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ON THE DISTRIBUTION OF EIGENVALUES OF NON-SELFADJOINT OPERATORS

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ABSTRACT. We prove quantitative bounds on the eigenvalues of non-selfadjoint bounded and unbounded operators. We use the perturbation determinant to reduce the problem to one of studying the zeroes of a holomorphic function.

1. INTRODUCTION

The aim of this work is to obtain some quantitative results on the structure of the discrete spectrum of wide classes of non-selfadjoint linear operators. While the study of the eigenvalues of selfadjoint operators is well-developed, much less is known in the non-selfadjoint case, since many of the methods used to study the discrete spectrum of selfadjoint operators, employing the variational characterization of the eigenvalues and the fact that the eigenvalues are real, do not apply in the non-selfadjoint setting.

Our approach is based on constructing a holomorphic function, in terms of perturbation determinants, whose zeroes are the eigenvalues of the operator we are interested in, and using complex analysis to obtain information on these zeroes, which in turn translates into information on the eigenvalues. Variants of this approach were used previously, e.g. in [1, 3].

We develop results in the bounded and unbounded settings, each of which is useful in applications to concrete operators.

In Section 4, we assume that A, B are bounded linear operators in a complex Hilbert space, where A is selfadjoint with $\sigma(A) = [a, b]$ and $B - A$ belongs to the Schatten class $\mathbf{S}_p, p > 0$. The aim is to obtain quantitative bounds on the discrete spectrum $\sigma_{disc}(B)$, i.e. on isolated eigenvalues of finite algebraic multiplicity, in terms of the p-Schatten norm of $B - A$. By the above assumptions, the essential spectrum $\sigma_{ess}(B) = [a, b]$, and $\sigma_{disc}(B) = \sigma(B) \cap (\mathbb{C} \setminus [a, b])$ consists of a sequence of eigenvalues which can only accumulate on $[a, b]$. Our results quantify the rate at which this approach to the essential spectrum must occur under the above assumptions. We prove that, for $\gamma > \max(1 + p, 2p)$,

$$(1) \qquad \sum_{\lambda \in \sigma_{disc}(B)} \frac{\mathrm{dist}(\lambda, [a, b])^\gamma}{|\lambda - a|^{\frac{\gamma}{2}} |\lambda - b|^{\frac{\gamma}{2}}} \leq C \|B - A\|^p_{\mathbf{S}_p},$$

where the constant C, which is given explicitly, depends only on p and on γ.

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In Section 5, we assume that H_0, H are (unbounded) closed operators in a complex Hilbert space, H_0 is selfadjoint with $\sigma(H_0) = [0, \infty)$ and the resolvent difference $R_s = (s - H)^{-1} - (s - H_0)^{-1}$ is in $\mathbf{S}_p$ for some $s < 0$. The eigenvalues of H may accumulate on $\sigma_{ess}(H) = [0, \infty)$, or at infinity. We obtain quantitative information on the rate of accumulation given by the following inequality: for $\gamma > \max(1 + p, 2p)$,

$$\sum_{\lambda \in \sigma_{disc}(H)} \frac{\operatorname{dist}(\lambda, [0, \infty))^\gamma}{|\lambda|^{\frac{\gamma}{2}}(1 + |\lambda|)^\gamma} \leq C \|R_s\|^p_{\mathbf{S}_p}, \tag{2}$$

where the constant C depends only on p, γ, and s.

The results for unbounded operators above are proved by reduction to the case of bounded operators, so that the results in the bounded case are fundamental. Our results for bounded operators are in the same spirit as the results of Borichev, Golinskii, and Kupin [1]. A chief difference is that while in the proof of their results, the above authors used a new result in complex analysis about zeroes of certain holomorphic functions, whose proof is quite involved, we use a result which is directly derived from Jensen's identity (see Section 3). As will be discussed in Remark 3, our complex analysis result is sometimes weaker than that of [1], but has the advantage that it enables us to derive explicit expressions for the constants involved. This allows us to derive the explicit form of the constant C in (1) and (2), and their dependence on the parameters γ and p.

Some results for unbounded operators H, in the case that H is selfadjoint and lower semi-bounded, were obtained previously in [3], by the perturbation-determinant method. These bounds were given in terms of Schatten norms of the semigroup difference $e^{-tH} - e^{-tH_0}$. In [7] it was shown that inequalities *stronger* than those obtained in [3] can be proven by a completely different method, based on the spectral shift function (and thus restricted to selfadjoint operators). Here we see that the perturbation-determinant method finds its natural place in the study of non-selfadjoint operators. In this work we develop results in terms of the resolvent difference rather than the semigroup difference, but we note that it is also possible to derive results in terms of semigroup differences, using the same procedure of reduction to the bounded case.

In the final section of this paper, we construct a counterexample which demonstrates that our results are sharp in a certain sense.

2. Preliminaries

For a seperable complex Hilbert space $\mathcal{H}$ let $\mathbf{C}(\mathcal{H})$ and $\mathbf{B}(\mathcal{H})$ denote the closed and bounded linear operators on $\mathcal{H}$ respectively. We denote the ideal of all compact operators on $\mathcal{H}$ by $\mathbf{S}_\infty$ and the ideal of all Schatten class operators by $\mathbf{S}_p, p > 0$, i.e. a compact operator $C \in \mathbf{S}_p$ if

$$\|C\|^p_{\mathbf{S}_p} = \sum_{n=0}^{\infty} \mu_n(C)^p < \infty \tag{3}$$

where $\mu_n(C)$ denotes the n-th singular value of C. Suppose that $A, B \in \mathbf{B}(\mathcal{H})$ and $B - A \in \mathbf{S}_p$ for some real $p > 0$. Since $B - A \in \mathbf{S}_{\lceil p \rceil}$, where

