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An estimate for the resolvent of a non-self-adjoint differential operator on the half-line

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We consider the operator defined by $(T_0 y)(x) = -y'' + q(x)y \ \ (x > 0)$ on the domain $Dom\ (T_0) = \{f \in L^2(0,\infty) : f'' \in L^2(0,\infty),\ f(0) = 0\}$. Here $q(x) = p(x) + ib(x)$, where $p(x)$ and $b(x)$ are real functions satisfying the following conditions: $b(x)$ is bounded on $[0,\infty)$, there exists the limit $b_0 := \lim_{x\to\infty} b(x)$ and $b(x) - b_0 \in L^2(0,\infty)$. In addition, $\inf_x p(x) > \sup_x |b_1(x)|$. We derive an estimate for the norm of the resolvent of T_0, as well as prove that $(T_0 - ib_0 I)^{-1}$ is a sum of a normal operator and a quasinilpotent one, and these operators have the same invariant subspaces. © *2011 American Institute of Physics*. [doi:10.1063/1.3578931]

I. INTRODUCTION AND STATEMENT OF THE BASIC LEMMA

The present paper is devoted to the resolvent of a non-self-adjoint differential operator on the half-line. The literature on the spectral theory of ordinary differential operators is very rich. In particular, the book[16] contains the classical results on scalar self-adjoint operators and Dirac systems. The monograph[24] considers the interplay between spectral and oscillatory properties of both finite and infinite systems of linear ordinary differential self-adjoint operators. These can be written as single differential equations with matrix-valued and (bounded) operator-valued coefficients, respectively. The book[19] studies non-self-adjoint boundary eigenvalue problems for first-order systems of ordinary differential equations and nth-order scalar differential equations. The treatment is based on functional analytic methods. The eigenvalues and completeness for scalar regular and simply irregular two-point higher order differential operator are deeply investigated in the monographs.[7,17,18] The paper[8] is concerned with the boundary value problem,

$$y^{(n)}(x) + p_2(x)y^{(n-2)}(x) + \cdots + p_{n-1}(x)y'(x) + p_n(x)y(x) = \lambda y(x), \quad 0 \le x \le 1,$$

under certain boundary conditions. Besides, the coefficients are nonsmooth in general. By using an iterative method, the authors obtain asymptotic formulas of any order for the simple eigenvalues λ_k and the eigenfunctions of the considered non-self-adjoint differential operator. The essential spectrum of various singular matrix differential operators is investigated in Refs. 6, 9, 20, 23, and 25. In Ref. 25 the authors consider the problem of localization of the spectrum of non-self-adjoint differential operators on unbounded domains with power coefficients. To find the location of spectrum points in the complex plane, they use isospectral deformations of differential operators and the properties of families of closed operators analytic in the Kato sense. Furthermore, in Ref. 10 the spectrum of a class of fourth-order left-definite differential operators is studied. The paper[26] deals with a self-adjoint differential operator in a Hilbert space; the author obtains a criterion for the discreteness of the spectrum of the operator T and a criterion for the uniform positivity of the considered operator. In Ref. 6, necessary and also simple sufficient conditions are given for self-adjoint operators associated with the second-order linear differential expression $\tau(y) = \frac{1}{w}(-(py')' + qy)$ on $[a, b)$ to have a discrete spectrum. Here, the coefficients of τ are non-negative and satisfy minimal smoothness conditions.

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Let $L^2 = L^2(0, \infty)$ be the complex Hilbert spaceof scalar functions defined on $[0, \infty)$ with the scalar product,

$$(f, h) = \int_0^\infty f(x)\overline{h}(x)dx \quad (f, h \in L^2),$$

and the norm $\|.\| = \|.\|_{L^2} = \sqrt{(.,.)}$. The first fundamental results in the spectral analysis of the operator,

$$(T_0 y)(x) = -y'' + q(x)y \quad (x > 0), \tag{1.1}$$

on the domain,

$$Dom\ (T_0) = Dom_0 := \{f \in L^2 : f'' \in L^2, f(0) = 0\},$$

belong to Naimark.[22] Here q is a measurable complex-valued function. In particular, he has proved that some of the poles of the resolvent kernel are not the eigenvalues of the operator. He has also shown that these poles, which are called spectral singularities, belong to the continuous spectrum. Besides, he has derived the following result: if

$$\int_0^\infty |q(x)|e^{\epsilon x}dx < \infty,$$

then the eigenvalues and the spectral singularities are of finite number, and each of them is of finite multiplicity. Similar problems for the non-self-adjoint differential operators on the whole axis have been considered in the paper.[2] Bounds for the spectrum of a non-self-adjoint matrix differential operator on a segment and norm estimates for its resolvent were investigated in Ref. 13. Certainly, we could not survey the whole subject here and refer the reader to the above listed publications and references given therein.

