SIAM J. MATH. ANAL.
Vol. 21, No. 6, pp. 1487-1510, November 1990

007

MATHEMATICAL ANALYSIS OF THE GUIDED MODES OF AN OPTICAL FIBER*

A. BAMBERGER† AND A. S. BONNET‡

Abstract. A mathematical formulation for the guided modes of an optical fiber is derived from Maxwell's equations: this formulation leads to an eigenvalue problem for a family of self-adjoint noncompact operators. The main spectral properties of these operators are established. Then the min-max principle provides an expression of the nonlinear dispersion relation, which connects the propagation constants of guided modes to the frequency. Various existence results are finally proved and a complete description of the dispersion curves (monotonicity, asymptotic behavior, existence of cutoff values) is carried out.

Key words. spectral analysis, guided modes, optical fiber

AMS(MOS) subject classifications. 35P, 78

0. Introduction. During the past 15 years, optical telecommunications and then integrated optics have undergone considerable development, which revived interest in the study of dielectrical guides.

Classically, the wave propagation in a cylindrical guide is based on the determination of the eigenmodes of the structure: these are electromagnetic waves of the form $\Phi(x)\, e^{i(\omega t-\beta z)}$, where x denotes the vector of transverse coordinates and z the longitudinal coordinate. The guided modes are of particular interest: they propagate without attenuation (β real) and the wave amplitude decays exponentially as the distance to the core of the guide increases.

The guided modes of a circular step-index fiber are well known (cf. [16], [17]). Indeed, in that case, the dispersion relation between the propagation constant β and the pulsation ω can be given explicitly by means of Bessel functions. The generic aspect of the dispersion curves for a circular step-index fiber is shown in Fig. 1.

Actually, in practice, there is a wide range of dielectrical guides (step-index fibers of arbitrary geometry, graded-index fibers, two cores fibers, $\cdots$) which cannot be treated analytically. Up until now, these cases have been studied by means of numerical computations (see, for example, [13], [18], [28], [29]). Only the one-dimensional problem of the planar waveguide has been considered in various theoretical studies (cf. [9], [27]).

In this work, we present a mathematical analysis of guided modes, for optical fibers having an arbitrary index profile in the core region.

Physical assumptions, equations, and notation are presented in § 1.

Then the first step consists in choosing between many mathematical formulations, resulting from Maxwell's equations. A previous study (cf. [3]) led us to a formulation on the magnetic field H, which we describe in § 2.

This formulation leads to an eigenvalue problem of the form $C_\beta H=\omega^2 H$, where C_β is an unbounded self-adjoint operator, with noncompact resolvent. Section 3 is concerned with a general study of the spectrum of the operator C_β. By deriving bounds for the eigenvalues, we establish a necessary condition for the existence of guided modes: the refraction index somewhere in the core region must be greater than the

* Received by the editors November 24, 1986; accepted for publication (in revised form) August 2, 1989.

† Institut Français du Pétrole, 1-4 avenue du Bois-Préau, BP 311, 92506 Rueil Malmaison Cedex, France.

‡ Groupe Hydrodynamique Navale, E.N.S.T.A. Centre de l'Yvette, Chemin de la Humière, 91120 Palaiseau, France.

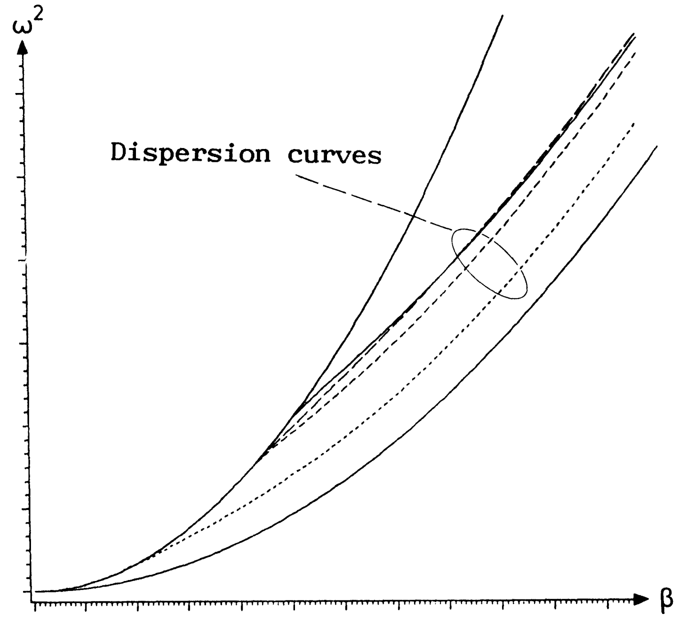

FIG. 1. *Dispersion curves of a circular step-index fiber.*

cladding index. Then after determining the essential spectrum, corresponding to a continuum of radiation modes, we apply the min-max principle to get an expression of the dispersion relation.

In § 4, we study the eigenvalues $\omega(\beta)$ as functions of β. A complete description of the dispersion curves (regularity, monotonicity, asymptotic behavior) is carried out and some existence results are derived. First we exhibit a category of index profiles such that the fiber supports at least two guided modes for every value of the propagation constant β. In the general case, we prove that the number of guided modes tends to infinity as β increases. This number varies at some special values of β which are called the cutoff values.

These cutoff values are studied in § 5. We prove that they are solutions of a nonlinear eigenvalue problem, set in a weighted Sobolev space, and that they form a sequence tending to infinity. For a given value of the propagation constant β, the fiber therefore admits at most a finite number of guided modes.

The results contained in this paper were announced in [2] and are part of the doctoral thesis [4] of A. S. Bonnet.

1. Modelling and notation.

1.1. The physical model. An optical fiber is a cylindrical structure which consists of a core of a dielectric material, surrounded by a cladding of another dielectric material (cf. Fig. 2). When the refractive index of the cladding is less than that of the core, the fiber is a waveguide. Electromagnetic waves can be propagated along the fiber without becoming deformed, their energy remaining confined in the core region.

We use a quite classical model (cf. [11], [16], [17], [20], [24], [25]): the fiber is assumed to be infinitely extended along its axis, denoted by Ox_3, and perfectly cylindrical (see [26] for the scattering theory of a deformed optical waveguide).

We suppose moreover that the cladding is infinitely extended in the transverse plane (O, x_1, x_2). This assumption is justified because the radius of the core is, in

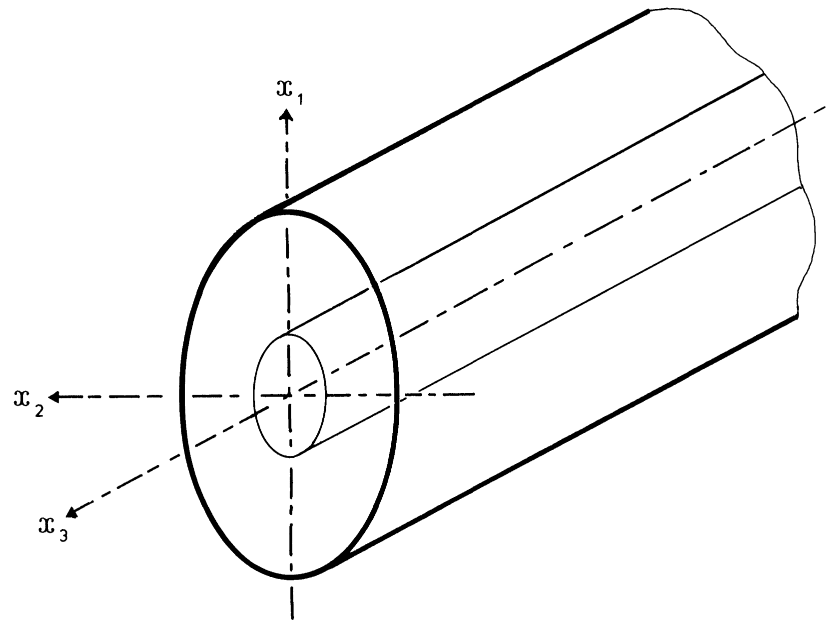

FIG. 2

practice, very small compared to the radius of the cladding, and because the guided modes have exponentially decaying fields in the cladding.

The fiber is completely determined by its index profile n, which is a bounded, positive function of the couple of transverse coordinates $x = (x_1, x_2)$. In the following, we just assume: $n \in L^\infty(\mathbb{R}^2)$ and $\inf\{n(x);\ x \in \mathbb{R}^2\} > 0$. Except for Proposition 3.1, it is not necessary for the index profile to be more regular.

In the following, the fiber cladding is supposed to be homogeneous (see [4] and [5] for some generalizations). More precisely, we suppose that there is a refractive index n_∞ and a bounded domain Ω such that $n(x) = n_\infty$ if $x \notin \Omega$. We say that Ω is the *core region*, the exterior domain $\mathbb{R}^2/\Omega$ is the *cladding*, and n_∞ is the refractive index of the cladding. We will also denote by D_a a disk of centre O and radius a which contains the core region Ω.

Some particular categories of fibers which are used in practice will be mentioned as references:

(a) A fiber whose index profile n is piecewise constant is called a step-index fiber. A step-index fiber is said to be circular (respectively, elliptic) if there is a circular (respectively, elliptic) domain Ω such that the index profile is constant in Ω and outside Ω.

(b) A fiber is called a graded-index fiber when its index profile n belongs to $\mathscr{C}^2(\mathbb{R}^2)$.

1.2. The equations for the guided modes. The cylindrical geometry of the fiber suggests that we look for particular solutions of the Maxwell equations

$$\text{(1.1)} \qquad \operatorname{Rot} \mathbb{H} = \varepsilon_0 n^2 \frac{\partial \mathbb{E}}{\partial t}, \qquad \operatorname{Rot} \mathbb{E} = -\mu_0 \frac{\partial \mathbb{H}}{\partial t},$$

which can be written as

$$\text{(1.2)} \qquad \begin{pmatrix} \mathbb{E} \\ \mathbb{H} \end{pmatrix}(x_1, x_2, x_3, t) = \begin{pmatrix} E \\ H \end{pmatrix}(x_1, x_2)\, e^{i(kc_0 t - \beta x_3)},$$

where $\mathbb{E}$ is the electric field, $\mathbb{H}$ the magnetic field, c_0 the velocity of light in the vacuum, and ε_0 and μ_0 the dielectric permittivity and the magnetic permeability of the vacuum, respectively. The variables k and β are called, respectively, the *wavenumber* and the *propagation constant.*

Such a solution (E, H, β, k) is said to be a *guided mode* if, moreover,

$$(\beta, k) \in \mathbb{R}^2, \quad (E, H) \neq (0, 0) \quad \text{and} \quad (E, H) \in [L^2(\mathbb{R}^2)]^3 \times [L^2(\mathbb{R}^2)]^3.$$

For fields of the form (1.2), system (1.1) becomes

$$\operatorname{Rot}_\beta H = i\varepsilon_0 c_0 k n^2 E, \qquad \operatorname{Rot}_\beta E = -i\mu_0 c_0 k H. \tag{1.3}$$

The index β associated with Rot operator means that the derivation with respect to x_3 is replaced by the multiplication by $(-i\beta)$.

