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Spectral Properties of Non-self-adjoint Operators in the Semi-classical Regime

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We give a spectral description of the semi-classical Schrödinger operator with a piecewise-linear, complex-valued potential. Moreover, using these results, we give an example to show how an arbitrarily small fixed perturbation of a non-self-adjoint operator can completely change the asymptotic spectrum of the operator in the semi-classical limit.

Key Words: Schrödinger operator; non-self-adjoint operator; Sturm–Liouville problem; Airy function; Stokes' lines; characteristic determinant; asymptotic analysis.

1. INTRODUCTION

This paper was initially motivated by the work of Shkalikov [7]. There it is shown rigorously that the spectrum of the Airy operator

$$H_{0,h} := -h^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix$$

acting in $L^2(-1, 1)$ subject to Dirichlet boundary conditions, $f(-1) = f(1) = 0$, has remarkable properties as the real parameter $h > 0$ approaches zero, i.e. in the semi-classical limit. Initially, when $h = 1$, we have a straightforward (non-self-adjoint) Sturm–Liouville problem, and the spectrum comprises countably many real eigenvalues accumulating at $+\infty$. However, as $h \to 0$, computer simulations of the associated discrete problem using the numerical package Matlab, suggest that the lowest two real eigenvalues coalesce and then split into complex conjugate pairs. Shkalikov confirms that as $h \to 0$, the spectrum becomes dense inside an arbitrarily small neighbourhood of the Y-shaped subset of **C** defined by

$$[i, 1/\sqrt{3}], [-i, 1/\sqrt{3}] \qquad \text{and} \qquad [1/\sqrt{3}, \infty).$$

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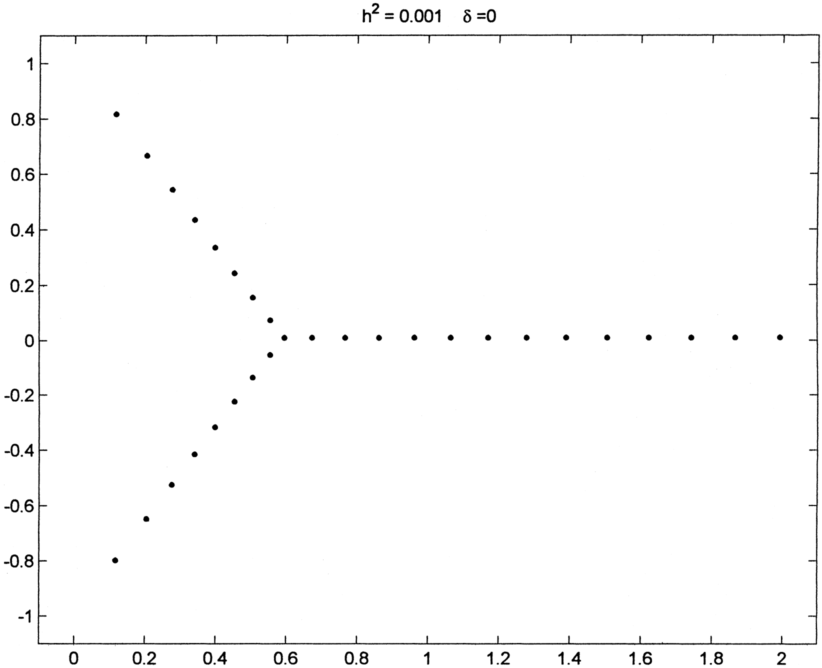

FIGURE 1

We use $[\alpha, \beta]$ to denote the line segment joining $\alpha, \beta \in \mathbf{C}$, and Fig. 1 shows the situation when $h^2 = 0.001$.

As mentioned by Shkalikov, this behaviour extends to more general analytic complex-valued potentials. Therefore we decided to examine the situation where the potential ix is perturbed in such a way that it is no longer analytic. Specifically, we examined the operator $H_{\delta, h}$ given formally by

$$H_{\delta, h} := -h^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\delta \qquad \text{on } L^2(-1, 1), \tag{1}$$

where

$$V_\delta(x) := \begin{cases} i(x+\delta) & \text{for} \quad x > 0 \\ i(x-\delta) & \text{for} \quad x < 0 \end{cases}$$

and both $\delta > 0$, $h > 0$ are small. The results were very surprising! For fixed $\delta > 0$, a further bifurcation of the spectrum occurs as $h \to 0$, and the limiting spectrum comprises *two* Y-shaped sets. See Fig. 2 and Corollary 5

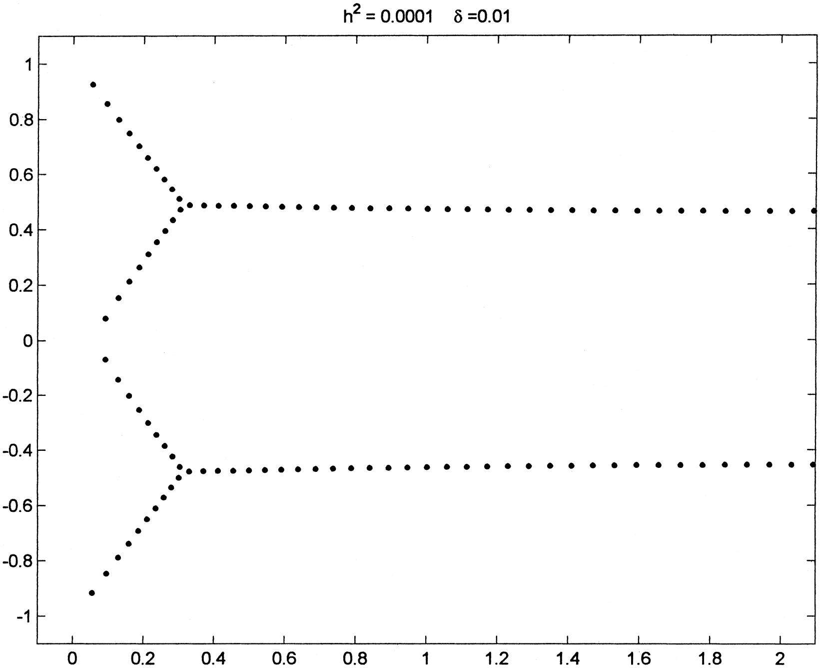

FIGURE 2

below. The operator appears to have decoupled at $x=0$ and to act as two separate operators: one on $L^2(-1, 0)$ and the other on $L^2(0, 1)$, the limiting spectrum being the union of the two separate spectra. This seemed to be the case even for arbitrarily small $\delta > 0$. Further investigation suggested that the same phenomena also occurs when the potential comprises two linear parts, joined continuously, but with an arbitrarily small change in slope. It was even seen when the ranges of the separate parts of the potential overlap, as in the example

$$V(x) := \begin{cases} 2ix+i & \text{for} \quad -1 \leqslant x < 0 \\ (i+1)\,x & \text{for} \quad 0 < x \leqslant 1 \end{cases}$$

shown in Fig. 3.

