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CORRESPONDENCE OF THE EIGENVALUES OF A NON-SELF-ADJOINT OPERATOR TO THOSE OF A SELF-ADJOINT OPERATOR

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Abstract. We prove that the eigenvalues of a certain highly non-self-adjoint operator that arises in fluid mechanics correspond, up to scaling by a positive constant, to those of a self-adjoint operator with compact resolvent; hence there are infinitely many real eigenvalues which accumulate only at $\pm\infty$. We use this result to determine the asymptotic distribution of the eigenvalues.

§1. *Introduction.* In a recent paper [**11**], we showed that the spectrum of the highly non-self-adjoint operator $-\mathrm{i}H$ is real, where H is the closure of the operator H_0 on $\mathrm{L}^2(-\pi, \pi)$ defined by

$$(H_0 f)(\theta) = \varepsilon \frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f}{\partial\theta}\right) + \frac{\partial f}{\partial\theta} \tag{1}$$

for any fixed $\varepsilon \in (0, 2)$ and all $f \in \mathrm{Dom}(H_0) = \mathcal{C}^2_{\mathrm{per}}[-\pi, \pi]$. Boulton *et al* subsequently proved in a recent paper [**2**] the existence of a wider class of operators possessing only real eigenvalues. However, they did not prove that any non-zero eigenvalues exist for these operators or that their spectra are real. The results obtained in this paper for the original operator (1) are much more detailed than those presented in [**2, 11**].

The operator H was first studied by Benilov *et al*, who argued in [**1**] that the equation

$$\frac{\partial f}{\partial t} = Hf \tag{2}$$

approximates the evolution of a liquid film inside a rotating horizontal cylinder. They also made several conjectures based on non-rigorous numerical analysis, including that the spectrum of H is purely imaginary and consists of eigenvalues which accumulate at $\pm\mathrm{i}\infty$.

Davies showed in [**7**] that $-\mathrm{i}H$ has compact resolvent by considering the unitarily equivalent operator A on $l^2(\mathbb{Z})$ defined by

$$(Av)_n = \frac{\varepsilon}{2}n(n-1)v_{n-1} - \frac{\varepsilon}{2}n(n+1)v_{n+1} + nv_n \tag{3}$$

for all $v \in \mathrm{Dom}(A) = \{v \in l^2(\mathbb{Z}) : Av \in l^2(\mathbb{Z})\}$. Here $A = \mathcal{F}^{-1}(-\mathrm{i}H)\mathcal{F}$, where $\mathcal{F} : \mathrm{L}^2(-\pi, \pi) \to l^2(\mathbb{Z})$ is the Fourier transform. If $\mathcal{F}f = v$, then $(v_n)_{n\in\mathbb{Z}}$ are the Fourier coefficients of f. This result was achieved by obtaining sharp bounds on the rate of decay of eigenvectors and resolvent kernels, and by determining

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the precise domains of the operators involved. Davies also showed that

$$A = A_- \oplus 0 \oplus A_+, \tag{4}$$

where A_- and A_+ are the restrictions of A to $l^2(\mathbb{Z}_-)$ and $l^2(\mathbb{Z}_+)$, respectively, and that A_- is unitarily equivalent to $-A_+$. Since the resolvent is compact and the adjoint has the same eigenvalues, the spectrum of $-\mathrm{i}H$ consists entirely of eigenvalues.

As previously mentioned, we proved in [**11**] that these eigenvalues, if they exist, must all be real. Eigenvalues of H or $-\mathrm{i}H$ have been calculated numerically in [**1, 5, 7**], but until now it has not been proven rigorously that any non-zero eigenvalues exist.

In this paper we prove rigorously that $-\mathrm{i}H$ has infinitely many eigenvalues which accumulate at $\pm\infty$ (Corollary 4.7). Our approach is to show that the eigenvalues of A_+ correspond, up to scaling by a positive constant, to those of a self-adjoint operator with compact resolvent (see Corollary 2.2 and Theorems 3.1 and 4.3). By analysing the self-adjoint operator, we determine the asymptotic distribution of the eigenvalues (Theorem 5.2). It was argued in [**5**] that the distribution of the eigenvalues, if they exist, should be quadratic, but no rigorous bounds were given. Here we prove rigorously that $\lambda_n \sim \varepsilon\pi^2 n^2 \beta^{-2}$ for a constant β which we determine.

The correspondence of the eigenvalues of $\mathrm{i}H$ to those of a self-adjoint operator Q might lead us to believe that $\mathrm{i}H$ is similar to Q in the sense that there exists a bounded linear operator S with bounded inverse such that $S\,\mathrm{Dom}(Q) = \mathrm{Dom}(H)$ and $\mathrm{i}H = SQS^{-1}$. However, it has recently been proven that $\mathrm{i}H$ is not similar in this sense to any self-adjoint operator; see [**4**, proof of Theorem 5.1].

§2. *Correspondence of eigenvalues to those of a Sturm–Liouville problem.* We have already shown in [**11**] that if λ is an eigenvalue of the operator A_+ defined on its natural maximal domain by

$$(A_+v)_n = \frac{\varepsilon}{2}n(n-1)v_{n-1} - \frac{\varepsilon}{2}n(n+1)v_{n+1} + nv_n,$$

then $\mu = 2\lambda/\varepsilon$ is an eigenvalue of the (singular) classical Sturm–Liouville problem

$$-(pu')' = \mu w u, \tag{5}$$

$$u \in \mathcal{C}^\infty[0, 1] \quad \text{and} \quad u(0) = 0, \tag{6}$$

where

$$p(x) = (1-x)^{1+1/\varepsilon}(x+1)^{1-1/\varepsilon}, \tag{7}$$

$$w(x) = x^{-1}(1-x)^{1/\varepsilon}(x+1)^{-1/\varepsilon}. \tag{8}$$

By $\mathcal{C}^\infty[0, 1]$ we mean the set of functions $f \in \mathcal{C}^\infty(0, 1) \cap \mathcal{C}[0, 1]$ for which one-sided derivatives of all orders exist at both endpoints. We also define $\mathcal{C}_c^\infty(0, 1)$ as the set of all compactly supported $\mathcal{C}^\infty$ functions on $(0, 1)$. We denote by L and L_c the symmetric operators defined by the same formula (5) on the domains (6) and $\mathcal{C}_c^\infty(0, 1)$, respectively. We shall see in §3 that for $0 < \varepsilon \leq 1$, both operators

L and L_c have unique self-adjoint extensions, which coincide with each other. On the other hand, if $\varepsilon \geq 1$, then L_c has infinitely many extensions whereas L has a unique self-adjoint extension (Corollary 4.6). This is guaranteed by the condition that $u \in \mathrm{Dom}(L)$ should remain bounded at the right-hand endpoint.