We are interested in the description of the set of all pairs (β, k) in $\mathbb{R}^2$ such that system (1.3) has nontrivial square integrable solutions. Our aim is therefore to establish and study *the dispersion relation* between β and k.

Note that for every fixed β, we get a *two-dimensional eigenvalue problem* where the wavenumber k is the eigenvalue and the electromagnetic field (E, H) the associated eigenvector.

Suppose that (E, H, β, k) is a guided mode. Then we can easily show that $(E, -H, \beta, -k)$, $(E', H', -\beta, k)$, and $(E', -H', -\beta, -k)$ are guided modes as well, where E' and H' denote the symmetrics of E and H with respect to the transverse plane (O, x_1, x_2). Thus we can restrict ourselves in the following to the pairs (β, k) of positive real numbers.

We first need to introduce some notation.

1.3. Notation. Denote by $\mathscr{D}(\mathbb{R}^2)$ the space of indefinitely differentiable complex-valued functions which have compact support in $\mathbb{R}^2$, and by $\mathscr{D}'(\mathbb{R}^2)$ the space of complex-valued distributions on $\mathbb{R}^2$.

Let $\varphi \in \mathscr{D}'(\mathbb{R}^2)$ and

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \in [\mathscr{D}'(\mathbb{R}^2)]^3.$$

We define

$$\operatorname{grad} \varphi = \begin{bmatrix} \dfrac{\partial \varphi}{\partial x_1} \\ \dfrac{\partial \varphi}{\partial x_2} \end{bmatrix}, \qquad \operatorname{rot} \varphi = \begin{bmatrix} \dfrac{\partial \varphi}{\partial x_2} \\ -\dfrac{\partial \varphi}{\partial x_1} \end{bmatrix},$$

$$\Delta \varphi = \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2}, \qquad \mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},$$

$$\operatorname{rot} \mathbf{F} = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}, \qquad \operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2},$$

and for $\beta \in \mathbb{R}$

$$\operatorname{Div}_\beta F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} - i\beta F_3, \qquad \operatorname{Rot}_\beta F = \begin{bmatrix} \dfrac{\partial F_3}{\partial x_2} + i\beta F_2 \\ -i\beta F_1 - \dfrac{\partial F_3}{\partial x_1} \\ \dfrac{\partial F_2}{\partial x_1} - \dfrac{\partial F_1}{\partial x_2} \end{bmatrix},$$

$$\operatorname{Grad}_\beta \varphi = \begin{bmatrix} \dfrac{\partial \varphi}{\partial x_1} \\ \dfrac{\partial \varphi}{\partial x_2} \\ -i\beta\varphi \end{bmatrix}.$$

The following identities hold:

$$\operatorname{rot}(\operatorname{rot} \mathbf{F}) = -\Delta \mathbf{F} + \operatorname{grad}(\operatorname{div} \mathbf{F}),$$

$$\operatorname{Rot}_\beta(\operatorname{Rot}_\beta F) = -\Delta F + \beta^2 F + \operatorname{Grad}_\beta(\operatorname{Div}_\beta F).$$

The inner product and norm in $[L^2(\mathbb{R}^2)]^j$ $(j = 1, 2, 3)$ will be denoted, without distinction, by $(\cdot, \cdot)$ and $|\cdot|_2$.

2. The choice of the variational formulation.

2.1. Classical formulations. Many mathematical formulations have already been established for problems of electromagnetics (cf. [3] and [7]). We briefly recall some of them, dwelling on the reasons which have motivated our choice.

At first, note that we can derive from system (1.3) some equivalent equations by eliminating either E or H.

LEMMA 2.1. *Let* (E, H) *be a solution of system* (1.3). *Then*

$$\operatorname{Rot}_\beta(\operatorname{Rot}_\beta E) = k^2 n^2 E, \tag{2.1}$$

$$\operatorname{Div}_\beta(n^2 E) = 0, \tag{2.2}$$

$$\operatorname{Rot}_\beta\left(\frac{1}{n^2}\operatorname{Rot}_\beta H\right) = k^2 H, \tag{2.3}$$

$$\operatorname{Div}_\beta H = 0. \tag{2.4}$$

Proof. Equations (2.1) and (2.3) are obtained by applying the operator Rot_β to (1.3). Equations (2.2) and (2.4) are deduced from (2.1) and (2.3) using the following identity:

$$\operatorname{Div}_\beta(\operatorname{Rot}_\beta F) = 0 \quad \forall F \in [\mathcal{D}(\mathbb{R}^2)]^3. \tag{2.5}$$ □

By (2.2), the electric field E does not belong to the space $[H^1(\mathbb{R}^2)]^3$ if the index profile n is not regular. Hence, a formulation on the magnetic field is more convenient: indeed, H belongs to the space $[H^1(\mathbb{R}^2)]^3$ for every index profile (see § 2.2).

Using (2.4), it is possible to eliminate the longitudinal component H_3 in system (2.3). The derived formulation involves only H_1 and H_2 as unknowns, but is no longer symmetrical. For that reason, we prefer keeping the three unknowns H_1, H_2, and H_3.

The natural variational formulation associated with (2.3) uses the following bilinear form:

$$b(\beta; H, H') = \int_{\mathbb{R}^2} \frac{1}{n^2} \operatorname{Rot}_\beta H \cdot \overline{\operatorname{Rot}_\beta H'}\, dx,$$

which is not elliptic on the space $[H^1(\mathbb{R}^2)]^3$, but on the subspace

$$\{H \in [H^1(\mathbb{R}^2)]^3;\ \operatorname{Div}_\beta H = 0\}.$$

This space depends on β, which is an unknown and may vary. To obviate this difficulty, a modification used for solving Maxwell's equations in bounded domains (cf. [14]) is to introduce the following equation:

$$\mathrm{Rot}_\beta\left(\frac{1}{n^2}\,\mathrm{Rot}_\beta\,H\right) - s\,\mathrm{Grad}_\beta\,(\mathrm{Div}_\beta\,H) = k^2 H, \tag{2.6}$$

where s is some arbitrary positive real.

Every solution of (2.3) satisfies (2.6). The converse will be established in the next section. Moreover, the bilinear form associated with (2.6) is elliptic on $(H^1(\mathbb{R}^2))^3$.

2.2. A formulation for the magnetic field. Let us denote by V the Sobolev space $(H^1(\mathbb{R}^2))^3$ equipped with the usual norm $\|F\|_V = [|F|_2^2 + |\mathrm{grad}\,F|_2^2]^{1/2}$, and by $V(\beta)$ the space of all square integrable fields F, such that $\mathrm{Rot}_\beta\,F$ and $\mathrm{Div}_\beta\,F$ are also square integrable. Equipped with the norm $\|F\|_{V(\beta)} = [|F|_2^2 + |\mathrm{Rot}_\beta\,F|_2^2 + |\mathrm{Div}_\beta\,F|_2^2]^{1/2}$, $V(\beta)$ is a Hilbert space (cf. [7]).

We first prove that the magnetic field belongs to V.

LEMMA 2.2. (i) *Let* (E, H) *be the solution of system* (3.1). *Then* $(E, H) \in (L^2(\mathbb{R}^2))^3 \times (L^2(\mathbb{R}^2))^3$ *if and only if* $H \in V(\beta)$.

(ii) *For every* F *in* V

$$\int_{\mathbb{R}^2} \{|\mathrm{Rot}_\beta\,F|^2 + |\mathrm{Div}_\beta\,F|^2\}\,dx = \int_{\mathbb{R}^2} \{|\mathrm{grad}\,F|^2 + \beta^2 |F|^2\}\,dx. \tag{2.7}$$

(iii) *Hilbert spaces* V *and* $V(\beta)$ *are isomorphic.*

Proof. Assertion (i) is a trivial consequence of (2.3) and (2.4).

(ii) For every F in $[\mathscr{D}(\mathbb{R}^2)]^3$:

$$\begin{aligned}\int_{\mathbb{R}^2} |\mathrm{Rot}_\beta\,F|^2\,dx &= \int_{\mathbb{R}^2} \mathrm{Rot}_\beta\,(\mathrm{Rot}_\beta\,F)\cdot \bar{F}\,dx \\ &= \int_{\mathbb{R}^2} \{-\Delta F + \beta^2 F + \mathrm{Grad}_\beta\,(\mathrm{Div}_\beta\,F)\}\cdot \bar{F}\,dx \\ &= \int_{\mathbb{R}^2} \{|\mathrm{grad}\,F|^2 + \beta^2|F|^2 - |\mathrm{Div}_\beta\,F|^2\}\,dx.\end{aligned}$$

Since $(\mathscr{D}(\mathbb{R}^2))^3$ is dense in V, this equality holds for every F in V.

(iii) By [7], the space $(\mathscr{D}(\mathbb{R}^2))^3$ is also dense in $V(\beta)$. Hence assertion (iii) is a consequence of identity (2.7). □

Remarks 2.1. (1) Identity (2.7) is analogous to the following classical one:

$$\int_{\mathbb{R}^2} \{|\mathrm{rot}\,\mathbf{F}|^2 + |\mathrm{div}\,\mathbf{F}|^2\}\,dx = \int_{\mathbb{R}^2} |\mathrm{grad}\,\mathbf{F}|^2\,dx. \tag{2.8}$$

(2) Note that the functional space V neither depends on β nor on the index profile n. □

We can now establish the following theorem.

THEOREM 2.1. *The following assertions are equivalent*:

(i) (E, H) *is a nontrivial solution of* (1.3) *and* $E, H \in [L^2(\mathbb{R}^2)]^3$.

(ii) H *is a nontrivial solution of* (2.6) *and* $H \in V$.

