Applied Mathematics and Computation 243 (2014) 728–739

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A dynamic IS–LM model with two time delays in the public sector

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ARTICLE INFO

Keywords:
Delay differential equations
Dynamic stability
Policy lag

ABSTRACT

Some recent contributions to Economic Dynamics have shown an increasing interest on the impact that fiscal policy lags may have on the income adjustment processes. Lags dealing with fiscal policy come from delays either in the government expenditure or in the tax revenues. These two lags yield jointly their macroeconomic effects. They are such that to make traditional fiscal policy rules ineffectual to control, and stabilize the GDP dynamics. Here we study a dynamic IS–LM model where the public expenditure and the tax revenues have a delayed functional form. We show that the equilibrium of the system may lose or gain its local stability depending either on the length of the lags or on their particular combinations. When instability arises, very complicated dynamics may characterize the national income time path.

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1. Introduction

In the last decade, several works studied the impact of fiscal policy lags on income adjustment process. Lags dealing with fiscal policy came from delays either in the government expenditure or in the tax system. The former delay pertains to the political process governing the public purchase decisions and the actual expenditures. The latter concerns the structure of the tax system and therefore the government tax revenues.

The question about the existence of collection lags in the tax system was originally of raised by Tanzi [18,19] with reference to typical arguments of public finance. More recently, Fanti and Manfredi [6,7] put the question about the effect that delayed tax revenues have on the GDP dynamics. The framework wherein this question has been studied is the well-known intermediate-run IS–LM model by Schinasi [16,17] successively generalized as a three dimensional ODEs system by Sasakura [15]. Despite the broad empirical evidence on the existence of finite lags between the accrual and the payment of taxes, Fanti and Manfredi treated the tax delay as an exponentially distributed lag. This assumption implies that their system remains a simple ODEs system involving the possibility to change the Sasakura limit cycle in deterministic chaos. Nevertheless, we know that, as really happens, for the same tax every taxpayer face the same lag, if it exists. The lag is institutionally fixed and any agent knows about it as, for example, it happens in the case of VAT payments, which are made at fixed date by all taxpayer. Looking at these considerations De Cesare and Sportelli [3] and Neamtu et al. [11] formalized a dynamic IS–LM model where a share of tax collection contains a finite lag. The analytical consequence was a dynamical system of DDEs able to give rise to a wide variety of dynamic behaviors. Therefore, since the lag cannot be completely manageable

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http://dx.doi.org/10.1016/j.amc.2014.06.051
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by policy makers, stabilization policies by means of the traditional fiscal policy rules take the risk of being ineffectual to control the GDP dynamics.

As far as the delay in the government's expenditure is concerned, we have to point out that this question goes back to Friedman [8][1] and successively formalized by Phillips [12,13]. Recently, Asada and Yoshida [1] re-proposed this theme by studying a Kaldorian type macro-dynamic model of trade cycle where the public expenditures responds to the changes of national income with a delay, which is assumed to be the inevitable consequence of political processes. Following the Asada and Yoshida approach, studies by Xiaofeng et al. [20], Yoshida and Asada [21] and Matsumoto [10] confirmed that policy lags may be such that to make fiscal policy outcomes inconsistent with their stabilization purposes.

Considering as a whole all the cited works, we acknowledge that they are able to shed new light on the consequence that fiscal policy may have on the income dynamics. However, we cannot help raising a crucial question completely disregarded by the literature dealing with fiscal policy lags: delays both in the government expenditure and in its tax revenues yield jointly their macroeconomic effects. This is the reason why here we put together these two lags by using as a framework a Sasakura type model. Following the De Cesare and Sportelli [4] and Zhou and Li [22] approach, we study the resulting system of differential equations with two delays analytically and numerically to check up how different pairs of lags may affect the system's dynamic behavior. We prove that the equilibrium point may lose or gain its local stability depending either on the length of the lags or on their particular combinations. When instability arises, very complicated dynamics may characterize the national income time path. This implies that fiscal policy purposes may become consistent with their real results only if policy makers are able to control either the structure of the tax system or the political processes governing the public expenditure. Furthermore, policy makers cannot neglect the influence that fiscal policy may have on the rate of interest behavior and consequently on the private investment dynamics.

We organized the paper as follows. Section 2 contains a formal description of the model and a qualitative study of its dynamic behavior. Section 3 is devoted to the numerical simulation of a specific version of the system by using different sets of fiscal policy parameters and different combinations of the lags. Section 4 provides some concluding remarks.

2. The model

2.1. Notations

In the model, we make use of the following notations:

$Y(t)$	$=$	the current national income(GDP);
τ_i	$=$	finite time lag $(i = 1, 2)$;
$I(t)$	$=$	the private investment;
$r(t)$	$=$	the interest rate;
$S(t)$	$=$	the private saving;
$T(t)$	$=$	the tax revenues;
$G(t)$	$=$	the public expenditure;
$Y^D(t) = Y(t) - T(t)$	$=$	the disposable income;
$L(t)$	$=$	the liquidity preference;
$M(t)$	$=$	the money supply.

All the variables are considered in real term. From now on, a dot on the variable will indicate the derivative with respect to time t.

2.2. The system

Looking at the Sasakura approach, we study the following time-delay system:

$$\begin{cases} \dot{Y}(t) = \alpha[I(Y(t), r(t)) + G(Y(t-\tau_2)) - S(Y^D(t)) - T(Y(t), Y(t-\tau_1))] \\ \dot{r}(t) = \beta[L(Y(t), r(t)) - M(t)] \\ \dot{M}(t) = \psi M(t)[G(Y(t-\tau_2)) - T(Y(t), Y(t-\tau_1))] \end{cases}, \tag{1}$$

where $\alpha > 0,\ \beta > 0$ and $0 < \psi < 1$.

The first equation represents the traditional disequilibrium adjustment in the product market, where the tax revenues depend on the current and the delayed income. The second equation represents the disequilibrium dynamic adjustment in the money market. The third equation is the government budget constraint. This equation means that the rate of growth of money supply must be increased (reduced) according to the budget deficit (surplus).

[1] According to Friedman, stabilization policy might be destabilizing because of the existence of the lags in policy response.

