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Spectral behaviour of a simple non-self-adjoint operator

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Abstract

We investigate the spectrum of a typical non-self-adjoint differential operator $AD = -d^2/dx^2 \otimes A$ acting on $L^2(0,1) \otimes \mathbb{C}^2$, where A is a 2×2 constant matrix. We impose Dirichlet and Neumann boundary conditions in the first and second coordinate, respectively, at both ends of $[0,1] \subset \mathbb{R}$. For $A \in \mathbb{R}^{2\times 2}$ we explore in detail the connection between the entries of A and the spectrum of AD, we find necessary conditions to ensure similarity to a self-adjoint operator and give numerical evidence that suggests a non-trivial spectral evolution.

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1. Introduction

In this paper, we investigate spectral properties of the linear operator AD acting on $L^2(0,1) \otimes \mathbb{C}^2$ where A is a 2×2 constant matrix and D denotes the

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ordinary differential operator

$$D\begin{pmatrix}\phi\\ \gamma\end{pmatrix} := -\begin{pmatrix}\phi''\\ \gamma''\end{pmatrix}, \qquad \begin{array}{l}\phi(0)=\phi(1)=0,\\ \gamma'(0)=\gamma'(1)=0.\end{array}$$

The apparently simple combination of Dirichlet and Neumann boundary conditions allows self-adjointness if, an only if, A is real and diagonal. If A is non-diagonal and upper-triangular the numerical range of AD is a large sector of $\mathbb{C}$. Otherwise it is the whole of $\mathbb{C}$ preventing us from applying the theory of sectorial sesquilinear forms in a straightforward manner.

Our main goal is to explore the connection between the entries of matrix A and the location of the spectrum of AD in the complex plane. Streater [8] considers the particular case

$$A = \begin{pmatrix}1 & \gamma\\ 1/2\gamma & 1\end{pmatrix}, \quad \gamma > 0,$$

in order to find necessary conditions for the stability of small perturbations about the stationary solution of certain non-linear system of parabolic equations. Streater's system represents a thermodynamical model for hot fluid in one dimension and the localization of the spectrum is achieved by constructing a non-unitary transformation that makes AD similar to a non-negative self-adjoint operator, hence the spectrum of AD is real and non-negative. This similarity transformation does not work for other matrices and a slight modification of the entries of A can destroy reality of the spectrum (cf. Sections 6 and 7) so the general case should be attacked by other methods.

Although this paper mainly concerns $A\in\mathbb{R}^{2\times 2}$, the results of Sections 2–5 refer to any complex 2×2 matrix. The core results are to be found in Section 6 where we present an exhaustive description of the spectrum of AD in terms of the entries of A. Among various other unexpected conclusions, the following three epitomize the complexity of the problem to be considered:

(a) When A is triangular and non-diagonalizable, AD is not similar to a self-adjoint operator but the spectrum of AD is real (Theorem 19).
(b) The spectrum of AD can be non-real even when both eigenvalues of A are positive and equal (Theorem 26).
(c) There is a continuous family of matrices A whose eigenvalues do not intersect the real line but such that the spectrum of AD is real (Theorem 23).

The last two assertions show that the spectra of A, D and AD are typically unrelated.

The crucial idea in Section 6 is to reduce the four-parameter problem of localizing the spectrum of AD in terms of the entries of A, to five two-parameter cases and describe separately each of these cases. Sections 2–5 are devoted to describing the various properties of AD we will use in Section 6, whereas Section 7 is devoted to numerical computations which illustrate some of the results reported. In Section 2, we find the boundary conditions associated to the adjoint of AD and compute the numerical range of AD. In Section 3, we show that the resolvent of AD is compact for all non-singular A. In Section 4, we explore the stability of the spectrum of AD in the sense of [4,9], and provide estimates which allow us to enclose the spectrum of AD in angular regions when A is subject to various different constraints. In Section 5, we use standard ODE methods to compute the transcendental function of the spectral problem associated to AD.

2. Definitions and notation

Let K be a linear operator whose domain is denoted by Dom(K). Throughout this paper Spec(K) stands for the spectrum of K and the numerical range of K is defined to be

$$\mathrm{Num}(K) := \{\langle Kf, f\rangle : f \in \mathrm{Dom}(K),\ \|f\| = 1\}.$$

We recall that the numerical range of any linear operator is convex and that if $\mathrm{Spec}(K) \neq \emptyset$, then

$$\mathrm{Spec}(K) \subset \overline{\mathrm{Num}(K)}.$$

If $K = K^*$ and $\mathrm{Spec}(K) \subset (0, \infty)$, we will say that K is positive and write $K > 0$. If $K = K^*$ and $\mathrm{Spec}(K) \subset [0, \infty)$, we will say that K is non-negative and write $K \geqslant 0$.

Below and elsewhere $|v|$ denotes the norm of a vector $v \in \mathbb{C}^2$. The norm of any

$$f \equiv \begin{pmatrix} \phi \\ \gamma \end{pmatrix} \in L^2(0,1) \otimes \mathbb{C}^2$$

is the standard Hilbert tensor product norm

$$\|f\|^2 = \langle f, f\rangle = \int_0^1 |f(x)|^2\, dx = \int_0^1 (|\phi(x)|^2 + |\gamma(x)|^2)\, dx.$$

Unless explicitly stated, we denote

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The complex numbers a_+, a_- denote the eigenvalues of A and the non-zero $\mathbb{C}^2$ vectors v_+, v_- denote the eigenvectors

$$Av_\pm = a_\pm v_\pm.$$

If a_+ and a_- are real and different, we adopt the convention $a_- < a_+$. Notice that the $v_\pm$ are not necessarily orthogonal.

Let $W^{2,2}$ be the Sobolev space of all $f \in L^2(0,1) \otimes \mathbb{C}^2$, such that the generalized derivative $f'' \in L^2(0,1) \otimes \mathbb{C}^2$. We define rigorously the domain of AD as

$$\mathrm{Dom}(D) = \{f \in W^{2,2} : \phi(0) = \phi(1) = 0,\ \gamma'(0) = \gamma'(1) = 0\}.$$

If A is invertible, it is standard to show that AD is always a closed densely defined linear operator acting on $L^2(0,1) \otimes \mathbb{C}^2$.

Lemma 1. *If A is singular, then AD is not closed in the domain* $\mathrm{Dom}(D)$.

Proof. Let $v \in \mathbb{C}^2$ be a non-vanishing vector such that $Av = 0$ and let $f(x) := vx \in L^2(0,1) \otimes \mathbb{C}^2$. Clearly $f \notin \mathrm{Dom}(D)$. Let ϕ_n be a sequence of smooth functions whose support is compact in $(0,1)$ and such that $\phi_n(x) \to x$ in $L^2(0,1)$. Then $\phi_n v \in \mathrm{Dom}(D)$ and $\phi_n v \to f$. Also

$$AD\phi_n(x)v = -\phi_n''(x)Av = 0,$$

so that $AD(\phi_n v)$ is a convergent sequence in $L^2(0,1) \otimes \mathbb{C}^2$. We complete the proof by noticing that if AD was closed, then we would have $f \in \mathrm{Dom}(D)$. $\square$

For the rest of this section and in Sections 3–5 we will assume without further mention that A is non-singular. In Section 6, we will consider again singular A.

In order to show that AD is in general non-self-adjoint, let us compute the adjoint $(AD)^*$. Let

$$P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the boundary conditions for D can be rewritten as

$$Pf(0) = Pf(1) = 0, \qquad (I - P)f'(0) = (I - P)f'(1) = 0.$$

Lemma 2. *The adjoint of AD is*

$$(AD)^* f = -A^* f'',$$

for $f \in W^{2,2}$ subject to the boundary conditions

$$\hat{P}f(0) = \hat{P}f(1) = 0,$$

$$(I - \hat{P})f'(0) = (I - \hat{P})f'(1) = 0, \tag{1}$$

where $\hat{P} = \hat{P}^2$ is the rank one projection such that

$$\mathrm{Ran}(\hat{P}) = \mathrm{Ran}(A(I - P))^\perp,$$

$$\mathrm{Ran}(I - \hat{P}) = \mathrm{Ran}(AP)^\perp. \tag{2}$$

Proof. For $f \in \mathrm{Dom}(D)$ and $g \in L^2(0,1) \otimes \mathbb{C}^2$,

$$\begin{aligned}\langle ADf, g\rangle &= -\int_0^1 \langle Af''(x), g(x)\rangle \, dx \\ &= \langle APf', g\rangle|_1^0 + \int_0^1 \langle Af'(x), g'(x)\rangle \, dx.\end{aligned}$$

We ought to find a complex 2×2 matrix B and impose boundary conditions on g, for

$$\begin{aligned}\langle f, (AD)^* g\rangle &= -\int_0^1 \langle f(x), Bg''(x)\rangle \, dx \\ &= \langle B^* f, g'\rangle|_1^0 + \int_0^1 \langle B^* f'(x), g'(x)\rangle \, dx \\ &= \langle B^*(I-P)f, g'\rangle|_1^0 + \int_0^1 \langle B^* f'(x), g'(x)\rangle \, dx\end{aligned}$$

and

$$\langle ADf, g\rangle = \langle f, (AD)^* g\rangle.$$

This must be true in particular for all f and g with compact support in $(0,1)$ so clearly $B = A^*$.

Let the boundary conditions for $(AD)^*$ be given by (1) where $\hat{P} = \hat{P}^2$ is a non-necessarily orthogonal projection on $\mathbb{C}^2$. We show (2). If f, g are smooth functions supported in $[0,1)$, then

$$\langle APf'(0), (I-\hat{P})g(0)\rangle = \langle AQf(0), \hat{P}g'(0)\rangle,$$

where $f(0)$, $f'(0)$, $g(0)$ and $g'(0)$ are arbitrary vectors in $\mathbb{C}^2$. If $f'(0) = 0$, the right-hand side should vanish for all $f(0), g'(0) \in \mathbb{C}^2$, so that

$$\mathrm{Ran}(\hat{P}) = \mathrm{Ran}(AQ)^{\perp}.$$

If $f(0) = 0$, the left-hand side should vanish for all $f'(0), g(0) \in \mathbb{C}^2$, so that

$$\mathrm{Ran}(I-\hat{P}) = \mathrm{Ran}(AP)^{\perp}.$$

Since A is non-singular these two spaces are one dimensional. □

Corollary 3. *AD is self-adjoint if, and only if, A is real and diagonal.*

Proof. Using the notation of Lemma 2, AD is self-adjoint if, and only if,

$$A = A^* \quad \text{and} \quad P = \hat{P}.$$

The latter occurs if, and only if,

$$A\begin{pmatrix}1\\0\end{pmatrix} \perp \begin{pmatrix}0\\1\end{pmatrix}.$$

These conditions ensure A real and diagonal. □

We now show that due to the boundary conditions we have chosen,

$$\mathrm{Num}(AD) = \mathbb{C}$$

for a large family of non-diagonal matrices A. This prevent us from employing the theory of sectorial sesquilinear forms in order to find the spectrum.

Theorem 4. *Let A be a non-singular matrix.*

(a) *If A is an upper triangular matrix (that is $c = 0$), then*

$$\overline{\mathrm{Num}(AD)} = \{rz\colon\ r \in [0, \infty),\ z \in \overline{\mathrm{Num}(A)}\}.$$

(b) *If A is not an upper triangular matrix (that is $c \neq 0$), then*

$$\mathrm{Num}(AD) = \mathbb{C}.$$

Proof. Since 0 is always an eigenvalue of AD (cf. Section 3), then $0 \in \mathrm{Num}(AD)$. For $f \in \mathrm{Dom}(D)$,

$$\begin{aligned} \langle ADf, f\rangle &= -\int_0^1 \langle Af''(x), f(x)\rangle\, dx \\ &= \langle Af', f\rangle|_1^0 + \int_0^1 \langle Af'(x), f'(x)\rangle\, dx \\ &= \left\langle A\begin{pmatrix}\phi' \\ \gamma'\end{pmatrix}, \begin{pmatrix}\phi \\ \gamma\end{pmatrix}\right\rangle\Bigg|_1^0 + \int_0^1 \langle Af'(x), f'(x)\rangle\, dx \\ &= \left\langle A\begin{pmatrix}\phi' \\ 0\end{pmatrix}, \begin{pmatrix}0 \\ \gamma\end{pmatrix}\right\rangle\Bigg|_1^0 + \int_0^1 \langle Af'(x), f'(x)\rangle\, dx. \end{aligned} \tag{3}$$

Case (a): Call

$$\Phi := \{rz\colon\ r \in [0, \infty), z \in \overline{\mathrm{Num}(A)}\}.$$

Then Φ is a convex set and

$$\begin{aligned} \Phi &= \overline{\{rz\colon\ r \in [0, \infty), z \in \mathrm{Num}(A)\}} \\ &= \overline{\{\langle Av, v\rangle\colon\ v \in \mathbb{C}^2\}}. \end{aligned}$$

If $c = 0$,

$$\left\langle A\begin{pmatrix}\phi' \\ 0\end{pmatrix}, \begin{pmatrix}0 \\ \gamma\end{pmatrix}\right\rangle\Bigg|_1^0 = \left\langle \begin{pmatrix}a\phi' \\ 0\end{pmatrix}, \begin{pmatrix}0 \\ \gamma\end{pmatrix}\right\rangle\Bigg|_1^0 = 0$$

so that

$$\langle ADf, f \rangle = \int_0^1 \langle Af'(x), f'(x) \rangle \, dx.$$

This and the fact that Φ is closed and convex, yield

$$\overline{\mathrm{Num}(AD)} \subseteq \Phi.$$

In order to prove the reverse inclusion, let $v \in \mathbb{C}^2$ be such that $|v| = 1$ and let

$$z := \langle Av, v \rangle \in \mathrm{Num}(A).$$

For all $t \geqslant 5$, let

$$\psi_t(x) := \begin{cases} \frac{1-\cos(\pi t x/2)}{\sqrt{4-10/t}} & \text{if } 0 \leqslant x \leqslant 2/t, \\ \frac{2}{\sqrt{4-10/t}} & \text{if } 2/t \leqslant x \leqslant 1 - 2/t, \\ \frac{1-\cos(\pi t(x-1)/2)}{\sqrt{4-10/t}} & \text{if } 1 - 2/t \leqslant x \leqslant 1. \end{cases}$$

Then $\psi_t(0) = \psi_t(1) = \psi_t'(0) = \psi_t'(1) = 0$,

$$\int_0^1 |\psi_t(x)|^2 \, dx = 1 \quad \text{and} \quad \int_0^1 |\psi_t'(x)|^2 \, dx = \frac{\pi^2 t^2}{8t - 20}.$$

Let $f_t := v\psi_t \in \mathrm{Dom}(D)$. By construction $\|f_t\| = 1$ and

$$\begin{aligned} \langle ADf_t, f_t \rangle &= \int_0^1 \langle Af_t'(x), f_t'(x) \rangle \, dx \\ &= \langle Av, v \rangle \int_0^1 |\psi_t'(x)|^2 \, dx \\ &= \frac{z\pi^2 t^2}{8t - 20}. \end{aligned}$$

Thus by taking $t \to \infty$, from the fact that $0 \in \mathrm{Num}(AD)$ and since $\mathrm{Num}(AD)$ is convex, we gather

$$\overline{\mathrm{Num}(AD)} \supseteq \Phi.$$

Case (b): Now $c \neq 0$. Let z be a fixed non-zero complex number. Our aim is to find functions $f_\varepsilon \in \mathrm{Dom}(D)$ parameterized by $\varepsilon > 0$, such that $\|f_\varepsilon\| = 1$ and $\langle Af_\varepsilon, f_\varepsilon \rangle$ is close to z for small ε.