Proof. By Lemmas 2.1 and 2.2, (ii) is a consequence of (i). Conversely, let H be a solution of (2.6). We first need to prove that H satisfies (2.3). By taking the divergence of (2.6), we get

$$-s\,\mathrm{Div}_\beta\,(\mathrm{Grad}_\beta\,\varphi)=k^2\varphi,$$

where $\varphi=\mathrm{Div}_\beta\,H$. Thus φ satisfies $-s\Delta\varphi=(k^2-\beta^2 s)\varphi$ and $\varphi\in L^2(\mathbb{R}^2)$. Consequently, φ must vanish everywhere and (2.3) holds.

Moreover, by (2.7), the wavenumber k cannot be equal to zero since H does not vanish identically. Therefore we can set

$$E=\frac{1}{i\varepsilon_0 c_0 k n^2}\,\mathrm{Rot}_\beta\,H,$$

and (E, H) is a nontrivial solution of (1.3). □

In order to choose the value of s, we note that in the outside region, with refractive index n_∞, equation (2.6) becomes

$$-\frac{1}{n_\infty^2}\Delta H+\frac{\beta^2}{n_\infty^2}H+\left(\frac{1}{n_\infty^2}-s\right)\mathrm{Grad}_\beta\,(\mathrm{Div}_\beta\,H)=k^2H. \tag{2.9}$$

By taking $s=1/n_\infty^2$, we get uncoupled Helmholtz equations in this domain. The convenience of this choice will be confirmed in the following section.

2.3. Variational formulation. To conclude this section, let us prove that equation (2.9) leads to a variational formulation with a V-elliptic bilinear form.

Hereafter, we set

$$c(\beta;\,H,H')=\int_{\mathbb{R}^2}\left\{\frac{1}{n^2}\mathrm{Rot}_\beta\,H\cdot\overline{\mathrm{Rot}_\beta\,H'}+\frac{1}{n_\infty^2}\mathrm{Div}_\beta\,H\,\overline{\mathrm{Div}_\beta\,H'}\right\}dx. \tag{2.10}$$

By Theorem 2.1, we can now consider the following problem:

(2.11) $(\mathcal{P})$

Find all pairs of reals (β, k) such that there exists H satisfying
(2.12) $\quad H\in V,\quad H\neq 0,$ $\qquad c(\beta;\,H,H')=k^2(H,H')\quad \forall H'\in V.$

The following lemma is a straightforward consequence of identity (2.7).

LEMMA 2.3. *The form $c(\beta)$ is Hermitian and satisfies, for every $H\in V$*

$$c(\beta;\,H,H)\geqq\frac{1}{n_+^2}\int_{\mathbb{R}^2}\{|\mathrm{grad}\,H|^2+\beta^2|H|^2\}\,dx, \tag{2.13}$$

$$c(\beta;\,H,H)\leqq\frac{1}{n_-^2}\int_{\mathbb{R}^2}\{|\mathrm{grad}\,H|^2+\beta^2|H|^2\}\,dx, \tag{2.14}$$

where n_+ and n_- are defined by

$$n_+=\sup_{x\in\mathbb{R}^2}n(x)\quad\text{and}\quad n_-=\inf_{x\in\mathbb{R}^2}n(x). \tag{2.15}$$

In particular, $c(\beta)$ is V-elliptic for nonzero values of β.

3. Derivation of the dispersion relation. The problem $(\mathcal{P})$, defined by (2.11), can be written equivalently as follows:

(3.1) $(\mathcal{P})$

$$\boxed{\begin{array}{l}\text{Find all pairs } (\beta, k) \in (\mathbb{R}^+)^2 \text{ such that there} \\ \text{exists } H \text{ satisfying} \\ \qquad H \in D(C_\beta), \quad H \neq 0, \quad C_\beta H = k^2 H,\end{array}}$$

where we denote by C_β the unbounded operator of $(L^2(\mathbb{R}^2))^3$, with domain $D(C_\beta)$, associated with the form $c(\beta)$. In other words, k^2 is an eigenvalue of C_β and H an associated eigenfield.

We therefore must study the spectrum $\sigma(\beta)$ of C_β and especially the set of its eigenvalues $\sigma_p(\beta)$.

In this section we first derive a lower bound for $\sigma(\beta)$ and an upper bound for $\sigma_p(\beta)$. The essential spectrum $\sigma_{\text{ess}}(\beta)$ is then determined. The dispersion relation between β and k is finally derived by using the min-max principle.

3.1. Lower and upper bounds for the eigenvalues. Using Lemma 2.3, we first establish Lemma 3.1.

LEMMA 3.1. *For every nonnegative real* β, *the operator* C_β *is self-adjoint. Moreover,*

(i) $\sigma(\beta) \subset [\beta^2/n_+^2, +\infty[$,

(ii) $\sigma_p(\beta) \subset]\beta^2/n_+^2, +\infty[$,

where n_+ *is defined by* (2.15).

Proof. The self-adjointness of C_β and inclusion (i) are consequences of (2.13) (cf. [23]).

Suppose now that $c(\beta; H, H) = (\beta/n_+)^2 |H|_2^2$ for some H in V. It therefore follows from (2.13) that

$$\int_{\mathbb{R}^2} |\text{grad } H|^2 \, dx = 0,$$

which implies that H is constant. Since H belongs to $[L^2(\mathbb{R}^2)]^3$, it must then vanish everywhere. Consequently, $(\beta/n_+)^2$ cannot be an eigenvalue of C_β. $\square$

Remark 3.1. By the previous lemma, every solution (β, k) of $(\mathcal{P})$ must satisfy

$$k^2 > \left(\frac{\beta}{n_+}\right)^2.$$

This means that the guided wave always propagates faster than a plane wave in an homogeneous medium of index n_+.

To derive an upper bound for the eigenvalues, the following additional hypothesis is required:

(3.2) There exists a finite collection $\{\Omega_1, \cdots, \Omega_m\}$ of open regular subsets of $\mathbb{R}^2$ such that

$$\mathbb{R}^2 = \bigcup_{j=1}^{m} \bar{\Omega}_j \quad \text{and} \quad n|_{\bar{\Omega}_j} \in \mathscr{C}^2(\bar{\Omega}_j) \quad \text{for } j = 1, \cdots, m.$$

In other words, the index profile is supposed to be piecewise regular. This hypothesis is sufficient but probably not necessary. However, assumption (3.2) is satisfied by all fibers used in practice (step-index fibers, graded-index fibers, two-fiber couplers).

PROPOSITION 3.1. *Assume that the index profile n satisfies* (3.2). *Then* $\sigma_p(\beta) \subset]-\infty, \beta^2/n_\infty^2]$.

Proof. (1) Let $H \in V$ and $\lambda > (\beta/n_\infty)^2$ such that $C_\beta H = \lambda H$. Then H satisfies $-\Delta H + (\lambda n_\infty^2 - \beta^2)H = 0$ outside the core region Ω. Hence we deduce from the uniqueness result proved in [21] that H vanishes in $\mathbb{R}^2 \backslash D_a$, where D_a is a disk containing the core region Ω.

(2) Under assumption (3.2), H must vanish identically (cf. [10, pp. 190–192]). □

Remarks 3.2. (1) Notice that condition (3.2) has been used only for the second part of the proof, which requires a continuation principle.

(2) By the previous lemma, every solution (β, k) of problem $(\mathcal{P})$, defined by (3.1), must satisfy the following inequality:

$$k^2 \leqq \left(\frac{\beta}{n_\infty}\right)^2.$$

This means that the guided wave always propagates slower than a plane wave in a homogeneous medium of index n_∞.

These first results are summarized in Corollary 3.2.

COROLLARY 3.2. *Assume that* (3.2) *holds*:

(i) *If* $n_+ = n_\infty$ *or* $\beta = 0$, *then* $\sigma_p(\beta) = \emptyset$.

(ii) *If* $n_+ > n_\infty$ *and* $\beta \neq 0$, *then* $\sigma_p(\beta) \subset]\beta^2/n_+^2, \beta^2/n_\infty^2]$.

In other words, the problem $(\mathcal{P})$, defined by (3.1), has no solution when n_+ is equal to n_∞. For this reason, we shall assume in the sequel that the index profile always satisfies the "guidance condition":

$$\boxed{n_+ > n_\infty}.$$

3.2. The essential spectrum. The aim of this section is to establish the following proposition.

PROPOSITION 3.2. *For every nonnegative real* β, *the essential spectrum of* C_β *is given by*

$$\sigma_{\text{ess}}(\beta) = \left[\frac{\beta^2}{n_\infty^2}, +\infty\right[.$$

Remark 3.3. The essential spectrum corresponds to the continuum of propagation constants of radiation modes. See, for example, [16] for a description of the radiation modes of a cylindrical step-index fiber.

To prove this proposition, we shall use another version of the bilinear form $c(\beta)$. Let us define

$$d_0(H, H') = \int_{\mathbb{R}^2} \left\{\frac{1}{n^2} \operatorname{rot} \mathbf{H} \operatorname{rot} \bar{\mathbf{H}}' + \frac{1}{n_\infty^2} \operatorname{div} \mathbf{H} \operatorname{div} \bar{\mathbf{H}}' + \frac{1}{n^2} \operatorname{grad} H_3 \cdot \operatorname{grad} \bar{H}_3'\right\} dx, \tag{3.3}$$

$$d_1(H, H') = i \int_{\mathbb{R}^2} \left(\frac{1}{n^2} - \frac{1}{n_\infty^2}\right) \{\operatorname{grad} H_3 \cdot \bar{\mathbf{H}}' - \mathbf{H} \cdot \operatorname{grad} \bar{H}_3'\}\, dx, \tag{3.4}$$

$$d_2(H, H') = \int_{\mathbb{R}^2} \left(\frac{1}{n^2} - \frac{1}{n_\infty^2}\right) \mathbf{H} \cdot \bar{\mathbf{H}}'\, dx, \tag{3.5}$$

$$d(\beta; H, H') = d_0(H, H') + \beta d_1(H, H') + \beta^2 d_2(H, H'). \tag{3.6}$$

Then we have the following lemma.

