A short introduction to Sobolev-spaces and applications for engineering students

Dag Lukkassen

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1. Preliminary definitions and results

1.1. Hilbert spaces

Let V be a vector-space (i.e. $x, y \in V \Rightarrow x + y \in V$, and $kx \in V$ for all $k \in R$) with a scalar product denoted $\cdot$ (also called inner product). This scalar product induces a norm $\|x\|_V = (x \cdot x)^{\frac{1}{2}}$. If V is complete with respect to this norm then V is called a Hilbert-space. Roughly speaking, we say that a space V is complete if for any sequence $\{x_h\}$ of elements in V converging to an element x, i.e. $\|x_h - x\|_V \to 0$, we have that x also belongs to V.

Example 1.1. *Let*

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

The scalar product $(\cdot)$ between vectors of this type is usually defined by

$$x \cdot y = x_1 y_1 + x_2 y_2 + ... + x_n y_n$$

and the norm of x is given by

$$\|x\| = \sqrt{x_1^2 + x_2^2 + ... x_n^2}\ ).$$

It is easy to check that the set of all such vectors forms a vector space. This space is usually denoted R^n. It is possible to show that this space is complete with respect to the given scalar product. Hence R^n is a Hilbert space. In particular, $R = R^1$ is a Hilbert space.

Example 1.2. *An example of a set which is not complete is the set Q of rational numbers (numbers which can be written on the form a/b where a and b are integers). This set is not complete with respect to the norm $\|x\| = \sqrt{x^2}$. For* *example we can find elements in Q which converge to the number $\sqrt{2} = 1.4142..$, (take e.g. the sequence $1,\ 1.4, 1.41, 1.414, ...$) but this number is not a member of Q.*

From now on we will reserve the symbol $(\cdot)$ for the usual scalar product

$$x \cdot y = x_1 y_1 + x_2 y_2 + ... + x_n y_n\ ,\quad x \in R^n \text{ and } \ y \in R^n.$$

Remark 1. *Let V be a m-dimensional vector space (with a scalar product $*$), i.e. there exist m vectors $v_1, ...v_m$ (so called basis vectors) such that each vector $w \in V$ can be written (uniquely) as:*

$$w = a_1 v_1 + ... + a_m v_m$$

for some real constants $a_1, ...a_m$. Then it is possible to prove that V is a Hilbert space. Hence vector spaces which are not Hilbert spaces must be infinitely dimensional.

Exercise 1.1. *The set Q is not complete and is therefore not a Hilbert space. There is also another reason why Q is not a Hilbert space, what is that?*

1.1.1. The precise definition of completeness (optional for the students)

Let V be a vector-space with a scalar product $*$. The corresponding norm is then defined by $\|x\|_V = (x{*}x)^{\frac{1}{2}}$. We say that $\{x_h\}$, $h = 1, 2, 3...$, is a *Cauchy sequence* if the difference $\|x_k - x_m\|_V$ tends to 0 as k and m goes to ∞ (or even more precisely: for any number $\varepsilon > 0$ there exists an integer N such that $\|x_k - x_m\|_V < \varepsilon$ for all $k > N$ and $m > N$). The space V is said to be *complete* (with respect to the given scalar product) if every Cauchy sequence converges to an element $x \in V$, i.e. $\|x_h - x\|_V \to 0$ as $h \to \infty$.

1.2. Sobolev- and Lebesgue-spaces

Sobolev- and Lebesgue-spaces are important examples of Hilbert-spaces. Let Ω be a bounded domain in R^n.

$L^2(\Omega)$ = all functions v that has the property so that

$$\int_\Omega |v|^2\, dx < \infty. \tag{1.1}$$

$W^{1,2}(\Omega) =$ all functions u where $u \in L^2(\Omega)$ and $\partial u/\partial x_i \in L^2(\Omega)$. (The number 1 is used to indicate that the partial derivatives of order 1 should belong to $L^2(\Omega)$. The number 2 refers to power of the integrand in (1.1)). Functions belonging to $W^{1,2}(\Omega)$ do not have to be differentiable at every point; e.g. it is enough if they are continuous with piecewise continuous partial derivatives in the domain of definition and satisfy the above conditions.

Let Y be a cell in R^n (a rectangle if $n = 2$, a box if $n = 3$). $W^{1,2}_{per}(Y) = \{u \in W^{1,2}(Y) : \ u \text{ is } Y-\text{periodic}\}$. (We read it like this: W- periodic, 1,2, of Y is the set of all functions u which is an element of the space W,1,2 of Y where u is $Y-$periodic.) Recall that u is $Y-$periodic means that u have the same values on opposite faces of the cell Y

$W^{1,2}_0(\Omega) = \{u \in W^{1,2}(\Omega) : \ u \text{ is 0 on the boundary } \partial\Omega\}$. (We read it like this: W,0,1,2 of Y) is the set of all functions u which is an element of the space W,1,2 of Y where u is zero on the boundary $\partial\Omega$.

In detail we see that:

$$W^{1,2}_0(Y) \subset W^{1,2}_{per}(Y) \subset W^{1,2}(Y) \subset L^2(Y)$$

and

$$W^{1,2}_0(\Omega) \subset W^{1,2}(\Omega) \subset L^2(\Omega).$$

The "W- spaces" are often called Sobolev-spaces. The space $L^2(\Omega)$ is called a Lebesgue-space.

The scalar product $(*)$ of $W^{1,2}(\Omega)$ is defined by

$$u * v = \int_\Omega uv \ dx + \int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \ dx$$

and hence the norm is defined as

$$\|u\|_{W^{1,2}(\Omega)} = \sqrt{\int_\Omega u^2 \ dx + \int_\Omega |\operatorname{grad} u|^2 \ dx}.$$

We use the same norm for the other Sobolev-spaces. Moreover the scalar product $(*)$ on $L^2(\Omega)$ is defined by

$$u * v = \int_\Omega uv \ dx.$$

The corresponding norm is then given by

$$\|u\|_{L^2(\Omega)} = \sqrt{\int_\Omega u^2 \ dx}.$$

1.3. An example of a Sobolev-function

If $\Omega \subset R^n$, we will always let $|\Omega|$ denote the value

$$|\Omega| = \int_\Omega 1 \, dx$$

(which equals the length, area or volume of Ω if $n = 1, 2$ or 3, respectively).

Let Ω be the square $Y = [0, 2\pi]^2$ and let

$$u(x) = u(x_1, x_2) = \sin x_1 + \cos x_2.$$

We see that $\partial u/\partial x_1 = \cos x_1$ and $\partial u/\partial x_2 = -\sin x_2$.

Moreover, we observe that

$$\int_Y |u|^2 \, dx = \int_Y (\sin x_1 + \cos x_2)^2 \, dx \leq \int_Y |1+1|^2 \, dx = \int_Y 4 \, dx = 4\,|Y| < \infty \,,$$

$$\int_Y \left|\frac{\partial u}{\partial x_1}\right|^2 dx = \int_Y |\cos x_1|^2 \, dx \leq \int_Y |1|^2 \, dx = |Y| < \infty$$

and

$$\int_Y \left|\frac{\partial u}{\partial x_2}\right|^2 dx = \int_Y |-\sin x_2|^2 \, dx \leq \int_Y |1|^2 \, dx = |Y| < \infty.$$

Thus,

$$u \in L^2(Y), \ \frac{\partial u}{\partial x_1} \in L^2(Y) \text{ and } \frac{\partial u}{\partial x_2} \in L^2(Y)$$

This shows that $u \in W^{1,2}(Y)$. Moreover, we observe that u is Y periodic, and therefore $u \in W^{1,2}_{per}(Y)$.

1.4. Example of a function which is not a Sobolev-function

Let Ω be the open unit-square $Y = \langle 0, 1\rangle^2$ and let

$$u(x) = u(x_1, x_2) = x_1^{\frac{1}{2}}.$$

We see that $\partial u/\partial x_1 = \frac{1}{2} x_1^{-\frac{1}{2}}$. Thus,

$$\int_Y \left|\frac{\partial u}{\partial x_1}\right|^2 dx = \int_0^1 \int_0^1 \left|\frac{1}{2} x_1^{-\frac{1}{2}}\right|^2 dx_1 dx_2 = \frac{1}{4} \int_0^1 \int_0^1 x_1^{-1} dx_1 dx_2$$

$$= \frac{1}{4} \int_0^1 (\ln 1 - \ln 0) dx_2 = \infty$$

This shows that $u \notin W^{1,2}(Y)$, i.e. u is not a Sobolev-function. Remark that for this example both $\int_Y \left| \frac{\partial u}{\partial x_2} \right|^2 dx < \infty$ and $\int_Y |u|^2 \, dx < \infty$.

Exercise 1.2. *Let Ω be the open interval $Y = \langle 0, 1 \rangle$ in R^1. Determine whether the following functions are in some of the spaces $L^2(Y), W^{1,2}(Y)$, $W^{1,2}_{per}(Y), W^{1,2}_0(Y)$:*

1. $u(x) = x^2$
2. $u(x) = e^{-x^2}$
3. $u(x) = \sin x$
4. $u(x) = \sin(2\pi x)$
5. $u(x) = \cos(2\pi x)$

Exercise 1.3. *Let Ω be the open square $Y = \langle 0, 1 \rangle^2$ in R^2. Compute the $L^2(Y)$ norm and the $W^{1,2}(Y)$ norm of the function $u(x_1, x_2) = x_1^2 + x_2^2$.*

Exercise 1.4. *Show $\int_Y |\partial u / \partial x_2|^2 \, dx < \infty$ and $\int_Y |u|^2 \, dx < \infty$ in the example above (where*

$$u(x) = u(x_1, x_2) = x_1^{\frac{1}{2}}).$$

Exercise 1.5. *Let Ω be the open unit-square $Y = \langle 0, 1 \rangle^2$ and let*

$$u(x) = u(x_1, x_2) = x_1^{\alpha}$$

where α is a constant. For which values of α is $u \in W^{1,2}(Y)$, and, for which values of α is $u \in L^2(Y)$?

Exercise 1.6. *Let Ω be the open unit-square $\langle 0, 1 \rangle^2$. Determine whether the functions v and w, defined by*

$$\begin{aligned} v(x_1, x_2) &= e^{-x_1} \sin(\pi x_1) \sin(\pi x_2), \\ w(x_1, x_2) &= x_1^{\frac{2}{3}} \sin(\pi x_2) \end{aligned}$$

are in some of the Lebesgue and Sobolev spaces $L^2(\Omega), W^{1,2}(\Omega), W^{1,2}_0(\Omega)$.

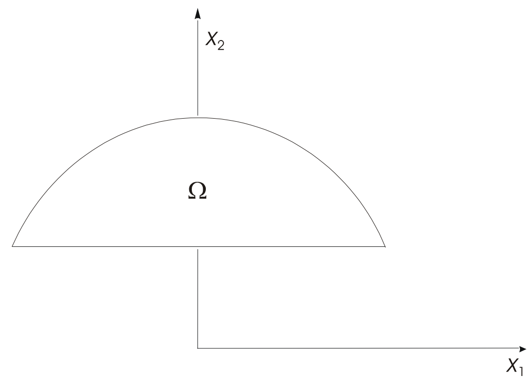


Figure 1.1: Ω in Exercise 1.7

Exercise 1.7. *Let $\Omega \subset R^2$ be the set which consists of all points above the line $x_2 = 1$ and below the curve $x_2 = \sqrt{4 - x_1^2}$ (see Figure 1.1). Determine whether the functions v and w, defined by*

$$\begin{aligned} v(x_1, x_2) &= (x_1^2 + x_2^2 - 4)\,(2x_2 - 2)\,, \\ w(x_1, x_2) &= x_2^{\frac{1}{3}} \end{aligned}$$

are in some of the Lebesgue and Sobolev spaces $L^2(\Omega), W^{1,2}(\Omega), W_0^{1,2}(\Omega)$.

1.5. Schwarz inequality

If V is a vector space with a scalar product $(*)$ then the Schwarz inequality takes the following form

$$|x * y| \le \|x\|_V \, \|y\|_V = (x * x)^{\frac{1}{2}} \, (y * y)^{\frac{1}{2}} \, .$$

Let V be the space of vector valued functions with n components with scalar product defined by

$$u * v = \int_\Omega \lambda \; u \cdot v \; dx, \tag{1.2}$$

where λ is a positive bounded function, such that

$$0 < \lambda^- \le \lambda(x) \le \lambda^+ < \infty,$$

where λ^- and λ^+ are constants. In this case the Schwarz inequality takes the form

$$\left|\int_\Omega \lambda\, u \cdot v\; dx\right| \le \left(\int_\Omega \lambda \left|u\right|^2\; dx\right)^{\frac{1}{2}} \left(\int_\Omega \lambda \left|v\right|^2\; dx\right)^{\frac{1}{2}}. \tag{1.3}$$

In particular we obtain that

$$\left|\int_\Omega uv\; dx\right| \le \left(\int_\Omega \left|u\right|^2\; dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|v\right|^2\; dx\right)^{\frac{1}{2}}, \tag{1.4}$$

where u and v are real functions.

Exercise 1.8. *Recall the definition of a scalar product (inner product). Let V be the space of continuous real valued functions defined on the interval $\Omega = [-1, 1]$, let λ be a function such that $1 \le \lambda(x) \le 2$ and let $*$ be defined by*

$$u * v = \int_{-1}^{1} \lambda(x)u(x)v(x)dx.$$

1. *Show that V is a vector space and show that $*$ defines a scalar product on V.*

2. *Let $\|u\|_V$ be the norm $\|u\|_V = (u * u)^{\frac{1}{2}}$. Find the number $\|u\|_V$ in the case when $u(x) = x$ and $\lambda = 1$.*

3. *Let u_h be defined by*

$$u_h(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ hx & \text{if } 0 < x \le 1/h \\ 1 & \text{if } 1/h \le x \le 1 \end{cases}.$$

   Plot the graph of u_h. Do we have that $u_h \in V$? What function does u_h converge to? Is V complete? Is V a Hilbert space?

1.6. Poincaré's and Friedrich's inequalities

We will not go into details on the precise definition of *Lipschitz continuous boundary* but we may mention that such boundaries must be continuous and do not always need to be differentiable at each point, e.g. all closed polygonal boundaries

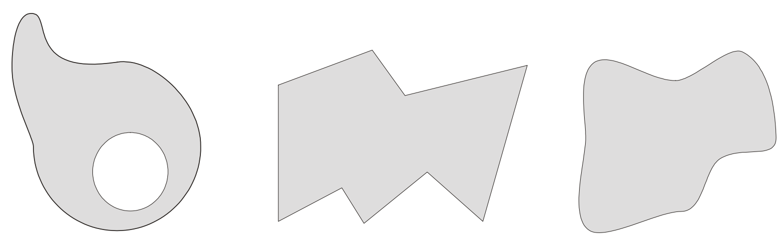


Figure 1.2: 3 examples of sets with Lipschitz continuous boundaries. Note that the smallest angle on the boundary is larger than 0.

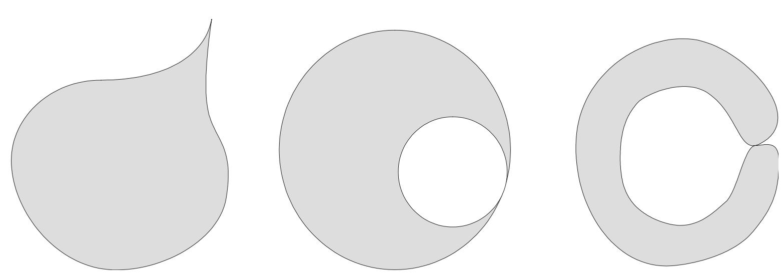


Figure 1.3: 3 examples of sets which boundaries are not Lipschitz continuous.

has this property and also boundaries of convex domains. See also Figure 1.2 and Figure 1.3.

Let Ω be a domain whose boundary is Lipschitz continuous. Then the *Friedrich's inequality*:

$$\int_\Omega v^2\, dx \le C_0 \int_\Omega |\mathrm{grad}\, v|^2\, dx, \quad \forall v \in W_0^{1,2}(\Omega)$$

is valid: If, in addition, Ω is connected then the *Poincaré's inequality* is also valid:

$$\int_\Omega u^2\, dx \le C_0 \left\{ \left(\int_\Omega u\, dx \right)^2 + \int_\Omega |\mathrm{grad}\, u|^2\ dx \right\} \quad \forall u \in W^{1,2}(\Omega).$$

(the symbol $\forall$ means: "for all"). Note that $C_0 > 0$ in both cases, and C_0 is a constant which only dependent of Ω (not the functions u or v).

Exercise 1.9. *Use the inequality (1.4) and show that*

$$\left(\int_\Omega u\, dx \right)^2 \le K \left(\int_\Omega |u|^2\ dx \right) \tag{1.5}$$

for some positive constant K which is independent of u. Prove Friedrich's inequality in the case $\Omega = [0, 1]$ (Hint: use that

$$\begin{aligned} |v(t) - v(0)| &= \left| \int_0^t v'(x)\, dx \right| \le \int_0^t |v'(x)\,|\, dx \\ &\le \int_0^1 |v'(x)\,|\, dx = \int_\Omega |\mathrm{grad}\, v|\, dx \end{aligned}$$

and then use (1.5))

1.7. Other useful inequalities

We also have the following inequality

$$\left| \int_\Omega f(x)\, dx \right| \le \int_\Omega |f(x)\,|\, dx,$$

which is valid for any real valued or vector valued function f (which is possible to integrate). In addition we sometimes use the *triangle inequality*

$$|a + b| \le |a| + |b|\,,$$

which is valid for any real number a and b. More generally, for elements a, b in inner product spaces with norm $\|(\cdot)\|$, the *triangle inequality* takes the form

$$\|a+b\| \leq \|a\| + \|b\| .$$

2. The heat conduction problem

We start with the *strong formulation* of the stationary heat conduction problem:

$$\operatorname{div}\left(\lambda(x) \operatorname{grad} u(x)\right) = -f(x), \;\; x \in \Omega, \tag{2.1}$$

where we have certain information of u on the boundary $\partial\Omega$ of the connected body Ω, see Figure 2.1 ($-f(x)$ is the heat source in the system and $\lambda(x)$ is the conductivity). We will assume that $f \in L^2(\Omega)$ and that

$$0 < \lambda^- \leq \lambda(x) \leq \lambda^+ < \infty,$$

where λ^- and λ^+ are constants. In addition we will assume that the boundary is sufficiently smooth (e.g. Lipschitz continuous, such that the Poincaré's inequality and Friedrich's inequality are valid). The solution u can for instance be the temperature, an electromagnetic potential or the displacement in certain elasticity problems. Let $\partial u/\partial \mathbf{n} = (\operatorname{grad} u) \cdot \mathbf{n}$ be the directional derivative ($\mathbf{n}$ is the outward unit normal to $\partial\Omega$) in the direction $\mathbf{n}$. If u is known on the boundary $\partial\Omega$ we have a *Dirichlet boundary condition.* If $\partial u/\partial n$ is known on the boundary we have a *Neumann boundary condition.*Recall that if

$$x = (x_1, ..., x_n),$$

and u is a real valued function (of x) then $\operatorname{grad} u$ is the vector function

$$\operatorname{grad} u = \left[\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right].$$

Moreover, if $\mathbf{w}$ is a vector function $\mathbf{w} = [w_1, ..., w_n]$, then

$$\operatorname{div} \mathbf{w} = \frac{\partial w_1}{\partial x_1} + \cdots + \frac{\partial w_n}{\partial x_n}.$$

Thus, since

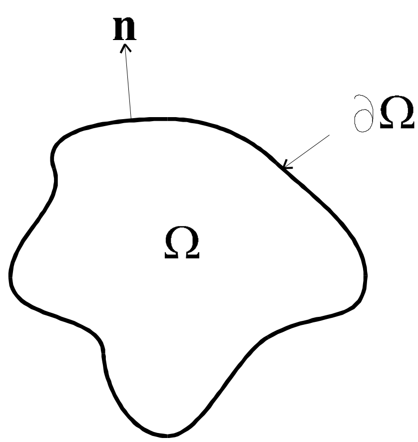


Figure 2.1: Body Ω

$$\lambda \operatorname{grad} u = \left[\lambda \frac{\partial u}{\partial x_1}, \cdots, \lambda \frac{\partial u}{\partial x_n}\right],$$

we obtain that

$$\operatorname{div}(\lambda \operatorname{grad} u) = \frac{\partial}{\partial x_1}\lambda \frac{\partial u}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}\lambda \frac{\partial u}{\partial x_n}.$$

In the two-dimensional case ($n = 2$) this means that 2.1 can be written as:

$$\frac{\partial}{\partial x_1}\lambda \frac{\partial u}{\partial x_1} + \frac{\partial}{\partial x_2}\lambda \frac{\partial u}{\partial x_2} = -f$$

We multiply with a test function ϕ on both sides of (2.1), and then integrate. This leads to the *weak formulation* of (2.1):

$$\int_\Omega \phi \operatorname{div}(\lambda \operatorname{grad} u)\, dx = -\int_\Omega \phi f\, dx. \qquad (2.2)$$

From the following version of *Greens formula* (see Exercise 2.1):

$$\int_\Omega \phi \operatorname{div} \mathbf{w} dx = -\int_\Omega \operatorname{grad} \phi \cdot \mathbf{w}\, dx + \int_{\partial\Omega} \phi \mathbf{w} \cdot n\, ds \qquad (2.3)$$

we obtain (putting $\mathbf{w} = \lambda \operatorname{grad} u = [\lambda \partial u/\partial x_1, .., \lambda \partial u/\partial x_n]$) that

$$\int_\Omega \phi \operatorname{div}(\lambda \operatorname{grad} u)\, dx =$$

$$= -\int_\Omega \operatorname{grad}\phi \cdot\, (\lambda \operatorname{grad} u)\, dx + \int_{\partial\Omega} \phi\, (\lambda \operatorname{grad} u) \cdot n\, ds.$$

The equation (2.2) can therefore be written as

$$-\int_\Omega \lambda\, (\operatorname{grad}\phi) \cdot \operatorname{grad} u\; dx + \int_{\partial\Omega} \phi\, (\lambda \operatorname{grad} u) \cdot n\, ds = -\int_\Omega \phi f\; dx. \tag{2.4}$$

We remark that the strong formulation implies the weak formulation.