$$\lceil p \rceil = \min\{n \in \mathbb{N} : n \geq p\},$$

the $\lceil p \rceil$-regularized perturbation-determinant of A by $B - A$ is well defined as

$$h^{(p)}_{A,B}(z) = \det_{\lceil p \rceil}(I - (z - A)^{-1}(B - A)), \tag{4}$$

and is analytic on $\varrho(A)$, the resolvent set of A. Furthermore, $z_0 \in \varrho(A)$ is an eigenvalue of B of algebraic multiplicity k_0 if and only if z_0 is a zero of $h^{(p)}_{A,B}$ of the same multiplicity. For more information on regularized perturbation-determinants we refer to the books by Dunford and Schwartz [4], Gohberg and Krein [6] or Simon [12]. Note that

$$\lim_{z \to \infty} h^{(p)}_{A,B}(z) = 1 \tag{5}$$

and

$$|h^{(p)}_{A,B}(z)| \leq \exp\left(\Gamma_p \|(z - A)^{-1}(B - A)\|^p_{\mathbf{S}_p}\right), \tag{6}$$

where Γ_p is some positive constant, see [4, page 1106]. We remark that $\Gamma_p = \frac{1}{p}$ for $p \leq 1$, which is a direct consequence of the definition of the determinant, $\Gamma_2 = \frac{1}{2}$ and $\Gamma_p \leq e(2 + \log p)$ in general, see [11, Simon].

Remark 1. Assuming $B - A \in \mathbf{S}_\infty$, we have $\sigma_{ess}(A) = \sigma_{ess}(B)$. Let G be the unbounded component of $\mathbb{C} \setminus \sigma_{ess}(A)$. Then $G \cap \sigma(B) \subset \sigma_{disc}(B)$ and all possible limit points of this set lie in $\sigma_{ess}(A)$. Here the discrete spectrum $\sigma_{disc}(B)$ consists of those eigenvalues of B that have finite algebraic multiplicity and are isolated from the rest of the spectrum. As a general reference for the mentioned definitions and results we refer to the book of Davies [2, Chapter 4.3], see Theorem 4.3.18 in particular.

3. Some complex analysis results

We have seen that the zeroes of the analytic function $h^{(p)}_{A,B}$ play an important role in the study of the eigenvalues of B in $\mathbb{C} \setminus \sigma(A)$. In the following, we use Jensen's identity to obtain some general results on the distribution of zeroes of functions analytic in the open unit disk U.

Lemma 1. *Let $\varphi \in C^2(0,1)$ be a nonnegative, nonincreasing function with*

$$\lim_{r \to 1} \varphi(r) = \lim_{r \to 1} \varphi'(r) = 0 \tag{7}$$

that obeys

$$\operatorname{supp}\left([r\varphi'(r)]'\right)_- \subset [0,1) \quad \text{and} \quad \sup_{r \in (0,1)} \left([r\varphi'(r)]'\right)_- < \infty, \tag{8}$$

where $f_- = -\min(f, 0)$ is the negative part of a function f. Let $h : U \to \mathbb{C}$ be an analytic function with $h(0) = 1$. Then

$$\sum_{z \in U,\, h(z) = 0} \varphi(|z|) = \frac{1}{2\pi} \int_0^1 dr\, [r\varphi'(r)]' \int_0^{2\pi} d\theta \log|h(re^{i\theta})| \tag{9}$$

where in the sum each zero of h is counted according to its multiplicity.

Remark 2. A short calculation shows that $\varphi_1(r) = |\log(r)|^\gamma$, $\varphi_2(r) = (1-r)^\gamma$ and $\varphi_3(r) = (r^{-1} - r)^\gamma$ fulfill the above assumptions in case that $\gamma > 1$.

Proof. Jensen's identity states that

$$0 \leq \int_0^r ds\, s^{-1} n(h; U_s) = \frac{1}{2\pi} \int_0^{2\pi} d\theta\, \log|h(re^{i\theta})|, \quad 0 < r < 1 \tag{10}$$

where $n(h; U_s)$ counts the number of zeroes of h (including multiplicities) in the disk of radius s, see e.g. Rudin [10, page 307]. Multiplying both sides of (10) by $[r\varphi'(r)]'$ and integrating over $r \in [0,1]$ leads to

$$\begin{aligned}
& \frac{1}{2\pi} \int_0^1 dr\, [r\varphi'(r)]' \int_0^{2\pi} d\theta\, \log|h(re^{i\theta})| \\
= & \int_0^1 dr\, [r\varphi'(r)]' \int_0^r ds\, s^{-1} n(h; U_s) \\
\overset{(\star)}{=} & \int_0^1 ds\, s^{-1} n(h; U_s) \int_s^1 dr\, [r\varphi'(r)]' = -\int_0^1 ds\, \varphi'(s) n(h; U_s) \\
= & \int_0^\infty dt\, \left[\frac{d}{dt}\varphi(e^{-t})\right] n(h; U_{e^{-t}}). \qquad (11)
\end{aligned}$$

The application of Fubini's theorem in $(\star)$ is allowed by assumption (8). We recall the layer cake representation, see Lieb and Loss [8, Theorem 1.13].

