_{co}$ | $h = R_{cl}$ | $d^2$ |
|---|---|---|---|---|---|---|
| 1.45 | 1.40 | 1.00 | $2\pi$ | 0.4 | 1.4 | 43.52 |

Table 5.5: Parameters of the waveguide.

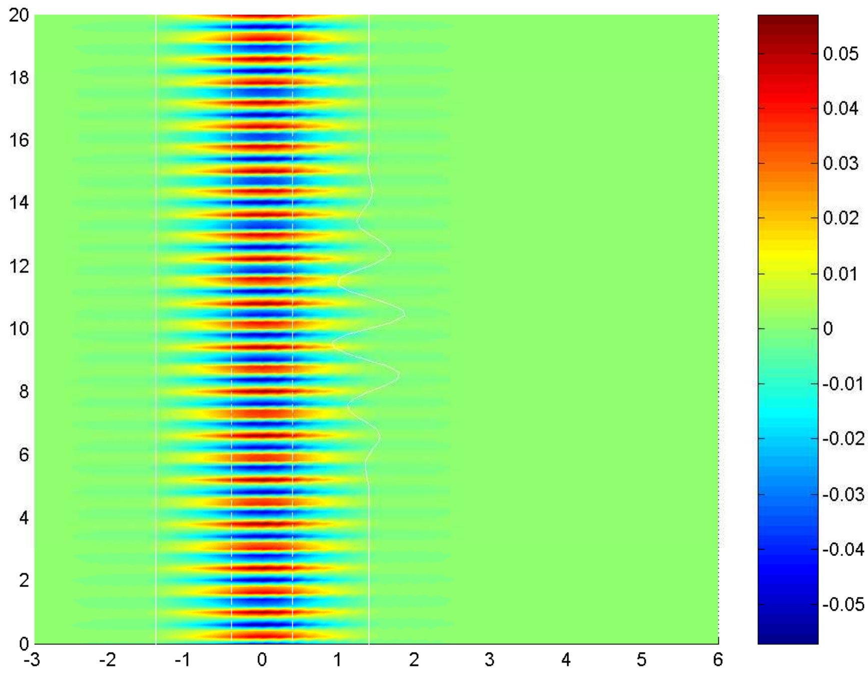


Figure 5.14: The first guided mode $w_0$.

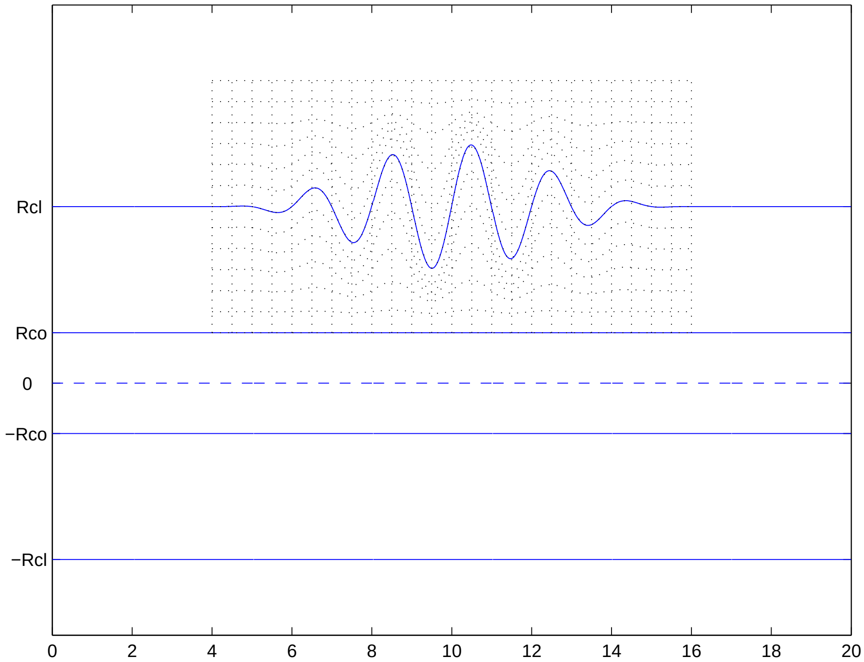


Figure 5.15: The "perturbed" grid.

In Figures 5.16-5.17 and 5.20-5.21 the real part and the modulus of $w_1$ are represented. To compute $w_1(t,s)$, i.e. the integral

$$w_1(t,s) = -\int_P G(t,s;t_1,s_1)L_1 w_0(t_1,s_1)dt_1 ds_1,$$

we use the trapezoidal rule on a rectangular grid in the $(t,s)$-plane (which corresponds to a "perturbed" grid in the $(x,z)$-plane, as shown in Fig. 5.15). The sampling intervals are been chosen by dividing the perturbed zone $P$ in $12 \times 24$ rectangles.

Now, we show how we computed the integrals defining $G$. Firstly, by changing the variables, we write $G^r$ and $G^e$ as

$$G^r = -i \sum_{j\in\{s,a\}} \int_0^{kn_{cl}} e^{i|z-\zeta|\mu} v_j(x, k^2n_*^2 - \mu^2) v_j(\xi, k^2n_*^2 - \mu^2)\sigma_j(k^2n_*^2 - \mu^2)d\mu,$$

and

$$G^e = -\sum_{j\in\{s,a\}} \int_0^{+\infty} e^{-|z-\zeta|\mu} v_j(x, k^2n_*^2 + \mu^2) v_j(\xi, k^2n_*^2 + \mu^2)\sigma_j(k^2n_*^2 + \mu^2)d\mu;$$

then we use the trapezoidal rule with sampling intervals of length $kn_{cl}/80$ in $G^r$ and of $kn_*/40$ in $G^e$, where we truncate the integral at $\mu = kn_*$.

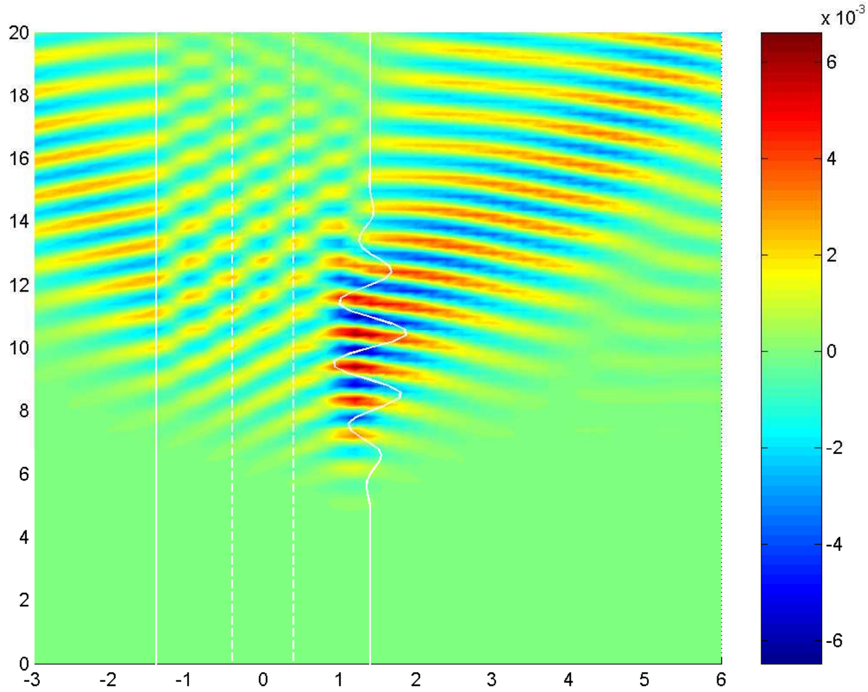


Figure 5.16: Real part of $w_1^{rad}$ for $a_0 = 0.5$.