In either case, the solution of (5) satisfying (6) is given by a power series with radius of convergence 1, namely

$$u(x) = \sum_{n=1}^{\infty} v_n x^n, \tag{9}$$

where v_n is the solution of the recurrence relation

$$n(n-1)v_{n-1} - n(n+1)v_{n+1} + 2\frac{n-\lambda}{\varepsilon} v_n = 0 \tag{10}$$

satisfying the initial conditions $v_1 = 1$ and $v_2 = (1-\lambda)/\varepsilon$.

We now show the converse.

THEOREM 2.1. *If μ is an eigenvalue of the classical Sturm–Liouville problem defined by (5) and (6), then $\lambda = \varepsilon\mu/2$ is an eigenvalue of A_+.*

Proof. We will make use of the standard theory of regular singular points of ordinary differential equations (ODE), which can be found in [3]. Accordingly, we consider the equation acting in the complex plane. If (v_n) is the solution of the recurrence relation (10) satisfying the stated initial conditions and u is defined by (9) on the unit disc centred at the origin, then u is a non-zero solution of (5). Equation (5) is equivalent to

$$u'' + \left(\frac{1+1/\varepsilon}{z-1} + \frac{1-1/\varepsilon}{z+1}\right)u' - \frac{\mu}{z(z-1)(z+1)}u = 0, \tag{11}$$

so we see that a second linearly independent solution is

$$u_1 = au(z)\log z + \sum_{n=0}^{\infty} b_n z^n,$$

with $b_0 \neq 0$. The series $\sum_{n=0}^{\infty} b_n z^n$ has radius of convergence 1, as does the series for u. At least one of the two series must be unbounded as $z \to 1$, to reflect the behaviour of the singular solution, but *a priori* we do not know which.

Suppose that μ is an eigenvalue of the Sturm–Liouville problem and y is a corresponding eigenvector. We proved in [11] that $\mu \in \mathbb{R}$. Now, y is a non-zero solution of (5) in $(0, 1)$ such that $\lim_{x\to 0+} y(x) = 0$ and $\lim_{x\to 1-} y(x)$ is finite. Since the space of solutions of (5) is two-dimensional, $y = \alpha u + \beta u_1$ for some $\alpha, \beta \in \mathbb{C}$. Considering the endpoint $x = 0$, we see that we must have $y = \alpha u$. Without loss of generality, we may assume $y = u$. Hence $u(x)$ converges to a finite limit as $x \to 1-$.

Suppose that λ is not an eigenvalue of A_+. Davies showed in [7] that for $\lambda \in \mathbb{R}$, (10) has two linearly independent solutions ϕ and ψ such that $\phi_n \geq n^{1/\varepsilon-1} \geq n^{-1}$ for all sufficiently large n and $|\psi_n| \sim n^{-1/\varepsilon-1}$ as $n \to \infty$. The space of solutions of (10) is two-dimensional, so $v_n = a\phi_n + b\psi_n$, and we have $a \neq 0$ since $\psi \in l^2(\mathbb{Z}_+)$ and $v \notin l^2(\mathbb{Z}_+)$. Without loss of generality, $a > 1$.

Hence there exists $N > 0$ such that $v_n \geq n^{-1}$ for all $n \geq N$. For $x \in (0, 1)$,

$$\begin{aligned} u(x) &\geq \sum_{n=1}^{N-1}(v_n - n^{-1})x^n + \sum_{n=1}^{\infty} n^{-1}x^n \\ &= \sum_{n=1}^{N-1}(v_n - n^{-1})x^n - \log(1-x) \\ &\to \infty \end{aligned}$$

as $x \to 1-$. This is a contradiction, so λ is an eigenvalue of A_+. □

COROLLARY 2.2. *A complex number λ is an eigenvalue of A_+ if and only if $\mu = 2\lambda/\varepsilon$ is an eigenvalue of the Sturm–Liouville problem (5).*

§3. *Self-adjointness.* We now show that the operator corresponding to the classical Sturm–Liouville problem has a self-adjoint extension with the same eigenvalues. Equation (5) can be written as

$$Lu = \mu u \tag{12}$$

where L is an operator on $\mathcal{H} = \mathrm{L}^2((0,1), w(x)\,dx)$ defined by

$$Lf = -w^{-1}(pf')' \tag{13}$$

on

$$\mathrm{Dom}(L) = \mathcal{C}_0^{\infty}[0,1] = \{f \in \mathcal{C}^{\infty}[0,1] : f(0) = 0\} \subset \mathrm{L}^2((0,1), w(x)\,dx).$$

Note that the conditions on the domain of L are the same as in (6), so the eigenvalues and eigenfunctions of the classical problem and those of the operator are identical. We define

$$\langle f, g\rangle_w = \int_0^1 f(x)\overline{g(x)}w(x)\,dx$$

and $\|f\|_w = \langle f, f\rangle_w^{1/2}$ for all $f, g \in \mathrm{L}^2((0,1), w(x)\,dx)$.

We also consider L_c, the restriction of L to $\mathcal{C}_c^{\infty}(0,1)$, which is the space of smooth, compactly supported functions on $(0, 1)$, and the Friedrichs extension L_F of L_c. We shall show that L_F is also an extension of L and that the two operators have the same eigenvalues.