Lemma 2. *Let ν be a Borel measure on $\mathbb{R}_+$ such that*

$$\Psi(t) = \nu([0,t))$$

is finite for every $t > 0$. Then for any Borel measure μ on $\mathbb{C}$ and any μ-measureable nonnegative function f

$$\int_{\mathbb{C}} \Psi(f(z))\mu(dz) = \int_0^\infty \mu(\{z : f(z) > t\})\nu(dt).$$

Applying the layer cake representation to the point measure

$$\mu_h(\{z\}) = \begin{cases} m(h; z) & , h(z) = 0,\ z \in U \\ 0 & , \text{else} \end{cases}$$

where $m(h; z)$ counts the multiplicity of a zero z of h, and to the measure

$$\nu_\varphi(dt) = \left[\frac{d}{dt}\varphi(e^{-t})\right] dt$$

we can reformulate the RHS of (11) as follows

$$\begin{aligned}
& \int_0^\infty dt\, \left[\frac{d}{dt}\varphi(e^{-t})\right] n(h; U_{e^{-t}}) \\
= & \int_0^\infty \nu_\varphi(dt)\, \mu_h(\{z : -\log|z| > t\}) = \int_{\mathbb{C}} \nu_\varphi([0, -\log|z|))\mu_h(dz) \\
= & \sum_{z \in U,\, h(z)=0} \int_0^{-\log|z|} dt\, \left[\frac{d}{dt}\varphi(e^{-t})\right] = \sum_{z \in U,\, h(z)=0} \varphi(|z|).
\end{aligned}$$

Now (11) yields the result. □

In order to find conditions on h and φ that ensure the RHS of (9) to be finite, we introduce the following class of functions.

Definition 1. Let $M(E,\alpha,\beta)$ denote the set of all functions $m : U \to \mathbb{R}_+$ that obey an estimate of the form

$$m(z) \leq \frac{C_0}{(1-|z|)^\alpha \operatorname{dist}(z,E)^\beta} \tag{12}$$

where $C_0 > 0$ and $E \subset \partial U$ is any finite subset of the unit circle.

We present a result on the finiteness of the RHS of (9) in case that $\log|h(z)| \in M(E,\alpha,\beta)$ for some $\alpha,\beta \geq 0$. We do not try to present the most general result in terms of the function φ but will restrict ourselves to one particular choice, namely $\varphi(r) = (1-r)^\gamma, \gamma > 1$.

Lemma 3. *Let $m \in M(E,\alpha,\beta)$ for some $\alpha,\beta \geq 0$ and some finite $E \subset \partial U$. Let $h : U \to \mathbb{C}$ be analytic with $h(0) = 1$ and*

$$|h(z)| \leq \exp(m(z)). \tag{13}$$

Then for every $\gamma > \max(1+\alpha, \alpha+\beta)$

$$\sum_{z\in U,\ h(z)=0} (1-|z|)^\gamma \leq C_\gamma(m) \tag{14}$$

where each zero of h is counted according to its multiplicity and

$$C_\gamma(m) = \frac{\gamma}{2\pi}\int_{\frac{1}{\gamma}}^1 dr \int_0^{2\pi} d\theta\, \frac{(r\gamma-1)}{(1-r)^{2-\gamma}} m(re^{i\theta}) \tag{15}$$

is a finite constant.

Remark 3. In [1] it has been shown that condition (13) implies that for every $\varepsilon > 0$

$$\sum_{z\in U,\ h(z)=0} (1-|z|)^{\alpha+1+\varepsilon} \operatorname{dist}(z,E)^{(\beta-1+\varepsilon)_+} \leq C, \tag{16}$$

where C depends on α,β,ε and m in a way that is not made explicit. In case that $\beta < 1$ the finiteness of the LHS of (16) for every $\varepsilon > 0$ is equivalent to the finiteness of the LHS of (14) for every $\gamma > 1+\alpha$, whereas in case that $\beta \geq 1$ the finiteness of (16) implies the finiteness of (14), but not vice versa. In this respect, the result obtained in [1] is stronger than ours.

Proof. For $\gamma > 1$ let $\varphi(r) = (1-r)^\gamma$. Since

$$[r\varphi'(r)]' = \gamma(1-r)^{\gamma-2}(r\gamma-1)$$

we obtain from (9) and our assumptions, using the non-negativity of $\int_0^{2\pi}\log|h(re^{i\theta})|d\theta$, which follows from (10),

$$\begin{aligned}
\sum_{z\in U,\ h(z)=0} (1-|z|)^\gamma &= \frac{\gamma}{2\pi}\int_0^1 dr \frac{(r\gamma-1)}{(1-r)^{2-\gamma}}\int_0^{2\pi} d\theta \log|h(re^{i\theta})| \\
&\leq \frac{\gamma}{2\pi}\int_{1/\gamma}^1 dr \frac{(r\gamma-1)}{(1-r)^{2-\gamma}}\int_0^{2\pi} d\theta\, m(re^{i\theta}) \\
&\leq \frac{C_0\gamma(\gamma-1)}{2\pi}\int_{1/\gamma}^1 dr \frac{1}{(1-r)^{2-\gamma+\alpha}}\int_0^{2\pi} \frac{d\theta}{\operatorname{dist}(re^{i\theta},E)^\beta}.
\end{aligned} \tag{17}$$

It remains to show that the integral on the RHS of (17) is finite whenever $\gamma > \max(1+\alpha, \alpha+\beta)$. To this end we denote $E = \{e^{i\theta_1}, \ldots, e^{i\theta_n}\}$ where $0 \leq \theta_1 < \ldots < \theta_n < 2\pi$. Let

$$\delta = \frac{1}{4} \min_{1 \leq k \leq n} |e^{i\theta_{k+1}} - e^{i\theta_k}| \quad , \theta_{n+1} := \theta_1.$$

We further define

$$G_k = \{\theta \in [0, 2\pi] : |e^{i\theta} - e^{i\theta_k}| < \delta\}, \quad k = 1, \ldots, n.$$