Exercise 2.1. *Prove (2.3) (Hint: Use the product rule*

$$\frac{\partial\, (\phi w_i)}{\partial x_i} = \frac{\partial \phi}{\partial x_i} w_i + \frac{\partial w_i}{\partial x_i} \phi$$

and the Divergence theorem of Gauss:

$$\int_\Omega \operatorname{div} \mathbf{v}\, dx = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, ds)$$

2.1. Dirichlet problem ($u = 0$ on the boundary)

In the case of the Dirichlet problem we let $\phi = 0$ on the boundary. Hence

$$\int_{\partial\Omega} \phi\, (\lambda \operatorname{grad} u) \cdot \mathbf{n}\, ds = 0,$$

and we end up with the weak formulation

$$\int_\Omega \lambda \operatorname{grad} u \cdot \operatorname{grad}\phi\; dx = \int_\Omega \phi f\; dx.$$

For the case $u = 0$ on the boundary $\partial\Omega$, the precise formulation of the weak problem takes the following form:

Find $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \lambda \operatorname{grad} u \cdot \operatorname{grad}\phi\; dx = \int_\Omega \phi f\; dx$$

for all $\phi \in W_0^{1,2}(\Omega)$. In other words

$$a(u,\phi) = L(\phi), \quad \forall \phi \in W_0^{1,2}(\Omega), \tag{2.5}$$

where

$$a(u,\phi) = \int_\Omega \lambda \operatorname{grad} u \cdot \operatorname{grad} \phi \, dx$$

and

$$L(\phi) = \int_\Omega \phi f \, dx.$$

It it possible to prove that $a(.,.)$ is a *bilinear form* on $W_0^{1,2}(\Omega)$, i.e. that for all functions u, v, w in $W_0^{1,2}(\Omega)$ and $k \in R$, it holds that $a(u,v)$ is a real number and

$$a(u+v,w) = a(u,w) + a(v,w),$$

$$a(ku,w) = ka(u,w),$$

$$a(w,u+v) = a(w,u) + a(w,v),$$

$$a(u,kw) = ka(u,w).$$

Exercise 2.2. *Show that*

$$a(u,v) = \int_\Omega \lambda \operatorname{grad} u \cdot \operatorname{grad} v \, dx$$

defines a bilinear form on $W_0^{1,2}(\Omega)$.

Exercise 2.3. *Determine whether* $a(.,.)$ *is a bilinear form on* V *:*

1. $a(u,v) = u+v$, $V = R$
2. $a(u,v) = uv$, $V = R$
3. $a(u,v) = u(0)v(0)$, $V = C([0,1])$, *i.e. the space of continuous functions on the interval* $[0,1]$.
4. $a(u,v) = \int_0^1 u(x)v(x)dx$, $V = C([0,1])$,
5. $a(u,v) = (u(0)v(0))^2$, $V = C([0,1])$,

2.2. Existence and uniqueness of the Dirichlet problem

As we know a given problem may not have a solution (e.g. the problem: Find $x \in R$ such that $x^2 + 1 = 0$). However the existence of a solution of the weak formulation follows by a well-known result in mathematics called the *Lax-Milgram Lemma* (see Theorem 3.1 and the discussion in the same section). Moreover the solution of the weak formulation is unique. This fact is proved as follows: Consider two solutions u_1 and u_2 of (2.5)

$$\int_\Omega \lambda \operatorname{grad} u \cdot \operatorname{grad} \phi \, dx = \int_\Omega \phi f \, dx, \ \forall \phi \in W_0^{1,2}(\Omega).$$

Then,

$$\int_\Omega \lambda \operatorname{grad} u_1 \cdot \operatorname{grad} \phi \, dx = \int_\Omega \phi f \, dx, \ \forall \phi \in W_0^{1,2}(\Omega)$$

and

$$\int_\Omega \lambda \operatorname{grad} u_2 \cdot \operatorname{grad} \phi \, dx = \int_\Omega \phi f \, dx, \ \forall \phi \in W_0^{1,2}(\Omega).$$

Thus, we get

$$\int_\Omega \lambda \operatorname{grad}(u_1 - u_2) \cdot \operatorname{grad} \phi \, dx = 0, \ \forall \phi \in W_0^{1,2}(\Omega).$$

Since this holds for $\forall \phi \in W_0^{1,2}$, it particularly holds for $\phi = u_1 - u_2$, which gives us that

$$\int_\Omega \lambda \left| \operatorname{grad}(u_1 - u_2) \right|^2 dx = 0. \tag{2.6}$$

Since $\lambda(x) > 0$ for all $x \in \Omega$ and

$$\left| \operatorname{grad}(u_1 - u_2) \right|^2 \geq 0, \quad \forall x \in \Omega,$$

(2.6) gives that

$$\operatorname{grad}(u_1 - u_2) = 0, \quad x \in \Omega,$$

and hence

$$(u_1 - u_2) = \text{constant in } \Omega.$$

Because $u_1 = u_2 = 0$ on the boundary $\partial\Omega$, the constant is 0, i.e. $u_1 = u_2$ for all $x \in \Omega$. This shows that the solution u in (2.5) is unique.

2.3. Equivalent minimum energy formulation

A function F defined on the set K has a minimum at x_0 with the minimum value $F(x_0)$ if

$$F(x_0) = \min_{x \in K} F(x).$$

This is the same as the statement:

$$F(x_0) \leq F(x) \quad \forall x \in K.$$

[For a simple analogy let us consider the function

$$F(x) = x^2.$$

($F(x)$ is shown in the curve below). We easily see that

$$\begin{array}{rcl} F(0) & = & \min_{x \in R} F(x) = 0 \\ & \Updownarrow & \\ F(0) & \leq & F(x) \text{ for } \forall x \in R \text{ .]} \end{array}$$

We will now show that (2.5) is equivalent with the following minimum. energy formulation: Find $u \in W_0^{1,2}(\Omega)$ such that

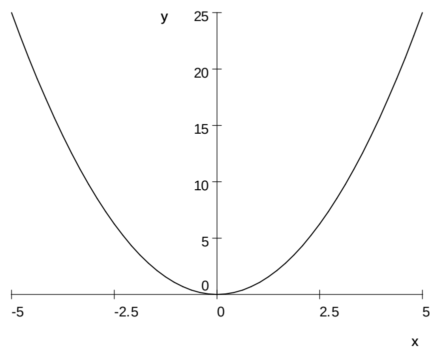
$$F(u) = \min_{\phi \in W_0^{1,2}(\Omega)} F(\phi).$$

which is the same as finding $u \in W_0^{1,2}(\Omega)$ such that

$$F(u) \leq F(\phi), \ \forall \ \phi \in W_0^{1,2}(\Omega), \tag{2.7}$$

where

$$F(\phi) = \frac{1}{2} a(\phi, \phi) - L(\phi). \tag{2.8}$$



Curve for x^2

To prove that (2.5) $\iff$ (2.7), we must check if the implication is true in the both ways. First we prove the implication (2.5) $\Rightarrow$ (2.7):

We note that

$$a(w,w) = \int_\Omega \lambda \operatorname{grad} w \cdot \operatorname{grad} w \; dx =$$

$$= \int_\Omega \lambda \left|\operatorname{grad} w\right|^2 dx \geq 0.$$

Letting $w = u - \phi$, we get that

$$a(u-\phi, u-\phi) \geq 0.$$

Since a is bilinear and $a(u,\phi) = a(\phi,u)$ we obtain that

$$\begin{aligned} a(u-\phi, u-\phi) = a(u, u-\phi) - a(\phi, u-\phi) = \\ = a(u,u) - a(u,\phi) - [a(\phi,u) - a(\phi,\phi)] = \\ = a(u,u) - a(u,\phi) - a(\phi,u) + a(\phi,\phi) = \\ = a(u,u) - 2a(\phi,u) + a(\phi,\phi). \end{aligned}$$

Thus,

$$a(u,u) - 2a(\phi,u) + a(\phi,\phi) = a(u-\phi, u-\phi) \geq 0. \tag{2.9}$$

i.e. (the tricky part)

$$\frac{1}{2}a(\phi,\phi) - a(\phi,u) \geq \frac{1}{2}a(u,u) - a(u,u).$$

Since u is a solution we have that $a(\phi, u) = L(\phi)$, and $a(u, u) = L(u)$. Hence,

$$\frac{1}{2}a(\phi, \phi) - L(\phi) \geq \frac{1}{2}a(u, u) - L(u),$$

which is the same as

$$F(\phi) \geq F(u).$$

This completes the proof of (2.5) $\Rightarrow$ (2.7).

Proof of (2.7) $\Rightarrow$ (2.5): Let $\phi \in W_0^{1,2}(\Omega)$ and $x \in R$ be arbitrary. Then $(u + x\phi) \in W_0^{1,2}(\Omega)$, and since u is a minimum point, we obtain that

$$F(u) \leq F(u + x\phi) \quad \forall x \in R.$$

Put $g(x) = F(u + x\phi)$, $x \in R$. Observe that $g(0) = F(u + 0\phi) = F(u)$. Therefore, we have that

$$g(0) \leq g(x) \quad \forall x \in R,$$

i.e. g has a minimum at $x = 0$. Hence $g'(0) = 0$ if the derivative $g'(x)$ exists at $x = 0$. But

$$g(x) = \frac{1}{2}a(u + x\phi, u + x\phi) - L(u + x\phi) =$$

$$= \frac{1}{2}a(u, u + x\phi) + \frac{1}{2}a(x\phi, u + x\phi) - L(u) - L(x\phi) =$$

$$= \frac{1}{2}a(u, u) + \frac{1}{2}a(u, x\phi) + \frac{1}{2}a(x\phi, u) + \frac{1}{2}a(x\phi, x\phi) - L(u) - L(x\phi) =$$

$$= \frac{1}{2}a(u, u) + \frac{x}{2}a(u, \phi) + \frac{x}{2}a(\phi, u) + \frac{x^2}{2}a(\phi, \phi) - L(u) - xL(\phi) =$$

$$= \frac{1}{2}a(u, u) - L(u) + xa(u, \phi) - xL(\phi) + \frac{x^2}{2}a(\phi, \phi),$$

where we used the symmetry of $a(.,.)$. Thus

$$g'(x) = a(u, \phi) - L(\phi) + xa(\phi, \phi)$$

and it follows that

$$0 = g'(0) = a(u, \phi) - L(\phi),$$

i.e.

$$a(u, \phi) = L(\phi)$$

This proves that (2.7) $\Rightarrow$ (2.5).

2.4. The Neumann problem ($\partial u/\partial n = 0$ on the boundary)

In the case of the Neumann problem we put no restriction on the boundary values of the test function ϕ in (2.4), and instead of using the space $W_0^{1,2}(\Omega)$ we use the whole space $W^{1,2}(\Omega)$. Thus using $(\lambda \operatorname{grad} u) \cdot \mathbf{n} = \lambda \partial u/\partial \mathbf{n} = 0$ in (2.4) we obtain the following weak formulation of the Neumann problem: Find $u \in W^{1,2}(\Omega)$ such that

$$a(u, \phi) = L(\phi), \quad \forall \phi \in W^{1,2}(\Omega), \tag{2.10}$$

where (as before)

$$a(u, \phi) = \int_\Omega \lambda \operatorname{grad} u \cdot \operatorname{grad} \phi \, dx$$

and

$$L(\phi) = \int_\Omega \phi f \, dx.$$

If, in addition, $f \in L^2(\Omega)$, $\int_\Omega f \, dx = 0$ and the boundary of Ω is Lipschitz continuous, then the solution of (2.10) exists and is unique within an additive constant (see Exercise 3.6).

3. Abstract formulation

There exists many problems in physics and mechanics and in order to avoid the necessity to give an "infinite" number of concrete examples we will give an abstract (i.e. more general) formulation in the case of so-called elliptic problems. Elliptic equations model for example stationary heat conduction and static problems in elasticity. A formulation in the abstract way will also make it easier to understand the basic structure of the finite element method.

Let V be a Hilbert space with scalar product $*$ and a corresponding norm $\|\cdot\|_V$ (defined as usual as $\|v\|_V = \sqrt{v * v}$). Assume that $a(.,.)$ is a bilinear form on V, i.e. for all u, v, w in V and $k \in R$, it holds that

$$a(u + v, w) = a(u, w) + a(v, w),$$

$$a(ku, w) = ka(u, w),$$

$$a(w, u + v) = a(w, u) + a(w, v),$$

$$a(u, kw) = ka(u, w).$$

Moreover, assume that L a linear form on V. This means that

$$L(u+v) = L(u) + L(v),$$

$$L(ku) = kL(u),$$

In addition assume that

1. $a(.,.)$ is symmetric, i.e. $a(\phi, v) = a(v, \phi)$, $\forall \phi, v \in V$
2. $a(.,.)$ is continuous, i.e. there is a constant $\gamma > 0$ such that $|a(\phi, v)| \leq \gamma \|\phi\|_V \|v\|_V \quad \forall \phi, v \in V$,
3. $a(.,.)$ is V-elliptic, i.e. there is a constant $\alpha > 0$ such that $a(\phi, \phi) \geq \alpha \|\phi\|_V^2 \quad \forall \phi \in V$.
4. L is continuous, i.e. there is a constant $\Lambda > 0$ such that $|L(\phi)| \leq \Lambda \|\phi\|_V$ $\forall \phi \in V$.

We will now consider the following abstract minimization problem: Find $u \in V$ such that

$$F(u) = \min_{\phi \in V} F(\phi) \tag{3.1}$$

(i.e. $F(u) \leq F(\phi)$, $\forall \phi \in V$) where

$$F(\phi) = \frac{1}{2} a(\phi, \phi) - L(\phi),$$

and we will also consider the following abstract weak formulation problem: Find $u \in V$ such that

$$a(u, \phi) = L(\phi), \quad \forall \phi \in V. \tag{3.2}$$

Theorem 3.1. *[Lax-Milgram Lemma] If the conditions 2, 3 and 4 are satisfied then there exists a unique solution $u \in V$ of the problem (3.2).*

Theorem 3.2. *If the conditions 1, 2, 3 and 4 are satisfied then there exists a unique solution $u \in V$ of the problem (3.1). In addition, the problems in (3.1) and (3.2) are equivalent, i.e. $u \in V$ satisfies (3.1) if and only if u satisfies (3.2).*

Proposition 3.3. *If the conditions 2 and 4 are satisfied then the function $F(\phi)$ is continuous, i.e.*

$$F(\phi_h) - F(\phi) \to 0, \text{ whenever } \|\phi_h - \phi\|_V \to 0.$$

Proposition 3.4. *If the conditions 3 and 4 are satisfied then the function $F(\phi)$ has the property:*

$$F(\phi) \to \infty \text{ if } \|\phi\|_V \to \infty.$$

We will not prove Theorem 3.1. The equivalence in Theorem 3.2 has been proved for the Dirichlet problem earlier. The proof in the general case is the same. The fact that the problem (3.1) has a unique solution then follows by Theorem 3.1.

Let us prove Proposition 3.3: We have that

$$F(\phi_h) - F(\phi) = (\frac{1}{2}a(\phi_h, \phi_h) - L(\phi_h)) - (\frac{1}{2}a(\phi, \phi) - L(\phi))$$

The trick is to add the "0"-term

$$-\frac{1}{2}a(\phi, \phi_h) + \frac{1}{2}a(\phi, \phi_h) = 0,$$

which give us

$$F(\phi_h) - F(\phi) = \frac{1}{2}(a(\phi_h, \phi_h) - a(\phi, \phi_h)) +$$

$$+\frac{1}{2}\left(a(\phi, \phi_h) - a(\phi, \phi)\right) + (L(\phi) - L(\phi_h)).$$

Next we use that a is bilinear and that L is linear:

$$F(\phi_h) - F(\phi) = \frac{1}{2}(a(\phi_h - \phi, \phi_h) + \frac{1}{2}\left(a(\phi, \phi_h - \phi)\right) + L(\phi - \phi_h).$$

Thus,

$$|F(\phi_h) - F(\phi)| \leq \frac{1}{2}\left|a(\phi_h - \phi, \phi_h)\right| + \frac{1}{2}\left|a(\phi, \phi_h - \phi)\right| + \left|L(\phi - \phi_h)\right|.$$

Condition 2) gives the inequalities

$$\frac{1}{2}\left|a(\phi_h - \phi, \phi_h)\right| \leq \frac{1}{2}\gamma \left\|\phi_h - \phi\right\|_V \left\|\phi_h\right\|_V$$

$$\frac{1}{2}\left|a(\phi, \phi_h - \phi)\right| \leq \frac{1}{2}\gamma \left\|\phi\right\|_V \left\|\phi_h - \phi\right\|_V$$

and condition 4) gives the inequality

$$\left|L(\phi - \phi_h)\right| \leq \Lambda \left\|\phi - \phi_h\right\|_V .$$

Thus

$$\left|F(\phi_h) - F(\phi)\right| \leq \frac{1}{2}\left|a(\phi_h - \phi, \phi_h)\right| + \frac{1}{2}\left|a(\phi, \phi_h - \phi)\right| + \left|L(\phi - \phi_h)\right| \leq$$

$$\leq \frac{1}{2}\gamma \left\|\phi_h - \phi\right\|_V \left\|\phi_h\right\|_V + \frac{1}{2}\gamma \left\|\phi\right\|_V \left\|\phi_h - \phi\right\|_V + \Lambda \left\|\phi - \phi_h\right\|_V =$$

$$\left(\frac{1}{2}\gamma \left\|\phi_h\right\|_V + \frac{1}{2}\gamma \left\|\phi\right\|_V + \Lambda\right) \left\|\phi_h - \phi\right\|_V \quad \to 0$$

as $\left\|\phi_h - \phi\right\|_V \to 0$, i.e.

$$\left|F(\phi_h) - F(\phi)\right| \to 0$$

as $\left\|\phi_h - \phi\right\|_V \to 0$.

We turn to the proof of Proposition 3.4: Condition 3 gives that

$$a(\phi, \phi) \geq \alpha \left\|\phi\right\|_V^2$$

and condition 4 gives that

$$-\left|L(\phi)\right| \geq -\Lambda \left\|\phi\right\|$$

Since

$$F(\phi) = \frac{1}{2}a(\phi, \phi) - L(\phi) \geq \frac{1}{2}a(\phi, \phi) - \left|L(\phi)\right| ,$$

we obtain that

$$F(\phi) \geq \frac{1}{2}a(\phi, \phi) - \left|L(\phi)\right| \geq \frac{1}{2}\alpha \left\|\phi\right\|_V^2 - \Lambda \left\|\phi\right\|_V .$$

Thus,

$$\frac{F(\phi)}{\left\|\phi\right\|_V} \geq \frac{\alpha}{2} \left\|\phi\right\|_V - \Lambda \to \infty$$

as $\left\|\phi\right\|_V \to \infty$. Thus $F(\phi) \to \infty$ as $\left\|\phi\right\|_V \to \infty$.

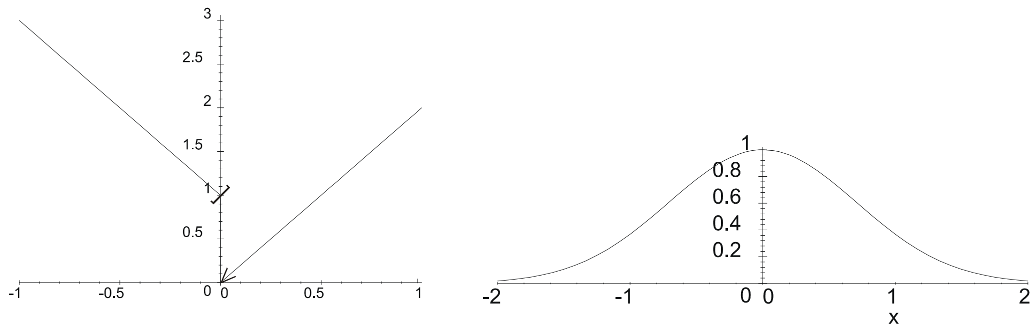


Figure 3.1: The functions f_1 and f_2.

3.1. Comparison with minimum problems on the real line

In Figure 3.1 we see the graphs of the function

$$f_1(x) = \begin{cases} 2x & \text{if } \; x > 0 \\ -2x + 1 & \text{if } \; x \le 0 \end{cases}$$

(to the left) and the function

$$f_2(x) = e^{-x^2}.$$

Note that non of these two functions has any minimum points. The first function f_1 has the "bad" property that is not continuous, but has the "good" property that $f_1(x)$ grows as $|x|$ grows. The second function f_2 has the opposite properties: the "good" property is that it is continuous at every point, but the "bad" property is that $f_2(x)$ does not grow as $|x|$ grows.

The function

$$f_3(x) = \left(x - \sqrt{2}\right)^2$$

(see Figure 3.2) possesses both these "good" properties and has a minimum point on the real line.

