WKB Analysis of Planar Surface Waveguides with Truncated Index Profiles

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Abstract—**We present a WKB analysis of planar surface waveguides with truncated arbitrary refractive index profiles. It is shown that the phase change at the turning point on the substrate side approaches zero near the cutoff. The validity of the conventional assumption of constant phase change of $\pi/2$ at the turning point is investigated, and the consequent errors near the mode cutoffs are analyzed for index profiles of practical interest.**

THE WKB method has been extensively used for the calculation of propagation constants of guided modes in optical waveguides [1]–[7]. In the limit of slow index variations across the guiding cross section, with the transverse waveguide dimensions much larger than the wavelength, this method predicts the propagation characteristics of multimode optical fibers [1] with reasonable accuracy. In the case of a step-index planar waveguide, the method gives exact results independent of the waveguide dimensions (represented by the V parameter), and the derived propagation constant values ($b - V$ curves) are the same as those obtained by solving the wave equation with appropriate boundary conditions [2]. This is attributed to the fact that in the case of the step-index planar waveguide, exact expressions for the phase change for the total internal reflection at the cover and substrate interfaces are available which are valid regardless of the propagation constant of the guided mode [3]. Motivated by the success of the WKB theory in step-index planar waveguides, researchers have applied it to the graded-index waveguides [4]–[7]. In this approach, it is generally assumed that the phase change $2\phi_t$ at the turning point on the substrate side is $\pi/2$. Moreover, since most surface waveguides have air as the surrounding cover medium, the mode cutoff condition is determined by the substrate refractive index. In this limit, the phase change upon total internal reflection at the cover surface is very nearly π. With these assumptions, $b - V$ curves and modal fields have been calculated [7] for graded-index waveguides and shown to be in agreement with exact analytical solutions for parabolic and exponential index profiles in the case of large V. The error caused by assuming a constant phase shift of π at the cover surface has also been examined [7]. When a mode is buried and one of the turning points lies near the film–cover interface, the phase change upon reflection at this turning point becomes dependent on the propagation constant. In this case, WKB approximation has been shown [5] to give accurate results for well-guided modes provided the above dependence is included in the analysis.

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In this paper we show that although the approximation of the $\pi/2$ phase shift at the turning point on the substrate side is acceptable for well-guided modes (large V), errors are caused in the calculated propagation constant at lower V-values, particularly near the cutoff. This problem is of paramount importance when the $b - V$ curves are used for calculation of the field profile of guided modes, especially in the case of single-mode waveguides. Furthermore, since the $b - V$ curves are often used to characterize the planar waveguides for the diffusion profile [8], [9] and the Δn value from mode index measurements [8]–[10], any error in these curves is likely to give erroneous results for the waveguide parameters. The quantitative evaluation of the error introduced by the assumption of $\phi_t = \pi/4$ for modes near cutoff was first alluded to in [5] for surface waveguides with parabolic profile. It was suggested that the correction to ϕ_t depends on the first derivative of the index at the film–substrate boundary and decreases as the mode effective index approaches the substrate index, i.e., as the mode approaches the cutoff. However, no analysis was presented to understand the behavior of ϕ_t near cutoff. Recognizing that in truncated profiles the phase shift $2\phi_t$ has to approach zero near the mode cutoff, we have obtained curves for ϕ_t as a function of the propagation constant by using the WKB theory and comparing it with $b - V$ curves as obtained by the numerical solution of the wave equation using the finite difference method [12]. The variation of ϕ_t is shown to be directly related to the change of the slope of the refractive index profile near the substrate interface, and the quantitative results are in agreement with those of [5]. To the best of our knowledge, this is the first report which considers quantitative variation of ϕ_t in the cutoff region using the WKB approximation.