As suggested by Sasakura, we assume the following signs about derivatives: $I_Y > 0$, $I_r < 0$, $0 < S'(Y^D(t)) < 1$, $L_Y > 0$, $L_r < 0$. As far as the public sector variables are concerned, we consider $\frac{dG(t)}{dY(t-\tau_2)} < 0$, according to the Keynesian postulate of a counter-cyclical changes of public expenditure, and the tax rates such that to be $0 < \frac{\partial T(t)}{\partial Y(t)} < 1$ and $0 < \frac{\partial T(t)}{\partial Y(t-\tau_1)} < 1$.

2.3. Specific functional form

In this section and from now on, we will make use of the following specific functions:

- Investments,

$$I(Y(t), r(t)) = A\frac{Y^a}{r^b} \quad A > 0 \text{ and } a, b > 0 \tag{2}$$

- Saving,

$$S(Y^D(t)) = sY^D(t) \quad 0 < s < 1 \tag{3}$$

Parameters in these equations have the usual meaning.

- Tax revenues,

$$T(Y(t), Y(t-\tau_1)) = \eta[(1-\epsilon)Y(t) + \epsilon Y(t-\tau_1)] \tag{4}$$

The existence of collection lags in the tax system implies that, at a given time t, tax revenues come from a weighted sum of a finite sequence of differently lagged incomes whose tax rate might be different. For sake of simplicity, but without losing of generality, like De Cesare and Sportelli [3, p. 234] here we assumed that tax revenues come from a weighted sum of only two differently dated incomes. Furthermore we assumed that the two components have the same tax rate $0 < \eta < 1$ such that to be $\eta > s$. The parameter ϵ in Eq. (4) denotes the share of the delayed tax revenue. In our view Eq. (4) is a practical compromise able to capture only the broad qualitative feature of time lags in tax collection.

- Public expenditure,

$$G(Y(t-\tau_2)) = G_0 + \frac{\gamma}{g_1 Y(t-\tau_2) + g_2}. \tag{5}$$

Unlike Asada and Yoshida [1, p. 290] we used a non-linear function, where $G_0 > 0$ is the autonomous public expenditure and $\gamma, g_1, g_2 > 0$ parameters governing the sensitivity of G_t to the change of $Y(t-\tau_2)$.

- Demand for money and liquidity preference,

$$L(Y(t), r(t)) = L_1(Y(t)) + L_2(r(t)) = mY(t) + \frac{\mu}{(r(t)-\bar{r})^\rho}. \tag{6}$$

This is a standard function of the money market. Specifically, $L_1(Y(t)) = mY(t)$ is the transaction demand for money, where $m > 0$ is one over the transaction velocity of money, and $L_2(r(t)) = \frac{\mu}{(r(t)-\bar{r})^\rho}$ is the liquidity preference function, where μ, $\rho > 0$ and $\bar{r}$ is a fixed very small rate of interest generating the liquidity trap as soon as $r(t)$ falls to the level $\bar{r}$ (i.e. $L_2(r(t)) \to +\infty$ as $r(t) \to \bar{r}$).

2.4. The qualitative analysis

To find the steady-state solutions of (1), by setting $\tau_1 = \tau_2 = 0$, we have to solve the following system:

$$\begin{cases} I(Y, r) = S(Y_D) \\ M = L(Y, r) \\ G(Y) = T(Y) \end{cases},$$

(M is nonzero otherwise $L(Y, r) = 0$).

Since G and T are respectively decreasing and increasing, the equation $G(Y) = T(Y)$ has only one positive solution Y^*. From the condition $I_r < 0$, the equation

$$I(Y^*, r) = S(Y_D^*),$$

has a unique positive solution r^*, if $I(Y^*, 0) > S(Y_D^*)$. Therefore system (1) has a unique critical point (Y^*, r^*, M^*) where $M^* = L(Y^*, r^*)$.

Taking into account the equations from (2)–(6), we have

$$Y^* = \frac{1}{2g_1\eta}\left[G_0 g_1 - \eta g_2 + \sqrt{(G_0 g_1 + \eta g_2)^2 + 4\eta g_1 \gamma}\right],$$

$$r^* = \left(\frac{AY^{*a-1}}{s(1-\eta)}\right)^{\frac{1}{b}},$$

$$M^* = mY^* + \frac{\mu}{(r^* - \bar{r})^{\rho}}.$$

When the lags are introduced, the equilibrium point is the same and therefore we can take the local coordinates

$$\widetilde{Y}(t) = Y(t) - Y^*, \tilde{r}(t) = r(t) - r^*, \widetilde{M}(t) = M(t) - M^*,$$

and centre the system's singular point at the origin. It follows that, in matrix form, the system (1) becomes:

$$\begin{pmatrix} \widetilde{Y}(t) \\ \tilde{r}(t) \\ \widetilde{M}(t) \end{pmatrix}' = A\begin{pmatrix} Y(t) - Y^* \\ r(t) - r^* \\ M(t) - M^* \end{pmatrix} + B\begin{pmatrix} Y(t-\tau_1) - Y^* \\ r(t-\tau_1) - r^* \\ M(t-\tau_1) - M^* \end{pmatrix} + C\begin{pmatrix} Y(t-\tau_2) - Y^* \\ r(t-\tau_2) - r^* \\ M(t-\tau_2) - M^* \end{pmatrix}, \tag{7}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha(I_Y - s - (1-s)(1-\epsilon)\eta) & \alpha I_r & 0 \\ \beta L_Y & \beta L_r & -\beta \\ -(1-\epsilon)\psi\eta M^* & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\alpha(1-s)\epsilon\eta & 0 & 0 \\ 0 & 0 & 0 \\ -\epsilon\psi\eta M^* & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & 0 & 0 \\ c_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha G_{Y_{\tau_2}} & 0 & 0 \\ 0 & 0 & 0 \\ \psi G_{Y_{\tau_2}} M^* & 0 & 0 \end{pmatrix}.$$

The derivatives are evaluated at the equilibrium point.