THEOREM 3.1. *The operator L_c is non-negative and symmetric, and has a self-adjoint extension L_F such that $\mathcal{C}_c^{\infty}(0,1)$ is a core for the quadratic form Q corresponding to L_F. Moreover, L_F is an extension of L, and the two operators have the same eigenvalues and eigenfunctions.*

Proof. Simple integration by parts tells us that L_c is symmetric and that $\langle L_c u, u\rangle = \int_0^1 p|u'|^2 \geq 0$ for all $u \in \mathrm{Dom}(L_c)$. By [**6**, Theorem 4.4.5], L_c has a self-adjoint extension L_F, which we call the Friedrichs extension, and the domain of L_c is a core for Q. It follows from a characterization of the Friedrichs extension by Kalf (see [**9**, Theorem 1 and Remark 3]) that L_F is defined by (13) on

$$\mathrm{Dom}(L_F) = \left\{ u \in \mathcal{D} : \int^1 pg^2 \left| \left(\frac{u}{g} \right)' \right|^2 < \infty, \lim_{x \to 1-} \frac{u(x)}{g(x)} = 0 \right\},$$

where

$$\mathcal{D} = \{u \in AC_{\mathrm{loc}}(0, 1) \cap \mathcal{H} : pu' \in AC_{\mathrm{loc}}(0, 1), -w^{-1}(pu')' \in \mathcal{H}\}$$

and $g(x) = (1-x)^{-1/\varepsilon}$. If $u \in \mathcal{D}$ is bounded, then $\lim_{x\to 1-}(u(x)/g(x)) = 0$ and

$$\begin{aligned}
\int^1 pg^2 \left| \left(\frac{u}{g} \right)' \right|^2 &= \int^1 pg^2 \left| \frac{u'}{g} + u \left(\frac{1}{g} \right)' \right|^2 \\
&\leq 2 \int^1 pg^2 \left\{ \left| \frac{u'}{g} \right|^2 + \left| u \left(\frac{1}{g} \right)' \right|^2 \right\} \\
&\leq 2 \int^1 p|u'|^2 + \frac{2}{\varepsilon^2} \int^1 (1-x)^{1/\varepsilon - 1}(1+x)^{1-1/\varepsilon}|u(x)|^2 \, dx \\
&< \infty
\end{aligned}$$

since $p \leq 2w$. Therefore $\mathrm{Dom}(L) \subset \mathrm{Dom}(L_F)$ and, since the two operators are both defined by (13) on $\mathrm{Dom}(L)$, L_F is an extension of L.

Clearly, all eigenvalues and eigenfunctions of L are eigenvalues and eigenfunctions of L_F. Suppose that μ is an eigenvalue of L_F and u is a corresponding eigenfunction. Then $(pu')' = -\mu wu \in \mathcal{C}(0, 1)$. This implies that $pu' \in \mathcal{C}^1(0, 1)$ and, since p is smooth, $u' \in \mathcal{C}^1(0, 1)$. Therefore $u \in \mathcal{C}^2(0, 1)$, and hence the method of Frobenius and other standard ODE theory is applicable. Considering the Frobenius expansions at 0 and the condition that $u \in \mathcal{H}$, we see that we may write

$$u(x) = x(a_0 + a_1 x + a_2 x^2 + \cdots)$$

for all $x \in (0, 1)$, where $a_0 \neq 0$. At the other endpoint, the Frobenius theory tells us that we may write

$$u(x) = (x-1)^\rho (b_0 + b_1(x-1) + b_2(x-1)^2 + \cdots)$$

for all $x \in (0, 1)$, where $b_0 \neq 0$ and ρ is either 0 or $-1/\varepsilon$. Since $(1-x)^{1/\varepsilon}u(x) \to 0$ as $x \to 1-$, we must have $\rho = 0$. It thus follows that $u \in \mathcal{C}^\infty[0, 1] = \mathrm{Dom}(L)$. □

Remark. If $0 < \varepsilon \leq 1$, then both endpoints are in the limit-point case (the point 0 is said to be in the limit-point case if for every $\lambda \in \mathbb{C}$ there is a solution of $(pu')' = \lambda wu$ such that $\int_0^c |u|^2 w = \infty$ for all $c \in (0, 1)$; the definition is similar for the point 1). [**10**, Satz 1] then tells us that L_c is essentially self-adjoint. The same theorem tells us, however, that this is not the case when either endpoint fails to be in the limit-point case, as happens at the right-hand endpoint when $1 < \varepsilon < 2$.

§4. *Compactness of the resolvent.* To prove that L_F has compact resolvent, we use the following theorem from [**8**].

THEOREM 4.1 [8, Theorem XIII.7.40]. *Let*

$$\tau = -\left(\frac{d}{dt}\right) r(t) \left(\frac{d}{dt}\right) + s(t)$$

be a real formally self-adjoint formal differential operator defined on an interval $I = (a, b)$. *Let* $r(t) > 0$ *for* $t \in I$. *Then:*

(a) *if* τ *is not bounded below, then all solutions of every equation* $\tau\sigma = \lambda\sigma$ *have infinitely many zeros in* I*;*

and, conversely,

(b) *if* τ *is bounded below and* λ_0 *is the smallest point in* $\sigma_e(\tau)$ *(so that* $\lambda_0 > -\infty$ *in the present case), then for* $\mu > \lambda_0$ *every solution of* $\tau\sigma = \mu\sigma$ *has infinitely many zeros, while for* $\mu < \lambda_0$ *no solution of* $\tau\sigma = \mu\sigma$ *has infinitely many zeros.*

As it stands, this theorem does not apply to our operator L_F because of the lack of a weight function, but we can circumvent this problem by making the following transformation.

LEMMA 4.2. *There exists a real function* $q \in \mathcal{C}^\infty(0, 1)$ *such that* $L_F = U\tilde{L}_F U^{-1}$, *where* $\tilde{L}_F$ *is the Friedrichs extension of the operator* $\tilde{L}$ *acting in* $\mathrm{L}^2((0, 1), dx)$ *and defined on* $\mathrm{Dom}(\tilde{L}) = \mathcal{C}_c^\infty(0, 1)$ *by*

$$\tilde{L} f = -\left(\frac{p}{w} f'\right)' + qf$$

and U *is the unitary operator from* $\mathrm{L}^2((0, 1), dx)$ *to* $\mathrm{L}^2((0, 1), w(x)\, dx)$ *defined by* $Uf = w^{-1/2} f$.