Consider the truncated graded-index profile $n(x)$ of the waveguide as shown in Fig. 1. The curved ray trajectories make an angle $\theta(x)$ with the x-axis. The propagation constant β and the transverse propagation constant $\kappa(x)$ are

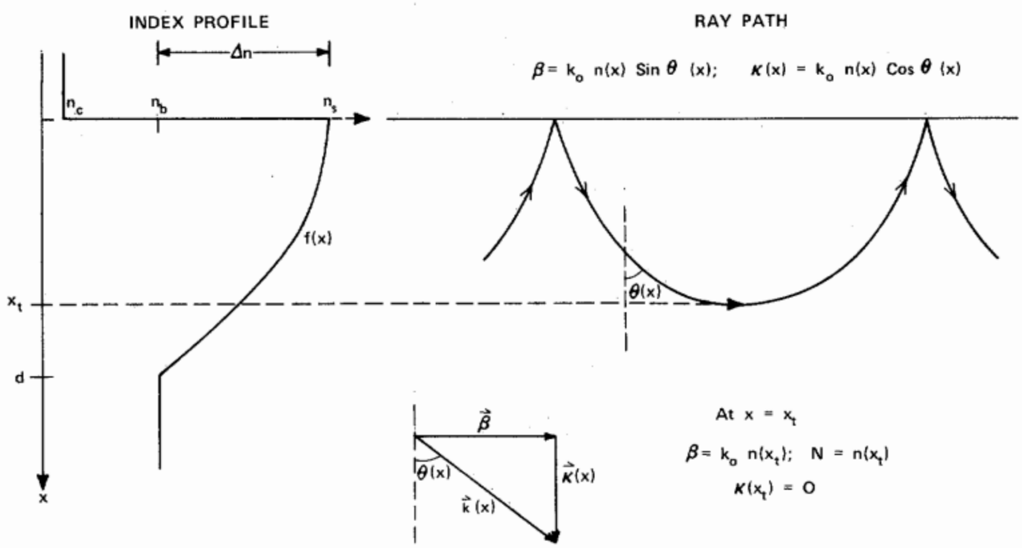

Fig. 1. Index profile and the ray trajectories showing the parameters used in the text.

given by:

$$\beta = k_0 n(x) \sin \theta(x) \tag{1}$$

$$\kappa(x) = k_0 n(x) \cos \theta(x) = \sqrt{k_0^2 n^2(x) - \beta^2}. \tag{2}$$

Using the WKB approach, the condition for the TE_m mode propagation is given by the characteristic equation [2]

$$\int_0^{x_t} \kappa(x)\, dx = m\pi + \phi_c + \phi_t \tag{3}$$

where ϕ_c is one-half the phase change at the cover surface. The expressions for ϕ_c and ϕ_t are given by [3], [6]

$$\phi_c = \tan^{-1}\left[\frac{\beta^2 - k_0^2 n_c^2}{k_0^2 n_s^2 - \beta^2}\right]^{1/2} \tag{4}$$

and

$$\phi_t = \lim_{\delta \to 0} \tan^{-2}\left[\frac{\beta^2 - k_0^2 n^2(x_t + \delta)}{k_0^2 n^2(x_t - \delta) - \beta^2}\right]^{1/2}$$

Upon binomial expansion of the RHS, the above equation becomes

$$\phi_t = \lim_{\delta \to 0} \tan^{-1}\sqrt{\left(\left.\frac{dn}{dx}\right|_{(x_t+\delta)} \Big/ \left.\frac{dn}{dx}\right|_{(x_t-\delta)}\right)}. \tag{5}$$

At the turning point

$$\theta(x_t) = \frac{\pi}{2}$$

and

$$\beta = k_0 n(x_t).$$

If the turning point x_t is "far" from the substrate interface, the wave does not "see" the discontinuity in the slope of the index profile at the interface. In this limit, it has been shown [6] that $\phi_t = \pi/4$ with the total phase change $2\phi_t = \pi/2$ at the turning point. This corresponds to the case of a well-guided mode ($\beta \to k_0 n_s$), or to any mode of an untruncated profile such as Gaussian or exponential. However, in truncated profiles, as the mode approaches its cutoff, $\beta \to k_0 n_b$ and $x_t \to$ film–substrate interface. At the cutoff, therefore, the slope discontinuity forces the numerator in (5) to be zero. Thus we obtain $\phi_t = 0$ for $b = 0$ and $\phi_t = \pi/4$ for $b \to 1$.