LEMMA 3.3. (i) *The bilinear form $c(\beta)$ admits the following expression*:

$$c(\beta; H, H') = d(\beta; H, H') + \frac{\beta^2}{n_\infty^2}(H, H'). \tag{3.7}$$

(ii) *The forms d_0, d_1, and d_2 are Hermitian and continuous on V.*

(iii) *The form d_0 satisfies*

$$\forall H \in V, \quad d_0(H, H) \geqq \frac{1}{n_+^2} \int_{\mathbb{R}^2} |\operatorname{grad} H|^2 \, dx. \tag{3.8}$$

(iv) *The forms d_1 and d_2 are compact perturbations of d_0.*

Proof. The decomposition (i) follows easily from the Green formula:

$$\int_{\mathbb{R}^2} \{\operatorname{grad} H_3 \cdot \bar{\mathbf{H}}' - \mathbf{H} \cdot \operatorname{grad} \bar{H}_3'\} \, dx$$

$$= \int_{\mathbb{R}^2} \{(\operatorname{div} \mathbf{H})\bar{H}_3' - H_3(\operatorname{div} \bar{\mathbf{H}}')\} \, dx.$$

To prove (ii), we first note that

$$d_0(H, H) \geqq \frac{1}{n_+^2} \int_{\mathbb{R}^2} \{|\operatorname{grad} H_3|^2 + |\operatorname{rot} \mathbf{H}|^2 + |\operatorname{div} \mathbf{H}|^2\} \, dx$$

and then we use equality (2.8).

Now, since $1/n(x)^2 - 1/n_\infty^2 = 0$ outside of Ω, (iv) becomes a straightforward consequence of the compact injection of $H^1(\Omega)$ into $L^2(\Omega)$. $\square$

We are now ready for the proof of Proposition 3.2.

Proof of Proposition 3.2. Let us denote by D_β the unbounded operator associated with the bilinear form $d(\beta; H, H)$ defined by (3.6) and by $\sigma_{\text{ess}}(D_\beta)$ its essential spectrum. By (3.7), we must establish the following identity:

$$\sigma_{\text{ess}}(D_\beta) = [0, +\infty[. \tag{3.9}$$

Now the proof of (3.9) involves two parts:

— First the essential spectrum of D_β is proved to be independent of β. Indeed, by the previous lemma the forms d_1 and d_2 are compact perturbations of d_0. By the Weyl theorem (cf. [23]), they do not modify the essential spectrum. In other words, $\sigma_{\text{ess}}(D_\beta) = \sigma_{\text{ess}}(D_0)$ for every value of β.

— It remains to prove that $\sigma_{\text{ess}}(D_0) = [0, +\infty[$.

By Lemma 3.1, the following inclusion holds: $\sigma_{\text{ess}}(D_0) \subset [0, +\infty[$, and we must just establish the converse inclusion. Moreover, $\sigma_{\text{ess}}(D_0)$ is closed. Hence we must actually prove that $]0, +\infty[\subset \sigma_{\text{ess}}(D_0)$.

We do it by using singular sequences (cf. [23]). Let γ be a strictly positive real. Let ψ be some function of $\mathcal{D}(\mathbb{R}^2)$ which vanishes in Ω, let $H^{(0)}$ be some arbitrary vector of $\mathbb{C}^3$, and let J_0 denote the Bessel function of first kind of order zero (cf. [1]). We define a sequence $(H^{(p)})$ as follows:

$$H^{(p)} = \frac{1}{\sqrt{p}} \psi\left(\frac{x}{p}\right) J_0(\sqrt{\gamma}\, x) H^{(0)}.$$

To prove that $(H^{(p)})$ is a singular sequence, we must establish the three following statements:

$$|H^{(p)}|_2 \geqq \alpha > 0 \quad \forall p \in \mathbb{N}, \tag{3.10a}$$

$$|D_0 H^{(p)} - \gamma H^{(p)}|_2 \xrightarrow[p \to +\infty]{} 0, \tag{3.10b}$$

$$H^{(p)} \xrightarrow[p \to +\infty]{} 0 \quad \text{for } (L^2(\mathbb{R}^2))^3, \text{ weakly.} \tag{3.10c}$$

Assertions (3.10a) and (3.10b) can be deduced from the asymptotic behavior of J_0 (cf. [1]), and (3.10c) is a direct consequence of the Lebesgue dominated convergence theorem. □

3.3. The min-max principle and the dispersion relation. By Propositions 3.1 and 3.2, if (3.2) holds, all eigenvalues of C_β—except perhaps $(\beta/n_\infty)^2$—are located below the essential spectrum. Hence we can apply the min-max principle to get expressions of the eigenvalues as functions of β.

Let us define

$$\lambda_1(\beta) = \inf_{\substack{H \in (L^2(\mathbb{R}^2))^3 \\ |H|_2 = 1}} c(\beta; H, H)$$

(3.11) and if $m > 1$

$$\lambda_m(\beta) = \sup_{H^{(1)},\cdots,H^{(m-1)} \in (L^2(\mathbb{R}^2))^3} \ \inf_{\substack{H \in [H^{(1)},\cdots,H^{(m-1)}]_V^\perp \\ |H|_2 = 1}} c(\beta; H, H),$$

where $[H^{(1)}, \cdots, H^{(m-1)}]_V^\perp = \{H \in V;\ (H, H^{(j)}) = 0;\ j = 1, m-1\}$.

By the min-max principle (cf. [22]), we have, for each fixed m

$$(\beta/n_+)^2 < \lambda_1(\beta) \leqq \lambda_2(\beta) \cdots \leqq \lambda_m(\beta) \leqq (\beta/n_\infty)^2 \tag{3.12}$$

and EITHER

(i) $\lambda_m(\beta) < (\beta/n_\infty)^2$.

In that case, there are m eigenvalues of C_β (counting them a number of times equal to their multiplicity) below $(\beta/n_\infty)^2$, and $\lambda_m(\beta)$ is the mth eigenvalue.

OR

(ii) $\lambda_m(\beta) = (\beta/n_\infty)^2$.

In that case,

$$\lambda_m(\beta) = \lambda_{m+1}(\beta) = \lambda_{m+2}(\beta) = \cdots = (\beta/n_\infty)^2$$

and there are at most $(m-1)$ eigenvalues of C_β (counting multiplicity) below $(\beta/n_\infty)^2$.

Note particularly that if (3.2) holds all eigenvalues of C_β—except perhaps $(\beta/n_\infty)^2$—have finite multiplicity and are isolated points of $\sigma(\beta)$.

It is now clear that the min-max principle provides an expression of the dispersion relation between the wavenumber k and the propagation constant β as follows.

THEOREM 3.1. *The solutions* (β, k) *of problem* $(\mathcal{P})$, *defined by* (3.1), *such that* $k < \beta/n_\infty$ *are the roots of the dispersion relation*:

$$\boxed{\begin{array}{ll} \lambda_m(\beta) = k^2, & \\ \lambda_m(\beta) < (\beta/n_\infty)^2, & m = 1, 2, \cdots. \end{array}} \tag{3.13}$$

Remarks 3.4. (1) The min-max principle does not allow us to take into account the solutions (β, k) of problem $(\mathcal{P})$ such that $k = \beta/n_\infty$. These solutions are different from the others: indeed the associated field does not decrease exponentially at infinity (it satisfies $\Delta H = 0$ outside the core region). From a physical viewpoint, the corresponding mode is not "guided." However, the case $k = \beta / n_\infty$ will be considered in § 4 for the study of cutoff values.

(2) The min-max principle is useful for comparing eigenvalues for various indices. Indeed, let n and n' be two index profiles such that

$$n(x) \leqq n'(x) \quad \text{a.e. in } \mathbb{R}^2. \tag{3.14}$$

Then $c(\beta, n'; H, H) \leqq c(\beta, n; H, H)$ for every H and every β. Consequently,

$$\lambda_m(\beta, n') \leqq \lambda_m(\beta, n) \quad \forall \beta > 0, \quad m = 1, 2, \cdots.$$

This obvious consequence of the min-max principle has many practical applications. Indeed, for every index profile n, a circular step-index profile n' can be defined by

$$n'(x) = n_+ \quad \text{a.e. in } D_a,$$
$$n'(x) = n_\infty \quad \text{a.e. in } \mathbb{R}^2 \backslash D_a$$

such that (3.14) is satisfied. Since the functions $\lambda_m(\beta, n')$ are well known, this comparison principle immediately provides information about the functions $\lambda_m(\beta, n)$. □

4. Study of the dispersion relation (3.13).

4.1. Differentiability and monotonicity results for the dispersion curves. The aim of this section is to establish Proposition 4.1.

PROPOSITION 4.1. (i) *The functions $\beta \to \lambda_m(\beta)$ are continuous and almost everywhere differentiable for $\beta \in \mathbb{R}^+$.*

(ii) *Suppose that*

$$\left(\frac{n_+}{n_\infty} + 1\right) \Delta(n) < \frac{1}{n_\infty^2}, \tag{4.1}$$

where

$$\Delta(n) = \sup_{x \in \mathbb{R}^2} \left| \frac{1}{n(x)^2} - \frac{1}{n_\infty^2} \right|. \tag{4.2}$$

Then the functions $\lambda_m(\beta)$ are strictly increasing in β for $\beta \in \mathbb{R}^+$.

This proposition is a straightforward consequence of the following lemma, which provides an estimate for the derivatives of the functions $\lambda_m(\beta)$.

LEMMA 4.1. *The function*

$$\Lambda_m(\beta) = \lambda_m(\beta) - (\beta / n_\infty)^2 \tag{4.3}$$

is continuous. It is differentiable almost everywhere and its derivative satisfies

$$\left| \frac{d\Lambda_m}{d\beta}(\beta) \right| \leqq 2\beta \left(\frac{n_+}{n_\infty} + 1\right) \Delta(n), \tag{4.4}$$

where $\Delta(n)$ is defined by (4.2).

Proof. Note first that $\Lambda_m(\beta)$ is the mth max-min associated with the form $d(\beta; H, H)$ defined by (3.6).

Let β and β' be two distinct positive reals and let $H \in V$ such that $|H|_2 = 1$. Then

$$d(\beta; H, H) - d(\beta'; H, H) = (\beta - \beta') d_1(H, H) + (\beta^2 - \beta'^2) d_2(H, H), \tag{4.5}$$

where d_1 and d_2 are defined by (3.4) and (3.5).

