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Schrödinger operators with slowly decaying potentials

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Abstract

Several recent papers have obtained bounds on the distribution of eigenvalues of non-self-adjoint Schrödinger operators and resonances of self-adjoint operators. In this paper we describe two new methods of obtaining such bounds when the potential decays more slowly than previously permitted.

Keywords: Schrödinger operators; Complex eigenvalues; Non-self-adjoint operators

0. Introduction

We consider the Schrödinger operator

$$Hf(x) := -\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + V(x)f(x) := H_0 f(x) + V(x)f(x) \tag{1}$$

acting in $L^2(\mathbb{R})$, where V is a complex-valued potential defined in $\mathbb{R}$ and H is defined by the quadratic form technique. This paper arose from earlier work by Abramov et al. [1,2] which considered the same operator in N-dimensional Euclidean space with potential $V(x)$ vanishing sufficiently rapidly as $|x| \to \infty$. They found bounds for eigenvalues when V is complex valued and also bounds for the resonances when V is real valued. A related paper of Brown and Eastham in one dimension also requires the potential to lie in $L^{-1}(\mathbb{R})$, [3]. The importance of such bounds is that they delimit the region in the complex plane within which one needs to search for eigenvalues by numerical complex analysis techniques. Efficient bounds may increase the speed of such a search by a substantial amount.

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In this paper we find bounds when the potential decays less rapidly at infinity than either of these papers permit. We confine our investigation to one dimension, but the basic idea can be extended to higher dimensions. In Section 1 we describe an L^2 method of finding a region in the complex plane which contains the spectrum of H. In Section 2 we investigate the same problem but consider the operator acting in L^1 and L^∞. We show that the spectrum of the operator in both these spaces lies in a certain region of the complex plane. We then interpolate to prove that this region contains the spectrum of the operator acting in L^2. In Section 3 we consider some examples of potentials for which we implement the ideas previously described. Our method yields better bounds than existing methods in the literature even when those other methods are applicable. The computations involved are not trivial, but they only involve the evaluation of integrals, not any special calculations. In Section 4 we compare the methods and also make some further comments.

1. Estimates using L^2 methods

We will consider potentials which decay slowly at infinity in the following sense.

Definition 1.1. We say $V(x)$ belongs to $L^1 + L_0^\infty$ if $V(x)$ can be expressed as $V(x) = W(x) + X(x)$ where $W(x)$ belongs to $L^1(\mathbb{R})$ and $X(x)$ in $L_0^\infty(\mathbb{R})$, where $X \in L_0^\infty$ if and only if X is bounded and measurable with $\lim_{|x|\to\infty} X(x) = 0$.

If $V \in L^1 + L_0^\infty$ and $\varepsilon > 0$ then there exists a decomposition $V = V_\varepsilon + X_\varepsilon$, where $V_\varepsilon \in L^1(\mathbb{R})$ and $\|X_\varepsilon\|_\infty \leqslant \varepsilon$. It is clear that it is possible to have many such decompositions for a particular $V(x)$ and a given ε. If $V(x)$ belongs to $L_1 + L_0^\infty$ then for any given $\varepsilon > 0$ there exists a positive constant C_ε defined by

$$C_\varepsilon = \inf\{\|W\|_1 : V = W + X \text{ and } \|X\|_\infty \leqslant \varepsilon\}, \tag{2}$$

where the infimum is taken over all such decompositions. This corresponds to the optimal decomposition, given by $V = W_\varepsilon + X_\varepsilon$, where

$$X_\varepsilon(x) = \begin{cases} \varepsilon \dfrac{V(x)}{|V(x)|} & \text{if } |V(x)| > \varepsilon, \\ V(x) & \text{if } |V(x)| \leqslant \varepsilon \end{cases} \tag{3}$$

and where $W_\varepsilon(x)$ is defined by

$$W_\varepsilon(x) = \begin{cases} V(x)\left(1 - \dfrac{\varepsilon}{|V(x)|}\right) & \text{if } |V(x)| > \varepsilon, \\ 0 & \text{if } |V(x)| \leqslant \varepsilon. \end{cases} \tag{4}$$

This is because the stated choice of X_ε reduces $|V(x)|$ to the maximum extent possible for every $x \in \mathbb{R}$. In that case $C_\varepsilon = \|W_\varepsilon\|_1$. Now if $V(x)$ is bounded then for $\varepsilon \geqslant \|V\|_\infty$

$$X_\varepsilon(x) = V(x) \quad \text{and} \quad W_\varepsilon(x) = 0$$

and so $C_\varepsilon = 0$. But for $\varepsilon < \|V\|_\infty$ we have $W_\varepsilon \neq 0$ and so $C_\varepsilon > 0$. If $V(x)$ is essentially unbounded then $0 < C_\varepsilon < \infty$ for all $\varepsilon > 0$. The following two lemmas give some more properties of C_ε.

Lemma 1.2. *C_ε is a non-negative, decreasing convex function of ε for $\varepsilon > 0$. It is bounded if and only if $V \in L^1(\mathbb{R})$ in which case*

$$\lim_{\varepsilon \to 0+} C_\varepsilon = \|V\|_1.$$

Also the equation $C_\varepsilon = k\sqrt{\varepsilon}$ has a unique solution for any positive real number k.

Proof. It is very easy to see that C_ε is a non-negative decreasing function of ε for $\varepsilon > 0$. For convexity, let ε_1 and ε_2 be any two positive real numbers and $\varepsilon_1 < \varepsilon_2$. For α in $[0,1]$ we have

$$\alpha W_{\varepsilon_1} + (1-\alpha) W_{\varepsilon_2}(x) = \begin{cases} V(x)\left(1 - \dfrac{\alpha\varepsilon_1 + (1-\alpha)\varepsilon_2}{|V(x)|}\right) & \text{if } |V(x)| > \varepsilon_2, \\ \alpha V(x)\left(1 - \dfrac{\varepsilon_1}{|V(x)|}\right) & \text{if } \varepsilon_2 \geqslant |V(x)| > \varepsilon_1, \\ 0 & \text{if } |V(x)| \leqslant \varepsilon_1. \end{cases}$$

According to (3) and (4) the expressions

$$W_{\alpha\varepsilon_1+(1-\alpha)\varepsilon_2}(x) = \begin{cases} V(x)\left(1 - \dfrac{\alpha\varepsilon_1 + (1-\alpha)\varepsilon_2}{|V(x)|}\right) & \text{if } |V(x)| > \alpha\varepsilon_1 + (1-\alpha)\varepsilon_2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$X_{\alpha\varepsilon_1+(1-\alpha)\varepsilon_2}(x) = \begin{cases} (\alpha\varepsilon_1 + (1-\alpha)\varepsilon_2)\dfrac{V(x)}{|V(x)|} & \text{if } |V(x)| > \alpha\varepsilon_1 + (1-\alpha)\varepsilon_2, \\ V(x) & \text{if } |V(x)| \leqslant \alpha\varepsilon_1 + (1-\alpha)\varepsilon_2 \end{cases}$$

gives the decomposition for the potential $V(x)$ corresponding to $\alpha\varepsilon_1 + (1-\alpha)\varepsilon_2$. Now comparing the definitions of $\alpha W_{\varepsilon_1} + (1-\alpha)W_{\varepsilon_2}$ and $W_{\alpha\varepsilon_1+(1-\alpha)\varepsilon_2}$ and noting that $\varepsilon_1 \leqslant \alpha\varepsilon_1 + (1-\alpha)\varepsilon_2 \leqslant \varepsilon_2$ we obtain

$$\begin{aligned} \alpha C_{\varepsilon_1} + (1-\alpha) C_{\varepsilon_2} &= \alpha\|W_{\varepsilon_1}\|_1 + (1-\alpha)\|W_{\varepsilon_2}\|_1 \\ &\geqslant \|\alpha W_{\varepsilon_1} + (1-\alpha) W_{\varepsilon_2}\|_1 \\ &\geqslant \|W_{\alpha\varepsilon_1+(1-\alpha)\varepsilon_2}\|_1 \\ &= C_{\alpha\varepsilon_1+(1-\alpha)\varepsilon_2}, \end{aligned}$$

which proves that C_ε is a convex function of ε.

If $V \in L^1(\mathbb{R})$ then with the decomposition described in (3) and (4) it is clear that

$$C_\varepsilon \leqslant \|V\|_1,$$

for every $\varepsilon > 0$. In this case we note that

$$\begin{aligned} C_\varepsilon &= \int_{\{x:\ |V(x)| > \varepsilon\}} \{|V(x)| - \varepsilon\}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f_\varepsilon(x)\,\mathrm{d}x, \end{aligned}$$

where $f_\varepsilon(x) = \chi_{\{x:\ |V(x)|>\varepsilon\}}(|V(x)| - \varepsilon)$ is a non-negative decreasing function of ε where χ_A denotes the characteristic function of the set A. So by the monotone convergence theorem we obtain

$$\lim_{\varepsilon\to 0+} C_\varepsilon = \|V\|_1.$$

Conversely, if V does not belong to L^1 then an application of the monotone convergence theorem results in $C_\varepsilon \to \infty$ as $\varepsilon \to 0+$. The last part of the lemma is obvious.

Lemma 1.3. *Let* $1 < p < \infty$ *and* $V \in L^p(\mathbb{R})$. *Then for any* $\varepsilon > 0$ *we have*

$$C_\varepsilon \leqslant k\varepsilon^{-\gamma}, \tag{5}$$

where $k = \|V\|_p^p$ *and* $\gamma = p - 1$.

Proof. From the decomposition of $V(x)$ into $W_\varepsilon(x)$ and $X_\varepsilon(x)$ we have

$$|W_\varepsilon(x)| \leqslant |V(x)|.$$

So

$$\begin{aligned}\|V\|_p^p &= \int_{\mathbb{R}} |V|^p \\ &\geqslant \int_{|V(x)|>\varepsilon} |V|^p \\ &\geqslant \int_{|V(x)|>\varepsilon} \varepsilon^{p-1}|V| \\ &\geqslant \varepsilon^{p-1}\|W_\varepsilon\|_1.\end{aligned}$$

This is equivalent to (5).