Proof. Let

$$q = p\big\{w^{-1/2}(w^{-1/2})' + ((w^{-1/2})')^2\big\}.$$

Then $\tilde{L}$ is a symmetric operator and

$$\begin{aligned}
\langle \tilde{L} f, f\rangle &= \int_0^1 \frac{p(x)}{w(x)} |f'(x)|^2\, dx \\
&\quad + \int_0^1 p(x)\{w^{-1/2}(x)(w^{-1/2})'(x) + ((w^{-1/2})'(x))^2\} |f(x)|^2\, dx \\
&= \int_0^1 p(x) |w^{-1/2}(x) f'(x) + (w^{-1/2})'(x) f(x)|^2\, dx \\
&= Q(w^{-1/2} f),
\end{aligned}$$

so it is also non-negative. By [6, Theorem 4.4.5], the form $\tilde{Q}_c$ defined on $\mathrm{Dom}(\tilde{L})$ by $\tilde{Q}_c(f) = \langle \tilde{L} f, f\rangle$ is closable and its closure $\tilde{Q}$ is associated with the (self-adjoint) Friedrichs extension $\tilde{L}_F$ of $\tilde{L}$. Since w is smooth and non-zero, $f \in \mathcal{C}_c^\infty(0, 1)$ if and only if $Uf \in \mathcal{C}_c^\infty(0, 1)$. By the above calculation, $\tilde{Q}(f) = Q(Uf)$ for all such f. The same holds for all f in $\mathrm{L}^2(0, 1)$ by an approximation argument. Finally, it follows from the polarization identity for sesquilinear forms and [6, Lemma 4.4.1] that $f \in \mathrm{Dom}(\tilde{L})$ if and only if $Uf \in \mathrm{Dom}(L_F)$ and that $\tilde{L}_F = U^{-1} L_F U$. □

THEOREM 4.3. *The operator L_F has compact resolvent.*

Proof. Suppose that λ lies in the essential spectrum of L_F. Then λ also lies in the essential spectrum of $\tilde{L}_F$. By [8, Theorem XIII.7.40], if $\mu > \lambda$, then every solution of

$$-\left(\frac{p}{w} f'\right)' + qf = \mu f$$

has infinitely many zeros. It follows that for such μ, every solution of

$$-(pf')' = \mu w f \tag{14}$$

has infinitely many zeros. Considering the ODE theory (see [3, §17]) of the regular singular point $x = 0$, we see that there is a non-zero solution f of (14) which can be written as $f(x) = \sum_{n=1}^{\infty} a_n x^n$ in a neighbourhood of $x = 0$ and hence is differentiable at $x = 0$. The ODE theory of the regular singular point $x = 1$ shows that f is non-zero in a neighbourhood of $x = 1$. There is at least one accumulation point α of the zeros of f, which must therefore lie in $[0, 1)$. We know that f is differentiable at α, and since α is an accumulation point of zeros of f we must have $f(\alpha) = f'(\alpha) = 0$. This implies that f vanishes identically, which is a contradiction. We may therefore conclude that the essential spectrum of L_F is empty and hence that L_F has compact resolvent. □

COROLLARY 4.4. *The Sturm–Liouville operator L_F is non-negative in the sense that* $\mathrm{Spec}(L_F) \subseteq (0, \infty)$.

Proof. Since L_F has compact resolvent, it has empty essential spectrum, and since it is self-adjoint, its spectrum is thus equal to the set of its eigenvalues. By Theorem 3.1, it is sufficient to show that all eigenvalues of L are non-negative. If μ is an eigenvalue of L and f is a corresponding eigenvector with $\|f\|_w = 1$, then

$$\mu = \langle Lf, f\rangle_w = -\int_0^1 (pf')'(x)\overline{f(x)}\, dx = \int_0^1 p(x)|f'(x)|^2\, dx > 0 \tag{15}$$

since p is non-negative on $[0, 1]$. Note that the inequality is strict, as $f' = 0$ a.e. would imply that f is constant and hence 0, since $f(0) = 0$. □

COROLLARY 4.5. *There exists a complete orthonormal set of eigenvectors $\{f_n\}_{n=1}^{\infty}$ of L with corresponding eigenvalues $\mu_n \geq 0$ which converge monotonically to $+\infty$ as $n \to \infty$.*

Proof. The corresponding result for L_F is standard, and the result for L follows from Theorem 3.1. □

COROLLARY 4.6. *The operator L is essentially self-adjoint, and hence $L_F = \bar{L}$.*

Proof. This follows immediately from Corollary 4.5 above and [6, Lemma 1.2.2]. □

COROLLARY 4.7. *The operator* $-\mathrm{i}H$ *defined in §1 has infinitely many eigenvalues which can be enumerated as* $\{\lambda_n\}_{n=-\infty}^{\infty}$, *in increasing order, such that* $\lambda_0 = 0$, $\lambda_{-n} = -\lambda_n$ *and* $\lambda_n \to \infty$ *as* $n \to \infty$.

Proof. The eigenvalues of $-\mathrm{i}H$ are the same as the eigenvalues of A, since the two operators are unitarily equivalent. If $\{\lambda_n\}_{n=1}^{\infty}$ are the eigenvalues of A_+ in increasing order, then (4) tells us that the eigenvalues of A are $\{\lambda_n\}_{n=-\infty}^{\infty}$, where $\lambda_0 = 0$ and $\lambda_{-n} = -\lambda_n$. It follows from Corollary 2.2, Theorem 3.1 and Corollary 4.5 that $\lambda_n = \varepsilon\mu_n/2 \to \infty$ as $n \to \infty$. □

§5. *Eigenvalue asymptotics.*

5.1. *Transformation to a Schrödinger operator.* We next use a change of variables called the Liouville transformation to convert $\bar{L}$ into a Schrödinger operator on a certain space $\mathrm{L}^2(0, \beta)$. This allows us to use the extensive range of standard techniques available for controlling the eigenvalues of Schrödinger operators.