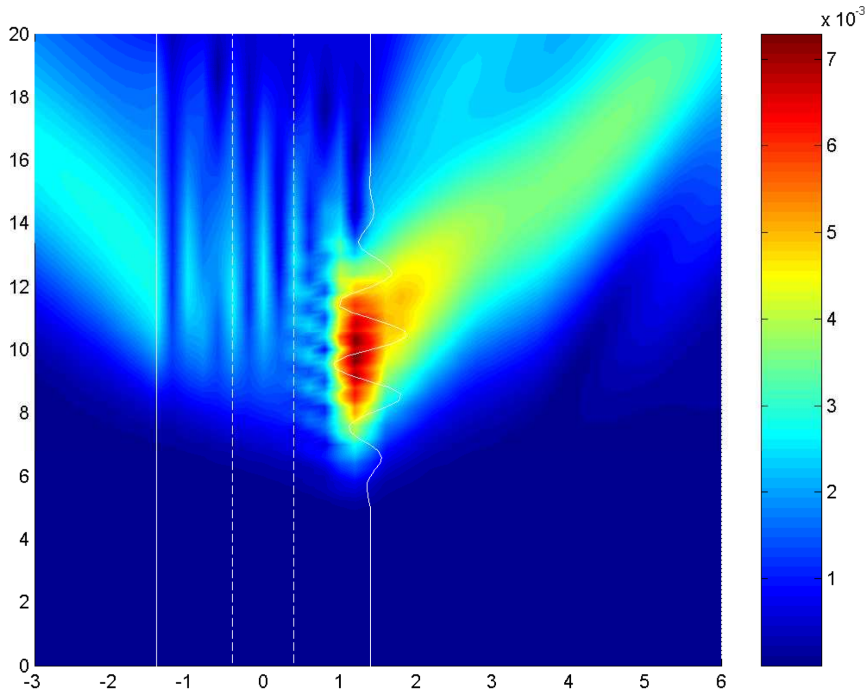


Figure 5.17: Absolute value of $w_1^{rad}$ for $a_0 = 0.5$.

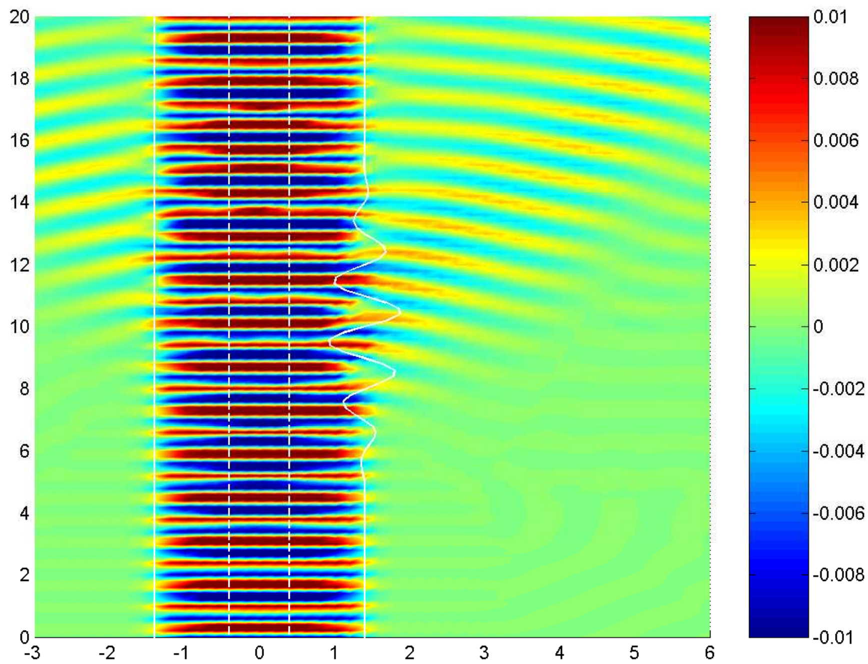


Figure 5.18: Real part of the near field of $w_0 + \varepsilon w_1^{rad}$, for $a_0 = 0.5$.

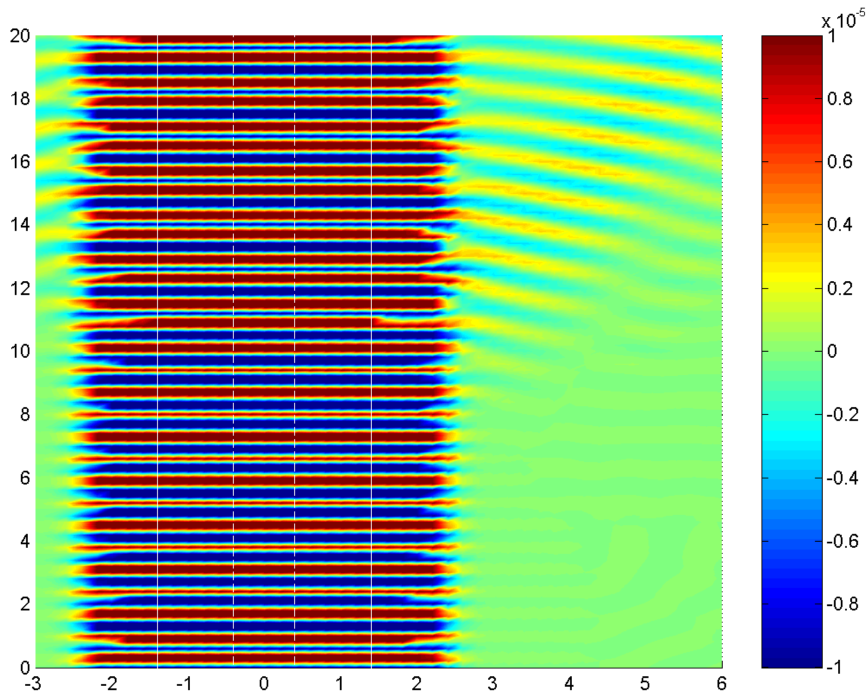


Figure 5.19: Real part of the near field of $w_0 + \varepsilon w_1^{rad}$ with $\varepsilon = 0.856 \cdot 10^{-5}$,when $a_0 = 0.5$.