Since for $r \geq 0$

$$\sup_{\theta \notin \cup_k G_k} \operatorname{dist}(re^{i\theta}, E) \geq C > 0,$$

the integral

$$\int_{1/\gamma}^1 dr \frac{1}{(1-r)^{2-\gamma+\alpha}} \int_{\theta \notin \cup_k G_k} \frac{d\theta}{\operatorname{dist}(re^{i\theta}, E)^\beta}$$

is finite whenever $\gamma > \alpha + 1$. It remains to show the finiteness of

$$\int_{1/\gamma}^1 dr \frac{1}{(1-r)^{2-\gamma+\alpha}} \int_{\cup_k G_k} \frac{d\theta}{\operatorname{dist}(re^{i\theta}, E)^\beta}. \tag{18}$$

But for $\theta \in G_k$

$$\operatorname{dist}(re^{i\theta}, E) = |re^{i\theta} - e^{i\theta_k}|$$

and hence

$$\int_{\cup_k G_k} \frac{d\theta}{\operatorname{dist}(re^{i\theta}, E)^\beta} \quad = \quad \sum_k \int_{G_k} \frac{d\theta}{|re^{i\theta} - e^{i\theta_k}|^\beta}. \tag{19}$$

It is not difficult to show that as $r \to 1$

$$\int_{G_k} \frac{d\theta}{|re^{i\theta} - e^{i\theta_k}|^\beta} = \begin{cases} O\left(\frac{1}{(1-r)^{\beta-1}}\right), & \beta > 1 \\ O\left(-\log(1-r)\right), & \beta = 1 \\ O\left(1\right), & \beta < 1. \end{cases} \tag{20}$$

We skip the elementary but technical calculation. Estimates (19) and (20) show that the integral in (18) is finite whenever $\gamma > \max(1+\alpha, \alpha+\beta)$. $\square$

Remark 4. In case that $m \in M(E, 0, \beta)$ for $\beta < 1$, the above proof actually shows that h is element of the Nevanlinna class, i.e.

$$\sup_{0<r<1} \int_0^{2\pi} \log_+ |h(re^{i\theta})| \, d\theta < \infty.$$

As is well-known this implies the stronger result that $\sum_{z \in U,\, h(z)=0} (1-|z|) < \infty$, see e.g. Rudin [10, page 311].

We conclude this section with the classical Koebe distortion theorem, which will be used later.

Theorem 1. *Let $f : U \to \mathbb{C}$ be conformal. Then*

$$\frac{1}{4}|f'(z)|(1-|z|^2) \leq \operatorname{dist}(f(z), \partial f(U)) \leq |f'(z)|(1-|z|^2), \quad z \in U.$$

For a proof we refer to Pommerenke [9, Cor. 1.4].

4. EIGENVALUE ESTIMATES FOR BOUNDED OPERATORS

Let $A, B \in \mathbf{B}(\mathcal{H})$. Assume that A is selfadjoint with

$$\sigma(A) = [a, b], \quad a < b$$

and

$$B - A \in \mathbf{S}_p \quad \text{for some } p > 0.$$

The last assumption and Remark 1 imply that

$$\sigma(B) \cap (\mathbb{C} \setminus [a, b]) = \sigma_{disc}(B).$$

To obtain information on $\sigma_{disc}(B)$ we define a conformal map $k : \hat{\mathbb{C}} \setminus [a, b] \to U$ as follows:

$$k = k_{a,b} = w^{-1} \circ g, \tag{21}$$

where $g : \hat{\mathbb{C}} \setminus [a, b] \to \hat{\mathbb{C}} \setminus [-1, 1]$ with

$$g(z) = \frac{1}{b-a}\left(2z - (b + a)\right), \quad g^{-1}(z) = \frac{1}{2}\left((b - a)z + (b + a)\right)$$

and $w : U \to \hat{\mathbb{C}} \setminus [-1, 1]$ with

$$w(z) = \frac{1}{2}\left(z + z^{-1}\right), \quad w^{-1}(z) = z - \sqrt{z^2 - 1}. \tag{22}$$

With $h^{(p)}_{A,B}$ as defined in (4) the composition

$$[h^{(p)}_{A,B} \circ k^{-1}](z) = \det\nolimits_{\lceil p \rceil}(I - (k^{-1}(z) - A)^{-1}(B - A))$$

is analytic on U, by (5) we have $[h^{(p)}_{A,B} \circ k^{-1}](0) = 1$, and z_0 is a zero of this function if and only if $k^{-1}(z_0)$ is an eigenvalue of B of the same multiplicity. Since

$$|[h^{(p)}_{A,B} \circ k^{-1}](z)| \leq \exp\left(\Gamma_p \|(k^{-1}(z) - A)^{-1}(B - A)\|^p_{\mathbf{S}_p}\right)$$

by estimate (6), the following theorem is a direct consequence of Lemma 3.

Theorem 2. *Let $m \in M(E, \alpha, \beta)$ for some finite $E \subset \partial U$, $\alpha, \beta \geq 0$ and suppose that for $z \in U$*

$$\|(k^{-1}(z) - A)^{-1}(B - A)\|^p_{\mathbf{S}_p} \leq m(z). \tag{23}$$

Then for $\gamma > \max(1 + \alpha, \alpha + \beta)$

$$\sum_{\lambda \in \sigma_{disc}(B)} (1 - |k(\lambda)|)^\gamma \leq \Gamma_p C_\gamma(m) \tag{24}$$

where the finite constant $C_\gamma(m)$ was defined in (15).

Remark 5. In the summation on the LHS of (24), each eigenvalue of B is counted according to its algebraic multiplicity. In the following results in this paper, this will be taken for granted whenever a sum involving eigenvalues is considered.