Intuitively, in the example (1), what seems to be happening is that the potential term V_δ acts as two separate operators, decoupled at the point $x=0$ where it is not continuously differentiable. Only the kinetic term $-h^2(\mathrm{d}^2/\mathrm{d}x^2)$ provides a coupling force to bind the whole operator $H_{\delta,h}$

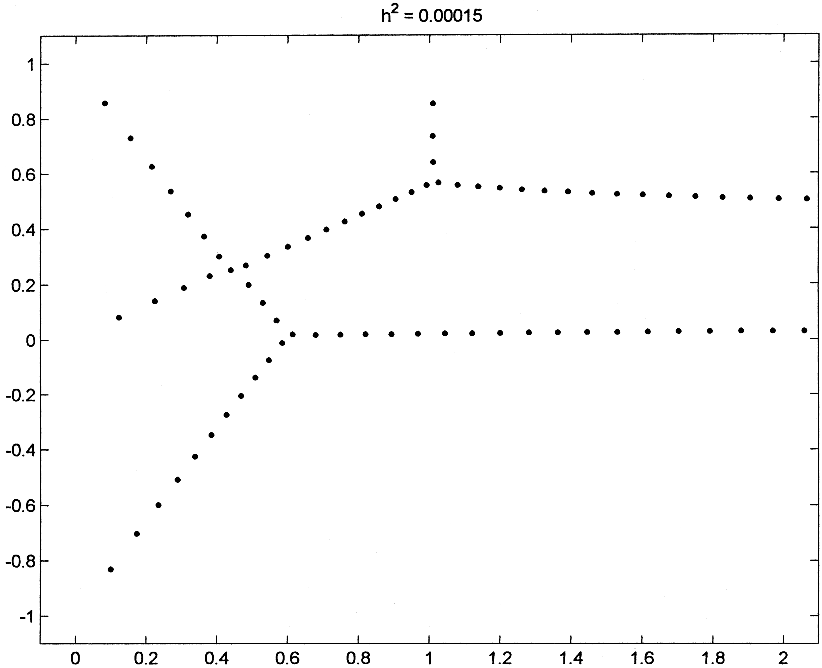

FIGURE 3

together, but as $h \to 0$ this coupling effect eventually becomes weaker and weaker relative to the decoupling action of the kink or discontinuity in the potential.

In this paper we attempt to analytically verify these computer experiments, shed some light on the mathematical machinery causing these surprising phenomena, and hence try to put the previous paragraph on a more rigorous footing. Several papers [6, 7, 8] have already been written about the operator $H_{0,h}$—a major motivation being that it is a model operator for the "Orr–Sommerfeld" problem [6]. The operator also defines the "Squire model for the Couette flow" in hydrodynamics; and in its own right, defines the semigroup which is the solution of the so-called "Torrey equation" [8], related to the diffusion of magnetic fields. Thus, despite its strange behaviour, this operator must not be dismissed as a "pathological" example from pure mathematics since it has important applications—for example, in magnetic resonance imaging devices. However, our conclusion must be that whilst the asymptotic spectrum is theoretically computable for an idealised linear potential; in practice, any *arbitrarily* small deviation from the ideal can *completely* change it.

There is a growing literature, for example [1, 2, 3, 4, 6, 9], demonstrating that for many non-self-adjoint operators the pseudospectral sets

$$\mathrm{Spec}_\epsilon(H) := \mathrm{Spec}(H) \cup \{z \in \mathbf{C} : \|(H-z)^{-1}\| \geqslant \epsilon^{-1}\},$$

i.e. the contour sets of the resolvent norm, become very large as some parameter varies, even though z may be far from the spectrum of the operator. The convention is adopted here that $z \in \mathrm{Spec}(H)$ implies

$$\|(H-z)^{-1}\| := \infty.$$

This could also be expressed by saying that the spectrum is computationally unstable. Our aim in this paper is to demonstrate for a relatively transparent case one mechanism behind this phenomenon, seemingly widespread in the case of non-self-adjoint operators, and in contrast to the stability theorems of the self-adjoint theory. We believe our results to be capable of extension to a more general class of piecewise analytic potentials.

In preparation for our main proofs, Section 2 gives some notation, and for completeness, lists some well-known properties of the Airy functions and Stokes' lines. For more details in these matters the reader could consult [5]. At the heart of all our analysis in this work lies the concept of the *characteristic determinant*, which we will define in Section 3. We give our main result in Section 5; and in Section 6 we return to the example (1) and provide an analysis of the simultaneous limit as $h \to 0$ and $\delta \to 0$ together, which gives further insight into the spectral behaviour observed in our main result, Theorem 3.

2. AIRY FUNCTIONS AND STOKES' LINES

First let us discuss some notation. In all that follows we let the argument function Arg take principal values i.e.

$$\mathrm{Arg}\colon \mathbf{C} \to (-\pi, \pi].$$

If $\mathrm{Arg}(\beta-\alpha) := \theta$, the subsets $Y(\alpha, \beta)$ of $\mathbf{C}$ are to be constructed as follows: take the lines

$$\alpha + re^{2\theta i/3} \qquad \text{and} \qquad \begin{cases} \beta + re^{2\theta i/3 + 2\pi i/3} & \text{for } \theta < 0 \\ \beta + re^{2\theta i/3 - 2\pi i/3} & \text{for } \theta \geqslant 0 \end{cases}$$

as r ranges in $[0, \infty)$, to their point of intersection, Γ say. Then, from Γ extend the infinite line defined by the set of $\lambda \in \mathbf{C}$ such that

$$\mathrm{Re}\{(\mathrm{e}^{-2\theta i/3}(\alpha-\lambda))^{3/2}\} = \mathrm{Re}\{(\mathrm{e}^{-2\theta i/3}(\beta-\lambda))^{3/2}\}. \tag{2}$$

The motivation for this set will become clear during our proofs; in fact, it will be seen to comprise a curve asymptotic to the line

$$\left\{ z \in \mathbf{C} : \operatorname{Im}(z) = \frac{\operatorname{Im}(\alpha) + \operatorname{Im}(\beta)}{2} \right\}.$$

Note for now, however, that (2) is h-independent. The ε-neighbourhood of any subset T of $\mathbf{C}$ will be defined by

$$\operatorname{Nhd}(T; \varepsilon) := \{ t + z : t \in T \text{ and } |z| < \varepsilon \}.$$

A basis for solutions of the so-called "Airy equation"

$$-f''(z) + z f(z) = 0, \tag{3}$$

where z is the complex independent variable, can be given by any two of the Airy functions $Ai(z)$, $Ai(e^{-2\pi i/3} z)$ and $Ai(e^{2\pi i/3} z)$. We will use the well-known [5] asymptotic expansion of the Airy function Ai, giving the WKB-type approximation:

$$Ai(z) = \frac{z^{-1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3} z^{3/2} \right) (1 + O(z^{-3/2})) \tag{4}$$

as $|z| \to \infty$, valid for $|\operatorname{Arg}(z)| < \pi$; and where the principal value of $z^{3/2}$ is taken. Following the notation of Olver (see [5, p. 413]), we define

$$S_0 := \{ z \colon |\operatorname{Arg}(z)| < \pi/3 \}$$
$$S_1 := \{ z \colon \pi/3 < \operatorname{Arg}(z) < \pi \}$$
$$S_{-1} := \{ z \colon -\pi/3 > \operatorname{Arg}(z) > -\pi \}$$