We see that the function $F(u)$ has similar "good" properties when the conditions 2) 3) and 4) are satisfied. The continuity property follows by Proposition 3.3 and the growth property follows by Proposition 3.4

Note the minimum point for the function f_3 (which equals $\sqrt{2}$) is not a rational number. If we had minimized over the set $Q = \{\text{all rational numbers}\}$ instead of R, then the function f_3 would not have had any minima. This motivates why

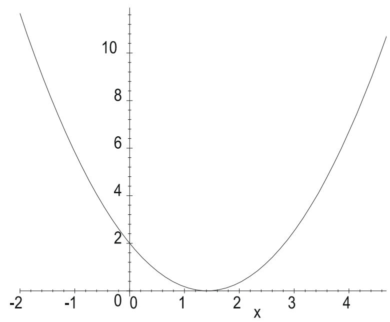


Figure 3.2: The function f_3

the space V, on which we are seeking the minimizer (the solution), has to be a complete space.

Exercise 3.1. *1. Show that all vectors u and v in R^2*

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

satisfy the inequality

$$|u_1v_1 + 2u_2v_2| \le \left(u_1^2 + 2u_2^2\right)^{\frac{1}{2}} \left(v_1^2 + 2v_2^2\right)^{\frac{1}{2}} \tag{3.3}$$

2. *Let $V = R^2$ with the usual scalar product. Consider the problem: Find $u \in V$ such that*

$$a(u, v) = L(v) \text{ for all } v \in V,$$

where $a(u, v) = u_1v_1 + 2u_2v_2$ and $L(v) = 2v_1 + 4v_2$. Use Lax Milgrams Lemma and (3.3) to verify that this problem has a unique solution.

3. *The purpose of this exercise is to give an example of a problem where the students easily may find the solution themselves (in contrast to many of the problems involving partial differential equations). Calculate explicitly the solution of the above problem.*

4. *Consider the minimum problem: Find $u \in V$ such that*

$$F(u) \le F(v) \text{ for all } v \in V,$$

where $F(v) = \frac{1}{2}a(v, v) - L(v)$. Calculate explicitly the solution of this problem. Compare with the solution you found above. Any comments?

Exercise 3.2. *The purpose of this exercise is to show what might happen if not all the conditions 2-4 are satisfied. As in Exercise 3.1, let* $V = R^2$. *Moreover let* $a(u,v) = u_1v_1 + u_1v_2 + u_2v_1 + u_2v_2$.

1. *Let* $L(v) = 0$ *for all* $v \in V$. *Show that both vectors*

$$u = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   are solutions to the problem:

$$a(u,v) = L(v) \text{ for all } v \in V.$$

2. *Let* $L(v) = v_1$ *for all* $v \in V$. *Show that there are no solutions* u *to the problem:*

$$a(u,v) = L(v) \text{ for all } v \in V.$$

3. *Apparently, the above two problems did not satisfy the conclusion of Lax-Milgrams Lemma. Show that* $a(u,v)$ *is not* V*-elliptic, i.e. that condition 3 is not satisfied.*

3.2. The abstract theory for the Dirichlet problem

Let us now show that the bilinear form $a(.,.)$ and L defined for the Dirichlet problem in (2.5) satisfies the conditions 1), 2), 3) and 4) Proof: Condition 1) is obvious because

$$a(u,\phi) = \int_\Omega \lambda \operatorname{grad} u \cdot \operatorname{grad} \phi \, dx = \int_\Omega \lambda \operatorname{grad} \phi \cdot \operatorname{grad} u \, dx = a(\phi, u).$$

Proof: Condition 2): By Schwarz inequality (1.3)

$$|a(\phi,v)| = \left| \int_\Omega \lambda (\operatorname{grad} \phi) \cdot \operatorname{grad} v \, dx \right| \leq$$

$$\leq \left(\int_\Omega \lambda \left|\operatorname{grad} \phi\right|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \lambda \left|\operatorname{grad} v\right|^2 \, dx \right)^{\frac{1}{2}} \leq$$

$$\leq \left(\int_\Omega \lambda^+ \left|\operatorname{grad} \phi\right|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \lambda^+ \left|\operatorname{grad} v\right|^2 \, dx \right)^{\frac{1}{2}} =$$

$$= \left(\lambda^+\right)^{\frac{1}{2}} \left(\int_\Omega |\operatorname{grad} \phi|^2 \ dx\right)^{\frac{1}{2}} \left(\lambda^+\right)^{\frac{1}{2}} \left(\int_\Omega |\operatorname{grad} v|^2 \ dx\right)^{\frac{1}{2}} =$$

$$= \lambda^+ \left(\int_\Omega |\operatorname{grad} \phi|^2 \ dx\right)^{\frac{1}{2}} \left(\int_\Omega |\operatorname{grad} v|^2 \ dx\right)^{\frac{1}{2}} \leq$$

$$\leq \lambda^+ \left(\int_\Omega \left(|\operatorname{grad} \phi|^2 + |\phi|^2\right) \ dx\right)^{\frac{1}{2}} \left(\int_\Omega \left(|\operatorname{grad} v|^2 + |v|^2\right) \ dx\right)^{\frac{1}{2}} =$$

$$= \lambda^+ \|\phi\|_{W_0^{1,2}(\Omega)} \ \|v\|_{W_0^{1,2}(\Omega)} .$$

By putting $\gamma = \lambda^+$ and $V = W_0^{1,2}(\Omega)$ we get that

$$|a(\phi, v)| \leq \gamma \|\phi\|_V \|v\|_V \quad \forall \phi, v \in V,$$

This proves 2).

Proof: Condition 3):

$$a(\phi,\phi) = \int_\Omega \lambda |\operatorname{grad} \phi|^2 \ dx \geq \int_\Omega \lambda^- |\operatorname{grad} \phi|^2 \ dx = \lambda^- \int_\Omega |\operatorname{grad} \phi|^2 \ dx.$$

By Friedrich's inequality we have that

$$\int_\Omega \phi^2 \ dx \leq C_0 \int_\Omega |\operatorname{grad} \phi|^2 \ dx.$$

Thus

$$\int_\Omega \phi^2 \ dx + \int_\Omega |\operatorname{grad} \phi|^2 \ dx \leq C_0 \int_\Omega |\operatorname{grad} \phi|^2 \ dx + \int_\Omega |\operatorname{grad} \phi|^2 \ dx =$$

$$= (C_0 + 1) \int_\Omega |\operatorname{grad} \phi|^2 \ dx,$$

which gives that

$$\frac{1}{C_0 + 1} \left(\int_\Omega \phi^2 \ dx + \int_\Omega |\operatorname{grad} \phi|^2 \ dx\right) \leq \int_\Omega |\operatorname{grad} \phi|^2 \ dx,$$

i.e.

$$\lambda^- \frac{1}{C_0 + 1} \left(\int_\Omega \phi^2 \ dx + \int_\Omega |\operatorname{grad} \phi|^2 \ dx\right) \leq \lambda^- \int_\Omega |\operatorname{grad} \phi|^2 \ dx.$$

Hence we obtain that

$$a(\phi,\phi) \geq \lambda^- \int_\Omega |\mathrm{grad}\,\phi|^2\ dx \geq \lambda^- \frac{1}{C_0+1}\left(\int_\Omega \phi^2\ dx + \int_\Omega |\mathrm{grad}\,\phi|^2\ dx\right) =$$

$$= \lambda^- \frac{1}{C_0+1} \|\phi\|^2_{W_0^{1,2}(\Omega)}\,.$$

By putting

$$\alpha = \lambda^- \frac{1}{C_0+1}$$

and

$$V = W_0^{1,2}(\Omega)$$

we obtain that

$$a(\phi,\phi) \geq \alpha \|\phi\|_V^2\,,$$

which proves 3).

Proof of Condition 4):

$$L(\phi) = \int_\Omega \phi\ f\ dx$$

$$|L(\phi)| = \left|\int_\Omega \phi\ f\ dx\right| \leq \int_\Omega |\phi\ f|\, dx \leq \left(\int_\Omega |\phi|^2\ dx\right)^{\frac{1}{2}} \left(\int_\Omega |f|^2\ dx\right)^{\frac{1}{2}} \leq$$

$$\leq \left(\int_\Omega |\phi|^2 + |\mathrm{grad}\,\phi|^2\ dx\right)^{\frac{1}{2}} \left(\int_\Omega |f|^2\ dx\right)^{\frac{1}{2}} = \|\phi\|_{W_0^{1,2}(\Omega)}\,\Lambda$$

where $\Lambda = \left(\int_\Omega |f|^2\ dx\right)^{\frac{1}{2}}$ (In the second inequality from the left we have used Schwarz inequality (1.4)). Let $W_0^{1,2}(\Omega) = V$, then $|L(\phi)| \leq \Lambda \|\phi\|_V$, which proves 4).

Exercise 3.3. *In this task we consider the following partial differential equation (strong formulation):*

$$\begin{cases} \dfrac{\partial^2 u(x)}{\partial x_1^2} + \dfrac{\partial^2 u(x)}{\partial x_2^2} - \dfrac{\partial u(x)}{\partial x_1} - \dfrac{\partial u(x)}{\partial x_2} = -\sin x_1, \quad x = (x_1,x_2) \in \Omega, \\ \\ u = 0 \ \mathit{on}\ \partial\Omega, \end{cases}$$

where Ω is the open square $\Omega = \langle 0,1\rangle^2$. Derive (this means: verify) the following weak formulation of the above problem: Find $u \in W_0^{1,2}(\Omega)$ such that

$$a(u,v) = L(v) \text{ for all } v \in W_0^{1,2}(\Omega), \tag{3.4}$$

where

$$\begin{aligned} a(u,v) &= \int_\Omega \operatorname{grad} u(x) \cdot \operatorname{grad} v(x)\ dx\ + \int_\Omega (\frac{\partial u(x)}{\partial x_1} + \frac{\partial u(x)}{\partial x_2}) v(x)\ dx, \\ L(v) &= \int_\Omega v(x) \sin x_1 dx. \end{aligned}$$

Next, prove that $L(v)$ is continuous.

Exercise 3.4. *In this task we consider the following Dirichlet problem (weak formulation): Find $u \in W_0^{1,2}(\Omega)$ such that*

$$a(u,v) = L(v) \text{ for all } v \in W_0^{1,2}(\Omega), \tag{3.5}$$

where

$$\begin{aligned} a(u,v) &= \int_\Omega \left[4\frac{\partial u}{\partial x_1}, 2\frac{\partial u}{\partial x_2}\right] \cdot \operatorname{grad} v\, dx\ (= \int_\Omega 4\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + 2\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\ dx), \\ L(v) &= \int_\Omega v dx \end{aligned}$$

and $\Omega = \langle 0,1\rangle^2$ .Show that there exists a unique solution of (3.5).

Exercise 3.5. *Given the fact that $W^{1,2}(\Omega)$ is a Hilbert space (this fact will not be proven in this report). Let V be the space:*

$$V = \left\{ u \in W^{1,2}(\Omega) : \int_\Omega u\, dx = 0 \right\}$$

(with the same norm as that for $W^{1,2}(\Omega)$).

1. *Show that V is a vector space.*
2. *Show that V is complete and conclude that V is a Hilbert space (this task is more difficult, hint: use the inequality (1.5)).*

3. *Assume $f \in L^2(\Omega)$ and that the boundary of Ω is Lipschitz continuous. Consider the "modified" Neumann problem: Find $u \in V$ such that*

$$a(u, \phi) = L(\phi), \quad \forall \phi \in V, \tag{3.6}$$

where (as before)

$$a(u, \phi) = \int_\Omega \lambda \operatorname{grad} u \cdot \operatorname{grad} \phi \, dx$$

and

$$L(\phi) = \int_\Omega \phi f \, dx.$$

Show that (3.6) has a unique solution.

Exercise 3.6. *Prove that if $\int_\Omega f \, dx = 0$ and the boundary of Ω is Lipschitz continuous, then the solution of the Neumann problem (2.10) exists and is unique within an additive constant (Hint: Observe first that if $\phi \in W^{1,2}(\Omega)$ then the function $v = \phi - (\frac{1}{|\Omega|} \int_\Omega \phi dx)$ belongs to the space V defined in the previous exercise, where $|\Omega|$ is the "area" of Ω, i.e. $|\Omega| = \int_\Omega dx$).*

Exercise 3.7. *It turns out that the condition $\int_\Omega f \, dx = 0$ in the previous exercise is a necessary condition to obtain a solution.*

1. *show this fact theoretically*

2. *give a physical explanation of this fact*

4. Effective properties of periodic structures

Consider the stationary heat-conduction equation for a periodic structure of the form

$$\operatorname{div} \lambda \operatorname{grad} u = f, \quad x \in R^2,$$

where $\lambda(x)$ is periodic relative to a small cell Y and bounded between the positive constants λ^- and λ^+ (see Figure 4.1).When the structure is symmetric it is possible to use so-called G-convergence results to show that the effective behavior of the above partial differential equation is close to the homogeneous equation

$$\operatorname{div} \begin{bmatrix} \lambda_{1,\text{eff}} & 0 \\ 0 & \lambda_{2,\text{eff}} \end{bmatrix} \operatorname{grad} u = f,$$

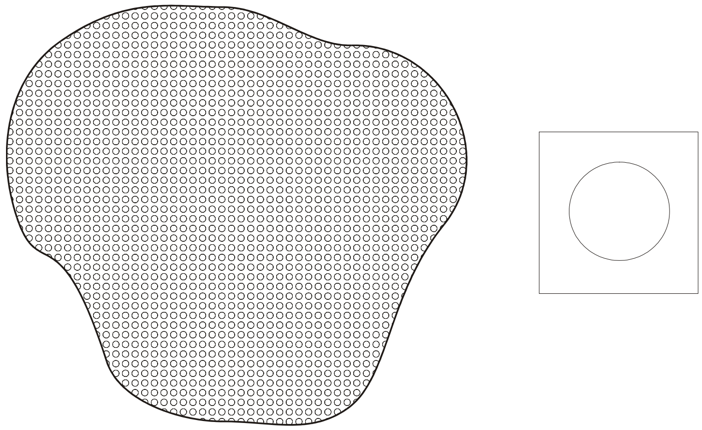


Figure 4.1: Periodic structure

where

$$\lambda_{i,\text{eff}} = \frac{1}{|Y|} \int_Y \lambda(x) \left(\frac{\partial u_{\text{per}}}{\partial x_i} + 1 \right) dx, \tag{4.1}$$

where u_{per} is the solution of the following periodic problem, called the *cell-problem*: Find $u_{\text{per}} \in W^{1,2}_{\text{per}}(Y)$ such that

$$\int_Y \lambda \ \operatorname{grad} u_{\text{per}} \cdot \operatorname{grad} \phi \ dx = - \int_Y \lambda \ e_i \cdot \operatorname{grad} \phi \ dx, \quad \forall \phi \in W^{1,2}_{\text{per}}(Y),$$

where e_i is the usual basis-vector in the *i-th* direction. The parameter $\lambda_{i,\text{eff}}$ is called *the effective conductivity* in the *i-th* direction.

Theorem 4.1. *The cell-problem has a solution $u_{per} \in W^{1,2}_{per}(Y)$. This solution is unique within an arbitrary constant, i.e. if $u_{\text{per},1}$ and $u_{\text{per},2}$ are solutions then $u_{\text{per},1} - u_{\text{per},2} =$ constant.*

Proof: We put

$$a(u_{\text{per}}, \phi) = \int_Y \lambda \operatorname{grad} u_{\text{per}} \cdot \operatorname{grad} \phi dx \tag{4.2}$$

and

$$L(\phi) = - \int_Y \lambda \ e_i \cdot \operatorname{grad} \phi \ dx. \tag{4.3}$$

Thus the cell problem can be written as: Find u_{per} such that

$$a(u_{\rm per}, \phi) = L(\phi), \quad \forall \phi \in W^{1,2}_{\rm per}(Y).$$

We first let V be the space

$$V = \left\{ \phi \in W^{1,2}_{per}(Y) : \int_Y \phi \, dx = 0 \right\}$$

with the usual norm in $W^{1,2}(Y)$ and show that the problem

$$a(u_{\rm per}, \phi) = L(\phi), \quad \forall \phi \in V \tag{4.4}$$

has a unique solution. This is done by first proving that $a(u_{\rm per}, \phi)$ and $L(\phi)$ have the properties 2), 3) and 4). Since V is a Hilbert space (see Exercise 3.5), the existence and uniqueness will then follow by Theorem 3.1.

Proof of condition 2): By Schwarz inequality (1.3)

$$\begin{aligned}
|a(\phi, v)| = \left| \int_\Omega \lambda \, (\operatorname{grad} \phi) \cdot \operatorname{grad} v \, dx \right| \leq \\
\leq \left(\int_\Omega \lambda \, |\operatorname{grad} \phi|^2 \; dx \right)^{\frac{1}{2}} \left(\int_\Omega \lambda \, |\operatorname{grad} v|^2 \; dx \right)^{\frac{1}{2}} \leq \\
\leq \left(\int_\Omega \lambda^+ \, |\operatorname{grad} \phi|^2 \; dx \right)^{\frac{1}{2}} \left(\int_\Omega \lambda^+ \, |\operatorname{grad} v|^2 \; dx \right)^{\frac{1}{2}} = \\
= (\lambda^+)^{\frac{1}{2}} \left(\int_\Omega |\operatorname{grad} \phi|^2 \; dx \right)^{\frac{1}{2}} (\lambda^+)^{\frac{1}{2}} \left(\int_\Omega |\operatorname{grad} v|^2 \; dx \right)^{\frac{1}{2}} = \\
= \lambda^+ \left(\int_\Omega |\operatorname{grad} \phi|^2 \; dx \right)^{\frac{1}{2}} \left(\int_\Omega |\operatorname{grad} v|^2 \; dx \right)^{\frac{1}{2}} \leq \\
\leq \lambda^+ \left(\int_\Omega \left(|\operatorname{grad} \phi|^2 + |\phi|^2 \right) \; dx \right)^{\frac{1}{2}} \left(\int_\Omega \left(|\operatorname{grad} v|^2 + |v|^2 \right) \; dx \right)^{\frac{1}{2}} = \\
= \lambda^+ \, \|\phi\|_V \; \|v\|_V \, .
\end{aligned}$$

Thus we get that

$$|a(\phi, v)| \leq \gamma \, \|\phi\|_V \, \|v\|_V \quad \forall \phi, v \in V,$$

This proves 2).

Proof of condition 3):

$$a(\phi,\phi) = \int_Y \lambda \left|\operatorname{grad}\phi\right|^2\ dx \geq \int_Y \lambda^- \left|\operatorname{grad}\phi\right|^2\ dx = \lambda^- \int_Y \left|\operatorname{grad}\phi\right|^2\ dx.$$

By Poincaré's inequality we have that

$$\int_Y \left|\phi^2\right|\ dx \leq C_0 \left\{ \left(\int_Y \phi\ dx \right)^2 + \int_Y \left|\operatorname{grad}\phi\right|^2\ dx \right\}.$$

Since $V = \left\{ \phi \in W^{1,2}_{\mathrm{per}}(Y) : \int_Y \phi\ dx = 0 \right\}$, we have

$$\int_Y \left|\phi\right|^2\ dx \leq C_0 \left\{ 0 + \int_Y \left|\operatorname{grad}\phi\right|^2\ dx \right\} = C_0 \int_Y \left|\operatorname{grad}\phi\right|^2\ dx,$$

and so

$$\int_Y \left|\phi\right|^2\ dx + \int_Y \left|\operatorname{grad}\phi\right|^2\ dx \leq C_0 \left\{ \int_Y \left|\operatorname{grad}\phi\right|^2\ dx \right\} + \int_Y \left|\operatorname{grad}\phi\right|^2\ dx,$$

i.e.

$$\int_Y \left|\phi\right|^2\ dx + \int_Y \left|\operatorname{grad}\phi\right|^2\ dx \leq (C_0+1) \left\{ \int_Y \left|\operatorname{grad}\phi\right|^2\ dx \right\},$$

which gives that

$$\frac{1}{C_0+1} \left(\int_Y \left|\phi\right|^2\ dx + \int_Y \left|\operatorname{grad}\phi\right|^2\ dx \right) \leq \int_Y \left|\operatorname{grad}\phi\right|^2\ dx.$$

Thus,

$$\lambda^- \frac{1}{C_0+1} \left(\int_Y \phi^2\ dx + \int_Y \left|\operatorname{grad}\phi\right|^2\ dx \right) \leq \lambda^- \int_Y \left|\operatorname{grad}\phi\right|^2\ dx.$$

Hence we obtain that

$$a(\phi,\phi) \geq \lambda^- \int_Y \left|\operatorname{grad}\phi\right|^2\ dx \geq \lambda^- \frac{1}{C_0+1} \left(\int_Y \left|\phi\right|^2\ dx + \int_Y \left|\operatorname{grad}\phi\right|^2\ dx \right) =$$

$$= \lambda^- \frac{1}{C_0+1} \left\|\phi\right\|_V^2.$$

By putting $\alpha = \lambda^{-} \frac{1}{C_0+1}$ we obtain that

$$a(\phi, \phi) \geq \alpha \|\phi\|_V^2$$

which proves 3).

