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SHORT COMMUNICATIONS

On the Spectral Characteristics of Non-Self-Adjoint Fourth-Order Operators with Matrix Coefficients

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Consider the Hilbert space $L_2[0,1]$ of measurable complex functions on $[0,1]$ taking values in $\mathbb{C}$ and having square integrable modulus. By $L_2^k[0,1]$ we denote the space

$$L_2^k[0,1] = L_2([0,1],\mathbb{C}^k) = \underbrace{L_2[0,1]\times\cdots\times L_2[0,1]}_{k \text{ times}}$$

with inner product

$$(f,g) = \sum_{j=1}^{k}\int_0^1 f_j(t)\overline{g_j(t)}\,dt,$$

where $f=(f_1,f_2,\dots,f_k)\in L_2^k[0,1]$ and $g=(g_1,g_2,\dots,g_k)\in L_2^k[0,1]$.

The goal in the present paper is to study some asymptotic formulas for eigenvalues of the fourth-order non-self-adjoint operator $L_\theta\colon D(L_\theta)\subset L_2^k[0,1]\to L_2^k[0,1]$ defined by the differential expression

$$l(y) = y^{IV} - \mathfrak{A}(t)y'' - \mathfrak{B}(t)y,$$

where $\mathfrak{A}(t)=(a_{pj}(t))_{p,j=1}^k$ and $\mathfrak{B}(t)=(b_{pj}(t))_{p,j=1}^k$ are $k\times k$ matrices with elements a_{pj} and b_{pj} belonging to $L_2[0,1]$. The domain

$$D(L_\theta) = \{y\in W_2^4([0,1],\mathbb{C}^k)\}\subset L_2^k[0,1]$$

of the operator L_θ is given by the quasiperiodic boundary conditions $y^{(j)}(1)=e^{i\pi\theta}y^{(j)}(0)$, $j=0,1,2,3$, where $\theta\in(0,2)$, $\theta\neq 1$.

Let $\mathfrak{A}_0$ denote the matrix

$$\mathfrak{A}_0 = (a_{0,pj})_{p,j=1}^k, \qquad a_{0,pj} = \int_0^1 a_{pj}(t)\,dt.$$

In what follows, we assume that the matrix $\mathfrak{A}_0$ is similar to a diagonal matrix.

The spectral analysis of differential operators with matrix coefficients has recently become one of the popular areas of research. Apparently, the study of asymptotic formulas for eigenvalues of higher-order differential operators with matrix coefficients and integrable elements originated in the paper [1] of Birkhoff and Langer. More recently, such a class of operators was studied by Naimark in [2, Chap. III]

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and Veliev in [3] and [4]. Note that, in [4], asymptotic formulas for eigenvalues were used to obtain conditions under which the system of root functions constitutes a Riesz basis in $L_2^k(0,1)$. General results on the Riesz basis property of higher-order ordinary differential operators and of more complicated boundary-value problems involving a nonlinear spectral parameter in the equation and the boundary conditions were obtained by Shkalikov in [5]. Later he generalized these results to the matrix case in [6].

In addition to the works mentioned above, we must also point out Veliev's paper [7] dealing with the study of the Sturm–Liouville operator with matrix potential and quasiperiodic boundary conditions. In that paper, asymptotic formulas for eigenvalues were obtained and conditions were written out under which the system of root functions constitutes a Riesz basis in $L_2^k(0,1)$. More recently, Uskova [8] strengthened the asymptotic formulas in these results. In her study, she considered the case of a square-integrable matrix potential.

In the present paper, we obtain asymptotic formulas for eigenvalues of the differential operator L_θ for $\theta \in (0,2)$, $\theta \neq 1$. In addition, we also consider the case $\theta \in \{0,1\}$ for $k=1$.