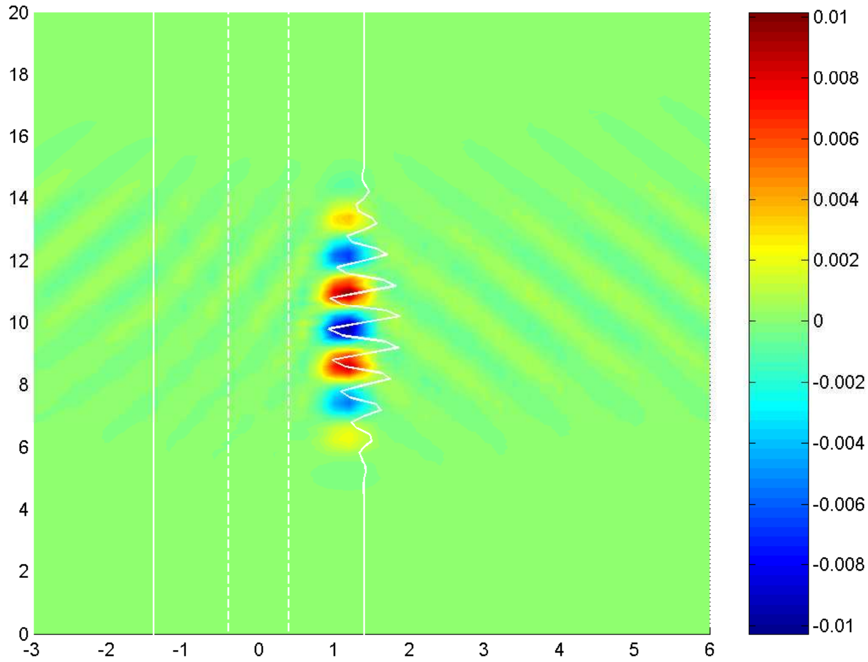


Figure 5.20: Real part of $w_1^{rad}$ for $a_0 = 1.0$.

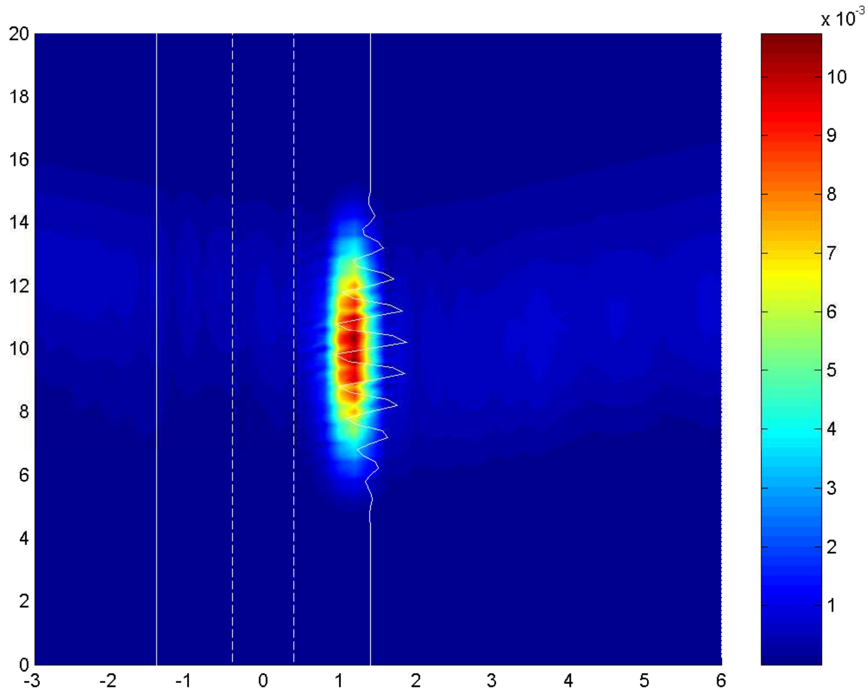


Figure 5.21: Absolute value of $w_1^{rad}$ for $a_0 = 1.0$.

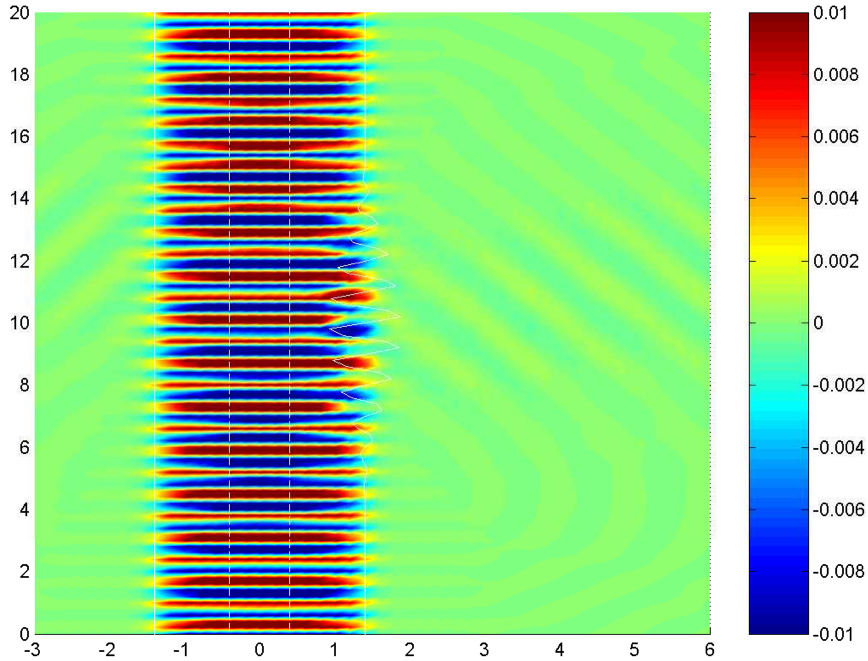


Figure 5.22: Real part of the near field of $w_0 + \varepsilon w_1^{rad}$, for $a_0 = 1.0$.