Proof: Condition 4):

$$L(\phi) = -\int_Y \lambda e_i \cdot \operatorname{grad} \phi \, dx$$

$$|L(\phi)| = \left| -\lambda \int_Y e_i \cdot \operatorname{grad} \phi \, dx \right| = \left| \int_Y \lambda e_i \cdot \operatorname{grad} \phi \, dx \right| \leq$$

$$\leq \left(\int_Y \lambda \, |e_i|^2 \, dx \right)^{\frac{1}{2}} \left(\int_Y \lambda \, |\operatorname{grad} \phi|^2 \, dx \right)^{\frac{1}{2}} \leq$$

$$\leq \left(\int_Y \lambda^+ \, |e_i|^2 \, dx \right)^{\frac{1}{2}} \left(\int_Y \lambda^+ \, |\operatorname{grad} \phi|^2 \, dx \right)^{\frac{1}{2}} =$$

$$= \left(\lambda^+\right)^{\frac{1}{2}} \left(\lambda^+\right)^{\frac{1}{2}} \left(\int_Y |e_i|^2 \, dx \right)^{\frac{1}{2}} \left(\int_Y |\operatorname{grad} \phi|^2 \, dx \right)^{\frac{1}{2}} =$$

$$= \left(\lambda^+\right) \left(\int_Y |e_i|^2 \, dx \right)^{\frac{1}{2}} \left(\int_Y |\operatorname{grad} \phi|^2 \, dx \right)^{\frac{1}{2}} \leq$$

$$\leq \left(\lambda^+\right) \left(\int_Y |e_i|^2 \, dx \right)^{\frac{1}{2}} \left(\int_Y |\operatorname{grad} \phi|^2 \, dx + \int_Y |\phi|^2 \, dx \right)^{\frac{1}{2}}$$

$$= \left(\lambda^+\right) \left(\int_Y |e_i|^2 \, dx \right)^{\frac{1}{2}} \|\phi\|_V^2$$

(in the first line we have used Schwarz inequality (1.4). By putting

$$\Lambda = \left(\lambda^+\right) \left(\int_Y |e_i|^2 \, dx \right)^{\frac{1}{2}},$$

we obtain

$$|L(\phi)| \leq \|\phi\|_V \Lambda,$$

which proves 4). Thus we have proved that the problem

$$a(u_{\text{per}}, \phi) = L(\phi), \quad \forall \phi \in V$$

has a unique solution. But note that the solution u_{per} must also be a solution to the cell problem, because if $\phi \in W^{1,2}_{\text{per}}(Y)$ then the function ϕ_0, defined by

$$\phi_0 = \phi - K,$$

where

$$K = \frac{1}{|Y|} \int_Y \phi dx,$$

must belong to V (since

$$\int_Y \phi_0 dx = \int_Y \phi dx - \int_Y K dx = \int_Y \phi dx - K\,|Y| = \int_Y \phi dx - \int_Y \phi dx = 0).$$

Hence,

$$a(u_{\text{per}}, \phi_0) = L(\phi_0).$$

Noting that

$$a(u_{\text{per}}, \phi) = a(u_{\text{per}}, \phi_0 + K) = \int_\Omega \lambda\,(\operatorname{grad} u_{\text{per}}) \cdot \operatorname{grad}(\phi_0 + K)\; dx$$

$$= \int_\Omega \lambda\,(\operatorname{grad} u_{\text{per}}) \cdot \operatorname{grad} \phi_0\; dx = a(u_{\text{per}}, \phi_0),$$

and similarly that

$$L(\phi) = L(\phi_0),$$

we find that

$$a(u_{\text{per}}, \phi) = L(\phi)$$

Next, let $u_{\text{per},1}$ and $u_{\text{per},2}$ be solutions to (4.4) then

$$a(u_{\text{per},1}, \phi) = L(\phi),$$

$$a(u_{\text{per},2}, \phi) = L(\phi),$$

so

$$a(u_{\text{per},1}, \phi) - a(u_{\text{per},2}, \phi) = 0, \quad \forall \phi \in W^{1,2}_{\text{per}}(Y).$$

Hence, using the linearity of $a(.,.)$ we obtain that

$$a\left((u_{\text{per},1} - u_{\text{per},2}), \phi\right) = 0, \quad \forall \phi \in W^{1,2}_{\text{per}}(Y).$$

Particularly this holds for

$$\phi = u_{\text{per},1} - u_{\text{per},2},$$

i.e.

$$a\left((u_{\text{per},1} - u_{\text{per},2}), (u_{\text{per},1} - u_{\text{per},2})\right) = 0,$$

and according to (4.2) we have that

$$a\left((u_{\text{per},1} - u_{\text{per},2}), (u_{\text{per},1} - u_{\text{per},2})\right) = \int_Y \lambda \ \left|\text{grad}(u_{\text{per},1} - u_{\text{per},2})\right|^2 = 0.$$

Thus

$$\int_Y \lambda \ \left|\text{grad}(u_{\text{per},1} - u_{\text{per},2})\right|^2 \ dx = 0,$$

which gives us that

$$\text{grad}\left(u_{\text{per},1} - u_{\text{per},2}\right) = 0, \quad \forall\, x \in Y.$$

Thus $u_{\text{per},1} - u_{\text{per},2} =$ constant. This completes the proof.

We also state the following theorem (without proof):

Theorem 4.2. *It holds that*

$$\lambda_{i,\text{eff}} = \frac{1}{|Y|} \min_{\phi \in W^{1,2}_{\text{per}}(Y)} \int_Y \lambda \left|\text{grad}\,\phi + e_i\right|^2 \ dx.$$

Exercise 4.1. *Show that*

$$\int_Y \lambda \left|\text{grad}\,\phi + e_i\right|^2 \ dx = 2F(\phi) + \int_Y \lambda dx$$

where

$$F(\phi) = \frac{1}{2} a(\phi, \phi) - L(\phi)$$

$a(\phi, \phi)$ and $L(\phi)$ being given by (4.2) and (4.3). Next, use this, Theorem 3.2 and the definition of $\lambda_{i,\text{eff}}$ (4.1) to prove Theorem 4.2.

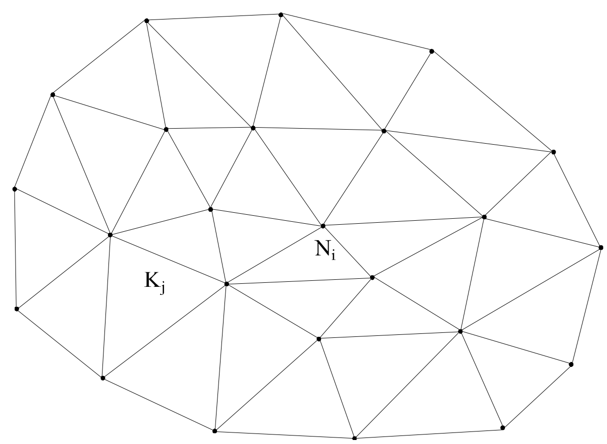


Figure 5.1: Finite triangulation of the body Ω.

5. Finite Element Method

In order to formulate the finite element method starting from the weak formulation we have to define/construct a finite-dimensional subspace V_h of V. For simplicity and the sake of illustration we consider the Dirichlet problem (where $V = W_0^{1,2}(\Omega)$) for the conductivity problem (2.1) and assume that the body Ω is divided into triangles, where $\partial\Omega$ is a polygonal curve, see Figure 5.1. We must also introduce a mesh-parameter

$$h = \max_{K \in T_h} \text{diam}\,(K), \text{diam}\,(K) = \text{diameter of } K = \text{longest side of } K,$$

where T_h is the set $T_h = \{K_1, ..., K_m\}$ of non-overlapping triangles K_i such that

$$\Omega = \bigcup_{K \in T_h} K = K_1 \cup K_2 ... \cup K_m.$$

We define V_h as

$$V_h = \{w \mid w = 0 \text{ on } \partial\Omega, \text{ continuous on } \Omega, \, w|_K \text{ is a first order polynomial for all } K \in T_h\}$$

Note that $w|_K$ denotes the restriction of w to K (i.e. the function which is defined only on K and which is equal to w on K), i.e. V_h consists of all continuous functions which is a first order polynomial on each triangle K and vanishes on $\partial\Omega$. Observe that $V_h \subset V = W_0^{1,2}(\Omega)$

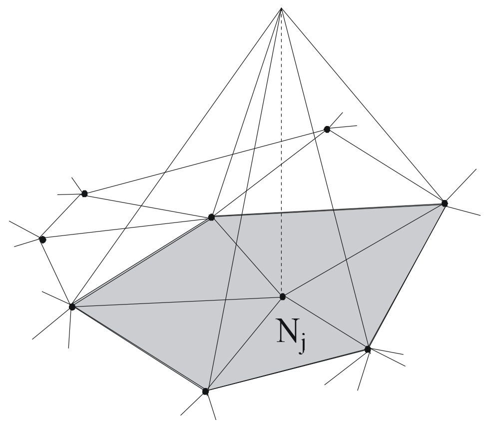


Figure 5.2: Basis function v_j.

Remark 2. *More generally V is a function space (Hilbert space) which takes into account the boundary conditions (in the case of elasticity problems these functions are vector valued), the "triangles" in T_h may be polygons and $w|_K$ may be a (possibly vector-valued) polynomial..*

To describe a function $w \in V_h$ we will use the values $w(N_i)$ of w as parameters, where $N_i,\ i = 1, ..., M$ are nodes (see Figure 5.1) of triangles T_h. Since $w = 0$ on $\partial\Omega$ (i.e. known values) we may exclude the nodes on the boundary $\partial\Omega$. Consider the basis function $v_j \in V_h$ illustrated in Figure 5.2, where $j = 1, ..., M$, defined as

$$v_j(N_i) = \delta_{ij} \equiv \left\{ \begin{array}{c} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{array} \right\} \ i, j = 1, ..., M.$$

The set of points x for which $v_j(x) \neq 0$ is called the support of v_j. For the function v_j this set consists of the triangles with common node N_j (see the shaded area in Figure 5.2). A function $w \in V_h$ has the representation

$$w(x) = \sum_{j=1}^{M} \xi_j v_j(x),\ \xi_j = w(N_j),\ \text{for } x \in \Omega \cup \partial\Omega.$$

We are now in the position to formulate the following finite element method for the variational formulation of the Dirichlet problem (based on the variational formulation 2.5): Find $u_h \in V_h$ such that

$$a(u_h, v) = L(v) \quad \forall v \in V_h, \tag{5.1}$$

Since $u_h = \xi_1 v_1 + ... + \xi_m v_m$, (5.1) holds and $a(.,.)$ is a bilinear form we obtain that

$$\begin{aligned} L(v_1) &= a(u_h, v_1) = a(\xi_1 v_1 + ... + \xi_m v_m, v_1) = \xi_1 a(v_1, v_1) + ... + \xi_m a(v_m, v_1) \\ L(v_2) &= a(u_h, v_2) = a(\xi_1 v_1 + ... + \xi_m v_m, v_2) = \xi_1 a(v_1, v_2) + ... + \xi_m a(v_m, v_2) \\ &\vdots \\ L(v_m) &= a(u_h, v_m) = a(\xi_1 v_1 + ... + \xi_m v_m, v_m) = \xi_1 a(v_1, v_m) + ... + \xi_m a(v_m, v_m) \end{aligned}$$

Thus

$$\begin{bmatrix} L(v_1) \\ L(v_2) \\ \vdots \\ L(v_m) \end{bmatrix} = \begin{bmatrix} a(v_1, v_1) & a(v_2, v_1) & \cdots & a(v_m, v_1) \\ a(v_1, v_2) & a(v_2, v_2) & & \\ \vdots & & \ddots & \\ a(v_1, v_m) & & & a(v_m, v_m) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix},$$

i.e.

$$A\xi = b, \tag{5.2}$$

where $A = (a_{ij})$ is the $m \times m$ "stiffness matrix" with elements $(a_{ij}) = a(v_i, v_j)$ and $\xi = (\xi_i)$ and $b = (b_i)$ are m-vectors with elements $\xi_i = u_h(N_i)$, $b_i = L(v_i)$. Thus solving this system with respect to ξ directly gives us the solution

$$u_h = \xi_1 v_1 + ... + \xi_m v_m. \tag{5.3}$$

Observe that problem (5.1) (which is equivalent with (5.2) has a unique solution. This follows by letting $V_h = V$ in Theorem 3.1. It is important to note that u_h is the best approximation in V_h of the solution u with respect to the energy norm. This means that

$$\|u - u_h\|_a \leq \|u - v\|_a \quad \forall v \in V_h \tag{5.4}$$

where

$$\|v\|_a = \sqrt{a(v, v)}.$$

Exercise 5.1. *1. Prove that $a(u - u_h, u_h - v) = 0$ for all $v \in V_h$.*

2. Use that $u - v = (u - u_h) + (u_h - v)$ to show that

$$\|u - v\|_a^2 = \|u - u_h\|_a^2 + 2a(u - u_h, u_h - v) + \|u_h - v\|_a^2$$

3. Use the above exercises to show the inequality (5.4).

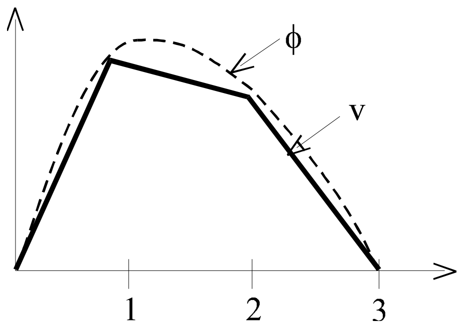


Figure 5.3: The functions ϕ and v.

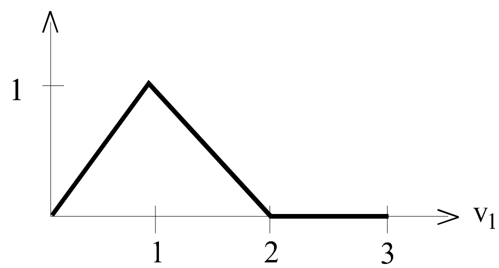


Figure 5.4: Basis function v_1.

5.1. Example 1 (dimension 1)

We will now show how to find the FEM-solution in the one-dimensional heat conduction problem with Dirichlet boundary condition ($V = W_0^{1,2}(\Omega)$). We put $\Omega = [0,3]$ and let $T_h = \{[0,1], [1,2], [2,3]\}$. In Figure 5.3 we have illustrated a general function $\phi \in V$ and a function $v \in V_h$. The (two) basis functions for V_h are illustrated in Figure 5.4 and 5.5.

We recall that

$$a(u,v) = \int_\Omega \lambda \operatorname{grad} u \cdot \operatorname{grad} v \, dx$$

and

$$L(v) = \int_\Omega f v \, dx$$

where f is an internal heating source which in our case has the form illustrated in Figure 5.6. We put $\lambda = 1$ for simplicity.Since this example is one-dimensional

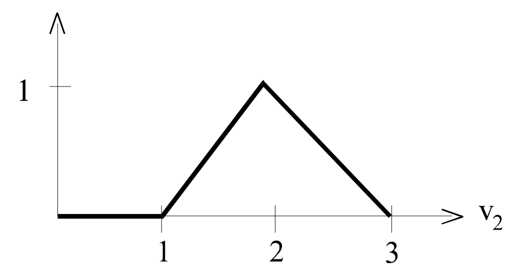


Figure 5.5: Basis function v_2.

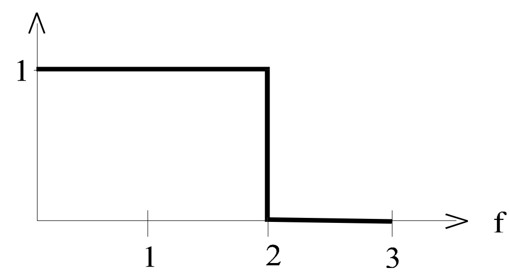


Figure 5.6: The internal source f.

$$a(u,v) = \int_\Omega \lambda \operatorname{grad} u \cdot \operatorname{grad} v \; dx = \int_\Omega \lambda \frac{du}{dx} \cdot \frac{dv}{dx} dx.$$

In this case the linear system

$$A\xi = b$$

takes the form:

$$\begin{bmatrix} a(v_1, v_1) & a(v_1, v_2) \\ a(v_2, v_1) & a(v_2, v_2) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} L(v_1) \\ L(v_2) \end{bmatrix}$$

We need the expressions for $a(v_1, v_1)$, $a(v_2, v_1)$, $a(v_1, v_2)$, $a(v_2, v_2)$, $L(v_1)$,and $L(v_2)$. By considering Figure 5.4 we see that

$$a(v_1, v_1) = \int_0^3 \lambda \frac{dv_1}{dx} \cdot \frac{dv_1}{dx} dx =$$

$$= \int_0^1 \lambda \frac{dv_1}{dx} \cdot \frac{dv_1}{dx} dx + \int_1^2 \lambda \frac{dv_1}{dx} \cdot \frac{dv_1}{dx} dx + \int_2^3 \lambda \frac{dv_1}{dx} \cdot \frac{dv_1}{dx} dx =$$

$$= \int_0^1 1 \cdot 1 \cdot 1 \; dx + \int_1^2 1 \cdot (-1) \cdot (-1) \; dx + \int_2^3 1 \cdot 0 \cdot 0 \; dx =$$

$$= \int_0^1 1 \; dx + \int_1^2 1 \; dx + \int_2^3 0 \; dx = [x]_0^1 + [x]_1^2 + 0 = 1 + 1 + 0 = 2.$$

Using both Figure 5.5 and 5.4 we obtain that

$$a(v_2, v_1) = \int_0^3 \lambda \frac{dv_2}{dx} \cdot \frac{dv_1}{dx} dx =$$

$$= \int_0^1 \lambda \frac{dv_2}{dx} \cdot \frac{dv_1}{dx} dx + \int_1^2 \lambda \frac{dv_2}{dx} \cdot \frac{dv_1}{dx} dx + \int_2^3 \lambda \frac{dv_2}{dx} \cdot \frac{dv_1}{dx} dx =$$

$$= \int_0^1 1 \cdot 0 \cdot 1 \; dx + \int_1^2 1 \cdot 1 \cdot (-1) \; dx + \int_2^3 1 \cdot 0 \cdot (-1) \; dx =$$

$$= \int_0^1 0 \; dx + \int_1^2 -1 \; dx + \int_2^3 0 \; dx = 0 + [-x]_1^2 + 0 = 0 - 1 + 0 = -1.$$

Using the symmetry fact of $a(v_1, v_2)$ (or by considering Figure 5.5 and 5.4) we find that

$$a(v_2, v_1) = a(v_1, v_2) = \int_0^3 \lambda \frac{dv_2}{dx} \cdot \frac{dv_1}{dx} dx = -1.$$

By considering Figure 5.5 we see that

$$a(v_2, v_2) = \int_0^3 \lambda \frac{dv_2}{dx} \cdot \frac{dv_2}{dx} dx =$$

$$= \int_0^1 \lambda \frac{dv_2}{dx} \cdot \frac{dv_2}{dx} dx + \int_1^2 \lambda \frac{dv_2}{dx} \cdot \frac{dv_2}{dx} dx + \int_2^3 \lambda \frac{dv_2}{dx} \cdot \frac{dv_2}{dx} dx =$$

$$= \int_0^1 1 \cdot 0 \cdot 0 \; dx + \int_1^2 1 \cdot 1 \cdot 1 \; dx + \int_2^3 1 \cdot (-1) \cdot (-1) \; dx =$$

$$= \int_0^1 0 \; dx + \int_1^2 1 \; dx + \int_2^3 1 \; dx = 0 + [x]_1^2 + [x]_2^3 = 0 + 1 + 1 = 2.$$

To find $L(v_1)$ we split the integral in 3 parts:

$$L(v_1) = \int_0^3 f \cdot v_1 \; dx = \int_0^1 f \cdot v_1 \; dx + \int_1^2 f \cdot v_1 \; dx + \int_2^3 f \cdot v_1 \; dx.$$

We need to know the expression for v_1 is in each interval, which is found as follows: From Figure 5.4 we see that

$$v_1 = x, \quad \text{for } 0 \leq x \leq 1$$

(equal to the linear line with the slope = 1). Since the slope is -1 in the next interval, we use the formula

$$\frac{y - y_1}{x - x_1} = -1$$

and the fact that the point $(x_1, y_1) = (1, 1)$ lies on the graph of v_1 to obtain

$$\frac{y - 1}{x - 1} = -1 \iff y = -x + 2.$$

This shows that

$$v_1 = -x + 2, \quad \text{for } 1 \leq x \leq 2.$$

Moreover,

$$v_1 = 0, \quad \text{for } 2 \leq x \leq 3.$$

By Figure 5.6 we see that

$$f = 1, \quad \text{for } 0 \leq x \leq 1,$$

$$f = 1, \quad \text{for } 1 \leq x \leq 2$$

and

$$f = 0, \quad \text{for } 2 \leq x \leq 3.$$

Thus,

$$L(v_1) = \int_0^3 f \cdot v_1 \, dx = \int_0^1 1 \cdot x \, dx + \int_1^2 1 \cdot (-x + 2) \, dx + \int_2^3 0 \cdot 0 \, dx =$$

$$= \int_0^1 x \, dx + \int_1^2 (-x + 2) \, dx + \int_2^3 0 \, dx = \left[\frac{1}{2}x^2\right]_0^1 + \left[-\frac{1}{2}x^2 + 2x\right]_1^2 + 0 =$$

$$= \frac{1}{2} + \frac{1}{2} + 0 = 1.$$

Similarly, we have that

$$L(v_2) = \int_0^3 f \cdot v_2 \, dx = \int_0^1 f \cdot v_2 \, dx + \int_1^2 f \cdot v_2 \, dx + \int_2^3 f \cdot v_2 \, dx.$$

We must find v_2 in each interval, and this done similarly as above for v_1: Summing up we obtain that

$$v_2 = 0, \quad \text{for } 0 \le x \le 1,$$
$$v_2 = x - 1, \quad \text{for } 1 \le x \le 2,$$
$$v_2 = -x + 3, \quad \text{for } 2 \le x \le 3.$$

Thus

$$L(v_2) = \int_0^3 f \cdot v_2 \, dx = \int_0^1 1 \cdot 0 \, dx + \int_1^2 1 \cdot (x-1) \, dx + \int_2^3 0 \cdot (-x+3) \, dx =$$

$$= \int_0^1 0 \, dx + \int_1^2 (x-1) \, dx + \int_2^3 0 \, dx = 0 + \left[\frac{1}{2} x^2 - x \right]_1^2 + 0 = 0 + \frac{1}{2} + 0 = \frac{1}{2}.$$

Now we have got all the parameters we need for solving the system

$$A\xi = b,$$

which takes the form

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix},$$

The solution is

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{4}{6} \end{bmatrix}.$$

$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right)$$

(Recall the formula

$$\begin{bmatrix} a & d \\ c & b \end{bmatrix}^{-1} = \frac{1}{ab - dc} \begin{bmatrix} b & -d \\ -c & a \end{bmatrix})$$

Thus, we have found the approximative solution

$$u_h = \xi_1 v_1 + \xi_2 v_2 = \frac{5}{6} v_1 + \frac{4}{6} v_2$$

of the problem, see Figure 5.7. The solution u_h is only an approximation of the exact solution u and can be improved by dividing Ω into more intervals. This

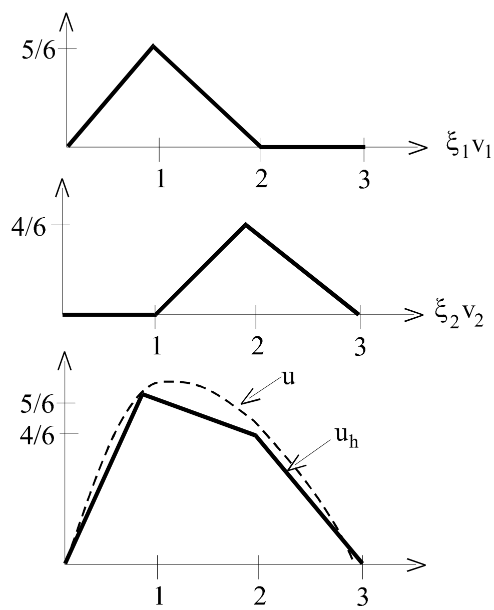


Figure 5.7: The solution of the problem.

means that we will have to deal with not only two basis functions but several more which gives us a more complicated system. We get m equations to solve. The larger m the more accurate the solution will be. In two or three dimensional problems the complexity becomes even greater. That is why we are very grateful for the powerful computers that can solve these kind of problems numerically very fast.