Now to locate the spectrum, $\sigma(H)$, of H we first note that the essential spectrum of H and H_0 are equal. This is because for any z with large negative $\mathrm{Re}(z)$ we have

$$(H - z)^{-1} - (H_0 - z)^{-1} = -(H - z)^{-1}V(H_0 - z)^{-1}.$$

If we write $V = W_\varepsilon + X_\varepsilon$ and $W_\varepsilon = W_1W_2$, where $W_1 = |W_\varepsilon|^{1/2}$ and $W_2 = W_\varepsilon/|W_1|^{1/2}$ then the right-hand side of the above equation can be written as

$$(H - z)^{-1} - (H_0 - z)^{-1} = A_\varepsilon + B_\varepsilon,$$

where

$$\begin{aligned}A_\varepsilon &= -(H - z)^{-1}W_\varepsilon(H_0 - z)^{-1}, \\ B_\varepsilon &= -(H - z)^{-1}X_\varepsilon(H_0 - z)^{-1}.\end{aligned}$$

Since

$$\|B_\varepsilon\|_2 \leqslant \|(H - z)^{-1}\|_2\, \|(H_0 - z)^{-1}\|_2\, \varepsilon \to 0$$

as $\varepsilon \to 0$, it is sufficient to prove that A_ε is compact for every $\varepsilon > 0$. Now we write $A_\varepsilon = G_\varepsilon^* D_\varepsilon$, where

$$G_\varepsilon = \bar{W}_1(H^* - \bar{z})^{-1},$$

$$D_\varepsilon = W_2(H_0 - z)^{-1}.$$

Then clearly D_ε is a Hilbert Schmidt operator. So we have finally to prove that G_ε is bounded. The operators H and H^* are defined by quadratic form techniques and their operator domains are continuously embedded in $W^{1,2}(\mathbb{R})$, where H^* is adjoint of H and $W^{1,2}(\mathbb{R})$ is Sobolev space. Therefore,

$$(H^* - \bar{z})^{-1} : L^2 \to \mathrm{Dom}(H^*) \to W^{1,2}(\mathbb{R}) \to L^\infty(\mathbb{R}),$$

all being continuous embeddings and $\mathrm{Dom}(H^*)$ denotes the domain of H^*, and

$$\bar{W}_1(H^* - \bar{z})^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

must be bounded.

After this observation it is left to locate the isolated eigenvalues. The following theorem provides the basis for the subsequent results. Here we recall that in [1] the authors proved that if $V(x)$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then

$$\sigma(H) \subset [0,\infty) \cup B\left(0, \frac{\|V\|_1^2}{4}\right),$$

where $B(z,r)$ denotes an open ball in the complex plane of radius r centred at z. The same proof works for $V \in L^1(\mathbb{R})$ subject to the type of domain consideration just presented.

Theorem 1.4. *If $\varepsilon > 0$ and $W_\varepsilon, X_\varepsilon$ are defined by (3) and (4) then z does not belong to $\sigma(H_0 + V)$ if $\|(H_0 + W_\varepsilon - z)^{-1}\|_2 < \varepsilon^{-1}$.*

Proof. From the decomposition of $V(x)$ in (3) and (4) we have

$$\begin{aligned}(H_0 + V - z)^{-1} &= (H_0 + W_\varepsilon - z + X)^{-1} \\ &= (H_0 + W_\varepsilon - z)^{-1} - (H_0 + W_\varepsilon - z)^{-1} X (H_0 + V - z)^{-1}.\end{aligned}$$

This implies that

$$(H_0 + V - z)^{-1} = \{1 + (H_0 + W_\varepsilon - z)^{-1} X\}^{-1} (H_0 + W_\varepsilon - z)^{-1}. \tag{6}$$

Now since z is such that

$$\|(H_0 + W_\varepsilon - z)^{-1}\|_2 < \varepsilon^{-1}$$

therefore we have

$$\|(H_0 + W_\varepsilon - z)^{-1} X_\varepsilon\|_2 < 1.$$

This implies that the right-hand side of (6) is well defined and so z does not belong into $\sigma(H_0 + V)$.

Our next theorem provides an explicit, but somewhat complicated, condition on z which implies the hypothesis of Theorem 1.4. We will subsequently show how to apply the condition, and also present a simplified but less effective version, Theorem 1.7.

Theorem 1.5. *Let H be defined as in (1) and $V \in L_0^1$. Let $\varepsilon > 0$. Then the resolvent set of H contains all $z = -\lambda^2$ such that $\mathrm{Re}(\lambda) > 0$ and*

$$\frac{1}{\mathrm{d}(z)} + \frac{C_\varepsilon}{2\,\mathrm{Re}(\lambda)|\lambda|(2\,|\lambda| - C_\varepsilon)} < \frac{1}{\varepsilon}, \tag{7}$$

and $C_\varepsilon < 2|\lambda|$ where $\mathrm{d}(z) := \mathrm{dist}(z, \sigma(H_0))$, C_ε is as defined by (4).

Proof. Let $V = W_\varepsilon + X_\varepsilon$ where $\|X_\varepsilon\|_\infty = \varepsilon$ and $W_\varepsilon \in L^1(\mathbb{R})$. Also we put $C_\varepsilon = \|W_\varepsilon\|_1$ following definition (3) and (4). Let z be such that $(H_0 + W_\varepsilon - z)^{-1}$ exists and write

$$(H_0 + W_\varepsilon - z)^{-1} = (H_0 - z)^{-1} - (H_0 - z)^{-1} W_\varepsilon (H_0 + W_\varepsilon - z)^{-1}.$$

Let us put

$$W_\varepsilon = W_1 W_2,$$

where $W_1 = |W_\varepsilon|^{1/2}$ and $W_2 = W_\varepsilon / W_1$. Using this decomposition in the above resolvent equation and then an iterative argument yields

$$(H_0 + W_\varepsilon - z)^{-1} = (H_0 - z)^{-1} - (H_0 - z)^{-1} W_1 D W_2 (H_0 - z)^{-1},$$

where

$$D = (1 + W_2 (H_0 - z)^{-1} W_1)^{-1}$$

and we assume that $\|W_2(H_0 - z)^{-1} W_1\| < 1$. Thus we have

$$\|(H_0 + W_\varepsilon - z)^{-1}\|_2 \leqslant \|(H_0 - z)^{-1}\|_2 + \|(H_0 - z)^{-1} W_1\|_2\, \|D\|_2\, \|W_2 (H_0 - z)^{-1}\|_2.$$

In the following we denote the Hilbert Schmidt norm by $\|\ \|_{\mathrm{HS}}$ and use the fact that Hilbert Schmidt norm is greater than equal to the operator norm. Noting also the equality

$$\|(H_0 - z)^{-1}\|_2 = \frac{1}{\mathrm{d}(z)},$$

we obtain that

$$\|(H_0 + W_\varepsilon - z)^{-1}\|_2 \leqslant \frac{1}{\mathrm{d}(z)} + \|(H_0 - z)^{-1} W_1\|_{\mathrm{HS}}\, \|D\|_2\, \|W_2 (H_0 - z)^{-1}\|_{\mathrm{HS}}. \tag{8}$$

Let us put $z = -\lambda^2$ where $\mathrm{Re}(\lambda) > 0$. Using this and observing that the integral kernel of the operator $(H_0 + \lambda^2)^{-1} W_1$ is

$$\frac{\mathrm{e}^{-\lambda\,|x-y|}}{2\lambda}\, W_1(y)$$

we get

$$\begin{aligned}\|(H_0 + \lambda^2)^{-1} W_1\|_{\mathrm{HS}}^2 &= \int_{\mathbb{R}^2} \frac{\mathrm{e}^{-2\mathrm{Re}(\lambda)|x-y|}}{4|\lambda|^2}\, |W_1(y)^2|\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \frac{\|W_\varepsilon\|_1}{4|\lambda|^2 \mathrm{Re}(\lambda)}.\end{aligned}$$

Similarly we obtain the following estimate

$$\|W_2(H_0+\lambda^2)^{-1}\|_{\mathrm{HS}}^2 \leqslant \frac{\|W_\varepsilon\|_1}{4|\lambda|^2\mathrm{Re}(\lambda)}.$$

For the estimate of $\|D\|_2$,

$$\begin{aligned}\|D\|_2 &= \|(I+E)^{-1}\|_2\\ &\leqslant \frac{1}{1-\|E\|_2}\\ &\leqslant \frac{1}{1-\|E\|_{\mathrm{HS}}}\end{aligned}$$

provided that $\|E\|_{\mathrm{HS}}<1$, where $E=W_2(H_0+\lambda^2)^{-1}W_1$, which is compact. Now using the integral kernel

$$W_2(x)\,\frac{\mathrm{e}^{-\lambda|x-y|}}{2\lambda}\,W_1(y)$$

of E we get

$$\begin{aligned}\|E\|_{\mathrm{HS}}^2 &= \int_{\mathbb{R}^2}|W_2(x)|^2\,\frac{\mathrm{e}^{-2\mathrm{Re}(\lambda)|x-y|}}{4|\lambda|^2}\,|W_1(y)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &\leqslant \frac{\|W_\varepsilon\|_1^2}{4|\lambda|^2}.\end{aligned}$$

Thus

$$\|D\|_2 \leqslant \frac{1}{1-\|W_\varepsilon\|_1/2|\lambda|},$$

provided that $\|W_\varepsilon\|_1<2|\lambda|$. Hence putting all the above estimates into (8) we get

$$\|(H_0+W_\varepsilon-z)^{-1}\| \leqslant \frac{1}{\mathrm{d}(z)}+\frac{C_\varepsilon}{2|\lambda|\mathrm{Re}(\lambda)(2|\lambda|-C_\varepsilon)}.$$

Now if z is such that it satisfies (7) with conditions associated with it then from the above inequality together with Theorem 1.4 we conclude that such a z belongs to the resolvent of H.