For $0 < \varepsilon < \frac{1}{2}$, let

$$\phi_\varepsilon(x) := \begin{cases} \frac{\varepsilon}{c\pi} \sin(x\pi/\varepsilon) & \text{if } 0 \leqslant x \leqslant \varepsilon/2, \\ \frac{\varepsilon}{2c\pi}[1 - \cos(2\pi(x/\varepsilon - 1))] & \text{if } \varepsilon/2 \leqslant x \leqslant \varepsilon, \\ 0 & \text{if } \varepsilon \leqslant x \leqslant 1. \end{cases}$$

Then, straightforward computations show $\phi_\varepsilon(0) = \phi_\varepsilon(1) = \phi'_\varepsilon(1) = 0$, $\phi'_\varepsilon(0) = c^{-1}$,

$$\int_0^1 |\phi_\varepsilon(x)|^2\, dx = \frac{11\varepsilon^3}{16c^2\pi^2} \quad \text{and} \quad \int_0^1 |\phi'_\varepsilon(x)|^2\, dx = \frac{\varepsilon}{2c^2}.$$

For all $\varepsilon > 0$ small enough, we define the required test function f_ε as

$$f_\varepsilon(x) := \begin{pmatrix} z\phi_\varepsilon(x) \\ \alpha(\varepsilon) \end{pmatrix},$$

where

$$\alpha(\varepsilon) := \sqrt{1 - |z|^2\|\phi_\varepsilon\|^2} = \sqrt{1 - \frac{11|z|^2\varepsilon^3}{16c^2\pi^2}}$$

is independent of x. By construction $f_\varepsilon \in \mathrm{Dom}(D)$ and

$$\|f_\varepsilon\|^2 = \|z\phi_\varepsilon\|^2 + \alpha(\varepsilon)^2 = 1.$$

According to (3),

$$\begin{aligned}
\langle ADf_\varepsilon, f_\varepsilon \rangle &= \left\langle A\begin{pmatrix} z\phi'_\varepsilon(0) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha(\varepsilon) \end{pmatrix} \right\rangle + \int_0^1 \langle Af'(x), f'(x) \rangle\, dx \\
&= z\alpha(\varepsilon) + \left\langle A\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \int_0^1 |z|^2|\phi'_\varepsilon(x)|^2\, dx \\
&= z\alpha(\varepsilon) + \frac{a\varepsilon}{2c^2}|z|^2.
\end{aligned}$$

Since $\alpha(\varepsilon) \to 1$ as $\varepsilon \to 0$, the above shows $\langle ADf_\varepsilon, f_\varepsilon \rangle \to z$ as $\varepsilon \to 0$, so that z is an accumulation point of $\mathrm{Num}(AD)$. By moving $z \in \mathbb{C}$, any complex number is an accumulation point of $\mathrm{Num}(AD)$. Since $\mathrm{Num}(AD)$ is convex, the only possibility for $\mathrm{Num}(AD)$ is to be the whole of $\mathbb{C}$. $\square$

3. The resolvent of *AD*

In this section we show that the resolvent of AD is compact for all non-singular A. In general it is false that the product of a bounded operator and an operator whose resolvent is compact has compact resolvent, however if we know in addition that the spectrum of the product is not the whole of $\mathbb{C}$, then the assertion is true.

We first show that the resolvent of D is compact by making use of its self-adjointness. Since the constant function

$$f_0 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is in Dom(D) and ADf_0 vanishes,

$$0 \in \mathrm{Spec}(AD).$$

Proposition 5. *If A is a diagonal matrix, then*

$$\mathrm{Spec}(AD) = \{a_-\pi^2k^2, a_+\pi^2k^2\}_{k=0}^{\infty}.$$

The zero eigenvalue is always non-degenerate and all the remaining eigenvalues are of multiplicity no greater than 2.

Proof. Let $f_0 \in \mathrm{Dom}(D)$ be as above. For all $n = 1, 2, \ldots$, let

$$f_{2n-1}(x) := \sqrt{2}\begin{pmatrix} \sin(\pi n x) \\ 0 \end{pmatrix} \quad \text{and} \quad f_{2n}(x) := \sqrt{2}\begin{pmatrix} 0 \\ \cos(\pi n x) \end{pmatrix}. \tag{4}$$

Then $f_k \in \mathrm{Dom}(D)$,

$$ADf_{2n-1} = (a_+\pi^2n^2)f_{2n-1}, \quad ADf_{2n} = (a_-\pi^2n^2)f_{2n}$$

and $\{f_k\}_{k=0}^{\infty}$ is a complete orthonormal set in $L^2(0,1)\otimes\mathbb{C}^2$. $\square$

According to Corollary 3 and the above proposition, $D = D^* \geqslant 0$ and

$$\mathrm{Spec}(D) = \{\pi^2k^2\}_{k=0}^{\infty}.$$

Since the eigenfunctions $\{f_k\}_{k=0}^{\infty}$ form a complete orthonormal set, the resolvent of D is compact.

Let us now rule out the possibility $\mathrm{Spec}(AD) = \mathbb{C}$.

Lemma 6. *For any non-singular $A \in \mathbb{C}$*

$$\mathrm{Spec}(AD) \neq \mathbb{C}.$$

Proof. Fix matrix A. Since

$$AD - \lambda = A(D - \lambda A^{-1}),$$

the complex number $\lambda \in \mathrm{Spec}(AD)$ if, and only in,

$$0 \in \mathrm{Spec}(D - \lambda A^{-1}).$$

Let $H(\lambda) := D - \lambda A^{-1}$. Then the family of operators $H(\lambda)$ with domain Dom(D) independent of λ is a holomorphic family of type (A) for all $\lambda \in \mathbb{C}$. Since 0 is a non-degenerate isolated eigenvalue of $H(0) = D$ and A^{-1} is bounded, there exist an open neighbourhood $0 \in U \subset \mathbb{C}$ such that $H(\lambda)$ has a non-degenerate isolated eigenvalue, (denoted by $\mu(\lambda)$) close to 0 for all $\lambda \in U$ and $\mu(\lambda)$ is a complex valued holomorphic function in U (cf. [7, Theorem XII.8]).

If there exists some $\lambda_0 \in U$ satisfying $\mu(\lambda_0) \neq 0$, then $0 \notin \mathrm{Spec}(H(\lambda_0))$ so that $\lambda_0 \notin \mathrm{Spec}(AD)$. Hence, in order to show that $\mathrm{Spec}(AD) \neq \mathbb{C}$, it is enough to show that $\mu \not\equiv 0$. For this we find the first coefficients in the Rayleigh–Schrödinger series expansion of μ about 0. Let

$$\mu(\lambda) = \mu_0 + \mu_1\lambda + \mu_2\lambda^2 + \cdots, \quad \lambda \in U.$$

Since $\mu(0) = 0$, $\mu_0 = 0$. Since $\|f_0\| = 1$ and $H(0)f_0 = Df_0 = 0$, we compute directly μ_1 (cf. [6, Remark 2.2, p. 80]) by

$$\begin{aligned}\mu_1 &= \langle A^{-1}f_0, f_0 \rangle \\ &= \left\langle A^{-1}\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\rangle.\end{aligned}$$

If A is such that $a \neq 0$,

$$\left\langle A^{-1}\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\rangle \neq 0$$

so that μ_1 does not vanish and hence $\mu \not\equiv 0$.

Let A be such that $a = 0$. Then $\mu_1 = 0$ so we compute μ_2. Let f_k be the eigenfunctions of D as in (4) so that $\|f_k\| = 1$ for all $k = 1, 2, \ldots$. Let $\lambda_{2n-1} = \lambda_{2n} := \pi^2 n^2$ for all $n = 1, 2, \ldots$ so that

$$H(0)f_k = Df_k = \lambda_k f_k.$$

Then (cf. [6, Remark 2.2, p. 80])

$$-\mu_2 = \sum_{k=1}^{\infty} \frac{\langle A^{-1}f_0, f_k \rangle \langle A^{-1}f_k, f_0 \rangle}{\lambda_k}.$$

We compute each term in the series. Since $a = 0$ and A is invertible, then b and c do not vanish and

$$A^{-1} = \begin{pmatrix} -d/(bc) & 1/c \\ 1/b & 0 \end{pmatrix}.$$

Hence

$$\langle A^{-1}f_0, f_k \rangle = \int_0^1 \left\langle A^{-1}\begin{pmatrix}0\\1\end{pmatrix}, f_k(x)\right\rangle dx = \int_0^1 \left\langle \begin{pmatrix}1/c\\0\end{pmatrix}, f_k(x)\right\rangle dx,$$

so that

$$\langle A^{-1}f_0, f_{2n} \rangle = \sqrt{2}\int_0^1 \left\langle \begin{pmatrix}1/c\\0\end{pmatrix}, \begin{pmatrix}0\\ \cos(\pi n x)\end{pmatrix}\right\rangle dx = 0$$

and

$$\begin{aligned}\langle A^{-1}f_0, f_{2n-1}\rangle &= \sqrt{2}\int_0^1 \left\langle \begin{pmatrix} 1/c \\ 0 \end{pmatrix}, \begin{pmatrix} \sin(\pi n x) \\ 0 \end{pmatrix} \right\rangle dx \\ &= \sqrt{2}/c \int_0^1 \sin(\pi n x)\, dx \\ &= \begin{cases} 0 & \text{if } n = 2m, \\ 2\sqrt{2}/(c\pi n) & \text{if } n = 2m-1, \end{cases}\end{aligned}$$

for m integer and $n = 1, 2, \ldots$. On the other hand,

$$\langle A^{-1}f_k, f_0\rangle = \int_0^1 \left\langle A^{-1}f_k(x), \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle dx,$$

so that

$$\begin{aligned}\langle A^{-1}f_{2n}, f_0\rangle &= \sqrt{2}\int_0^1 \left\langle A^{-1}\begin{pmatrix} 0 \\ \cos(\pi n x) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle dx \\ &= \sqrt{2}\int_0^1 \left\langle \begin{pmatrix} \cos(\pi n x)/c \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle dx = 0\end{aligned}$$

and

$$\begin{aligned}\langle A^{-1}f_{2n-1}, f_0\rangle &= \sqrt{2}\int_0^1 \left\langle A^{-1}\begin{pmatrix} \sin(\pi n x) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle dx \\ &= \sqrt{2}\int_0^1 \left\langle \begin{pmatrix} d\sin(\pi n x)/(bc) \\ \sin(\pi n x)/b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle dx \\ &= \sqrt{2}/(b)\int_0^1 \sin(\pi n x)\, dx \\ &= \begin{cases} 0 & \text{if } n = 2m, \\ 2\sqrt{2}/(b\pi n) & \text{if } n = 2m-1 \end{cases}\end{aligned}$$

for m integer and $n = 1, 2, \ldots$. This yields

$$\langle A^{-1}f_0, f_k\rangle \langle A^{-1}f_k, f_0\rangle = \begin{cases} 0 & \text{if } k \neq 4m-3, \\ 8/(bc\pi^2 n^2) & \text{if } k = 4m-3 \end{cases}$$

for $m = 1, 2, \ldots$. Thus

$$-\mu_2 = \sum_{m=1}^{\infty} \frac{8}{bc\pi^4(2m-1)^4} \neq 0$$

so that $\mu \neq 0$ as we required. $\square$

Theorem 7. *For all $z \notin \mathrm{Spec}(AD)$, the resolvent $(AD - z)^{-1}$ is compact.*

Proof. Since D is non-negative and it has compact resolvent,

$$AD + A = A(D + 1)$$

has a compact inverse. Let $z \notin \mathrm{Spec}(AD)$, then

$$\begin{aligned} AD - z &= AD + A - A - z \\ &= (I - (A + z)(AD + A)^{-1})(AD + A). \end{aligned}$$

Hence

$$(AD + A)^{-1} = (AD - z)^{-1}(I - (A + z)(AD + A)^{-1}),$$

so that

$$\begin{aligned} (AD - z)^{-1} &= (AD - z)^{-1}(A + z)(AD + A)^{-1} + (AD + A)^{-1} \\ &= ((AD - z)^{-1}(A + z) + 1)(AD + A)^{-1}. \end{aligned}$$

Thus $(AD - z)^{-1}$ is compact as needed. □

Theorem 7 shows that the spectrum of AD consists entirely of isolated eigenvalues of finite multiplicity. Since the eigenvalue problem $ADf = \lambda f$ is a constant coefficient system of second-order ordinary differential equations, due to the fact that we have a combination Dirichlet and Neumann boundary condition at both ends of the interval, the multiplicity of each eigenvalue is never greater than 2.

4. Asymptotics of the resolvent

We now investigate the asymptotic behaviour of the resolvent norm of AD. The results we discuss in this section are connected with the stability of the heat semigroup e^{-ADt}. They are also relevant from the computational point of view and they are closely related to both local and global stability of the spectrum (cf. [1,4,9] and the reference therein). The present approach is motivated by analogous reports on non-self-adjoint Schrödinger operators in [2,4,5].

Let

$$J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Below and elsewhere we will denote by $\tilde{D} := JD$. According to Lemma 3, $\tilde{D} = \tilde{D}^*$. According to Lemma 5,

$$\mathrm{Spec}(\tilde{D}) = \{\pm\pi^2 n^2\}_{n=0}^{\infty}$$

each eigenvalue being of multiplicity 1. We will employ part (b) of the following theorem in the proof of Theorem 20(b).