Exercise 5.2. *Put* $\lambda = 1$, $\Omega = [0, 6]$ *and let* $T_h = \{[0, 2], [2, 4], [4, 6]\}$. *Moreover, let* V_h *be the two dimensional function space with basis functions* v_1 *and* v_2 *as illustrated in Figure 5.8. Find the best approximation* u_h *in* V_h *of the exact solution* u *for the Dirichlet problem when* f *is given as in Figure 5.8.*

Exercise 5.3. *Let* λ, Ω *and* f *be the same as in the previous exercise, but now let*

$$T_h = \{[0, 1], [1, 2], [2, 3], [3, 4], [4, 5], [5, 6]\}$$

and V_h *be the 5-dimensional function space with basis functions* $\{v_i\}$ *as illustrated in Figure 5.9. Formulate the corresponding linear system:*

$$A\xi = b.$$

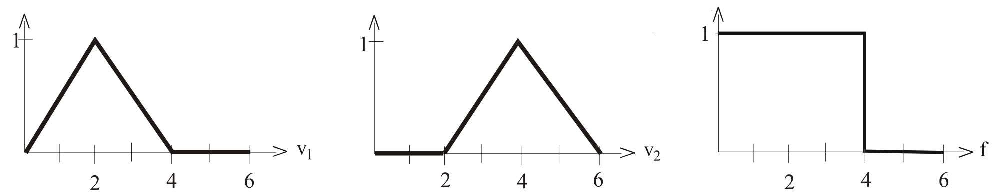


Figure 5.8: Basis functions and source.

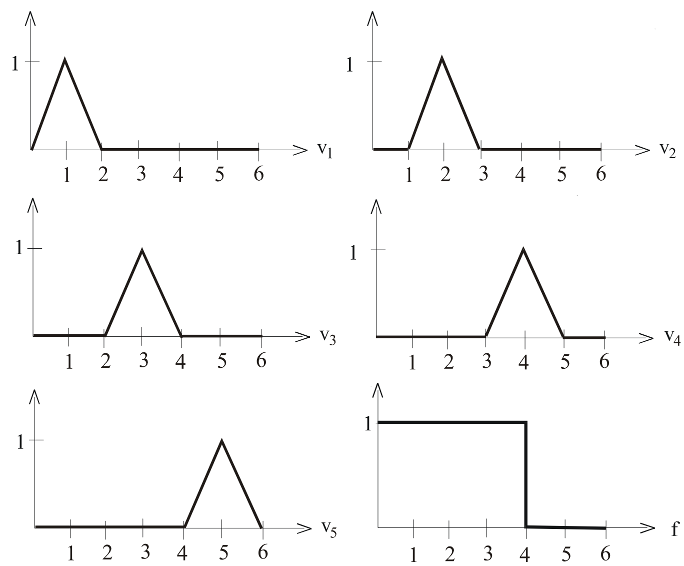


Figure 5.9: Basis functions and source.

Exercise 5.4. *It is important to remark that for the one dimensional case one usually can find the exact solution of the conductivity problem without any FEM-calculation (the reason why we still do this is usually only for illustration purposes). Find the exact solution u for the Dirichlet problem when $f = -1$, $\Omega = [0,1]$ and $\lambda = 1$ by solving the strong formulation of the problem directly.*

Exercise 5.5. *Try to do the same as in Exercise 5.4 but for the Neumann problem. What happens? What if $f = -\sin x$, $\Omega = [0, 2\pi]$. Discuss your observations in view of Exercise 3.7).*

5.2. Example 2 (dimension 2)

Now let Ω be the square $[0,1]^2$ and $\lambda = 1$. Again we consider the Dirichlet problem and let T_h be the triangulation illustrated in Figure 5.10 with $N \times N$ (interior) nodes. V_h consist of all functions which are 0 on the boundary, continuous and a first order polynomial on each $K \in T_h$. The basis function v_i has support in the shaded area in Figure 5.10 and $v_i(N_i) = 1$. We have that

$$a(v_i, v_j) = \int_\Omega \lambda \operatorname{grad} v_i \cdot \operatorname{grad} v_j \, dx =$$

$$= \int_\Omega \lambda \left[\frac{\partial v_i}{\partial x_1}, \frac{\partial v_i}{\partial x_2}\right] \cdot \left[\frac{\partial v_j}{\partial x_1}, \frac{\partial v_j}{\partial x_2}\right] \, dx =$$

$$= \int_\Omega \lambda \left(\frac{\partial v_i}{\partial x_1}\frac{\partial v_j}{\partial x_1} + \frac{\partial v_i}{\partial x_2}\frac{\partial v_j}{\partial x_2}\right) \, dx$$

Observe that $a(v_i, v_j) = 0$ unless when N_i and N_j are nodes of the same triangle.

As a concrete example we will use Ω with triangulation and $3 \times 3 = 9$ nodes illustrated in Figure 5.11.To find the matrix A in 5.2 where the components are $A_{ij} = a(v_i, v_j)$we need to know the gradients for each triangle in the supported area around each node in Ω. All other gradients outside the shaded area are 0. We see from Figure 5.12 that each supported area for one node, which consists of six triangles (see Figure 5.12), looks the same for all nodes in Ω . We will now find the gradients on each triangle in the supported area.

Triangle I:

$$\frac{\partial v_1}{\partial x_1} = \frac{1}{\frac{1}{4}} = 4$$

$$\frac{\partial v_1}{\partial x_2} = 0$$

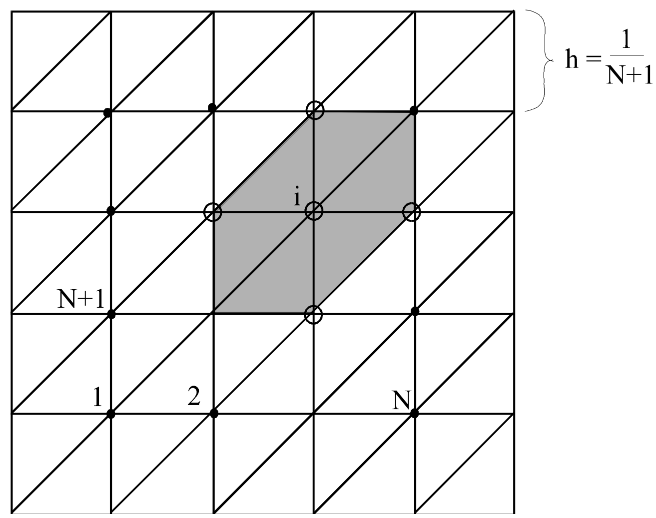


Figure 5.10: Two-dimensional Ω.

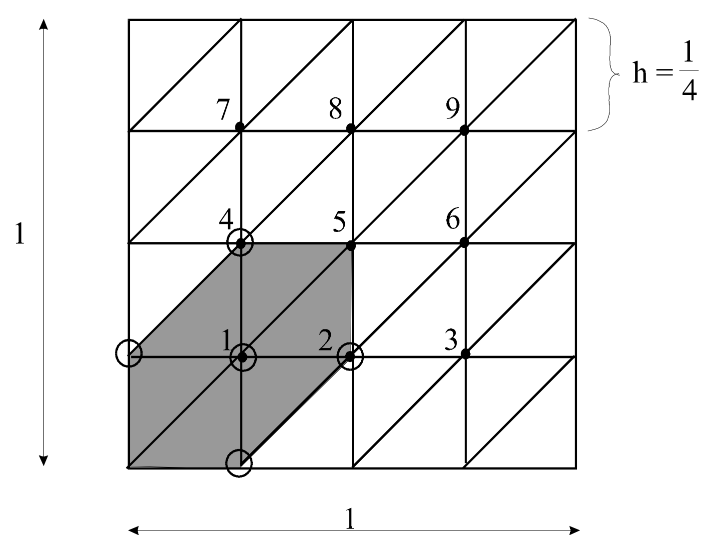


Figure 5.11: Ω with nine nodes.

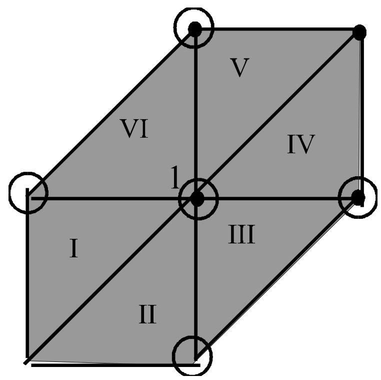


Figure 5.12: Supported area for a node.

For the other triangles we argue similarly and obtain the following table:

Triangle	$\frac{\partial v_1}{\partial x_1}$	$\frac{\partial v_1}{\partial x_2}$
I	4	0
II	0	4
III	-4	4
IV	-4	0
V	0	-4
VI	4	-4

Now, we can find each component of A from

$$A_{ij} = a(v_i, v_j) = \int_\Omega \lambda \left(\frac{\partial v_i}{\partial x_1} \frac{\partial v_j}{\partial x_1} + \frac{\partial v_i}{\partial x_2} \frac{\partial v_j}{\partial x_2} \right) \, dx.$$

The area of each triangle is

$$\frac{1}{2(N+1)^2} = \frac{1}{2(3+1)^2} = \frac{1}{32}$$

Therefore,

$$A_{11} = a(v_1, v_1) = \sum_{B=I}^{VI} \left[\left(\frac{\partial v_1}{\partial x_1} \right)_B \left(\frac{\partial v_1}{\partial x_1} \right)_B + \left(\frac{\partial v_1}{\partial x_2} \right)_B \left(\frac{\partial v_1}{\partial x_2} \right)_B \right] \frac{1}{32} =$$

$$= \frac{1}{32} [(4 \cdot 4 + 0 \cdot 0) + (0 \cdot 0 + 4 \cdot 4) + ((-4) \cdot (-4) + 4 \cdot 4) +$$

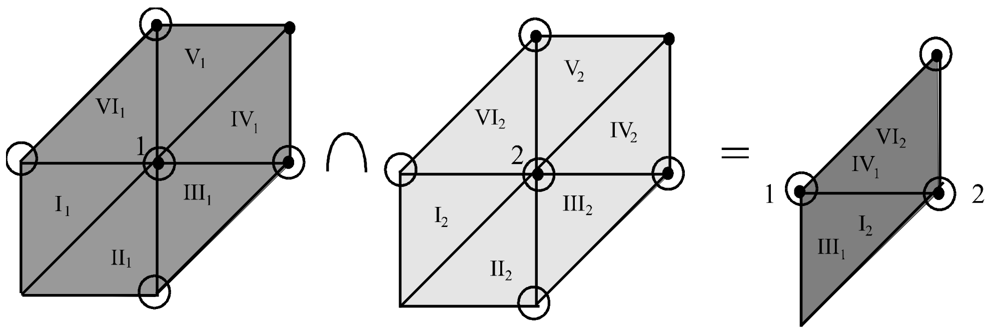


Figure 5.13: The two first figures shows the support of the basis function corresponding to node 1 and node 2 respectively. The last figure shows the intersection between these supports.

$$+((-4)\cdot(-4)+0\cdot 0)+(0\cdot 0+(-4)\cdot(-4))+(4\cdot 4+(-4)\cdot(-4))] =$$
$$= \frac{1}{32}[16+16+32+16+16+32] = 4.$$

To find A_{12} we must consider the support of the basis function of the two nodes 1 and 2 and multiply the gradients in the intersection of the two shaded areas (see Figure 5.13) with each other. The intersection $((III_1 \cap I_2) \cup (IV_1 \cap VI_2))$ is the only contributing triangles $(v_1, v_2 \neq 0)$ in the element A_{12}.

$$A_{12} = a(v_1,v_2) = \left[\left(\frac{\partial v_1}{\partial x_1}\right)_{III_1}\left(\frac{\partial v_2}{\partial x_1}\right)_{I_2} + \left(\frac{\partial v_1}{\partial x_2}\right)_{III_1}\left(\frac{\partial v_2}{\partial x_2}\right)_{I_2}\right]\frac{1}{32}+$$
$$+\left[\left(\frac{\partial v_1}{\partial x_1}\right)_{IV_1}\left(\frac{\partial v_2}{\partial x_1}\right)_{VI_2} + \left(\frac{\partial v_1}{\partial x_2}\right)_{IV_1}\left(\frac{\partial v_2}{\partial x_2}\right)_{VI_2}\right]\frac{1}{32} =$$
$$= \frac{1}{32}[((-4)\cdot 4+4\cdot 0)] + \frac{1}{32}[(-4)\cdot 4+0\cdot(-4)] =$$
$$= \frac{1}{32}[-16+0] + \frac{1}{32}[-16+0] = \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) = -1.$$

Element $A_{13} = 0$ since none of the triangles in node 1 and 3 are in contact with each other. Element A_{14} must be checked because the triangles V_1 and III_4, and the triangles VI_1 and II_4 is connected to each other. Therefore

$$A_{14} = a(v_1, v_4) = \left[\left(\frac{\partial v_1}{\partial x_1} \right)_{V_1} \left(\frac{\partial v_4}{\partial x_1} \right)_{III_4} + \left(\frac{\partial v_1}{\partial x_2} \right)_{V_1} \left(\frac{\partial v_4}{\partial x_2} \right)_{III_4} \right] \frac{1}{32} +$$

$$+ \left[\left(\frac{\partial v_1}{\partial x_1} \right)_{VI_1} \left(\frac{\partial v_4}{\partial x_1} \right)_{II_4} + \left(\frac{\partial v_1}{\partial x_2} \right)_{VI_1} \left(\frac{\partial v_4}{\partial x_2} \right)_{II_4} \right] \frac{1}{32} =$$

$$= \frac{1}{32}[(0 \cdot (-4) + (-4) \cdot 4)] + \frac{1}{32}[4 \cdot 0 + (-4) \cdot 4] = \left(-\frac{1}{2} \right) + \left(-\frac{1}{2} \right) = -1$$

Element A_{15} must also be checked because the triangles IV_1 and II_5, and the triangles V_1 and I_5 is connected to each other. We obtain that

$$A_{15} = a(v_1, v_5) = \left[\left(\frac{\partial v_1}{\partial x_1} \right)_{IV_1} \left(\frac{\partial v_5}{\partial x_1} \right)_{II_5} + \left(\frac{\partial v_1}{\partial x_2} \right)_{IV_1} \left(\frac{\partial v_5}{\partial x_2} \right)_{II_5} \right] \frac{1}{32} +$$

$$+ \left[\left(\frac{\partial v_1}{\partial x_1} \right)_{V_1} \left(\frac{\partial v_4}{\partial x_1} \right)_{I_5} + \left(\frac{\partial v_1}{\partial x_2} \right)_{V_1} \left(\frac{\partial v_4}{\partial x_2} \right)_{I_5} \right] \frac{1}{32} =$$

$$= \frac{1}{32}[((-4) \cdot 0 + 0 \cdot 4)] + \frac{1}{32}[0 \cdot 4 + (-4) \cdot 0] = 0 + 0 = 0.$$

Element A_{15} turns out to be 0 even if it these triangles are connected. Note that whenever triangles connect we have to examine the element further, because it has a potential to be different from 0. Node 1 has no further triangles from other nodes that connect. This means that element A_{13}, A_{16}, A_{17}, A_{18} and A_{19} are all 0. In the same way we find all other elements by examine each supported area at each node where the triangles connect. All elements at the diagonal have the same coefficient, this means $A_{11} = A_{22} = A_{33} = A_{44} = A_{55} = A_{66} = A_{77} = A_{88} = A_{99} = 4$. We also see that $A_{12} = A_{21} = A_{14} = A_{41} = -1$. So far we have found the following part of the matrix:

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & & & & & & & \\ 0 & & 4 & & & & & & \\ -1 & & & 4 & & & & & \\ 0 & & & & 4 & & & & \\ 0 & & & & & 4 & & & \\ 0 & & & & & & 4 & & \\ 0 & & & & & & & 4 & \\ 0 & & & & & & & & 4 \end{bmatrix}.$$

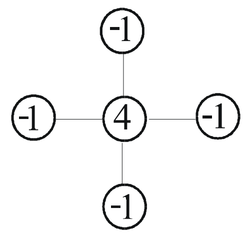


Figure 5.14: Coeffisients.

If we now consider node 2 we see that it has triangles that connect with node 1,2,3,5 and 6. The elements A_{12}, A_{21} and A_{22} are already found, and we see that $A_{26} = A_{62}$ have the same relation to each other with respect to the triangles as element $A_{15} = 0$. We also see that $A_{25} = A_{52}$ and $A_{23} = A_{32}$ have the same relation to each other with respect to the triangles as element $A_{14} = -1$ and $A_{12} = -1$, respectively. If we continue with these kind of comparisons we find that the 9x9 matrix will look like this:

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix}.$$

Each node has a connection to four other nodes where the coefficient turns out to be different from zero in addition to their own "pairing" in the system like the one in Figure 5.14.Something different happens for the elements that is affected by the nodes on $r(N)$ and $r(N+1)$, where r describes the rows in Ω, see Fig.5.10. In our case $N = 3$ this happens for relations between node 3 and 4, and 6 and 7. This will affect the elements $A_{34} = A_{43}$ and $A_{67} = A_{76}$, by turning them into 0 instead of -1 what we were likely to believe from the Figure 5.14. This happens because the shaded areas of the pair of nodes 3 and 4, and 6 and 7 do not connect to each other. If N in Figure 5.10 had been 10, we would have that the elements A_{11}, $A_{12} \neq 0$ and the elements $A_{13}, ..., A_{1\,10} = 0$ while suddenly $A_{1\,11} \neq 0$, and all other elements would be 0.

Exercise 5.6. *Find the stiffness-matrix A when $\lambda = 1$, $\Omega = [0,1]^2$ and $N = 2$ (i.e. 2×2 interior nodes and $h = 1/3$)*

Exercise 5.7. *Find the stiffness-matrix A when $\lambda = 1$, $\Omega = \left[0, \frac{5}{4}\right]$ with 4×4 interior nodes (i.e. the shortest side length of each triangle $h = 1/4$).*

5.3. Example 3 (Effective conductivity of periodic structures)

We have in example 1 and 2 considered the Dirichlet problem. Let us now consider the periodic problem given in 3.4. For simplicity we only consider the two-phase laminate case, i.e. the case when Y can be described according to Figure 5.15. Let

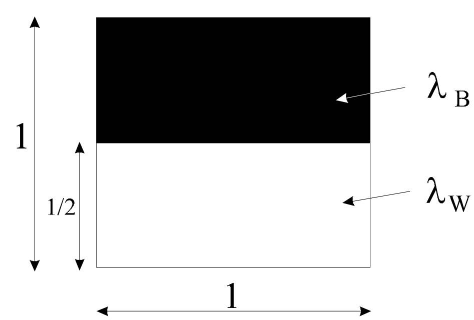


Figure 5.15: The Y-cell.

us first find $\lambda_{2,\text{eff}}$. We use the subspace V_h of $V = W^{1,2}_{per}(Y)$ consisting of periodic "roof"-functions (see Figure 5.16) which are 0 at two of the boundaries.According

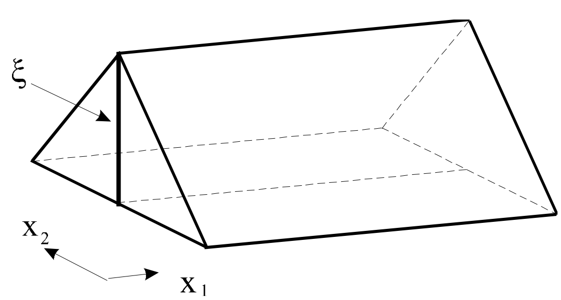


Figure 5.16: The "roof"-function.