Now let us pass to the study of the operator L_θ, $\theta \in (0,2)$, $\theta \neq 1$. For brevity, in what follows, we shall denote the space $L_2^k[0,1]$ by $\mathscr{H}$. Let us represent this operator as

$$L_\theta = L_\theta^0 - B,$$

where

$$L_\theta^0\colon D(L_\theta^0) = D(L_\theta) \subset \mathscr{H} \to \mathscr{H}, \qquad L_\theta^0 y = y^{IV},$$
$$B\colon D(B) \subset \mathscr{H} \to \mathscr{H}, \qquad (By)(t) = \mathfrak{A}(t)y''(t) + \mathfrak{B}(t)y(t).$$

The operator L_θ^0 is an operator with well-known spectral properties. The spectrum of the operator L_θ^0 is discrete, and its eigenvalues are of the form

$$\lambda_{n,j} = \pi^4(2n+\theta)^4, \qquad n \in \mathbb{Z}, \quad j = 1,\dots,k.$$

The corresponding eigenvectors are the functions

$$e_{n,j}(t) = e^{i\pi(2n+\theta)t} \cdot f_j(t), \qquad n \in \mathbb{Z}, \quad j = 1,\dots,k, \quad t \in [0,1],$$

where the vectors f_j, $j=1,\dots,k$, constitute an orthonormal basis in $\mathbb{C}^k$. In addition, for any $x \in \mathscr{H}$, we define the Riesz projection P_n, $n \in \mathbb{Z}$, as follows:

$$P_n x = \sum_{j=1}^{k} (x, e_{n,j}) e_{n,j}, \qquad n \in \mathbb{Z}.$$

Our study is based on a method from [9] and [10]. The scheme of study of the operator L_θ is related to the notion of similarity of operators (see [9, Definition 1.5]). It is well known that similar operators possess a number of coinciding spectral properties (see [10, Lemma 1]). In particular, the spectra of similar operators coincide. The main idea of the method is to reduce the study of the operator $L_\theta^0 - B$ to that of a similar operator of block-diagonal form (the concrete form of this operator will be given below) with respect to a basis in which the operator L_θ^0 is of "diagonal" form. Thus, according to [9], the first (and most involved) step in the application of the method is the construction of an admissible triplet: the space of study $\mathfrak{U}$ and two transformers J_m and Γ_m (being linear operators on the space of linear operators). For the space $\mathfrak{U}$ we choose a well-known Banach space so that either the operator B belongs to $\mathfrak{U}$ or an operator $L_\theta^0 - \widetilde{B}$, where $\widetilde{B} \in \mathfrak{U}$, similar to the operator L_θ has a simple structure. The transformer J_m will be responsible for the resulting similarity operator. Therefore, the transformer J_m is defined so that, in addition to being similar, these operators are of block-diagonal form. The introduction of the operator Γ_m is related to the construction of the similarity transformation operator.

First, we construct the space $\mathfrak{U}$. Consider the operator $(L_\theta^0)^{1/2}\colon D((L_\theta^0)^{1/2}) \subset \mathscr{H} \to \mathscr{H}$ defined by

$$(L_\theta^0)^{1/2} x = \sum_{n\in\mathbb{Z}} \lambda_{n,j}^{1/2} P_n x, \qquad j = 1,\dots,k, \quad x \in \mathscr{H},$$

with domain

$$D((L_\theta^0)^{1/2}) = \left\{ x \in \mathscr{H} : \sum_{n\in\mathbb{Z}} |\lambda_{n,j}| \|P_n x\|^2 < \infty \right\}.$$

The Banach space $\mathfrak{U}$ will consist of operators X representable as $X = X_0(L_\theta^0)^{1/2}$, where X_0 is a Hilbert–Schmidt operator, i.e., belongs to $\mathfrak{S}_2(\mathscr{H})$ (see [11, Chap. 3, Sec. 9]). The norm on this space will be denoted by $\|\cdot\|_2$. For the norm $\|X\|_*$ of the operator X in the space $\mathfrak{U}$ we take the quantity $\|X_0\|_2$.

For any $m \in \mathbb{Z}_+$, we define the family of transformers J_m as follows:

$$J_m X = J_0(X - P_{(m)} X P_{(m)}) + P_{(m)} X P_{(m)} = P_{(m)} X P_{(m)} + \sum_{|n|\ge m+1} P_n X P_n, \qquad X \in \mathfrak{U},$$

where

$$P_{(m)} = \sum_{|j|\le m} P_j, \qquad J_0 X = \sum_{s\in\mathbb{Z}} P_s X P_s, \quad X \in \mathfrak{U}.$$

Since the operator X belongs to the space $\mathfrak{U}$, it follows that the series in the last equalities are convergent.