By solving

$$\det(\lambda I - A - Be^{-\tau_1\lambda} - Ce^{-\tau_2\lambda}) = 0$$

we obtain the following characteristic equation

$$\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + (q_2\lambda^2 + q_1\lambda + q_0)e^{-\tau_1\lambda} + (r_2\lambda^2 + r_1\lambda + r_0)e^{-\tau_2\lambda} = 0, \tag{8}$$

where

$$\begin{array}{lll} p_0 = -a_{12}a_{23}a_{31} & q_0 = -a_{12}a_{23}b_{31} & r_0 = -a_{12}a_{23}c_{31} \\ p_1 = a_{11}a_{22} - a_{12}a_{21} & q_1 = a_{22}b_{11} & r_1 = a_{22}c_{11} \\ p_2 = -(a_{11} + a_{22}) & q_2 = -b_{11} & r_2 = -c_{11} \end{array}$$

We note that $a_{12}, a_{22}, a_{23}, a_{31}, b_{11}, b_{31}, c_{11}, c_{31}$ are all negative and a_{21} is positive. It follows that $p_0, q_0, r_0, q_1, r_1, q_2, r_2$ are positive and if $a_{11} < 0$ then also p_1 and p_2 are positive.

The origin is stable if all the solutions of Eq. (8) have negative real parts.

To check the existence of limit cycles (either stable or unstable) generated by Hopf bifurcations, with reference to system (7), following Beretta and Kuang [2] , Ruan and Wei [14], and Zhou and Li [22], we distinguish three cases.

2.5. The case $\tau_1 = \tau_2 = 0$

The system (7) becomes a system of ODEs whose characteristic equation is

$$\lambda^3 + (p_2 + q_2 + r_2)\lambda^2 + (p_1 + q_1 + r_1)\lambda + p_0 + q_0 + r_0 = 0.$$

By the Routh–Hurwitz stability criterion, the equilibrium solution is stable if and only if

$$p_2 + q_2 + r_2 > 0, \quad (p_1 + q_1 + r_1)(p_2 + q_2 + r_2) > p_0 + q_0 + r_0, \tag{9}$$

(note that $p_0 + q_0 + r_0 > 0$).

Let us notice that, if $a_{11} < 0$ then $p_2 + q_2 + r_2 > 0$ and the equilibrium solution is stable if and only if $(p_1 + q_1 + r_1)(p_2 + q_2 + r_2) > p_0 + q_0 + r_0$.

2.6. The case $\tau_1 = 0$ and $\tau_2 > 0$

If $\tau_1 = 0$ the characteristic equation of system (7) becomes

$$D(\lambda, \tau_2) = P(\lambda) + Q(\lambda)e^{-\tau_2\lambda} = 0, \tag{10}$$

$$P(\lambda) = \lambda^3 + h_2\lambda^2 + h_1\lambda + h_0,$$

$$Q(\lambda) = r_2\lambda^2 + r_1\lambda + r_0,$$

where $h_i = p_i + q_i$. The steady state solution is stable if all solutions (eigenvalues) of $D(\lambda, \tau_2)$ have negative real part. From standard results on exponential polynomial, the number of complex solutions are infinite but only a finite number or zero have positive real part. The sum of the orders of the zeros of $D(\lambda, \tau_2)$ with positive real part can change only if a zero appears on or crosses the imaginary axis.

To find pure imaginary solutions of (10), let $\lambda = i\omega$ be a root of (10), with $\omega > 0$. Obviously if $i\omega$ is a root also the conjugate $-i\omega$ is a root of (10). Because $p_0 + q_0 + r_0 \neq 0$, then $\lambda = 0$ is not a solution, i.e. the imaginary axis cannot be crossed by real values. We suppose that

$$\forall \omega > 0, \quad P(i\omega) + Q(i\omega) \neq 0.$$

This condition is equivalent to

$$(p_1 + q_1 + r_1)(p_2 + q_2 + r_2) \neq p_0 + q_0 + r_0.$$

Therefore, if $\lambda = i\omega$ is a solution of (10), then it is not a solution of $Q(\lambda) = 0$, otherwise $Q(i\omega) = P(i\omega) = 0$.

Separating the real and imaginary parts, ω is a solution of the equation

$$e^{-i\omega\tau_2} = \frac{P(i\omega)}{Q(i\omega)} = \frac{P(i\omega)\overline{Q(i\omega)}}{|Q(i\omega)|^2}, \tag{11}$$

if it is a solution of the system:

$$\begin{cases} \cos\omega\tau_2 = \mathrm{Re}\,\frac{P(i\omega)\overline{Q(i\omega)}}{|Q(i\omega)|^2} \\ -\sin\omega\tau_2 = \mathrm{Im}\,\frac{P(i\omega)\overline{Q(i\omega)}}{|Q(i\omega)|^2} \end{cases},$$

which gives

$$\begin{cases} \cos\omega\tau_2 = \frac{(r_2h_2 - r_1)\omega^4 + (h_1r_1 - h_0r_0 - r_0h_2)\omega^2 + r_0h_0}{(r_2\omega^2 - r_0)^2 + r_1^2\omega^2} \\ \sin\omega\tau_2 = -\frac{\omega(r_2\omega^4 + (r_0 + h_1r_2 - r_1h_2)\omega^2 + r_1h_0 - h_1r_0)}{(r_2\omega^2 - r_0)^2 + r_1^2\omega^2} \end{cases}.$$

From (11) we have

$$\frac{|P(i\omega)|^2}{|Q(i\omega)|^2} = 1,$$

which leads to the following equation

$$|P(i\omega)|^2 - |Q(i\omega)|^2 = \omega^6 + a\omega^4 + b\omega^2 + c = 0, \tag{12}$$

where

$$a = h_2^2 - 2h_1 - r_2^2,$$

$$b = h_1^2 - 2h_0h_2 + 2r_0r_2 - r_1^2,$$

$$c = h_0^2 - r_0^2,$$

By setting $y = \omega^2$, the previous equation becomes

$$f(y) := y^3 + ay^2 + by + c = 0, \tag{13}$$

As it is known, we can distinguish these following cases

- Eq. (13) has no positive solution. In this case the characteristic Eq. (10) has no purely imaginary solutions.
- Eq. (13) has only one positive solution y_1.
- Eq. (13) has two distinct positive solution y_2 and y_3. We set $y_2 < y_3$.

- Eq. (13) has three distinct positive solution y_1, y_2 and y_3. We set $y_1 < y_2 < y_3$.