If no further information on the operators A and B is available, the obvious way to show the validity of (23) is to use the estimate

$$\|(k^{-1}(z) - A)^{-1}(B - A)\|^p_{\mathbf{S}_p} \leq \|(k^{-1}(z) - A)^{-1}\|^p \|B - A\|^p_{\mathbf{S}_p} \tag{25}$$

and the identity (here we use the assumption that A is selfadjoint)

$$\|(k^{-1}(z)-A)^{-1}\| = \frac{1}{\mathrm{dist}(k^{-1}(z),[a,b])} = \frac{2}{b-a}\frac{1}{\mathrm{dist}(w(z),[-1,1])}. \tag{26}$$

The proof of the following Lemma is provided in the appendix.

Lemma 4. *Let $w(z)$ be defined by (22). For $z \in U$, we have*

$$\frac{1}{4}\frac{|z^2-1|(1-|z|)}{|z|} \leq \mathrm{dist}(w(z),[-1,1]) \leq \frac{1+\sqrt{2}}{4}\frac{|z^2-1|(1-|z|)}{|z|}.$$

Theorem 3. *For $\gamma > \max(1+p, 2p)$ and $k = k_{a,b}$ as above we have*

$$\sum_{\lambda\in\sigma_{disc}(B)} (1-|k(\lambda)|)^\gamma \leq \Gamma_p \left(\frac{2}{b-a}\right)^p C_{\gamma,p}\|B-A\|_{\mathbf{S}_p}^p,$$

where

$$C_{\gamma,p} = \frac{\gamma}{2\pi}\int_{\frac{1}{\gamma}}^1 dr \frac{(r\gamma-1)}{(1-r)^{2-\gamma}}\int_0^{2\pi} d\theta\, \frac{1}{\mathrm{dist}(w(re^{i\theta}),[-1,1])^p} \tag{27}$$

is a finite constant.

Proof. From (25) and (26) we obtain

$$\|(k^{-1}(z)-A)^{-1}(B-A)\|_{\mathbf{S}_p}^p \leq \left(\frac{2}{b-a}\right)^p \|B-A\|_{\mathbf{S}_p}^p \frac{1}{\mathrm{dist}(w(z),[-1,1])^p}.$$

Since $z \mapsto (\mathrm{dist}(w(z),[-1,1]))^{-p} \in M(\{-1,1\}, 1+p, 2p)$ by Lemma 4, we obtain from Theorem 2 that for $\gamma > \max(1+p,2p)$

$$\sum_{\lambda\in\sigma_{disc}(B)} (1-|k(\lambda)|)^\gamma \leq \Gamma_p \left(\frac{2}{b-a}\right)^p C_{\gamma,p}\|B-A\|_{\mathbf{S}_p}^p.$$

Here the finite constant $C_{\gamma,p}$ is defined as in (27). □

Lemma 4 can be used to obtain a more transparent formulation of Theorem 3.

Theorem 4. *Let $\gamma > \max(1+p,2p)$. Then*

$$\sum_{\lambda\in\sigma_{disc}(B)} \frac{\mathrm{dist}(\lambda,[a,b])^\gamma}{|\lambda-a|^{\frac{\gamma}{2}}|\lambda-b|^{\frac{\gamma}{2}}} \leq \Gamma_p\left(\frac{2}{b-a}\right)^p\left(\frac{1+\sqrt{2}}{2}\right)^\gamma C_{\gamma,p}\|B-A\|_{\mathbf{S}_p}^p$$

where the finite constant $C_{\gamma,p}$ was defined in (27).

Proof. From Lemma 4 we get for $z = k(\lambda) = w^{-1}(g(\lambda))$

$$\begin{aligned}
\mathrm{dist}(\lambda,[a,b]) &= \mathrm{dist}(g^{-1}(w(z)),[a,b]) \\
&= \frac{b-a}{2}\mathrm{dist}(w(z),[-1,1]) \leq \frac{b-a}{2}\frac{1+\sqrt{2}}{4}\frac{|z^2-1|}{|z|}(1-|z|) \\
&= \frac{b-a}{2}\frac{1+\sqrt{2}}{4}\frac{|k(\lambda)^2-1|}{|k(\lambda)|}(1-|k(\lambda)|) \\
&= \frac{1+\sqrt{2}}{2}|(\lambda-a)(\lambda-b)|^{1/2}(1-|k(\lambda)|),
\end{aligned}$$

so that

$$(1-|k(\lambda)|)^\gamma \geq \left(\frac{2}{1+\sqrt{2}}\frac{\mathrm{dist}(\lambda,[a,b])}{|(\lambda-a)(\lambda-b)|^{\frac{1}{2}}}\right)^\gamma,$$

and an application of Theorem 3 concludes the proof. □

5. Eigenvalue estimates for unbounded operators

Let $H_0, H \in \mathbf{C}(\mathcal{H})$ and suppose that H_0 is selfadjoint with $\sigma(H_0) = [0,\infty)$. To apply the results of the last section, we assume that

$$R_s = (s-H)^{-1} - (s-H_0)^{-1} \in \mathbf{S}_p \tag{28}$$

for some $p > 0$ and $s \in \varrho(H_0) \cap \varrho(H) \cap \mathbb{R}_-$. The last assumption, together with the spectral mapping theorem for resolvents, implies that

$$\sigma(H) \cap (\mathbb{C} \setminus [0,\infty)) = \sigma_{disc}(H).$$

Remark 6. Given assumption (28) there might exist a sequence of eigenvalues of H that diverges to infinity, i.e. the points of $\sigma_{disc}(H)$ can accumulate in $[0,\infty) \cup \{\infty\}$. However, (28) implies some restrictions on the rate of divergence as can be seen from the next theorem.