(suffixes enumerated modulo 3). One can check that for all complex z (and taking principal values), we have

$$\operatorname{Re}\{(e^{-2\pi i/3} z)^{3/2}\} = \begin{cases} -\operatorname{Re}\{(e^{2\pi i/3} z)^{3/2}\} & \text{for} \quad z \in S_1 \cup S_{-1} \\ \operatorname{Re}\{(e^{2\pi i/3} z)^{3/2}\} & \text{for} \quad z \in S_0 \end{cases} \tag{5}$$

$$\operatorname{Re}\{(e^{-2\pi i/3} z)^{3/2}\} = \begin{cases} -\operatorname{Re}\{(z)^{3/2}\} & \text{for} \quad z \in S_1 \cup S_0 \\ \operatorname{Re}\{(z)^{3/2}\} & \text{for} \quad z \in S_{-1} \end{cases} \tag{6}$$

and

$$\operatorname{Re}\{(e^{2\pi i/3} z)^{3/2}\} = \begin{cases} -\operatorname{Re}\{(z)^{3/2}\} & \text{for} \quad z \in S_0 \cup S_{-1} \\ \operatorname{Re}\{(z)^{3/2}\} & \text{for} \quad z \in S_1. \end{cases} \tag{7}$$

Then, putting

$$Ai_k(z) := Ai(e^{-2k\pi i/3}z) \tag{8}$$

the asymptotics (4) show that as $|z| \to \infty$, $|Ai_k(z)|$ decreases exponentially for $z \in S_k$, and increases exponentially for $z \in S_{k-1} \cup S_{k+1}$. The boundaries of the sectors S_k i.e. the rays $te^{\pm\pi i/3}$ and $te^{\pi i}$ for $t \in [0, \infty)$, are known as the Stokes' lines (or principal curves) of the problem [5, p. 503]. Indeed, for the Airy equation

$$-f''(z) + zf(z) = 0$$

the Stokes' lines are defined to be the values of z such that

$$\mathrm{Re} \int_0^z \sqrt{t}\, \mathrm{d}t = \mathrm{Re}\{\tfrac{2}{3} z^{3/2}\} = 0$$

and denote the boundaries of the *principal subdomains* S_1 etc., inside of which the asymptotic expression (4) is valid for each k.

We will call the suffix k "allowable" for any given $z \in \mathbf{C}$, if

$$|\mathrm{Arg}(e^{-2k\pi i/3}z)| < \pi. \tag{9}$$

3. THE CHARACTERISTIC DETERMINANT

In this section we describe the characteristic determinant of the operator H_h defined by

$$H_h f(x) := -h^2 \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + V(x)\, f(x)$$

acting on $L^2(-1, 1)$ with Dirichlet boundary conditions, $h > 0$ small, and $V(x)$ the complex valued, n-times piecewise linear function

$$V(x) := \begin{cases} m_1 x + l_1 & x_0 \leqslant x < x_1 \\ m_2 x + l_2 & x_1 < x < x_2 \\ \vdots & \vdots \\ m_n x + l_n & x_{n-1} < x \leqslant x_n \end{cases}$$

with $-1 = x_0 < x_1 < \cdots < x_n = 1$, and the m_i, l_i $i = 1, \ldots, n$ complex constants. The domain of the operator is given precisely by

$$\begin{aligned} \mathrm{Dom}(H_h) = \{ f \in C[-1, 1] : f(-1) = f(1) = 0,\ f' \in C[-1, 1] \\ \text{and } f'' \in L^2(-1, 1) \} \end{aligned} \tag{10}$$

where the primes $'$ denote differentiation with respect to x, and f'' is initially to be interpreted in the distributional sense. A direct substitution shows that a basis of solutions for the differential equation

$$-h^2 f''(x) + (V(x) - \lambda)\, f(x) = 0,$$

where $V(x) := mx + l$; and l, m are complex constants, is given by any two of the Airy functions $Ai(w)$ and $Ai(e^{\pm 2\pi i/3} w)$, where

$$w := h^{-2/3} m^{-2/3} (V(x) - \lambda).$$

It follows that, in order to construct an eigenfunction of the operator H_h, we seek constants α_{i1}, α_{i2} $i = 1, \ldots, n$ such that the function

$$f(x) := \begin{cases} \alpha_{11} u_{11}(x) + \alpha_{12} u_{12}(x) & x_0 \leqslant x < x_1 \\ \alpha_{21} u_{21}(x) + \alpha_{22} u_{22}(x) & x_1 < x < x_2 \\ \vdots & \vdots \\ \alpha_{n1} u_{n1}(x) + \alpha_{n2} u_{n2}(x) & x_{n-1} < x \leqslant x_n \end{cases}$$

satisfies all of the domain conditions (10), where

$$u_{i1}(x) := Ai(e^{-2k\pi i/3} h^{-2/3} m_i^{-2/3} ((m_i x + l_i) - \lambda)) \tag{11}$$

with $k \in \{-1, 0, 1\}$. For each $i = 1, \ldots, n$, the functions u_{i2} are defined similarly, except that a different choice of k is to be taken from $\{-1, 0, 1\}$.

In addition to satisfying the boundary conditions $f(-1) = f(1) = 0$, f must also be continuously differentiable, even at the points x_i. From the power series definition [5, p. 54], it is clear that the Airy functions Ai are analytic on the whole of $\mathbf{C}$, and so the requirement that f be continuously differentiable reduces to the $2(n-1)$ simultaneous "matching" conditions

$$\alpha_{i1} u_{i1}(x_i -) + \alpha_{i2} u_{i2}(x_i -) - \alpha_{(i+1)\,1} u_{(i+1)\,1}(x_i +) - \alpha_{(i+1)\,2} u_{(i+1)\,2}(x_i +) = 0$$

and

$$\alpha_{i1} u'_{i1}(x_i -) + \alpha_{i2} u'_{i2}(x_i -) - \alpha_{(i+1)\,1} u'_{(i+1)\,1}(x_i +) - \alpha_{(i+1)\,2} u'_{(i+1)\,2}(x_i +) = 0.$$

Remark 1. As we shall see, these matching conditions are the crucial elements in the mathematical process which causes the spectral behaviour of the operator to decouple at the non-linearities of the potential.

The boundary conditions $f(-1) = f(1) = 0$ demand that

$$\alpha_{11} u_{11}(-1) + \alpha_{12} u_{12}(-1) = 0$$

and

$$\alpha_{n1} u_{n1}(1) + \alpha_{n2} u_{n2}(1) = 0.$$

Thus finding a solution of the differential equation

$$-h^2 f''(x) + V(x)\, f(x) = \lambda f(x)$$

which satisfies all of the domain conditions (10), involves solving the matrix equation

$$\begin{pmatrix} u_{11}(-1) & u_{12}(-1) & 0 & 0 & 0 & \cdots & 0 \\ u_{11}(x_1) & u_{12}(x_1) & -u_{21}(x_1) & -u_{22}(x_1) & 0 & \cdots & 0 \\ u'_{11}(x_1) & u'_{12}(x_1) & -u'_{21}(x_1) & -u'_{22}(x_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & u_{(n-1)\,1}(x_{n-1}) & u_{(n-1)\,2}(x_{n-1}) & -u_{n1}(x_{n-1}) & -u_{n2}(x_{n-1}) \\ 0 & \cdots & 0 & u'_{(n-1)\,1}(x_{n-1}) & u'_{(n-1)\,2}(x_{n-1}) & -u'_{n1}(x_{n-1}) & -u'_{n2}(x_{n-1}) \\ 0 & \cdots & 0 & 0 & 0 & u_{n1}(1) & u_{n2}(1) \end{pmatrix}$$

$$\times \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{21} \\ \alpha_{22} \\ \vdots \\ \alpha_{n1} \\ \alpha_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

for the constants α_{i1}, α_{i2}. Note that we are taking the left- and right-hand limits at the points x_i, although here and subsequently we will abuse the notation in order to add clarity, and simply write $u_{i1}(x_i)$ etc. It is the determinant of the matrix on the left-hand side of (12) that we shall call the characteristic determinant of the eigenvalue problem defined by H_h.

