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Lecture 4 Notes

These notes correspond to Section 5.3 in the text.

Operators

There are many applications in which some function is applied to a vector $\mathbf{v}$ in a given vector space V to obtain another vector $\mathbf{w}$ that is also in V. That is, the *domain* and *range* of the function are the same–the vector space V. Such a function is called an *operator* on V. We will be studying operators on Hilbert spaces. If f is a function in a Hilbert space $\mathcal{H}$, and A is an operator on $\mathcal{H}$, then the function $g \in \mathcal{H}$ obtained by applying A to f is denoted by $g = Af$.

In particular, we will be interested in *linear* operators on a Hilbert space. If A and B are linear operators on a Hilbert space $\mathcal{H}$, then, for all functions f and g in $\mathcal{H}$, and any scalar c, the operators A and B satisfy

$$(A + B)f = Af + Bf, \quad A(f + g) = Af + Ag, \quad A(cf) = c(Af).$$

A particularly useful type of linear operator is a *differential operator*, which is an operator that involves differentiation and multiplication by specific functions, known as *coefficients*.

Example An example of a linear differential operator is

$$\mathcal{L}y = x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y,$$

where n is an integer. We say that $\mathcal{L}$ is a *variable-coefficient* operator because the coefficients, x^2, x and $x^2 - n^2$, are not constants.

This operator allows the well-known *Bessel's equation* to be described very concisely as

$$\mathcal{L}y = 0$$

where $\mathcal{L}$ is the linear operator described above. It can be verified directly that this operator is linear, using the linearity of the *differentiation operator* and the distributive property of multiplication of functions. □

Commutation of Operators

If two operators A and B are applied in succession to a function f, it is important to keep in mind that the order in which the operators are applied is relevant. That is, it is generally not the case that $ABf = BAf$. When this is true for *every* vector f in the Hilbert space H on which A and B are defined, then we say that A and B *commute*. Equivalently, A and B commute if and only if the operator

$$[A, B] = AB - BA,$$

known as the *commutator* of A and B, is equal to zero.

Example Consider the operators

$$Ay = \frac{dy}{dx}, \quad By = xy.$$

Then, using the Product Rule for differentiation, we obtain

$$\begin{aligned}
[A,B]y &= ABy - BAy \\
&= \frac{d}{dx}[xy] - x\frac{dy}{dx} \\
&= y + x\frac{dy}{dx} - x\frac{dy}{dx} \\
&= y.
\end{aligned}$$

That is, while A and B both change the function y to which they are applied, their commutator does not. □

The following properties are useful for working with commutators:

$$[A,B] = -[B,A], \quad [A,B+C] = [A,B] + [A,C], \quad c[A,B] = [cA,B] = [A,cB].$$

Example Let $A = \frac{d}{dx}$ and $B = x$ as in the previous example. Then

$$B^2y = B(By) = x(xy) = x^2y,$$

and

$$\begin{aligned}
[A,B^2] &= AB^2 - B^2A \\
&= AB^2 - BAB + BAB - B^2A \\
&= [A,B]B + B[A,B] \\
&= B + B \\
&= 2B.
\end{aligned}$$

That is,

$$[A,B^2]y = 2xy.$$

□

Identity, Inverse, Adjoint

In the preceding discussion, we observed that for $A = \frac{d}{dx}$ and $B = x$, that

$$[A,B]y = y$$

for any differentiable function y. In other words, $[A,B]$ is the *identity operator* I, defined by

$$Iy = y$$

for any function y in the Hilbert space $\mathcal{H}$ under consideration.

An operator A has an *inverse* if there exists an operator B such that

$$AB = BA = I.$$

This operator B is the inverse of A, and is denoted by A^{-1}. We also say that A is *invertible*. Not every operator has an inverse. In particular, if there exists any nonzero function y such that $Ay = 0$, then A cannot have an inverse, because if it did, then we would have $A^{-1}Ay = A^{-1}(0) = 0$ and $AA^{-1}y = Iy = y \neq 0$, which is a contradiction.

Given an operator A, a particularly useful operator is the *adjoint* of A, which is denoted by $A^\dagger$. Many texts denote the adjoint by A^*. It is defined to be the unique operator such that for all functions f and g in the Hilbert space on which A is defined,

$$\langle f|Ag\rangle = \langle A^\dagger f|g\rangle.$$

That is, the value of the scalar product is is the same regardless of whether A is applied to the right member of the scalar product or $A^\dagger$ is applied to the left member. While an operator may not have an inverse, it always has an adjoint.

Adjoints have the following properties:

- $(A^\dagger)^\dagger = A$. That is, the adjoint of the adjoint of A is A.
- $(AB)^\dagger = B^\dagger A^\dagger$. That is, the adjoint of a composition of operators is the composition of the adjoints, in reverse order.

Note that similar properties apply to adjoints of matrices, as they are linear operators on physical vector spaces. To see that the first property holds, note that

$$\langle f|A^\dagger g\rangle = \langle (A^\dagger)^\dagger f|g\rangle$$

but

$$\langle f|A^\dagger g\rangle = \langle A^\dagger g|f\rangle^* = \langle g|Af\rangle^* = \langle Af|g\rangle,$$

which proves that $(A^\dagger)^\dagger = A$.