If y_j is a solution of (13), the characteristic Eq. (10) has imaginary solutions $\pm\omega_j i$, where $\omega_j = \sqrt{y_j}$. Let $\phi_j \in [0, 2\pi[$ the solution of

$$\begin{cases} \cos\phi_j = \frac{(r_2h_2 - r_1)\omega_j^4 + (h_1r_1 - h_0r_0 - r_0h_2)\omega_j^2 + r_0h_0}{(r_2\omega_j^2 - r_0)^2 + r_1^2\omega_j^2} \\ \sin\phi_j = -\frac{\omega_j(r_2\omega_j^4 + (r_0 + h_1r_2 - r_1h_2)\omega_j^2 + r_1h_0 - h_1r_0)}{(r_2\omega_i^2 - r_0)^2 + r_1^2\omega_j^2} \end{cases}.$$

For the following values

$$\tau_2^{j,n} = \frac{\phi_j + 2n\pi}{\omega_j} \quad n = 0, 1, \ldots, \tag{14}$$

the equation $D(\lambda, \tau_2) = 0$ has pure imaginary roots $\lambda = \pm i\omega_j$. We can also prove that

Lemma 1. *If ω_j is a solution of (12) and $\tau_2^{j,n}$ is given by (14) then*

$$\mathrm{sign}\left(\frac{d\mathrm{Re}\lambda}{d\tau_2}\bigg|_{\tau=\tau_2^{j,n}}\right) = \mathrm{sign} f'(\omega_j^2) = \mathrm{sign}\left(3\omega_j^4 + 2a\omega_j^2 + b\right). \tag{15}$$

Proof. Let $\lambda(\tau_2) = \alpha(\tau_2) + i\omega(\tau_2)$ a solution of (10). Differentiating with respect to τ_2, we obtain

$$\frac{d\lambda}{d\tau_2} = -\frac{D'_{\tau_2}}{D'_\lambda} = \frac{\lambda Q e^{-\lambda\tau_2}}{P'_\lambda + e^{-\lambda\tau_2}(Q'_\lambda - \tau_2 Q)}.$$

For simplicity we omit functional dependency. P'_λ and Q'_λ are the derivatives with respect to λ. By using $e^{-\lambda\tau_2} = -P/Q$, the previous expression becomes

$$\frac{d\lambda}{d\tau_2} = \frac{\lambda}{\frac{Q'_\lambda}{Q} - \frac{P'_\lambda}{P} - \tau_2}.$$

Therefore,

$$\begin{aligned}\mathrm{sign}\left(\frac{d\mathrm{Re}\lambda}{d\tau_2}\bigg|_{\tau=\tau_2^{j,n}}\right) &= \mathrm{sign}\left(\mathrm{Re}\left(\frac{i\omega_j}{\frac{Q'_\lambda}{Q} - \frac{P'_\lambda}{P} - \tau_2^{j,n}}\right)\right) = \mathrm{sign}\left(\mathrm{Re}\left(\frac{1}{i\omega_j}\left(\frac{Q'_\lambda}{Q} - \frac{P'_\lambda}{P} - \tau_2^{j,n}\right)\right)\right) = \mathrm{sign}\left(\mathrm{Im}\left(\frac{Q'_\lambda}{Q} - \frac{P'_\lambda}{P}\right)\right) \\ &= \mathrm{sign}\left(\mathrm{Im}\left(Q'_\lambda\overline{Q} - P'_\lambda\overline{P}\right)\right),\end{aligned}$$

because of $|P|^2 = |Q|^2$. By considering the derivatives P'_λ and Q'_λ, the substitution in the previous expression yields the formula (15). □

Taking into account the monotonicity of f, we have the following transversality conditions.

Theorem 1.

$$sign\left(\frac{dRe\lambda}{d\tau_2}\bigg|_{\tau=\tau_2^{1,n}}\right) > 0, \quad sign\left(\frac{dRe\lambda}{d\tau_2}\bigg|_{\tau=\tau_2^{2,n}}\right) < 0, \quad sign\left(\frac{dRe\lambda}{d\tau_2}\bigg|_{\tau=\tau_2^{3,n}}\right) > 0.$$

Proof. If Eq. (13) has only one positive solution y_1, then for all $y \in]0, y_1[, f(y)$ is negative and, for all $y \in]y_1, +\infty[, f(y)$ is positive. It follows that $f'(y_1) > 0$ and, by the previous lemma, the first transversality condition holds.

If Eq. (13) has two distinct positive solution y_2 and y_3 ($y_2 < y_3$), then, for all $y \in]y_2, y_3[, f(y)$ is negative and, for all $y \in]0, y_2[\cup]y_3, +\infty[, f(y)$ is positive. It follows that $f'(y_2) < 0$, $f'(y_3) > 0$ and, by the previous lemma, the last two transversality conditions hold.

If Eq. (13) has three distinct positive solution y_1, y_2 and y_3 ($y_1 < y_2 < y_3$), then for all $y \in]0, y_1[\cup]y_2, y_3[, f(y)$ is negative and for all $y \in]y_1, y_2[\cup]y_3, +\infty[, f(y)$ is positive. It follows that $f'(y_1) > 0$, $f'(y_2) < 0$, $f'(y_3) > 0$ and, by the previous lemma, the transversality conditions hold. □

Theorem 2. *If condition (9) holds:*

(a) *if* Eq. (13) *has no positive solution, then the origin is locally asymptotically stable for all* $\tau_2 \geqslant 0$*;*
(b) *if* Eq. (13) *has only one positive solution* ω_1*, then the origin is locally asymptotically stable for any* $0 \leqslant \tau_2 < \tau_2^{1,0}$ *and unstable for any* $\tau_2 > \tau_2^{1,0}$*. Therefore* $\tau_2^{1,0}$ *is a stability switch, and if* $\tau_2 = \tau_2^{1,0}$ *we have a Hopf bifurcation;*
(c) *if* Eq. (13) *has two distinct positive solutions* ω_2 *and* ω_3*, then there is a positive integer k such that the equilibrium is stable if* τ_2 *is an element of the following set*

$$[0, \tau_2^{3,0}[\cup]\tau_2^{2,0}, \tau_2^{3,1}[\cup, \dots, \cup]\tau_2^{2,k-1}, \tau_2^{3,k}[,$$

and is unstable if τ_2 *is an element of the following set*

$$]\tau_2^{3,0}, \tau_2^{2,0}[\cup]\tau_2^{3,1}, \tau_2^{2,1}[\cup, \dots, \cup]\tau_2^{3,k-1}, \tau_2^{2,k-1}[.$$

In this last case we have $2k+1$ *stability switches, and either if* $\tau_2 = \tau_2^{2,n},\ n = 0, 1, \dots, k-1$*, or* $\tau_2 = \tau_2^{3,n},\ n = 0, 1, \dots, k$*, we have a Hopf bifurcation.*
(d) *if* Eq. (13) *has three distinct positive solutions, there exists at least one stability switch and a corresponding Hopf bifurcation.*

Proof.