Figures 5.18 and 5.22 show $w_0+\varepsilon w_1^{rad}$ corresponding to the two different perturbations considered. Here we set $\varepsilon = 1$ in order to bring $w_1^{rad}$ out. As already mentioned, our results hold for $\varepsilon \leq \varepsilon_0$, where $\varepsilon_0$ is given by $(CK)^{-1}$. In Tables 5.3 and 5.4 we computed several estimates of $\varepsilon_0$ according to what described in Section 5.2. Figure 5.19 shows the real part of $w_0 + \varepsilon w_1$ with $\varepsilon = 0.856 \cdot 10^{-5}$ (the best estimate for $\varepsilon_0$ we obtained) for the case in which $a_0 = 0.5$.

We notice that in the first example, a small perturbation in the profile of the cladding determines a sort of plane wave going out from the waveguide. In the second example, the different shape (in frequency) of the perturbation does not create an important outgoing wave, but the intensity of $w_1$ is mostly confined in the region close to the perturbation.

We want to stress that Fig. 5.16-5.22 represent the *near field* of $w_1^{rad}$ and $w_0 + \varepsilon w_1^{rad}$. The computations in a wider region of the plane would require a large increase in terms of time. This problem is due to the oscillatory behaviour of the functions defining $G$.

### 5.4.2 Far-field

As well as in the near field, we are interested in the behaviour of the *far field*, which describes the behaviour of the solution far from the fiber. Here, formulas from Corollary

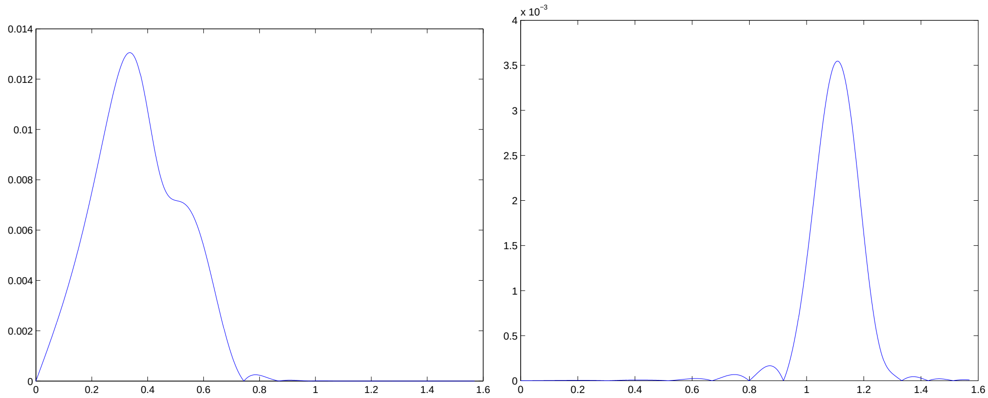


Figure 5.23: Absolute value of the angular component of the far-field as a function of the angular variable $\vartheta$.

2.18 have been used.

Figure 5.23 shows the absolute value of the angular component of the far-field as a function of the angular variable $\vartheta$, with $\vartheta \in [0, \pi/2]$. The two pictures in Figure 5.23 correspond to the two cases described in the previous subsection. We notice that different kind of perturbations may affect the far-field in a remarkable different way.

## 5.5 Coupling between guided modes

In this section we study the coupling between guided modes (see Section 1.3 for a brief introduction to this topic).

Before showing our numerical results, we describe how the coupling mode theory can be reformulated by using our mathematical framework. In particular, the coupled mode equations will be replaced by an iterative method which gives an approximation of the amplitude of the forward and backward propagating modes.

This approximation is yielded by the iterative method described in Chapter 3 and recalled in (4.4).

For simplicity, we will consider the case in which $w_0$ is a forward propagating guided mode and thus assume

$$w_0(x, z) = v(x, \lambda_0)e^{iz\beta_0},$$

where $\lambda_0 \in \{\lambda_m^j : \ j \in \{s, a\}, \ m = 1, \ldots, M_j\}$.

As shown in (4.4), we consider the equation $L_\varepsilon w = f$, and then the first order approximation of the solution $w$ is given by solving

$$L_0 w_1(x, z) = -L_1 w_0,$$

where $L_1$ is an operator of the following form:

$$L_1 w = \sum_{i,j}^{2} a_{ij}^1 w_{x_i x_j} + \sum_{i=1}^{2} b_i^1 w_{x_i} + c^1 w,$$

with all the coefficients having support contained in $P$.

We set $g_0 = -L_1 w_0$ and then we have

$$w_1(x,z) = \int_P G(x,z;\xi,\zeta) g_0(\xi,\zeta) d\xi d\zeta. \tag{5.8}$$

Hence, the guided part of $w_1$ is given by

$$w_1^g(x,z) = \sum_{j\in\{s,a\}} \sum_{m=1}^{M_j} \int_P \frac{e^{i|z-\zeta|\beta_m^j}}{2i\beta_m^j} v_j(x,\lambda_m^j) v_j(\xi,\lambda_m^j) r_m^j g_0(\xi,\zeta) d\xi d\zeta.$$

As in Section 1.3, we denote the amplitude and phase-dependence of the forward and backward propagating modes by $b_{j,m}^+$ and $b_{j,m}^-$, and express such quantities by using their Neumann series:

$$b_{j,m}^{\pm} = b_{j,m}^{\pm,0} + \varepsilon b_{j,m}^{\pm,1} + \dots$$

(notice that $b_{j,m}^{\pm,0} = 0$ if $\lambda_m^j \neq \lambda_0$).

In order to obtain a simpler and nicer formulation, we derive a formula for $b_{j,m}^{\pm}$ outside the perturbation, that is, since $b_{j,m}^+$ and $b_{j,m}^-$ correspond to the forward and backward propagation, we can assume that $z > \zeta$ for $b_{j,m}^+$ and $z < \zeta$ for $b_{j,m}^-$. Thus, we have

$$b_{j,m}^{\pm,1}(z) = B_{j,m}^{\pm}[g_0]\, e^{\pm iz\beta_m^j},$$

with

$$B_{j,m}^{\pm}[g_0] = -\sum_{j\in\{s,a\}} \sum_{m=1}^{M_j} \frac{r_j^m}{2i\beta_m^j} \hat{G}_0(\lambda_m^j, \mp\beta_m^j),$$

where $\hat{G}_0(\lambda,\beta)$ is defined by (2.71).

Higher order approximations of $b_{j,m}^{\pm}$ can be obtained by iterating the above method.

### 5.5.1 Cross-coupling

In this section we refer to Figures 5.24-5.29. Three waveguides with the same index of refraction are close to each other and a pure guided mode is propagating in the central one (see Figure 5.24); a perturbation of this waveguide excites the guided modes supported by the nearby waveguides (Figures 5.27-5.29).

We notice that such a configuration can be still studied by using our mathematical framework. In fact, we can think of the whole (rectilinear) configuration as a “big” waveguide containing three smaller waveguides.