At the same time, norm estimates for the resolvent of a non-self-adjoint differential operator on the half-line are almost not investigated although they are very important for various applications.

The aim of this paper is to derive a norm estimate for the non-self-adjoint differential operator defined by (1.1) on the domain $Dom\ (T_0)$. So the boundary conditions,

$$y(0) = 0 \text{ and } y \in L^2, \tag{1.2}$$

are imposed. Put $p(x) = Re\ q(x)$ and $b(x) = Im\ q(x)$. So

$$(T_0 y)(x) = -y'' + (p(x) + ib(x))y \quad (x > 0;\ y \in\ Dom_0).$$

In the present paper it is assumed that $b(x)$ is bounded on $[0, \infty)$, *there exists the limit* $b_0 := \lim_{x\to\infty} b(x)$ *and*

$$b_1(x) := b(x) - b_0 \in L^2. \tag{1.3}$$

In addition, $p(x)$ is bounded from below on $[0, \infty)$. Without loss of generality assume that

$$m := \inf_x p(x) > \sup_x\ |b_1(x)|. \tag{1.4}$$

If $\inf_x p(x) \le \sup_x\ |b_1(x)|$, then below, instead of T_0 we can consider the operators $T_0 + cI$ with $c > -\inf_x p(x) + \sup_x\ |b_1(x)|$. By I we denote the unit operator in the corresponding space. We will prove that operator $T_0 - ib_0 I$ has a bounded inverse, whose imaginary Hermitian component is a Hilbert–Schmidt operator. This fact allows us to derive an estimate for the norm of the resolvent of T_0, as well as to prove that $(T_0 - ib_0 I)^{-1}$ can be represented as a sum of a normal operator and a Hilbert–Schmidt quasinilpotent operator, and these operators have the same invariant subspaces. Besides, we have followed the ideas of the papers.[3,5] Finally, we present some special expansion for the operator $(T_0 - ib_0 I)^{-1}$.

Introduce the notations. For a linear operator A, $\sigma(A)$ is the spectrum, $R_\lambda(A) := (A - I\lambda)^{-1}$ is the resolvent, $\lambda_k(A)$ $(k = 1, 2, \ldots)$ are the eigenvalues with their multiplicities, A^* is the adjoint operator, $A_R = (A + A^*)/2$ and $A_I = (A - A^*)/2i$ are the real and imaginary Hermitian components, respectively; $\|A\|$ is the operator norm; $N_p(A)$ $(1 \le p < \infty)$ is the Schatten-von Neumann norm: $N_p(A) = [Trace\ (AA^*)^{p/2}]^{1/p}$; S_p is the ideal of the Schatten-von Neumann operators.

We have

$$T_0 = T + ib_0 I, \tag{1.5}$$

where T is the operator defined by

$$Ty(x) = -y'' + p(x)y + ib_1(x)y \quad (y \in Dom_0).$$

As it was already mentioned, we will prove that operator T has the bounded inverse T^{-1}. Put $(T^{-1})_I = (T^{-1} - (T_0^{-1})^*)/2i$, $(T^{-1})_R := (T^{-1} + (T^{-1})^*)/2$,

$$\eta := \frac{1}{m - \sup_x |b_1(x)|},\ \zeta := \eta(1 + \eta \sup_x |b_1(x)|) \text{ and } \tau := \frac{\zeta}{2\sqrt{m}}\|b_1\|.$$

Lemma 1.1: Let conditions (1.3) and (1.4) hold. Then T is invertible and its inverse satisfies the inequalities,

$$\|T^{-1}\| \le \eta, \tag{1.6}$$

$$\|(T^{-1})_I\| \le \eta^2 \sup_x |b_1(x)|, \tag{1.7}$$

and

$$\|(T^{-1})_R\| \le \zeta. \tag{1.8}$$

Moreover, $(T^{-1})_I$ *is a Hilbert–Schmidt operator and*

$$N_2((T^{-1})_I) \le \tau. \tag{1.9}$$

The proof of this lemma is presented in Sec. II.

In Sec. II we also show that Lemma 1.1 implies the following result.