By the Cauchy-Schwarz inequality

$$(4.6)\qquad \begin{aligned} &|d_1(H, H)| \leqq 2\Delta(n)|\text{grad } H_3|_2^2, \\ &|d_2(H, H)| \leqq \Delta(n), \end{aligned}$$

and by (2.13)

$$(4.7)\qquad \int_{\mathbb{R}^2} |\text{grad } H_3|^2 \, dx \leqq n_+^2(d(\beta'; H, H) + (\beta'/n_\infty)^2).$$

Using (4.5)-(4.7), we obtain

$$(4.8)\qquad \begin{aligned} d(\beta; H, H) \leqq{} & d(\beta'; H, H) + |\beta - \beta'|\Delta(n) \\ & \cdot \left\{ 2n_+ \left(d(\beta'; H, H) + \frac{\beta'^2}{n_\infty^2} \right)^{1/2} + (\beta + \beta') \right\}. \end{aligned}$$

Therefore, for every integer m

$$\begin{aligned} \Lambda_m(\beta) \leqq{} & \Lambda_m(\beta') + |\beta - \beta'|\Delta(n) \\ & \cdot \{2n_+(\Lambda_m(\beta') + (\beta'/n_\infty)^2)^{1/2} + (\beta + \beta')\}. \end{aligned}$$

By (3.12), $\Lambda_m(\beta')$ is always negative, so that we have

$$\Lambda_m(\beta) \leqq \Lambda_m(\beta') + |\beta - \beta'|\Delta(n) \left\{ 2 \frac{n_+}{n_\infty} \beta' + (\beta + \beta') \right\}$$

and, by inverting β and β'

$$|\Lambda_m(\beta') - \Lambda_m(\beta)| \leqq |\beta - \beta'| 2\Delta(n) \max(\beta, \beta') \left\{ \frac{n_+}{n_\infty} + 1 \right\}.$$

Consequently, the function $\Lambda_m(\beta)$ is locally Lipschitz and hence almost everywhere differentiable. By the previous inequality, its derivative satisfies (4.4). □

Remarks 4.1. (i) Note that Lemma 4.1 cannot be improved, since, in the case of the circular step-index fiber, the functions $\lambda_m(\beta)$ are not everywhere differentiable. In fact, we can show that the eigenvalues are analytic functions of the propagation constant β (cf. [12]). However, since the analytic curves may intersect, the functions $\lambda_m(\beta)$ are just piecewise analytic.

(ii) By Theorem 4.1, if (4.1) holds, the functions $\lambda_m(\beta)$ are one to one and the dispersion relation (3.13) equivalently reads:

$$\boxed{\begin{aligned} &\lambda_m^{-1}(k^2) = \beta, \\ &\lambda_m^{-1}(k^2) < kn_\infty, \end{aligned} \qquad m = 1, 2, \cdots.}$$

Consequently, all the results concerning the direct problem, which consists in studying the solutions k for a given value of β, may be transposed to the inverse problem.

(iii) A simple calculation shows that if

$$(4.9)\qquad n_+/n_\infty < Q^+ \quad \text{and} \quad n_\infty/n_- < Q^-,$$

where Q^+ is the greatest root of the equation $x^3 - x - 1 = 0$ and $Q^- = (1/(1+Q^+)+1)^{1/2}$, then the functions $\lambda_m(\beta)$ are strictly increasing in β for $\beta \in \mathbb{R}^+$.

The computation gives the following approximate values:

$$Q^+ \simeq 1.33, \qquad Q^- \simeq 1.20$$

and condition (4.9) is satisfied by all the fibers used in practice. □

4.2. Existence of eigenvalues: a sufficient condition. Let us recall that (see, e.g., [17]) a circular or elliptic step-index fiber has, for every nonzero value of β, at least two guided modes.

In the circular case $\lambda_1(\beta)=\lambda_2(\beta)<(\beta/n_\infty)^2$, and in the elliptic case $\lambda_1(\beta)<\lambda_2(\beta)<(\beta/n_\infty)^2$, for every nonzero value of the propagation constant β.

However, it may happen if there are some regions in D_a where the index profile n is lower than n_∞, that the fiber supports no guided modes for small values of β. This is, for example, the case of the so-called "W-profiles" (cf. [11], [20]).

The following theorem provides a sufficient condition on the index profile which ensures the existence of two guided modes for every nonzero value of β.

THEOREM 4.1. *Suppose*

$$\int_{\mathbb{R}^2}\left(\frac{1}{n_\infty^2}-\frac{1}{n^2}\right)dx \geqq 0; \tag{4.10}$$

then $\lambda_1(\beta)\leqq\lambda_2(\beta)<(\beta/n_\infty)^2$ *for all* $\beta>0$. *Thus for every nonzero value of* β *there are two eigenvalues of* C_β *below* $(\beta/n_\infty)^2$.

Proof. By [8] the functions $\lambda_m(\beta)$ admit the following equivalent expression:

$$\lambda_m(\beta)=\inf_{V_m\in\mathcal{V}_m}\ \sup_{\substack{H\in V_m\\ |H|_2=1}} c(\beta;H,H), \tag{4.11}$$

where $\mathcal{V}_m$ is the set of all m-dimensional subspaces of V.

If $v\in H^1(\mathbb{R}^2)$, it results from (4.11) that

$$\lambda_2(\beta)\leqq \sup_{\substack{H\in V_2\\ |H|_2=1}} c(\beta;H,H),$$

where

$$V_2=\left\{\begin{pmatrix}\gamma_1 v\\ \gamma_2 v\\ 0\end{pmatrix};(\gamma_1,\gamma_2)\in\mathbb{C}^2\right\}.$$

Since the function v can be chosen arbitrarily, this implies that

$$\lambda_2(\beta)\leqq\left(\frac{\beta}{n_\infty}\right)^2+\mu(\beta), \tag{4.12}$$

where

$$\mu(\beta)=\inf_{\substack{v\in H^1(\mathbb{R}^2)\\ |v|_2=1}}\left\{\frac{1}{n_-^2}\int_{\mathbb{R}^2}|\mathrm{grad}\, v|^2\,dx-\beta^2\int_{\mathbb{R}^2}\left(\frac{1}{n_\infty^2}-\frac{1}{n^2}\right)|v|^2\,dx\right\}.$$

Suppose first that (4.10) is a strict inequality and consider the following test functions (cf. [19]):

$$v_N(x)=\begin{cases}1 & \text{if } |x|<a,\\ \dfrac{\log|x|-\log N}{\log a-\log N} & \text{if } a<|x|<N,\\ 0 & \text{if } |x|>N,\end{cases}$$

where a is the radius of a disk containing the core region Ω. Then, by (4.12), we have

$$\mu(\beta)\leqq\frac{2\pi\left(n_-^2\log\left(\dfrac{N}{a}\right)\right)^{-1}-\beta^2\displaystyle\int_{D_a}\left(\frac{1}{n_\infty^2}-\frac{1}{n^2}\right)dx}{|v_N|_2^2}.$$

Consequently, by taking N large enough, we see that $\mu(\beta)$ is strictly negative for every nonzero value of β.

If (4.10) is an equality, we introduce some function $w \in H^1(\mathbb{R}^2)$ such that

$$\int_{D_a} \left(\frac{1}{n_\infty^2} - \frac{1}{n^2}\right) w\, dx > 0,$$

and we consider the following test functions:

$$v_N^\alpha = v_N + \alpha w,$$

where α denotes a strictly positive real function. Then, by (4.12), we obtain

$$\mu(\beta) \leqq \frac{4\pi\left(n_-^2 \log\left(\dfrac{N}{a}\right)\right) + \alpha^2 K(\beta, w) - \alpha\beta^2 \displaystyle\int_{D_a}\left(\frac{1}{n_\infty^2} - \frac{1}{n^2}\right) w\, dx}{|v_N^\alpha|_2^2},$$

where $K(\beta, w)$ neither depends on α or on N. The required result follows by taking N large enough and α small enough. $\square$

Remarks 4.2. (i) A similar result has been established for the scalar equation of the weak-guidance approximation (cf. [6]).

(ii) The converse will be discussed in § 5 (cf. Proposition 5.1).

4.3. Asymptotic behavior of the dispersion curves. Definition of the cutoff values. In this section we prove that, for every integer m, the mth guided mode, associated with the mth eigenvalue $\lambda_m(\beta)$, exists for β large enough. Indeed, we have the following proposition.

PROPOSITION 4.2. *For every integer m there is a value β_m^* such that*

$$\lambda_m(\beta) < (\beta/n_\infty)^2 \quad \forall \beta > \beta_m^* \quad \text{and} \quad \lambda_m(\beta_m^*) = (\beta_m^*/n_\infty)^2.$$

This value is called, by definition, the mth upper cutoff value.

This proposition can be deduced from Lemma 4.2.

LEMMA 4.2. *For every integer m, we have*

$$\lim_{\beta \to +\infty} \frac{\lambda_m(\beta)}{\beta^2} = \frac{1}{n_+^2}. \tag{4.13}$$

Proof. Let $\eta > 0$. By the definition of n_+ there exist $x_0 \in \mathbb{R}^2$ and $\rho > 0$ such that

$$\frac{1}{\rho^2}\int_{D_\rho}\left(\frac{1}{n^2} - \frac{1}{n_+^2}\right) dx < \eta,$$

where $D_\rho = \{x \in \mathbb{R}^2;\ |x - x_0| < \rho\}$.

We denote by $\mu_\rho^{(1)}, \cdots, \mu_\rho^{(m)}$ the m first eigenvalues of the Laplacian operator in D_ρ with Dirichlet boundary conditions and by $w_\rho^{(m)}$ the associated eigenfunctions extended by zero out of D_ρ and satisfying $|w_\rho^i|_2 = 1$, $i = 1, \cdots, m$.

Let $\tilde{V}_\rho$ be the m-dimensional subspace of $H^1(\mathbb{R}^2)$ spanned by $w_\rho^{(1)}, \cdots, w_\rho^{(m)}$. We set

$$V_\rho = \left\{\begin{pmatrix} v \\ w \\ 0 \end{pmatrix};\ v, w \in \tilde{V}_\rho\right\}.$$

By (4.11), we have

$$\lambda_{2m}(\beta) \leqq \sup_{\substack{H \in V_\rho \\ |H|_2 = 1}} c(\beta; H, H),$$

and thus

$$0<\frac{\lambda_{2m}(\beta)}{\beta^2}-\frac{1}{n_+^2}\leqq \alpha_m(\beta,\rho)$$

where

$$\alpha_m(\beta,\rho)=\sup_{\substack{v\in \tilde{V}_\rho\\ |v|_2=1}}\left\{\frac{1}{\beta^2 n_-^2}\int_{\mathbb{R}^2}|\text{grad } v|^2\,dx+\int_{\mathbb{R}^2}\left(\frac{1}{n^2}-\frac{1}{n_+^2}\right)|v|^2\,dx\right\}.$$

Finally we derive the following estimate:

$$\left|\frac{\lambda_{2m}(\beta)}{\beta^2}-\frac{1}{n_+^2}\right|\leqq\frac{\mu_\rho^{(m)}}{\beta^2 n_-^2}+\left\{\sup_{\substack{v\in \tilde{V}_\rho\\ |v|_2=1}}\|v\|^2_{L^\infty(D_\rho)}\right\}\rho^2\eta.$$

To conclude, we just note that for every integer l

$$\mu_\rho^{(l)}=\frac{1}{\rho^2}\mu_1^{(l)}\quad\text{and}\quad \|w_\rho^{(l)}\|_{L^\infty(D_\rho)}=\frac{1}{\rho}\|w_1^{(l)}\|_{L^\infty(D_1)},$$

so that, for $\beta>1/\rho$

$$\left|\frac{\lambda_{2m-1}(\beta)}{\beta^2}-\frac{1}{n_+^2}\right|\leqq\left|\frac{\lambda_{2m}(\beta)}{\beta^2}-\frac{1}{n_+^2}\right|\leqq K(m)\eta,$$

where $K(m)$ is independent of β, ρ, and η. □

Remarks 4.3. (i) Lemma 4.2 has its own interest. It means that when β tends to infinity the phase velocity of every mode tends to a limit, independent of m and equal to c_0/n_+.