Theorem 8. *Assume that there exists a non-singular diagonal matrix B such that $B^{-1}AB = (B^{-1}AB)^* > 0$. Then*

(a) *AD is similar to a non-negative self-adjoint operator.*
(b) *$A\tilde{D}$ is similar to a self-adjoint operator whose numerical range is the whole real line.*

Proof. Let $C := B^{-1}AB$ so that $C = C^* > 0$. Since diagonal matrices commute with the boundary conditions, AD is similar to CD. For the same reason and since diagonal matrices also commute with J, $A\tilde{D}$ is similar to $C\tilde{D}$.

By hypothesis, the square root $C^{1/2} = (C^{1/2})^* > 0$. Then

$$CD = C^{1/2}(C^{1/2}DC^{1/2})C^{-1/2} = C^{1/2}KC^{-1/2},$$

where

$$K = C^{1/2}DC^{1/2},$$

$$\mathrm{Dom}(K) = \{f \in L^2(0,1) \otimes \mathbb{C}^2 : C^{1/2}f \in \mathrm{Dom}(D)\},$$

so that CD is similar to K. Since $D = D^* \geqslant 0$, then $K = K^* \geqslant 0$.

Analogously, $C\tilde{D}$ is similar to

$$\tilde{K} := C^{1/2}\tilde{D}C^{1/2},$$

where $\mathrm{Dom}(\tilde{K}) = \mathrm{Dom}(K)$. Since $\tilde{D} = \tilde{D}^*$, then $\tilde{K} = \tilde{K}^*$. Furthermore, since

$$\mathrm{Num}(\tilde{D}) = \mathbb{R}$$

and

$$\langle \tilde{K}f, f \rangle = \langle \tilde{D}C^{1/2}f, C^{1/2}f \rangle,$$

the numerical range of $\tilde{K}$ the whole real line. $\square$

Let A be as in the hypothesis. The similarity to a self-adjoint operator ensures the existence of a constant $k_A \geqslant 1$ such that

$$\|(AD - z)^{-1}\| \leqslant \frac{k_A}{\mathrm{dist}(z, [0, \infty))}, \quad z \notin \mathrm{Spec}(AD)$$

and

$$\|(A\tilde{D} - z)^{-1}\| \leqslant \frac{k_A}{\mathrm{dist}(z, \mathbb{R})}, \quad z \notin \mathrm{Spec}(A\tilde{D}).$$

These identities show that although the numerical range of AD and $A\tilde{D}$ are in general the whole complex plane, the eigenvalues of these operators are stable in the sense of [9].

If we assume the weaker condition $C + C^* > 0$, we show how to recover part of the above estimate. We start with a preliminary lemma.

Lemma 9. *Let A be such that* $\mathrm{Num}(A) \subset \{\mathrm{Re}(z) > 0\}$. *Then* $\mathrm{Spec}(AD) \subset \{\mathrm{Re}(z) \geqslant 0\}$ *and there exists $k > 0$ independent of z, such that*

$$\|(AD - z)^{-1}\| \leqslant \frac{k}{|z|}, \quad \mathrm{Re}(z) < 0. \tag{5}$$

Proof. Let $r > 0$ and let $z \notin [0, \infty)$. Then

$$\begin{aligned} AD - z &= A(D - zA^{-1}) \\ &= A\left[(D - rz) + z(r - A^{-1})\right] \\ &= A\left[1 + (r - A^{-1})z(D - rz)^{-1}\right](D - rz). \end{aligned}$$

Therefore $z \notin \mathrm{Spec}(AD)$, whenever

$$\|(r - A^{-1})z(D - rz)^{-1}\| < 1. \tag{6}$$

We show that there is always $r > 0$ independent of z, such that this holds for all $\mathrm{Re}(z) < 0$.

Since $D \geqslant 0$ and $0 \in \mathrm{Spec}(AD)$,

$$\|(D - rz)^{-1}\| = \frac{1}{r|z|}.$$

Thus

$$\|(r - A^{-1})z(D - rz)^{-1}\| \leqslant \|1 - r^{-1}A^{-1}\|.$$

The hypothesis we imposed on A is equivalent to saying

$$A + A^* > 0,$$

then

$$A^{-1} + (A^{-1})^* = A^{-1}(A^* + A)(A^{-1})^* > 0.$$

For all $v \in \mathbb{C}^2$,

$$\begin{aligned} \|(I - r^{-1}A^{-1})v\|^2 &= \langle (I - r^{-1}(A^{-1} + (A^{-1})^*) + r^{-2}(A^{-1})^*A^{-1})v, v \rangle \\ &= |v|^2 - r^{-1}\langle (A^{-1} + (A^{-1})^* + r^{-1}(A^{-1})^*A^{-1})v, v \rangle. \end{aligned}$$

Hence there exists a constant $k_0 > 0$ independent of r (and z), such that

$$\|I - r^{-1}A^{-1}\| < 1 - r^{-1}k_0$$

when r is large enough. For such an r, identity (4) holds for any $\mathrm{Re}(z) < 0$. This shows that $\mathrm{Spec}(AD)$ must be enclosed in the right-hand plane.

Furthermore,

$$\begin{aligned}\|(AD-z)^{-1}\| &\leqslant \|A^{-1}\|\,\|(D-zA^{-1})^{-1}\| \\ &\leqslant \|A^{-1}\|\,\|(D-rz)^{-1}\|\,\|(1+z(D-rz)^{-1}(r-A^{-1}))^{-1}\| \\ &\leqslant \frac{\|A^{-1}\|}{r|z|}\sum_{k=0}^{\infty}\|z(D-rz)^{-1}(r-A^{-1})\|^k \\ &\leqslant \frac{\|A^{-1}\|}{r|z|}\sum_{k=0}^{\infty}\|1-r^{-1}A^{-1}\|^k \\ &\leqslant \frac{k}{|z|}\end{aligned}$$

so (5) is also proven. □

Below and elsewhere we denote by Ω the set of non-singular diagonal matrices and

$$S(\alpha,\beta) := \{z\in\mathbb{C}:\ \alpha\leqslant\arg(z)\leqslant\beta\},\quad \alpha\leqslant\beta.$$

Theorem 10. *If there exists* $B\in\Omega$ *such that*

$$\mathrm{Num}(B^{-1}AB)\subset S(\alpha,\beta),\quad \beta-\alpha<\pi,$$

then $\mathrm{Spec}(AD)\subset S(\alpha,\beta)$ *and for any small enough* $\varepsilon>0$ *there exists* $k_\varepsilon>0$ *independent of* z, *such that*

$$\|(AD-z)^{-1}\|\leqslant\frac{k_\varepsilon}{|z|},\quad z\notin S(\alpha-\varepsilon,\beta+\varepsilon).$$

Proof. Let $C := B^{-1}AB$, so that

$$\mathrm{Num}(C)\subset S(\alpha,\beta).$$

Since B commutes with the boundary conditions, AD is similar to CD and so it is enough to show the theorem for CD. Now, for all $-(\alpha+\pi/2)<\vartheta<\pi/2-\beta$

$$\mathrm{Num}(e^{i\vartheta}C)\subset\{\mathrm{Re}(z)>0\},$$

so we just have to apply Lemma 9 to $e^{i\vartheta}C$. □

The constant k_ε of this theorem is in general strictly greater than 1, therefore this is weaker than the similar condition for m-sectorial operators in [6, p. 279].

If A is triangular, the hypothesis of the above theorem does not necessarily hold. For instance if

$$A=\begin{pmatrix} a & 0 \\ 1 & a\end{pmatrix},\quad a>0,$$

then

$$\mathrm{Num}(A) = \{a + z : |z| < \tfrac{1}{2}\}$$

and so for small a the numerical range contains the origin. Nonetheless by using a similarity transformation and an approximation argument, we can show positivity of the spectrum whenever both of the eigenvalues of A are positive ($a > 0$ in our example). The conclusion about the spectrum of the following result will be improved in Theorem 19.

Corollary 11. *Let A be either upper or lower triangular. If $a \geqslant d > 0$, then*

$$\mathrm{Spec}(AD) \subset [0, \infty)$$

and for all $\varepsilon > 0$ there exists $k_\varepsilon > 0$ independent of z, such that

$$\|(AD - z)^{-1}\| < \frac{k_\varepsilon}{|z|}$$

for all $z \notin S(-\varepsilon, \varepsilon)$.

Proof. If A is upper triangular the proof is similar so let us assume that

$$A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}.$$

Let

$$A(r) := \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ rc & d \end{pmatrix}.$$

Then AD is similar to $A(r)D$ for all $r \neq 0$. Put

$$C(r) := A(r) + A(r)^* = \begin{pmatrix} 2a & r\bar{c} \\ rc & 2d \end{pmatrix}.$$

Then $C(r) = C(r)^*$. The eigenvalues of $C(r)$ are

$$a + d \pm \sqrt{(a-d)^2 + r^2|c|^2},$$

thus for small $r > 0$, $C(r) > 0$. The numerical range of $A(r)$ is an ellipse with focus at a, d and principal axis in the vertical direction of the order of r. By taking $r \to 0$, Theorem 10 completes the proof. $\square$

If A is as in the hypothesis of Corollary 11, there does not exist $B \in \Omega$ such that $B^{-1}AB = (B^{-1}AB)^*$ or $B^{-1}(AJ)B = (B^{-1}(AJ)B)^*$ so Theorem 8 is not applicable. We show that at least in one case AD fails to be similar to self-adjoint.

Theorem 12. *Let*

$$A = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}, \quad a > 0.$$

Let $\varepsilon > 0$ *and* $z(r) := 4a\pi^2 r^2 \pm i\varepsilon$. *Then there exists a constant* $k_\varepsilon > 0$ *independent of* r, *such that*

$$\|(AD - z(r))^{-1}\| > k_\varepsilon r^{1/2}, \quad r = 1, 2, \ldots .$$

Proof. Fix $\varepsilon > 0$ and let $z(r) := 4a\pi^2 r^2 - i\varepsilon$. Without loss of generality, we can assume $r = 3, 4, \ldots$. Throughout the proof the constants l_j are assumed to be positive, possibly depending upon ε but independent of r. In order to show the desired conclusion, it is enough to find $f_r \in \mathrm{Dom}(D)$ and l_0, such that

$$\frac{\|ADf_r - z(r)f_r\|}{\|f_r\|} \leqslant l_0 r^{-1/2} \tag{7}$$

for all large enough r.

Let

$$f = \begin{pmatrix} \phi \\ \gamma \end{pmatrix} \in \mathrm{Dom}(D).$$

Then

$$\begin{aligned} ADf - z(r)f &= -\begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}\begin{pmatrix} \phi'' \\ \gamma'' \end{pmatrix} - z(r)\begin{pmatrix} \phi \\ \gamma \end{pmatrix} \\ &= \begin{pmatrix} -a\phi'' - z(r)\phi \\ -\phi'' - a\gamma'' - z(r)\gamma \end{pmatrix}. \end{aligned}$$

Hence

$$\|f\|^2 = \|\phi\|^2 + \|\gamma\|^2$$

and

$$\|ADf - z(r)f\|^2 = \|a\phi'' + z(r)\phi\|^2 + \|a\gamma'' + z(r)\gamma + \phi''\|^2.$$

We now define the appropriate $f_r \in \mathrm{Dom}(D)$ satisfying (7). Let

$$\gamma_r(x) := \cos(2\pi r x).$$

Then $\|\gamma_r\|^2 = \frac{1}{2}$. Let

$$\phi_r(x) := \begin{cases} -i\varepsilon\cos(2\pi rx)/(4\pi^2r^2) & \text{if } x \in (1/r, 1-1/r), \\ 0 & \text{if } x \notin (1/(2r), 1-1/(2r)) \end{cases}$$

be such that ϕ_r is smooth and

(a) $|\phi_r(x)| \leqslant \varepsilon/(4\pi^2r^2)$ for all $x \in [0,1]$,
(b) $|\phi_r'(x)| \leqslant l_1/r$ for all $x \notin (1/r, 1-1/r)$,
(c) $|\phi_r''(x)| \leqslant l_2$ for all $x \notin (1/r, 1-1/r)$.

Then

$$\|\phi_r\|^2 = \int_0^1 |\phi_r(x)|^2\,dx \leqslant \varepsilon^2/(16\pi^4r^4) \leqslant l_3 r^{-4}$$

and

$$\|\phi_r\|^2 \geqslant \int_{1/r}^{1-1/r} \frac{\varepsilon^2\cos^2(2\pi rx)}{16\pi^4r^4}\,dx \geqslant l_4 r^{-4}.$$

Hence

$$f_r = \begin{pmatrix} \phi_r \\ \gamma_r \end{pmatrix} \in \mathrm{Dom}(D)$$

and

$$\tfrac{1}{4} \leqslant \|f_r\|^2 = \|\phi_r\|^2 + \|\gamma_r\|^2 \leqslant 1 \tag{8}$$

for all large enough r. If $1/r < x < 1-1/r$,

$$\begin{aligned} a\phi''(x) + z(r)\phi(x) &= a\phi(x)'' + 4a\pi r^2\phi(x) - i\varepsilon\phi(x) \\ &= [ai\varepsilon\cos''(2\pi rx) + 4a\pi^2r^2 i\varepsilon\cos(2\pi rx) \\ &\quad + \varepsilon^2\cos(2\pi rx)]/(4\pi^2r^2) \\ &= \varepsilon^2\cos(2\pi rx)/(4\pi^2r^2). \end{aligned}$$

Then, (a) and (c) yield

$$\begin{aligned} \|a\phi'' + z(r)\phi\|^2 &= \int_0^1 |a\phi''(x) + z(r)\phi(x)|^2\,dx \\ &\leqslant \int_{1/r}^{1-1/r} l_5/r^4\,dx + \int_{x\notin[1/r,1-1/r]} l_6 + l_7/r^4\,dx \\ &\leqslant l_6 r^{-1} + l_5 r^{-4} + l_7 r^{-5}. \end{aligned} \tag{9}$$

Also,

$$\begin{aligned} a\gamma''(x) + z(r)\gamma(x) + \phi''(x) &= a\gamma''(x) + 4a\pi^2 r^2\gamma(x) - i\varepsilon\gamma(x) + \phi''(x) \\ &= a\cos''(2\pi rx) + 4a\pi^2 r^2 \cos(2\pi rx) \\ &\quad - i\varepsilon\cos(2\pi rx) + \phi''(x) \\ &= \phi''(x) - i\varepsilon\cos(2\pi rx). \end{aligned}$$

Then for $1/r < x < 1 - 1/r$,

$$a\gamma''(x) + z(r)\gamma(x) + \phi''(x) = i\varepsilon\cos(2\pi rx) - i\varepsilon\cos(2\pi rx) = 0$$

and thus (c) yields

$$\begin{aligned} \|a\gamma'' + z(r)\gamma + \phi''\|^2 &= \int_0^1 |a\gamma''(x) + z(r)\gamma(x) + \phi''(x)|^2\, dx \\ &= \int_{x\notin[1/r,1-1/r]} |i\varepsilon\cos(2\pi rx) - \phi''(x)|^2\, dx \\ &\leqslant \int_{x\notin[1/r,1-1/r]} l_8\, dx \\ &\leqslant l_8 r^{-1}. \end{aligned} \tag{10}$$

In order to complete the proof for $z(r) \coloneqq 4a\pi r^2 - i\varepsilon$, notice that (8)–(10), show (7). On the other hand, if $z(r) \coloneqq 4a\pi r^2 + i\varepsilon$ it is enough to substitute ϕ_r by $-\phi_r$ and repeat the above computations. □

This result is still valid for

$$A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

Indeed, it is enough to put $\phi_r(x) \coloneqq \sin(2\pi rx)$,

$$\gamma_r(x) \coloneqq \begin{cases} \pm i\varepsilon\sin(2\pi rx)/(4\pi^2 r^2) & \text{if } x\in(1/r, 1-1/r), \\ 0 & \text{if } x\notin(1/(2r), 1-1/(2r)) \end{cases}$$

and carry out similar calculations. Since the resolvent norm of self-adjoint operators remains bounded in horizontal lines, the above AD cannot be similar to any self-adjoint operator.