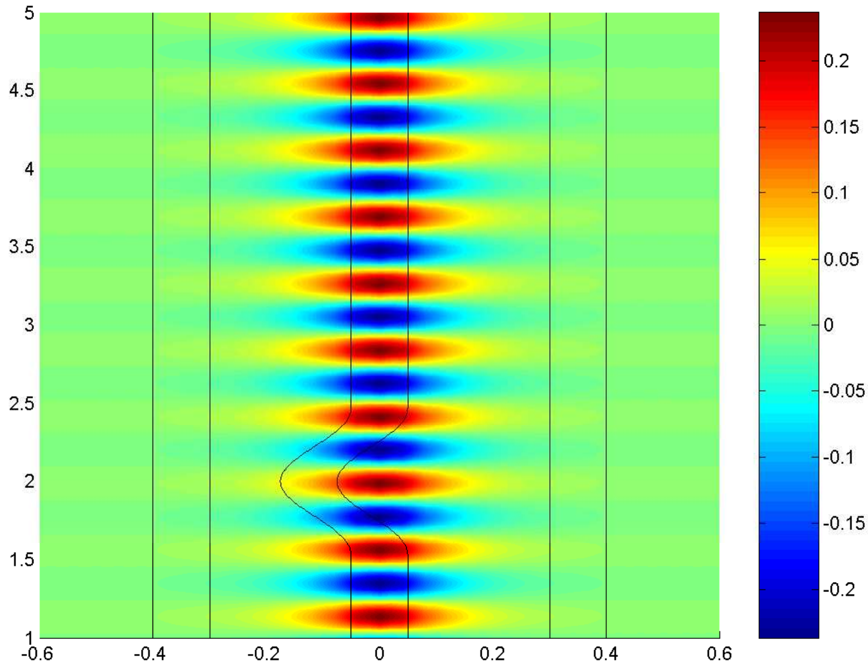


Figure 5.24: Real part of $w_0$. In the example considered, $w_0$ is a guided mode propagating in the central waveguide

In our numerical example, $n = 2.0$ in the cental waveguide, $n = 1.4$ in the two side waveguides and $n = 1.0$ in the rest of the plane. Here, $k = 10.0$ and thus the entire configuration supports three guided modes: $\lambda_1^s = 182.43$, $\lambda_2^s = 288.38$ and $\lambda_1^a = 284.2763$.

We notice that a guided mode propagating in one of the side waveguide can be approximated by a linear combination of those corresponding to the values $\lambda_2^s$ and $\lambda_1^a$ (see [Ma] for further details).

As already mentioned, in this section we neglect the radiating energy and focus our attention on the guided part of the solution.

This numerical example differs from the ones shown in the previous sections because the whole central waveguide is bent. In Figure 5.25 we show how we choose the functions $S$ and $T$ and in Figure 5.26 the resulting effect of such functions.

The guided part of the first order approximation $w_1^g$ is computed by using (5.8) and the integration over $P$ is performed by using a 20×24 rectangular grid.

Figures 5.27 and 5.28 represent the real part of $w_0 + \varepsilon w_1^g$ by using a different scale in the colormap.

In Figure 5.29 we show the details of the real part of $w_1^g$, by plotting the real part of each guided mode supported by the three waveguides.

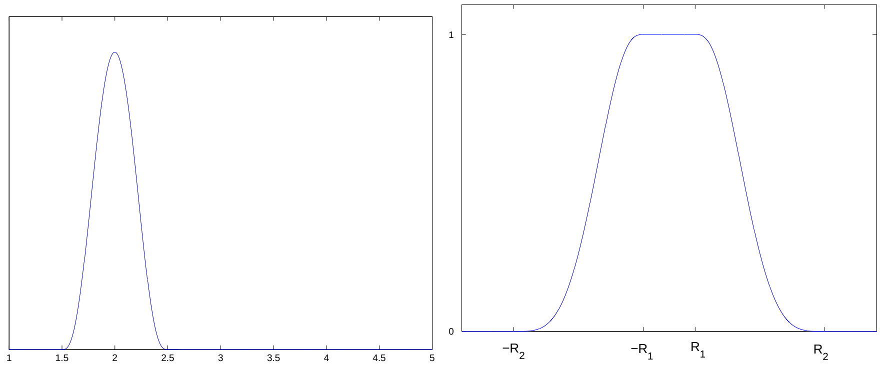


Figure 5.25: In the first figure we show the function $S$ which describe the perturbation. In the latter, our choice of the function $T$ is represented.

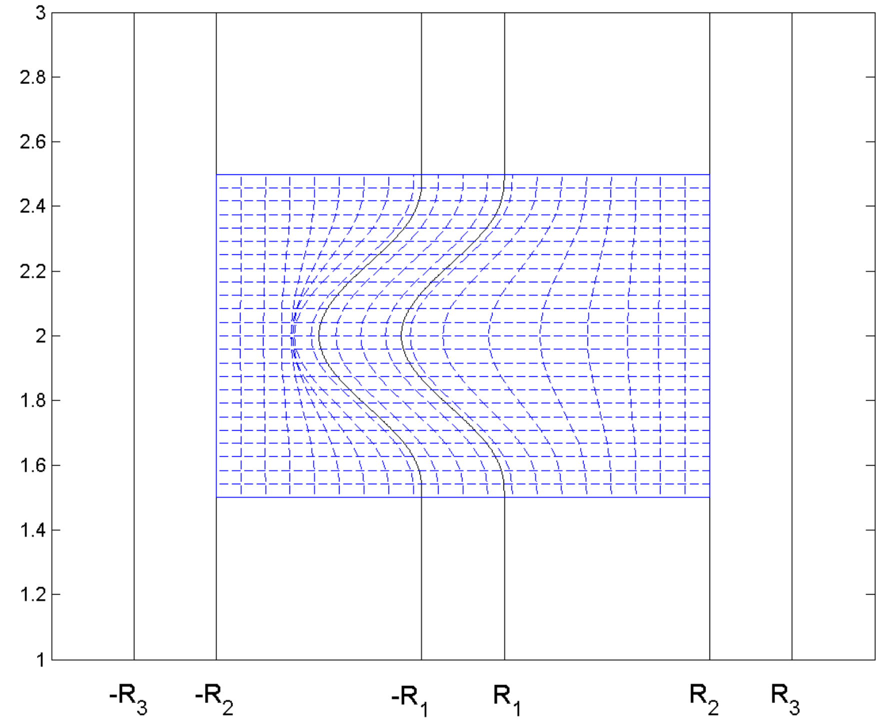


Figure 5.26: The effect of $\Gamma$ on the $(x, z)$-plane.