(ii) Other results about the asymptotic behavior of the functions $\lambda_m(\beta)$ have been carried out in [4] and [5].

Proposition 4.2 does not ensure that $\lambda_m(\beta)=(\beta/n_\infty)^2$ when $\beta<\beta_m^*$. That is the reason we set Definition 4.1.

DEFINITION 4.1. For every integer m,

$$\beta_m^0=\sup\left\{\beta_m\in\mathbb{R}^+\,\middle|\,\forall\beta\leqq\beta_m,\ \lambda_m(\beta)=\frac{\beta^2}{n_\infty^2}\right\}.$$

The value β_m^0 is called the mth lower cutoff value.

Obviously, the following inequalities hold:

$$\beta_m^*\leqq\beta_{m+1}^*,\quad \beta_m^0\leqq\beta_{m+1}^0,\quad 0\leqq\beta_m^0\leqq\beta_m^*,$$

and we have (cf. Fig. 3) $\lambda_m(\beta)=\beta^2/n_\infty^2$ for $\beta\leqq\beta_m^0$ and $\lambda_m(\beta)<\beta^2/n_\infty^2$ for $\beta>\beta_m^*$.

These cutoff values will be studied in § 5.

4.4. The case $n\geqq n_\infty$. We can improve these results in the important case where

(4.14) $$n(x)\geqq n_\infty\quad\text{a.e. in } \mathbb{R}^2.$$

Indeed, we have the following theorem.

THEOREM 4.2. *If* (4.14) *holds, then, for every integer m,* $\beta_m^0=\beta_m^*$, *and the number of eigenvalues of* C_β, *located below* $(\beta/n_\infty)^2$, *is monotone nondecreasing in* β.

This theorem, illustrated by Fig. 4, is a straightforward consequence of the next lemma.

LEMMA 4.3. *Assume that* (4.14) *holds. Then the functions* $\Lambda_m(\beta)$, *defined by* (4.3), *are monotone nonincreasing in* β *for* $\beta\in\mathbb{R}^+$.

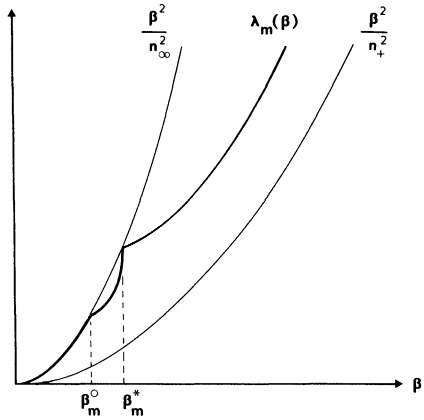

FIG. 3

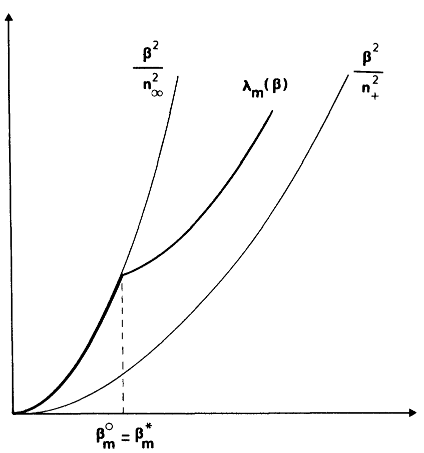

FIG. 4

Proof. We extend to the quadratic case a technique of [22] applied to the linear case.

By (3.12), $\Lambda_m(\beta)$ is always negative, so that the following identity holds:

$$\Lambda_m(\beta) = \sup_{H^{(1)},\cdots,H^{(m-1)}\in(L^2(\mathbb{R}^2))^3} \inf_{\substack{|H|_2=1\\ H\in[H^{(1)},\cdots,H^{(m-1)}]_V^\perp}} \min(0, d(\beta; H, H)),$$

where $d(\beta; H, H)$ is defined by (3.6). Moreover, for every fixed H, the function which associates $\min(0, d(\beta; H, H))$ to β is monotone nonincreasing in β for $\beta\in\mathbb{R}^+$ (cf. Fig. 5). Indeed, by (4.14), we have

$$d_2(H, H) = \int_{\mathbb{R}^2}\left(\frac{1}{n^2}-\frac{1}{n_\infty^2}\right)|\mathbf{H}|^2\,dx \leqq 0.$$

The required result follows. $\square$

The structure of the curves $\lambda_m(\beta)$ is, in this case, similar to that of the circular step-index fiber.

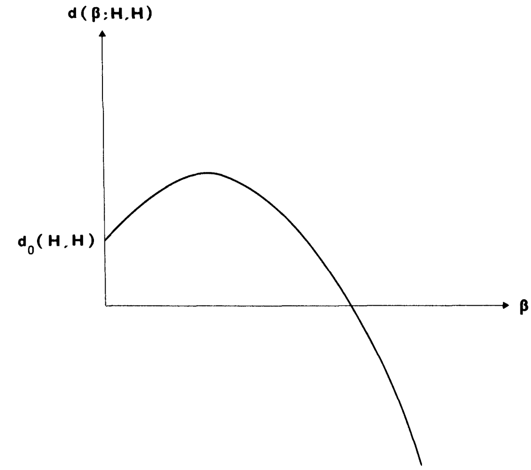

FIG. 5

5. Study of the cutoff equation. The cutoff values, which we introduced in § 4.3, are important features of the guided modes. They are especially useful for counting the number of guided modes which can be propagated in the fiber for a given value of β. Therefore we need a characterization of these cutoff values.

After deriving some a priori estimates, we study the eigenvalue equation

$$c(\beta;\, H_m(\beta),\, H') = \lambda_m(\beta)(H_m(\beta),\, H')$$

for β in a neighborhood of β_m^0 or β_m^*. So we obtain the cutoff equation which is set in a weighted Sobolev space.

Finally, we prove that the sequences (β_m^0) and (β_m^*) tend to infinity, and consequently, that the number of guided modes, for a given value of β, is finite.

5.1. A priori estimates. We first establish some a priori estimates for the eigenfield H.

LEMMA 5.1. *Let* $\lambda \in \mathbb{R}^+$ *and* $H \in V$ *such that*

$$c(\beta;\, H,\, H) = \lambda |H|^2, \qquad \lambda \leqq (\beta/n_\infty)^2. \tag{5.1}$$

Then

$$\int_{\mathbb{R}^2} |\text{grad } H|^2\, dx \leqq \beta^2 M(n) \int_\Omega |\mathbf{H}|^2\, dx, \tag{5.2}$$

$$((\beta/n_\infty)^2 - \lambda) \int_{\mathbb{R}^2} |H|^2\, dx \leqq \beta^2 \Delta(n)(1 + 2M(n)^{1/2}) \int_\Omega |\mathbf{H}|^2\, dx, \tag{5.3}$$

$$\int_{\mathbb{R}^2} |H_3|^2\, dx \leqq M(n) \int_\Omega |\mathbf{H}|^2\, dx, \tag{5.4}$$

where $M(n) = [\Delta(n) n_+^2 + (\Delta(n) n_+^2 + \Delta(n)^2 n_+^4)^{1/2}]^2$, $\Delta(n)$ *is defined by* (4.2), *and* Ω *denotes the core region.*

Proof. By hypothesis, the field H satisfies

$$d_0(H,\, H) \leqq \beta |d_1(H,\, H)| + \beta^2 |d_2(H,\, H)|, \quad \text{and} \tag{5.5}$$

$$((\beta/n_\infty)^2 - \lambda)|H|_2^2 \leqq \beta |d_1(H,\, H)| + \beta^2 |d_2(H,\, H)|. \tag{5.6}$$

Moreover, by the Cauchy-Schwarz inequality

$$|d_1(H,H)| \leqq 2\Delta(n)\left(\int_\Omega |\text{grad } H_3|^2\,dx\right)^{1/2}, \quad \text{and} \tag{5.7}$$

$$|d_2(H,H)| \leqq \Delta(n)\int_\Omega |\mathbf{H}|^2\,dx. \tag{5.8}$$

By inserting (5.6), (5.7), and (2.7) in (5.4) and (5.5), we obtain

$$\begin{aligned}\frac{1}{n_+^2}\int_{\mathbb{R}^2} |\text{grad } H|^2\,dx \leqq \beta^2\Delta(n)\int_\Omega |\mathbf{H}|^2\,dx + 2\beta\Delta(n)\\ \cdot\left(\int_\Omega |\text{grad } H|^2\,dx\right)^{1/2}\left(\int_\Omega |\mathbf{H}|^2\,dx\right)^{1/2}\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}((\beta/n_\infty)^2-\lambda)\int_{\mathbb{R}^2} |H|^2\,dx \leqq \beta^2\Delta(n)\int_\Omega |\mathbf{H}|^2\,dx + 2\beta\Delta(n)\\ \cdot\left(\int_\Omega |\text{grad } H|^2\,dx\right)^{1/2}\left(\int_\Omega |\mathbf{H}|^2\,dx\right)^{1/2}.\end{aligned} \tag{5.10}$$

Now we set $X^2=\int_{\mathbb{R}^2}|\text{grad } H|^2\,dx$ and $Y^2=\int_\Omega |\mathbf{H}|^2\,dx$. With this notation, (5.7) reads

$$X^2-2\beta\Delta(n)n_\infty^2XY-\beta^2\Delta(n)n_+^2Y^2\leqq 0,$$

which implies

$$X\leqq\{\beta\Delta(n)n_+^2+(\beta^2\Delta(n)^2n_+^4+\beta^2\Delta(n)n_+^2)^{1/2}\}Y.$$

Inequality (5.2) is exactly the square of the previous inequality. Then we deduce inequality (5.3) from (5.2) and (5.10). Eventually, we obtain inequality (5.4) by using identity (2.2). □

Remarks 5.1. (i) These estimates are satisfied, in particular, if λ is an eigenvalue of the operator C_β and H an associated eigenvector.