Let Ω_r be the set of all non-degenerate real diagonal matrices. If A does not satisfy the hypothesis of Theorem 10 (for instance the numerical range of A is an ellipse centered at the origin), but A is "close" in some sense to Ω_r, an alternative to Theorem 10 can be established. We will employ this result in the proof of Theorem 19.

Theorem 13. *Let there exist* $B \in \Omega_r$ *such that*

$$\|AB - I\| < 1.$$

Let $\omega := \arcsin(\|AB - I\|)$ *with* $0 \leqslant \omega < \pi/2$. *Then*

$$\mathrm{Spec}(AD) \subset S(-\omega, \omega) \cup S(-\pi - \omega, \omega - \pi)$$

and for any small enough $\varepsilon > 0$ *there exist* $k_\varepsilon > 0$ *independent of* z, *such that*

$$\|(AD - z)^{-1}\| \leqslant \frac{k_\varepsilon}{|z|}$$

for all $z \notin S(-\omega - \varepsilon, \omega + \varepsilon) \cup S(-\pi - \omega - \varepsilon, \omega - \pi + \varepsilon)$.

Proof. If $\omega = 0$, $A \in \Omega_r$ so the conclusion is a consequence of Corollary 3. Let $\omega > 0$, let $l := \|AB - I\|$ and put $C := B^{-1} \in \Omega_r$. Then $CD = (CD)^*$ and according to the hypothesis $0 < l < 1$.

Let $z \in \mathbb{C}$ be such that $z \notin S(-\omega, \omega) \cup S(-\pi - \omega, \omega - \pi)$. Then

$$\begin{aligned}(AD - z) &= CD + (A - C)D - z \\ &= [I + (AB - I)CD(CD - z)^{-1}](CD - z).\end{aligned}$$

Since CD is self-adjoint and by definition $w = \arcsin(l)$,

$$\begin{aligned}\|(AB - I)CD(CD - z)^{-1}\| &\leqslant l\|CD(CD - z)^{-1}\| \\ &\leqslant l \sup_{x \in \mathbb{R}} \left|\frac{x}{x - z}\right| \\ &\leqslant \sup_{x \in \mathbb{R}} \frac{l}{|1 - \frac{z}{x}|} < 1, \end{aligned} \tag{11}$$

so that

$$[I + (AB - I)CD(CD - z)^{-1}]$$

is invertible. Hence

$$z \notin \mathrm{Spec}(AD)$$

and

$$(AD - z)^{-1} = (CD - z)^{-1}[I + (AB - I)CD(CD - z)^{-1}]^{-1} \tag{12}$$

for all $\omega < |\arg(z)| \leqslant \pi$. This encloses $\mathrm{Spec}(AD)$.

In order to show the second part, let

$$z \notin S(-\omega - \varepsilon, \omega + \varepsilon) \cup S(-\pi - \omega - \varepsilon, \omega - \pi + \varepsilon),$$

for small $\varepsilon > 0$. Then there exist a constant $l_1(\varepsilon) > 0$ independent of z, such that

$$\|(CD - z)^{-1}\| \leqslant \frac{l_1(\varepsilon)}{|z|}.$$

Also, there exist a constant $0 < l_2(\varepsilon) < 1$ independent of z, such that

$$\sup_{x \in \mathbb{R}} \frac{l}{|1 - \frac{z}{x}|} < l_2(\varepsilon).$$

These two estimates, (11) and (12) yield

$$\|(AD - z)^{-1}\| \leqslant \frac{l_1(\varepsilon) \sum_{n=0}^{\infty} l_2(\varepsilon)^n}{|z|} = \frac{k_\varepsilon}{|z|}. \qquad \square$$

This shows that if $A_n \in \mathbb{C}^{2\times 2}$ is a sequence of non-singular matrices and there exists $B \in \Omega_r$ such that

$$\|A_n - B\| \to 0$$

as $n \to \infty$, then

$$\mathrm{Spec}(A_n D) \to \mathbb{R}.$$

Corollary 14. *Let A be either upper or lower triangular. If $a, d \in \mathbb{R}$ and $ad < 0$, then*

$$\mathrm{Spec}(AD) \subset \mathbb{R}$$

and for all $\varepsilon > 0$ there exists $k_\varepsilon > 0$, such that

$$\|(AD - z)^{-1}\| < \frac{k_\varepsilon}{|z|}$$

for all $z \notin S(-\varepsilon, \varepsilon) \cup S(-\pi - \varepsilon, \varepsilon - \pi)$.

Proof. It is similar to the proof of Corollary 11. Assume without loss of generality that $b = 0$ and let

$$A(r) = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ rc & d \end{pmatrix}.$$

Then AD is similar to $A(r)D$ for all $r > 0$. Put

$$C = \begin{pmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix} \in \Omega_r,$$

then

$$\|A(r)C - I\| = \left\| \begin{pmatrix} 0 & 0 \\ rc/a & 0 \end{pmatrix} \right\| = r|c/a|.$$

Let $\omega_r := \arcsin(r|c/a|)$. According to Theorem 13, for all $0<r<|a/c|$

$$\begin{aligned}\operatorname{Spec}(AD) &= \operatorname{Spec}(A(r)D)\\ &\subset S(-\omega_r,\omega_r)\cup S(-\pi-\omega_r,\omega_r-\pi).\end{aligned}$$

By taking r small enough, Theorem 13 yields the desired estimate for the resolvent norm. By taking $r\to 0$, a fortiori $\operatorname{Spec}(AD)\subset\mathbb{R}$. $\square$

5. The Hamiltonian ode system

In this section we find an entire function whose zeros coincide with $\operatorname{Spec}(AD)$. This is made by computing the transcendental function of the 2×2 system of ordinary differential equations associated to AD via standard ODE arguments.

Let the 2×2 constant coefficients second-order eigenvalue problem

$$-Af'' = \lambda^2 f, \tag{13}$$

$$Pf(0) + Qf'(0) = 0,$$

$$Pf(1) + Qf'(1) = 0. \tag{14}$$

We will say that the complex number λ is an eigenvalue of system (13)–(14), if there exist a non-vanishing $f\in C^\infty(0,1)\otimes\mathbb{C}^2$ satisfying (13) and the boundary conditions (14). By regularity, λ^2 is an eigenvalue of AD if, and only if, λ is an eigenvalue of (13)–(14). Our aim is to find a holomorphic function, denoted by $EV(x)$ below, whose zeros coincide with the eigenvalues of (13)–(14).

We proceed in the classical manner. Let the decomposition in Jordan canonical form of A be

$$A =: VCV^{-1},$$

where the Jordan matrix C is either

$$C = \begin{pmatrix} a_+ & 0\\ 0 & a_-\end{pmatrix} \quad\text{or}\quad C = \begin{pmatrix} a_+ & 0\\ 1 & a_+\end{pmatrix}$$

and

$$V := \begin{pmatrix} v_1 & v_2\\ v_3 & v_4\end{pmatrix}.$$

Then (13)–(14) is equivalent to the 2×2 system

$$-Cg'' = \lambda^2 g, \tag{15}$$

$$PVg(0) + QVg'(0) = 0,$$
$$PVg(1) + QVg'(1) = 0. \tag{16}$$

In order to solve (15)–(16), we reduce it to a first order 4×4 system as follows. For all $\lambda \in \mathbb{C}$, let

$$B_\lambda = \begin{pmatrix} 0 & I \\ -\lambda^2 C^{-1} & 0 \end{pmatrix} \in \mathbb{C}^{4\times 4}$$

and let

$$\Psi := \begin{pmatrix} v_1 & v_2 & 0 & 0 \\ 0 & 0 & v_3 & v_4 \end{pmatrix} \in \mathbb{C}^{2\times 4}.$$

By regarding

$$\Phi = \begin{pmatrix} g \\ g' \end{pmatrix} \in \mathbb{C}^4,$$

one sees that (15)–(16) is equivalent to

$$\Phi' = B_\lambda \Phi, \tag{17}$$

$$\Psi\Phi(0) = \Psi\Phi(1) = 0. \tag{18}$$

In order to solve (17) and (18) we must find a fundamental system of solutions. Let e_1, e_2, e_3, e_4 be the standard orthonormal basis of the Euclidean space $\mathbb{C}^4$. A straightforward computations show that

$$\exp(B_\lambda x)e_j, \quad x \in [0,1], \; j = 1,2,3,4$$

is indeed a linearly independent fundamental system for (17) and (18). Hence, λ is an eigenvalue of this system if, and only if, there exist k_1, k_2, k_3, k_4, such that

$$\Phi(x) = \sum_{j=1}^{4} k_j \exp(B_\lambda x) e_j \tag{19}$$

is non-vanishing and satisfies the boundary conditions.

We now proceed to compute $EV(x)$. The exponential of $B_\lambda x$ is given by

$$\exp(B_\lambda x) = \begin{pmatrix} \cos(\lambda C^{-1/2} x) & \lambda^{-1} C^{1/2} \sin(\lambda C^{-1/2} x) \\ -\lambda C^{-1/2} \sin(\lambda C^{-1/2} x) & \cos(\lambda C^{-1/2} x) \end{pmatrix}$$

for $x \in [0,1]$. In Theorems 15 and 16, we split our computation into two cases depending upon the Jordan matrix C.

Theorem 15. *When*

$$C = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix},$$

λ is an eigenvalue of system (17), (18) *if, and only if, $EV(\lambda) = 0$ for*

$$\begin{aligned} EV(x) &:= \left(2\prod_{j=1}^{4} v_j\right)\left[1 - \cos\left(\frac{x}{\sqrt{a_+}}\right)\cos\left(\frac{x}{\sqrt{a_-}}\right)\right] \\ &\quad - \left[v_1^2 v_4^2 \frac{\sqrt{a_+}}{\sqrt{a_-}} + v_2^2 v_3^2 \frac{\sqrt{a_-}}{\sqrt{a_+}}\right] \sin\left(\frac{x}{\sqrt{a_+}}\right)\sin\left(\frac{x}{\sqrt{a_-}}\right). \end{aligned}$$

Proof. Notice that $EV(0) = 0$. Assume $\lambda \neq 0$. According to the hypothesis,

$$C^{1/2} = \begin{pmatrix} a_+^{1/2} & 0 \\ 0 & a_-^{1/2} \end{pmatrix} \quad \text{and} \quad C^{-1/2} = \begin{pmatrix} a_+^{-1/2} & 0 \\ 0 & a_-^{-1/2} \end{pmatrix}.$$

Then

$$\exp(B_\lambda x) = \begin{pmatrix} \cos\frac{\lambda x}{\sqrt{a_+}} & 0 & \frac{\sqrt{a_+}}{\lambda}\sin\frac{\lambda x}{\sqrt{a_+}} & 0 \\ 0 & \cos\frac{\lambda x}{\sqrt{a_-}} & 0 & \frac{\sqrt{a_-}}{\lambda}\sin\frac{\lambda x}{\sqrt{a_-}} \\ -\frac{\lambda}{\sqrt{a_+}}\sin\frac{\lambda x}{\sqrt{a_+}} & 0 & \cos\frac{\lambda x}{\sqrt{a_+}} & 0 \\ 0 & -\frac{\lambda x}{\sqrt{a_-}}\sin\frac{\lambda x}{\sqrt{a_-}} & 0 & \cos\frac{\lambda x}{\sqrt{a_-}} \end{pmatrix}.$$

Let $\Phi(x)$ be a particular solution given as in (19), where the complex parameters k_j are to be determined. Then

$$\Psi\Phi(0) = \begin{pmatrix} k_1 v_1 + k_2 v_2 \\ k_3 v_3 + k_4 v_4 \end{pmatrix}$$

and

$$\Psi\Phi(1) = \begin{pmatrix} k_1 v_1 \cos\frac{\lambda}{\sqrt{a_+}} + k_2 v_2 \cos\frac{\lambda}{\sqrt{a_-}} \\ + k_3 v_1 \frac{\sqrt{a_+}}{\lambda}\sin\frac{\lambda}{\sqrt{a_+}} + k_4 v_2 \frac{\sqrt{a_-}}{\lambda}\sin\frac{\lambda}{\sqrt{a_-}} \\ -k_1 v_3 \frac{\lambda}{\sqrt{a_+}}\sin\frac{\lambda}{\sqrt{a_+}} - k_2 v_4 \frac{\lambda}{\sqrt{a_-}}\sin\frac{\lambda}{\sqrt{a_-}} \\ +k_3 v_3 \cos\frac{\lambda}{\sqrt{a_+}} + k_4 v_4 \cos\frac{\lambda}{\sqrt{a_-}} \end{pmatrix}.$$