By analogy with the transformers J_m, for any $m \in \mathbb{Z}_+$, we define the family of transformers Γ_m of the form

$$\Gamma_m X = \Gamma_0 X - \Gamma_0(P_{(m)} X P_{(m)}) = \Gamma_0 X - P_{(m)}(\Gamma_0 X) P_{(m)},$$

where the operator $\Gamma_0 \colon \mathfrak{U} \to \mathfrak{S}_2(\mathscr{H})$ is constructed as follows. For each operator block X_{sr}, $s, r \in \mathbb{Z}$, $s \ne r$, of the form

$$X_{sr} = P_s X P_r = P_s X_0 (L_\theta^0)^{1/2} P_r = \lambda_{r,j}^{1/2} P_s X_0 P_r, \qquad X \in \mathfrak{U},$$

the transformer Γ_0 is defined by $\Gamma_0 X_{sr} = Y_{sr}$, where Y_{sr} is a solution of the equation

$$L_\theta^0 Y_{sr} - Y_{sr} L_\theta^0 = X_{sr}, \qquad s \ne r$$

and $Y_{ss} = 0$ for any $s \in \mathbb{Z}$. Note that the last equation can be rewritten as

$$A_s Y_{sr} - Y_{sr} A_r = X_{sr},$$

where A_s is the restriction of the operator L_θ^0 to the subspace $\operatorname{Im} P_s$ for any $s \in \mathbb{Z}$. Since we have $\sigma(A_s) \cap \sigma(A_r) = \varnothing$ for all $s, r \in \mathbb{Z}$, it follows that each of these equations is solvable (for greater detail, see [8]), and

$$\|Y_{sr}\|_* \le \frac{C\|X_{sr}\|_*}{\operatorname{dist}(\sigma(A_s), \sigma(A_r))}, \qquad Y_{ss} = 0, \quad s, r \in \mathbb{Z}.$$

Here and in what follows, by the common symbol $C > 0$ we denote various positive constants.

Thus, we have constructed the triplet $(\mathfrak{U}, J_m, \Gamma_m)$. Using arguments similar to those in [12, Lemma 4], we readily see that this triplet is admissible (for this, we need to verify five properties). Note that the admissible triplet is determined not uniquely. In constructing the triplet, we only use "nice" properties of the operators under consideration and the chosen space.

Further similarity theory (see [9, Theorem 1.5]) can only be applied if B belongs to $\mathfrak{U}$. However, it is often difficult to construct a space containing the needed perturbation. In this case, it is necessary to carry out a preliminary similarity transformation, i.e., reduce, by similarity, the study of the operator L_θ to that of the operator $L_\theta^0 - \widetilde{B}$, where $\widetilde{B} \in \mathfrak{U}$. This is the situation that we must now consider. To carry out the preliminary similarity transformation, it is necessary to verify five properties resembling the properties of the admissible triplet (see [10, Assumption]). The following statement holds.

Lemma 1. *There exists a number $m \in \mathbb{Z}_+$ such that $\|\Gamma_m B\|_* \le 1/2$. Then the operator L_θ is similar to the operator $L_\theta^0 - J_m B - \widetilde{B}$, where $\widetilde{B}$ belongs to the space $\mathfrak{U}$ and can be represented as*

$$\widetilde{B} = B\Gamma_m B - (\Gamma_m B) J_m B - (\Gamma_m B)(I + \Gamma_m B)^{-1}(B\Gamma_m B - (\Gamma_m B) J_m B). \tag{1}$$

Lemma 1 makes it possible to use the main theory of the method (see [9, Theorem 1.5]).

Lemma 2. *There exists a number $m \in \mathbb{Z}_+$ such that the operator $L_\theta^0 - J_m B - \widetilde{B}$ is similar to the operator $L_\theta^0 - J_m B - J_m X_*$, where X_* is a solution of the nonlinear equation*

$$X = \widetilde{B}\Gamma_m X - (\Gamma_m X)(J_m \widetilde{B}) - (\Gamma_m X)J_m(\widetilde{B}\Gamma_m X) + \widetilde{B}$$

considered in $\mathfrak{U}$. The similarity transformation is carried out by the operator $I + \Gamma_m X_*$ and the operator $\widetilde{B}$ defined by formula (1).

Lemmas 1 and 2 allow us to reduce the study of the operator L_θ to that of the block-diagonal operator $L_\theta^0 - J_m B - J_m X_*$. It follows from their similarity that the spectra of these operators coincide. Thus, using the asymptotic formulas for the eigenvalues of the operator $L_\theta^0 - J_m B - J_m X_*$, we can write out an asymptotics for the operator L_θ as well. This is reflected by the following intermediate result.