(a) If Eq. (13) has no positive solution, there are no zeroes crossing the imaginary axis from left to right. It follows that the origin remain stable for all $\tau_2 \geqslant 0$.
(b) If Eq. (13) has only one positive solution ω_1, when $t_2 = \tau_2^{1,0}$ the characteristic Eq. (10) admits pure imaginary solutions $\lambda = \pm\omega_1 i$. By the previous theorem, there are a couple of zeroes crossing the imaginary axis from left to right for $t_2 = \tau_2^{1,0}$. Hence $t_2 = \tau_2^{1,0}$ is a stability switch. Moreover the system exhibits a Hopf bifurcation (see ([9])).
(c) If Eq. (13) has two distinct positive solutions ω_2 and ω_3 ($\omega_2 < \omega_3$), by the previous transversality conditions, when $t_2 = \tau_2^{2,n}$, a couple of zeroes crosses the imaginary axis from right to left and when $t_2 = \tau_2^{3,n}$, a couple of zeroes crosses the imaginary axis from left to right.
Notice that $\tau_2^{3,0} < \tau_2^{2,0}$, otherwise a couple of zeros crosses the imaginary axes from right to left for $\tau_2 = \tau_2^{2,0}$. This is impossible because the origin should be stable for $\tau_2 \in]0, \tau_2^{2,0}[$.
From (14), let us define the sequences

$$x_n := \tau_2^{3,n} = \tau_2^{3,0} + nh_3 \qquad n = 0, 1, \dots,$$

$$y_n := \tau_2^{2,n} = \tau_2^{2,0} + nh_2 \qquad n = 0, 1, \dots,$$

where $h_i = \frac{2\pi}{\omega_i}\ i = 2, 3$. We observe that $x_0 < y_0$ and $h_3 < h_2$. Easily, we have that $x_{n+1} < y_n$ iff $n > \frac{x_0 - y_0 + h_3}{h_2 - h_3}$. If $k = \left[\frac{x_0 - y_0 + h_3}{h_2 - h_3}\right] + 1$, where $[x]$ denote the biggest integer smaller than x, we have

$$x_0 < y_0 < x_1 < y_1 < \dots < x_{k-1} < y_{k-1} < x_k < x_{k+1} < y_k.$$

Summarizing, when $x_i,\ i = 0, \dots, k$, the origin's stability switches from stable to unstable and, when $y_i,\ i = 0, \dots, k-1$, the origin's stability switches from unstable to stable.
(d) In this case the proof is similar to the previous one. □

Similar results hold for $\tau_1 > 0$ and $\tau_2 = 0$.

2.7. The case $\tau_1 > 0$ *and* $\tau_2 > 0$

By using τ_2 as a parameter, if the equilibrium of system (1) is stable for $\tau_1 = 0$ and $\tau_2 > 0$ there exists $\overline{\tau}_1(\tau_2)$ such that the system remains stable for $\tau_1 \in [0, \overline{\tau}_1(\tau_2)[$ (see [14]).

We have

Theorem 3. *If condition (9) holds,*

- *if* Eq. (13) *has no positive solution, for any* $\tau_2 > 0$ *there exists* $\overline{\tau}_1(\tau_2)$ *such that the origin is locally asymptotically stable for* $\tau_1 \in [0, \overline{\tau}_1(\tau_2)[$*;*
- *if* Eq. (13) *has only one positive solution, for any* $0 \leqslant \tau_2 < \tau_2^{1,0}$*, there exists* $\overline{\tau}_1(\tau_2)$ *such that the origin is locally asymptotically stable for* $\tau_1 \in [0, \overline{\tau}_1(\tau_2)[$*;*
- if Eq. (13) *has two distinct positive solutions, there is a positive integer k such that for any*

$$\tau_2 \in [0, \tau_2^{3,0}[\cup]\tau_2^{2,0}, \tau_2^{3,1}[\cup, \dots, \cup]\tau_2^{2,k-1}, \tau_2^{3,k}[,$$

there exists $\overline{\tau}_1(\tau_2)$ *such that the origin is locally asymptotically stable for* $\tau_1 \in [0, \overline{\tau}_1(\tau_2)[$.

Proof. The proof easily follows from theorem (2). □

3. Parameters and numerical simulations

3.1. Parameters

In this section, we used the following set of parameters:

$\alpha = 1$	$\psi = 0.02$	$A = 0.01$	$a = 1.03$
$s = 0.08$	$\epsilon \in [0.05, 0.7]$	$\eta \in \{0.35, 0.4\}$	$b = 0.37$
$G_0 = 40$	$\gamma = 100$	$g_1 = 0.0098$	$g_2 = 0.001$
$m = 0.05$	$\mu = 0.0001$	$\rho = 1$	$\bar{r} = 0.0015$

Like Sasakura [15], we assumed β as critical parameter when $\tau_1 = \tau_2 = 0$. By the Routh–Hurwitz stability criterion (9), we obtained that the steady state solution is stable if and only if

$$\beta > \beta_0 = \frac{-\alpha\chi - I_r\psi M(G_Y - \eta)}{L_r(\chi L_r - L_y I_r)},$$

where $\chi = s(a-1)(1-\eta) - \eta + G_Y$.