THEOREM 5.1. *Define*

$$\psi(t) = \int_0^t \sqrt{\frac{w(y)}{p(y)}}\, dy$$

for $t \in [0, 1]$, *and let* $\beta = \psi(1)$. *Then* $\beta < \infty$ *and* $\psi : [0, 1] \to [0, \beta]$ *is invertible. Let* $\phi = \psi^{-1}$ *and define*

$$\begin{aligned} c(s) &= (w(\phi(s))p(\phi(s)))^{-1/4}, \\ V(s) &= -c(s)c''(s)\frac{p(\phi(s))}{\phi'(s)} - c(s)c'(s)\frac{d}{ds}\frac{p(\phi(s))}{\phi'(s)}, \\ \hat{Q}_c(g) &= \int_0^\beta \{|g'(s)|^2 + V(s)|g(s)|^2\}\, ds \end{aligned}$$

for all $g \in \mathcal{C}_c^\infty(0, \beta) \subset \mathrm{L}^2(0, \beta)$. *Then* $\hat{Q}_c$ *is a closable quadratic form and its closure* $\hat{Q}$ *is associated with a self-adjoint operator* H, *which is unitarily equivalent to* $\bar{L}$. *The potential* V *is smooth on* $(0, 1)$, $V(s) \sim \frac{3}{4}s^{-2}$ *as* $s \to 0+$ *and* $V(s) \sim (1/\varepsilon^2 - 1/4)(\beta - s)^{-2}$ *as* $s \to \beta-$.

Proof. We have

$$\begin{aligned} \beta &= \int_0^1 y^{-1/2}(1-y)^{-1/2}(1+y)^{-1/2}\, dy \\ &\le \sqrt{2}\int_0^{1/2} y^{-1/2}\, dy + \sqrt{2}\int_{1/2}^1 (1-y)^{-1/2}\, dy \\ &= 2\sqrt{2}\int_0^{1/2} y^{-1/2}\, dy = 4. \end{aligned}$$

Since $w(t)^{1/2}p(t)^{-1/2}$ is smooth and positive for all $t \in (0, 1)$, it is immediate from its definition that ψ is smooth on $(0, 1)$ and continuous and strictly monotone increasing on $[0, 1]$. Thus ψ is injective. Since ψ is continuous,

$\psi(0) = 0$ and $\psi(1) = \beta$, ψ must be surjective. Hence ψ is invertible. It is easy to show that ϕ and ϕ' are smooth and non-zero.

We define

$$(Uf)(s) = c(s)^{-1} f(\phi(s))$$

for all $f \in \mathrm{L}^2((0,1), w(x)\,dx)$ and all $s \in [0, \beta]$. It follows from the definitions of ψ and ϕ that $\psi'(t) = w(t)^{1/2} p(t)^{-1/2}$ and $\phi'(s) = p(\phi(s))^{1/2} w(\phi(s))^{-1/2}$. Hence, upon making the change of variables $x = \phi(s)$, we have

$$\begin{aligned}
\|f\|_w^2 &= \int_0^1 |f(x)|^2 w(x)\,dx \\
&= \int_{\psi(0)}^{\psi(1)} |f(\phi(s))|^2 w(\phi(s))\phi'(s)\,ds \\
&= \int_0^\beta |f(\phi(s))|^2 (w(\phi(s))p(\phi(s)))^{1/2}\,ds \\
&= \int_0^\beta |c(s)^{-1} f(\phi(s))|^2\,ds \\
&= \|Uf\|_{\mathrm{L}^2}^2
\end{aligned}$$

for all $f \in \mathrm{L}^2((0,1), w(x)\,dx)$. It follows that U is a unitary operator from $\mathrm{L}^2((0,1), w(x)\,dx)$ to $\mathrm{L}^2(0,\beta)$.

Since p and w are smooth and non-zero on $(0, 1)$ and ϕ is smooth and non-zero on $(0, \beta)$, c is smooth and non-zero on $(0, \beta)$. Hence $Uf \in \mathcal{C}_c^\infty(0, \beta)$ if and only if $f \in \mathcal{C}_c^\infty(0, 1)$. Making the change of variables $x = \phi(s)$ as above gives

$$\begin{aligned}
Q(f) &= \int_0^1 |f'(x)|^2 p(x)\,dx \\
&= \int_0^\beta |c'(s)(Uf)(s) + c(s)(Uf)'(s)|^2 \psi'(\phi(s))^2 p(\phi(s))\phi'(s)\,ds \\
&= \int_0^\beta |c'(s)(Uf)(s) + c(s)(Uf)'(s)|^2 \phi'(s)^{-1} p(\phi(s))\,ds \\
&= \int_0^\beta \{c(s)^2|(Uf)'(s)|^2 + c'(s)^2|(Uf)(s)|^2\} \frac{p(\phi(s))}{\phi'(s)}\,ds \\
&\quad + \int_0^\beta c(s)c'(s)\frac{p(\phi(s))}{\phi'(s)} \frac{d}{ds}(|(Uf)(s)|^2)\,ds \\
&= \int_0^\beta \{c(s)^2|(Uf)'(s)|^2 + (c'(s)^2 - (cc')'(s))|(Uf)(s)|^2\} \frac{p(\phi(s))}{\phi'(s)}\,ds \\
&\quad - \int_0^\beta c(s)c'(s)\frac{d}{ds}\left\{\frac{p(\phi(s))}{\phi'(s)}\right\}|(Uf)(s)|^2\,ds \\
&= \int_0^\beta \{|(Uf)'(s)|^2 + V(s)|(Uf)(s)|^2\}\,ds \\
&= \hat{Q}_c(Uf)
\end{aligned}$$

for all $f \in \mathcal{C}_c^\infty(0, 1)$.

Since $\hat{Q}_c$ is the form arising from the symmetric operator

$$H_c = -\Delta + V$$

with domain $\mathrm{Dom}(H_c) = \mathcal{C}_c^\infty(0, \beta)$ and

$$\langle H_c f, f\rangle = \hat{Q}_c(f) = Q(U^{-1} f) \geq 0$$

for all $f \in \mathcal{C}_c^\infty(0, \beta)$, $\hat{Q}_c$ is closable and its closure, $\hat{Q}$, is associated with a non-negative self-adjoint extension H of H_c by [**6**, Theorem 4.4.5].