The fact that ϕ_t is zero at the cutoff is not surprising. In the case of an asymmetric step-index waveguide, the phase change at the substrate interface is given by

$$\phi_s = \tan^{-1}\left[\frac{\beta^2 - k_0^2 n_b^2}{k_0^2 n_s^2 - \beta^2}\right]^{1/2}$$

which approaches zero at the cutoff. Interpreted differently, it reflects the fact that as the ray approaches the critical angle of incidence at the interface, the phase change drops to zero. We return to this point later when we discuss the case of a buried waveguide with a symmetric profile. For the sake of completeness, we also write the expression for ϕ_c at the cutoff point for the asymmetric surface waveguide of the type shown in Fig. 1:

$$(\phi_c)_{\text{cutoff}} = \tan^{-1}\sqrt{a_E} \tag{6}$$

where a_E is the asymmetry parameter defined by [2]:

$$a_E = \left(\frac{n_b^2 - n_c^2}{n_s^2 - n_b^2}\right). \tag{7}$$

For an ion-exchanged glass waveguide, $n_c = 1$, $n_b \simeq 1.5$ and $n_s \sim 1.52$ and (8) gives $\phi_c = \tan^{-1}\sqrt{20.7} = 0.43\,\pi$. In $LiNbO_3$ and GaAs based waveguides, ϕ_c is larger due to the higher asymmetry and is therefore close to $\pi/2$.

In dealing with waveguides, it is convenient to write the dispersion relations in terms of the normalized frequency V and the normalized propagation constant b given by [2]

$$V = k_0 d\sqrt{n_s^2 - n_b^2} \tag{8}$$

and

$$b = \frac{\left(\frac{\beta^2}{k_0^2} - n_b^2\right)}{(n_s^2 - n_b^2)}. \tag{9}$$

Here d is some measure of the diffusion depth depending on the profile.

Now let us write the characteristic equation (3) in terms of the normalized parameters [7]. We write

$$x' = \frac{x}{d} \tag{10}$$

and

$$f(x') = \left(n(x') - n_b\right)/\Delta n. \tag{11}$$

Thus (3) for the case $\Delta n << n_b$ becomes

$$\int_0^{x_t'} V\sqrt{f(x') - b}\, dx' = m\pi + \phi_c + \phi_t. \tag{12}$$

Let us consider the following index profiles:

$$\begin{aligned} f(x') &= 1 && \text{(step-index)} \\ &= e^{-x'} && \text{(exponential)} \\ &= e^{-x'^2} && \text{(Gaussian)} \\ &= 1 - x'^2 && \text{(parabolic)} \\ &= 1 - x' && \text{(linear)}. \end{aligned} \tag{13}$$

From (12), assuming $\phi_c = \pi/2$ and $\phi_t = \pi/4$ [4]–[7], we get the cutoff condition

$$V_c \int_0^{x_t'} \sqrt{f(x')}\, dx' = m\pi + \tfrac{3}{4}\pi. \tag{14a}$$

On the other hand, using the expression for ϕ_c as given by (4) and using $\phi_t = 0$ at the cutoff, we get

$$V_c \int_0^{x_t'} \sqrt{f(x')}\, dx' = m\pi + \tan^{-1}\left(\frac{n_b^2 - n_c^2}{n_s^2 - n_b^2}\right)^{1/2} \tag{14b}$$

showing that the error in the cutoff normalized frequency V_c of any mode is given by $(\frac{3}{4}\pi - 0.43\pi)/(\int_0^{x_t'} \sqrt{f(x')}\, dx')$ for the ion exchange case described previously. The value of the integral in the denominator is 2, $\sqrt{\pi/2}$, $\pi/4$, and $2/3$ for untruncated exponential, untruncated Gaussian, parabolic, and linear profiles, respectively. Thus the error in V_c amounts to 0.44, 0.70, 1.12, and 1.32 for the four profiles, respectively. The waveguides of practical interest have a Gaussian or an ERFC profile or a combination of the two. Therefore, theoretically $\phi_t = \pi/4$, independent of b, due to the infinite width of these functions ($x_t' = \infty$). However, in practice the diffused waveguides have truncated profiles and the function $f(x')$ does go to zero at a finite distance. Nevertheless, the errors cited above will not change significantly due to the truncation of these profiles as the contribution to the integral on the LHS of (14) is negligible beyond $x' = 5$.