4. A LEMMA

We have seen that the analytic investigation of the asymptotic spectrum of the semi-classical Schrödinger operator with a complex-valued linear potential involves examining the asymptotic behaviour of certain Airy functions. We will show that the eigenvalues lie inside a certain subset of the complex plane, which is intimately related to the Stokes' lines of the problem. The proof will depend upon showing that for all λ outside this subset, the eigenvalue problem

$$H_h f(x) = \lambda f(x)$$

has a Green's function which is uniformly bounded as $h \to 0$. As in [7], our analysis uses the concept of the characteristic determinant, described in Section 3. In fact, the following is a generalisation of the argument used there for the potential $V(x) = ix$, and essentially forms a lemma for our main theorem.

PROPOSITION 2. *Let V be the complex valued linear potential given by*

$$V(x) = mx + l \qquad x \in [-1, 1],$$

where m and l are complex constants; $u_{11}(x)$, $u_{12}(x)$ are as defined in (11), *and $a, b \in [-1, 1]$, $a < b$.*

Let $\varepsilon > 0$ be given and $\lambda \in \mathbf{C}$. If

$$\lambda \notin \mathrm{Nhd}(Y(V(a), V(b)); \varepsilon),$$

then

$$u_{11}(b)\, u_{12}(a) = o(u_{11}(a)\, u_{12}(b)) \tag{13}$$

as $h \to 0$.

Proof. A simple scaling and translation of the operator H_h allows us, without loss of generality, to assume that $a := -1$, $b := 1$ and $l = 0$. That is, we assume

$$V(x) := x\mathrm{e}^{i\theta}, \qquad \text{where} \quad \theta := \mathrm{Arg}(m).$$

By elementary trigonometry, one can check that we then have

$$\Gamma = \mathrm{e}^{i\theta} + \frac{4}{\sqrt{3}} \sin\left(\frac{|\theta|}{3}\right) \mathrm{e}^{2(\theta - \pi)\, i/3}$$

or

$$\Gamma = -e^{i\theta} + \frac{4}{\sqrt{3}} \sin\left(\frac{2\pi}{3} + \frac{|\theta|}{3}\right) e^{2\theta i/3}.$$

Recalling our definition of the Airy functions $u_{11}(x)$ and $u_{12}(x)$ (11), we put

$$z(h, \lambda, x) := h^{-2/3} e^{-2\theta i/3}(x e^{\theta i} - \lambda),$$

and can rewrite (4) explicitly in terms of h. Then, taking the modulus we obtain

$$|Ai(z(h, \lambda, x))| \\ = h^{1/6} \frac{|x e^{\theta i} - \lambda|^{-1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3h} \mathrm{Re}(e^{-2\theta i/3}(x e^{\theta i} - \lambda))^{3/2}\right)(1 + O(h)) \qquad (14)$$

as $h \to 0$, valid for $|\mathrm{Arg}(z(h, \lambda, x))| < \pi$. Therefore, in order to estimate the moduli of the Airy functions $Ai_k(z(h, \lambda, x))$ in the limit $h \to 0$, it is sufficient to examine the behaviour of the functions

$$x \mapsto \mathrm{Re}\{(e^{-2k\pi i/3} z(h, \lambda, x))^{3/2}\}, \qquad x \in \mathbf{R}$$

for $k = -1, 0, 1$.

The basic idea of our proof is to show that for λ outside an arbitrarily small ε-neighbourhood of $Y(V(-1), V(1))$, one can assign allowable values of j and k from $\{-1, 0, 1\}$ (in the sense of (9)) such that the inequalities

$$\mathrm{Re}\{(e^{-2j\pi i/3} z(h, \lambda, -1)^{3/2}\} < \mathrm{Re}\{(e^{-2j\pi i/3} z(h, \lambda, 1)^{3/2}\} \qquad (15)$$

and

$$\mathrm{Re}\{(e^{-2k\pi i/3} z(h, \lambda, 1)^{3/2}\} < \mathrm{Re}\{(e^{-2k\pi i/3} z(h, \lambda, -1)^{3/2}\} \qquad (16)$$

hold in the limit $h \to 0$. This will then be enough, by (14), to ensure that $u_{11}(1)\, u_{12}(-1)$ and $u_{11}(-1)\, u_{12}(1)$ are of different orders of magnitude as $h \to 0$, thus implying (13).

Using the statements of the previous section; for all values of λ such that

$$z(h, \lambda, \pm 1) := h^{-2/3} e^{-2\theta i/3}(\pm e^{\theta i} - \lambda)$$

does not lie within an ε-neighbourhood of any of the Stokes' lines, and $z(h, \lambda, -1)$, $z(h, \lambda, 1)$ lie in different principal domains, one can always obtain (15) and (16), and the asymptotics (14) will be valid. However, for λ lying in the sector having its apex at Γ, and bounded by the rays

$$\Gamma + re^{2\theta i/3} \qquad \text{and} \qquad \begin{cases} \Gamma + re^{2\theta i/3 + 2\pi i/3} & \text{for} \quad \theta < 0 \\ \Gamma + re^{2\theta i/3 - 2\pi i/3} & \text{for} \quad \theta \geqslant 0 \end{cases}$$

$r \in [0, \infty)$, it is easy to check that $e^{-2k\pi i/3} z(h, \lambda, \pm 1)$ both lie in the same principal domain, for each $k \in \{-1, 0, 1\}$. Then it is also straightforward to check that as x ranges from -1 to 1, the function

$$x \mapsto \operatorname{Re}\{(e^{-2k\pi i/3} z(\varepsilon, \lambda, x))^{3/2}$$

which has been at the heart of our analysis, has a single maximum/minimum. Together with the identities (5), (6) and (7), this means that there *will* be values of λ such that equality holds in both (15) and (16)—no matter what choices of j and k are made. Thus, (and without loss of generality assuming $j = k = 0$,) the set of λ satisfying

$$\operatorname{Re}\{(e^{-2\theta i/3}(e^{\theta i} - \lambda))^{3/2}\} = \operatorname{Re}\{(e^{-2\theta i/3}(-e^{\theta i} - \lambda))^{3/2}\} \tag{17}$$

lies in $Y(V(-1), V(1))$. We now examine this set in more detail. Expanding the Taylor series, we have

$$(e^{-2\theta i/3}(e^{\theta i} - \lambda))^{3/2} = -i\lambda^{3/2} e^{-\theta i} + \tfrac{3}{2} i\lambda^{1/2} - \tfrac{3}{8} i\lambda^{-1/2} e^{\theta i} - \tfrac{1}{16} i\lambda^{-3/2} e^{2\theta i} + O(\lambda^{-5/2})$$

and

$$(e^{-2\theta i/3}(-e^{\theta i} - \lambda))^{3/2} = -i\lambda^{3/2} e^{-\theta i} - \tfrac{3}{2} i\lambda^{1/2} - \tfrac{3}{8} i\lambda^{-1/2} e^{\theta i} + \tfrac{1}{16} i\lambda^{-3/2} e^{2\theta i} + O(\lambda^{-5/2})$$

as $|\lambda| \to \infty$. Dividing through by $-i$, this means that (17) will hold if and only if

$$\operatorname{Im}\left\{\frac{3}{2}\lambda^{1/2} - \frac{\lambda^{-3/2} e^{2\theta i}}{16}\right\} = O(\lambda^{-5/2})$$