The solution Φ satisfies the boundary conditions (18) if, and only if,

$$\begin{cases} k_1 v_1 + k_2 v_2 = 0, \\ k_3 v_3 + k_4 v_4 = 0, \\ k_1 v_1 \cos \frac{\lambda}{\sqrt{a_+}} + k_2 v_2 \cos \frac{\lambda}{\sqrt{a_-}} + k_3 \frac{v_1 \sqrt{a_+}}{\lambda} \sin \frac{\lambda}{\sqrt{a_+}} + k_4 \frac{v_2 \sqrt{a_-}}{\lambda} \sin \frac{\lambda}{\sqrt{a_-}} = 0, \\ -k_1 \frac{v_3 \lambda}{\sqrt{a_+}} \sin \frac{\lambda}{\sqrt{a_+}} - k_2 \frac{v_4 \lambda}{\sqrt{a_-}} \sin \frac{\lambda}{\sqrt{a_-}} + k_3 v_3 \cos \frac{\lambda}{\sqrt{a_+}} + k_4 v_4 \cos \frac{\lambda}{\sqrt{a_-}} = 0. \end{cases}$$

The determinant of this 4×4 system of linear equations in k_j is precisely $EV(\lambda)$. $\square$

Theorem 16. *When*

$$C = \begin{pmatrix} a_+ & 0 \\ 1 & a_+ \end{pmatrix},$$

λ is an eigenvalue of system (17), (18) *if, and only if,* $EV(\lambda) = 0$ *for*

$$EV(x) := \left(\frac{v_2^2 v_4^2}{4a_+^3} \right) x^2 - \left(\det V + \frac{v_2 v_4}{2a_+} \right)^2 \sin^2 \frac{x}{\sqrt{a_+}}.$$

Proof. Notice that $EV(0) = 0$. Assume $\lambda \neq 0$. One can verify directly that

$$C^{1/2} = \begin{pmatrix} a_+^{1/2} & 0 \\ \frac{1}{2\sqrt{a_+}} & a_+^{1/2} \end{pmatrix} \quad \text{and} \quad C^{-1/2} = \begin{pmatrix} a_+^{-1/2} & 0 \\ -\frac{1}{\left(2a_+^{3/2}\right)} & a_+^{-1/2} \end{pmatrix}.$$

Then the four 2×2 blocks of matrix $\exp(B_\lambda x)$ are

$$\cos(\lambda C^{-1/2} x) = \begin{pmatrix} \cos \frac{\lambda x}{\sqrt{a_+}} & 0 \\ \frac{\lambda x}{(2a_+^{3/2})} \sin \frac{\lambda x}{\sqrt{a_+}} & \cos \frac{\lambda x}{\sqrt{a_+}} \end{pmatrix},$$

$\lambda^{-1} C^{1/2} \sin(\lambda C^{-1/2} x)$ equal to

$$\begin{pmatrix} \frac{a_+^{1/2}}{\lambda} \sin \frac{\lambda x}{\sqrt{a_+}} & 0 \\ \left[\frac{1}{2\lambda \sqrt{a_+}} \sin \frac{\lambda x}{\sqrt{a_+}} - \frac{x}{2a_+} \cos \frac{\lambda x}{\sqrt{a_+}} \right] & \frac{a_+^{1/2}}{\lambda} \sin \frac{\lambda x}{\sqrt{a_+}} \end{pmatrix}$$

and $-\lambda C^{-1/2} \sin(\lambda C^{-1/2} x)$ equal to

$$\begin{pmatrix} -\frac{\lambda}{a_+^{1/2}} \sin \frac{\lambda x}{\sqrt{a_+}} & 0 \\ \left[\frac{\lambda}{(2a_+^{3/2})} \sin \frac{\lambda x}{\sqrt{a_+}} + \frac{\lambda^2 x}{2a_+^2} \cos \frac{\lambda x}{\sqrt{a_+}} \right] & -\frac{\lambda}{a_+^{1/2}} \sin \frac{\lambda x}{\sqrt{a_+}} \end{pmatrix}.$$

Let $\Phi(x)$ be a particular solution given as in (19), where the complex parameters k_j are to be determined. Then

$$\Psi\Phi(0) = \begin{pmatrix} k_1 v_1 + k_2 v_2 \\ k_3 v_3 + k_4 v_4 \end{pmatrix}$$

and

$$\Psi\Phi(1) = \begin{pmatrix} \Psi\Phi(1)_1 \\ \Psi\Phi(1)_2 \end{pmatrix},$$

where

$$\begin{aligned}\Psi\Phi(1)_1 = {} & k_1\left(v_1 \cos\frac{\lambda}{\sqrt{a_+}} + \frac{v_2\lambda}{2a_+^{3/2}}\sin\frac{\lambda}{\sqrt{a_+}}\right) + k_2 v_2 \cos\frac{\lambda}{\sqrt{a_+}} \\ & + k_3\left(\frac{v_1\sqrt{a_+}}{\lambda}\sin\frac{\lambda}{\sqrt{a_+}} + \frac{v_2}{2\lambda\sqrt{a_+}}\sin\frac{\lambda}{\sqrt{a_+}} - \frac{v_2}{2a_+}\cos\frac{\lambda}{\sqrt{a_+}}\right) \\ & + k_4\frac{v_2\sqrt{a_+}}{\lambda}\sin\frac{\lambda}{\sqrt{a_+}}\end{aligned}$$

and

$$\begin{aligned}\Psi\Phi(1)_2 = {} & k_1\left(-\frac{v_3\lambda}{\sqrt{a_+}}\sin\frac{\lambda}{\sqrt{a_+}} + \frac{v_4\lambda}{2a_+^{3/2}}\sin\frac{\lambda}{\sqrt{a_+}} + \frac{v_4\lambda^2}{2a_+}\cos\frac{\lambda}{\sqrt{a_+}}\right) \\ & + k_2\frac{v_4\lambda}{\sqrt{a_+}}\sin\frac{\lambda}{\sqrt{a_+}} + k_3\left(v_3\cos\frac{\lambda}{\sqrt{a_+}} + \frac{v_4\lambda}{2a_+^{3/2}}\sin\frac{\lambda}{\sqrt{a_+}}\right) \\ & + k_4 v_4 \cos\frac{\lambda}{\sqrt{a_+}}.\end{aligned}$$

The solution Φ satisfies the boundary conditions (18) if, and only if,

$$\begin{cases} k_1 v_1 + k_2 v_2 = 0, \\ k_3 v_3 + k_4 v_4 = 0, \\ \Psi\Phi(1)_1 = 0, \\ \Psi\Phi(1)_2 = 0. \end{cases}$$

A rather long but straightforward computation shows that the determinant of this 4×4 system of linear equations in k_j is $EV(\lambda)$. $\square$

We show that AD can have non-real eigenvalues even when the spectrum of A is positive.

Example 1. Put

$$A := \begin{pmatrix} 2/5 + 3i/10 & 3/5 - 3i/10 \\ 3/20 + 3i/10 & 17/20 - 3i/10 \end{pmatrix}.$$

Then the eigenvalues of A are $a_+ = 1$, $a_- = \frac{1}{4}$, and the eigenvectors

$$v_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_- = \begin{pmatrix} 2i \\ 1 \end{pmatrix}.$$

Thus

$$EV(x) = 4i(1 - \cos(x)\cos(2x)) = 4i(1 - 2\cos^3(x) + \cos(x))$$

so that $EV(\lambda) = 0$ if, and only if,

$$\cos(\lambda) = 1 \quad \text{or} \quad \cos(\lambda) = -\tfrac{1}{2} \pm i/2.$$

Hence

$$\mathrm{Spec}(AD) = \{4k^2\pi^2, (\lambda_\pm + 2k\pi)^2\}_{k \in \mathbb{Z}},$$

where $\lambda_\pm = \arccos(-\frac{1}{2} \pm i/2) \approx 2.02 \pm 0.53i$.

6. Real matrices

In this section we explore some connections between the entries of matrix A and the global behaviour of $\mathrm{Spec}(AD)$ when $A \in \mathbb{R}^{2\times 2}$. Alongside we discuss conditions to ensure similarity to a self-adjoint operator. For completeness of the picture, below and elsewhere we allow $\det(A) = 0$.

Our first task is to reduce to two parameters the four that are initially given as entries of A. This leads us to five different types of matrices to deal with. For $a, d \in \mathbb{R}$, let

$$A_0 := \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \qquad A_1 := \begin{pmatrix} a & 1 \\ 1 & d \end{pmatrix}, \qquad A_2 := \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix},$$

$$A_3 := \begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix} \qquad \text{and} \qquad A_4 := \begin{pmatrix} a & -1 \\ 1 & d \end{pmatrix}.$$

We show that the A_jD generate any $AD, A \in \mathbb{R}^{2\times 2}$ via similarity transformations.

Lemma 17. *If $A \in \mathbb{R}^{2\times 2}$, then AD is similar to $\alpha A_j D$ for some $\alpha, a, d \in \mathbb{R}$ and $j = 0, \ldots, 4$.*

Proof. Let

$$A = \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix}.$$

If $bc = 0$, the proof is trivial. Let

$$A(r) := \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} = \begin{pmatrix} \tilde{a} & r^{-1}b \\ rc & \tilde{d} \end{pmatrix}.$$

Then, AD is similar to $A(r)D$ for all $r \neq 0$. If $b/c > 0$,

$$A(\sqrt{b/c}) = \begin{pmatrix} \tilde{a} & \sqrt{bc} \\ \sqrt{bc} & \tilde{d} \end{pmatrix} = \alpha A_1$$

for $\alpha = \sqrt{bc}$, $a = \tilde{a}/\sqrt{bc}$ and $d = \tilde{d}/\sqrt{bc}$. If $b/c < 0$,

$$A(\sqrt{-b/c}) = \begin{pmatrix} \tilde{a} & \mp\sqrt{-bc} \\ \pm\sqrt{-bc} & \tilde{d} \end{pmatrix} = \pm\alpha A_4$$

for $\alpha = \sqrt{-bc}$, $a = \pm\tilde{a}/\sqrt{-bc}$ and $d = \pm\tilde{d}/\sqrt{-bc}$. $\square$

The case $j = 0$ was already described in Corollary 3. Indeed if $ad \neq 0$ then $A_0 D$ is similar to a self-adjoint operator and

$$\mathrm{Spec}(A_0 D) = \{an^2\pi^2, dn^2\pi^2\}_{n=0}^{\infty} \subset \mathbb{R}.$$

6.1. *Matrix A_1*

Since a and d are real, $A_1 = A_1^*$. Let $b_\pm$ be the eigenvalues of A_1. Then

$$b_\pm = \frac{a + d \pm \sqrt{(a-d)^2 + 4}}{2},$$

so that

(i) $b_+ \geqslant b_- > 0$ if, and only if, $ad > 1$ and $a, d > 0$,
(ii) $b_- \leqslant b_+ < 0$ if, and only if, $ad > 1$ and $a, d < 0$,
(iii) b_+ and b_- have opposite signs if, and only if, $ad < 1$.

Theorem 18. *The following statements are true*:

(a) *If* $ad = 1$ *then* $\mathrm{Spec}(A_1 D) = \mathbb{C}$.
(b) *If* $ad > 1$ *and* $a, d > 0$ *then* $A_1 D$ *is similar to a non-negative operator so that* $\mathrm{Spec}(A_1 D) \subset [0, \infty)$.
(c) *If* $ad > 1$ *and* $a, d < 0$ *then* $-A_1 D$ *is similar to a non-negative self-adjoint operator so that* $\mathrm{Spec}(A_1 D) \subset (-\infty, 0]$.
(d) *If* $ad < 1$ *then* $\mathrm{Spec}(A_1 D) \subset \mathbb{R}$.

Proof. If $ad = 1$, matrix A_1 is singular so according to Lemma 1, A_1D is not a closed operator. This shows (a). Statement (b) is a consequence of (i) and Theorem 8, and statement (c) is a consequence of (ii) and Theorem 8.

Let us show (d). For $\varepsilon \in \mathbb{R}$, let

$$B(\varepsilon) := A_1 + i\varepsilon.$$

Then

$$\mathrm{Num}(B(\varepsilon)) \subset \{\mathrm{Im}(z) > 0\}, \quad \varepsilon > 0$$

and

$$\mathrm{Num}(B(\varepsilon)) \subset \{\mathrm{Im}(z) < 0\}, \quad \varepsilon < 0.$$

According to Theorem 10,

$$\mathrm{Spec}(B(\varepsilon)D) \subset \{\mathrm{Im}(z) \geqslant 0\}, \quad \varepsilon > 0$$

and

$$\mathrm{Spec}(B(\varepsilon)D) \subset \{\mathrm{Im}(z) \leqslant 0\}, \quad \varepsilon < 0.$$

Since $B(\varepsilon)D$ is a holomorphic family of type (A) in a neighbourhood of $\varepsilon = 0$ and $B(0) = A_1$, *a fortiori*

$$\mathrm{Spec}(A_1D) \subset \mathbb{R}. \qquad \square$$

Although $A_1 = A_1^*$, it is unclear to us whether A_1D is similar to self-adjoint in the latter case.

6.2. Matrices A_2 and A_3

Since the results for matrix A_3 are analogous and shown in a similar manner as for A_2, we will only consider the latter.

Theorem 19. *The following statements are true*:

(a) *If* $ad = 0$ *then* $\mathrm{Spec}(A_2D) = \mathbb{C}$.

(b) *If* $ad \neq 0$ *then* $\mathrm{Spec}(A_2D) = \{a\pi^2n^2, d\pi^2n^2\}_{n=0}^{\infty}$.

(c) *If* $ad > 0$, *for all* $\varepsilon > 0$ *there exists* $k_\varepsilon > 0$ *independent of* z, *such that*

$$\|(A_2D - z)^{-1}\| \leqslant \frac{k_\varepsilon}{|z|}, \quad z \notin \pm S(-\varepsilon, \varepsilon),$$

where the symbol $\pm$ *is chosen according to the symbol of* a.

(d) *If* $ad < 0$, *then for all* $\varepsilon > 0$ *there exists* $k_\varepsilon > 0$ *independent of* z, *such that*

$$\|(A_2D - z)^{-1}\| \leqslant \frac{k_\varepsilon}{|z|}$$

for all $z \notin S(-\varepsilon, \varepsilon) \cup S(-\pi - \varepsilon, -\pi + \varepsilon)$.

(e) *If* $a = d \neq 0$, *let* $\varepsilon > 0$ *and* $z_r = 4a\pi^2 r^2 \pm i\varepsilon$. *Then there exists a constant* $k_\varepsilon > 0$ *independent of* r, *such that*

$$\|(A_2 D - z_r)^{-1}\| \geqslant k_\varepsilon |z_r|^{1/4}$$

for all $r = 1, 2, \ldots$.

Proof. If $ad = 0$, matrix A_2 is singular according to Lemma 1, $A_2 D$ is not a closed operator. This shows (a).