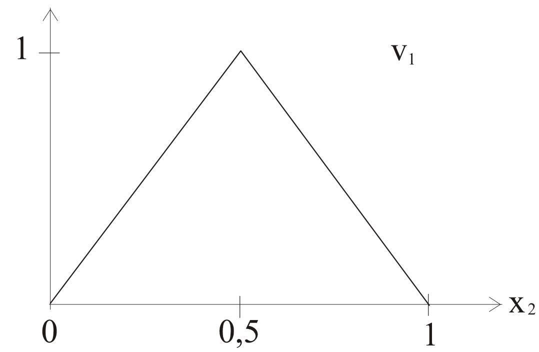


Figure 5.17: Basisfunction with height 1.

to the description in 3.4 we have to find the FEM-solution of the following problem: Find $u_h \in V_h$ such that

$$\int_Y \lambda \operatorname{grad} u_h \cdot \operatorname{grad} \phi \, dx = -\int_Y e_2 \cdot \operatorname{grad} \phi \, dx \quad \forall \phi \in V_h.$$

We continue as in example 1 and 2. Put

$$a(u_h, \phi) = \int_Y \lambda \operatorname{grad} u_h \cdot \operatorname{grad} \phi \, dx$$

and

$$L(\phi) = -\int_Y e_2 \cdot \operatorname{grad} \phi \, dx.$$

As before we have to find

$$a_{ij} = a(v_i, v_j)$$

and

$$b_i = L(v_i)$$

where v_i are basis functions.. In this case we only have one basis function, v_i. This function is 1 on the top of the "roof", see also Fig5.17. We have to find $a_{11} = a(v_1, v_1)$ and $b_1 = L(v_1)$. (That's it.) We put

$$a(u_h, v_1) = \int_Y \lambda \operatorname{grad} u_h \cdot \operatorname{grad} v_1 \, dx,$$

and then we obtain that

$$a(v_1, v_1) = \int_Y \lambda \operatorname{grad} v_1 \cdot \operatorname{grad} v_1 dx = \int_Y \lambda \left[0, \frac{\partial v_1}{\partial x_2}\right] \cdot \left[0, \frac{\partial v_1}{\partial x_2}\right] dx =$$

$$= \int_Y \lambda \frac{\partial v_1}{\partial x_2} \cdot \frac{\partial v_1}{\partial x_2} dx = \int_W \lambda_W \frac{\partial v_1}{\partial x_2} \cdot \frac{\partial v_1}{\partial x_2} dx + \int_B \lambda_B \frac{\partial v_1}{\partial x_2} \cdot \frac{\partial v_1}{\partial x_2} dx =$$

$$= \int_W \lambda_W (2 \cdot 2)\, dx + \int_B \lambda_B ((-2) \cdot (-2))\, dx = \lambda_W \cdot 4 \cdot \frac{1}{2} + \lambda_B \cdot 4 \cdot \frac{1}{2} = \lambda_W \cdot 2 + \lambda_B \cdot 2.$$

We also have that

$$L(v_1) = -\int_Y \lambda\, e_2 \operatorname{grad} v_1\, dx = -\int_Y \lambda\ [0,1] \left[0, \frac{\partial v_1}{\partial x_2}\right]\ dx = -\int_Y \lambda\, \frac{\partial v_1}{\partial x_2}\, dx =$$

$$= -\int_W \lambda_W\, \frac{\partial v_1}{\partial x_2}\, dx - \int_B \lambda_B\, \frac{\partial v_1}{\partial x_2}\, dx = -\lambda_W \cdot 2 \cdot \frac{1}{2} - \lambda_B \cdot (-2) \cdot \frac{1}{2}\ = -\lambda_W - \lambda_B.$$

In this case where $A = [a_{11}]\,,\ \xi = \xi_1$ and $b = b_1$ we observe that

$$A\xi = B \implies a_{11}\xi_1 = b_1$$

which is the equation we have to solve. Then

$$(\lambda_W \cdot 2 + \lambda_B \cdot 2)\, \xi_1 = -\lambda_W - \lambda_B$$

and

$$\xi_1 = \frac{-\lambda_W - \lambda_B}{(\lambda_W \cdot 2 + \lambda_B \cdot 2)}.$$

Because

$$u_h = \xi_1 \cdot v_1$$

thus

$$\frac{\partial u_h}{\partial x_2} = \xi_1 \cdot \frac{\partial v_1}{\partial x_2}.$$

Finally from Section 4 we have that

$$\lambda_{2,\text{eff}} = \frac{1}{|Y|} \int_Y \lambda(x) \left(\frac{\partial u_h}{\partial x_2} + 1 \right)\ dx =$$

$$= \frac{1}{|Y|} \left(\int_W \lambda_W \left(\xi_1 \cdot 2 + 1\right)\ dx + \int_B \lambda_B \left(\xi_1 \cdot (-2) + 1\right)\ dx \right) =$$

$$= 1\left(\frac{1}{2}\lambda_W\left(\xi_1\cdot 2+1\right)\ +\frac{1}{2}\lambda_B\left(\xi_1\cdot(-2)+1\right)\ dx\right) =$$

$$= \frac{1}{2}\lambda_W\left(\frac{-\lambda_W+\lambda_B}{\lambda_W+\lambda_B}+\frac{\lambda_W+\lambda_B}{\lambda_W+\lambda_B}\right)+\frac{1}{2}\lambda_B\left(\frac{\lambda_W-\lambda_B}{\lambda_W+\lambda_B}+\frac{\lambda_W+\lambda_B}{\lambda_W+\lambda_B}\right) =$$

$$= \frac{\lambda_W\cdot\lambda_B}{\lambda_W+\lambda_B}+\frac{\lambda_W\cdot\lambda_B}{\lambda_W+\lambda_B} = 2\frac{\lambda_W\cdot\lambda_B}{\lambda_W+\lambda_B} = \frac{1}{\frac{1}{2}\frac{1}{\lambda_W}+\frac{1}{2}\frac{1}{\lambda_B}},$$

which is the harmonic mean. Similarly we can derive that

$$\lambda_{1,\mathrm{eff}} = \frac{1}{2}\lambda_W + \frac{1}{2}\lambda_B,$$

which is the arithmetic mean.

Remark 3. *In this example it turns out that the FEM-solution is the same as the exact solution.*

6. Further exercises

Exercise 6.1. *What is a Hilbert space?*

Exercise 6.2. *Let Ω be the open square $Y = \langle 0, 2\pi\rangle^2$ in R^2. Compute the $L^2(Y)$ norm and the $W^{1,2}(Y)$ norm of the function $u(x_1,x_2) = \cos x_1$ (recall that $\int_0^{2\pi}\sin^2 t\,dt = \int_0^{2\pi}\cos^2 t\,dt = \pi$)*

Exercise 6.3. *Let Ω be the open square $Y = \langle 0, 2\pi\rangle^2$. Determine whether the following functions are in some of the Lebesgue and Sobolev spaces $L^2(Y), W^{1,2}(Y)$, $W^{1,2}_{per}(Y), W^{1,2}_0(Y)$:*

1. $u(x_1,x_2) = x_1^2 + x_2^2$
2. $u(x_1,x_2) = x_1^{\frac{3}{2}}$
3. $u(x_1,x_2) = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$
4. $u(x_1,x_2) = x_1^{-\frac{1}{2}}$
5. $u(x_1,x_2) = \cos x_1 + \sin x_2$

6. $u(x_1, x_2) = \cos \frac{x_1}{4}$

7. $u(x_1, x_2) = \sin x_1 \sin x_2$

Exercise 6.4. *State (i.e. formulate) the Poincaré's inequality and the Friedrich's inequality*

In the exercises below we consider the strong formulation of the following Dirichlet problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial x_1^2} + k\frac{\partial^2 u}{\partial x_2^2} &= -f(x), \ \ x \in \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{6.1}$$

where $\Omega = \langle 0, 1\rangle^2$, k is a constant $k \geq 1$ and $f \in L^2(\Omega)$.

Exercise 6.5. *Show that the strong formulation (6.1) can be written on the form*

$$\operatorname{div}\left[\frac{\partial u(x)}{\partial x_1}, k\frac{\partial u(x)}{\partial x_2}\right] = -f(x), \ \ x \in \Omega,$$

Exercise 6.6. *Use this and the Greens formula*

$$\int_\Omega \phi \operatorname{div} \mathbf{w} dx = -\int_\Omega \operatorname{grad}\phi \cdot \mathbf{w}\, dx + \int_{\partial\Omega} \phi\mathbf{w} \cdot n\, ds$$

to verify the following weak formulation of the above problem: Find $u \in W_0^{1,2}(\Omega)$ such that

$$a(u, v) = L(v) \text{ for all } v \in W_0^{1,2}(\Omega)$$

where

$$\begin{aligned} a(u, v) &= \int_\Omega \left[\frac{\partial u(x)}{\partial x_1}, k\frac{\partial u(x)}{\partial x_2}\right] \cdot \operatorname{grad} v\, dx = \\ &= \int_\Omega \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + k\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2} dx \end{aligned}$$

and

$$L(v) = \int_\Omega f v dx$$

Exercise 6.7. *Show that the above weak formulation has a unique solution by using the Lax-Milgram lemma.*

Exercise 6.8. *Show that the above weak formulation is equivalent with the following minimum energy principle: Find $u \in W_0^{1,2}(\Omega)$ such that*

$$F(u) = \min_{u \in W_0^{1,2}(\Omega)} F(v),$$

where

$$F(v) = \int_\Omega \frac{1}{2}\left(\frac{\partial v}{\partial x_1}\right)^2 + \frac{k}{2}\left(\frac{\partial v}{\partial x_2}\right)^2 - fv\,dx.$$

Exercise 6.9. *Let $g \in W^{1,2}(\Omega)$ and consider the following problem: Find $w \in W_0^{1,2}(\Omega)$ such that*

$$a(w, v) = G(v) \text{ for all } v \in W_0^{1,2}(\Omega)$$

where

$$G(v) = L(v) - a(g, v)$$

($L(v)$ and $a(u, v)$ is defined above). Prove that this problem has a unique solution.

Exercise 6.10. *Let $g \in W^{1,2}(\Omega)$ and consider the following Dirichlet problem (corresponding to the boundary condition $u = g$ on $\partial\Omega$): Find $u \in W^{1,2}(\Omega)$, such that $u - g \in W_0^{1,2}(\Omega)$ and such that*

$$a(u, v) = L(v) \text{ for all } v \in W_0^{1,2}(\Omega). \tag{6.2}$$

Show that this problem has an unique solution. (Hint: Verify that $u = w + g$ is a solution, where w is given in Exercise 6.9).

Exercise 6.11. *Do the same as in Exercise 5.3, with*

$$T_h = \{[0,1], [1,2], [2,3], [3,4], [4,6]\}$$

and where V_h is the corresponding 4-dimensional space.

Exercise 6.12. *Let $\lambda = 1/2, \Omega = [0,1]$ and $f = 1$. Moreover, let*

$$T_h = \{[0,h], [h,2h], ..., [1-h,1]\}$$

with the corresponding FEM-space $V_h \subset V = W_0^{1,2}(\Omega)$. The FEM-solution u_h converges to a function u as $h \to 0$. Find the function u (without finding u_h).

References

[1] Johnson, C. (1987) *Numerical solution of partial differential equations by the finite element method*, Studentlitteratur, Lund.

[2] Jikov, V.V., Kozlov, S.M. and Oleinik, O.A. (1994) *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin.

[3] Marti, J.T. (1986) *Introduction to Sobolev Spaces and Finite Element Solution of Elliptic Boundary Value Problems.* Academic Press, London.

[4] Meidell, A. (1998) *Anvendt Homogeniseringsteori*, ISBN-82-7823-037-4,HiN-rapport (51s.), Høgskolen i Narvik, Narvik.

[5] Persson, L-E., Persson, L., Svanstedt, N., Wyller, J. (1993) *The Homogenization Method*, Studentlitteratur, Lund.

7. Solutions to exercises

Exercise 1.1 Q is not a vector space since e.g. $1 \in Q$, but $1 \cdot k \notin Q$ for any scalar k which is irrational.

Exercise 1.2

1.
$$\int_Y |u(x)|^2\,dx = \int_0^1 (x)^4\,dx \le \int_0^1 (1)^4\,dx = 1 < \infty, \text{i.e. } u \in L^2(Y).$$
$$\int_Y \left|\frac{\partial u(x)}{\partial x}\right|^2 dx = \int_Y |2x|^2\,dx \le \int_0^1 (2)^2\,dx = 4 < \infty\ i.e. \frac{\partial u(x)}{\partial x} \in L^2(Y).$$
Thus, $u \in W^{1,2}(Y)$. Since $u(0) \ne u(1)$ we get that $u \notin W^{1,2}_{per}(Y)$ and $u \notin W^{1,2}_0(Y)$.

2.
$$\int_Y |u(x)|^2\,dx = \int_0^1 \left|e^{-x^2}\right|^2 dx \le \int_0^1 e^0 dx = 1 < \infty, \text{ i.e. } u \in L^2(Y)$$
$$\int_Y \left|\frac{\partial u(x)}{\partial x}\right|^2 dx = \int_0^1 \left|-2xe^{-x^2}\right|^2 dx \le \int_0^1 2e^0 dx = 2 < \infty, \text{ i.e. } \frac{\partial u(x)}{\partial x} \in L^2(Y).$$
Thus, $u \in W^{1,2}(Y)$. Since $u(0) \ne u(1)$ we get that $u \notin W^{1,2}_{per}(Y)$ and $u \notin W^{1,2}_0(Y)$ $u \in W^{1,2}(Y)$.

3.
$$\int_Y |u(x)|^2\,dx = \int_0^1 |\sin x|^2\,dx \le \int_0^1 1dx = 1 < \infty, \text{ i.e. } u \in L^2(Y),$$
$$\int_Y \left|\frac{\partial u(x)}{\partial x}\right|^2 dx = \int_0^1 |\cos x|^2\,dx \le \int_0^1 1dx = 1 < \infty,\ i.e. \frac{\partial u(x)}{\partial x} \in L^2(Y).$$
Thus, $u \in W^{1,2}(Y)$. Since $u(0) \ne u(1)$ we get that $u \notin W^{1,2}_{per}(Y)$ and $u \notin W^{1,2}_0(Y)$.

4. Similarily we prove that $u \in L^2(Y)$ and for $u \in W^{1,2}(Y)$ for $u(x) = \sin(2\pi x)$, but here we have that $u(0) = 0 = u(1)$,so $u \in W^{1,2}_0(Y)$ (and hence $u \in W^{1,2}_{per}(Y)$).

5. Similarly we prove that $u \in L^2(Y)$ and for $u \in W^{1,2}(Y)$ for $u(x) = \cos(2\pi x)$, but here we have that $u(0) = u(1) = 1$,so $u \in W^{1,2}_{per}(Y)$ but $u \notin W^{1,2}_0(Y)$).

Exercise 1.3:

$$\|u\|_{L^2(Y)} = \left(\int_Y u^2 dx\right)^{\frac{1}{2}} = \left(\int_0^1 \int_0^1 \left(x_1^2 + x_2^2\right)^2 dx_1 dx_2\right)^{\frac{1}{2}} = \sqrt{\frac{28}{45}}$$

$$\begin{aligned}\|u\|_{W^{1,2}(Y)} &= \left(\int_Y u^2 + |\mathrm{grad}\, u|^2 \, dx\right)^{\frac{1}{2}} \\ &= \left(\int_0^1 \int_0^1 \left(\left(x_1^2 + x_2^2\right)^2 + (2x_1)^2 + (2x_2)^2\right) dx_1 dx_2\right)^{\frac{1}{2}} = \sqrt{\frac{148}{45}}\end{aligned}$$

Exercise 1.4: For $u(x) = u(x_1, x_2) = x_1^{1/2}$ we have that

$$\int_0^1 |\partial u / \partial x_2|^2 \, dx = 0 < \infty$$

and

$$\int_Y |u|^2 \, dx = \int_0^1 \left|x_1^{\frac{1}{2}}\right|^2 dx \leq \int_0^1 \left|1^{\frac{1}{2}}\right|^2 dx < \infty.$$

Exercise 1.5:

$$u(x) = u(x_1, x_2) = x_1^\alpha$$

$$\int_Y u^2 dx = \int_0^1 \int_0^1 x_1^{2\alpha} dx_1 dx_2 = \begin{cases} \frac{1}{2\alpha+1} \lim_{a \to 0+} \left(1 - a^{2\alpha+1}\right) = \frac{1}{2\alpha+1} & \text{if } \alpha > -\frac{1}{2} \\ \lim_{a \to 0+} \left(\ln 1 - \ln a\right) = \infty & \text{if } \alpha = -\frac{1}{2} \\ \frac{1}{2\alpha+1} \lim_{a \to 0+} \left(1 - a^{2\alpha+1}\right) = \infty & \text{if } \alpha < -\frac{1}{2} \end{cases}.$$

Thus $u \in L^2(Y)$ when $\alpha > -\frac{1}{2}$.

$\partial u / \partial x_2 = 0$. Thus $\partial u / \partial x_2 \in L^2(Y)$ for all α. Moreover, if $\alpha \neq 0$

$$\int_Y \left(\frac{\partial u}{\partial x_1}\right)^2 dx = \int_0^1 \int_0^1 \left(\alpha x_1^{\alpha-1}\right)^2 dx_1 dx_2 = \int_0^1 \int_0^1 \alpha^2 x_1^{2(\alpha-1)} dx_1 dx_2$$

and similarly as above we obtain that $\partial u/\partial x_2 \in L^2(Y)$ for all $\alpha > \frac{1}{2}$. If $\alpha = 0$, then $\partial u/\partial x_1 = 0$ and $\partial u/\partial x_2 \in L^2(Y)$. Summing up this means that $u \in W^{1,2}(Y)$ when $\alpha > \frac{1}{2}$ or $\alpha = 0$.

Exercise 1.6: $v \in ?$:

The three functions v,

$$\begin{aligned}
\partial v/\partial x_1 &= \left(-e^{-x_1}\sin(\pi x_1) + \pi e^{-x_1}\cos(\pi x_1)\right)\sin(\pi x_2),\\
\partial v/\partial x_2 &= \pi e^{-x_1}\sin(\pi x_1)\cos(\pi x_2)
\end{aligned}$$

are all bounded functions (cannot grow infinitely large in any point in Ω). For example,

$$|\partial v/\partial x_1| \leq \left(e^0 1 + \pi e^0 1 \cdot 1\right) \cdot 1 = 1 + \pi,$$

which gives that

$$\int_\Omega \left(\frac{\partial v}{\partial x_1}\right)^2 dx \leq \int_\Omega (1+\pi)^2\, dx \leq (1+\pi)^2 < \infty.$$

Hence $\partial v/\partial x_1 \in L^2(\Omega)$. Similarly, we obtain that $v \in L^2(\Omega)$ and $\partial v/\partial x_2 \in L^2(\Omega)$, which means that $v \in W^{1,2}(\Omega)$. Moreover, $v(0,x_2) = v(1,x_2) = v(x_1,0) = v(x_1,1) = 0$, so $v = 0$ on $\partial\Omega$. Hence $v \in W_0^{1,2}(\Omega)$.

$w \in ?$:

The functions w and $\partial w/\partial x_2 = \pi x_1^{\frac{2}{3}}\cos(\pi x_2)$ are bounded in Ω since $|w| \leq 1^{\frac{2}{3}} \cdot 1 = 1$ and $|\partial w/\partial x_2| \leq \pi \cdot 1^{\frac{2}{3}} \cdot 1 = \pi$. Thus (as for the function v) we directly see that $w \in L^2(\Omega)$ and $\partial w/\partial x_2 \in L^2(\Omega)$. However, the function $\partial w/\partial x_1 = \frac{2}{3}x_1^{-\frac{1}{3}}\sin(\pi x_2)$ is not bounded! Here we may do as follows:

$$\begin{aligned}
\int_\Omega \left(\frac{\partial w}{\partial x_1}\right)^2 dx &= \int_\Omega \left(\frac{2}{3}x_1^{-\frac{1}{3}}\sin(\pi x_2)\right)^2 dx \leq \int_\Omega \left(\frac{2}{3}x_1^{-\frac{1}{3}} \cdot 1\right)^2 dx = \\
\int_0^1 \int_0^1 \frac{4}{9}x_1^{-\frac{2}{3}}dx_1dx_2 &= \frac{4}{3} < \infty.
\end{aligned}$$

Thus we also obtain that $\partial w/\partial x_1 \in L^2(\Omega)$. Accordingly, $w \in W^{1,2}(\Omega)$. We have that $w(x_1,1) = v(x_1,0) = w(0,x_2) = 0$, but $w(1,x_2) = \sin(\pi x_2) \neq 0$, so $w \notin W_0^{1,2}(\Omega)$.