We now show how Theorem 1.5 can directly be used to find a region numerically containing the spectrum of the operator H. By Theorem 1.5 the resolvent of H contains the set of all $z=-\lambda^2$ satisfying (7) for some $\varepsilon>0$ such that $C_\varepsilon<2|\lambda|$. Let us define the real-valued function f by

$$f(z,\varepsilon)=\frac{1}{\mathrm{d}(z)}+\frac{C_\varepsilon}{2\mathrm{Re}(\lambda)\,|\lambda|(2|\lambda|-C_\varepsilon)}-\frac{1}{\varepsilon},$$

for $z\notin\mathbb{R}_+$ and $\varepsilon>0$, where $\mathbb{R}_+$ is the set of non-negative real numbers, and $\mathrm{d}(z)$ and C_ε are as defined in Theorem 1.5. If we define

$$g(z)=\inf_{\varepsilon>0}\{f(z,\varepsilon)\colon\ C_\varepsilon<2|\lambda|\}$$

then clearly

$$\sigma(H) \subset \{z\colon\, g(z) \geqslant 0\}.$$

The boundary of the right-hand side is the contour on which $g(z) = 0$.

Example 1.6. We demonstrate this method for the potential

$$V(x) = c|x|^{\alpha-1}, \tag{9}$$

where c is a complex number and $0 < \alpha < 1$. Then for $\varepsilon > 0$ we have

$$C_\varepsilon = \|W_\varepsilon\|_1 = \frac{2(1-\alpha)}{\alpha}\,|c|^{1/(1-\alpha)}\varepsilon^{-\alpha/(1-\alpha)}. \tag{10}$$

For simplicity we take $\alpha = \frac{1}{4}$ and c be such that $|c| = 1$. Since we will use MATLAB to find the contour, we consider a finite set S of values of ε and then we write

$$g_1(z) = \inf_{\varepsilon>0}\{f(z,\varepsilon)\colon\, C_\varepsilon < 2|\lambda| \text{ and } \varepsilon \in S\}.$$

Clearly, $g(z) \leqslant g_1(z)$ for every z and $\sigma(H) \subset \{z\colon\, g_1(z) \geqslant 0\}$. The quality of the bound obtained on $\sigma(H)$ clearly depends on the number of points in S and how they are chosen.

To draw the contour using MATLAB we first took a finite number of values for $\mathrm{Re}(z) = -a, -a+\delta, \ldots, a$ for $a > 0$ and $\mathrm{Im}(z) = -b, -b+h, \ldots, b$ for $b > 0$, where $\delta = 2a/m$, $h = 2b/n$ and m, n are positive integers. This gave us a $(m+1) \times (n+1)$ matrix of values of z and consequently λ and $\mathrm{d}(z)$. The contour in Fig. 1 was obtained by putting $a = 15$, $b = 13$ and $m = n = 200$ and choosing $S = \{(1.1)^{(j-15)},\ j = 1, 2, \ldots, 200\}$.

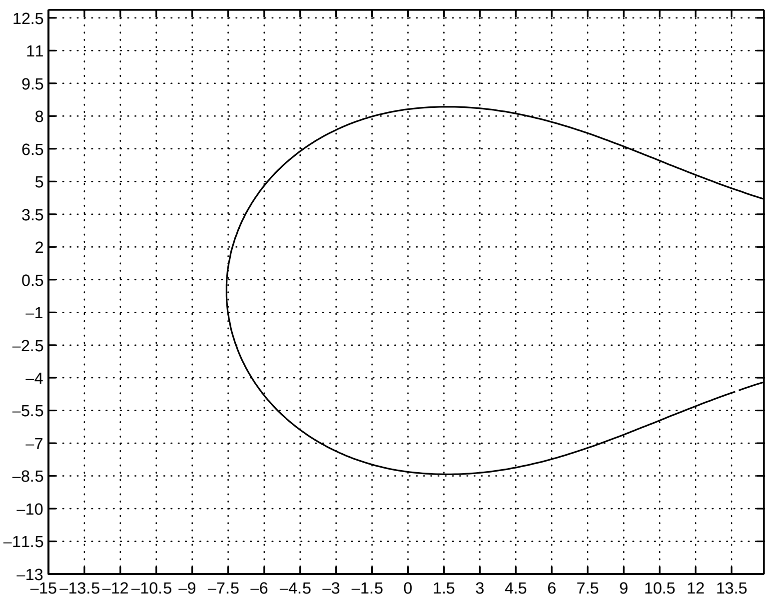

Fig. 1. Region obtained by using Theorem 1.5 for $V(x) = |x|^{-1/4}$.

We next describe a simpler condition, which we will later prove is weaker than that obtained using the L^1-method of Section 2. The following theorem is a simplification of condition (7).

Theorem 1.7. *The resolvent of the operator H defined by* (1) *contains the set* $\{z=-\lambda^2\colon \lambda\in\bigcup_{\varepsilon>0} U_\varepsilon\}$, *where U_ε is given by*

$$U_\varepsilon=\{\lambda\in\mathbb{C}\colon\ \mathrm{Re}(\lambda)\geqslant\sqrt{\varepsilon}\ \textit{and}\ |\lambda|\geqslant\max\{C_\varepsilon/2+\sqrt{\varepsilon},\sqrt{2\varepsilon}\}\}.$$

Proof. From Theorem 1.5 we know that the resolvent of H contains all z in complex plane for which condition (7) is satisfied with the conditions imposed in that theorem. Let $z=-\lambda^2$ where $\lambda\in U_\varepsilon$ for some $\varepsilon>0$. It is clear from the definition of U_ε that for λ in U_ε we have $C_\varepsilon<2|\lambda|$.

First, we establish that $\mathrm{d}(z)\geqslant 2\varepsilon$. We observe that if $\mathrm{Re}(z)\geqslant 0$ then we always have $|\mathrm{Im}(z)|\geqslant 2\varepsilon$. This is because if we put $\lambda=\lambda_1+\mathrm{i}\lambda_2$ then

$$|\lambda|\geqslant\sqrt{2\varepsilon}$$

together with $z=-\lambda^2$ and $\mathrm{Re}(z)=(\lambda_2^2-\lambda_1^2)\geqslant 0$ implies that

$$|\lambda_2|\geqslant\sqrt{\varepsilon}.$$

From this we get

$$|\mathrm{Im}(z)|=2\lambda_1|\lambda_2|\geqslant 2\varepsilon, \tag{11}$$

which proves our observation.

Since $\sigma(H_0)=[0,\infty)$, if $\mathrm{Re}(z)<0$ then we get $\mathrm{d}(z)=|z|=|\lambda|^2\geqslant 2\varepsilon$ and if $\mathrm{Re}(z)\geqslant 0$ then we get $\mathrm{d}(z)=|\mathrm{Im}(z)|\geqslant 2\varepsilon$ by (11). Using this fact and the definition of U_ε in the left-hand side of inequality (7) we get

$$\frac{1}{\mathrm{d}(z)}+\frac{C_\varepsilon}{2|\lambda|\,\mathrm{Re}(\lambda)(2|\lambda|-C_\varepsilon)}<\frac{1}{\varepsilon}.$$

Thus, by using Theorem 1.5, we conclude that the set of all $z\in\mathbb{C}$ such that $z=-\lambda^2$ with $\lambda\in U_\varepsilon$ is a part of the resolvent. And since this is true for every $\varepsilon>0$ so the statement of the theorem is proved.

In order to describe the set $\bigcup_{\varepsilon>0}U_\varepsilon$ in more detail we examine the set S of points $\lambda=\lambda_1+\mathrm{i}\lambda_2$ such that

$$\mathrm{Re}(\lambda)=\sqrt{\varepsilon}\quad\text{and}\quad|\lambda|=\max\{\tfrac12 C_\varepsilon+\sqrt{\varepsilon},\sqrt{2\varepsilon}\}$$

for some $\varepsilon>0$. By Lemma 1.2 there is a unique solution $\varepsilon=\varepsilon_0$ of

$$\tfrac12 C_\varepsilon+\sqrt{\varepsilon}=\sqrt{2\varepsilon}$$

and one must consider the cases $0<\varepsilon\leqslant\varepsilon_0$, $\varepsilon>\varepsilon_0$ separately. If we put

$$R(\lambda_1)=\begin{cases}\tfrac12 C_{\lambda_1^2}+\lambda_1 & \text{if }\ 0<\lambda_1\leqslant\sqrt{\varepsilon_0},\\ \sqrt{2}\lambda_1 & \text{if }\ \lambda_1>\sqrt{\varepsilon_0}\end{cases} \tag{12}$$

then the derivative, $R'(\lambda_1)$, of $R(\lambda_1)$ is

$$R'(\lambda_1) = \begin{cases} \lambda_1 C'_{\lambda_1^2} + 1 & \text{if } 0 < \lambda_1 \leqslant \sqrt{\varepsilon_0}, \\ \sqrt{2} & \text{if } \lambda_1 > \sqrt{\varepsilon_0}. \end{cases} \tag{13}$$

If $\lambda \in S$ then

$$\lambda_1^2 + \lambda_2^2 = R(\lambda_1)^2.$$

So

$$|\lambda_2| = \Gamma(\lambda_1),$$

where

$$\Gamma(\lambda_1) = \sqrt{R(\lambda_1)^2 - \lambda_1^2}. \tag{14}$$

Hence

$$\Gamma'(\lambda_1) = \{\Gamma(\lambda_1)\}^{-1}\{R(\lambda_1)R'(\lambda_1) - \lambda_1\}. \tag{15}$$

The assumption of the following theorem is easily verified in examples.

Theorem 1.8. *Assume that there exists a largest $\mu > 0$ such that $R(\lambda_1)$ is monotonically decreasing on $[0, \mu]$. Then $\bigcup_{\varepsilon>0} U_\varepsilon$ contains the region to the right of the curve*

$$|\lambda_2| = \begin{cases} \Gamma(\lambda_1) & \textit{if } 0 < \lambda_1 \leqslant \mu, \\ \sqrt{R(\mu)^2 - \lambda_1^2} & \textit{if } \mu < \lambda_1 \leqslant R(\mu). \end{cases} \tag{16}$$

Proof. It is immediate from (13) and the definition of μ that $0 < \mu \leqslant \sqrt{\varepsilon_0}$. It follows from (15) that $\Gamma'(\lambda_1) < 0$ for all $0 < \lambda_1 \leqslant \mu$ and from (14) that $\Gamma(\lambda_1) = \lambda_1$ for all $\lambda_1 \geqslant \sqrt{\varepsilon_0}$. These results are illustrated in Fig. 2. A geometrical argument now establishes that

$$\bigcup_{\varepsilon>0} U_\varepsilon \supseteq \bigcup_{0<\varepsilon<\mu^2} U_\varepsilon$$
$$= \{\lambda\colon\ 0 < \lambda_1 \leqslant \mu \text{ and } |\lambda_2| > \Gamma(\lambda_1)\} \cup \{\lambda\colon\ \lambda_1 > \mu \text{ and } |\lambda| > R(\mu)\}.$$

This is exactly the region to the right of the curve (16), shown in Fig. 3.