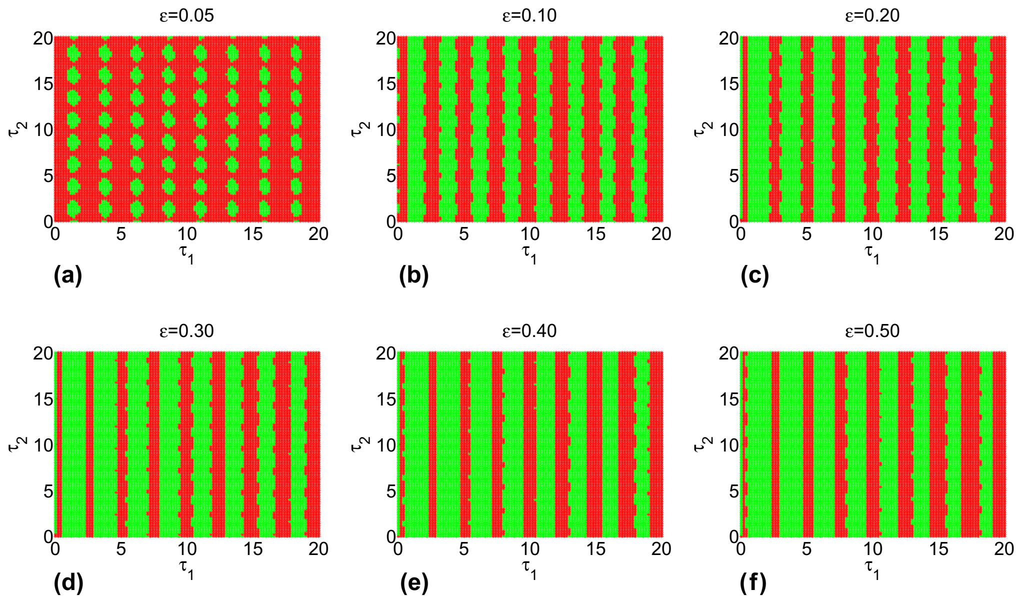

Fig. 1. Stable/Unstable area with fixed $\eta = 0.35$ and $\beta = 1$ and different values of ϵ. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

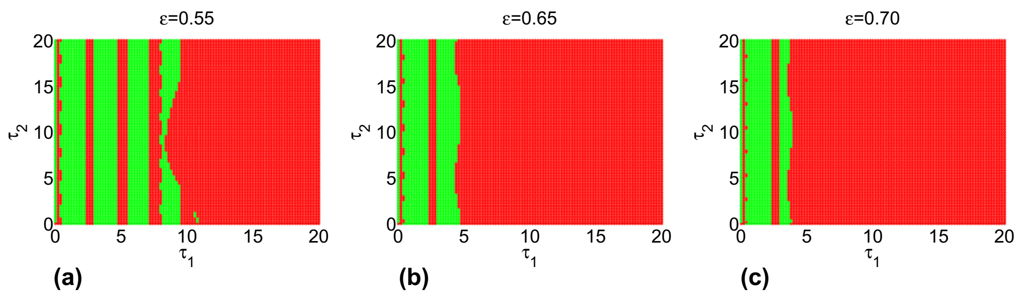

Fig. 2. Stable/Unstable area with fixed $\eta = 0.35$ and $\beta = 1$ and different values of ϵ. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

With the tax rate $\eta = 0.35$ and the parameters given above, we obtained the bifurcation value $\beta_0 \approx 1.206$, while if $\eta = 0.4,\ \beta_0 \approx 1.564$. This implies that, if $\beta < \beta_0$, the system is unstable, but such that to generate a stable limit cycle near the equilibrium point. On the contrary, if $\beta > \beta_0$, the equilibrium point is stable. From the economic point of view, this means that instability prevails only when the rate of interest reacts slowly to the money market disequilibrium. Later on, we shall see that, with $\tau_1, \tau_2 > 0$, the system may gain or lose its local stability.

3.2. Numerical simulations

Preliminarily we have to remind the reader that in this model the income level Y^* depends on the government choices about the public expenditure and the income tax rate. Therefore, the system's dynamics strictly reflects those public decisions.

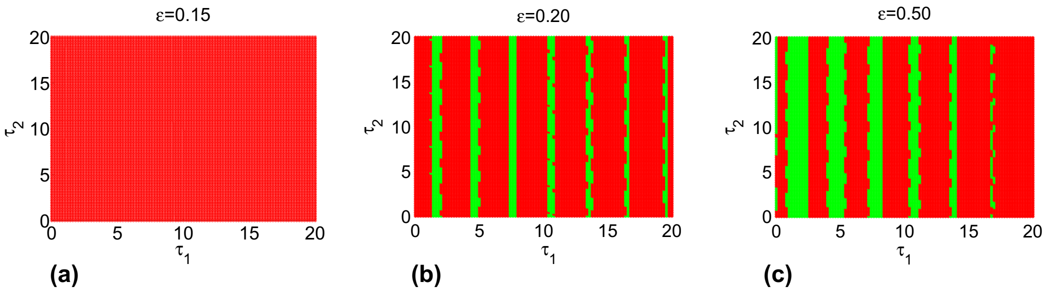

Fig. 3. Stable/Unstable area with fixed $\eta = 0.4$ and $\beta = 1$ and different values of ϵ.

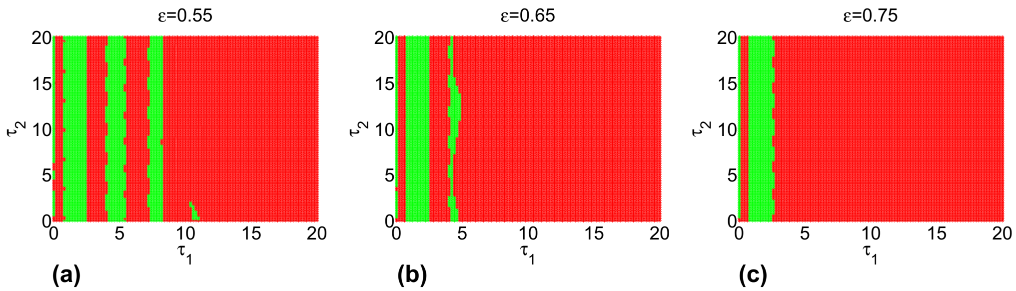

Fig. 4. Stable/Unstable area with fixed $\eta = 0.4$ and $\beta = 1$ and different values of ϵ.