We have proven that $f \in \mathcal{C}_c^\infty(0, 1)$ if and only if $Uf \in \mathcal{C}_c^\infty(0, \beta)$, and that

$$Q(f) = \hat{Q}_c(Uf) \tag{16}$$

for all $f \in \mathcal{C}_c^\infty(0, 1)$. It now follows from an approximation argument that $Uf \in \mathrm{Dom}(\hat{Q})$ if and only if $f \in \mathrm{Dom}(Q)$, and that

$$Q(f) = \hat{Q}(Uf) \tag{17}$$

for all $f \in \mathrm{Dom}(Q)$. By the polarization identity for sesquilinear forms,

$$\hat{Q}'(Uf, Ug) = Q'(f, g)$$

for all $f, g \in \mathrm{Dom}(Q)$. An application of [**6**, Theorem 4.4.1] shows that $Uf \in \mathrm{Dom}(H)$ if and only if $f \in \mathrm{Dom}(\bar{L})$, and that $H = U\bar{L}U^{-1}$. Since c, c', c'', p, ϕ and $1/\phi'$ are smooth on $(0, 1)$, it follows that V is smooth on $(0, 1)$. For the asymptotics of V, see Appendix A. □

5.2. *Eigenvalue asymptotics.* Throughout this section, $(\mu_n)_{n=1}^\infty$ shall be the set of eigenvalues of L, or equivalently of $\bar{L}$ by Theorem 3.1 and Corollary 4.6, listed in increasing order and repeated according to multiplicity as in Corollary 4.5. By Theorem 5.1, $\bar{L}$ is unitarily equivalent to a Schrödinger operator H with Dirichlet boundary conditions. We shall use the Rayleigh–Ritz variational formula to obtain bounds on μ_n in terms of the Dirichlet eigenvalues of $-\Delta$ on various intervals. Recall that the Dirichlet eigenvalues of $-\Delta$ on the interval $[a, b]$ are $\{n^2\pi^2(b-a)^{-2} : n \in \mathbb{N}\}$, with corresponding eigenfunctions $\{\sin(n\pi(x-a)/(b-a)) : n \in \mathbb{N}\}$. We quote the variational formula from [**6**], as follows.

If K is a non-negative self-adjoint operator on a Hilbert space $\mathcal{H}$ and M is a finite-dimensional subspace of $\mathrm{Dom}(K)$, then we define

$$\lambda(M) = \sup\{\langle Kf, f\rangle : f \in M \text{ and } \|f\| = 1\}$$

and

$$\lambda_n = \inf\{\lambda(M) : M \subseteq \mathrm{Dom}(K) \text{ and } \dim(M) = n\}.$$

If $\lambda_n \to +\infty$ as $n \to \infty$, then K has compact resolvent and the numbers λ_n coincide with the eigenvalues of K written in increasing order and repeated according to multiplicity. Since K is non-negative and self-adjoint, it is associated with a closed quadratic form Q. If $\mathcal{D}$ is a core for Q, i.e. a subspace of the domain of Q such that the closure of Q restricted to $\mathcal{D}$ is Q, then we have

$$\lambda_n = \inf\{\lambda(M) : M \subseteq \mathcal{D} \text{ and } \dim(M) = n\}$$

for all $n \in \mathbb{N}$.

THEOREM 5.2. *Let*

$$\beta = \int_0^1 y^{-1/2}(1-y)^{-1/2}(1+y)^{-1/2}\, dy$$

as in Theorem 5.1. Then

$$\lim_{n\to\infty} n^{-2}\mu_n = \frac{\pi^2}{\beta^2}.$$

Indeed, if $\alpha = \min\{V(s) : s \in (0,\beta)\}$, *where* V *is as in Theorem 5.1, then*

$$\mu_n \geq \frac{n^2\pi^2}{\beta^2} + \alpha$$

for all $n \in \mathbb{N}$ *and*

$$\mu_n \leq \frac{n^2\pi^2}{\beta^2} + O(n^{4/3})$$

as $n \to \infty$.

Proof. Define

$$Q_\alpha(g) = \langle Hg, g\rangle - \alpha\langle g, g\rangle = \hat{Q}_c(g) - \alpha\langle g, g\rangle \geq 0$$

for all $g \in \mathcal{C}_c^\infty(0,\beta)$. Then Q_α is a closable form and its closure $\overline{Q}_\alpha$ is the form associated with a non-negative self-adjoint extension H_α of $H_c - \alpha I$, where $H_c = -\Delta + V$ with $\mathrm{Dom}(H_c) = \mathcal{C}_c^\infty(0,\beta)$. Since $\mathcal{C}_c^\infty(0,\beta)$ is a core for both $\hat{Q}$ and $\bar{Q}_\alpha$, the variational formula implies that the nth eigenvalue of H_α is $\mu_n - \alpha$.

We now define

$$\tilde{Q}(g) = \int_0^\beta |g'(s)|^2\, ds \leq \int_0^\beta |g'(s)|^2 + (V(s)-\alpha)|g(s)|^2\, ds = Q_\alpha(g)$$

for all $g \in \mathcal{C}_c^\infty(0,\beta)$. The closure $\overline{\tilde{Q}}$ of $\tilde{Q}$ is the form associated with the operator $-\Delta$ on $\mathrm{L}^2(0,\beta)$ with Dirichlet boundary conditions. Since $\mathcal{C}_c^\infty(0,\beta)$ is a core for both $\overline{\tilde{Q}}$ and $\bar{Q}_\alpha$, the variational formula implies that

$$\frac{n^2\pi^2}{\beta^2} \leq \mu_n - \alpha$$

for all $n \in \mathbb{N}$.