Corollary 1.2: Let conditions (1.3) and (1.4) hold. Then the nonreal spectrum of operator $T^{-1} = (T - ib_0 I)^{-1}$ *consists of isolated eigenvalues* $\lambda_k(T^{-1})$ $(k = 1, 2, \ldots,)$ *counted with multiplicities and satisfying the inequality,*

$$\sum_{k=1}^{\infty} |Im\ \lambda_k(T^{-1})|^2 = \sum_{k=1}^{\infty} \left| Im\ \frac{1}{\lambda_k(T_0) - ib_0} \right|^2 \le \tau^2. \tag{1.10}$$

Moreover,

$$|Im\ \frac{1}{\lambda_k(T)}| = \left| Im\ \frac{1}{\lambda_k(T_0) - ib_0} \right| \le \eta^2 \sup_x |b_1(x)| \quad (k = 1, 2, \ldots) \tag{1.11}$$

and any point $\mu \in \sigma(T)$ *satisfies the inequalities,*

$$0 \le Re\ \frac{1}{\mu} \le \zeta. \tag{1.12}$$

II. PROOF OF LEMMA 1.1

The real Hermitian component E of T is defined by

$$Ey := -y'' + p(x)y,$$

and the imaginary Hermitian component of T is the bounded operator B_1 defined by $(B_1 f)(x) = b_1(x)f(x)$ $(f \in L^2)$. Define on the domain Dom_0, also the operator E_m by $E_m y := -y'' + my$.

Lemma 2.1: Let $m = \inf_{x\ge 0} p(x) > 0$. Then E is invertible, and

$$\|E^{-1}h\| \le \|E_m^{-1}h\| = \frac{\|h\|}{m} \quad (h \in L^2). \tag{2.1}$$

Proof: For any $y \in Dom_0$, we obtain

$$(Ey, y) \ge -(y'', y) + m(y, y) = (E_m y, y) \ge m(y, y).$$

Since E is self-adjoint, hence (2.1) follows. □

We have

$$(E_m^{-1} f)(x) = \int_0^\infty G_m(x, s) f(s) ds \quad (f \in L^2),$$

where G_m is the Green function to operator E_m,

$$G_m(x, s) = \begin{cases} \phi_m(x)\psi_m(s) & \text{if } s < x, \\ \phi_m(s)\psi_m(x) & \text{if } x \le s, \end{cases}$$

with

$$\phi_m(x) = \frac{sh\ (\nu x)}{\nu} \text{ and } \psi_m(x) = e^{-\nu x} \quad (\nu = +\sqrt{m};\ sh\ (x) = (e^x - e^{-x})/2).$$

Lemma 2.2: Let $b_1 \in L^2$. Then $B_1 E_m^{-1}$ is a Hilbert–Schmidt operator. Moreover,

$$N_2(B_1 E_m^{-1}) \le \frac{\|b_1\|}{2\sqrt{m}}.$$

Proof: We have

$$N_2^2(B_1 E_m^{-1}) = \int_0^\infty \int_0^\infty b_1^2(x) G_m^2(x, s) ds\ dx$$

$$= \frac{1}{\nu} \int_0^\infty b_1^2(x)[sh^2\ (\nu x) \int_x^\infty e^{-2\nu s} ds + e^{-2\nu x} \int_0^x sh^2\ (\nu s) ds] dx.$$

Take into account that

$$sh^2\ (\nu x) = \frac{1}{4}(e^{2\nu x} - 2 + e^{-2\nu x}) \le \frac{1}{2} ch\ (2\nu x) \quad (ch\ (x) = (e^x + e^{-x})/2).$$

So

$$N_2^2(B_1 E_m^{-1}) \le \frac{1}{4\nu^2} \int_0^\infty b_1^2(x) e^{-2\nu x}[sh\ (2\nu x) + ch\ (2\nu x)] dx \le \frac{1}{4m}\|b_1\|^2 < \infty.$$

The lemma is proved. □

Lemma 2.3: Let A_1, A_2, and C be bounded operators in a separable Hilbert space H with a norm $\|.\|_H$, and $\|A_1 h\|_H \le \|A_2 h\|_H$ for all $h \in H$. If, in addition, $N_2(A_2 C) < \infty$, then $N_2(A_1 C) \le N_2(A_2 C)$.

Proof: We have

$$N_2^2(A_2 C) = \sum_{k=1}^\infty \|A_2 C e_k\|_H^2 \ge \sum_{k=1}^\infty \|A_1 C e_k\|_H^2 = N_2^2(A_1 C),$$

where $\{e_k\}$ is an arbitrary orthonormal basis. As claimed. □

Now Lemmas 2.2 and 2.3 imply.

Corollary 2.4: Let $b_1 \in L^2$ and $m = \inf_{x\geq 0} p(x) > 0$. Then $B_1 E^{-1}$ is a Hilbert–Schmidt operator. Moreover,

$$N_2(B_1 E^{-1}) \leq \frac{\|b_1\|_{L^2}}{2\sqrt{m}}.$$

Proof of Lemma 1.1: For any $y \in Dom_0$, due to Lemma 2.1, the inequalities,

$$\|Ty\| = \|(E + iB_1)y\| \geq m\|y\| - \|B_1 y\| \geq (m - \sup_x |b_1(x)|)\|y\| = \|y\|/\eta$$

are valid. But $Ty = (I + B_1 E^{-1})E$ and thanks to the previous corollary $B_1 E^{-1}$ is compact. Thus T is invertible and (1.6) holds.