Lemma 3. *The spectrum of the operator L_θ can be represented as*

$$\sigma(L_\theta) = \sigma_{(m)} \cup \biggl(\bigcup_{|n|\ge m+1} \sigma_n\biggr), \tag{2}$$

where $\sigma_{(m)}$ is a finite set and the sets σ_n, $|n| \ge m+1$, consist of at most k points. In addition, the spectrum of each of the sets σ_n coincides with the spectrum of the operator L_θ defined by the matrix $\mathscr{A}_n$ of the form

$$\mathscr{A}_n = \pi^4(2n+\theta)^4 E - \mathscr{B}_n - \mathscr{C}_n + \pi^2(2n+\theta)^2\mathscr{D}_n, \qquad |n| \ge m+1, \tag{3}$$

where E is the identity matrix, $\mathscr{B}_n$ is the $k\times k$ matrix consisting of the elements $(Be_{n,j}, e_{n,j})$, $j = 1,\dots,k$, and $\mathscr{C}_n$ is the $k\times k$ matrix consisting of the elements $(\widetilde{B}e_{n,j}, e_{n,j})$, $j = 1,\dots,k$. In addition, for $n_1 = \max\{m+1, 3\|B\|_*/(4\pi^2) + (1+\theta)/2\}$, the norm of the matrices $\mathscr{D}_n$ satisfies the estimates

$$\begin{aligned}\|\mathscr{D}_n\|_* = \|P_n(X_* - \widetilde{B})P_n\|_* &\le 2d_n\|P_n\widetilde{B} - P_n\widetilde{B}P_n\|_*\|\widetilde{B}P_n - P_n\widetilde{B}P_n\|_* \\ &\le \frac{C\eta_n}{|2|n|-1-\theta|(\theta-1)^2(1-|\theta-1|)^2}, \qquad |n| \ge n_1,\end{aligned}$$

where (η_n) is a summable sequence and $d_n = (4\pi^2|2|n|-1-\theta|)^{-1}$.

Note that this lemma is also valid when $\theta \in \{0,1\}$ and $k = 1$. In this case, all matrices from (3) are of size 2×2, $d_n = (4\pi^2(2n-1+\theta))^{-1}$, $n \in \mathbb{Z}_+$, and $\|\mathscr{D}_n\|_* \le C\widetilde{\eta}_n/n$, where $(\widetilde{\eta}_n)$ is a summable sequence.

Lemmas 2 and 3 will yield all main results of the paper.

Theorem 1. *There exists a number $m \in \mathbb{Z}_+$ for which the spectrum of the operator L_θ can be represented as* (2). *Then, for the arithmetic mean of the eigenvalues $\widehat{\lambda}_n$ of the operator L_θ, the following asymptotic representation holds*:

$$\widehat{\lambda}_n = \sum_{j=1}^{k}\widetilde{\lambda}_{n,j} = \pi^4(2n+\theta)^4 + \frac{\pi^2(2n+\theta)^2}{k}\sum_{j=1}^{k}\mu_j + \mathscr{O}(|n|), \qquad |n| \ge m+1,$$

where μ_j, $j = 1,\dots,k$, are the eigenvalues of the matrix $\mathfrak{A}_0$.

Theorem 2. *If, under the assumptions of Theorem* 1, *the eigenvalues μ_j, $j = 1,\dots,k$, of the matrices $\mathfrak{A}_0$ are simple, then the following asymptotic expansion holds*:

$$\widetilde{\lambda}_{n,j} = \pi^4(2n+\theta)^4 + \pi^2(2n+\theta)^2\mu_j + \mathscr{O}(|n|), \qquad j = 1,\dots,k, \quad |n| \ge m+1.$$

The asymptotic formulas given in Theorems 1 and 2, refine the remainder in the corresponding results (Theorems 1 and 2) of [4].

Let the matrices $\mathfrak{A}$ and $\mathfrak{B}$ of the operator L_θ be of size 1×1 and consist of the single elements a and b, respectively. Since $a, b \in L_2[0,1]$, it follows that the following expansions are valid:

$$a(t) = \sum_{s\in\mathbb{Z}} a_s e^{i2\pi st}, \quad b(t) = \sum_{s\in\mathbb{Z}} b_s e^{i2\pi st}, \qquad t \in [0,1],$$

where the a_s and b_s are the Fourier coefficients of the functions a and b. Then we have the following theorem.