(ii) Inequality (5.3) provides a lower bound for the following factor:

$$\frac{\int_\Omega |H|^2\,dx}{\int_{\mathbb{R}^2}|H|^2\,dx},$$

which gives an estimation of the confinement of the field in the core region.

We can deduce from the previous lemma some results concerning the existence of guided modes for small values of the propagation constant β as follows.

PROPOSITION 5.1. (i) *The lower cutoff value of the third guided mode β_0^3 is strictly positive for every index profile.*

(ii) *If the index profile satisfies*

$$\frac{1}{|\Omega|}\int_\Omega\left(\frac{1}{n_\infty^2}-\frac{1}{n^2}\right)dx \geqq -\Delta(n)^2n_+^2, \tag{5.11}$$

where $|\Omega|$ denotes the measure of Ω, the lower cutoff value of the first mode β_0^1 is strictly positive.

Remarks 5.2. (1) By (i), a fiber supports at most two guided modes at low frequencies.

(2) Assertion (ii) provides a partial converse part for Theorem 4.1.

Proof. (i) Suppose that $\beta_0^3=0$ and consider a sequence β_p tending to zero as p tends to infinity and satisfying

$$\lambda_3(\beta_p)<(\beta_p/n_\infty)^2 \tag{5.12}$$

for every p. By (3.11), we have for every value of β

$$\lambda_3(\beta)\geqq \inf_{\substack{H\in\tilde V\\ H\neq 0}}\frac{c(\beta;H,H)}{|H|_2^2}, \tag{5.13}$$

where $\tilde V$ is the subspace of V defined as follows:

$$\tilde V=\left\{H\in V;\int_\Omega H_1\,dx=\int_\Omega H_2\,dx=0\right\}.$$

Consequently, by (5.12) and (5.13), there exists a sequence $H^{(p)}$ in $\tilde V$ such that

$$c(\beta_p;H^{(p)},H^{(p)})<(\beta/n_\infty)^2|H^{(p)}|_2^2.$$

By Lemma 5.1, sequence $H^{(p)}$ can be normalized as follows:

$$\int_\Omega |\mathbf{H}^{(p)}|^2\,dx=1. \tag{5.14}$$

By (5.2), the sequence $\mathbf{H}^{(p)}$ is bounded in $(H^1(\Omega))^2$, and there exists a subsequence converging to $\mathbf{H}$, weakly in $(H^1(\Omega))^2$ and strongly in $(L^2(\Omega))^2$. The limit $\mathbf{H}$ satisfies

$$\int_\Omega |\mathbf{H}|^2\,dx=1 \quad\text{and}\quad \int_\Omega H_1\,dx=\int_\Omega H_2\,dx=0. \tag{5.15}$$

Since the sequence β_p tends to zero, it results from (5.2) that H is constant on Ω. The contradiction follows eventually from (5.15).

(ii) Suppose likewise that $\beta_0^1=0$ and consider a sequence (β_p), converging to zero and satisfying $\lambda_1(\beta_p)<(\beta_p/n_\infty)^2$ for every p. As in the first part of the proof, we can prove that there exists a sequence (H^p) of V such that

$$c(\beta_p,H^p,H^p)<(\beta_p/n_\infty)^2|H^p|_2^2 \quad\text{and}\quad \int_\Omega |\mathbf{H}^p|^2\,dx=1,$$

for every p, and such that sequence $(\mathbf{H}^{(p)})$ converges strongly to a constant field $\mathbf{H}$ in $(L^2(\Omega))^2$. Moreover, by (5.5) and (5.6), we have

$$\int_{\mathbb{R}}\left(\frac{1}{n_\infty^2}-\frac{1}{n^2}\right)|\mathbf{H}^p|^2\,dx\geqq\frac{1}{\beta^2n_+^2}|\text{grad } H^p|_2^2-\frac{2\Delta(n)}{\beta}|\text{grad } H^p|_2,$$

where $\Delta(n)$ is defined by (4.2) and the term at the right of the inequality is always greater than $(-\Delta(n)^2n_+^2)$. To conclude, we take the limit of previous inequality when p tends to infinity. □

5.2. The cutoff equation. Assume that $\beta>\beta_m^*$, let $H_m(\beta)$ be an eigenfunction of C_β associated with the eigenvalue $\lambda_m(\beta)$, and consider the eigenvalue equation

$$c(\beta;H_m(\beta),H')=\lambda_m(\beta)(H_m(\beta),H'). \tag{5.16}$$

Formally, when β tends to β_m^*, this equation becomes

$$d(\beta;H_m^*,H')=0,$$

where we denote by H_m^* the limit, in some sense, of the family $(H_m(\beta))_\beta$. This equation is precisely the so-called cutoff equation.

Now our aim is to make this formal reasoning rigorous by introducing some appropriate functional space.

Note that the limit H_m^* does not generally belong to V since the energy $\int_{\mathbb{R}^2} |\mathbf{H}|^2\, dx$ is generally infinite. Indeed, for $\beta > \beta_m^*$, $H_m(\beta)$ has an exponential decay at infinity (cf. [11]):

$$H_m(\beta)(x) \underset{|x|\to+\infty}{\sim} K \frac{1}{|x|^{1/2}} \exp\{-((\beta/n_\infty)^2 - \lambda_m(\beta))^{1/2}|x|\}$$

but it decreases less and less when $((\beta/n_\infty)^2 - \lambda_m(\beta))$ becomes smaller. At cutoff, it may happen that the field does not decrease outside the core region Ω.

In view of the estimates (5.1) and (5.3), we are led to consider the following functional norm:

$$[H] = \left(\int_\Omega |\mathbf{H}|^2\, dx + \int_{\mathbb{R}^2} |H_3|^2\, dx + \int_{\mathbb{R}^2} |\text{grad } H|^2\, dx \right)^{1/2}. \tag{5.17}$$

We denote by W the completion of $\mathscr{D}(\mathbb{R}^2)$ with respect to this norm. In fact, the space W is equal to the following product space:

$$W = W_0^1(\mathbb{R}^2) \times W_0^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2), \tag{5.18}$$

where $W_0^1(\mathbb{R}^2)$ is a weighted Sobolev space (cf. [15]) defined by

$$W_0^1(\mathbb{R}^2) = \{\varphi;\ \rho\varphi \in L^2(\mathbb{R}^2) \quad \text{and} \quad \text{grad } \varphi \in [L^2(\mathbb{R}^2)]^2\}$$

with $\rho(x) = ((1+|x|^2)^{1/2} \log(2+|x|^2))^{-1}$.

Now we set the following definition.

DEFINITION 5.1. We say that β satisfies the cutoff equation if and only if there exists $\mathrm{H} \in \mathrm{W}$, $\mathrm{H} \neq 0$, such that

$$d(\beta; H, H') = 0 \quad \forall H' \in W,$$

where W is defined by (5.18) and $d(\beta; H, H')$ by (3.6).

This terminology is justified by Theorem 5.1.

THEOREM 5.1. *The cutoff values β_m^0 and β_m^* satisfy the cutoff equation for every integer m.*

Remark 5.3. It is not proved that every solution of the cutoff equation is, conversely, a cutoff value. Nevertheless, this result has been established for the scalar equation (cf. [4]).

Proof. Let $(\beta^p)_{p\in\mathbb{N}}$ be a decreasing sequence tending to β_m^* as p tends to infinity. By the definition of β_m^*, since β^p is greater than β_m^*, $\lambda_m(\beta^p)$ is an eigenvalue of $C_{(\beta^p)}$, and we denote by $H^{(p)}$ an associated eigenfield.

By Lemma 5.1 we can choose the sequence $(H^{(p)})_{p\in\mathbb{N}}$ such that $\int_\Omega |\mathbf{H}^{(p)}|^2\, dx = 1$. Moreover, by (5.1) and (5.3), the sequence $(H^{(p)})$ is bounded in W. Consequently, there exists a subsequence $(H^{(p')})$ and an element H^* in W such that $H^{(p')} \to H^*$ weakly in W and strongly in $(L^2(\Omega))^3$.

We can now take the limit of (5.16) for $H' \in [\mathscr{D}(\mathbb{R}^2)]^3$ and we obtain

$$d(\beta_m^*; H^*, H') = 0,$$

which is valid by density for every H' in W. Finally, H^* does not vanish everywhere since

$$\int_{D_a} |\mathbf{H}^*|^2\, dx = 1.$$

We have proved the lemma for β_m^*. The proof is similar for β_m^0. Indeed, by the definition of β_m^0, there exists a sequence (β^p) such that $\beta^p > \beta_m^0$, $\lambda_m(\beta^p) < (\beta_p/n_\infty)^2$ and $\lim_{p\to+\infty} \beta^p = \beta_m^0$. □

The following lemma is a useful generalization of Lemma 5.1.

LEMMA 5.2. *Let $\beta \in \mathbb{R}^+$ and $H \in W$ such that $d(\beta; H, H) = 0$. Then*

$$\int_{\mathbb{R}^2} |\text{grad } H|^2 \, dx \leqq \beta^2 M(n) \int_\Omega |\mathbf{H}|^2 \, dx,$$

$$\int_{\mathbb{R}^2} |H_3|^2 \, dx \leqq M(n) \int_\Omega |\mathbf{H}|^2 \, dx.$$

We deduce from Lemma 5.2 the following corollary.

COROLLARY 5.3. *For every solution β of the cutoff equation, the associated eigenspace $W_\beta = \{H \in W;\ d(\beta; H, H') = 0,\ \forall H' \in W\}$ is finite-dimensional.*

Remark 5.4. By the previous lemma, if $(\beta/n_\infty)^2$ is an eigenvalue of C_β, it has necessarily finite multiplicity.