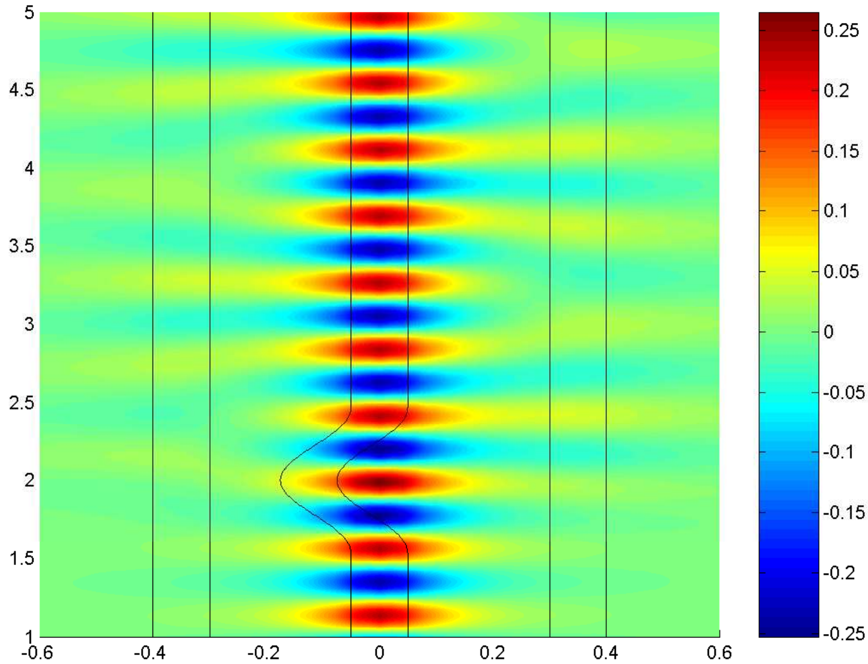


Figure 5.27: Real part of the guided component of $w_0 + \varepsilon w_1^g$.

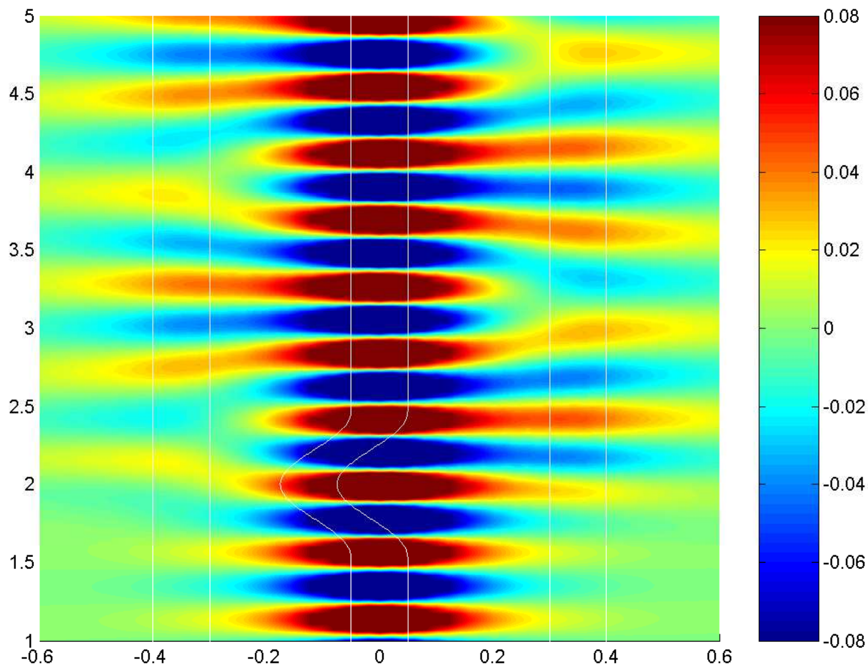


Figure 5.28: Real part of the guided component of $w_0 + \varepsilon w_1^g$ with a different scale in the colormap with respect of the previous figure.

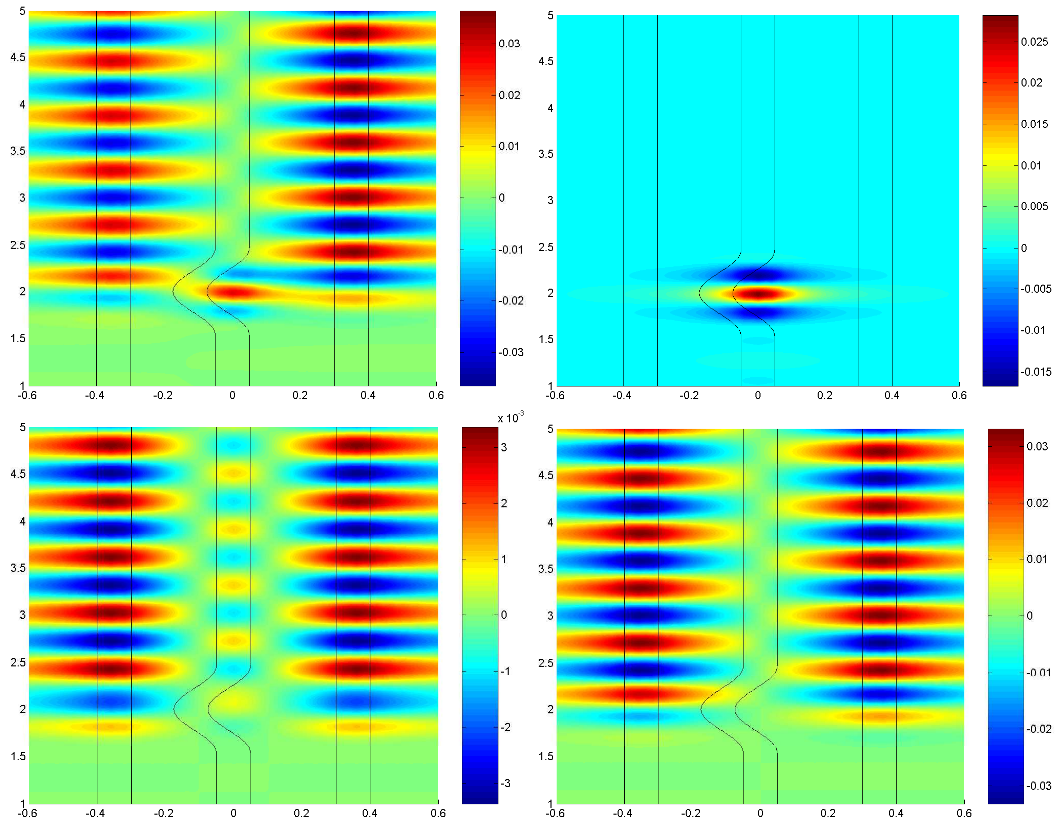


Figure 5.29: In the first figure the guided part of $w_1$ is represented. In the remaining figures we show the different components of $w_1^g$ corresponding to the guided modes supported by the whole configuration of waveguides.