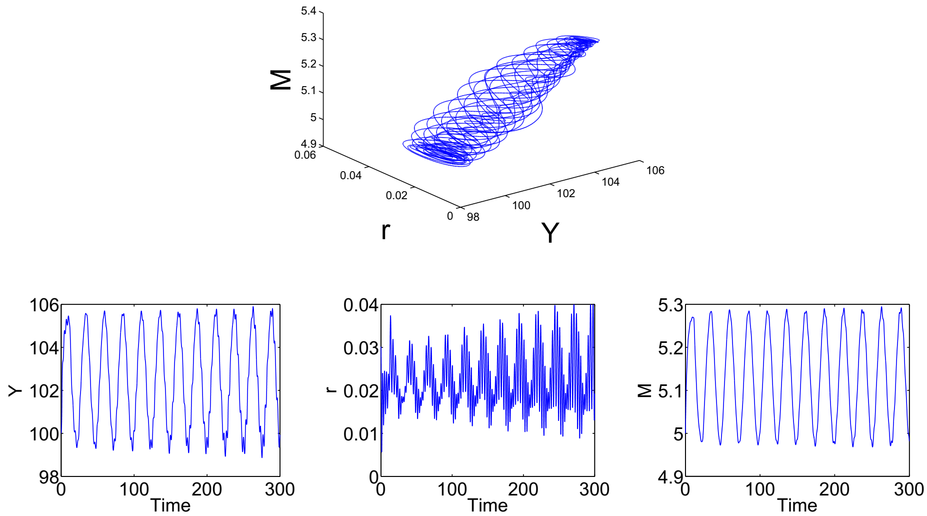

Fig. 5. System's trajectory for $\tau_1 = 10.5,\ \tau_2 = 2,\ \epsilon = 0.55,\ \beta = 1$ and $\eta = 0.4$.

As the set of parameters governing our model may give rise to different dynamic behaviors, we grouped the innumerable possible cases as follows:

i. Local instability without lags, fixed public expenditure and tax rate $\eta = 0.35$ and $\eta = 0.4$.
ii. Local stability without lags, fixed public expenditure and tax rate $\eta = 0.35$ and $\eta = 0.4$.

Numerical simulations are performed by using Matlab and the Matlab package DDE-BIFTOOL v. 2.03 by [5].

3.2.1. 1st case study: instability without lags

When the system is unstable with no time delay and the tax rate is $\eta = 0.35$, the instability persists with $\tau_1, \tau_2 > 0$, but a low share of delayed tax revenues is able to generate a lot of islands, where particular pairs of (τ_1, τ_2) make the system stable. Looking at the Fig. 1(a), we can see that a lag in the tax revenue $\tau_1 > 1$ is enough to stabilize the system (green area in the figure). When the share of postponed taxes increases, the stability extends in the plane (τ_1, τ_2) like a sequence of streaks

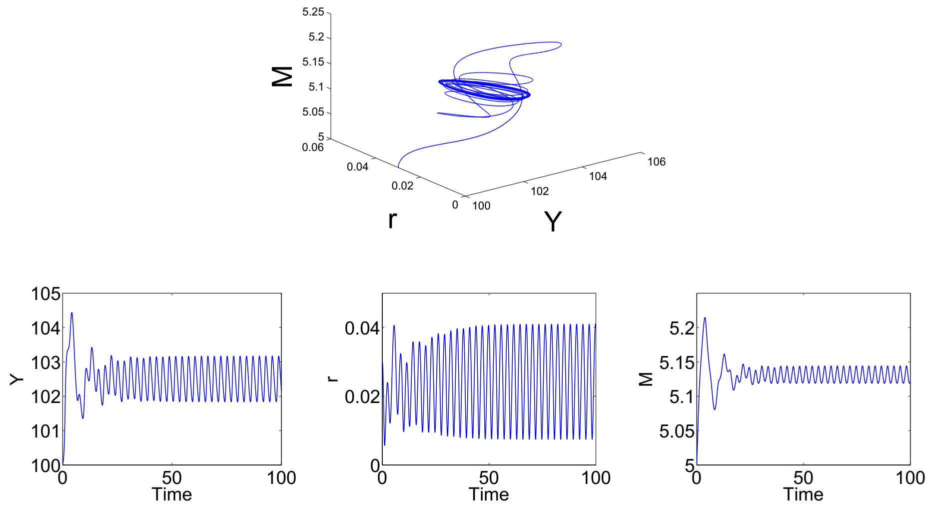

Fig. 6. System's trajectory for $\tau_1 = 2.9,\ \tau_2 = 2,\ \epsilon = 0.55,\ \beta = 1$ and $\eta = 0.4$.

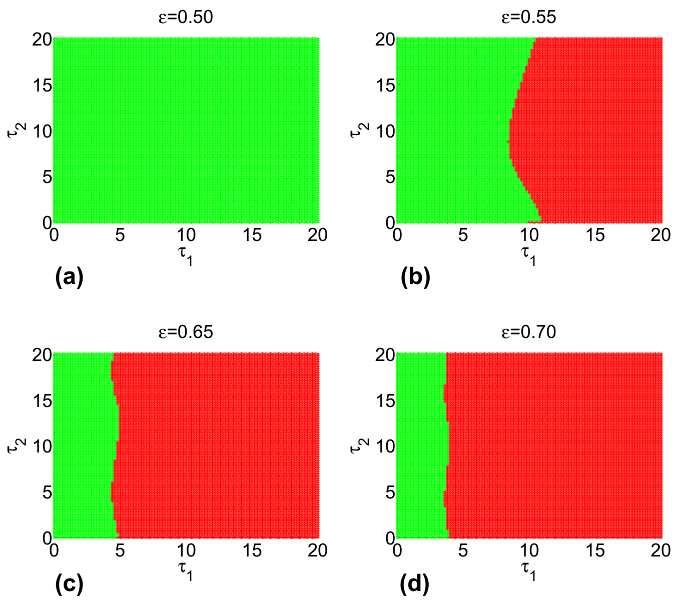

Fig. 7. Stable/Unstable area with fixed $\eta = 0.35$ and $\beta = 1.7$ and different values of ϵ.