Let us show (b). If $a \neq d$, matrix A_2 is diagonalizable and

$$A_2 = \begin{pmatrix} a-d & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} (a-d)^{-1} & 0 \\ 1-(a-d)^{-1} & (a-d)^{-1} \end{pmatrix}.$$

Then, according to Theorem 15,

$$EV(x) = k_0 \sin\left(\frac{x}{\sqrt{a_+}}\right) \sin\left(\frac{x}{\sqrt{a_-}}\right),$$

where k_0 is constant in x. If $a = d$, A_2 is already in Jordan form and according to Theorem 16,

$$EV(x) = -\sin\left(\frac{x}{\sqrt{a}}\right)^2.$$

Hence in both cases

$$\mathrm{Spec}(A_2 D) = \{a\pi^2 n^2, d\pi^2 n^2\}_{n=0}^{\infty}.$$

Statements (c) is a consequence of Corollary 11 and statement (d) is a consequence of Corollary 14. For statement (e) use Theorem 12 and the fact that $|z_r|$ is of order r^2. $\square$

6.3. Matrix A_4

Formally speaking, so far the spectrum of $A_j D$ for $j = 0, \ldots, 3$ reproduces the spectrum of A_j in the following sense: if A_j is non-degenerated and both eigenvalues of A_j are positive (negative) then $\mathrm{Spec}(A_j D)$ is non-negative (non-positive), and if the eigenvalues are of opposite sign then $A_j D$ possess both positive and negative spectrum. There is no reason to expect the same for $j = 4$, in fact this case is less simpler due to the way the entries of A_4 interact with the boundary conditions.

The eigenvalues of A_4 are given by

$$b_{\pm} := \frac{a + d \pm \sqrt{(a - d)^2 - 4}}{2}. \tag{20}$$

Then

(i) $b_+ = b_-$ if, and only if, $|a - d| = 2$. In this case A_4 is not a diagonalizable matrix.
(ii) $b_{\pm}$ are real and have opposite signs if, and only if, $ad < -1$.
(iii) $b_+ > b_- > 0$ if, and only if, $ad > -1$, $|a - d| > 2$ and $a + d > 0$.
(iv) $b_- < b_+ < 0$ if, and only if, $ad > -1$, $|a - d| > 2$ and $a + d < 0$.
(v) $b_{\pm}$ are non-real with $b_+ = \overline{b_-}$ if, and only if, $|a - d| < 2$.
(vi) A_4 is singular if, and only if, $ad = -1$.

Motivated by this and for simplicity, we can divide the plane into 6 disjoint regions R_k,

$$R_1 := \{(a, d) \in \mathbb{R}^2 : |a - d| = 2,\ a \neq \pm 1\},$$

$$R_2 := \{(a, d) \in \mathbb{R}^2 : ad < -1\},$$

$$R_3 := \{(a, d) \in \mathbb{R}^2 : ad > -1,\ |a - d| > 2,\ a + d > 0\},$$

$$R_4 := \{(a, d) \in \mathbb{R}^2 : ad > -1,\ |a - d| > 2,\ a + d < 0\},$$

$$R_5 := \{(a, d) \in \mathbb{R}^2 : |a - d| < 2\},$$

$$R_6 := \{(a, d) \in \mathbb{R}^2 : ad = -1\}.$$

Clearly $\mathbb{R}^2 = \bigcup R_k$. Below we establish the spectral results for $A_4 D$ separately in each region R_k.

Two cases are similar to what we have found so far.

Theorem 20. *The following statements are true*:

(a) *If* $(a, d) \in R_6$, *then* $\mathrm{Spec}(A_4 D) = \mathbb{C}$.
(b) *If* $(a, d) \in R_2$, *then* $\mathrm{Spec}(A_4 D) \subset \mathbb{R}$ *and* $A_4 D$ *is similar to a self-adjoint operator whose numerical range is the whole real line*.

Proof. If $ad = -1$, matrix A_4 is singular so according to Lemma 1, $A_4 D$ is not a closed operator. This shows (a).

Let us show (b). Let J be as in Section 4. Then

$$A_4 D = (A_4 J)(JD) = \begin{pmatrix} a & 1 \\ 1 & -d \end{pmatrix} \tilde{D} = \tilde{A} \tilde{D}.$$

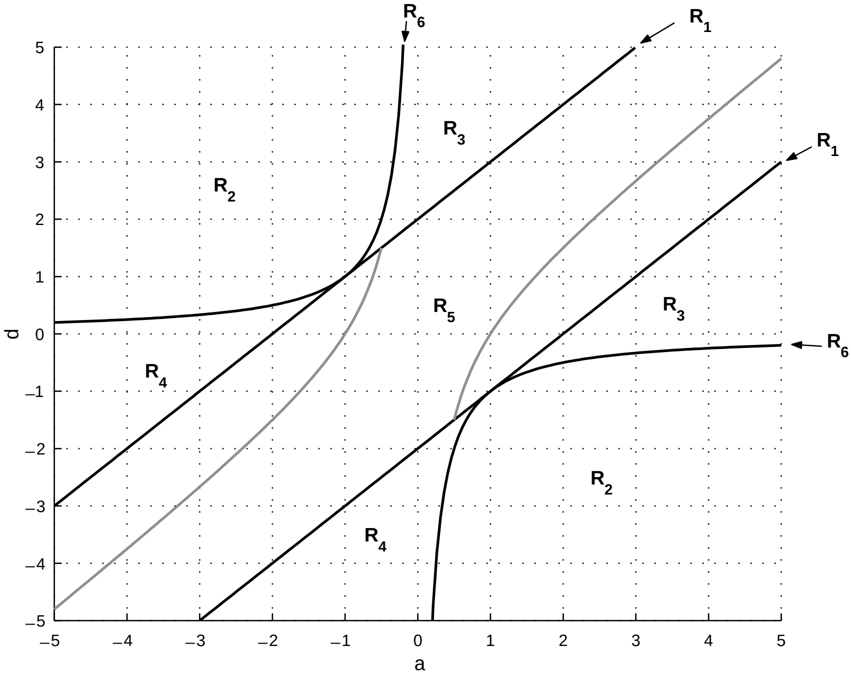

Fig. 1. Different regions of the plane in which $\mathrm{Spec}(A_4D)$ exhibits a similar behaviour. The grey line is $\{a^2 - ad - 1 = 0\} \cap R_5$. See Theorems 20–26.

Here $\tilde{A} = \tilde{A}^*$ and the eigenvalues of $\tilde{A}$ are

$$\tilde{b}_{\pm} = \frac{a - d \pm \sqrt{(a+d)^2 + 4}}{2}.$$

Since $ad < -1$, $\tilde{b}_{\pm}$ are either both positive or both negative. If they are both positive, $\tilde{A} > 0$ so that Theorem 8(b) provides the desired conclusion. If they are both negative apply the above argument to $-A_4D$ (Fig. 1). $\square$

In order to find $\mathrm{Spec}(A_4D)$ in R_k for $k = 1, 3, 4, 5$, we ought to rely on properties of the transcendental function $EV(x)$. Nonetheless, Theorem 21 provides some indication of what we should expect, it bases on the observation that if both a and d are positive,

$$A_4 + A_4^* = \begin{pmatrix} 2a & 0 \\ 0 & 2d \end{pmatrix} > 0,$$

so by virtue of Lemma 9, $\mathrm{Spec}(A_4D) \subset \{\mathrm{Re}(z) \geqslant 0\}$.

Theorem 21. *If both a and d are positive, then*

$$\mathrm{Spec}(A_4D)\subset S(-\omega,\omega),$$

where $\sin\omega = 1/\sqrt{ad+1}$ *for* $0<\omega<\pi/2$.

Proof. The numerical range of A_4 is an ellipse whose foci are $b_\pm$ and largest diameter is of length $|a-d|$. It is easy to see that $S(-\omega,\omega)$ is the minimal sector that contains such an ellipse. Use Theorem 10 to complete the proof. $\square$

Since

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -a & -1 \\ 1 & -d \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} a & -1 \\ 1 & d \end{pmatrix}$$

and because of diagonal matrices commute with the boundary conditions, $\mathrm{Spec}(A_4D)\subset -S(-\omega,\omega)$ where both a and d are negative. This also shows that the spectral results for A_4D are symmetric with respect to the axis $a+d=0$. Below we will employ this symmetry often without mention.

In order to describe $\mathrm{Spec}(A_4D)$ in R_5, we will make use of the following technical result.

Lemma 22. *Let* $\alpha\in\mathbb{C}$ *be such that* $\mathrm{Re}(\alpha^2)\geqslant 0$, *let* $-1\leqslant c\leqslant 1$ *and let*

$$F(x) := 1-\cos(\alpha x)\cos(\bar{\alpha}x) - c\sin(\alpha x)\sin(\bar{\alpha}x), \quad x\in\mathbb{C}.$$

Then $F(x)$ *has an infinite number of zeros in the complex plane and*

(a) *if* $c=-1$, *then* $F(x)=0$ *if, and only if,* $\sin(\mathrm{Re}(\alpha)x)=0$,
(b) *if* $c=1$, *then* $F(x)=0$ *if, and only if,* $\sinh(\mathrm{Im}(\alpha)x)=0$,
(c) *if* $-1<c<1$, *then* $F(x)$ *only has a finite number of zeros lying on the real and imaginary axis.*

Proof. Let $\alpha =: \rho + i\mu$ so that $\rho\geqslant\mu>0$ and let $x =: x_1 + ix_2$ for $x_1,x_2\in\mathbb{R}$.

In order to show (a), assume $c=-1$. Then

$$\begin{aligned}|F(x)|^2 &= |1-\cos[\alpha(x_1+ix_2)]\cos[\bar{\alpha}(x_1+ix_2)] \\ &\quad + \sin[\alpha(x_1+ix_2)]\sin[\bar{\alpha}(x_1+ix_2)]|^2 \\ &= 4[\cos^2(\rho x_1) - \cosh^2(\rho x_2)]^2.\end{aligned}$$

Hence

$$F(x)=0$$

if, and only if, $\cosh(\rho x_2)=1$ and $\cos(\rho x_1)=1$. This gives (a).

Similarly for (b), assume $c = 1$. Then

$$\begin{aligned}|F(x)|^2 &= |1 - \cos[\alpha(x_1 + ix_2)]\cos[\bar{\alpha}(x_1 + ix_2)] \\ &\qquad - \sin[\alpha(x_1 + ix_2)]\sin[\bar{\alpha}(x_1 + ix_2)]|^2 \\ &= 4[\cosh^2(\mu x_1) - \cos^2(\mu x_2)]^2.\end{aligned}$$

Hence

$$F(x) = 0$$

if, and only if, $\cosh(\mu x_1) = 1$ and $\cos(\mu x_2) = 1$.

Let us show assertion (c). If $x \in \mathbb{R}$, then

$$\begin{aligned}F(x) &= 1 - \cos(\alpha x)\overline{\cos(\alpha x)} - c\sin(\alpha x)\overline{\sin(\alpha x)} \\ &= 1 - |\cos(\alpha x)|^2 - c|\sin(\alpha x)|^2 \\ &= 1 - \cos^2(\rho x) - c\sin^2(\rho x) - (1 + c)\sinh^2(\mu x)\end{aligned}$$

and

$$\begin{aligned}F(ix) &= 1 - \cos(-\overline{i\bar{\alpha}}x)\cos(i\bar{\alpha}x) - c\sin(-\overline{i\bar{\alpha}}x)\sin(i\bar{\alpha}x) \\ &= 1 - \overline{\cos(i\bar{\alpha}x)}\cos(i\bar{\alpha}x) + c\,\overline{\sin(\bar{\alpha}x)}\sin(i\bar{\alpha}x) \\ &= 1 - |\cos(i\bar{\alpha}x)|^2 - c|\sin(i\bar{\alpha}x)|^2 \\ &= 1 - \cos^2(\mu x) + c\sin^2(\mu x) - (1 - c)\sinh^2(\rho x).\end{aligned}$$

Hence, if $-1 < c < 1$,

$$\lim_{x \to \pm\infty} F(x) = -\infty \quad \text{and} \quad \lim_{x \to \pm\infty} F(ix) = -\infty.$$

Since $F(x)$ is a smooth function, (c) follows.

Finally, let us show that $F(x)$ has a infinite number of zeros. Suppose that F only has a finite number of zeros $0, z_1, \ldots, z_n$ where the z_j repeats as many times as its order. Then

$$G(x) = \frac{F(x)}{x^2 \prod_{j=1}^n (x - z_j)}$$

is an entire function with no zeros. By virtue of the Weierstrass factorization theorem, there is an entire function $g(x)$ such that $G(x) = e^{g(x)}$. Then

$$F(x) = \left[x^2 \prod_{j=1}^n (x - z_j)\right] e^{g(x)} =: p(x)e^{g(x)}.$$

Since it is a combination of sines and cosines, the order (cf. [3, p. 285]) in the sense of entire functions of $F(x)$ is $\lambda = 1$. Thus by virtue of Hadamard's factorization theorem, $g(x)$ is a polynomial of degree 1 in x and so

$$F(x) = p(x)e^{kx+l}$$

for suitable $k, l \in \mathbb{C}$. Since $p(x)$ is a polynomial, this is clearly a contradiction, so $F(x)$ must have an infinite number of zeros. $\square$

Theorem 23. *Let* $(a,d)\in R_5$.

(a) *If* $(a,d)\in\{a^2-ad-1=0\}\cap\{-2<a-d<0\}$, *then*
$$\mathrm{Spec}(A_4D)=\{-k^2\pi^2/\mathrm{Im}(b_+^{-1/2})^2\}_{k\in\mathbb{Z}}\subset(-\infty,0].$$

(b) *If* $(a,d)\in\{a^2-ad-1=0\}\cap\{0<a-d<2\}$, *then*
$$\mathrm{Spec}(A_4D)=\{k^2\pi^2/\mathrm{Re}(b_+^{-1/2})^2\}_{k\in\mathbb{Z}}\subset[0,\infty).$$

(c) *If* $(a,d)\notin\{a^2-ad-1=0\}$, *then* $\mathrm{Spec}(A_4D)$ *is infinite but it only intersects the real line in a finite number of points.*

Proof. By virtue of (v), A_4 is diagonalizable. We assume $a+d\geqslant 0$, so that $\{b_\pm\}\subset\{\mathrm{Re}(z)\geqslant 0\}$.