Further, take into account that

$$B_1 T^{-1} = B_1 E^{-1} E T^{-1} = B_1 E^{-1} E(E + iB_1)^{-1} \text{ and } E(E + iB_1)^{-1} = I - iB_1 T^{-1}.$$

Therefore,

$$\|ET^{-1}\| = \|I - iB_1 T^{-1}\| \leq 1 + \|B_1\|\|T^{-1}\| \leq 1 + \eta \sup_x |b_1(x)|, \tag{2.2}$$

and by the previous corollary,

$$N_2(B_1 T^{-1}) \leq N_2(B_1 E^{-1})\|I - iB_1 E^{-1}\| \leq \frac{\|b_1\|}{2\sqrt{m}}(1 + \|B_1\|\|T^{-1}\|) \leq \tag{2.3}$$

$$\frac{\|b_1\|}{2\sqrt{m}}(1 + \eta \sup_x |b_1(x)|).$$

But

$$(T^{-1})_I := (T^{-1} - (T^{-1})^*)/2i = (T^{-1})^* B_1 T^{-1}. \tag{2.4}$$

Hence, $N_2(T_I^{-1}) \leq \|T^{-1}\| N_2(B_1 T^{-1})$. Now (2.3) implies that $(T^{-1})_I$ is a Hilbert–Schmidt operator and (1.9) holds.

Furthermore, by (2.4) and (1.6) we have inequality (1.7).

Finally, take into account that $(T^{-1})_R := (T^{-1} + (T^{-1})^*)/2 = (T^{-1})^* E T^{-1}$. Hence by (2.2) and (1.6) we get (1.8). The proof is complete. □

Proof of Corollary 1.2: Inequality (1.10) is due to (1.9) and the Weyl inequalities.[14] Inequality (1.11) is due to (1.7) and the just mentioned Weyl inequalities. To check (1.12) note that E is positively defined self-adjoint operator and therefore $(T^{-1})_R := (T^{-1})^* E T^{-1}$ is positively defined. Thus T^{-1} is a dissipative operator, therefore its spectrum is in the open right half-plane and by (1.8), any point $\mu \in \sigma(T)$ satisfies inequality (1.12). □

III. THE MAIN RESULT

Denote

$$\psi(\lambda) := \inf_{s \in \sigma(T_0)} \frac{|\lambda - s|}{|s - ib_0|} \text{ and } \Phi(y) := y\sqrt{e}\eta\, e^{y^2\tau^2} \quad (y > 0).$$

Now we are in a position to formulate our main result.

Theorem 3.1: *Let conditions (1.3) and (1.4) hold. Then*

$$\|(I\lambda - T_0)^{-1}\| \leq \Phi(1/\psi(\lambda)) \quad (\lambda \notin \sigma(T_0)). \tag{3.1}$$

Proof: We need the following result proved in Ref. 11, Theorem 7.7.1]: let A be a bounded operator in a separable Hilbert space H with the norm $\|.\|_H$ and $A_I \in S_2$. Then

$$\|(A - \lambda I)^{-1}\|_H \leq \frac{1}{\rho(A, \lambda)} exp\, [\frac{1}{2}(1 + \frac{u^2(A)}{\rho^2(A, \lambda)})] \quad (\lambda \notin \sigma(A)), \tag{3.2}$$

where $\rho(A,\lambda) = inf_{s\in\sigma(A)}|s-\lambda|$ and

$$u(A) := \sqrt{2[N_2^2(A)_I) - \sum_{k=1}^{\infty}|Im\ \lambda_k(A)|^2]} \leq \sqrt{2}N_2(A).$$

Here, $\lambda_k(A), k = 1, 2, \ldots$ are nonreal eigenvalues of A counted with their multiplicities. From (3.2) it follows that

$$\|(I-\lambda A)^{-1}\|_H \leq \frac{1}{\rho(1,A\lambda)} exp\ [\frac{1}{2}(1+\frac{u^2(A)}{\rho^2(1,A\lambda)})]\ \ (\frac{1}{\lambda}\notin\sigma(A)), \tag{3.3}$$

where $\rho(1, A\lambda) := \inf_{s\in\sigma(A)}|1-s\lambda|$. But $(T-I\lambda)^{-1} = T^{-1}(I-\lambda T)^{-1}$ and

$$\rho(1,\lambda T^{-1}) = \inf_{s\in\sigma(T^{-1})}|1-\lambda s| = \inf_{z\in\sigma(T)}|1-\frac{\lambda}{z}|.$$