Proof. Suppose that W_β is infinite-dimensional and let $(H^{(p)})_p$ be a basis of W_β. By Lemma 5.2, this sequence may be orthogonalized as follows:

$$\int_\Omega \mathbf{H}^{(p)} \cdot \mathbf{H}^{(q)} \, dx = \delta_{pq}. \tag{5.19}$$

Moreover, it is bounded in W and there exists a subsequence of $(\mathbf{H}^{(p)})$ which converges strongly in $(L^2(\Omega))^2$. But this is in contradiction to (5.19). □

5.3. The finite number of eigenvalues. We will now prove that the number of solutions k of the dispersion relation (3.13), for a given value of β, is finite. We first establish the result for a particular category of profiles and then generalize it to arbitrary profiles.

THEOREM 5.2. *Assume that* (4.14) *holds. Then* $\lim_{m\to+\infty} \beta_m^* = +\infty$, *and for each fixed β, C_β has at most a finite number of eigenvalues below $(\beta/n_\infty)^2$.*

Proof. We will prove this result by contradiction. Suppose that sequence (β_m^*) is bounded. Since it is a nondecreasing sequence, there exists β^* such that $\lim_{m\to+\infty} \beta_m^* = \beta^*$.

(1) First we prove that every β greater than β^* satisfies the cutoff equation.

Indeed, if $\beta > \beta^*$, then $\beta > \beta_m^*$ for every integer m. Consequently, the operator C_β has an infinite sequence of eigenvalues $(\lambda_m(\beta))_{m\geqq 1}$ which converges to $(\beta/n_\infty)^2$ as m tends to infinity. Then, using a sequence of associated eigenfields, we can prove, by following the demonstration of Theorem 5.1, that β satisfies the cutoff equation.

(2) Let $(\beta^p)_{p\in\mathbb{N}}$ be a strictly decreasing sequence tending to β^* as p tends to infinity. By the previous paragraph of the proof, β^p satisfies the cutoff equation for every p. Let us denote by $H^{(p)}$ an associated eigenfield which is assumed to satisfy (cf. Lemma 5.2)

$$\int_\Omega |\mathbf{H}^{(p)}|^2 \, dx = 1.$$

By Lemma 5.1, there exists a subsequence, still denoted $(H^{(p)})_{p\in\mathbb{N}}$, and an element H^* in W such that $H^{(p)} \to H^*$ weakly in W and strongly in $(L^2(\Omega))^3$. The field H^* satisfies

$$\int_\Omega |\mathbf{H}^*|^2 \, dx = 1 \quad \text{and} \quad d(\beta^*; H^*, H') = 0 \quad \forall H' \in W. \tag{5.20}$$

(3) Let $p, q \in \mathbb{N}$. By definition of $H^{(p)}$, we have $d(\beta^p; H^{(p)}, H^{(q)}) = 0$ and $d(\beta^q; H^{(q)}, H^{(p)}) = 0$.

By subtracting the conjugate part of the second equation from the first one and by dividing the result by $(\beta_p - \beta_q)$, we get $d_1(H^{(p)}, H^{(q)}) + (\beta^p + \beta^q) d_2(H^{(p)}, H^{(q)}) = 0$, which converges to

$$d_1(H^*, H^*) + 2\beta^* d_2(H^*, H^*) = 0, \tag{5.21}$$

as p tends to infinity. By (5.20) we have, in addition,

$$d_0(H^*, H^*) + \beta^* d_1(H^*, H^*) + (\beta^*)^2 d_2(H^*, H^*) = 0. \tag{5.22}$$

Using (5.20)-(5.22), we eventually obtain

$$d_0(H^*, H^*) = (\beta^*)^2 d_2(H^*, H^*).$$

But we have $d_0(H^*, H^*) \geqq 0$ and under assumption (4.14), $d_2(H^*, H^*) < 0$. Eventually, by Proposition 5.1, $\beta^* > 0$. □

Using a comparison technique, we can generalize this result as follows.

THEOREM 5.3. *The sequences of cutoff values* (β_m^0) *and* (β_m^*) *satisfy*

$$\lim_{m \to +\infty} \beta_m^0 = +\infty, \qquad \lim_{m \to +\infty} \beta_m^* = +\infty$$

and for each fixed β, C_β *has at most a finite number of eigenvalues below* $(\beta / n_\infty)^2$.

Proof. Assume that the index profile n does not satisfy (4.14) and let us define the index profile $\bar{n}$ of a step-index fiber by

$$\bar{n}(x) = n_+ \quad \text{if } x \in \Omega,$$
$$\bar{n}(x) = n_\infty \quad \text{if } x \in \mathbb{R}^2 \backslash \Omega.$$

Since $\bar{n}(x) \geqq n(x)$, almost everywhere $x \in \mathbb{R}^2$, by Remark 3.3, $\lambda_m(\beta, \bar{n}) \leqq \lambda_m(\beta, n)$ for every integer m and every value of β. Consequently, by definition of the cutoff values:

$$\beta_m^0(\bar{n}) \leqq \beta_m^*(\bar{n}) \leqq \beta_m^0(n) \leqq \beta_m^*(n).$$

Moreover, $\bar{n}$ satisfies (4.14). Therefore, by Proposition 4.2 $\beta_m^0(\bar{n}) = \beta_m^*(\bar{n})$, and by Theorem 5.2, $\lim_{m \to +\infty} \beta_m^*(\bar{n}) = +\infty$. □

6. Conclusion. This work provides a relatively complete description of the dispersion relation of a fiber, whose index profile is arbitrary.

The results are more precise when the index is everywhere in the core greater than the index of the cladding. In that case, we showed that the structure of the guided modes is similar to that of a circular step-index fiber.

When the above condition is not satisfied, some points must still be investigated, especially the existence of index profiles such that $\beta_m^0 < \beta_m^*$ for some m, and, in this case, the behavior of $\lambda_m(\beta)$ in the interval $[\beta_m^0, \beta_m^*]$. The cutoff equation therefore requires specific study.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions*, Dover, New York, 1964.

[2] A. BAMBERGER AND A. S. BONNET, *Calcul des modes guidés d'une fibre optique. Deuxième partie: analyse mathématique*, Rapport interne 143, Centre de Mathématiques Appliquées, Ecole Polytechnique, Palaiseau, France, 1986.

[3] A. BAMBERGER, A. S. BONNET, AND R. DJELLOULI, *Calcul des modes guidés d'une fibre optique. Première partie: différentes formulations mathématiques du problème*, Rapport interne 142, Centre de Mathématiques Appliquées, Ecole Polytechnique, Palaiseau, France, 1986.

[4] A. S. BONNET, *Analyse mathématique de la propagation de modes guidés dans les fibres optiques*, Thèse d'analyse numérique de l'Université Pierre et Marie Curie, Paris, France, 1988.

[5] A. S. BONNET AND R. DJELLOULI, *Etude mathématique des modes guidés d'une fibre optique. Résultats complémentaires et extension au cas de couplages*, Rapport interne 182, Centre de Mathématiques Appliquées, Ecole Polytechnique, Palaiseau, France, 1988.

[6] G. COPPA AND P. DI VITTA, *Cut-off condition of the fundamental mode in monotone fibers*, Optics Comm., 49 (1984), pp. 409–412.

[7] R. DAUTRAY AND J. L. LIONS, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, Tome 2, Masson, Paris, 1985.

[8] N. DUNFORD AND J. SCHWARTZ, *Linear Operators. Part* II: *Spectral Theory*, Interscience, New York, 1963.

[9] J. C. GUILLOT, *Complétude des modes T.E. et T.M. pour un guide d'ondes optiques planaire*, Rapport de Recherche INRIA, Le Chesnay, France, 385, 1985.

[10] L. HORMANDER, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1976.

[11] L. B. JEUNHOMME, *Single-Mode Fiber Optics: Principles and Applications*, Marcel Dekker, New York, 1983.

[12] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, New York, 1976.

[13] M. KOSHIBA, K. HAYATA, AND M. SUZUKI, *Vectorial finite-element method without spurious solutions for dielectric waveguide problems*, Electron. Lett., 20 (1984), pp. 402–410.

[14] R. LEIS, *Zur Theorie Elektromagnetischer Schwingungen in Anisotropen Medien*, Math. Z., 106 (1968), pp. 213–224.

[15] M. N. LE ROUX, *Thèse de 3ème cycle*, Université de Rennes, Rennes, France, 1974.

[16] D. MARCUSE, *Theory of Dielectric Optical Waveguide*, Academic Press, New York, 1974.

[17] ———, *Light Transmission Optics*, Van Nostrand, New York, 1982.

[18] N. MABAYA, P. E. LAGASSE, AND P. VANDENBULCKE, *Finite element analysis of optical waveguides*, IEEE Trans. Microwave Theory Tech., (1981), pp. 600–605.

[19] H. PICQ, *Détermination et calcul numérique de la première valeur propre d'opérateurs de Schrödinger dans le plan*, Thèse, Université de Nice, Nice, France, 1982.

[20] J. P. POCHOLLE, *Propagation dans les fibres optiques monomodes*, Revue technique de Thomson-CSF, 15, 1983.

[21] F. RELLICH, *Über das asymptotische Verhalten der Lösungen von* $\Delta u + \lambda u = 0$ *in unendlichen Gebieten*, Über. Deutsch. Math. Verein, 53 (1943), pp. 57–65.

[22] M. REED AND B. SIMON, *Methods of Modern Mathematical Physics*, Vol. 4, Academic Press, New York, 1978.

[23] M. SCHECHTER, *Operator Methods in Quantum Mechanics*, North Holland, Amsterdam, 1981.

[24] A. W. SNYDER AND J. D. LOVE, *Optical Waveguide Theory*, Chapman and Hall, London, 1983.

[25] C. VASSALO, *Théorie des guides d'ondes électromagnétiques*, Tomes 1 et 2, Editions Eyrolles et CNET-ENST, Paris, 1985.

[26] R. WEDER, *Spectral and scattering theory in deformed optical wave guides*, J. Reine Angew. Math., 390 (1988), pp. 130–169.

[27] C. WILCOX, *Sound Propagation in Stratified Fluids*, Applied Mathematical Sciences, Vol. 50, Springer-Verlag, Berlin, New York, 1984.

[28] C. YEH, K. HA, S. B. DONG, AND W. P. BROWN, *Single-mode optical waveguides*, Appl. Optics, 18 (1979), pp. 1490–1504.

[29] H. ZIANI AND C. DEVYS, *Méthode intégrale pour le calcul des modes guidés d'une fibre optique*, Rapport Interne, Centre de Mathématiques Appliquées, 129, Ecole Polytechnique, Palaiseau, France, 1985.