Theorem 5. *Let H_0, H be as above and let $\gamma > \max(1+p, 2p)$. Then*

$$\sum_{\lambda \in \sigma_{disc}(H)} \frac{\operatorname{dist}(\lambda,[0,\infty))^\gamma}{|\lambda|^{\frac{\gamma}{2}}(1+|\lambda|)^\gamma} \leq 2^p C_s^\gamma |s|^{(\frac{\gamma}{2}+p)} \Gamma_p C_{\gamma,p} \|R_s\|_{\mathbf{S}_p}^p,$$

where the finite constant $C_{\gamma,p}$ was defined in (27) and

$$C_s = 4 \sup_{z \in U} \frac{1+|z|}{|z-1|^2 + |s||z+1|^2}. \tag{29}$$

For the proof of this theorem we will need the contents of the next lemma.

Lemma 5. *Let $s \in \mathbb{R}_-$ and define*

$$l_s : \mathbb{C} \setminus [0,\infty) \to U, \quad l_s(\lambda) = k_{s^{-1},0}((s-\lambda)^{-1}), \tag{30}$$

where $k_{s^{-1},0}$ was defined in (21). Then the following holds for $\lambda \in \mathbb{C} \setminus [0,\infty)$

$$\frac{1}{4}\left|\frac{\lambda}{s}\right|^{1/2} \frac{|\lambda - s|(1-|l_s(\lambda)|^2)}{|l_s(\lambda)|} \leq \operatorname{dist}(\lambda,[0,\infty)) \leq \left|\frac{\lambda}{s}\right|^{1/2} \frac{|\lambda - s|(1-|l_s(\lambda)|^2)}{|l_s(\lambda)|}.$$

Proof. We note that by definition

$$l_s(\lambda) = \left(\frac{\lambda+s}{\lambda-s}\right) - \sqrt{\left(\frac{\lambda+s}{\lambda-s}\right)^2 - 1}, \quad l_s^{-1}(z) = s\left(\frac{z+1}{z-1}\right)^2$$

and l_s^{-1} is a conformal map of U onto $\mathbb{C} \setminus [0,\infty)$. For $z = l_s(\lambda)$ we can thus use the Koebe theorem (Theorem 1) to obtain

$$\begin{aligned}
\operatorname{dist}(\lambda,[0,\infty)) &= \operatorname{dist}(l_s^{-1}(z), \partial l_s^{-1}(U)) \leq |[l_s^{-1}]'(z)|(1-|z|^2) \\
&= 4|s|\left|\frac{z^2-1}{(z-1)^4}\right|(1-|z|^2) = 4|s|\left|\frac{l_s(\lambda)^2-1}{(l_s(\lambda)-1)^4}\right|(1-|l_s(\lambda)|^2) \\
&= \left|\frac{\lambda}{s}\right|^{1/2} \frac{|\lambda-s|(1-|l_s(\lambda)|^2)}{|l_s(\lambda)|}.
\end{aligned}$$

Here the last equality follows by some algebraic manipulations. The lower bound on $\operatorname{dist}(\lambda,[0,\infty))$ is obtained in exactly the same manner. ☐

Proof of Theorem 5. Let $\gamma > \max(1+p, 2p)$. Since $\sigma((s-H_0)^{-1}) = [\frac{1}{s}, 0]$ we can apply Theorem 3 to the bounded operators $A = (s-H_0)^{-1}$ and $B = (s-H)^{-1}$ to obtain

$$\sum_{\mu \in \sigma_{disc}((s-H)^{-1})} (1 - |k_{s^{-1},0}(\mu)|)^\gamma \leq \Gamma_p 2^p |s|^p C_{\gamma,p} \|R_s\|^p_{\mathbf{S}_p} \tag{31}$$

where $k_{s^{-1},0}$ and $C_{\gamma,p}$ were defined in (21) and (27) respectively. Since $\mu \in \sigma_{disc}((s-H)^{-1})$ if and only if $s - \frac{1}{\mu} \in \sigma_{disc}(H)$ we can reformulate the LHS of (31) as follows

$$\sum_{\mu \in \sigma_{disc}((s-H)^{-1})} (1 - |k_{s^{-1},0}(\mu)|)^\gamma = \sum_{\lambda \in \sigma_{disc}(H)} (1 - |l_s(\lambda)|)^\gamma, \tag{32}$$

where the function l_s was defined in (30). From Lemma 5 we have

$$1 - |l_s(\lambda)| \geq \left[\frac{|l_s(\lambda)|(1+|\lambda|)}{|\lambda - s|(1+|l_s(\lambda)|)} \right] \left[\frac{|s|^{\frac{1}{2}} \operatorname{dist}(\lambda, [0,\infty))}{|\lambda|^{\frac{1}{2}}(1+|\lambda|)} \right]. \tag{33}$$

Furthermore, a short computation shows that

$$\inf_{\lambda \in \mathbb{C} \setminus [0,\infty)} \frac{|l_s(\lambda)|(1+|\lambda|)}{|\lambda - s|(1+|l_s(\lambda)|)} = \inf_{z \in U} \frac{|z|(1+|l_s^{-1}(z)|)}{|l_s^{-1}(z) - s|(1+|z|)} = \frac{1}{|s|C_s}$$

where $C_s \in (0,\infty)$ was defined in (29). From (33) we thus obtain

$$1 - |l_s(\lambda)| \geq \frac{1}{|s|^{1/2} C_s} \frac{\operatorname{dist}(\lambda, [0,\infty))}{|\lambda|^{1/2}(1+|\lambda|)}.$$

With (31) and (32) this concludes the proof of the theorem. □

6. A COUNTEREXAMPLE

In this section we present a counterexample which shows that, in one respect, Theorem 3 and Theorem 4 are optimal: For $\gamma < 1$ it is not possible to conclude the finiteness of

$$\sum_{\lambda \in \sigma_{disc}(B)} \frac{\operatorname{dist}(\lambda, [a,b])^\gamma}{|\lambda - a|^{\gamma/2} |\lambda - b|^{\gamma/2}}$$

in terms of Schatten class properties of $B - A$.