Due to (3.3), we thus get

$$\|(I-\lambda T^{-1})\| \leq \frac{1}{\rho(1,\lambda T^{-1})} exp\ [\frac{1}{2}(1+\frac{u^2(T^{-1})}{\rho^2(1,\lambda T^{-1})})]\ \ (\lambda\notin\sigma(T)).$$

By (1.9) $u^2(T^{-1}) \leq 2\tau^2$. So

$$\|(I-\lambda T^{-1})^{-1}\| \leq \frac{\sqrt{e}}{\rho(1,\lambda T^{-1})} exp\ [\frac{\tau^2}{\rho^2(1,\lambda T^{-1})}].$$

Hence (1.6) implies

$$\|(I\lambda - T)^{-1}\| = \|T^{-1}(I-\lambda T)^{-1}\| \leq \frac{\eta}{\rho(1,\lambda T^{-1})} exp\ [\frac{1}{2}+\frac{\tau^2}{\rho^2(1,\lambda T^{-1})}]\ \ (\lambda\notin\sigma(T)). \tag{3.4}$$

Take into account that $(I\lambda - T_0)^{-1} = (I(\lambda - ib_0) - T)^{-1}$. Then by (3.4),

$$\|(I\lambda - T_0)^{-1}\| \leq \frac{\eta}{\rho(1,(\lambda-ib_0)T^{-1})} exp\ \left[\frac{1}{2}+\frac{\tau^2}{\rho^2(1,(\lambda-ib_0)T^{-1})}\right]. \tag{3.5}$$

According to (1.5), $z\in\sigma(T)$ implies that $z+ib_0\in\sigma(T_0)$ and

$$\inf_{z\in\sigma(T)}|1-\frac{\lambda}{z}| = \inf_{s\in\sigma(T_0)}|1-\frac{\lambda}{s-ib_0}|.$$

Hence,

$$\inf_{z\in\sigma(T)}|1-\frac{\lambda-ib_0}{z}| = \psi(\lambda).$$

Now (3.5) proves the theorem. □

The previous theorem implies

Corollary 3.2: Let conditions (1.3) and (1.4) hold, and C be a linear operator in L^2 with the domain Dom_0 and operator $C - T_0$ be bounded. If $\lambda\notin\sigma(T_0)$ and

$$\|C - T_0\|\Phi(1/\psi(\lambda)) < 1,$$

then λ is a regular point for C, and

$$\|R_z(C)\| \leq \frac{\Phi(1/\psi(\lambda))}{1-\|C-T_0\|\Phi(1/\psi(\lambda))}.$$

So the spectrum of C lies in the set $\{z\in\mathbb{C} : \|C-T_0\|\Phi(1/\psi(z)) \geq 1\}$.

IV. TRIANGULAR STRUCTURE OF OPERATORS WITH SCHATTEN-VON NEUMANN HERMITIAN COMPONENTS

The aim of this section is to prove an auxiliary theorem by which in Sec. V we obtain the triangular representation of the resolvent.

Theorem 4.1: *Let A be a bounded linear operator in a separable Hilbert space H with $A_I \in S_p$ $(2 \le p < \infty)$. Then there are an orthogonal resolution of the identity E_t $(0 \le t \le 1)$ in H, a E-measurable bounded function $\gamma(t)$ defined on $[0, 1]$, and a quasinilpotent operator $V \in S_p$, such that*

$$A = \int_0^1 \gamma(t) dE_t + V \tag{4.1}$$

and

$$E_t V E_t = V E_t. \tag{4.2}$$

Proof: Let $C_n = C_n^*$ $(n = 1, 2, \ldots)$ be the operators having n-dimensional ranges and converging to A_I in the norm N_p, and $B_n = B_n^*$ the operators having n-dimensional ranges and converging strongly to $A_R = (A + A^*)/2$. Then $A_n = B_n + iC_n \to A$ strongly as $n \to \infty$. By the Schur theorem $A_n = Z_n + V_n$ $(\sigma(A_n) = \sigma(Z_n))$, where Z_n is a normal matrix and V_n is a nilpotent matrix.

Let S_n and K_n be the real and imaginary Hermitian components of Z_n, respectively. Since the norms of V_n, S_n, and K_n $(n = 1, 2, \ldots)$ are uniformly bounded, there is a subsequence ν of natural numbers, such that operators V_n, S_n, and K_n $(n \in \nu)$ weakly converge. Denote their limits by V, S, and K, respectively. So $A = S + iK + V$ with $S = S^*$ and $K = K^*$.