For all sufficiently small $\delta > 0$, define

$$K_\delta(f) = (-\Delta + c_\delta I)(f)$$

for all $f \in \mathcal{C}_c^\infty(\delta, \beta-\delta)$, where

$$c_\delta = \sup\{V(s) : s \in (\delta, \beta-\delta)\}.$$

Then K_δ is a non-negative symmetric operator and hence has a non-negative self-adjoint extension $\tilde{K}_\delta$, called the Friedrichs extension of K_δ, associated with the closure of the form Q_δ defined by

$$Q_\delta(f) = \langle K_\delta f, f\rangle$$

for all $f \in C_c^\infty(\delta, \beta - \delta)$. For all finite-dimensional subspaces M of $\mathrm{Dom}(K_\delta)$, define

$$\tilde{\lambda}_\delta(M) = \sup\{\langle K_\delta f, f\rangle : f \in M \text{ and } \|f\| = 1\}$$

and

$$\tilde{\lambda}_{\delta,n} = \inf\{\tilde{\lambda}_\delta(M) : M \subseteq C_c^\infty(\delta, \beta - \delta) \text{ and } \dim(M) = n\}.$$

Then, for $M \subseteq C_c^\infty(\delta, \beta - \delta)$,

$$\begin{aligned}\tilde{\lambda}_\delta(M) &= \sup\{\langle(-\Delta + c_\delta I)f, f\rangle : f \in M \text{ and } \|f\| = 1\}\\ &= \sup\{\langle(-\Delta)f, f\rangle : f \in M \text{ and } \|f\| = 1\} + c_\delta\end{aligned}$$

and hence

$$\tilde{\lambda}_{\delta,n} = \frac{n^2\pi^2}{(\beta - 2\delta)^2} + c_\delta$$

by the variational formula for the Dirichlet eigenvalues of $-\Delta$ on $(\delta, \beta - \delta)$. Since $\tilde{\lambda}_{\delta,n} \to +\infty$ as $n \to \infty$ and $C_c^\infty(\delta, \beta - \delta)$ is a core for the quadratic form associated with $\tilde{K}_\delta$, $\tilde{K}_\delta$ has compact resolvent and its eigenvalues coincide with $\tilde{\lambda}_{\delta,n}$. For all $M \subseteq \mathrm{Dom}(H)$, define

$$\mu(M) = \sup\{\langle Hf, f\rangle : f \in M \text{ and } \|f\| = 1\}.$$

If $f \in C_c^\infty(\delta, \beta - \delta)$, then

$$\langle Hf, f\rangle = \langle(-\Delta + V)f, f\rangle \le \langle(-\Delta + c_\delta I)f, f\rangle = \langle K_\delta f, f\rangle,$$

so $\mu(M) \le \tilde{\lambda}_\delta(M)$ for all $M \subseteq C_c^\infty(\delta, \beta - \delta)$. Since $C_c^\infty(0, \beta)$ is a core for $\hat{Q}$,

$$\begin{aligned}\mu_n &= \inf\{\mu(M) : M \subseteq C_c^\infty(0, \beta) \text{ and } \dim(M) = n\}\\ &\le \inf\{\mu(M) : M \subseteq C_c^\infty(\delta, \beta - \delta) \text{ and } \dim(M) = n\}\\ &\le \inf\{\tilde{\lambda}_\delta(M) : M \subseteq C_c^\infty(\delta, \beta - \delta) \text{ and } \dim(M) = n\}\\ &= \frac{n^2\pi^2}{(\beta - 2\delta)^2} + c_\delta\end{aligned}$$

for all $n \in \mathbb{N}$. From the asymptotics of V, $c_\delta\delta^2 \to \max\{3/4, 1/\varepsilon^2 - 1/4\} =: c$ as $\delta \to 0$. Hence, given $\nu > 0$, $c_\delta \le (c + \nu)\delta^{-2}$ for all sufficiently small δ. For such δ,

$$\mu_n - \frac{n^2\pi^2}{\beta^2} \le n^2\pi^2\left(\frac{1}{(\beta - 2\delta)^2} - \frac{1}{\beta^2}\right) + \frac{c + \nu}{\delta^2} =: F_n(\delta)$$

for all $n \in \mathbb{N}$. Hence, for all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}\mu_n - \frac{n^2\pi^2}{\beta^2} &\leq F_n(n^{-2/3}) \\ &= n^2\pi^2\left(\frac{1}{(\beta - 2n^{-2/3})^2} - \frac{1}{\beta^2}\right) + (c+\nu)n^{4/3} \\ &= n^2\pi^2\frac{4n^{-2/3}(\beta - n^{-2/3})}{\beta^2(\beta - 2n^{-2/3})^2} + (c+\nu)n^{4/3} \\ &\sim \left(\frac{4\pi^2}{\beta^3} + c + \nu\right)n^{4/3}\end{aligned}$$

as $n \to \infty$. □

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A. *Appendix. Asymptotics of V.* In order to complete the proof of Theorem 5.2, we need to analyse the asymptotic behaviour of

$$V(s) = -c(s)c''(s)\frac{p(\phi(s))}{\phi'(s)} - c(s)c'(s)\frac{\phi'(s)^2 p'(\phi(s)) - p(\phi(s))\phi''(s)}{\phi'(s)^2} \tag{A1}$$

as $s \to 0+$ and as $s \to \beta-$. We have

$$c = k|_{(0,1)} \circ \phi, \tag{A2}$$

$$c' = (k'|_{(0,1)} \circ \phi)\phi', \tag{A3}$$

$$c'' = (k''|_{(0,1)} \circ \phi)\phi'^2 + (k'|_{(0,1)} \circ \phi)\phi'', \tag{A4}$$

where k is the analytic function defined by

$$k(z) = z^{1/4}(1-z)^{-1/2\varepsilon - 1/4}(1+z)^{1/2\varepsilon - 1/4}$$

for all $z \in \mathbb{C}\backslash((-\infty, 0] \cup [1, +\infty))$.

LEMMA A.1. *Asymptotically,*

$$\begin{aligned}k(z) &\sim z^{1/4}, \\ k'(z) &\sim \tfrac{1}{4}z^{-3/4}, \\ k''(z) &\sim -\tfrac{3}{16}z^{-7/4}\end{aligned}$$

as $z \to 0$ and

$$\begin{aligned}k(z) &\sim 2^{1/2\varepsilon - 1/4}(1-z)^{-1/2\varepsilon - 1/4}, \\ k'(z) &\sim \left(\frac{1}{2\varepsilon} + \frac{1}{4}\right)2^{1/2\varepsilon - 1/4}(1-z)^{-1/2\varepsilon - 5/4}, \\ k''(z) &\sim \left(\frac{1}{2\varepsilon} + \frac{1}{4}\right)\left(\frac{1}{2\varepsilon} + \frac{5}{4}\right)2^{1/2\varepsilon - 1/4}(1-z)^{-1/2\varepsilon - 9/4}\end{aligned}$$

as $z \to 1$.