(Fig. 1(b)–(f)) as long as the share of delayed tax revenues (ϵ) is near 50%. Further increasing values of ϵ (55%) imply instead persistent instability if $\tau_1 > 10$ whatever may be the lag of public expenditure (Fig. 2(a)). This case shows that the extension of the unstable area (red zone in the figures) enlarges with rising values of τ_1, while if τ_1 is low, local stability conditions are preserved in spite of very high the ϵ value may be (see Fig. 2(b) and (c)).

When we consider a tax rate $\eta = 40\%$, the scenario changes drastically. The dynamical system shows stability areas only if the share of delayed tax revenues (ϵ) is at least equal to 20% (see Fig. 3(a)–(c)). As soon as ϵ increases further on the 50% the existence of stability areas require a decreasing tax lag τ_1 (see Fig. 4(a)–(c)).

If instability prevails over the plane (τ_1, τ_2), the system may give rise either to periodic or aperiodic oscillations depending on the value assumed by τ_2. As it is shown in the Figs. 5 and 6, fixed $\tau_1 = 10.5$, a $\tau_2 = 2$ displays chaotic movements prevalently deriving from the interest rate dynamics, while a $\tau_1 = 2.9$ generates system's trajectories approaching a limit cycle. A similar system's behavior can be observed with different pairs of (τ_1, τ_2) along the frontier of the stable zone. We have to

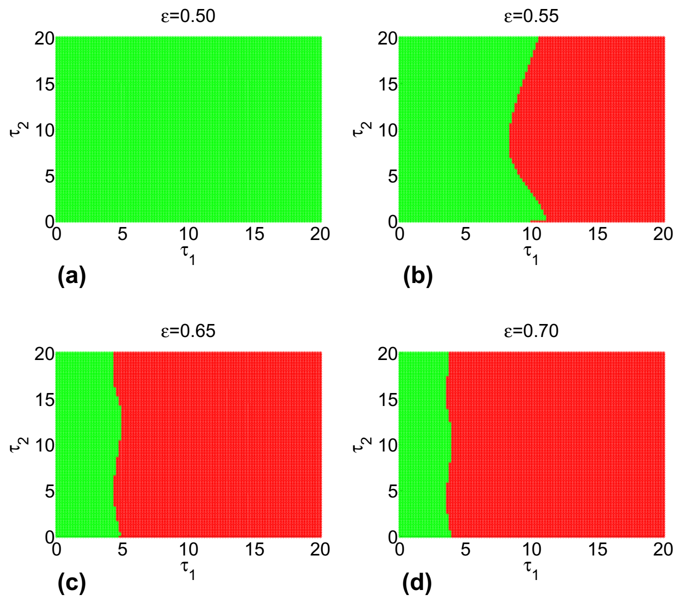

Fig. 8. Stable/Unstable area with fixed $\eta = 0.4$ and $\beta = 1.7$ and different values of ϵ.

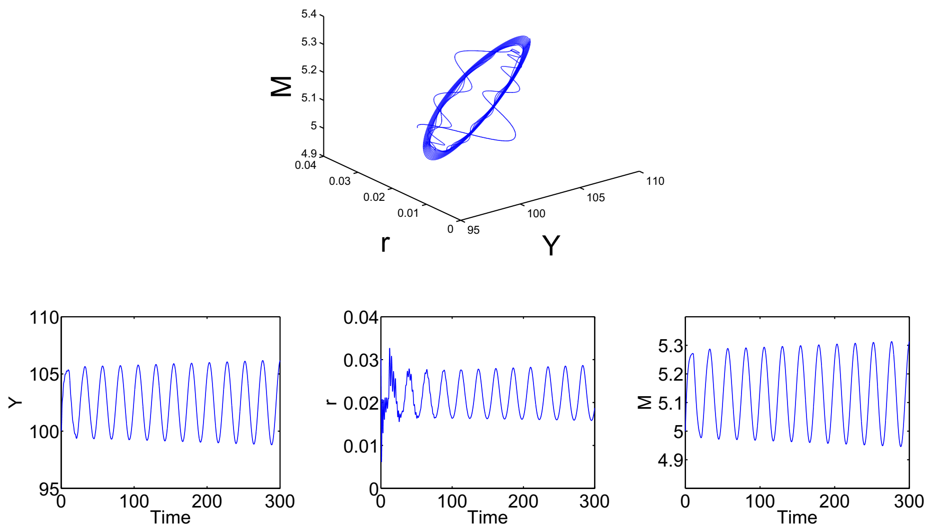

Fig. 9. System's trajectory for $\tau_1 = 10,\ \tau_2 = 3.8,\ \epsilon = 0.55,\ \beta = 1.7$ and $\eta = 0.4$.

point out that oscillations of the interest rate always have an impact on the income behavior through the investment function. The impact intensity depends on the sensitivity of investment to the changes of the interest rate.

3.2.2. 2nd case study: stability without lags

Independently of the tax rate, when the system is stable with no time delay, and the share of delayed tax revenues is no more than 50%, local stability conditions are preserved whatever may be the pairs of lags characterizing both the tax structure and the public expenditure. As soon as the share of delayed tax revenue becomes higher than 50% the stability area comes down progressively and survives on the left hand side of $\tau_1 \leqslant 4$ (see Figs. 7(a)–(d) and 8(a)–(d)).

Along the right hand side of the stability frontier, system's trajectories initially wander near a limit cycle before approaching the closed orbit directly. Fig. 9 shows this behavior. It seems to derive from an initial irregular oscillation of the interest rate.

Finally, we have to notice that with different parameter values, always preserving their economic meaning, results are qualitatively similar to the ones previously discussed.

4. Concluding remarks

We cannot forget that the results of numerical simulations we carried out in the previous section depend on the specification of the functional form here adopted. Nevertheless, we think that our results seem to show that a Keynesian fiscal policy aimed to stabilize the GDP dynamics may generate few Keynesian effects in the economic system. In our model, this happens not only because of the existence of fiscal policy lags, but also because in our model the government deficit is purely money financed. This last assumption implies an impact on the interest rate that destabilizes the investment behavior. Consequently, the national income undergoes either regular or irregular oscillations notwithstanding the government purpose is a stable dynamic path of the economy. Only if policy makers control the structure of either the tax system or the public expenditure delay, and take into account the induced reaction of the interest rate, then fiscal policy outcomes might be consistent with their purposes.

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