Let V_{nI} and A_{nI} be the imaginary components of V_n and A_n, respectively. The Weyl inequalities,[14] imply that $N_p(K_n) \le N_p(A_{nI})$. Since $N_p(K_n + V_{nI}) = N_p(A_{nI})$, we obtain the inequality $N_p(V_{nI}) \le N_p(A_{nI}) + N_p(K_n) \le 2N_p(A_{nI})$. But by Theorem III.6.2 (Ref. 15) (see also Ref. 12, Corollary 1.3), we have $N_p(V_n) \le c_p N_p(V_{nI})$, where the constant c_p depends on p, only. Hence, $N_p(V_n) \le 2c_p N_p(A_{nI})$. But $N_p(A_{nI}) \to N_p(A_I)$ and therefore, $V \in S_p$. We thus can assert that $V_n \to V$ in the operator norm (more exactly-in the norm N_p). By the well-known Theorem II.17.1 (Ref. 4), operator V is quasinilpotent. Moreover, $K \in S_p$ and therefore $K_n \to K$ in the operator norm (more exactly, in the norm N_p). Hence, the operators $S_n = A_n - V_n - iK_n$ $(n \in \nu)$ strongly converge to S. Besides,

$$S_n = \sum_{k=1}^{n} \lambda_{kn} \Delta Q_{kn} \text{ and } Q_{kn} V_n Q_{kn} = V_n Q_{kn} \quad (k = 1, \ldots, n).$$

Here λ_{kn} are the eigenvalues of S_n, with their multiplicities, enumerated in the increasing way and $\Delta Q_{kn} = (., e_{kn}) e_{kn}$, where $(., .)$ is the scalar product in H, and e_{kn} are the normed eigenvectors of S_n, and

$$Q_{jn} = \sum_{k=1}^{j} \Delta Q_{kn}.$$

Let $a = \inf_{k,n} \lambda_{kn} - \epsilon$ $(\epsilon > 0)$ and $b = \sup_{k,n} \lambda_{kn}$ and $\alpha(t) = (b - a)t + a$ $(0 \le t \le 1)$. For a λ_{kn} $(k = 1, .., n)$, put $t_k^{(n)} = (\lambda_{kn} - a)/(b - a)$; so $\alpha(t_k^{(n)}) = \lambda_{kn}$.

Furthermore, denote $E_n(-\epsilon) = 0$, $t_0^{(n)} = 0$, and

$$E_n(t) = Q_{kn} \quad (t_k^{(n)} < t \le t_{k+1}^{(n)};\ k = 0, \ldots, n - 1).$$

Then

$$S_n = \sum_{k=1}^{n} \lambda_{kn} \Delta Q_{kn} = \int_0^1 \alpha(t) dE_n(t) = \alpha(W_n) = (b - a)W_n + aI,$$

where

$$W_n = \int_0^1 t dE_n(t).$$

Since S_n strongly converge, the operators $W_n = (b-a)^{-1}(S_n - aI)$ strongly converge to $W = (b-a)^{-1}(S - aI)$ $(n \in \nu)$. By the well-known Theorem 8.1.15 (Ref. 21), $E_n(t) \to E_t$ strongly, where E_t is the orthogonal resolution of the identity of W. Thus,

$$S = \int_0^1 \alpha(t) dE_t.$$

Since K is compact and $E_n(t)K_n = K_n E_n(t)$, passing to the limit, we have $E_t K = K E_t$. By the von Neumann theorem, (Ref. 1, Sec. 92), K is a function of W. So there is a function $\beta(.)$, such that

$$K = \int_0^1 \beta(t) dE_t \text{ and thus } Z = \int_0^1 (\alpha(t) + i\beta(t)) dE_t.$$

Since $V \in S_p$, we have $E_n(t)V_n E_n(t) \to E_t V E_t$. Hence, $E_t V E_t = V E_t$. Taking into account that $A = Z + V$, we complete the proof. □

Note that the result similar to the previous theorem was derived in Ref. 3 in the more general situation but in a form inconvenient for us. Besides, our proof is absolutely different from the proof by L. de Branges.

Lemma 2.1 and Theorem 4.1 imply our next result.

Corollary 4.2: Let conditions (1.3) and (1.4) hold. Then $T_0 - ib_0 I$ is invertible and there are an orthogonal resolution of the identity $\hat{E}_t$ $(0 \le t \le 1)$ in L^2, a $\hat{E}_t$-measurable bounded scalar function $\hat{\gamma}(t)$ defined on [0, 1], and a quasinilpotent operator $V_{T_0} \in S_2$, such that

$$(T_0 - ib_0 I)^{-1} = \int_0^1 \hat{\gamma}(t) d\hat{E}_t + V_{T_0}$$

and $V_{T_0}\hat{E}_t = \hat{E}_t V_{T_0} \hat{E}_t \quad (0 \le t \le 1)$.