Let
$$y:=\sqrt{4-(a-d)^2}\quad\text{and}\quad \gamma_\pm=a-d\pm iy.$$
Then
$$A_4=\begin{pmatrix}\gamma_+ & \gamma_-\\ 2 & 2\end{pmatrix}\begin{pmatrix}b_+ & 0\\ 0 & b_-\end{pmatrix}\begin{pmatrix}\frac{1}{2iy} & -\frac{\gamma_-}{4iy}\\ -\frac{1}{4iy} & \frac{\gamma_+}{2iy}\end{pmatrix}.$$
Let $\vartheta:=\arg\gamma_+$ and $\alpha:=1/\sqrt{b_+}$ so that $\bar\alpha=1/\sqrt{b_-}$. Then
$$\begin{aligned}\frac{EV(x)}{4\gamma_-^2}&=\frac{2\gamma_+}{\gamma_-}[1-\cos(\alpha x)\cos(\bar\alpha x)]-\left(\frac{\gamma_+^2}{\gamma_-^2}\sqrt{\frac{b_+}{b_-}}+\sqrt{\frac{b_-}{b_+}}\right)\sin(\alpha x)\sin(\bar\alpha x)\\ &=\frac{2\gamma_+}{\overline{\gamma_+}}[1-\cos(\alpha x)\cos(\bar\alpha x)]-\left(\frac{\gamma_+^2}{\overline{\gamma_+}^2}\frac{\bar\alpha}{\alpha}+\frac{\alpha}{\bar\alpha}\right)\sin(\alpha x)\sin(\bar\alpha x)\\ &=2e^{i2\vartheta}[1-\cos(\alpha x)\cos(\bar\alpha x)]-\left(e^{i4\vartheta}\frac{\bar\alpha}{\alpha}+\frac{\alpha}{\bar\alpha}\right)\sin(\alpha x)\sin(\bar\alpha x)\\ &=2e^{i2\vartheta}[1-\cos(\alpha x)\cos(\bar\alpha x)-c\sin(\alpha x)\sin(\bar\alpha x)]\\ &=2e^{i2\vartheta}F(x),\end{aligned}\tag{21}$$
where $F(x)$ and
$$\begin{aligned}c&:=\frac{(\bar\alpha e^{i2\vartheta}/\alpha)+(\alpha e^{-i2\vartheta}/\bar\alpha)}{2}\\ &=\frac{e^{i(2\vartheta-2\arg(\alpha))}+e^{-i(2\vartheta-2\arg(\alpha))}}{2}\\ &=\cos(2\vartheta-2\arg(\alpha))=\cos(2\vartheta+\arg b_+)\end{aligned}$$
are as in Lemma 22.

Let us show (a). The hypothesis $a-d<0$ ensures $-1<c\leqslant 1$. Furthermore $c=1$ if, and only if,

$$\frac{\mathrm{Im}(\gamma_+^2)}{\mathrm{Re}(\gamma_+^2)}=-\frac{\mathrm{Im}(b_+)}{\mathrm{Re}(b_+)}.$$

The latter occurs if, and only if,

$$\frac{y(a-d)}{(a-d)^2-2}=-\frac{y}{a+d}.$$

By simplifying this identity, we gather that $c=1$ for $a^2-ad-1=0$ which is precisely our assumption. Then, Lemma 22(b) and (21) complete the proof of (a).

For (b), notice that since $a-d>0$, the constant c is now such that $-1\leqslant c<1$ and $c=-1$ if, and only if,

$$\frac{\mathrm{Im}(\gamma_+^2)}{\mathrm{Re}(\gamma_+^2)}=-\frac{\mathrm{Im}(b_+)}{\mathrm{Re}(b_+)}.$$

Therefore a similar argument as for (a) and Lemma 22(a) show this case. In order to prove (c) use the fact that $-1<c<1$ in

$$R_5\backslash\{a^2-ad-1=0\},$$

Lemma 22(c) and (21). $\square$

Theorem 24. *In the regions* R_3 *and* R_4, Spec(A_4D) *is infinite, and*

$$\mathrm{Spec}(A_4D)\subset\{(r+iy_0)^2 : r\in\mathbb{R}\}+[0,\infty),\quad (a,d)\in R_3,$$

$$\mathrm{Spec}(A_4D)\subset\{-(r+iy_0)^2 : r\in\mathbb{R}\}+(-\infty,0],\quad (a,d)\in R_4,$$

where in both cases the constant $y_0>0$ *only depends upon* (a,d).

Proof. We show the result only for R_3. According to (iii), in this case $0<b_-<b_+$ and A_4 is diagonalizable. By expressing the trigonometric functions in exponential form,

$$\begin{aligned}EV(x) &= k_1-k_1\cos(\alpha x)\cos(\beta x)-k_2\sin(\alpha x)\sin(\beta x)\\ &= k_1+\frac{K_2-K_2}{4}\left[e^{i(\alpha+\beta)x}+e^{-i(\alpha+\beta)x}\right]\\ &\quad -\frac{k_2+k_1}{4}\left[e^{i(\alpha-\beta)x}+e^{-i(\alpha-\beta)x}\right],\end{aligned}$$

where $k_1,k_2\in\mathbb{R}$ and $0<\beta<\alpha$ are constants we do not need to specify here. A similar argument involving Hadamard's theorem as in the proof of Lemma 22 shows that Spec(A_4D) is infinite.

By putting $x = r + iy$ where $r, y \in \mathbb{R}$, $\gamma := \alpha + \beta > 0$ and $\delta := \alpha - \beta > 0$,

$$EV(r + iy) = k_1 + \frac{K_2 - K_2}{4}[e^{-\gamma y}e^{i\gamma r} + e^{\gamma y}e^{-i\gamma r}] \\ - \frac{k_2 + k_1}{4}[e^{-\delta y}e^{i\delta r} + e^{\delta y}e^{-i\delta r}].$$

Since $\gamma > \delta > 0$, if we chose $y \gg 0$, the term $e^{\gamma y}$ dominates the expression and so $|EV(r + iy)| \geqslant c > 0$ for a suitable c independent of r. If we chose $y \ll 0$, the term $e^{-\gamma y}$ is the one that dominates and again $|EV(r + iy)|$ is large. This shows that all the zeros of $EV(x)$ must be contained in a band $\{-y_0 \leqslant \mathrm{Re}(x) \leqslant y_0\}$. $\square$

The above theorem does not rule out the possibility of negative eigenvalues when $ad < 0$. We will see in the numerical examples, evidence of points in this region such that $A_4 D$ has indeed negative spectrum.

With regard to finding the minimal y_0. We will see in Section 7 an argument involving Chebyshev polynomial that allows us to compute in closed form $\mathrm{Spec}(A_4 D)$ for a certain dense subset of R_3. We will also illustrate this technique in various examples where the parabolic region is found explicitly.

If $(a, d) \in R_1$, matrix A_4 is not diagonalizable and so $EV(x)$ is given by Theorem 16 instead of Theorem 15. Nevertheless, similar techniques to the ones we have seen so far apply to this case.

Lemma 25. *Let* $0 \neq c \in \mathbb{R}$ *and let*

$$F(x) = x^2 + c\sin(x)^2, \quad x \in \mathbb{C}.$$

Then $F(x)$ *has an infinite number of zeros in the complex plane but only a finite number of them lie on* $\mathbb{R}$ *and on* $i\mathbb{R}$.

Proof. See the proofs of Lemma 22 and Theorem 24. $\square$

Theorem 26. *Let* $(a, d) \in R_1$. *If* $(a, d) = (\pm\frac{1}{2}, \mp\frac{3}{2})$, then $\mathrm{Spec}(A_4 D) = \{0\}$. *Otherwise* $\mathrm{Spec}(A_4 D)$ *is infinite but it only intersects the real line in a finite number of points.*

Proof. If $a - d = 2$,

$$A_4 = \begin{pmatrix} b_+ + 1 & -1 \\ 1 & b_+ - 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_+ & 0 \\ 1 & b_+ \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and if $a - d = -2$,

$$A_4 = \begin{pmatrix} b_+ - 1 & -1 \\ 1 & b_+ + 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_+ & 0 \\ 1 & b_+ \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$EV(x) = \frac{x^2}{4b_+^3} - \left(1 \pm \frac{1}{2b_+}\right)^2 \sin\left(\frac{x}{\sqrt{b_+}}\right)^2, \quad a - d = \pm 2.$$

The first statement follows from the fact that if $(a, d) = (\pm\frac{1}{2}, \mp\frac{3}{2})$, then $b_+ = \mp\frac{1}{2}$ and so the trigonometric term disappear. The second follows from Lemma 25. $\square$

Notice that the curve $a^2 - ad - 1 = 0$ meets the region R_1 at $(\pm\frac{1}{2}, \mp\frac{3}{2})$. These are the only points where $\mathrm{Spec}(A_4D)$ is finite. Since all self-adjoint operators with compact resolvent must have an infinite number of eigenvalues, A_4D is not similar to self-adjoint. All this suggests that for (a, d) in a small neighbourhood of these points, $\mathrm{Spec}(A_4D)$ must be highly unstable. In the next section we explore closely this idea.

7. Some numerical results

As mentioned previously, this section is devoted to investigating some aspects of the global spectral evolution of AD when we move the entries of matrix A. To be more precise, we consider $A = A_4$ (see Section 6) and compute $\mathrm{Spec}(A_4D)$ as (a, d) moves along various lines inside $R_1 \cup R_3 \cup R_5 \subset \mathbb{R}^2$. We also introduce a technique that allows us to find explicitly $\mathrm{Spec}(A_4D)$ when (a, d) are in a certain dense subset of R_3 by computing the roots of certain polynomial $G(w)$.

Our first task is to decompose R_3 into a disjoint union of curves in order to find the dense subset. For $\alpha > 1$, let

$$d_\pm(a) := \frac{a(\alpha^4 + 1) \pm \sqrt{(\alpha^4 - 1)^2 a^2 + 4\alpha^2(\alpha^2 + 1)^2}}{2\alpha^2}$$

and let

$$\Lambda_\pm(\alpha) := \{(a, d_\pm(a)) : a > \mp 1\}.$$

Then

$$R_3 \cap \{a - d < 0\} = \bigcup_{\alpha > 1} \Lambda_+(\alpha) \quad \text{and} \quad R_3 \cap \{a - d > 0\} = \bigcup_{\alpha > 1} \Lambda_-(\alpha).$$

The motivation for this decomposition is found by observing that for

$$A_4 = \begin{pmatrix} a & -1 \\ 1 & d_\pm(a) \end{pmatrix},$$

$\sqrt{b_+/b_-} = \alpha$, where $0 < b_- < b_+$ are the eigenvalues of A_4. That is, $\Lambda_\pm$ are level curves of $\sqrt{b_+/b_-}$ in the (a,d)-plane. Notice that

$$R_3 = \overline{\bigcup_{1<\alpha\in\mathbb{Q}} \Lambda_+(\alpha) \cup \Lambda_-(\alpha)}.$$

The key idea behind finding $G(w)$ is that for $(a,d)\in\Lambda_\pm(\alpha)$ where $1<\alpha\in\mathbb{Q}$, the zeros of the transcendental function are periodic in the horizontal direction. We show how to construct this polynomial. The transcendental function for A_4D is

$$EV(x) = k_1\left[1 - \cos\left(\frac{x}{\sqrt{b_+}}\right)\cos\left(\frac{x}{\sqrt{b_-}}\right)\right] - k_2 \sin\left(\frac{x}{\sqrt{b_+}}\right)\sin\left(\frac{x}{\sqrt{b_-}}\right),$$

where k_1 and k_2 are two real constants depending upon a and d which we do not need to specify here. Since

$$\sqrt{b_+/b_-} = \alpha = p/q, \quad p,q\in\mathbb{Z}^+,$$

$\sqrt{b_\pm}$ are rationally related and so the zeros of $EV(x)$ appear periodically in lines parallel to the real axis. By putting $z = x/(q\sqrt{b_+})$,

$$EV(z) = k_1[1 - \cos(pz)\cos(qz)] - k_2\sin(pz)\sin(qz)$$
$$= k_1 + \frac{k_2 - k_1}{4}\cos[(p+q)z] - \frac{k_2 + k_1}{4}\cos[(p-q)z],$$

where $p - q < p + q \in \mathbb{Z}^+$. Standard computations show that,

$$\cos(mz) = T_m(\cos(z)), \quad m = 1, 2, \ldots,$$

where T_m a polynomial of degree m (the mth Chebyshev polynomial of first order). Then by letting

$$G(w) := k_1 + \frac{k_2 - k_1}{2}T_{(p+q)}(w) - \frac{k_2 + k_1}{2}T_{(p-q)}(w),$$

$EV(z) = 0$ if, and only if, $G(\cos(z)) = 0$. Hence all the zeros of $EV(x)$ are of the form

$$(\pm\arccos(w_0) + 2n\pi)q\sqrt{b_+}\in\mathbb{C}, \quad n\in\mathbb{Z},$$

where w_0 is a root of $G(w)$. In this manner, Spec(A_4D) is generated by translations of the roots of $G(w)$.

Although the above method computes Spec(A_4D) explicitly for $(a,d)\in\Lambda_\pm(\alpha)$, $1<\alpha\in\mathbb{Q}$, its numerical implementation for large $p+q$ (>20 in a PC) is highly unstable due to the well-known instability of the roots of polynomials of high degree. Nevertheless, no other procedure tried so far, has proven to be more efficient for estimating large eigenvalues in R_3. Figs. 2, 5 and 6 were produced via this approach.

7.1. Spectral behaviour of A_4 for (a,d) close to $(-\frac{1}{2},\frac{3}{2})$

By virtue of Theorem 26, $\mathrm{Spec}(A_4D)=\{0\}$ for $(a,d)=(-\frac{1}{2},\frac{3}{2})$. In any small neighbourhood of this point, the spectrum of A_4D is infinite so high instability is to be expected. Since A_4D is holomorphic in a and d, every non-zero eigenvalue of A_4D either concentrates at zero or diverges to ∞ for $(a,d)\to(-\frac{1}{2},\frac{3}{2})$. We explore this phenomenon in some detail.

According to Theorem 23(a), if $(a,d)\in R_5$ satisfy $a^2-ad-1=0$ and $-2<a-d<0$,

$$\mathrm{Spec}(A_4D)=\{-k^2\pi^2/\mathrm{Im}(b_+^{-1/2})^2\}_{k\in\mathbb{Z}},$$

where b_+ is the larger eigenvalue of A_4. By taking $a\to-\frac{1}{2}$ and $d\to\frac{3}{2}$,

$$b_+=\frac{a+d+\sqrt{(a-d)^2-4}}{2}\to\tfrac{1}{2}\in\mathbb{R},$$

so that $\mathrm{Im}(b_+^{-1/2})\to 0$. Hence, all non-zero eigenvalues of A_4D remain negative and escape to $-\infty$ as $(a,d)\in R_5$ approach the critical point on the curve $a^2-ad-1=0$.