The conclusion that the phase shift at the turning point near the cutoff goes to zero can also be reached by analyzing the case of a buried waveguide with a symmetric profile with x_{t_1}' and x_{t_2}' as turning points. The characteristic equation in this case becomes

$$\int_{x_{t_1}'}^{x_{t_2}'} V\sqrt{f(x') - b}\, dx' = m\pi + 2\phi_t. \tag{15}$$

In the limit $b \to 0$, it is required that for the fundamental mode, the cutoff V value be zero. This is achieved only if $\phi_t \to 0$.

So far we have discussed only the two extreme cases, $b = 0$ and $b \to 1$. The picture in the intermediate range (of practical interest) is not so clear and exact analytical expressions for ϕ_t are difficult to obtain. In order to overcome this problem, we have numerically calculated the $b - V$ curves for the step-index and all the four graded-index profiles, using the finite difference method [11]. The high accuracy of the finite difference method was checked by comparing the calculated $b - V$ curves and the mode field profiles with those obtained by the exact analytical solution of the characteristic equation derived from the field continuity conditions in the case of step-index and exponential profiles. We have observed that the discontinuity in the index profile does not appear to affect the numerical field solutions to any detectable extent. A detailed study of the finite difference method will be reported elsewhere [12]. In order to calculate the b-dependence of ϕ_t, ϕ_t was used as a parameter and for a given b its value was chosen to be that which gave agreement with the $b - V$ curves obtained by the finite difference method. Fig. 2 shows the results for the two lowest order modes in the case of linear and parabolic profiles. It is observed that for b values down to 0.5, ϕ_t remains constant and equals $\pi/4$. It then starts decreasing, reaching a very large slope near $b \sim 0.1$, and approaching zero as $b \to 0$. It is this steepness of the curve for lower b-values which is responsible for the flatness of the $b - V$ curves near the cutoff. In this region, therefore, the conventional WKB gives large errors. No calculations were made for $b < 0.01$ as a very large number of grid points is necessary to give accurate results in the finite difference method. In practice, the mode becomes highly attenuated for $b < 0.1$ and the cutoff is measured at higher V-values than those predicted theoretically. This behavior also suggests that the $b - V$ curves should be used with caution to estimate the theoretical cutoff values.

A few words regarding the shape of the curves in Fig. 2 are in order at this point. In particular, (5) predicts an abrupt drop in ϕ_t from $\pi/4$ to zero at $b = 0$, whereas Fig. 2 shows a gradual decrease in ϕ_t. The explanation of this behavior lies in the proximity of the turning points to the film–substrate interface given by $x' = 1$ for these profiles. The positions of these turning points for linear and parabolic profiles are given by $x_t' = (1 - b)$ and $x_t' = \sqrt{1 - b}$, respectively. The corresponding normalized distances from the film–substrate interface are b and $1 -$

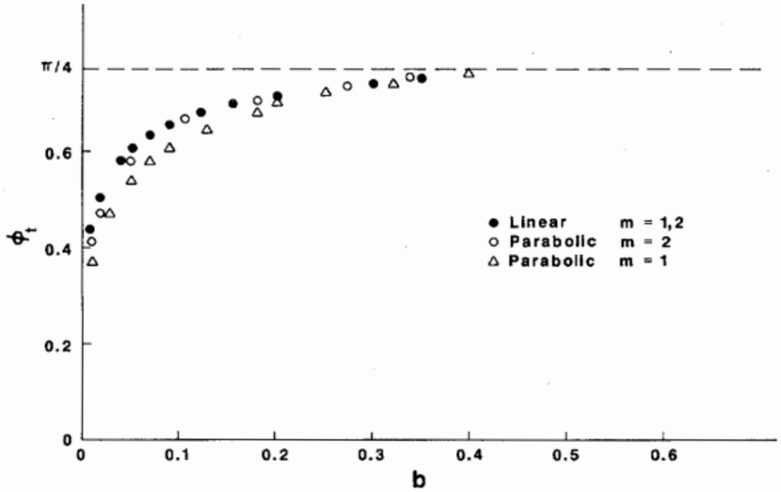

Fig. 2. Phase shift dependence on the propagation constant.