To demonstrate this simplified method we consider the potential given by (9) in Example 1.6. In this case we obtain

$$C_\varepsilon = \|W_\varepsilon\|_1 = \frac{2(1-\alpha)}{\alpha} |c|^{1/(1-\alpha)} \varepsilon^{-\alpha/(1-\alpha)}$$

and

$$\sqrt{\varepsilon_0} = \left(\frac{\alpha^{-1} - 1}{\sqrt{2} - 1}\right)^{(1-\alpha)/(1+\alpha)} |c|^{1/(1+\alpha)}.$$

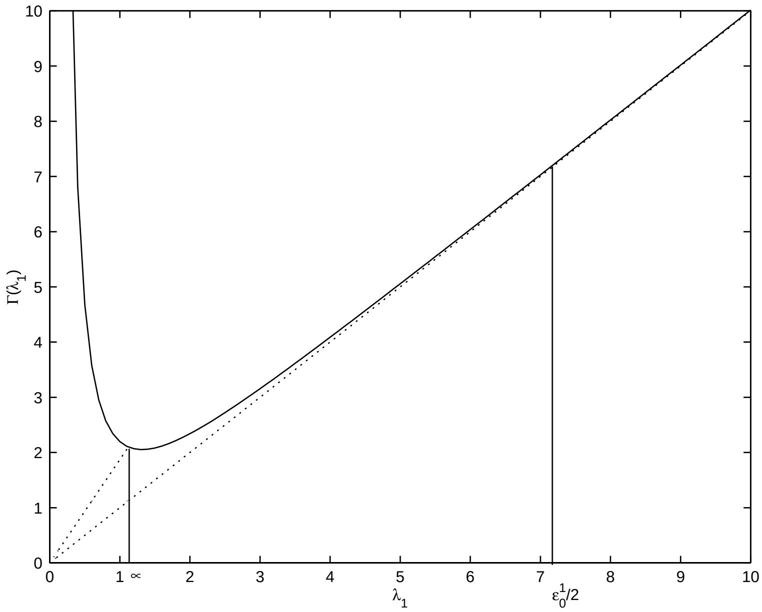

Fig. 2. The graph of the function $\Gamma(\lambda_1)$.

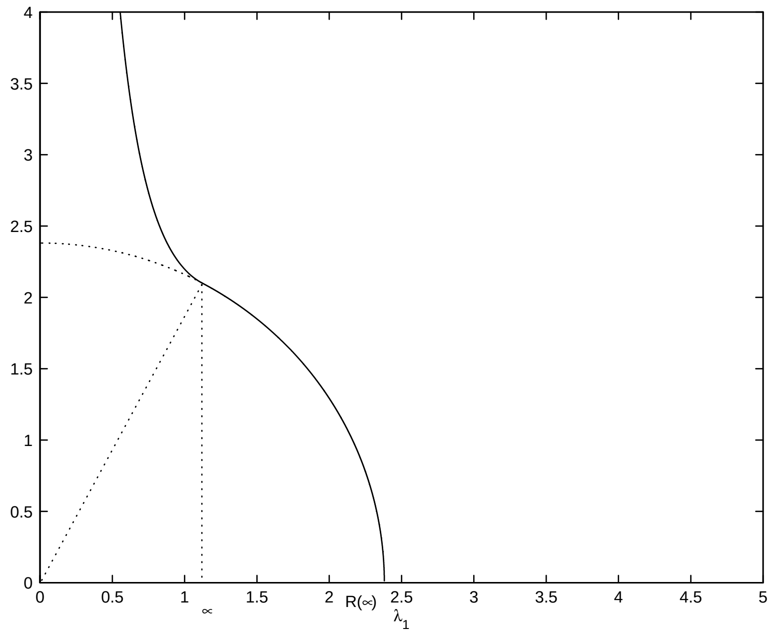

Fig. 3. The curve given by (16).

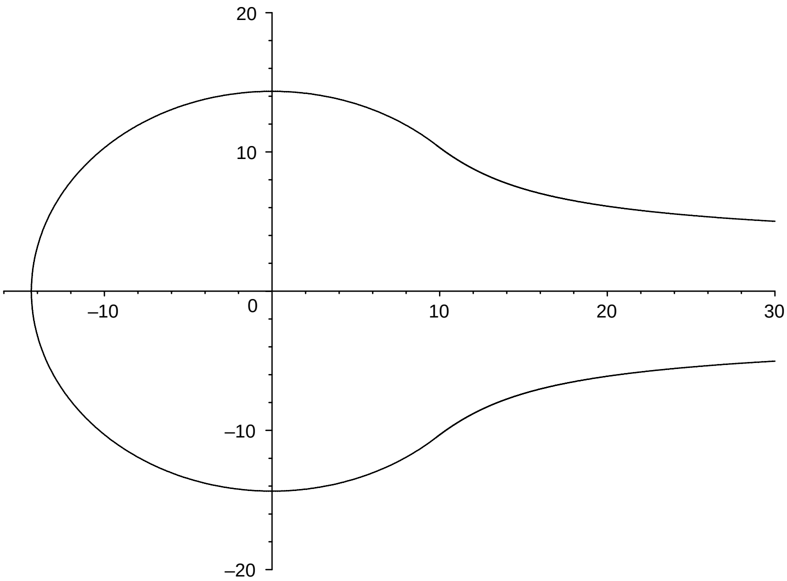

Fig. 4. The curve in the λ-plane.

By evaluating and then minimising $R(\lambda_1)$ we discover that $\mu=\sqrt{\varepsilon_0}$ if $\alpha \geqslant (2\sqrt{2}-1)^{-1}$ but $\mu = 2^{(1-\alpha)/(1+\alpha)}|c|^{1/(1+\alpha)} < \sqrt{\varepsilon_0}$ if $0<\alpha<(2\sqrt{2}-1)^{-1}$.

For $\alpha=\frac{1}{4}$ and $|c|=1$ we get $\mu=2^{3/5}$ and $R(\mu)=5\times 2^{-2/5}$ and the curve obtained in Theorem 1.8 is given by

$$|\lambda_2| = \begin{cases} \sqrt{9\lambda_1^{-4/3}+6\lambda_1^{1/3}} & \text{if } 0<\lambda_1\leqslant \mu, \\ \sqrt{R(\mu)^2-\lambda_1^2} & \text{if } \mu\leqslant\lambda_1\leqslant R(\mu). \end{cases}$$

Now if we apply the transformation $z=-\lambda^2$ then we obtain the corresponding curve in the z-plane.

So the region in the z-plane which contains all the isolated eigenvalues of the operator H is the region enclosed by the curve shown in Fig. 4. It is worth mentioning that the region obtained by this simplified method is larger than the one obtained by using Theorem 1.5, as is obvious from Figs. 1 and 4.

2. Estimates using L^1 methods

In this section we approach the problem by a different method. We consider the operator

$$H_1 = H_0 + V$$

on $L^1(\mathbb{R})$, where

$$H_0 = -\frac{d^2}{dx^2}$$

and V is a complex-valued potential in $L_0^1(\mathbb{R})$. It is known that H_1 is a densely defined operator with the same domain as H_0; see [10] for details.

Let V be a potential of the above-mentioned class. Then let us define the function F on $(0,\infty)$ as follows:

$$F(s) = \sup_y \int_{\mathbb{R}} |V(x)| e^{-s|x-y|} \, dx. \tag{17}$$

The following lemma describes some of the simple properties of the function $F(s)$.

Lemma 2.1. *$F(s)$ is a positive decreasing convex function of s for $s > 0$. It is bounded if and only if $V \in L^1(\mathbb{R})$ in which case*

$$\|V\|_1 = \lim_{s\to 0+} F(s).$$

The equation $F(s) = 2s$ has a unique solution.

Proof. Clearly, for $s > 0$ $F(s)$ is a positive decreasing function. We note that for a fixed value of y and compactly supported $V(x)$ the integral

$$\int_{\mathbb{R}} |V(x)| e^{-s|x-y|} \, dx$$

is a convex function of s; this can be proved by differentiating twice under the integral sign with respect to s. For a general potential we consider the cut-off potential and the corresponding integral and then by a limiting argument we conclude that the integral is convex for any $V(x)$. This uses the fact that the limit of a family of convex functions is a convex function. Also it is easy to see that

$$F(0+) = \begin{cases} \|V\|_1 & \text{if } V \in L^1(\mathbb{R}), \\ +\infty & \text{if } V \notin L^1(\mathbb{R}). \end{cases}$$

Thus, if $V \in L^1(\mathbb{R})$ then we have

$$F(s) \leqslant \|V\|_1$$

for all $s > 0$. Conversely, if V does not belong to $L^1(\mathbb{R})$ then an application of the monotone convergence theorem implies that $F(s) \to +\infty$ as $s \to 0+$. The last statement of the theorem is obvious from the graph of $F(s)$ and $2s$.

The following theorem establishes that the simplified L^2-method obtained by Theorem 1.8 cannot be better than the method we will describe in this section and this we show in Section 4. We will compare the two sides of the inequality (18) in Example 3.3.

Theorem 2.2. *If C_ε is as defined by (3) and (4) then we have*

$$F(s) \leqslant \inf_{\varepsilon>0} \{C_\varepsilon + 2\varepsilon/s\}, \tag{18}$$

for all $s > 0$.