# Conclusions and future work

In this thesis we studied the wave propagation in 2-D optical waveguides. Most of the results presented in this thesis are based on the knowledge of a Green's function for the problem of the wave propagation in rectilinear waveguides. Such a Green's function was already obtained in [MS] and it has been generalized in Chapter 2 by using a technique adopted in [AC1].

The main goal of this thesis was to propose a rigorous method for the study of wave propagation in non-rectilinear waveguides. In particular, we studied the case of small perturbations by proving an existence theorem and by proposing a resolution scheme. Such a scheme was used to perform numerical simulations of cases which are interesting for the applications.

Another remarkable result concerns the study of the uniqueness of the solutions for the kind of Helmholtz equation considered in this thesis. In particular, we proposed a condition which says that only the radiating part of the solution must satisfies a Sommerfeld-like radiation condition and we have to add a different condition for the guided part of the solution.

Most of the results presented in this thesis deal with 2-D optical waveguides. The extension to the 3-D case will be object of future work.

In the numerical scheme proposed for the non-rectilinear waveguides, we assumed that the solution has a Neumann series. What remains to prove is that the solution in analytical in the parameter $\varepsilon$, i.e. that the solution has a converging Neumann series. This will be the object of future research.

Finally, we will investigate how to generalize the uniqueness theorem proved in Chapter 3 to the 3-D and non-rectilinear case. Furthermore, we believe that such results will be useful for the construction of absorbing boundary conditions for waveguide problems, i.e. to impose artificial boundary conditions on a finite domain in order to approximate the problem in the whole space.

# Appendix A

# Asymptotic methods

In this appendix, we give a description of the techniques we use to find the asymptotic expansions in Section 2.7. More details and a rigorous derivation of these methods can be found in [BO], [Mu], [Er1] and [Co].

## A.1 Integration by parts

We consider integrals of the form:

$$\int_a^b e^{Rh(\mu)} g(\mu) d\mu, \quad R \to +\infty, \tag{A.1}$$

where $b$ can be also $+\infty$. When $h'(\mu) \neq 0$ in the domain of integration, we can use integration by parts to evaluate the asymptotic behaviour of (A.1):

$$\int_a^b e^{Rh(\mu)} g(\mu) d\mu = \frac{e^{Rh(\mu)} g(\mu)}{Rh'(\mu)}\Big|_a^b - \frac{1}{R}\int_a^b e^{Rh(\mu)} \left(\frac{g(\mu)}{h'(\mu)}\right)' d\mu.$$

By iterating integration by parts, we find

$$\int_a^b e^{Rh(\mu)} g(\mu) d\mu = \frac{e^{Rh(\mu)} g(\mu)}{Rh'(\mu)}\Big|_a^b + \mathcal{O}\left(\frac{1}{R^2}\right).$$

Higher order approximations can be derived by iterating integration by parts.

## A.2 The method of the stationary phase

### A.2.1 The leading order term

When integration by parts fails, the integrals in this thesis can be always expanded by using the method of the stationary phase.

Consider and integral of the following form:

$$\int_a^b e^{iRh(\mu)} g(\mu) d\mu, \quad R \to +\infty. \tag{A.2}$$

Under suitable assumptions, if $h'(\mu) = 0$ only for $\mu = \mu_0 \in (a, b)$, the method of the stationary phase gives

$$\int_a^b e^{iRh(\mu)} g(\mu) d\mu = g(\mu_0) \left[ \frac{2\pi}{R|h''(\mu_0)|} \right]^{\frac{1}{2}} e^{i(Rh(\mu_0) \pm \pi/4)} + \mathcal{O}\left( \frac{1}{R} \right), \quad R \to +\infty, \tag{A.3}$$

where the positive and negative signs in the exponential corresponds respectively to $h''(\mu_0) > 0$ and $h''(\mu_0) < 0$.

If $\mu_0 = a$, the above formula has a slightly different form:

$$\int_a^b e^{iRh(\mu)} g(\mu) d\mu = e^{iRh(a)} \left( \frac{\pi}{2|f''(a)|R} \right)^{\frac{1}{2}} e^{i \, \mathrm{sign}(f''(a))\pi/4} g(a), \quad R \to +\infty. \tag{A.4}$$

### A.2.2 Higher order terms

In a very general setting, Erdélyi (see [Er1] and [Er2]) wrote a formula which gives the higher order terms in the method of the stationary phase. Unfortunately, such a formula is very difficult to use. In [Co], the author was able to write down explicit formulas for the first three order terms of the asymptotic expansion of (A.2). Furthermore, in a simpler (but commonest) case, he was able to give explicit formulas for the first four order terms. Those results are reported in the following.

We consider an integral as in (A.2) and suppose that there exists only one $\mu_0 \in [a, b]$ such that $h'(\mu_0) = 0$. Furthermore, we will suppose that $h''(\mu_0) \neq 0$ and that $g$ is not singular in $[a, b]$. With such assumptions we give the asymptotic expansion formulas for three different cases: $\mu_0 = a$, $\mu_0 = b$, $\mu_0 \in (a, b)$.

We set $\varsigma = \mathrm{sign}(h''(\mu_0))$ and

$$\begin{aligned}
a_0 &= \sqrt{2} g / |h''|^{\frac{1}{2}} \big|_{\mu=\mu_0}, \\
a_1 &= 2\varsigma (3g'h'' - gh''') / 3(h'')^2 \big|_{\mu=\mu_0}, \\
a_2 &= \sqrt{2} \big[ 5g(h''')^2 - 3gh''h^{(iv)} - 12g'h''h''' + 12g''(h'')^2 \big] / 6|h''|^{\frac{7}{2}} \big|_{\mu=\mu_0}, \\
a_3 &= \big[ -(32/9)g(h''')^3 + 4gh''h'''h^{(iv)} - (4/5)g(h'')^2 h^{(v)} + 8g'h''(h''')^2 \\
&\qquad - 4g'(h'')^2 h^{(iv)} - 8g''(h'')^2 h''' + 4g'''(h'')^3 \big] / |h''|^5 \big|_{\mu=\mu_0}.
\end{aligned}$$

If $\mu_0 = a$, we have:

$$(A.5)\quad \int_a^b e^{iRh(\mu)} g(\mu) d\mu = \left[ \frac{1}{iR} \frac{g e^{iRh}}{h'} - \frac{1}{(iR)^2} \frac{1}{h'} \frac{d}{d\mu} \left( \frac{g}{h'} \right) e^{iRh} + \dots \right]_{\mu=b} +$$

$$\frac{e^{iRh(\mu_0)}}{2} \left( \frac{\pi}{R} \right)^{\frac{1}{2}} e^{\varsigma i\pi/4} \left\{ a_0 + \frac{a_2}{1!(4\varsigma iR)} + \frac{a_4}{2!(4\varsigma iR)^2} - \dots \right\} -$$

$$\frac{e^{iRh(\mu_0)}}{2} \left\{ \frac{a_1}{\varsigma iR} - \frac{a_3}{3!(\varsigma iR)^2} + \frac{2! a_5}{5!(\varsigma iR)^3} - \dots \right\}.$$

as $R \to +\infty$. If $\mu_0 = b$:

(A.6)

$$\int_a^b e^{iRh(\mu)} g(\mu) d\mu = \frac{e^{iRh(\mu_0)}}{2} \left( \frac{\pi}{R} \right)^{\frac{1}{2}} e^{\varsigma i\pi/4} \left\{ a_0 + \frac{a_2}{1!(4\varsigma iR)} + \frac{a_4}{2!(4\varsigma iR)^2} - \dots \right\} +$$

$$\frac{e^{iRh(\mu_0)}}{2} \left\{ \frac{a_1}{\varsigma iR} - \frac{a_3}{3!(\varsigma iR)^2} + \frac{2! a_5}{5!(\varsigma iR)^3} - \dots \right\} -$$

$$\left[ \frac{1}{iR} \frac{g e^{iRh}}{h'} - \frac{1}{(iR)^2} \frac{1}{h'} \frac{d}{d\mu} \left( \frac{g}{h'} \right) e^{iRh} + \dots \right]_{\mu=a}$$

as $R \to +\infty$. Finally, if $\mu_0 \in (a, b)$, by adding the above two results we have:

$$(A.7)\quad \int_a^b e^{iRh(\mu)} g(\mu) d\mu = \left[ \frac{1}{iR} \frac{g e^{iRh}}{h'} - \frac{1}{(iR)^2} \frac{1}{h'} \frac{d}{d\mu} \left( \frac{g}{h'} \right) e^{iRh} + \dots \right]_a^b +$$

$$e^{iRh(\mu_0)} \left( \frac{\pi}{R} \right)^{\frac{1}{2}} \left\{ a_0 + \frac{a_2}{1!(4\varsigma iR)} + \frac{a_4}{2!(4\varsigma iR)^2} - \dots \right\}.$$

# Appendix B

# A change of coordinates

## B.1 The operator $L_\varepsilon$

In this appendix we describe how to obtain a different expression for the operator $L_\varepsilon$ by using a general change of coordinates. This is useful for what we do in Chapter 5, when we make a geometric transformation of the plane, which "adapt" to the profile of the index of refraction $n$.

In this section, we will use the following notations: $(x_1, x_2)$ and $(\xi_1, \xi_2)$ are two system of coordinates in $\mathbb{R}^2$ and

$$f_{i;j_1,\dots,j_r} := \frac{\partial^r f_i}{\partial x_{j_1} \dots \partial x_{j_r}}, \quad g_{x_j} := \frac{\partial g}{\partial x_j}.$$

As usually, $\delta_{ij}$ will denote the Kronecker delta.

We use a $C^2$ invertible function

$$\Gamma^\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$$

to change the coordinates:

$$(x_1, x_2) = \Gamma^\varepsilon(\xi_1, \xi_2) : \begin{cases} x_1 = \gamma_1^\varepsilon(\xi_1, \xi_2), \\ x_2 = \gamma_2^\varepsilon(\xi_1, \xi_2). \end{cases} \tag{B.1}$$

In order to simplify the notations, we will omit the dependence on the parameter $\varepsilon$, i.e.:

$$x_i = \gamma_i(\xi_1, \xi_2), \quad i = 1, 2.$$

We differentiate the above equations with respect to $x_j$, $j = 1, 2$, and obtain

$$\delta_{ij} = \sum_{r=1}^{2} \xi_{r;j} \gamma_{i;r}, \quad i, j = 1, 2.$$

We note that the Jacobian is

$$J := \frac{\partial(\xi_1,\xi_2)}{\partial(x_1,x_2)} = \gamma_{1;1}\gamma_{2;2} - \gamma_{1;2}\gamma_{2;1}.$$

We suppose $J > 0$. We have

$$\nabla_{(x_1,x_2)}\xi_1 = \frac{1}{J}(\gamma_{2;2}, -\gamma_{1;2}), \quad \nabla_{(x_1,x_2)}\xi_2 = \frac{1}{J}(-\gamma_{2;1}, \gamma_{1;1}), \tag{B.2}$$

and then

$$|\nabla\xi_i|^2 = \frac{1}{J^2}\sum_{r=1}^{2}\gamma^2_{r;3-i}, \quad i = 1,2. \tag{B.3}$$

From (B.2) we can find the second derivatives of $\xi_i$, $i = 1,2$. In particular we have:

$$\Delta\xi_1 = \tfrac{1}{J^2}\textstyle\sum_{r=1}^{2}(-1)^{r+1}[\gamma^2_{3-r;2,r}J - \gamma^2_{3-r;2}J_{x_r}],$$

$$\Delta\xi_2 = \tfrac{1}{J^2}\textstyle\sum_{r=1}^{2}(-1)^{r}[\gamma^2_{3-r;1,r}J - \gamma^2_{3-r;1}J_{x_r}],$$

i.e.

$$\Delta\xi_i = \frac{1}{J^2}\sum_{r=1}^{2}(-1)^{r+i}[\gamma^2_{3-r;3-i,r}J - \gamma^2_{3-r;3-i}J_{x_r}], \quad i = 1,2. \tag{B.4}$$

We set $w(\xi_1,\xi_2) = u(x_1,x_2)$ and have:

$$\Delta u = \sum_{i,j=1}^{2} w_{\xi_i\xi_j}(\nabla\xi_i\cdot\nabla\xi_j) + \sum_{i=1}^{2} w_{\xi_i}\Delta\xi_i. \tag{B.5}$$

Finally, from $L_\varepsilon u = f$, we find that $w$ satisfies

$$L^\varepsilon w := \sum_{i,j=1}^{2} w_{\xi_i\xi_j}(\nabla\xi_i\cdot\nabla\xi_j) + \sum_{i=1}^{2} w_{\xi_i}\Delta\xi_i + k^2\tilde{n}(\xi_1,\xi_2)^2 w = \tilde{f}(\xi_1,\xi_2), \qquad (\xi_1,\xi_2)\in\mathbb{R}^2, \tag{B.6}$$

where $\nabla\xi_i$ and $\Delta\xi_i$ are given by (B.2), (B.3) and (B.4) and

$$\begin{aligned}\tilde{n}(\xi_1,\xi_2) &= n(x_1(\xi_1,\xi_2), x_2(\xi_1,\xi_2)),\\ \tilde{f}(\xi_1,\xi_2) &= f(x_1(\xi_1,\xi_2), x_2(\xi_1,\xi_2)).\end{aligned}$$

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