as $|\lambda| \to \infty$. Putting $\lambda := \rho e^{\phi i}$, this is equivalent to the requirement

$$\sin\left(\frac{\phi}{2}\right) - \frac{1}{24\rho^2} \sin\left(2\theta - \frac{3\phi}{2}\right) = O(\rho^{-6})$$

as $\rho \to \infty$. But then

$$\begin{aligned} \mathrm{Im}(\lambda) &= \rho \sin(\phi) \\ &= 2\rho \sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) \\ &= 2\rho \cos\left(\frac{\phi}{2}\right)\left\{\frac{1}{24\rho^2}\sin\left(2\theta-\frac{3\theta}{2}\right)+O(\rho^{-6})\right\} \\ &= O(\rho^{-1}) \end{aligned}$$

as $\rho \to \infty$. By our definition of Γ, $z(h, \Gamma, \pm 1)$ lies at the intersection of two Stokes' lines, and so

$$\mathrm{Re}\{(z(h, \Gamma, \pm 1))^{3/2}\} = 0$$

showing that Γ certainly lies in the set of λ satisfying (17). Therefore, we deduce that $Y(V(-1), V(1))$ contains a curve from Γ asymptotic to the positive real-axis.

Finally, we must examine what happens when z does lie on one of the Stokes' lines.

Firstly, suppose $\mathrm{Arg}(z(h, \lambda, 1)) = \pi/3$, corresponding to λ lying on the ray centred at $e^{\theta i}$ and passing through Γ. Then $k = 0, 1$ are allowable, and one checks that if λ lies on the segment $[e^{\theta i}, \Gamma)$, we have $z(h, \lambda, -1) \in S_{-1}$. It follows by (6) that

$$\mathrm{Re}\{(e^{-2\pi i/3} z(h, \lambda, -1))^{3/2}\} = \mathrm{Re}\{(z(h, \lambda, -1))^{3/2}\},$$

and so (15) and (16) cannot hold. However, if λ lies on that part of the ray which extends past Γ (but not $\lambda = \Gamma$ itself,) then $z(h, \lambda, -1) \in S_1$, and

$$\mathrm{Re}\{(e^{-2\pi i/3} z(h, \lambda, -1))^{3/2}\} = -\mathrm{Re}\{(z(h, \lambda, -1))^{3/2}\}$$

causing (15) and (16) to hold for $j = 0$, $k = 1$.

An entirely similar argument holds when $\mathrm{Arg}(z(h, \lambda, -1)) = \pi$, corresponding to λ lying on the ray centred at $-e^{\theta i}$ and passing through Γ using (5), with $j = 1$ and $k = -1$.

Finally, the case where $\mathrm{Arg}(z(h, \lambda, \pm 1)) = -\pi/3$ is taken care of using (7), which shows that we may use allowable values -1 and 0 to obtain (15) and (16).

This completes the proof.

In the case $\theta = \pi/2$; $\Gamma = 1/\sqrt{3}$ lies on the real-axis, and the figure $Y(-i, i)$ has three *linear* "arms". When $\theta = 0$, the symmetric case, $Y(-1, 1)$ is the semi-infinite interval $[-1, \infty)$, as is well-known from the theory of self-adjoint operators.

5. OUR MAIN RESULT

We are now ready to give our main result. In all the analysis of the proof it should be born in mind that all we are really doing is rearranging the determinant (18) below, and then examining the asymptotic limit of the resulting expression as $h \to 0$. It is then seen that the matching conditions, imposed by the kinks or discontinuities in the potential V, remain as *essential* conditions, in the form of (23), to determine whether or not λ lies outside the asymptotic spectrum. Accordingly, it does not matter how small the non-linearities of the potential are, eventually they will become significant as $h \to 0$. See also the comments prior to Proposition 6.

THEOREM 3. *Let*

$$H_h f(x) := -h^2 \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + V(x) f(x)$$

act on $L^2(-1, 1)$ *with Dirichlet boundary conditions, where* $h > 0$ *is small, and* $V(x)$ *is the complex valued n-times piecewise linear function*

$$V(x) := \begin{cases} m_1 x + l_1 & x_0 \leqslant x < x_1 \\ m_2 x + l_2 & x_1 < x < x_2 \\ \vdots & \vdots \\ m_n x + l_n & x_{n-1} < x \leqslant x_n \end{cases}$$

with $-1 = x_0 < x_1 < \cdots < x_n = 1$ *and the* m_i, l_i, $i = 1, \ldots, n$ *complex constants. We assume for each i that if* $m_i x_i + l_i = m_{i+1} x_i + l_{i+1}$, *then* $m_i \neq m_{i+1}$. *Put* $\theta_i := \mathrm{Arg}(m_i)$, *and, using our earlier notation*

$$T := \bigcup_{i=1}^{n} Y(V(x_i), V(x_{i+1})).$$

Let $\varepsilon > 0$ *and* $N \in \mathbf{Z}^+$ *be given. Then*

$$\mathrm{Spec}(H_h) \cap \{z\colon |z| \leqslant N\} \subset \mathrm{Nhd}(T; \varepsilon)$$

for all small enough $h > 0$.

Proof. Our proof involves an analysis of the behaviour of the characteristic-determinant, i.e. the left-hand side of (12), as $h \to 0$. We give a proof for the case $n=3$; the general case follows by a similar argument. For $n=3$, the characteristic-determinant is given by

$$\begin{vmatrix} u_{11}(-1) & u_{12}(-1) & 0 & 0 & 0 & 0 \\ u_{11}(x_1) & u_{12}(x_1) & -u_{21}(x_1) & -u_{22}(x_1) & 0 & 0 \\ u'_{11}(x_1) & u'_{12}(x_1) & -u'_{21}(x_1) & -u'_{22}(x_1) & 0 & 0 \\ 0 & 0 & u_{21}(x_2) & u_{22}(x_2) & -u_{31}(x_2) & -u_{32}(x_2) \\ 0 & 0 & u'_{21}(x_2) & u'_{22}(x_2) & -u'_{31}(x_2) & -u'_{32}(x_2) \\ 0 & 0 & 0 & 0 & u_{31}(1) & u_{32}(1) \end{vmatrix} \tag{18}$$

and we must prove that for certain values of $\lambda \in \mathbf{C}$, this determinant does not vanish as $h \to 0$. Expanding (18), one obtains

$$\begin{aligned} &\{(u_{11}(-1)\,u_{12}(x_1)-u_{12}(-1)\,u_{11}(x_1))(u'_{22}(x_1)\,u_{21}(x_2)-u'_{21}(x_1)\,u_{22}(x_2)) \\ &\qquad \times (u_{31}(1)\,u'_{32}(x_2)-u'_{31}(x_2)\,u_{32}(1))\} \\ &\quad -\{(u_{11}(-1)\,u_{12}(x_1)-u_{12}(-1)\,u_{11}(x_1))(u'_{22}(x_1)\,u'_{21}(x_2)-u'_{21}(x_1)\,u'_{22}(x_2)) \\ &\qquad \times (u_{31}(1)\,u_{32}(x_2)-u_{31}(x_2)\,u_{32}(1))\} \\ &\quad +\{(u_{11}(-1)\,u'_{12}(x_1)-u_{12}(-1)\,u'_{11}(x_1))(u_{22}(x_1)\,u'_{21}(x_2)-u_{21}(x_1)\,u'_{22}(x_2)) \\ &\qquad \times (u_{31}(1)\,u_{32}(x_2)-u_{31}(x_2)\,u_{32}(1))\} \\ &\quad -\{(u_{11}(-1)\,u'_{12}(x_1)-u_{12}(-1)\,u'_{11}(x_1))(u_{22}(x_1)\,u_{21}(x_2)-u_{21}(x_1)\,u_{22}(x_2)) \\ &\qquad \times (u_{31}(1)\,u'_{32}(x_2)-u'_{31}(x_2)\,u_{32}(1))\}, \end{aligned} \tag{19}$$

where so far, no asymptotics are involved.