V. EXPANSIONS OF THE RESOLVENT VIA INVARIANT RESOLUTIONS OF THE IDENTITY

Let Σ be the σ-algebra of the Borel sets of [0, 1]. An orthogonal resolution $P(\delta)$ $(\delta \in \Sigma)$ of the identity in a separable Hilbert space H is said to be a simple resolution of the identity, if there is a (generating) vector $g \in H$, such that the linear span of the vectors $P(\delta)g$, when δ is running Σ, is dense in H, cf. Ref. 1, Sec. 83. Put $M(\delta) = P(\delta)g$ and $\mu(t) = (M(\delta), g)_H$, where $(.,.)_H$ is the scalar product in H. Thanks to Theorem 2 from Sec. 83 (Ref. 1), for any $x \in H$, there is a $\tilde{x} \in L^2_\mu(0, 1)$, such that

$$x = \int_0^1 \tilde{x}(t) M(dt).$$

In the sequel $\tilde{x}(.)$ will be called *the coordinate function* of x [with respect to $M(.)$]. Everywhere in this section, the limits are understood in the sense of the strong topology. Let

$$x_n = \sum_{k=1}^n \tilde{x}_{kn} M(\Delta_k) \quad (\Delta_k = (t_{k-1}^{(n)}, t_k^{(n)}];\ \ 0 = t_0 \le t_1^{(n)} < t_2^{(n)} < \ldots < t_n^{(n)} = 1),$$

be a sequence converging to x. For a bounded linear operator A, we have

$$Ax_n = \sum_{k=1}^n \tilde{x}(t_k) AM(\Delta_k).$$

Let $K(\Delta_j, .)$ be the coordinate function of $AM(\Delta_j)$,

$$AM(\Delta_j) = \int_0^1 K(\Delta_j, t)M(dt). \text{ Then } Ax_n = \sum_{k=1}^{n} \tilde{x}_{kn} \int 0^1 K(\Delta_k, t)M(dt).$$

Denoting the limit of the right hand part of this equality by

$$\int_0^1 \tilde{x}(s) \int_0^1 K(ds, t)M(dt),$$

we get the following result.

Lemma 5.1: Let $P_t = P([0, t])$ be a simple resolution of the identity. Then

$$Ax = \int_0^1 \tilde{x}(s) \int_0^1 K(ds, t)M(dt),$$

where $K(\Delta, .) \in L^2_\mu(0, 1)$ for any $\Delta \in \Sigma$.

Introduce in H the functional $\theta(t)$ by the formula $\theta(t)x := \tilde{x}(t)$ $(t \in [0, 1];\ x \in H)$. Certainly, it is linear and $\theta(t)x \in L^2_\mu(0, 1)$ for any $x \in H$. We have

$$I = \int_0^1 \theta(t)dM(t) \text{ in the sense } x = \int_0^1 \tilde{x}(t)dM(t) = \int_0^1 \theta(t)dM(t)x.$$

We will call $\theta(t)$ the *representing functional* [with respect to $M(.)$]. By the previous lemma,

$$A = \int_0^1 \theta(s) \int_0^1 K(ds, t)M(dt), \tag{5.1}$$

in the sense that

$$Ax = \int_0^1 \theta(s)x \int_0^1 K(ds, t)M(dt).$$

Furthermore, assume that A has a simple invariant resolution of the identity $P_t = P([0, t])$: $P_t A P_t = A P_t$ $(t \in [0, 1])$ and thus $(I - P_t)AP_t = 0$. For any $y \in [0, 1]$, we have

$$P_y x = \int_0^y \tilde{x}(t)M(dt).$$

So, $P_y H$ is the space of elements x whose coordinate functions $\tilde{x}(t)$ are equal to zero for $t \geq y$. By Lemma 5.1,

$$AP_y x = \int_0^y \tilde{x}(s) \int_0^1 K(ds, t)M(dt).$$

In particular, let $x = x_0$, where x_0 has the coordinate function $\tilde{x}_0(s) \equiv 1$,

$$x_0 = \int_0^1 M(dt) = M([0, 1]).$$

Then

$$AP_y x_0 = \int_0^y \int_0^1 K(ds, t)M(dt) = \int_0^1 K([0, y], t)M(dt).$$

So $K([0, y], .)$ is the coordinate function of $AP_y x_0 = P_y A P_y x_0$. Therefore, $K([0, y], t) = 0$ for $t > 0$. Taking into account (5.1) we arrive at the following result.

Lemma 5.2: Let A have a simple invariant resolution of the identity P_t. Then

$$A = \int_0^1 \theta(s) \int_0^s K(ds, t)M(dt), \tag{5.2}$$

where $K(\Delta, .)$ is the coordinate function of $AP(\Delta)g$ with respect to $P(\Delta)g$, where g is a generating vector.