Proof. Let $\varepsilon > 0$ be arbitrary. If $V = W_\varepsilon + X_\varepsilon$, where $W_\varepsilon \in L^1(\mathbb{R})$ and $\|X_\varepsilon\|_\infty = \varepsilon$ and $\|W_\varepsilon\|_1 = C_\varepsilon$ then

$$\begin{aligned} F(s) &= \sup_y \int_{\mathbb{R}} |V(x)| \mathrm{e}^{-s|x-y|} \, \mathrm{d}x \\ &\leqslant \sup_y \left\{ \int_{\mathbb{R}} |W_\varepsilon(x)| \mathrm{e}^{-s|x-y|} \, \mathrm{d}x + \int_{\mathbb{R}} |X_\varepsilon(x)| \mathrm{e}^{-s|x-y|} \, \mathrm{d}x \right\} \\ &\leqslant \|W_\varepsilon\|_1 + 2\|X_\varepsilon\|_\infty / s \\ &= C_\varepsilon + 2\varepsilon/s. \qquad \square \end{aligned}$$

The statement of the theorem follows from this immediately.

Theorem 2.2 determines the growth rate of the function $F(s)$ as $s \to 0+$.

Corollary 2.3. *For a potential which does not belong to* $L^1(\mathbb{R})$ *the function* $F(s)$ *cannot diverge faster than* s^{-1} *as* $s \to 0+$. *In fact* $F(s) = \mathrm{o}(s^{-1})$ *as* $s \to 0+$.

Proof. This is obvious from Theorem 2.2. $\square$

Lemma 2.4. *If* $\mathrm{Re}(\lambda) > 0$ *then*

$$\|V(H_0 + \lambda^2)^{-1}\|_1 = \frac{F(\mathrm{Re}(\lambda))}{2|\lambda|}.$$

Proof. We have

$$V(H_0 + \lambda^2)^{-1} f(x) = \int_{\mathbb{R}} K(x, y) f(y) \, \mathrm{d}y,$$

where

$$K(x, y) = V(x) \frac{\mathrm{e}^{-\lambda|x-y|}}{2|\lambda|}.$$

Then by [6, Theorem 10, p. 507]

$$\begin{aligned} \|V(H_0 + \lambda^2)^{-1}\|_1 &= \sup_y \int_{\mathbb{R}} |K(x, y)| \, \mathrm{d}x \\ &= \frac{F(\mathrm{Re}(\lambda))}{2|\lambda|}. \end{aligned} \tag{19}$$

The following two lemmas will be essential to prove Theorem 2.8 on the essential spectrum of the operator H_p.

Lemma 2.5. *If $K(x)$ is uniformly continuous in $\mathbb{R}$ and $\|K\|_\infty = \sup_x |K(x)| < \infty$ then the operator T defined by*

$$Tf(x) = \int_{\mathbb{R}} K(x-y)f(y)\,\mathrm{d}y,$$

where $x \in [a,b]$, is compact from $L^1(\mathbb{R})$ to the set of all continuous functions on $[a,b]$.

Proof. We note that for $f \in L^1(\mathbb{R})$ and $x \in [a,b]$

$$\begin{aligned} |Tf(x)| &\leqslant \int_{\mathbb{R}} |K(x-y)|\,|f(y)|\,\mathrm{d}y \\ &\leqslant \|K\|_\infty \,\|f\|_1. \end{aligned}$$

This implies that for $\|f\|_1 \leqslant 1$ we have

$$\begin{aligned} \|Tf\|_\infty &\leqslant \|K\|_\infty \\ &< \infty. \end{aligned}$$

Now if we show that the set $\{Tf\colon \|f\|_1 \leqslant 1\}$ is an equicontinuous family of functions then by using Arzela–Ascoli theorem [6, Theorem 7, p. 266] we obtain that the operator T is a compact operator. Let ε be any arbitrary positive real number. By the uniform continuity of K there exists a $\delta > 0$ such that $|K(u) - K(v)| < \varepsilon$ whenever $|u - v| < \delta$. If $|x - x'| < \delta$ then we conclude that

$$\begin{aligned} |Tf(x) - Tf(x')| &\leqslant \int_{\mathbb{R}} |K(x-y) - K(x'-y)|\,|f(y)|\,\mathrm{d}y \\ &\leqslant \varepsilon. \end{aligned}$$

Lemma 2.6. *If $V(x) \in L^1(\mathbb{R})$ and $K(x)$ satisfies the conditions of Lemma 2.5 then the operator T defined by*

$$Tf(x) = \int_{\mathbb{R}} V(x)K(x-y)f(y)\,\mathrm{d}y$$

is compact from $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$.

Proof. Put

$$V_n(x) = \begin{cases} V(x) & \text{if } |x| \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Define $T_n := \mathbf{V}_n S_n$, where $S_n : L^1(\mathbb{R}) \to C[-n,n]$ and $\mathbf{V}_n : C[-n,n] \to L^1(\mathbb{R})$ are defined by

$$S_n f(x) = \int_{\mathbb{R}} K(x-y)f(y)\,\mathrm{d}y$$

and

$$\mathbf{V}_n f(x) = V_n(x)f(x).$$

Now S_n is compact by Lemma 2.5 and $\mathbf{V}_n$ is bounded and therefore the operator T_n defined from $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$ is a compact operator. But then it is obvious that $\|T_n - T\|_1 \to 0$ as $n \to \infty$ which implies that T is a compact operator.

Note 2.7. The result corresponding to Lemma 2.6 is true in $L^\infty(\mathbb{R})$ by a duality argument.

The following theorem provides us with the information about the essential spectrum of the operator H_1.

Theorem 2.8. *The essential spectrum of the operator* H_1 *is* $[0, \infty)$.

Proof. Let λ be such that $\mathrm{Re}(\lambda) > 0$ and very large. Since

$$V(H_0 + \lambda^2)^{-1} f(x) = \int_{\mathbb{R}} V(x)\, \mathrm{e}^{-\lambda|x-y|} f(y)\, \mathrm{d}y,$$

is a compact operator if V is in $L^1(\mathbb{R})$ by Lemmas 2.5 and 2.6 so for $V \in L^1 + L_0^\infty$ we deduce by an approximation argument that the operator $V(H_0 + \lambda^2)^{-1}$ is a compact operator from $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$. Since $(H + \lambda^2)^{-1}$ is a bounded operator so we have that

$$(H + \lambda^2)^{-1} - (H_0 + \lambda^2)^{-1} = (H + \lambda^2)^{-1} V (H_0 + \lambda^2)^{-1}$$

is a compact operator from $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$ if $V \in L^1(\mathbb{R})$ and for $V \in L^1 + L_0^\infty$ we conclude the same by a limiting argument as $\varepsilon \to 0$. Therefore by [7, Problems 3, p. 244] the operators H and H_0 have same essential spectrum.

To determine the spectrum of the operator H_1 completely one need only find the discrete spectrum. Relevant bounds are obtained in the following theorem.

Theorem 2.9. *If* $z = -\lambda^2$, *where* $\lambda = \lambda_1 + \mathrm{i}\lambda_2$ *and* $\lambda_1 > 0$, *is an eigenvalue of* H_1 *then*

$$\|\lambda_2\| \leqslant \sqrt{\frac{F(\lambda_1)^2}{4} - \lambda_1^2} \tag{20}$$

and $0 < \lambda_1 \leqslant \mu$ *where* μ *is determined by* $F(\mu) = 2\mu$.

Proof. Let $z = -\lambda^2$ be such that

$$(H_1 + \lambda^2)^{-1} = (H_0 + \lambda^2)^{-1} - (H_1 + \lambda^2)^{-1} V (H_0 + \lambda^2)^{-1}$$

and

$$\|V(H_1 + \lambda^2)^{-1}\|_1 < 1.$$

This implies that

$$\begin{aligned}\|(H_1 + \lambda^2)^{-1}\|_1 &\leqslant \frac{\|(H_0 + \lambda^2)^{-1}\|_1}{1 - \|V(H_0 + \lambda^2)^{-1}\|_1} \\ &< \infty.\end{aligned}$$

Thus if $\mathrm{Re}(\lambda) > 0$ and $F(\mathrm{Re}(\lambda)) < 2|\lambda|$ then $(H_1 + \lambda^2)^{-1}$ exists by (19). So if we put $\lambda = \lambda_1 + \mathrm{i}\lambda_2$ with $\lambda_1 > 0$ then the condition $F(\mathrm{Re}(\lambda)) < 2|\lambda|$ becomes

$$F(\lambda_1) < 2\sqrt{\lambda_1^2 + \lambda_2^2}.$$

Hence for $z = -\lambda^2$ to be in the spectrum of H_1 and not in $[0, \infty]$ we must have

$$\|\lambda_2\| \leqslant \sqrt{\frac{F(\lambda_1)^2}{4} - \lambda_1^2}$$

with $\lambda_1 \in (0, \mu]$ where $F(\mu) = 2\mu$. This proves the theorem.

Note 2.10. If $\tilde{F}(s)$ is an upper bound of $F(s)$ then one may replace F by $\tilde{F}$ in (20) and one has $\tilde{\mu} \geqslant \mu$.

Note 2.11. Combining Theorems 2.8 and 2.9 we conclude that

$$\sigma(H_1) \subset [0, \infty) \cup \left\{ -(\lambda_1 + \mathrm{i}\lambda_2)^2 \in \mathbb{C}\colon |\lambda_2| \leqslant \sqrt{\frac{F(\lambda_1)^2}{4} - \lambda_1^2} \text{ for } 0 < \lambda_1 \leqslant \mu \right\}, \tag{21}$$

where μ is given by the solution of the equation $F(s) = 2s$.

Now let us define H_∞ to be the operator associated with the resolvent family

$$R_\infty(z) = \{(H_1 - z)^{-1}\}^*$$

acting on $L^\infty(\mathbb{R})$; its domain is not norm dense in $L^\infty(\mathbb{R})$ for well-known reasons. See [4, Section 1.4, p.22; Problem 2.5, p. 37].

Note 2.12. By Note 2.7 the essential spectrum of the operator H_∞ is also $[0, \infty)$.

The following theorem describes the region in the complex plane which contains the eigenvalues of H_∞.

Theorem 2.13. *A statement analogous to that of Theorem* 2.9 *holds for the operator* H_∞.

Proof. We note that if $z = -\lambda^2$ with $\mathrm{Re}(\lambda) > 0$ is in the resolvent of H_∞ then

$$\|(H_0 + \lambda^2)^{-1} V\|_\infty = \sup_x \int \frac{\mathrm{e}^{-\mathrm{Re}(\lambda)|x-y|}}{2|\lambda|} |V(y)|\, \mathrm{d}y$$

the right-hand side of which is same as the expression on the right-hand side of (19). The rest of the proof similar to that of Theorem 2.9.