Now, let $\varepsilon > 0$ and $N \in \mathbf{Z}^+$ be as given in the statement of the theorem. Taking any

$$\lambda \in \{z\colon |z| \leqslant N\} \setminus \mathrm{Nhd}(T;\varepsilon),$$

we can use the results of Proposition 2 to show that (18) is non-zero in the limit as $h \to 0$. Indeed, by the proof of Proposition 2, we can ensure that the asymptotic estimates

$$u_{12}(-1)\,u_{11}(x_1) = o(u_{11}(-1)\,u_{12}(x_1)),$$
$$u_{21}(x_1)\,u_{22}(x_2) = o(u_{22}(x_1)\,u_{21}(x_2))$$

and

$$u_{31}(x_2)\,u_{32}(1) = o(u_{32}(x_2)\,u_{31}(1))$$

hold, as $h \to 0$. Then, using the standard asymptotic expansions of the Airy functions [5], which give

$$Ai(z) = \frac{z^{-1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3} z^{3/2}\right)(1+O(z^{-3/2})) \tag{20}$$

and

$$Ai'(z) = -\frac{z^{1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3} z^{3/2}\right)(1+O(z^{-3/2})) \tag{21}$$

as $|z| \to \infty$, valid for all z such that $|\mathrm{Arg}(z)| < \pi$; we see that, if

$$z(h, \lambda, x_i) := h^{-2/3} m_i^{-2/3}((m_i x_i + l_i) - \lambda),$$

then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} Ai(z) &= \frac{\mathrm{d}z}{\mathrm{d}x} Ai'(z) \\ &= -\frac{\mathrm{d}z}{\mathrm{d}x} \frac{z^{1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3} z^{3/2}\right)(1+O(z^{-3/2})) \\ &= -\frac{\mathrm{d}z}{\mathrm{d}x} z^{1/2} \frac{z^{-1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3} z^{3/2}\right)(1+O(z^{-3/2})) \end{aligned}$$

as $|z| \to \infty$. Comparing this last expression with (20), and using $f \sim g$ to mean that

$$\frac{f(h)}{g(h)} \to 1 \qquad \text{as} \quad h \to 0,$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x} Ai(z(h, \lambda, x_i)) \sim -h^{-1}((m_i x_i + l_i) - \lambda)^{1/2} Ai(z(h, \lambda, x_i)) \qquad \text{as} \quad h \to 0.$$

Moreover, similar calculations show that

$$\frac{\mathrm{d}}{\mathrm{d}x} Ai(\mathrm{e}^{\pm 2\pi i/3} z(h, \lambda, x_i)) \sim h^{-1}((m_i x_i + l_i) - \lambda))^{1/2} Ai(z(h, \lambda, x_i))$$
$$\text{as} \quad h \to 0.$$

Reverting to our notation of (11), we will write

$$u'_{i1}(x_i) \sim h^{-1} c_{i1}(x_i)\, u_{i1}(x_i) \qquad \text{etc.} \tag{22}$$

as $h \to 0$, where it is important to note that the $c_{ij}(x_i)$, $i = 1, \ldots, (n-1)$, $j = 1, 2$ are independent of h. Then, since the constant terms $c_{ij}(x_i)$ are negligible in magnitude compared with the exponential terms $u_{ij}(x_i)$ as $h \to 0$, the relations (22) imply that we also have the estimates

$$\begin{aligned}
u_{12}(-1)\, u'_{11}(x_1) &= o(u_{11}(-1)\, u'_{12}(x_1)),\\
u'_{21}(x_1)\, u_{22}(x_2) &= o(u'_{22}(x_1)\, u_{21}(x_2))\\
u'_{31}(x_2)\, u_{32}(1) &= o(u'_{32}(x_2)\, u_{31}(1))\\
u'_{21}(x_1)\, u'_{22}(x_2) &= o(u'_{22}(x_1)\, u'_{21}(x_2))
\end{aligned}$$

and

$$u_{21}(x_1)\, u'_{22}(x_2) = o(u_{22}(x_1)\, u'_{21}(x_2))$$

as $h \to 0$. Returning to (19), we first use the above estimates (since we may ignore the sub-dominant term in each round-bracketed expression), and then the relations (22) again, to obtain the asymptotic estimate on the first of the curly-bracketed terms:

$$\begin{aligned}
&\{(u_{11}(-1)\, u_{12}(x_1) - u_{12}(-1)\, u_{11}(x_1))(u'_{22}(x_1)\, u_{21}(x_2) - u'_{21}(x_1)\, u_{22}(x_2))\\
&\qquad \times (u_{31}(1)\, u'_{32}(x_2) - u'_{31}(x_2)\, u_{32}(1))\}\\
&\quad \sim u_{11}(-1)\, u_{12}(x_1)\, u'_{22}(x_1)\, u_{21}(x_2)\, u_{31}(1)\, u'_{32}(x_2)\\
&\quad \sim u_{11}(-1)\, u_{12}(x_1)\, \varepsilon^{-1/2} c_{22}(x_1)\, u_{22}(x_1)\, u_{21}(x_2)\, u_{31}(1)\, \varepsilon^{-1/2} c_{32}(x_2)\, u_{32}(x_2)\\
&\quad = h^{-2}[c_{22}(x_1)\, c_{32}(x_2)](u_{11}(-1)\, u_{12}(x_1)\, u_{22}(x_1)\, u_{21}(x_2)\, u_{31}(1)\, u_{32}(x_2))
\end{aligned}$$

as $h \to 0$. Similar estimates apply to each of the remaining three terms in (19) i.e.

$$\begin{aligned}
&\{(u_{11}(-1)\, u_{12}(x_1) - u_{12}(-1)\, u_{11}(x_1))(u'_{22}(x_1)\, u'_{21}(x_2) - u'_{21}(x_1)\, u'_{22}(x_2))\\
&\qquad \times (u_{31}(1)\, u_{32}(x_2) - u_{31}(x_2)\, u_{32}(1))\}\\
&\quad \sim h^{-2}[c_{22}(x_1)\, c_{21}(x_2)](u_{11}(-1)\, u_{12}(x_1)\, u_{22}(x_1)\, u_{21}(x_2)\, u_{31}(1)\, u_{32}(x_2)),\\
&\{(u_{11}(-1)\, u'_{12}(x_1) - u_{12}(-1)\, u'_{11}(x_1))(u_{22}(x_1)\, u'_{21}(x_2) - u_{21}(x_1)\, u'_{22}(x_2))\\
&\qquad \times (u_{31}(1)\, u_{32}(x_2) - u_{31}(x_2)\, u_{32}(1))\}\\
&\quad \sim h^{-2}[c_{12}(x_1)\, c_{21}(x_2)](u_{11}(-1)\, u_{12}(x_1)\, u_{22}(x_1)\, u_{21}(x_2)\, u_{31}(1)\, u_{32}(x_2))
\end{aligned}$$