Exercise 1.7 The three functions v,

$$\begin{aligned}
\partial v/\partial x_1 &= 2x_1(2x_2 - 2),\\
\partial v/\partial x_2 &= 2x_2(2x_2 - 2) + 2(x_1^2 + x_2^2 - 4),
\end{aligned}$$

are all bounded functions (cannot grow infinitely large in any point in Ω). For example,

$$|\partial v/\partial x_1| \leq (2 \cdot 2\,(2 \cdot 2 + 2)) = 24,$$

which gives that

$$\int_\Omega \left(\frac{\partial v}{\partial x_1}\right)^2 dx \leq \int_\Omega (24)^2\, dx \leq (24)^2\, |\Omega| < \infty.$$

Hence $\partial v/\partial x_1 \in L^2(\Omega)$. Similarly, we obtain that $v \in L^2(\Omega)$ and $\partial v/\partial x_2 \in L^2(\Omega)$, which means that $v \in W^{1,2}(\Omega)$. We now check the boundary-values, i.e. the values on $\partial\Omega$. On the line $x_2 = 1$ we have that $v(x) = v(x_1, 1) = (x_1^2 + 1^2 - 4)\,(2 \cdot 1 - 2) = 0$. On the curve $x_2 = \sqrt{4 - x_1^2}$ we have that $v(x) = v(x_1, x_2) = = (x_1^2 + \left(\sqrt{4 - x_1^2}\right)^2 - 4)\,(2x_2 - 2) = 0$. Thus, $v = 0$ on $\partial\Omega$. Hence $v \in W_0^{1,2}(\Omega)$.

$w \in ?$:

The function w is bounded in Ω since $|w| \leq 2^{\frac{1}{3}} < \infty$; so is $\partial w/\partial x_1 = 0$. Moreover, since $1 \leq x_2$ in Ω we even have that

$$|\partial w/\partial x_2| = \frac{1}{3} x_2^{-\frac{2}{3}} = \frac{1}{3x_2^{\frac{2}{3}}} \leq \frac{1}{3 \cdot 1^{\frac{2}{3}}} = \frac{1}{3}$$

Thus, as above we obtain that $w \in L^2(\Omega)$, $\partial w/\partial x_1 \in L^2(\Omega)$ and $\partial w/\partial x_1 \in L^2(\Omega)$. (note that if Ω had contained 0 then $\partial w/\partial x_2 \notin L^2(\Omega)$). which means that $w \in W^{1,2}(\Omega)$. We now check the boundary-values, i.e. the values on $\partial\Omega$. We observe that w is not equal to 0 on all boundary-points, e.g. $w(x_1, 1) = x_2^{\frac{1}{3}} = 1^{\frac{1}{3}} = 1$ on the boundary-line $x_2 = 1$. Hence, $w \notin W_0^{1,2}(\Omega)$

Exercise 1.8:

1. If u and v is continuous real valued functions and $k \in R$, then $u+v$ and ku are continuous real valued functioned (see e.g. Kreyszig concerning properties of continuous functions). Hence, V is a vector space. That $*$ defines a scalar product is proved as follows: Let $u, w, v \in V$ and $a, b \in R$, then

$$\begin{aligned}
(au + bw) * v &= \int_{-1}^{1} \lambda(x)\,(au(x) + bw(x))\,v(x)dx = \\
&= a\int_{-1}^{1} \lambda(x)\,(u(x)v(x))\,dx + b\int_{0}^{1} \lambda(x)\,(w(x)v(x))\,dx = \\
&= a\,(u * v) + b\,(w * v)
\end{aligned}$$

$$u * v = \int_{-1}^{1} \lambda(x)u(x)v(x)dx = \int_{-1}^{1} \lambda(x)v(x)u(x)dx = v * u$$

$$u * u = \int_{-1}^{1} \lambda(x) |u(x)|^2 \, dx \geq \int_{-1}^{1} 1 \, |u(x)|^2 \, dx \geq 0$$

and, thus, if $u * u = 0$ then $\int_{-1}^{1} |u(x)|^2 \, dx = 0$ which implies that $u = 0$.

2. $\|u\|_V = \sqrt{\int_{-1}^{1} 1x^2 dx} = \sqrt{\frac{2}{3}}$

3. $u_h \in V$ and $u_h \to u$ as $h \to \infty$, where u is given by

$$u(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0 \\ 1 & \text{if } 0 < x \leq 1 \end{cases} .$$

In order to see this we observe that

$$\|u_h - u\|_V = \sqrt{\int_{-1}^{1} \lambda \, |u_h - u|^2 \, dx} =$$

$$= \sqrt{\underbrace{\int_{-1}^{0} \lambda \, |u_h - u|^2 \, dx}_{0} + \int_{0}^{1/h} \lambda \, |u_h - u|^2 \, dx + \underbrace{\int_{1/h}^{1} \lambda \, |u_h - u|^2 \, dx}_{0}} =$$

$$= \sqrt{\int_{0}^{1/h} \lambda \, |u_h - u|^2 \, dx} \leq \sqrt{\int_{0}^{1/h} 2 \, |1|^2 \, dx} = \sqrt{\frac{1}{h} 2} \to 0 ash \to \infty.$$

We observe that $u \notin V$ (not continuous at $x = 0$). Thus V is not complete and V is therefore not a Hilbert space.

Exercise 1.9: We put $v = 1$ in (1.4) and obtain

$$\left| \int_\Omega u1 \, dx \right| \leq \left(\int_\Omega |u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega 1 dx \right)^{\frac{1}{2}} .$$

By taking the square of each side we obtain

$$\left(\int_\Omega u dx \right)^2 \leq K \left(\int_\Omega |u|^2 \, dx \right) \tag{7.1}$$

where $K = \int_\Omega 1dx$. By the hint and the fact that $v(0) = 0,$ we have that

$$|v(t)| \leq \int_0^1 |\mathrm{grad}\, v|\, dx.$$

Hence, by (7.1) (with $u = |\mathrm{grad}\, v|$) we obtain that

$$\int_0^1 |v(t)|^2\, dt \leq \int_0^1 \left(\int_\Omega |\mathrm{grad}\, v|\, dx \right)^2 dt = \left(\int_\Omega |\mathrm{grad}\, v|\, dx \right)^2 \leq K \left(\int_\Omega |\mathrm{grad}\, v|^2\, dx \right),$$

i.e.

$$\int_\Omega v^2 dx \leq K \left(\int_\Omega |\mathrm{grad}\, v|^2\, dx \right).$$

Exercise 2.1: We add the following n identities:

$$\begin{aligned} \frac{\partial (\phi w_1)}{\partial x_1} &= \frac{\partial \phi}{\partial x_1} w_1 + \phi \frac{\partial w_1}{\partial x_1} \\ &\vdots \\ \frac{\partial (\phi w_n)}{\partial x_n} &= \frac{\partial \phi}{\partial x_n} w_n + \phi \frac{\partial w_n}{\partial x_n} \end{aligned}$$

and obtain

$$\mathrm{div}\, (\phi \mathbf{w}) = \mathrm{grad}\, \phi \cdot \mathbf{w} + \phi\, \mathrm{div}\, \mathbf{w}$$

the Divergence theorem of Gauss (with $\mathbf{v} = \phi \mathbf{w}$)

$$\int_\Omega \mathrm{div}\, \mathbf{v}\, dx = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, ds$$

then yields that

$$\int_\Omega \mathrm{grad}\, \phi \cdot \mathbf{w} dx + \int_\Omega \phi\, \mathrm{div}\, \mathbf{w} dx = \int_{\partial\Omega} \phi \mathbf{w} \cdot \mathbf{n}\, ds,$$

i.e. that

$$\int_\Omega \phi\, \mathrm{div}\, \mathbf{w} dx = -\int_\Omega \mathrm{grad}\, \phi \cdot \mathbf{w} dx + \int_{\partial\Omega} \phi \mathbf{w} \cdot \mathbf{n}\, ds,$$

which is (2.3).

Exercise 2.2:

$$\begin{aligned} a(u+v,w) &= \int_\Omega \lambda\, \mathrm{grad}\,(u+v) \cdot\ \mathrm{grad}\, w\ dx = \\ &= \int_\Omega \lambda\, \mathrm{grad}\, u \cdot \mathrm{grad}\, w dx + \int_\Omega \lambda\, \mathrm{grad}\, v \cdot\ \mathrm{grad}\, w dx \\ &= a(u,w) + a(v,w) \end{aligned}$$

Similarly we prove that

$$a(ku,w) = ka(u,w),$$

$$a(w,u+v) = a(w,u) + a(w,v),$$

$$a(u,kw) = ka(u,w),$$

Thus a is a bilinear form on $W_0^{1,2}(\Omega)$.

Exercise 2.3:

1. If $w \neq 0$, then $a(u+v,w) = u+v+w \neq (u+w)+(v+w) = a(u,w)+a(v,w)$ i.e. $a(u+v,w) \neq a(u,w)+a(v,w)$. Thus a is not a bilinear form
2. Is a bilinear form
3. Is a bilinear form
4. Is a bilinear form
5. Is not a bilinear form (if $u(0)v(0) \neq 0$ and $k > 1$ then $a(ku,v) = ((ku(0))\,v(0))^2 = k^2\,(u(0)v(0))^2 \neq k\,(u(0)v(0))^2 = ka(u,v)$, i.e. $a(ku,v) \neq ka(u,v)$)

Exercise 3.1

1. Using the scalar product $u * v = u_1v_1 + 2u_2v_2$ we obtain from the Schwarz inequality

   $$|u * v| \leq (u * u)^{\frac{1}{2}}\,(v * v)^{\frac{1}{2}}$$

   that

   $$|u_1v_1 + 2u_2v_2| \leq \left(u_1^2 + 2u_2^2\right)^{\frac{1}{2}}\left(v_1^2 + 2v_2^2\right)^{\frac{1}{2}}$$

2. We observe first that $a(u,v)$ is a bilinear form and want to check if the conditions 2-4 are satisfied: Condition 2: We must prove that $|a(u,v)| \leq \gamma \|u\| \|v\|$, for some constant $\gamma > 0$, which in this case means that

$$|a(u,v)| \leq \gamma \left(u_1^2 + u_2^2\right)^{\frac{1}{2}} \left(v_1^2 + v_2^2\right)^{\frac{1}{2}}$$

By (3.3) we have that

$$|a(u,v)| = |u_1 v_1 + 2u_2 v_2| \leq \left(u_1^2 + 2u_2^2\right)^{\frac{1}{2}} \left(v_1^2 + 2v_2^2\right)^{\frac{1}{2}}$$

$$\leq \left(2u_1^2 + 2u_2^2\right)^{\frac{1}{2}} \left(2v_1^2 + 2v_2^2\right)^{\frac{1}{2}} = (2)^{\frac{1}{2}} \left(u_1^2 + u_2^2\right)^{\frac{1}{2}} (2)^{\frac{1}{2}} \left(v_1^2 + v_2^2\right)^{\frac{1}{2}}$$

$$= 2 \left(u_1^2 + u_2^2\right)^{\frac{1}{2}} \left(v_1^2 + v_2^2\right)^{\frac{1}{2}} = 2 \|u\| \|v\|.$$

Thus

$$|a(u,v)| \leq 2 \|u\| \|v\|,$$

so the condition 2 is satisfied for $\gamma = 2$. We turn to condition 3:

$$|a(v,v)| = v_1 v_1 + 2v_2 v_2 = v_1^2 + 2v_2^2 \geq v_1^2 + v_2^2 = \|v\|^2.$$

Thus

$$|a(v,v)| \geq \|v\|^2,$$

so the condition 3 is satisfied for $\alpha = 1$.Finally we prove condition 4:

$$|L(v)| = |2v_1 + 4v_2| = |[2,4] \cdot [v_1, v_2]| \leq$$

$$\leq |[2,4]| \cdot |[v_1, v_2]| = \left(2^2 + 4^2\right)^{\frac{1}{2}} \|v\| = 2\sqrt{5} \|v\|,$$

where the last inequality follows by Schwarz inequality. Thus, $|L(v)| \leq 2\sqrt{5} \|v\|$, so the condition 4 is satisfied for $\Lambda = 2\sqrt{5}$. Hence, conditions 2-4 are satisfied, and the problem has a unique solution by Lax Milgram lemma.

3. We have that

$$a(u,v) = L(v) \text{ for all } v \in V,$$

where $a(u,v) = u_1 v_1 + 2u_2 v_2$ and $L(v) = 2v_1 + 4v_2$. Putting

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we obtain the two equations

$$\begin{aligned} u_1 \cdot 1 + 2u_2 \cdot 0 &= 2 \cdot 1 + 4 \cdot 0, \\ u_1 \cdot 0 + 2u_2 \cdot 1 &= 2 \cdot 0 + 4 \cdot 1, \end{aligned}$$

which gives the solution

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

4. We have that

$$\begin{aligned} F(v) &= \frac{1}{2}a(v,v) - L(v) = \frac{1}{2}\left(v_1 v_1 + 2v_2 v_2\right) - \left(2v_1 + 4v_2\right) = \\ &= \left(\frac{1}{2}v_1^2 - 2v_1\right) + \left(v_2^2 - 4v_2\right). \end{aligned}$$

In this simple case we can easily find the minimum point of $F(v)$ by finding the minimum points of each of the two terms $\frac{1}{2}v_1^2 - 2v_1$ and $v_2^2 - 4v_2$, separately. This is done by differentiating with respect to v_1 and v_2 in the first and second term, respectively:

$$\begin{aligned} \frac{d\left(\frac{1}{2}v_1^2 - 2v_1\right)}{dv_1} &= v_1 - 2 = 0 \Rightarrow v_1 = 2, \\ \frac{d\left(v_2^2 - 4v_2\right)}{dv_2} &= 2v_2 - 4 = 0 \Rightarrow v_2 = 2. \end{aligned}$$

Thus, the solution (which we as usual call) u is

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Comments: This is the same solution as we found above, which agrees with the minimum theorem (Theorem 3.2) since $a(u,v)$ and $L(v)$ satisfy all conditions 1-4. Indeed, above we have verified that the conditions 2,3,4 are satisfied, but we see directly that even condition 1 is satisfied, since $a(u,v) = u_1v_1 + 2u_2v_2 = v_1u_1 + 2v_2u_2 = a(v,u)$, i.e. $a(u,v) = a(v,u)$.

Exercise 3.2

1. For $u = 0$ we obtain that

$$a(u, v) = 0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_1 + 0 \cdot v_2 = 0$$

for all $v \in V$. Since $L(v) = 0$ for all $v \in V$, we see that $a(u, v) = L(v)$ for all $v \in V$, so $u = 0$ is therefore a solution. For the other vector

$$u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

we obtain that

$$\begin{aligned} a(u, v) &= u_1 v_1 + u_1 v_2 + u_2 v_1 + u_2 v_2 \\ &= 1 \cdot v_1 + 1 \cdot v_2 + (-1) \cdot v_1 + (-1) \cdot v_2 = 0, \end{aligned}$$

so that $a(u, v) = L(v)$ for all $v \in V$, which means that u is a solution.

2. Let $L(v) = v_1$ and assume that the problem $a(u, v) = L(v)$ has a solution u (as we soon will see, this will lead to a contradiction). Then, $a(u, v) = L(v)$, particularly for

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which gives the two equations

$$\begin{aligned} u_1 \cdot 1 + u_1 \cdot 0 + u_2 \cdot 1 + u_2 \cdot 0 &= 1, \\ u_1 \cdot 0 + u_1 \cdot 1 + u_2 \cdot 0 + u_2 \cdot 1 &= 0. \end{aligned}$$

In other words,

$$\begin{aligned} u_1 + u_2 &= 1, \\ u_1 + u_2 &= 0, \end{aligned}$$

which is impossible. Thus there are no solutions u to the above problem.

3. Suppose that condition 3 is satisfied (as we soon will see, this will lead to a contradiction) , i.e. that there is a constant $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|_V^2 \quad \forall v \in V.$$

In our case this means that

$$v_1 v_1 + v_1 v_2 + v_2 v_1 + v_2 v_2 \geq \alpha \left(v_1^2 + v_2^2\right),$$

Since this inequality holds for all $v \in V$, it also holds for example when $v_1 = 1$ and $v_2 = -1$. But this implies that

$$1 \cdot 1 + 1 \cdot (-1) + (-1) \cdot 1 + (-1) \cdot (-1) \geq \alpha \left(1^2 + (-1)^2\right),$$

i.e. that

$$\alpha \leq 0,$$

which contradicts the assumption that $\alpha > 0$. Thus we can conclude that condition 3 is not satisfied.

Exercise 3.3:**Solution:** We replace

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

with $\operatorname{div} \operatorname{grad} u$, multiply with $v \in W_0^{1,2}(\Omega)$ on both sides of the strong formulation and integrate:

$$\int_\Omega v \operatorname{div} \operatorname{grad} u dx - \int_\Omega (\frac{\partial u(x)}{\partial x_1} + \frac{\partial u(x)}{\partial x_2}) v(x)\ dx = -\int_\Omega v(x) \sin x_1 dx. \tag{7.2}$$

Greens formula with $\mathbf{w} = \operatorname{grad} u$ gives

$$\int_\Omega v \operatorname{div} \operatorname{grad} u dx = -\int_\Omega \operatorname{grad} v \cdot \operatorname{grad} u dx + \int_{\partial\Omega} v \operatorname{grad} u \cdot n\, ds.$$

Inserting this into (7.2) we obtain

$$-\int_\Omega \operatorname{grad} v \cdot \operatorname{grad} u dx + \int_{\partial\Omega} v \operatorname{grad} u \cdot n\, ds - \int_\Omega (\frac{\partial u(x)}{\partial x_1} + \frac{\partial u(x)}{\partial x_2}) v(x)\ dx = -\int_\Omega v(x) \sin x_1 dx$$

The second term on the left side vanishes because $v = 0$ on $\partial\Omega$, and we obtain

$$\int_\Omega \operatorname{grad} v \cdot \operatorname{grad} u dx + \int_\Omega (\frac{\partial u(x)}{\partial x_1} + \frac{\partial u(x)}{\partial x_2}) v(x)\ dx = \int_\Omega v(x) \sin x_1 dx,$$

which is the same as (3.4).

Next, we prove that $L(v)$ is continuous:

$$|L(v)| = \left| \int_\Omega v(x) \sin x_1 dx \right| \underbrace{\leq}_{\text{Schwartz}} \sqrt{\int_\Omega (v(x))^2\, dx} \sqrt{\int_\Omega (\sin x_1)^2\, dx}$$

$$\leq \sqrt{\int_\Omega (v(x))^2\,dx}\sqrt{\int_\Omega (1)^2\,dx} = 1\cdot\sqrt{\int_\Omega (v(x))^2\,dx}$$

$$\leq 1\cdot\sqrt{\int_\Omega \left((v(x))^2 + |\mathrm{grad}\,v|^2\right)dx} = 1\cdot\|v\|\,.$$

Hence, $|L(v)| \leq 1\cdot\|v\|$ for all $v \in W_0^{1,2}(\Omega)$, which means that L is continuous.

Exercise 3.4:We observe first that $a(u,v)$ is a bilinear form and want to check if the conditions 2-4 are satisfied: Condition 2:

$$|a(u,v)| = \left|\int_\Omega \left[4\frac{\partial u(x)}{\partial x_1}, 2\frac{\partial u(x)}{\partial x_2}\right]\cdot \mathrm{grad}\,v\,dx\right|$$

$$\underset{\text{Scwarz ineq.}}{\leq} \sqrt{\int_\Omega \left|\left[4\frac{\partial u(x)}{\partial x_1}, 2\frac{\partial u(x)}{\partial x_2}\right]\right|^2 dx}\sqrt{\int_\Omega |\mathrm{grad}\,v|^2\,dx} =$$

$$= \sqrt{\int_\Omega 4^2\left(\frac{\partial u(x)}{\partial x_1}\right)^2 + 2^2\left(\frac{\partial u(x)}{\partial x_2}\right)^2 dx}\sqrt{\int_\Omega |\mathrm{grad}\,v|^2\,dx} \leq$$

$$\underset{\text{since } 2^2\leq 4^2}{\leq} \sqrt{\int_\Omega 4^2\left(\frac{\partial u(x)}{\partial x_1}\right)^2 + 4^2\left(\frac{\partial u(x)}{\partial x_2}\right)^2 dx}\sqrt{\int_\Omega |\mathrm{grad}\,v|^2\,dx} =$$

$$= 4\sqrt{\int_\Omega \left(\frac{\partial u(x)}{\partial x_1}\right)^2 + \left(\frac{\partial u(x)}{\partial x_2}\right)^2 dx}\sqrt{\int_\Omega |\mathrm{grad}\,v|^2\,dx} =$$

$$= 4\sqrt{\int_\Omega |\mathrm{grad}\,u|^2\,dx}\sqrt{\int_\Omega |\mathrm{grad}\,v|^2\,dx} \leq$$

$$\leq 4\sqrt{\int_\Omega |\mathrm{grad}\,u|^2 + |u|^2\,dx}\sqrt{\int_\Omega |\mathrm{grad}\,v|^2 + |v|^2\,dx} = 4\,\|u\|\,\|v\|$$

i.e.

$$|a(u,v)| \leq 4\,\|u\|\,\|v\|$$

Condition 3:

$$a(v,v) = \left|\int_\Omega \left[4\frac{\partial v(x)}{\partial x_1}, 2\frac{\partial v(x)}{\partial x_2}\right]\cdot \mathrm{grad}\,v\,dx\right| =$$

$$\left|\int_\Omega 4\left(\frac{\partial v(x)}{\partial x_1}\right)^2+2\left(\frac{\partial v(x)}{\partial x_2}\right)^2 dx\right| \underset{\text{since } 4\geq 2}{\geq}$$

$$\geq \left|\int_\Omega 2\left(\frac{\partial v(x)}{\partial x_1}\right)^2+2\left(\frac{\partial v(x)}{\partial x_2}\right)^2 dx\right| = 2\left|\int_\Omega \left(\frac{\partial v(x)}{\partial x_1}\right)^2+\left(\frac{\partial v(x)}{\partial x_2}\right)^2 dx\right|$$

$$= 2\int_\Omega |\mathrm{grad}\, v\,|^2\, dx \geq 2\frac{1}{C_0+1}\|v\|^2$$

[The last inequality follows by Friedrich's inequality:

$$\int_\Omega v^2\ dx \leq C_0 \int_\Omega |\mathrm{grad}\, v|^2\ dx.$$

which gives

$$\int_\Omega v^2\ dx + \int_\Omega |\mathrm{grad}\, v|^2\ dx \leq C_0\int_\Omega |\mathrm{grad}\, v|^2\ dx + \int_\Omega |\mathrm{grad}\, v|^2\ dx =$$

$$= (C_0+1)\int_\Omega |\mathrm{grad}\, v|^2\ dx.$$

Thus

$$\frac{1}{C_0+1}\left(\int_\Omega v^2\ dx + \int_\Omega |\mathrm{grad}\, v|^2\ dx\right) \leq \int_\Omega |\mathrm{grad}\, v|^2\ dx$$

i.e.

$$\frac{1}{C_0+1}\left(\|v\|^2\right) \leq \int_\Omega |\mathrm{grad}\, v|^2\ dx.]$$

Thus $|a(v,v)| \geq \alpha \|v\|^2$ where $\alpha = 2\frac{1}{C_0+1}$.