Note 2.14. By the above theorem and Note 2.12 we note that a conclusion analogous to that of Note 2.11 is also valid for the operator H_∞.

The potentials we are considering $V \in L^1 + L_0^\infty$ lie in the Kato class of [10]. If H_p is defined as the generator of the appropriate semigroup in $L^p(\mathbb{R})$ as in [10] then if $\lambda > 0$ is large enough we have $-\lambda^2 \notin \sigma(H_p)$ for any $1 \leqslant p \leqslant \infty$ and the resolvents

$$R_p = (H_p + \lambda^2)^{-1} : L^p \to L^p$$

are all consistent in the sense of [5], namely

$$R_1 f = R_p f = R_\infty f \tag{22}$$

for all $f \in L^1 \cap L^\infty \subset L^p$. Now since $\lambda \to R_p(\lambda)$ is an analytic function of λ for all $-\lambda^2 \notin \sigma(H_p)$ so (22) still holds as long as $-\lambda^2$ does not lie in the spectrum of any of the operators.

If $z_n = -\lambda_n^2$ converges to $z \in \sigma(H_p)$ then $\|(z_n - H_p)^{-1}\|_p \to \infty$. So by the interpolation estimate

$$\|(z_n - H_p)^{-1}\|_p \leqslant \|(z_n - H_1)^{-1}\|_1^{1/p} \|(z_n - H_\infty)^{-1}\|_\infty^{1-1/p},$$

we deduce that $\|(z_n - H_1)^{-1}\|_1 \to \infty$ or $\|(z_n - H_\infty)^{-1}\|_\infty \to \infty$. This implies that $z \in \sigma(H_1)$ or $z \in \sigma(H_\infty)$; these two are equal by the fact that $(H_\infty - v)^{-1} = \{(H_1 - v)^{-1}\}^*$ for all sufficiently large negative v, and the fact that the spectrum of an operator is determined by the spectrum of its (bounded) resolvent operator [4, Lemma 2.11, p. 39]. (note that the density of the domain is not used in the proof). Therefore, every eigenvalue of H_p is also an eigenvalue of H_1. Thus, we conclude that

Theorem 2.15. *The spectrum, $\sigma(H_p)$, of H_p for $1 \leqslant p \leqslant \infty$ is contained in the same set as described in Note* 2.11.

From the above theorems we immediately obtain a slight technical improvement of Theorem 4 of [1]. A similar type of bound was obtained in [3].

Corollary 2.16. *If $V \in L^1(\mathbb{R})$ and $z = -\lambda^2$ is an eigenvalue of* (1) *with* $\mathrm{Re}(\lambda) > 0$ *then*

$$|z| \leqslant \frac{\|V\|_1^2}{4}.$$

Proof. We simply use the fact that

$$F(s) \leqslant \|V\|_1.$$

Corollary 2.17. *If $V \in L^p(\mathbb{R})$, $p > 1$ and $z = -\lambda^2$ is an eigenvalue of H_1, where $\lambda = \lambda_1 + \mathrm{i}\lambda_2$ and $\lambda_1 > 0$ then $\lambda_1 \leqslant \mu = k^{q/2(q+1)}$ and*

$$|\lambda_2| \leqslant \sqrt{\frac{k^2}{4}\lambda_1^{-2/q} - \lambda_1^2},$$

where $k = \|V\|_p (2/q)^{1/q}$ and $1/p + 1/q = 1$.

Proof. We note that

$$\begin{aligned}
F(s) &= \sup_y \int_{\mathbb{R}} |V(x)|\, e^{-s|x-y|}\, dx \\
&\leqslant \|V\|_p \left(\int_R e^{-s|t|q}\, dt \right)^{1/q} \\
&= \|V\|_p \left(\frac{2}{qs} \right)^{1/q} \\
&= ks^{-1/q}.
\end{aligned}$$

We use this estimate in Theorem 2.9 and obtain the result. To obtain the value of μ we simply solve $(k^2/4)\mu^{-2/q} - \mu^2 = 0$.

The above theorems and lemmas provide bounds on the spectrum of H_p provided we show how to compute the function $F(s)$. Even before doing this it is evident that if $\|V\|_1$ is finite but very large, while $\|V\|_2$ is quite small, one gets stronger restriction on possible $z \in \sigma(H_2)$ when $|z| = O(1)$ as $|z| \to \infty$ by applying the fact that $V \in L^2(\mathbb{R})$ and Corollary 2.17 than one does from the bound of Theorem 4 of [1]. This observation is applicable to potentials such as

$$V(x) = (1 + |x|)^{-1-\varepsilon}$$

for small enough $\varepsilon > 0$. Computing $F(s)$ involves computing integrals and maximising over a range of values of y. If $V(x)$ has a single maximum at $x = a$, then one may start by putting $y = a$, and then consider values of y moving away from this until the integral becomes negligible. For large values of s the optimal value of y will be very close to a. For very small values of s, $F(s) \approx \|V\|_1$ if $V \in L^1(\mathbb{R})$ while $F(s)$ is very large if $V \notin L^1(\mathbb{R})$.

If we compute $F(s)$ for a finite number of values of s, then its convexity allows us to give an explicit upper bound in the intermediate intervals, and hence to obtain an upper bound on the set of λ for which $z = -\lambda^2 \in \sigma(H)$.

Note 2.18. All theorems restricting the complex eigenvalues of H_p on $L^p(\mathbb{R})$ where $V(-x) = V(x)$ for all x also imply the same bounds for H_p in $L^p(0, \infty)$ subject to Dirichlet boundary condition or Neumann boundary condition.

We now describe the simplifications when we make estimates of the basic functions which only depend on the symmetric decreasing rearrangement V^* of $|V|$. This includes all the estimates expressed in terms of L^p norms of V and $L^1_0(\mathbb{R})$ decompositions.

Theorem 2.19. *We have*

$$F(s) \leqslant F^*(s)$$

for all $s>0$, *where* F^* *is defined by*

$$F^*(s) = 2\int_0^\infty V^*(x)\,\mathrm{e}^{-sx}\,\mathrm{d}x$$
$$= 2\mathscr{L}(V^*)(s),$$

where $\mathscr{L}(V^*)$ *denotes the Laplace transform of the function* V^*.

Proof. Taking $f=|V|$, $g(x)=\mathrm{e}^{-s|x|}$ and $h(x)=\delta(x-y)$ in Lemma 3.6 of [8], where $\delta(x-y)$ denotes the delta function of the variable x at y, noting that the symmetric decreasing rearrangement of $\delta(x-y)$ is $\delta(x)$ and also that the convolution of two symmetric decreasing non-negative functions is also a symmetric decreasing non-negative function we obtain that

$$F(s) = \sup_y (|V| * \mathrm{e}^{-s|x|})(y)$$
$$\leqslant (V^* * \mathrm{e}^{-s|x|})(0)$$
$$= \int_{\mathbb{R}} V^*(x)\,\mathrm{e}^{-s|x|}\,\mathrm{d}x$$
$$= 2\int_0^\infty V^*(x)\,\mathrm{e}^{-s|x|}\,\mathrm{d}x$$
$$= F^*(s).$$

Corollary 2.20. *If the potential* $V(x)$ *is such that* $|V(x)|$ *is symmetric and decreasing then*

$$F(s) = 2\mathscr{L}(|V|)(s).$$

Proof. The proof is the same as that of the above theorem except that inequality will be replaced by equality because $|V| = V^*$.

3. Examples

In this section we consider some examples, use the results we obtained so far and also compare our observations with the theorems obtained in [1,3].

Example 3.1. Let $V(x) = c\mathrm{e}^{-|x|}$, for $c\in\mathbb{C}$ with $|c|=1$. Then $\|V\|_1 = 2$ and so the estimate of [1] concludes that if $z=-\lambda^2$ is an eigenvalue then $|\lambda^2| = |z| \leqslant \|V\|_1^2/4$, which becomes $|\lambda|\leqslant 1$. Since $|V|$ is a symmetric, decreasing and positive, we can use Corollary 2.20 to show that

$$F(s) = 2\int_0^\infty |V(x)|\,\mathrm{e}^{-s|x|}\,\mathrm{d}x$$
$$= \frac{2}{1+s}.$$

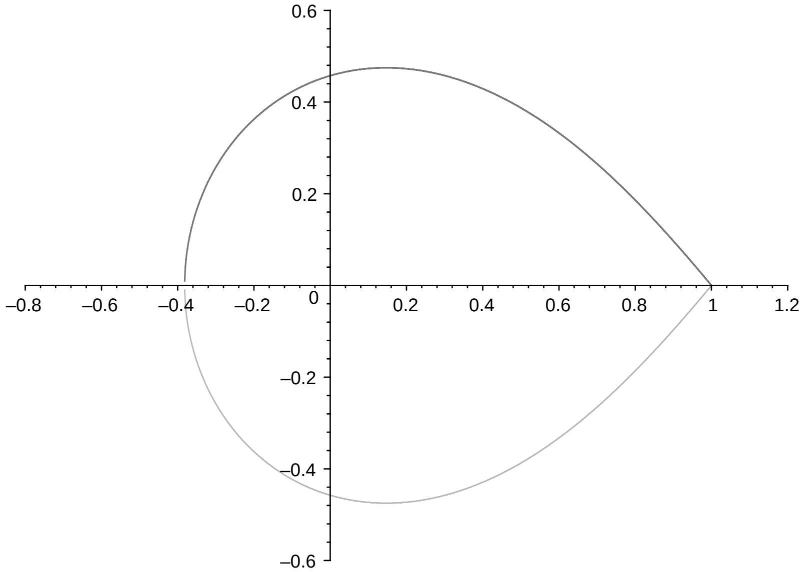

Fig. 5. Region in z-plane for Example 3.1.