We work on the space $l^2(\mathbb{Z})$, and denote its natural basis by $\{\delta_j\}_{j \in \mathbb{Z}}$, where δ_j is defined by $\delta_j(j) = 1$, $\delta_j(k) = 0$ for $k \neq j$.

We define $A : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ to be the discrete free Schrödinger operator:

$$(Au)(k) = u(k-1) + u(k+1), \quad u \in l^2(\mathbb{Z}), \quad k \in \mathbb{Z}.$$

The spectrum of A is $[-2,2]$.

Proposition 1. *Given any sequence $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{C} \setminus [-2,2]$ which satisfies*

$$\sum_k \frac{\operatorname{dist}(\lambda_k, [-2,2])}{|\lambda_k + 2|^{1/2} |\lambda_k - 2|^{1/2}} < \infty,$$

there exists a rank-one operator M such that, setting $B = A + M$, we have $\{\lambda_k\} \subset \sigma_{disc}(B)$.

Since we may choose λ_k in Lemma 1 to be, e.g., $\lambda_k = k^{-(1+\delta)} i$, with $\delta > 0$ arbitrarily small, we immediately get

Proposition 2. *For any $\gamma < 1$, there exists a rank-one operator M such that the eigenvalues of $B = A + M$ satisfy*

$$\sum_{\lambda \in \sigma_{disc}(B)} \frac{\mathrm{dist}(\lambda, [-2,2])^\gamma}{|\lambda+2|^{\frac{\gamma}{2}}|\lambda-2|^{\frac{\gamma}{2}}} = +\infty.$$

Since a rank-one perturbation belongs to all Schatten classes $\mathbf{S}_p$, $p > 0$, this shows that there is no hope to obtain the results of Theorem 3 and Theorem 4 for $\gamma < 1$, under an assumption of the form $B - A \in \mathbf{S}_p$.

We also note that the above propositions demonstrate a striking difference between the selfadjoint and non-selfadjoint cases. Given a selfadjoint operator with no eigenvalues, it is well-known that a selfadjoint rank-one perturbation of this operator can have at most one eigenvalue outside of its essential spectrum. Here we see that a non-selfadjoint rank-one perturbation of a selfadjoint operator can give birth to infinitely many eigenvalues.

Proof of Proposition 1. The rank-one perturbation M is defined by:

$$Mu = \Big[\sum_{j=-\infty}^{\infty} \alpha_j u(j)\Big]\delta_0, \quad u \in l^2(\mathbb{Z}),$$

where α_j are to be determined below. For M to be bounded, we need to assume that

$$\sum_{j=-\infty}^{\infty} |\alpha_j|^2 < \infty. \tag{34}$$

We now look for eigenvectors $u_z \in l^2(\mathbb{Z})$ of $B = A + M$ of the form

$$u_z(k) = z^{|k|},$$

with $|z| < 1$. Note that

$$|k| \geq 1 \ \Rightarrow \ (Bu_z)(k) = z^{|k|}(z^{-1} + z) \tag{35}$$

$$(Bu_z)(0) = 2z + \sum_{j=-\infty}^{\infty} \alpha_j z^{|j|} = \alpha_0 + (\alpha_1 + \alpha_{-1} + 2)z + \sum_{j=2}^{\infty}(\alpha_j + \alpha_{-j})z^j. \tag{36}$$

By (35), we see that if u_z is an eigenvector then the corresponding eigenvalue is $\lambda = z + z^{-1}$. From (36) we then get that a necessary and sufficient condition for u_z to be an eigenvector is that

$$\alpha_0 + (\alpha_1 + \alpha_{-1} + 2)z + \sum_{j=2}^{\infty}(\alpha_j + \alpha_{-j})z^j = \lambda = z + z^{-1},$$

which we can write as $\phi(z) = 0$ where $\phi(z)$ is defined by

$$\phi(z) = -1 + \alpha_0 z + (\alpha_1 + \alpha_{-1} + 1)z^2 + \sum_{j=3}^{\infty}(\alpha_{j-1} + \alpha_{-j+1})z^j. \tag{37}$$

Thus the numbers of the form $\lambda = z + z^{-1}$, where z are the zeroes of ϕ in U, are eigenvalues of B. Note that by assumption (34), $\phi(z) \in H^2(U)$.

Let $\{\lambda_k\} \subset \mathbb{C} \setminus [-2,2]$ be any sequence that satisfies

$$\sum_{k=1}^{\infty} \frac{\operatorname{dist}(\lambda_k, [-2,2])}{|\lambda_k^2 - 4|^{1/2}} < \infty. \tag{38}$$

In the following, we will select a specific sequence $\{\alpha_j\}$ such that $\{\lambda_k\} \subset \sigma_{disc}(B)$, where $B = B(\{\alpha_j\})$ as defined above. To this end, we define the sequence $\{z_k\} \subset U \setminus \{0\}$ by

$$\lambda_k = z_k + z_k^{-1}.$$

As in the proof of Theorem 4 one can use Lemma 4 to check that condition (38) on λ_k is equivalent to

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty. \tag{39}$$

By a well-known result from complex analysis, see e.g. Rudin [10, page 310], (39) implies that one can construct a function $g \in H^2(U)$ (in fact even $g \in H^\infty(U)$) whose zeroes are $\{z_k\}$.