and

$$\begin{aligned}&\{(u_{11}(-1)\,u'_{12}(x_1)-u_{12}(-1)\,u'_{11}(x_1))(u_{22}(x_1)\,u_{21}(x_2)-u_{21}(x_1)\,u_{22}(x_2))\\&\qquad\times(u_{31}(1)\,u'_{32}(x_2)-u'_{31}(x_2)\,u_{32}(1))\}\\&\qquad\sim h^{-2}[c_{12}(x_1)\,c_{32}(x_2)](u_{11}(-1)\,u_{12}(x_1)\,u_{22}(x_1)\,u_{21}(x_2)\,u_{31}(1)\,u_{32}(x_2))\end{aligned}$$

as $h\to 0$. Collecting these estimates together, we see that the characteristic determinant (18) tends asymptotically towards

$$\begin{aligned}&h^{-2}\{(c_{22}(x_1)-c_{12}(x_1))(c_{32}(x_2)-c_{21}(x_2))\}\\&\qquad\times(u_{11}(-1)\,u_{12}(x_1)\,u_{22}(x_1)\,u_{21}(x_2)\,u_{31}(1)\,u_{32}(x_2))\end{aligned}$$

as $h\to 0$. The Airy functions $Ai(z)$ have countably many negative real zeros, [5]; and so by our choice of λ outside $\mathrm{Nhd}(T;\varepsilon)$ together with the proof of Proposition 2, we are assured that none of the Airy functions $u_{ij}(x_i)$ vanishes. Therefore, the determinant (18) does not vanish in the limit as $h\to 0$, provided the "constant" terms

$$c_{22}(x_1)\neq c_{12}(x_1) \qquad \text{and} \qquad c_{32}(x_2)\neq c_{21}(x_2). \tag{23}$$

Our choice of λ ensures that each of the individual constant terms $c_{ij}(x_i)$ is non-zero. Moreover, reviewing the proof of Proposition 2 and the identities (5)–(7), we see that the choices for j and k are not uniquely determined. Therefore, it is always possible to ensure that (23) holds, even when V is continuous at some or all of the x_is. For example, if it happens that $V(x_1-)=V(x_1+)$, then we choose j and k so that the constants $c_{12}(x_1)$ and $c_{22}(x_1)$ take different signs (by the calculations immediately above (22)). Thus, we deduce that such λ cannot be an eigenvalue.

It now just requires the following compactness argument to complete the proof. Let $B(z;\varepsilon)$ denote the open ball centred at z, with radius ε. Our argument so far shows that for any

$$\lambda\in\{z\in\mathbf{C}:|z|\leqslant N\}$$

such that

$$B(\lambda;\varepsilon)\cap T=\varnothing$$

we have

$$B(\lambda;\varepsilon)\cap\mathrm{Spec}(H_h)=\varnothing$$

for all $0<h<E_\lambda$, where E_λ is some positive constant dependent upon λ. Let

$$M:=\{z\in \mathbf{C}:|z|\leqslant N,\ \text{and}\ \operatorname{dist}(z,T)\geqslant 2\varepsilon\},$$

so that M is compact. Then for all $\lambda\in M$

$$B(\lambda;\varepsilon)\cap \operatorname{Nhd}(T;\varepsilon)=\varnothing$$

and so

$$M\subseteq \bigcup_{\lambda\in M} B(\lambda;\varepsilon).$$

But by compactness this means that there exists a finite sub-covering

$$M\subseteq \bigcup_{r=1}^{n} B(\lambda_r;\varepsilon_{\lambda_r}).$$

Taking E to be $\min(E_{\lambda_1},\ldots,E_{\lambda_n})>0$, we deduce that for all $0<h<E$ we have

$$\operatorname{Spec}(H_h)\cap M=\varnothing$$

and this is equivalent to the statement of the theorem.

Remark 4. An important but subtle point, to note is that the zeros of

$$(u_{11}(-1)\,u_{12}(x_1)\,u_{22}(x_1)\,u_{21}(x_2)\,u_{31}(1)\,u_{32}(x_2)) \tag{24}$$

as a function of λ, are *not* the same as the zeros of (18). However, by a similar argument to that of Shkalikov [7] (i.e. using (20)), one can readily show that along each of the bounded arms of the Y-shaped figures making up T, the zeros (eigenvalues) do converge as $h\to 0$ to form a dense set. Finding an asymptotic expression for the density of spectral points along the infinite lines (in the direction of the positive real-axis) appears to be a much more difficult problem; and we have no results yet in that direction.

6. SIMULTANEOUS LIMITS FOR $H_{\delta,h}$

We can now state more precisely our observed numerical results, obtained with Matlab, which began our investigations. Thus we return to the operator $H_{\delta,h}$ defined in the first section.

COROLLARY 5. *Let $H_{\delta,h}$ be the non-self-adjoint operator defined by*

$$H_{\delta,h} := -h^2 \frac{d^2}{dx^2} + V_\delta(x)$$

acting on $L^2(-1, 1)$ with Dirichlet boundary conditions, $h > 0$, and

$$V_\delta(x) := \begin{cases} i(x-\delta) & \text{for} \quad x < 0 \\ i(x+\delta) & \text{for} \quad x > 0 \end{cases}$$

with $\delta > 0$. Define $S \subset \mathbf{C}$ to be the double Y-shaped figure given by the line segments

$$[i\delta, 1/2\sqrt{3} + i(1+2\delta)/2]$$
$$[i(1+\delta), 1/2\sqrt{3} + i(1+2\delta)/2]$$

together with

$$[1/2\sqrt{3} + i(1+2\delta)/2, +\infty),$$

and

$$[-i\delta, 1/2\sqrt{3} - i(1+2\delta)/2]$$
$$[-i(1+\delta), 1/2\sqrt{3} - i(1+2\delta)/2]$$

together with

$$[1/2\sqrt{3} - i(1+2\delta)/2, +\infty).$$

Then, given any $\varepsilon > 0$ and $N \in \mathbf{Z}^+$, we have

$$\operatorname{Spec}(H_{\delta,h}) \cap \{z \colon |z| \leqslant n\} \subset \operatorname{Nhd}(S; \varepsilon)$$

for small enough $h > 0$ (see Fig. 2).

By analyticity, however, for fixed $h > 0$ we have

$$\lim_{\delta \to 0} \operatorname{Spec}(H_{\delta,h}) = \operatorname{Spec}(H_{0,h}).$$

Hence, $\lim_{h\to 0} \lim_{\delta\to 0} \operatorname{Spec}(H_{\delta,h})$ is contained within an arbitrarily small neighbourhood of the line segments

$$[i, 1/\sqrt{3}], [-i, 1/\sqrt{3}] \qquad \text{and} \qquad [1/\sqrt{3}, \infty)$$

(see Fig. 1). Thus, the operations of taking limits do not commute, in the sense that as sets

$$\lim_{h\to 0}\lim_{\delta\to 0}\ \mathrm{Spec}(H_{\delta,h}) \neq \lim_{\delta\to 0}\lim_{h\to 0}\ \mathrm{Spec}(H_{\delta,h}).$$

This is essentially the mystery which first set us on our way. In order to gain some insight, let us examine the situation in which δ and h of Corollary 5 are no longer independent of each other. Recall that we are seeking (continuously differentiable) eigenfunctions of the form

$$f(x) := \begin{cases} \alpha_{11}u_{11}(x)+\alpha_{12}u_{12}(x) & -1\leqslant x<0 \\ \alpha_{21}u_{21}(x)+\alpha_{22}u_{22}(x) & 0<x\leqslant 1 \end{cases}$$

which satisfy

$$-h^2\,\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}+(V_\delta(x)-\lambda)\,f(x)=0$$

and that the eigenvalues are the zeros of the associated characteristic determinant.