Proof. We may write
$$k(z) = z^{1/4} k_0(z)$$
where
$$k_0(z) = (1-z)^{-1/2\varepsilon-1/4}(1+z)^{1/2\varepsilon-1/4} \tag{A5}$$
is analytic in a neighbourhood of 0. The function k_0 has a power series expansion
$$k_0(z) = \sum_{n=0}^{\infty} a_n z^n$$
valid in some disc $D(0;\ R_0)$. Putting $z = 0$ in (A5), we see that $a_0 = 1$. We thus have
$$\begin{aligned} k(z) &= z^{1/4} + \sum_{n=1}^{\infty} a_n z^{n+1/4}, \\ k'(z) &= \frac{1}{4} z^{-3/4} + \sum_{n=1}^{\infty}\left(n+\frac{1}{4}\right) a_n z^{n-3/4}, \\ k''(z) &= -\frac{3}{16} z^{-7/4} + \sum_{n=1}^{\infty}\left(n+\frac{1}{4}\right)\left(n-\frac{3}{4}\right) a_n z^{n-7/4} \end{aligned}$$
in the cut disc $D(0;\ R_0)\backslash(-R_0,\ 0]$, and the stated asymptotics as $z \to 0$ follow.

Similarly, by writing
$$k(z) = (1-z)^{-1/2\varepsilon-1/4} k_1(z)$$
where
$$k_1(z) = z^{1/4}(1+z)^{1/2\varepsilon-1/4}$$
is analytic in a neighbourhood of 1, we obtain the stated asymptotics as $z \to 1$. □

LEMMA A.2. *Asymptotically,*
$$\begin{aligned} \phi(s) &\sim 2^{-2} s^2, \\ \phi'(s) &\sim 2^{-1} s, \\ \phi''(s) &\sim 2^{-1} \end{aligned}$$
as $s \to 0+$ *and*
$$\begin{aligned} 1-\phi(s) &\sim 2^{-1}(\beta - s)^2, \\ \phi'(s) &\sim \beta - s, \\ \phi''(s) &\sim -1 \end{aligned}$$
as $s \to \beta-$.

Proof. The inverse ψ of ϕ is given by
$$\psi(t) = \int_0^t w(y)^{1/2} p(y)^{-1/2}\, dy$$
for all $t \in [0,\ 1]$. Since
$$w(y)^{1/2} p(y)^{-1/2} = y^{-1/2}(1-y)^{-1/2}(1+y)^{-1/2}$$

for all $y \in (0, 1)$,

$$\psi(t) \sim 2t^{1/2}$$

as $t \to 0+$ and

$$\beta - \psi(t) = \int_t^1 y^{-1/2}(1-y)^{-1/2}(1+y)^{-1/2}\,dy \sim 2^{1/2}(1-t)^{1/2}$$

as $t \to 1-$. Putting $t = \phi(s)$, we obtain the stated asymptotics for ϕ.

The stated asymptotics for ϕ' follow immediately from the asymptotics of ϕ and the fact that

$$\phi'(s) = \frac{1}{\psi'(\phi(s))} \tag{A6}$$

$$= p(\phi(s))^{1/2} w(\phi(s))^{-1/2} \tag{A7}$$

$$= \phi(s)^{1/2}(1-\phi(s))^{1/2}(1+\phi(s))^{1/2} \tag{A8}$$

for all $s \in (0, \beta)$. Differentiating (A8) and using the asymptotics we have calculated for ϕ and ϕ', we obtain the stated asymptotics for ϕ''. □

COROLLARY A.3. *Asymptotically,*

$$c(s) \sim 2^{-1/2}s^{1/2},$$
$$c'(s) \sim 2^{-3/2}s^{-1/2},$$
$$c''(s) \sim -2^{-5/2}s^{-3/2}$$

as $s \to 0+$ *and*

$$c(s) \sim 2^{1/\varepsilon}(\beta - s)^{-1/\varepsilon-1/2},$$
$$c'(s) \sim \left(\frac{1}{\varepsilon} + \frac{1}{2}\right)2^{1/\varepsilon}(\beta - s)^{-1/\varepsilon-3/2},$$
$$c''(s) \sim \left(\frac{1}{\varepsilon} + \frac{1}{2}\right)\left(\frac{1}{\varepsilon} + \frac{3}{2}\right)2^{1/\varepsilon}(\beta - s)^{-1/\varepsilon-5/2}$$

as $s \to \beta-$.

Proof. This follows immediately from equations (A2)–(A4) and Lemmas A.1 and A.2. □

COROLLARY A.4. *Asymptotically,*

$$p(\phi(s)) \sim 1,$$
$$p'(\phi(s)) \sim -\frac{2}{\varepsilon}$$

as $s \to 0+$ *and*

$$p(\phi(s)) \sim 2^{-2/\varepsilon}(\beta - s)^{2+2/\varepsilon},$$
$$p'(\phi(s)) \sim -\left(1 + \frac{1}{\varepsilon}\right)2^{1-2/\varepsilon}(\beta - s)^{2/\varepsilon}$$

as $s \to \beta-$.

Proof. The asymptotics of p and p' are:

$$p(x) \sim 1,$$
$$p'(x) \sim -\frac{2}{\varepsilon}$$

as $x \to 0$ and

$$p(x) \sim 2^{1-1/\varepsilon}(1-x)^{1+1/\varepsilon},$$
$$p'(x) \sim -\left(1+\frac{1}{\varepsilon}\right)2^{1-1/\varepsilon}(1-x)^{1+1/\varepsilon}$$

as $x \to 1$. The stated asymptotics follow from this and Lemma A.2. □

THEOREM A.5. *Asymptotically,*

$$V(s) \sim \tfrac{3}{4}s^{-2}$$

as $s \to 0+$ *and*

$$V(s) \sim \left(\frac{1}{\varepsilon^2}-\frac{1}{4}\right)(\beta - s)^{-2}$$

as $s \to \beta-$.

Proof. This follows from (A1), Lemma A.2 and Corollaries A.3 and A.4. □

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