Then using Theorem 2.15 we get that

$$|\lambda_2| \leqslant \sqrt{\frac{1}{(1+\lambda_1)^2} - \lambda_1^2}$$
$$= f(\lambda_1) \quad \text{say}$$

for $0 < \lambda_1 \leqslant \mu = (\sqrt{5}-1)/2 = 0.618$. This gives $-\mu^2 \approx -0.38$ at which the curve intersects the negative real axis and hence gives a smaller region in the left half plane than the region obtained using the result of [1]. If we put

$$\begin{aligned} z &= -(\lambda_1 + \mathrm{i}\lambda_2)^2 \\ &= \lambda_2^2 - \lambda_1^2 \pm 2\mathrm{i}\lambda_1\lambda_2 \\ &= f(\lambda_1)^2 - \lambda_1^2 - 2\mathrm{i}\lambda_1 f(\lambda_1) \end{aligned}$$

for $\lambda_2 = \pm f(\lambda_1)$, then we get the curve enclosing the spectrum in the z-plane which is shown in Fig. 5.

If one puts $V(x) = \gamma c\, \mathrm{e}^{-\gamma|x|}$ where $|c| = 1$ and $\gamma > 0$ then the intersection of the curve with the negative real axis (-0.38 in the above example) tends to zero as $\gamma \to 0$. However, in this case one still has $\|V\|_1 = 2$ so the classical method only gives that $|\lambda| \leqslant 1$.

Example 3.2. Let $V(x) = c(\mathrm{e}^{-|x|} + \mathrm{e}^{-\gamma|x|})$, where $c \in \mathbb{C}$ with $|c| = 1$ and $\gamma > 0$ is small. Here

$$\begin{aligned} \|V\|_1 &= 2\int_0^\infty (\mathrm{e}^{-x} + \mathrm{e}^{-\gamma x})\,\mathrm{d}x \\ &= 2(1 + 1/\gamma). \end{aligned}$$

So by the method of [1] we get that the eignevalues lie within the circle $|\lambda| \leqslant 1 + 1/\gamma$ in λ-plane. Now since $|V|$ is symmetric decreasing so

$$F(s) = 2\int_0^\infty (e^{-x} + e^{-\gamma x})\, e^{-sx}\, dx$$
$$= 2\left(\frac{1}{1+s} + \frac{1}{\gamma + s}\right)$$

and the new bound is

$$|\lambda_2| \leqslant \sqrt{\left(\frac{1}{1+\lambda_1} + \frac{1}{\gamma+\lambda_1}\right)^2 - \lambda_1^2}$$

for $0 < \lambda_1 \leqslant \mu$, where μ is given by the solution of

$$\frac{1}{1+s} + \frac{1}{\gamma+s} = s$$

which can be simplified to a cubic polynomial giving $1 < \mu \approx 1.1 < 2$ for $\gamma = \frac{1}{2}$ which is much better than $|\lambda| \leqslant 1 + 1/\gamma = 3$.

By Theorem 2.15, the curve of interest is given by

$$\lambda_1^2 + \lambda_1^2 = \frac{F(\lambda_1)^2}{4}.$$

It is better in some contexts to work in polar coordinates. So if $\lambda = s e^{i\phi}, -\pi/2 < \phi < \pi/2$ then

$$s^2 = \frac{F(s\cos\phi)^2}{4}.$$

Now $z = -\lambda^2 = -s^2 e^{2i\theta}$, where $r = s^2$ and $\theta = 2\phi + \pi$. So $0 < \theta < 2\pi$. Hence, in the z-plane the curve is

$$r = F(r^{1/2}\cos(\pi/2 - \theta/2))^2/4,$$

or

$$4r = F(r^{1/2}\sin(\theta/2))^2. \tag{23}$$

Clearly if $\theta = 0$ or 2π and $V \in L^1(\mathbb{R})$ then $|z| = F(0)^2/4$ and for all other θ, the value of r is smaller, because F is a monotonically decreasing function.

Example 3.3. Consider the potential $V(x) = c|x|^{\alpha-1}$ where $0 < \alpha < 1$ and c is a complex number. Clearly, $|V(x)|$ is symmetric decreasing so

$$F(s) = 2\int_0^\infty |c| x^{\alpha-1} e^{-sx}\, dx$$
$$= 2|c|\Gamma(\alpha)s^{-\alpha}.$$

The above value of $F(s)$ and (10) allow us to verify Theorem 2.2 here. The infimum in the right-hand side of inequality (18) is obtained at $\varepsilon = |c|s^{1-\alpha}$ and the infimum is $2|c|\alpha^{-1}s^{-\alpha}$ which when compared

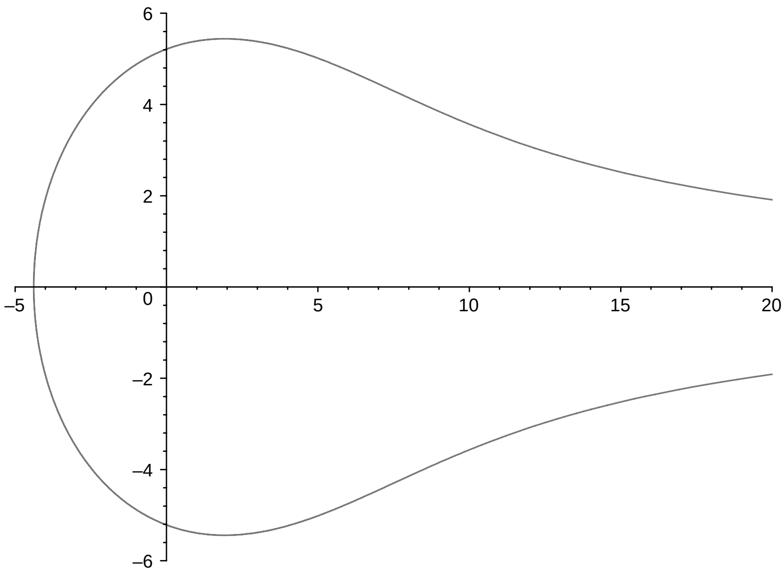

Fig. 6. $r = \Gamma(1/3)^{3/2}(\sin(\theta/2))^{-1/2}$.

to the value of $F(s)$ certainly justifies inequality (18) because of the fact that $0 < \alpha\Gamma(\alpha) = \Gamma(\alpha+1) \leqslant 1$ if $0 < \alpha < 1$. So in the polar coordinates the curve in the z-plane is

$$4r = 4|c|^2\Gamma(\alpha)^2\{r^{1/2}\sin(\theta/2)\}^{-2\alpha}$$

or

$$r^{1+\alpha} = |c|^2\Gamma(\alpha)^2\{\sin(\theta/2)\}^{-2\alpha}$$

or

$$r = k\{\sin(\theta/2)\}^{-2\alpha/(1+\alpha)},$$

where $k = \{|c|\Gamma(\alpha)\}^{2/(1+\alpha)} > 0$. For c such that $|c| = 1$ and $\alpha = \frac{1}{3}$ the graph of the above curve in z-plane is shown in Fig. 6.

Examples such as $V(x) = c\tilde{V}(x)$, where $|c| = 1$ and $\tilde{V} \geqslant 0$ is symmetric decreasing are not trivial. For $c = -1$

$$H = -\frac{d^2}{dx^2} - \tilde{V}$$

is self-adjoint with at least one negative eigenvalue [9, Problem 7.4, p. 89]. If we put $c = e^{i\theta}$ then

$$H_\theta = -\frac{d^2}{dx^2} + e^{i\theta}\tilde{V}$$

has eigenvalues $\lambda_{n,\theta}$, which move around in the complex plane as θ varies, but must always stay inside the curve. They are probably absorbed into the positive real axis as $\theta \to \pm\pi$.

Example 3.4. Consider

$$V(x) = \begin{cases} c\mathrm{e}^{-x} & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The symmetric decreasing function corresponding to $|V(x)|$ is given by

$$V^*(x) = |c|\,\mathrm{e}^{-2|x|}.$$

Then using Theorem 2.19 and noting that $F^*(s) = 2|c|/(s+2)$ and also using Note 2.10 we obtain a region where the eigenvalues lie. Consider the following integral:

$$\begin{aligned} F_-(s) &= \sup_{y \leqslant 0} \int_0^\infty \mathrm{e}^{-x}\,\mathrm{e}^{-s|x-y|}\,\mathrm{d}x \\ &= \sup_{y \leqslant 0} \int_0^\infty \mathrm{e}^{-x-s(x+|y|)}\,\mathrm{d}x \\ &= \sup_{y \leqslant 0} \frac{\mathrm{e}^{-s|y|}}{1+s} \\ &= 1/(1+s). \end{aligned}$$

Since the supremum in $F_-(s)$ is obtained at zero so $F(s) = |c|F_+(s)$, where $F_+(s)$ is given by

$$F_+(s) = \sup_{y \geqslant 0} \int_0^\infty \mathrm{e}^{-x}\,\mathrm{e}^{-s|x-y|}\,\mathrm{d}x.$$

But

$$\int_0^\infty \mathrm{e}^{-x}\,\mathrm{e}^{-s|x-y|}\,\mathrm{d}x = \begin{cases} \mathrm{e}^{-y}\dfrac{2s}{s^2-1} - \dfrac{\mathrm{e}^{-sy}}{s-1} & \text{for } s \neq 1, \\ (y+1/2)\,\mathrm{e}^{-y} & \text{if } s = 1. \end{cases}$$

It is not very difficult to find that

$$F(s) = \begin{cases} |c|\left(\dfrac{1+s}{2}\right)^{s/(1-s)} & \text{for } s \neq 1, \\ |c|/\sqrt{e} & \text{for } s = 1. \end{cases}$$

For c with $|c| = 1$ the μ of Theorem 2.9 is 0.3951 (approx.). And the corresponding region in the z-plane is given by the interior of the curve shown in Fig. 7. It is obvious that although $\|V\|_1 = 1$ the region described by our method is far smaller than the region described by [1].