In general, not every eigenvalue of A_4D need to be in the left-hand plane when (a,d) is close to $(-\frac{1}{2},\frac{3}{2})$. In Fig. 2 we consider the evolution of the first

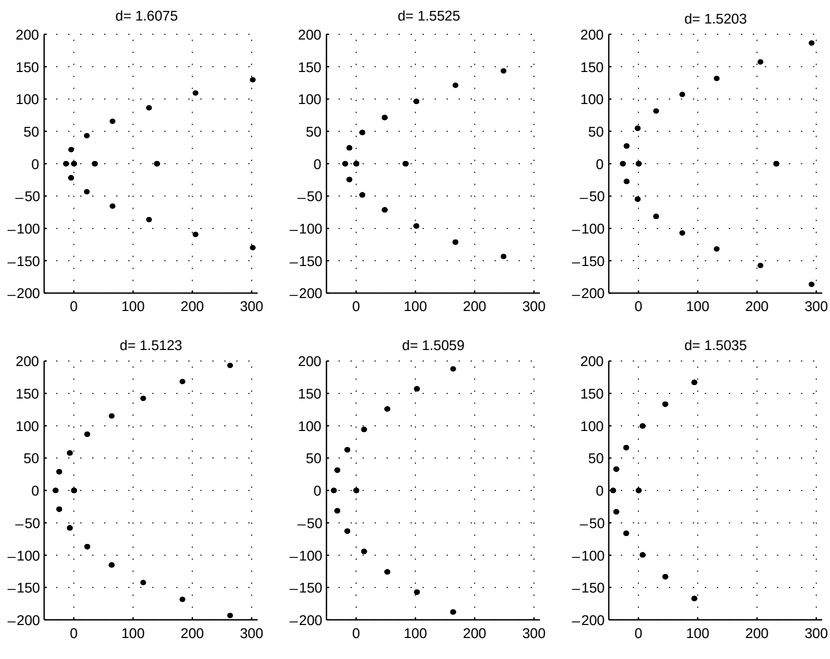

Fig. 2. Evolution of the first 16 eigenvalues of A_4D for $a=-\frac{1}{2}$ fixed and selected values of d close to $\frac{3}{2}$.

16 eigenvalues of A_4D for $a = -\frac{1}{2}$ fixed and 6 different values of d from $d = 1.6075$ to 1.5035. The awkward choice of d correspond to the sensible values of $\alpha \in \mathbb{Q}$; each pair $(-\frac{1}{2}, d) \in \Lambda_+(\alpha)$ for $\alpha = 2, \frac{8}{5}, \frac{4}{3}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}$. Notice that for large p, q the polynomial $G(w)$ has $p + q$ roots and nonetheless all these roots but 0 lie on the same curve. This curve moves away from the origin and there is always a negative eigenvalue. The positive eigenvalues also escape rapidly to $+\infty$ and there are infinitely many of them.

In Fig. 3 we isolate the negative eigenvalue for $a = -\frac{1}{2}$ against 100 different values of d close to $d = \frac{3}{2}$. This provides indication of how rapidly it escapes to $-\infty$. In order to produce this picture, we made use of the algorithm that Matlab provides to find the zero of $EV(x)$ for x on the imaginary axis. Comparing with the comment we made earlier in Section 6.3, this provides points in R_3 such that A_4D has a negative eigenvalue of arbitrarily large modulus.

7.2. Non-real eigenvalues in R_1

We now explore the transition from real to non-real spectrum by considering the spectral evolution of A_4 on the line

$$\{(0, d) \in R_3 : d > 2\}$$

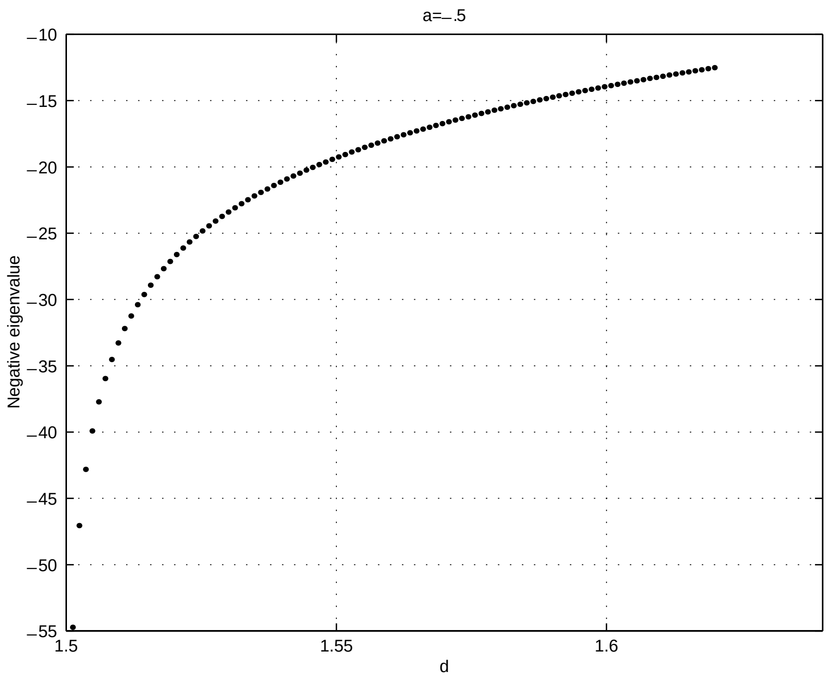

Fig. 3. Evolution of the negative eigenvalue for $a = -\frac{1}{2}$ and 100 different values of d linearly distributed on the segment [1.5012, 1.6200].

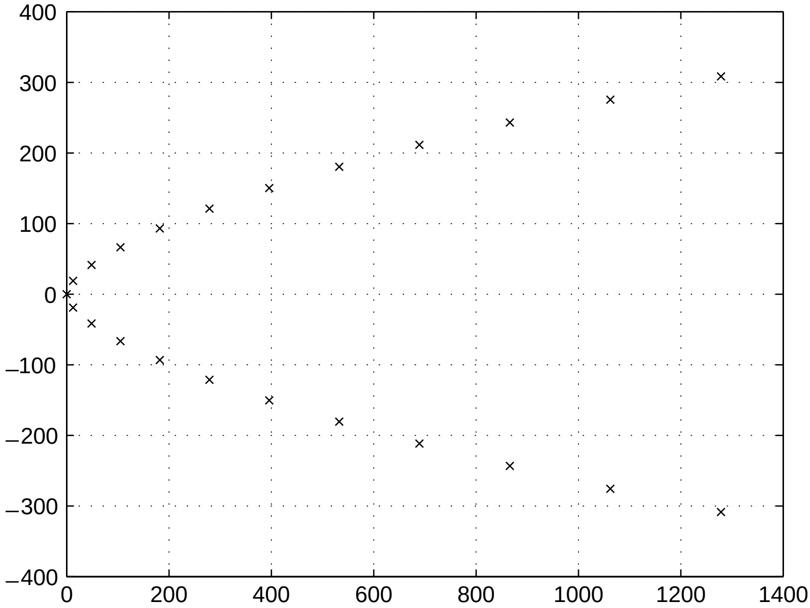

Fig. 4. First 23 eigenvalues of A_4D for $a = 0$ and $d = 2$.

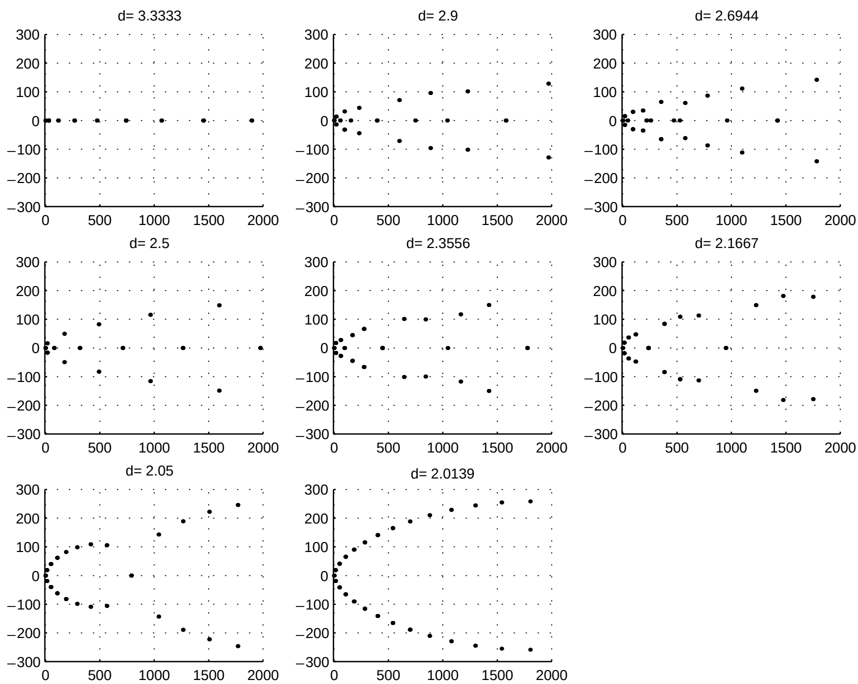

Fig. 5. Evolution of the first 17 eigenvalues (counting multiplicity) of A_4D for $a = 0$ and $d > 2$ close to $d = 2$.

close to $(0,2) \in R_1$. In Fig. 4 we show the first 23 eigenvalues of A_4D for $a = 0$ and $d = 2$. We produced this graphic by reducing the equation $EV(x) = 0$ to a single real variable and then making use of the algorithm that Maple provides to find zeros of real functions. According to Theorem 26, we know that Spec(A_4D) is infinite but there is only finite intersection with the real line. As the picture suggests, in this case the origin seems to be the only real eigenvalue.

Fig. 5 shows the evolution of the first 17 eigenvalues (counting multiplicity) of A_4 when $a = 0$ for various different values of d from $d =$ 3.3333 to 2.0139. Each pair $(0, d) \in \Lambda_+(\alpha)$, respectively, for $\alpha = 3, \frac{5}{2}, \frac{9}{4}, 2, \frac{9}{5}, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}$. The numerical evidence suggests that for $d = 3.3333$ the spectrum is close to the real line and each eigenvalue is of multiplicity 2. Each of these operators has infinitely many real eigenvalues. Unfortunately, the method we employed to find the roots of $G(w)$, is unable to deal with a finer partition of the d-interval. Nonetheless, the global behaviour of the spectrum can be appreciated, as d approaches to 2, each real eigenvalue eventually splits into two conjugate non-real single eigenvalues stabilizing close to the region in Fig. 4 (see the step $d = 2.0139$). Notice that there is no spectrum in the left-hand plane and compare with Theorem 21.

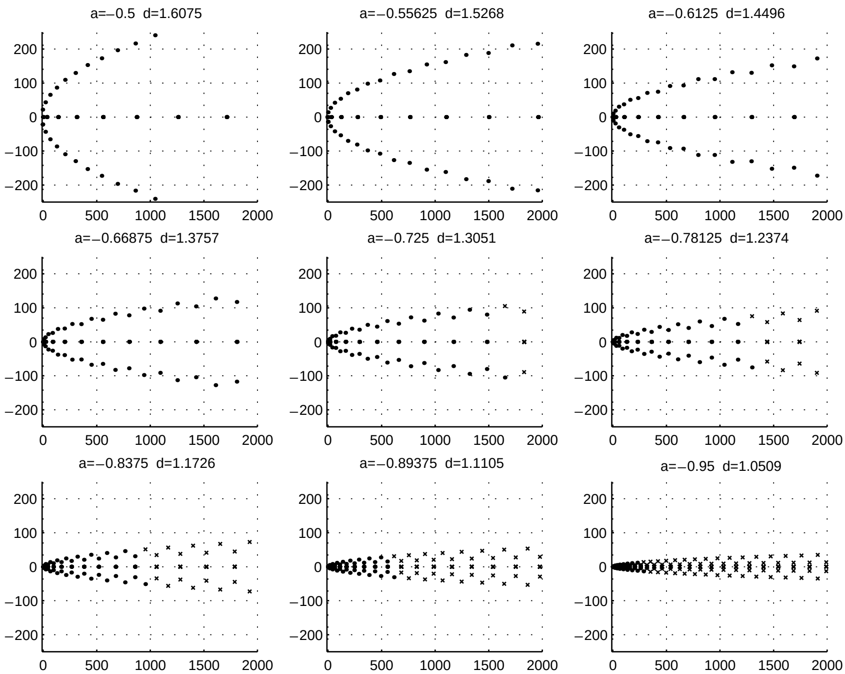

Fig. 6. Evolution of the first 100 eigenvalues of A_4D as $(a,d) \to (-1,1) \in R_6$ on $\Lambda_+(2)$. The dots are the first 100 eigenvalues while the crosses the remaining spectrum.

7.3. Spectral evolution close to R_6

Another type of peculiar behaviour can be observed as $(a, d) \in R_3$ approach the region R_6, where matrix A_4 is singular and $\mathrm{Spec}(A_4 D) = \mathbb{C}$. Here we concentrate on the point $(-1, 1) \in R_6$.

Fig. 6 shows the evolution of the first 100 eigenvalues of $A_4 D$ (represented by dots) as $(a, d) \in \Lambda_+(2)$ approaches to $(-1, 1) \in R_6$. Alongside we also picture the remaining eigenvalues (represented by crosses) that lie on the box $[0, 2000] \times [-300, 300]$. A very similar behaviour occurs for $(a, d) \in \Lambda_\pm(\alpha)$ as $(a, d) \to (\mp 1, \pm 1) \in R_6$ for other values of $\alpha \in \mathbb{Q}$. It cannot be appreciated in the graph provided but there are two conjugate eigenvalues whose real part is negative. These eigenvalues approach to the origin as $(a, d) \to (-1, 1)$. All the remaining spectrum concentrates on the real line suggesting that $\mathrm{Spec}(A_4 D) \to [0, \infty)$ as $(a, d) \to (-1, 1)$ this is in contrast with the fact that $\mathrm{Spec}(AD) = \mathbb{C}$ at $(-1, 1)$.

Here we have chosen $p = 2$ and $q = 1$. This means that $G(w)$ is only of order 3 and so the spectrum is always generated by 3 points. It is not difficult to show analytically that all three roots converge to 0 and then rigorously prove that $\mathrm{Spec}(A_4 D) \to [0, \infty)$.

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