Now let a resolution of the identity P_t have a finite range $r > 1$, that is, there are vectors $g_j \in H$ $(j = 1, \ldots, r)$, such that the linear span of the vectors $P(\delta)g_j$, when δ is running Σ, is dense in H. Then for any $x \in H$ we have

$$x = \sum_{j=1}^{r} \int_0^1 \tilde{x}_j(t)P(dt)g_j, \tag{5.3}$$

where $\tilde{x}_k \in L^2_{\mu_r}(0, 1)$ with

$$\mu_r(t) = (P_t \sum_{k=1}^{r} g_k, \sum_{j=1}^{r} g_j),$$

cf. (Ref. 1). In the sequel $\tilde{x}_j(.)$ will be called the coordinate function of x with respect to $P(.)g_j$. Let

$$x_n = \sum_{j=1}^{r} \sum_{k=1}^{n} \tilde{x}_{jkn} P(\Delta_k)g_j,$$

be a sequence converging to x. For a bounded operator A, we have

$$Ax_n = \sum_{j=1}^{r} \sum_{k=1}^{n} \tilde{x}_{jkn} AP(\Delta_k)g_j.$$

Let $K_{mj}(\Delta, .) \in L^2_{\mu_r}(0, 1)$; $(m = 1, \ldots, r)$ be the coordinate function of $AP(\Delta)g_j$ with respect to $P(\Delta)g_m$,

$$AP(\Delta)g_j = \sum_{m=1}^{r} \int_0^1 K_{mj}(\Delta, t)P(dt)g_m.$$

Then

$$Ax_n = \sum_{m=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{n} \tilde{x}_{jkn} \int_0^1 K_{mj}(\Delta_k, t)P(dt)g_m.$$

Passing to the limit as $n \to \infty$, we arrive at the following result.

Lemma 5.3: Let P be a resolution of the identity of a range $r < \infty$. Then

$$Ax = \sum_{m=1}^{r} \sum_{j=1}^{r} \int_0^1 \int_0^1 K_{mj}(ds, t)\tilde{x}_j(t)P(dt)g_m.$$

Introduce the functional θ_j by $\theta_j(s)x = \tilde{x}_j(s)$. Certainly, it is linear and at some points $\theta_j(t)x$ can be infinite. Then

$$I = \sum_{j=1}^{r} \int_0^1 \theta_j(t)P(dt)g_j,$$

in the sense that

$$Ix = x = \sum_{j=1}^{r} \int_0^1 \theta_j(t)xP(dt)g_j = \sum_{j=1}^{r} \int_0^1 \tilde{x}_j(t)P(dt)g_j.$$

We will call $\theta_j(t)$ the *coordinate representing functional* (with respect to $P_t g_j$). By the previous lemma,

$$A = \sum_{m=1}^{r} \sum_{j=1}^{r} \int_0^1 \int_0^1 K_{jm}(ds, t)\theta_j(s)P(dt)g_m. \tag{5.4}$$

Repeating the proof of Lemma 5.2, we arrive at the following result.

Lemma 5.4: Let A be a bounded operator in H, having an invariant resolution of the identity $P(.)$ whose range is $r < \infty$. Then

$$A = \sum_{m=1}^{r} \sum_{j=1}^{r} \int_0^1 \int_0^s \theta_j(s) K_{mj}(ds, t) P(dt) g_m,$$

where $K_{mj}(\Delta, .) \in L^2_{\mu_r}(0, 1)$; $(j, m = 1, \ldots, r)$ is the coordinate function of $AP(\Delta)g_j$ with respect to $P(\Delta)g_m$.

The previous lemma and Corollary 4.2 imply.

Corollary 5.5: Let conditions (1.3) and (1.4) hold. Then $T_0 - ib_0 I$ is boundedly invertible, $(T_0 - ib_0 I)^{-1}$ has the invariant resolution of the identity $\hat{E}$ and

$$(T_0 - ib_0 I)^{-1} = \sum_{m=1}^{r} \sum_{j=1}^{r} \int_0^1 \int_0^s \hat{\theta}_j(s) \hat{K}_{mj}(ds, t) \hat{E}(dt) \hat{g}_m,$$

where $r \geq 1$ is the range of $\hat{E}$, $\hat{g}_1, \ldots, \hat{g}_r \in L^2$ are the generating vectors, $\hat{K}_{mj}(\Delta, .)$ is the coordinate function of $(T_0 - ib_0 I)^{-1} \hat{E}(\Delta) g_j$ with respect to $\hat{E}(\Delta) g_m$ and $\hat{\theta}_j(t)$ is the coordinate representing functional with respect to $\hat{E}(\Delta) g_j$ $(j, m = 1, \ldots, r)$.

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