Example 3.5. In this example we consider a potential which cannot be dealt with by any other existing method. Let

$$V(x) = \mathrm{e}^{\mathrm{i}\theta}\frac{a^3x^2}{(x^2+a^2)^{3/2}},$$

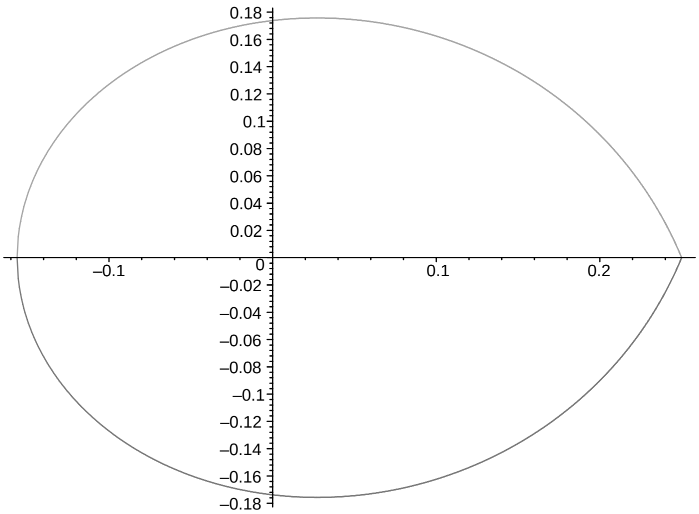

Fig. 7. Figure in the z-plane for Example 3.4.

where $0 \leqslant \theta < 2\pi$ and $a > 0$. We know that for $\theta = \pi$ there exists at least one negative eigenvalue of the Schrödinger operator in $L^2(\mathbb{R})$. For all other θ except zero the operator has complex eigenvalues. Since $|V_{(x)}|$ is not decreasing and also does not belong to $L^1(\mathbb{R})$ the results for symmetric decreasing potentials and for $L^1(\mathbb{R})$ potentials cannot be used to describe this case. So for $a = 1$ we used MATLAB to calculate the values of $F(s)$ numerically for a range of values of s. To evaluate the constant μ of Theorem 2.9 we use the bisection method in our program. In this case $\mu \approx 0.52$. We then plot the curve $z = -\lambda^2$, where $\lambda = \lambda_1 + i\lambda_2$ and $\lambda_2^2 = F(\lambda_1)^2/4 - \lambda_1^2$ in the z-plane for $0 \leqslant \lambda_1 \leqslant \mu$. The resulting graph is shown in Fig. 8 above which is what we expect and our theorems above predict.

Now we consider the resonance problem for $V(x) = a^3x^2/(x^2 + a^2)^{3/2}$, where a is a positive real number. Clearly, the potential $V(x)$ can be analytically continued to the sector containing z such that $-\pi/2 < \arg(z) < 0$. Then for $-\pi/2 < \theta < 0$ we know that those eigenvalues of the operator

$$e^{-2i\theta}H_0 + V(e^{i\theta})$$

which lie in the sector

$$\{z:\ -\theta < \arg(z) \leqslant 0\}$$

are independent of θ. The set of all such eigenvalues as θ varies over the permitted range is called the set of resonances of the self-adjoint operator

$$-\frac{d^2}{dx^2} + V.$$

Therefore, the resonances are given by the eigenvalues of

$$H_0 + W$$

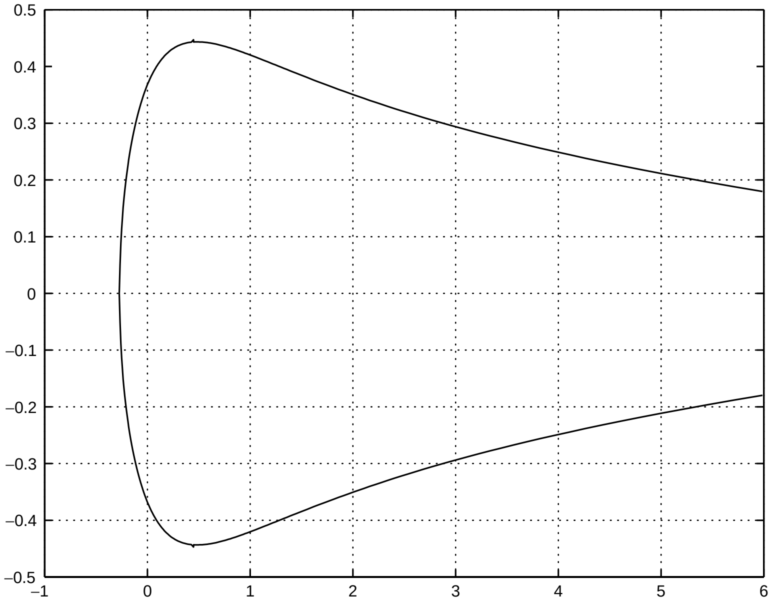

Fig. 8. The curve in the z-plane for Example 3.5.

multiplied by $e^{-2i\theta}$, where W is given by

$$W(x) = \frac{a^3 x^2 e^{4i\theta}}{(a^2 + x^2 e^{2i\theta})^{3/2}}.$$

To find the region where the eigenvalues of the operator $H_0 + W$ lie one may again use MATLAB to find values of the corresponding function $F(s)$ for a range of values of s for a particular value of θ and a and do the same as we did for eigenvalue problem. The resonance-free region is then the union of the eigenvalue-free regions as θ varies. Finding an explicit expression for the boundary of the resulting region requires further work.

We make a few comments about the numerical computations, performed using MATLAB. To calculate the values of $F(s)$, defined by (17), one needs to identify the relevant values of y in the definition in $F(s)$. If s is large then the supremum will be attained when y is very close to the point of maximum of $|V(x)|$ because of the exponential decay factor; one should only consider a narrow range of integration around this maximum. On the other hand, for small s the function $|V(x)\,e^{-s|x-y|}$ decreases slowly, making it necessary to consider a wide interval of integration when evaluating $F(s)$. An independent point is that one should integrate separately over $\{x\colon x < y\}$ and $\{x\colon x > y\}$. This makes the functions integrated much smoother, and either speeds up the integration or yields a more accurate value, depending on whether or not the integration algorithm is adaptive.

4. Comparison of the two methods

In this section, we compare the efficiency of the methods which we have discussed.

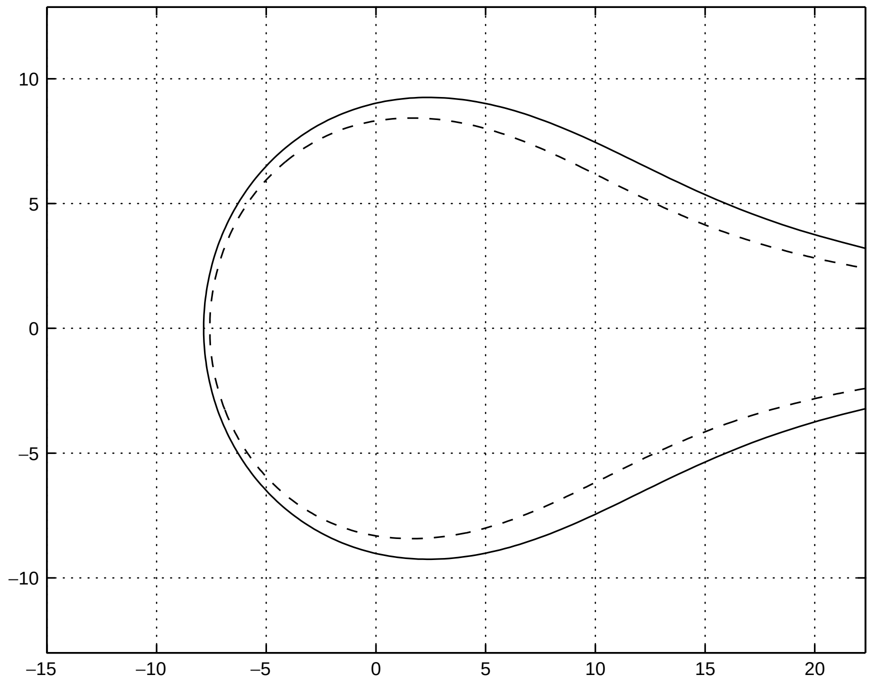

Fig. 9. Comparison of L^1- and L^2-methods. The dotted curve is obtained by using Theorem 1.5 of L^2-method.

The bound on the spectrum obtained in Theorem 1.8 is always less good than that obtained by the L^1-method, using (20). This is a consequence of the inequality

$$\frac{F(\lambda_1)}{2} \leqslant \frac{C_{\lambda_1^2}}{2} + \lambda_1$$

for all $\lambda_1 > 0$. This inequality follows from (18) if one puts $s = \lambda_1$ and $\varepsilon = \lambda_1^2$. In addition, the region given by Theorem 1.8 need not be contained in a finite width strip around the real axis. Consider the example $V(x) = c|x|^{\alpha-1}$, where $\frac{1}{3} < \alpha < 1$.

Now if we consider the L^1-method and the direct L^2-method of Theorem 1.5, we have no general theorems concerning which method is better. However, for the following example we have obtained a better result from the L^2-method. Fig. 9 shows the two boundary curves obtained by the L^1-method and Theorem 1.5 for the potential $V(x) = |x|^{-1/4}$.

It is also clear that a constant potential cannot be dealt with by the L^2-method. But the L^1-method can deal with this case although we do not get as good a result as is known from the classical theory. This is made clear by the following theorem.

Theorem 4.1. *If V is a constant potential then*

$$F(s) = \frac{2\|V\|_\infty}{s}$$

and $\sigma(H_p)$ is contained in the region bounded by the curve given in polar coordinates by

$$r = \frac{\|V\|_\infty}{\sin(\theta/2)}.$$

Proof. The proof of this theorem is immediate from (17) and (23).

It follows from this theorem that for $x \geqslant 0$ the distance of the curve from the x-axis is $2\|V\|_\infty$. But we know that if a normal operator N is perturbed by a bounded operator A then the spectrum of the perturbed operator $N + A$ is contained in the closed $\|A\|$-neighbourhood of the spectrum of N. Hence compared to the known result our L^1-method is very poor for constant potentials.

If we have a potential $V(x)$ which belongs to $L^\infty(\mathbb{R})$ then although one can use the estimate $F(s) \leqslant 2\|V\|_\infty/s$ to determine a boundary curve of the region of interest, it may be substantially larger than the region obtained by our computational method. This is obvious by Example 3.5 which is a bounded potential with $\|V\|_\infty \approx 0.39$ whereas the region we obtained by our numerical method is much smaller. □

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