$\sqrt{1-b}$, respectively. Thus for $b \sim 0.1$, these distances are of the order of wavelength of the radiation, and the profile slope discontinuity at $x' = 1$ therefore is likely to influence the phase shift. As b decreases further, the effect of the slope discontinuity is larger, causing a steep decline in the $\phi_t(b)$ curve shown in Fig. 2. This also explains the fact that for a given b value, ϕ_t for the linear profile is larger than that for the parabolic profile, and the exact form of the ϕ_t versus b curve depends on the index profile. The larger discrepancy at lower b-values between the conventional WKB ($\phi_t = \pi/4$) and the analysis presented here is in contrast with the results of [5] where the error was found to decrease for higher order modes in the case of the exponential profile. This is related to the fact that in [5] ϕ_t was assumed to be constant ($= \pi/4$), which is a perfectly valid assumption for untruncated profiles. For the case of an unburied parabolic profile, however, the WKB results for the TE_3 mode in [5] show an error similar to that discussed here.

Although the results for truncated Gaussian and ERFC profiles are not presented here, similar behavior ws observed in these cases. For untruncated profiles, on the other hand, $f(x')$ never goes to zero at a finite value of x' and therefore x_t for any mode theoretically occurs well within the guiding region. Therefore ϕ_t remains constant at a value of $\pi/4$. This was shown quantitatively for the exponential profile even for the cutoff of the fundamental mode [13]. This is consistent with the requirement that ϕ_t go to zero at cutoff only when the turning point is at the discontinuity of the slope of the refractive index, where dn/dx becomes zero beyond the turning point as required by (5).

In conclusion, we have analyzed the error in the conventional WKB theory caused by the assumption of a constant phase shift of $\pi/2$ at the turning point on the substrate side of a planar graded-index surface waveguide. It is shown that in untruncated profiles the phase shift at the turning point on the substrate side depends on the propagation constant and approaches zero at the cutoff. The exact variation of the phase change depends on the index profile.

References

[1] D. Gloge and E. A. J. Marcatili, "Multimode theory of graded-core-fibers," *Bell Syst. Tech. J.*, vol. 52, p. 1563, 1973.

[2] H. Kogelnik and V. Ramaswamy, "Scaling rules for thin film optical waveguides," *Appl. Opt.*, vol. 13, p. 1857, 1974.

[3] P. K. Tien, "Light waves in thin films and integrated optics," *Appl. Opt.*, vol. 10, p. 2395, 1971.

[4] A. Gedeon, "Comparison between rigorous theory and WKB analysis of modes in graded-index waveguides," *Opt. Commun.*, vol. 12, p. 329, 1974.

[5] J. Janta and J. Ctyroky, "On the accuracy of WKB analysis of TE and TM modes in planar-graded-index waveguides," *Opt. Commun.*, vol. 25, p. 49, 1978.

[6] G. B. Hocker and W. K. Burns, "Modes in diffused optical waveguides of arbitrary index profiles," *IEEE J. Quantum Electron.*, vol. QE-11, p. 270, 1975.

[7] R. V. Ramaswamy and R. K. Lagu, "Numerical field solution for an arbitrary asymmetrical graded-index planar waveguide," *J. Lightwave Tech.*, vol. LT-1, p. 408, 1985.

[8] R. K. Lagu and R. V. Ramaswamy, "Process and waveguide parameter relationships for the design of planar silver ion-diffused glass waveguides," *J. Lightwave Technol.*, vol. LT-4, p. 176, 1986.

[9] S. I. Najafi and R. V. Ramaswamy, "Ag^+-diffused graded-index glass waveguides: Diffusion and modal characterization," presented at Gradient-Index Optical Imaging Systems Conf., Palermo, Italy, Sept. 1985.

[10] G. C. Yip and J. Albert, "Characterization of planar optical waveguides by K^+-ion exchange in glass," *Opt. Lett.*, vol. 10, p. 151, 1985.

[11] E. Schweig and W. B. Bridges, "Computer analysis of dielectric waveguides: A finite-difference method," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, p. 531, 1984.

[12] R. V. Ramaswamy, E. Rodrigues, and R. Srivastava, "Normalized finite-difference solutions to symmetrical and asymmetrical arbitrary graded-index slab waveguides," unpublished.

[13] E. M. Conwell, "WKB approximation for optical guide modes in a medium with exponentially varying index," *J. Appl. Phys.*, vol. 46, p. 1407, 1975.

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