Theorem 3. *The operator L_θ is an operator with discrete spectrum and there exists a number $m \in \mathbb{Z}_+$ whose spectrum can be represented in the form* (2)*. The eigenvalues $\widetilde{\lambda}_{n,1}$, $|n| \geq m+1$, admit the following asymptotic expansion*:

$$\widetilde{\lambda}_{n,1} = \pi^4(2n+\theta)^4 + \pi^2(2n+\theta)^2 a_0 - (2n+\theta)^2 \sum_{\substack{s\in\mathbb{Z}\\ s\neq n}} \frac{(2s+\theta)^2 a_{n-s}a_{s-n}}{(2s+\theta)^4 - (2n+\theta)^4} + \frac{|n|\gamma_n}{(\theta-1)^2(1-|\theta-1|)^2}, \qquad |n| \geq m+1,$$

where (γ_n) is a summable sequence.

This techniques allows us to obtain the asymptotics of eigenvalues also in the periodic (antiperiodic) case. Let $\theta \in \{0,1\}$. In view of certain difficulties, we will restrict ourselves only to the case $k=1$. Then the matrices $\mathfrak{A}$ and $\mathfrak{B}$ are of size 1×1 and have elements a and b from $L_2[0,1]$. The study of the spectral properties of the operator L_θ, $\theta \in \{0,1\}$, is of interest in itself (see the papers [12] of the author and [13], [14] of Badanin and Korotyaev). Also we note Veliev's paper [15], in which the asymptotics of eigenvalues for a differential operator of arbitrary order with periodic boundary conditions was obtained.

Let us pass to the formulation of the main result of this part. In contrast to [12, Theorem 1] and [15, Theorem 1, 2], the following theorem refines the form of the second approximation, as well as the formula for the remainder.

Theorem 4. *The operator L_θ, $\theta \in \{0,1\}$, is an operator with discrete spectrum, and there exists a number $m \in \mathbb{Z}_+$ whose spectrum is expressed as* (2)*. Further, $\sigma_{(m)}$ is a finite set with number of points not exceeding $2m+1$, and the set σ_n is defined by $\sigma_n = \{\widetilde{\lambda}_n^+\} \cup \{\widetilde{\lambda}_n^-\}$. The eigenvalues $\widetilde{\lambda}_n^\mp$, $n \geq m+1$, of the operator L_θ, $\theta \in \{0,1\}$, admit the following asymptotic representation*:

$$\begin{aligned}\widetilde{\lambda}_n^\mp = {}& \pi^4(2n+\theta)^4 - 2(2n+\theta)^2 \sum_{s=1,\, s\neq n}^{\infty} \frac{(2s+\theta)^2(a_{n-s}a_{s-n} + a_{n+s+\theta}a_{-n-s-\theta})}{(2s+\theta)^4 - (2n+\theta)^4} + \pi^2(2n+\theta)^2 a_0 \\ & \mp (2n+\theta)^2 \Biggl(\pi^2 a_{-2n-\theta} - 2 \sum_{\substack{s\in\mathbb{Z}\\ s\neq n,\, s\neq -n-\theta}} \frac{(2s+\theta)^2 a_{s-n}a_{-n-s-\theta}}{(2s+\theta)^4 - (2n+\theta)^4}\Biggr)^{1/2} \\ & \times \Biggl(\pi^2 a_{2n+\theta} - 2 \sum_{\substack{s\in\mathbb{Z}\\ s\neq n,\, s\neq -n-\theta}} \frac{(2s+\theta)^2 a_{n-s}a_{n+s+\theta}}{(2s+\theta)^4 - (2n+\theta)^4}\Biggr)^{1/2} + n\widetilde{\gamma}_n, \qquad n \geq m+1,\end{aligned}$$

where $(\widetilde{\gamma}_n)$ is a summable sequence.

In addition, from Theorems 3 and 4 it is easy to obtain the asymptotics of the eigenvalues of the operator L_θ, $\theta \in [0,2)$, for the case in which the functions a and b are smooth or real. These results significantly improve those known earlier (see [12, Theorem 2]).

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