We shall see in the proof below that if the perturbation δ is smaller than $O(h)$ as $h\to 0$, then the (classical) solutions to the differential equation are asymptotically continuous at the non-linearity $(x=0)$ of the potential V_δ, in the sense that

$$\frac{u_{22}(0)}{u_{12}(0)}\to 1 \qquad \text{and} \qquad \frac{u_{21}(0)}{u_{11}(0)}\to 1$$

as $h\to 0$. Therefore, in this case the matching conditions at $x=0$ are then *automatically* satisfied in the semi-classical limit, and the operator behaves as though it were unperturbed: the only conditions to be satisfied are the boundary conditions $f(-1)=f(1)=0$. However, if the perturbation is $O(h)$ or larger as $h\to 0$, we shall see that the solutions are *not* asymptotically continuous in the above sense. In other words, if eventually h is small compared with δ, the matching conditions at $x=0$ are *necessary* conditions. Thus, the limiting spectra of the perturbed and unperturbed operators appear as the zeros of different characteristic determinants: one containing the matching conditions and the other comprising only the boundary conditions. The following result together with its proof makes all this more precise.

Proposition 6. *Defining the operator $H_{\delta, h}$ as above, and putting*

$$\delta := h^{1/p}$$

we have

$$\lim_{h \to 0} \operatorname{Spec}(H_{h^{1/p}, h}) = \lim_{h \to 0} \operatorname{Spec}(H_{0, h}) \qquad \text{if} \quad 0 < p < 1$$

and

$$\lim_{h \to 0} \operatorname{Spec}(H_{h^{1/p}, h}) = \lim_{\delta \to 0} \lim_{h \to 0} \operatorname{Spec}(H_{\delta, h}) \qquad \text{if} \quad p \geqslant 1.$$

Proof. Referring to (12) and expanding, we see that the characteristic determinant of the (perturbed) operator $H_{\delta, h}$ is given by

$$\begin{aligned} &\{(u_{11}(-1)\, u_{12}(0) - u_{12}(-1)\, u_{11}(0))(u'_{22}(0)\, u_{21}(1) - u'_{21}(0)\, u_{22}(1))\} \\ &\quad - \{(u_{11}(-1)\, u'_{12}(0) - u_{12}(-1)\, u'_{11}(0))(u_{22}(0)\, u_{21}(1) - u_{21}(0)\, u_{22}(1))\}, \end{aligned} \tag{25}$$

whereas the characteristic determinant of the (unperturbed) operator $H_{0, h}$ is given by

$$u_{21}(1)\, u_{12}(-1) - u_{22}(1)\, u_{11}(-1). \tag{26}$$

Now, putting

$$u_{12}(0) := Ai_k(h^{-2/3} \mathrm{e}^{-\pi i/3}(-i\delta - \lambda)) \quad \text{and} \quad u_{22}(0) := Ai_k(h^{-2/3} \mathrm{e}^{-\pi i/3}(i\delta - \lambda))$$

it is clear by analyticity, that

$$u_{12}(0) \sim u_{22}(0) \qquad \text{and} \qquad u_{11}(0) \sim u_{21}(0) \tag{27}$$

as $\delta \to 0$. Moreover, the calculations preceding (22) show that

$$u'_{i1}(0) \sim -\delta^{-p}(-\lambda)^{1/2}\, u_{i1}(0) \qquad \text{and} \qquad u'_{i2}(0) \sim \delta^{-p}(-\lambda)^{1/2}\, u_{i2}(0) \tag{28}$$

as $\delta \to 0$, for $i = 1, 2$. Therefore, using first (28) and then (27), the characteristic determinant (25) tends asymptotically towards

$$\begin{aligned} &2\delta^{-p}(-\lambda)^{1/2}\, (u_{11}(-1)\, u_{12}(0)\, u_{21}(0)\, u_{22}(1) - u_{12}(-1)\, u_{11}(0)\, u_{22}(0)\, u_{21}(1)) \\ &\quad \sim 2\delta^{-p}(-\lambda)^{1/2}\, (u_{22}(1)\, u_{11}(-1) - u_{21}(1)\, u_{12}(-1)) \end{aligned}$$

as $\delta \to 0$. Then the zeros of (25) tend asymptotically toward the zeros of (26) by *Rouché*'s theorem, explaining the behaviour of

$$\lim_{h\to 0} \lim_{\delta\to 0} \operatorname{Spec}(H_{\delta,h}).$$

Substituting $\delta = h^{1/p}$, the character of $\lim_{h\to 0} \operatorname{Spec}(H_{h^{1/p},h})$ therefore depends upon the range of p for which

$$\frac{u_{22}(0)}{u_{12}(0)} \to 1 \qquad \text{and} \qquad \frac{u_{21}(0)}{u_{11}(0)} \to 1$$

as $h \to 0$. Now, without loss of generality, and using our earlier notation, let

$$\frac{u_{22}(0)}{u_{12}(0)} := \frac{Ai_{-1}(z_1)}{Ai_{-1}(z_2)},$$

where

$$z_1 := h^{-2/3} e^{\pi i/3}(ih^{1/p} - \lambda) \qquad \text{and} \qquad z_2 := h^{-2/3} e^{\pi i/3}(-ih^{1/p} - \lambda)$$

so that, using the standard asymptotics (4)

$$\begin{aligned}
\frac{u_{22}(0)}{u_{12}(0)} &= \frac{z_1^{-1/4}\exp(-\frac{2}{3} z_1^{3/2})(1+O(z_1^{-3/2}))}{z_2^{-1/4}\exp(-\frac{2}{3} z_2^{3/2})(1+O(z_2^{-3/2}))} \\
&= \left(\frac{z_1}{z_2}\right)^{-1/4} \exp\left(-\frac{2}{3}[z_1^{3/2} - z_2^{3/2}]\right)(1+O(z_1^{-3/2})) \\
&\sim \exp\left(-\frac{2}{3}[z_1^{3/2} - z_2^{3/2}]\right)
\end{aligned}$$

as $h \to 0$. But

$$\begin{aligned}
z_1^{3/2} - z_2^{3/2} &= h^{-1} e^{\pi i/3}\{(ih^{1/p} - \lambda)^{3/2} - (-ih^{1/p} - \lambda)^{3/2}\} \\
&= h^{-1} e^{\pi i/3}(-\lambda)^{3/2}\{(1 - ih^{1/p}/\lambda)^{3/2} - (1 + ih^{1/p}/\lambda)^{3/2}\} \\
&= h^{-1} e^{\pi i/3}(-\lambda)^{3/2}\{(1 - 3ih^{1/p}/2\lambda + \cdots) - (1 + 3ih^{1/p}/2\lambda + \cdots)\} \\
&= h^{-1} e^{\pi i/3}(-\lambda)^{3/2}(-3ih^{1/p}/\lambda + \cdots) \\
&\to 0
\end{aligned}$$

as $h \to 0$ if and only if $0 < p < 1$. So, provided $0 < p < 1$

$$\frac{u_{22}(0)}{u_{12}(0)} \to 1 \qquad \text{as} \quad h \to 0$$

and a similar calculation shows that we then also have

$$\frac{u_{21}(0)}{u_{11}(0)} \to 1 \qquad \text{as} \quad h \to 0,$$

as required.

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