Condition 4:

$$L(v) = \int_\Omega v\ \ dx$$

$$|L(v)| = \left|\int_\Omega v\ \cdot 1\ dx\right| \leq \int_\Omega |v\cdot 1|\, dx \underbrace{\leq}_{\text{Schwarz}} \left(\int_\Omega |v|^2\ dx\right)^{\frac{1}{2}}\left(\int_\Omega |1|^2\ dx\right)^{\frac{1}{2}} \leq$$

$$\leq \left(\int_\Omega |v|^2+|\mathrm{grad}\, v|^2\ dx\right)^{\frac{1}{2}}\left(\int_\Omega |1|^2\ dx\right)^{\frac{1}{2}} = \|v\|_{W_0^{1,2}(\Omega)}\Lambda$$

where $\Lambda = \left(\int_\Omega |1|^2\ dx\right)^{\frac{1}{2}} = 1$. Thus $|L(v)| \leq \Lambda \|v\|_V$ which proves condition 4.

All conditions are satisfied, and hence, existence and uniqueness of the solution follow by Lax-Milgrams lemma

Exercise 3.5:

1. V is a vector space: Indeed if $u \in V$ and $v \in V$ and $k \in R$ we have that $u + v \in W^{1,2}(\Omega)$ and $ku \in W^{1,2}(\Omega)$ (simply because $W^{1,2}(\Omega)$ is a vector space). Moreover, $\int_\Omega u + v\, dx = \int_\Omega u dx + \int_\Omega v\, dx = 0 + 0 = 0$ and $\int_\Omega ku\, dx = k \int_\Omega u\, dx = k0 = 0$, i.e. $u + v \in V$ and $ku \in V$.

2. Next we show that V is complete: Let $\{u_h\}$ be a sequence in V converging to a function u. Then $u \in W^{1,2}(\Omega)$ since $W^{1,2}(\Omega)$ itself is complete. By the inequality (1.5) we obtain that

$$\left(\int_\Omega (u - u_h)\ dx\right)^2 \leq K \left(\int_\Omega |u - u_h|^2\ dx\right)$$

$$\leq K \left(\int_\Omega |u - u_h|^2\ dx + \int_\Omega |\text{grad}\,(u - u_h)|^2\ dx\right) = K \left\|u - u_h\right\|^2_{W^{1,2}(\Omega)},$$

i.e.

$$\left(\int_\Omega u dx - \underbrace{\int_\Omega u_h dx}_{=0}\right)^2 \leq K \left\|u - u_h\right\|^2_{W^{1,2}(\Omega)} \to 0,$$

Thus $\int_\Omega u dx = 0$ and we obtain that $u \in V$. This shows that V is complete.

3. We prove that $a(.,.)$ and L satisfies the conditions 2, 3 and 4. This is done exactly as for the Dirichlet problem, except that we must use the Poincaré's inequality instead of Friedrich's inequality. Thus existence of a solution follows by Lax-Milgram lemma (Theorem 3.1).

Exercise 3.6:

If $\phi \in W^{1,2}(\Omega)$, then the function $v = \phi - (\frac{1}{|\Omega|} \int_\Omega \phi\, dx)$ belongs to V (since $\frac{1}{|\Omega|} \int_\Omega v dx = 0$). If u is the solution of (3.6) then

$$a(u, v) = L\,(v)\,. \tag{7.3}$$

Since

$$\text{grad}\,\phi = \text{grad}\,v,$$

we have that

$$a(u,\phi) = \int_\Omega (\lambda \operatorname{grad} u \cdot \operatorname{grad} \phi) dx = \int_\Omega (\lambda \operatorname{grad} u \cdot \operatorname{grad} v) dx = a(u,v) \tag{7.4}$$

Moreover,

$$\begin{aligned} L(v) &= \int_\Omega f v dx = \int_\Omega f \left(\phi - (\frac{1}{|\Omega|} \int_\Omega \phi \, dx) \right) dx \\ &= \int_\Omega f \phi dx - (\frac{1}{|\Omega|} \int_\Omega \phi \, dx) \underbrace{\int_\Omega f dx}_{=0} \\ &= \int_\Omega f \phi dx = L(\phi) \end{aligned} \tag{7.5}$$

Thus, (7.3), (7.4) and (7.5) gives that

$$a(u,\phi) = L(\phi),$$

and this holds for all $\phi \in W^{1,2}(\Omega)$. This shows that u is a solution of the Neumann problem (2.10).

Next, let u_2 be an other solution to (2.10). Then

$$a(u,\phi) = L(\phi)$$

$$a(u_2,\phi) = L(\phi)$$

then

$$a(u,\phi) - a(u_2,\phi) = 0, \quad \forall \phi \in W^{1,2}(Y).$$

Hence, using the linearity of $a(.,.)$ we obtain that

$$a((u-u_2),\phi) = 0, \quad \forall \phi \in W^{1,2}(Y).$$

Particularly this holds for

$$\phi = u - u_2$$

i.e.

$$a((u-u_2),(u-u_2)) = 0,$$

Thus

$$\int_Y \lambda \, |\operatorname{grad}(u-u_2)|^2 \, dx = 0,$$

which gives us that

$$\operatorname{grad}(u-u_2)=0, \quad \forall\, x\in\Omega.$$

Thus $u-u_2=$ constant.

Exercise 3.7:

1. By inserting $\phi=1$ in $a(u,\phi)=L(\phi)$ we obtain that

   $$\int_\Omega \lambda \operatorname{grad} u \underbrace{\operatorname{grad}\phi}_{0} dx=\int_\Omega \underbrace{\phi}_{1} f dx,$$

   i.e. that

   $$\int_\Omega f dx=0$$

2. The Neumann condition $\partial u/\partial n=0$ on the boundary implies no heat transfer through the boundary, i.e. the body is insulated. If the total heat delivered to the body

   $$\int_\Omega f dx$$

   is larger than 0 then temperature will go to ∞, if it is less than 0 the temperature will never stop to fall. Both these extreme cases of solutions do not belong to $W^{1,2}(\Omega)$.

Exercise 4.1:

$$\int_Y \lambda \left|\operatorname{grad}\phi+e_i\right|^2\ dx=\int_Y \lambda\left(\operatorname{grad}\phi+e_i\right)\cdot\left(\operatorname{grad}\phi+e_i\right)\ dx=$$

$$=\int_Y \lambda\left|\operatorname{grad}\phi\right|^2 dx+2\int_Y \lambda e_i\cdot\operatorname{grad}\phi\ dx+\int_Y \lambda\left|e_i\right|^2 dx=a(\phi,\phi)-2L(\phi)+\int_Y \lambda dx$$

$$2F(\phi)+\int_Y \lambda dx$$

Thus, we obtain

$$\min_{\phi\in W^{1,2}_{\text{per}}(Y)}\frac{1}{|Y|}\int_Y \lambda\left|\operatorname{grad}\phi+e_i\right|^2\ dx=\frac{2}{|Y|}\min_{\phi\in W^{1,2}_{\text{per}}(Y)}F(\phi)+\frac{1}{|Y|}\int_Y \lambda dx$$

$$=(\text{according to Theorem 3.2})=$$

$$= \frac{2}{|Y|}F(u_{\text{per}}) + \frac{1}{|Y|}\int_Y \lambda dx = \frac{2}{|Y|}\left(\frac{1}{2}a(u_{\text{per}},u_{\text{per}}) - L(u_{\text{per}})\right) + \frac{1}{|Y|}\int_Y \lambda dx$$
$$= \frac{2}{|Y|}\left(\frac{1}{2}L(u_{\text{per}}) - L(u_{\text{per}})\right) + \frac{1}{|Y|}\int_Y \lambda dx = \frac{1}{|Y|}\left(-L(u_{\text{per}}) + \int_Y \lambda dx\right)$$
$$= \frac{1}{|Y|}\left(\int_Y \lambda e_i \cdot \operatorname{grad} u_{\text{per}}\, dx + \int_Y \lambda dx\right) = \frac{1}{|Y|}\left(\int_Y \lambda(\frac{\partial u_{\text{per}}}{\partial x_i} + 1)dx\right) = \lambda_{i,\text{eff}}.$$

Exercise 5.1: Since $\phi = v - u_h \in V_h \subset V$, we get that

$$a(u - u_h, \phi) = a(u, \phi) - a(u_h, \phi) = L(\phi) - L(\phi) = 0.$$

Moreover, since a is bilinear and symmetric, we derive that

$$\begin{aligned} \|x+y\|_a^2 &= a(x+y, x+y) = a(x,x) + 2a(x,y) + a(y,y) = \\ &= \|x\|_a^2 + 2a(x,y) + \|y\|_a^2 \end{aligned}$$

Thus

$$\|u - v\|_a^2 = \|(u - u_h) + (u_h - v)\| = \|u - u_h\|_a^2 + 2a(u - u_h, u_h - v) + \|u_h - v\|_a^2.$$

Since $a(u - u_h, u_h - v) = 0$, we obtain that

$$\|u - v\|_a^2 = \|u - u_h\|_a^2 + \|u_h - v\|_a^2$$

Thus,

$$\|u - u_h\|_a^2 = \|u - v\|_a^2 - \|u_h - v\|_a^2 \le \|u - v\|_a^2$$

Exercise 5.2:

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

The solution is

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{10}{3} \\ \frac{8}{3} \end{bmatrix}.$$

Exercise 5.3:

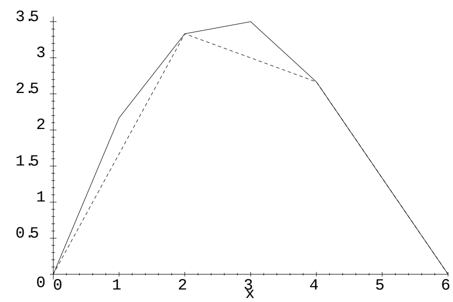


Figure 7.1: The plot of u_h in the case of two and five basis-functions

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

The solution is (not a part of the exercise, see also Figure 7.1):

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix} = \begin{bmatrix} \frac{13}{6} \\ \frac{10}{3} \\ \frac{7}{2} \\ \frac{8}{3} \\ \frac{4}{3} \end{bmatrix}$$

Exercise 5.4: Strong formulation:

$$\operatorname{div} 1 \operatorname{grad} u = \operatorname{div} \operatorname{grad} u = \frac{d}{dx}\left(\frac{du}{dx}\right) = 1$$

Integration gives:

$$\frac{du}{dx} = x + c_1$$

$$u(x) = \frac{1}{2}x^2 + c_1 x + c_2$$

Since $u(0) = u(1) = 0$, $c_2 = 0$ and $0 = \frac{1}{2}1^2 + c_1 1$, i.e. $c_1 = -\frac{1}{2}$. Thus

$$u(x) = \frac{1}{2}x^2 - \frac{1}{2}x.$$

Exercise 5.5: From above we have that

$$\frac{du}{dx} = x + c_1$$

Since we are dealing with the Neumann problem we have that

$$\frac{du}{dx}(0) = 0 + c_1 = 0,$$

(i.e. $c_1 = 0$) and

$$\frac{du}{dx}(1) = 1 + c_1 = 0,$$

(i.e. $c_1 = -1$) which is a contradiction, so this is impossible, there exists no solution in this case. On the other hand if $f = -\sin x$, $\Omega = [0, 2\pi]$, then

$$\frac{d}{dx}\left(\frac{du}{dx}\right) = \sin x,$$

which gives that

$$\frac{du}{dx} = -\cos x + c_1.$$

The Neumann condition gives that

$$\frac{du}{dx}(0) = -\cos 0 + c_1 = 0$$

(i.e. $c_1 = 1$) and

$$\frac{du}{dx}(2\pi) = -\cos 2\pi + c_1 = 0$$

(i.e. $c_1 = 1$). Thus there are no contradictions as above where $f = 1$. Integrating $-\cos x + 1$ we obtain that

$$u = -\sin x + x + c_2,$$

where c_2 is an arbitrary constant. We note that

$$\int_\Omega f dx = \int_0^{2\pi} \sin x dx = 0.$$

According to Exercise 3.7) this is in general a necessary condition for obtaining a solution for the Neumann problem. In the previous exercise

$$\int_\Omega f dx = \int_0^{1} 1 dx = 1 \neq 0.$$

Thus, we conclude that our observations agrees with the general theory stated in Exercise 3.7.

Exercise 5.6:

$$A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

Exercise 5.7:

$$A = $$

$$\begin{bmatrix}
4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4
\end{bmatrix}$$

7.1. Solution to further exercises

Exercise 6.2:
$\|u\|_{L^2} = \sqrt{\int_0^{2\pi}\int_0^{2\pi} (\cos x_1)^2 \, dx_1 dx_2} = \sqrt{2}\pi,$
$\|u\|_{W^{1,2}} = \sqrt{\int_0^{2\pi}\int_0^{2\pi} \left((\cos x_1)^2 + (-\sin x_1)^2\right) dx_1 dx_2} = 2\pi$
Exercise 6.3: The smallest space u belongs to is: 1) $W^{1,2}(Y)$ 2)$W^{1,2}(Y)$ 3)$L^2(Y)$ 4)non of these spaces 5)$W^{1,2}_{per}(Y)$ 6)$W^{1,2}(Y)$ 7)$W^{1,2}_0(Y)$

Exercise 6.6: We multiply a test function $\phi \in W_0^{1,2}(\Omega)$ on both sides, and integrate

$$\int_\Omega \phi \operatorname{div}\left[\frac{\partial u(x)}{\partial x_1}, k\frac{\partial u(x)}{\partial x_2}\right] dx = -\int_\Omega \phi f \, dx.$$

We do as before except that we use

$$\mathbf{w} = \left[\frac{\partial u(x)}{\partial x_1}, k\frac{\partial u(x)}{\partial x_2}\right]$$

and obtain the weak formulation.

Exercise 6.7 and 6.8: We observe first that $a(u, v)$ is a bilinear form and want to check if the conditions 1-4 are satisfied: condition 1)

$$a(u,v) = \int_\Omega \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + k\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2} dx = \int_\Omega \frac{\partial v}{\partial x_1}\frac{\partial u}{\partial x_1} + k\frac{\partial v}{\partial x_2}\frac{\partial u}{\partial x_2} dx = a(v,u).$$

Condition 2:

$$|a(u,v)| = \left|\int_\Omega \left[\frac{\partial u(x)}{\partial x_1}, k\frac{\partial u(x)}{\partial x_2}\right] \cdot \operatorname{grad} v \, dx\right|$$

$$\underset{\text{Scwarz ineq.}}{\leq} \sqrt{\int_\Omega \left|\left[\frac{\partial u(x)}{\partial x_1}, k\frac{\partial u(x)}{\partial x_2}\right]\right|^2 dx}\sqrt{\int_\Omega |\operatorname{grad} v|^2 \, dx} =$$

$$= \sqrt{\int_\Omega \left(\frac{\partial u(x)}{\partial x_1}\right)^2 + k^2\left(\frac{\partial u(x)}{\partial x_2}\right)^2 dx}\sqrt{\int_\Omega |\operatorname{grad} v|^2 \, dx} \leq$$

$$\underset{\text{since } 1\le k}{\le} \sqrt{\int_\Omega k^2 \left(\frac{\partial u(x)}{\partial x_1}\right)^2 + k^2 \left(\frac{\partial u(x)}{\partial x_2}\right)^2 dx} \sqrt{\int_\Omega |\text{grad}\, v|^2 \, dx} =$$

$$= k\sqrt{\int_\Omega \left(\frac{\partial u(x)}{\partial x_1}\right)^2 + \left(\frac{\partial u(x)}{\partial x_2}\right)^2 dx} \sqrt{\int_\Omega |\text{grad}\, v|^2 \, dx} =$$

$$= k\sqrt{\int_\Omega |\text{grad}\, u|^2 \, dx} \sqrt{\int_\Omega |\text{grad}\, v|^2 \, dx} \le$$

$$\le k\sqrt{\int_\Omega |\text{grad}\, u|^2 + |u|^2 \, dx} \sqrt{\int_\Omega |\text{grad}\, v|^2 + |v|^2 \, dx} = k\,\|u\|\,\|v\|$$

i.e.

$$|a(u,v)| \le k\,\|u\|\,\|v\|$$

Condition 3:

$$a(v,v) = \left| \int_\Omega \left[\frac{\partial v(x)}{\partial x_1}, k\frac{\partial v(x)}{\partial x_2}\right] \cdot \text{grad}\, v \, dx \right| =$$

$$\left| \int_\Omega \left(\frac{\partial v(x)}{\partial x_1}\right)^2 + k\left(\frac{\partial v(x)}{\partial x_2}\right)^2 dx \right| \underset{\text{since } k\ge 1}{\ge} \left| \int_\Omega \left(\frac{\partial v(x)}{\partial x_1}\right)^2 + \left(\frac{\partial v(x)}{\partial x_2}\right)^2 dx \right| =$$

$$\int_\Omega |\text{grad}\, v\,|^2 \, dx \ge \frac{1}{C_0+1}\,\|v\|^2$$

The last inequality follows by Friedrich's inequality [This is shown earlier in the text for the Dirichlet problem]. Thus $|a(v,v)| \ge \alpha\,\|v\|^2$ for some positive constant α.

Condition 4: The proof is the same as for the Dirichlet problem (see the text).

Since conditions 2-4 are satisfied there exists a unique solution by Lax-Milgram lemma (Theorem 3.1), moreover since conditions 1-4 are satisfied the equivalence of the two formulations follows by Theorem 3.2.

Exercise 6.9: Again we must show that conditions 2-4 are satisfied. We have already proved condition 2) and 3) for $a(w,v)$ above.

Condition 4: We have that:

$$|G(v)| = |L(v) - a(g,v)| \le |L(v)| + |a(g,v)| \tag{7.6}$$

Since $L(v)$ satisfies condition 4 ($|L(v)| \leq c \|v\|$ for some positive constant c) and $a(g, v)$ satisfies condition 2 ($|a(g, v)| \leq \alpha \|g\| \|v\|$ for some positive constant α) we obtain from (7.6) that

$$|G(v)| \leq c \|v\| + \alpha \|g\| \|v\| = (c + \alpha \|g\|) \|v\|$$

i.e.

$$|G(v)| \leq \Lambda \|v\|, \ \forall v$$

for some positive constant Λ ($= c + \alpha \|g\|$). Thus the existence and uniqueness of the solution follows by Lax-Milgram lemma.

Exercise 6.10: $u - g = (w + g) - g = w \in W_0^{1,2}(\Omega)$. Moreover,

$$\begin{aligned} a(u, v) &= a(w + g, v) \underset{\text{bilinear}}{=} a(w, v) + a(g, v) = G(v) + a(g, v) \\ &= (L(v) - a(g, v)) + a(g, v) = L(v). \end{aligned}$$

Summing up: $u - g \in W_0^{1,2}(\Omega)$ and $a(u, v) = L(v)$ for all $v \in W^{1,2}(\Omega)$. Hence u is a solution of the problem. The uniqueness of the solution is proved as follows: Let u_1 and u_2 be solutions $w_1 = u_1 - g$ and $w_2 = u_2 - g$ are solutions of the problem given in Exercise 6.9 (since $a(w_1, v) = a(u_1, v) - a(g, v) = L(v) - a(g, v) = G(v)$ and similarly that $a(w_2, v) = G(v)$). But since these are unique, we have that $w_1 = w_2$, i.e. $u_1 = u_2$)

Exercise 6.11: u_h converges to the exact solution u, which can be found by solving the problem (as before)

$$\operatorname{div} \frac{1}{2} \operatorname{grad} u = \frac{1}{2} \operatorname{div} \operatorname{grad} u = \frac{1}{2} \frac{d}{dx} \left(\frac{du}{dx} \right) = 1$$

$$u(x) = x^2 - x.$$