We can normalize g so that $g(0) = -1$. Denoting the Taylor expansion of g by

$$g(z) = -1 + \sum_{j=1}^{\infty} \beta_j z^j,$$

we can choose $\alpha_0 = \beta_1$, $\alpha_1 = \beta_2 - 1$, $\alpha_j = \beta_{j+1}$ for $j \geq 2$ and $\alpha_j = 0$ for $j < 0$, so that from (37) we obtain $\phi = g$. From the considerations above, this implies that $\lambda_k = z_k + z_k^{-1}$ are eigenvalues of B. We have thus proven Proposition 1. $\square$

7. Appendix

Proof of Lemma 4. For $w = \frac{1}{2}\left(z + z^{-1}\right)$ we define

$$Z_1 = \{z : \operatorname{Re} w \leq -1\}, \quad Z_2 = \{z : \operatorname{Re} w \geq 1\}, \quad Z_3 = \{z : |\operatorname{Re} w| < 1\}$$

where $\operatorname{Re} w = \frac{\operatorname{Re} z}{2}\left(\frac{1+|z|^2}{|z|^2}\right)$. Then

$$\operatorname{dist}(w, [-1,1]) = \begin{cases} |w+1| = \frac{1}{2}\frac{|1+z|^2}{|z|} & , z \in Z_1 \\ |w-1| = \frac{1}{2}\frac{|1-z|^2}{|z|} & , z \in Z_2 \\ |\operatorname{Im} w| = \frac{|\operatorname{Im} z|}{2}\frac{1-|z|^2}{|z|^2} & , z \in Z_3. \end{cases} \tag{40}$$

We first show that for $z \in Z_3$ the following holds

$$\frac{\sqrt{2}}{4}\frac{|z^2-1|(1-|z|)}{|z|} \leq \operatorname{dist}(w, [-1,1]) \leq \frac{1+\sqrt{2}}{4}\frac{|z^2-1|(1-|z|)}{|z|}. \tag{41}$$

With (40) this is equivalent to

$$\frac{1}{\sqrt{2}} \leq |\operatorname{Im} z| \frac{1+|z|}{|z||z^2-1|} \leq \frac{1+\sqrt{2}}{2}. \tag{42}$$

Switching to polar coordinates we see that $re^{i\theta} \in Z_3$ if $\cos^2(\theta) < 4\frac{r^2}{(1+r^2)^2}$ and (42) can be rewritten as follows

$$\frac{1}{\sqrt{2}} \leq \frac{(1+r)\sqrt{1-\cos^2(\theta)}}{\sqrt{(1+r^2)^2 - 4r^2\cos^2(\theta)}} \leq \frac{1+\sqrt{2}}{2}. \tag{43}$$

For $x = \cos^2(\theta)$ and fixed r we define

$$f(x) = \frac{1-x}{(1+r^2)^2 - 4r^2 x} \quad , 0 \leq x < 4\frac{r^2}{(1+r^2)^2}.$$

It is easy to see that f is monotonically decreasing. We thus obtain

$$\frac{1}{1+6r^2+r^4} = f\left(4\frac{r^2}{(1+r^2)^2}\right) \leq f(x) \leq f(0) = \frac{1}{(1+r^2)^2}.$$

The last chain of inequalities implies the validity of (43) and (42) since

$$\sup_{r\in[0,1]} \frac{1+r}{1+r^2} = \frac{1+\sqrt{2}}{2} \quad \text{and} \quad \inf_{r\in[0,1]} \frac{1+r}{\sqrt{1+6r^2+r^4}} = \frac{1}{\sqrt{2}}.$$

Next, we show that for $z \in Z_1 \cup Z_2$

$$\frac{1}{4}\frac{|z^2-1|(1-|z|)}{|z|} \leq \operatorname{dist}(w,[-1,1]) \leq \frac{1+\sqrt{2}}{4}\frac{|z^2-1|(1-|z|)}{|z|}.$$

By symmetry, it is sufficient to show it for $z \in Z_1$, i.e. we have to show

$$\frac{1}{4}\frac{|z^2-1|(1-|z|)}{|z|} \leq \frac{1}{2}\frac{|z+1|^2}{|z|} \leq \frac{1+\sqrt{2}}{4}\frac{|z^2-1|(1-|z|)}{|z|} \quad , z \in Z_1. \tag{44}$$

In polar coordinates this is equivalent to

$$\frac{1}{2} \leq \frac{1}{1-r}\sqrt{\frac{r^2+1+2r\cos(\theta)}{r^2+1-2r\cos(\theta)}} \leq \frac{1+\sqrt{2}}{2} \tag{45}$$

for $\cos(\theta) \leq -2\frac{r}{1+r^2}$. For $y = \cos(\theta)$ and fixed r we define

$$q(y) = \frac{r^2+1+2ry}{r^2+1-2ry} \quad , -1 \leq y \leq -2\frac{r}{1+r^2}. \tag{46}$$

A short calculation shows that q is monotonically increasing and we obtain that

$$\left(\frac{1-r}{1+r}\right)^2 = q(-1) \leq q(y) \leq q\left(-2\frac{r}{1+r^2}\right) = \frac{(1-r^2)^2}{1+6r^2+r^4}. \tag{47}$$

(46) and (47) imply the validity of (45) and (44) since

$$\inf_{r\in[0,1]} \frac{1}{1+r} = \frac{1}{2} \quad \text{and} \quad \sup_{r\in[0,1]} \frac{1+r}{\sqrt{1+6r^2+r^4}} \leq \frac{1+\sqrt{2}}{2}. \tag{48}$$

This concludes the proof of the lemma. □

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