The following definitions arise from the adjoint:

- An operator A is *self-adjoint*, or *Hermitian*, if $A = A^\dagger$.
- A is *anti-Hermitian* if $A^\dagger = -A$.
- A is *unitary* if A is invertible, and $A^\dagger = A^{-1}$. If A is a real operator, then A is also called *orthogonal*.

Example Consider the Hilbert space $\mathcal{H}$ consisting of twice continuously differentiable, 2π-periodic functions, denoted by $C_p^2[0, 2\pi]$, with the scalar product

$$\langle f|g\rangle = \int_0^{2\pi} f^*(s)g(s)\,ds.$$

Let $A = \frac{d^2}{dx^2}$. Then for functions $f, g \in C_p[0, 2\pi]$, we have, from the periodicity of f and g and integration by parts,

$$\begin{aligned}
\langle f|Ag\rangle &= \langle f|g''\rangle \\
&= \int_0^{2\pi} f^*(s)g''(s)\,ds \\
&= f^*(s)g'(s)\Big|_0^{2\pi} - \int_0^{2\pi} (f^*)'(s)g'(s)\,ds \\
&= -\int_0^{2\pi} (f^*)'(s)g'(s)\,ds \\
&= -(f^*)'(s)g(s)\Big|_0^{2\pi} + \int_0^{2\pi} (f^*)''(s)g(s)\,ds \\
&= \int_0^{2\pi} (f^*)''(s)g(s)\,ds \\
&= \langle Af|g\rangle.
\end{aligned}$$

We conclude that A is self-adjoint. $\Box$

It is important to keep in mind that the adjoint of an operator depends on a scalar product being used, and the Hilbert space on which it is defined. Therefore, an operator may be self-adjoint with respect to one scalar product, but not another. For example, the operator from the preceding example, $A = \frac{d^2}{dx^2}$, is *not* necessarily self-adjoint with respect to a scalar product whose weight function is not constant.

Basis Expansions of Operators

Suppose that two vectors ψ and χ in a Hilbert space $\mathcal{H}$ are expanded in the same orthonormal basis $\{\varphi_n\}$:

$$\psi = \sum_n c_n \varphi_n, \quad \chi = \sum_m b_m \varphi_m.$$

Furthermore, suppose that ψ and χ are related by $\chi = A\psi$, where A is a linear operator on $\mathcal{H}$. We would like to be able to express the coefficients $\{b_n\}$ of χ in terms of the coefficients $\{c_n\}$ of ψ.

Taking the scalar product of both sides of the equation $\chi = A\psi$ with a basis function φ_k, we obtain

$$\begin{aligned}
b_k &= \langle \varphi_k | \chi \rangle \\
&= \langle \varphi_k | A\psi \rangle \\
&= \left\langle \varphi_k \middle| A \sum_n c_n \varphi_n \right\rangle \\
&= \sum_n c_n \langle \varphi_k | A\varphi_n \rangle \\
&= \sum_n a_{kn} c_n,
\end{aligned}$$

where the quantities

$$a_{kn} = \langle \varphi_k | A\varphi_n \rangle$$

define the entries of a matrix A. It follows that the coefficients of $A\psi$ in the basis $\{\varphi_n\}$ can be obtained from those of ψ by matrix-vector multiplication. That is, if $\mathbf{c}$ is the vector of coefficients of ψ, and $\mathbf{b}$ is the vector of coefficients of χ, then

$$\mathbf{b} = \mathsf{A}\mathbf{c}.$$

In addition, we can use the matrix A and basis functions $\{\varphi_n\}$ to obtain a representation of the operator A. Using Dirac notation, we obtain

$$\begin{aligned}
|\chi\rangle &= \sum_m |\varphi_m\rangle b_m \\
&= \sum_m |\varphi_m\rangle \left(\sum_n a_{mn} c_n \right) \\
&= \sum_m |\varphi_m\rangle \left(\sum_n a_{mn} \langle \varphi_n | \psi \rangle \right) \\
&= \sum_m \sum_n |\varphi_m\rangle a_{mn} \langle \varphi_n | \psi \rangle
\end{aligned}$$

$$\begin{aligned} &= \left(\sum_{m,n} |\varphi_m\rangle a_{mn}\langle\varphi_n|\right)|\psi\rangle \\ &= A|\psi\rangle. \end{aligned}$$

It follows that

$$A = \sum_{m,n} |\varphi_m\rangle a_{mn}\langle\varphi_n|.$$

This is a generalization of the resolution of the identity seen previously, in which the matrix A was simply the identity matrix I.

Example Let $\mathcal{H} = C_p[0, 2\pi]$ be the space of continuous 2π-periodic functions, and let $A = \frac{d}{dx}$. We use the orthonormal basis $\{\varphi_n\}_{n=-\infty}^{\infty}$, where $\varphi_n(x) = \frac{1}{\sqrt{2pi}}e^{inx}$ for each integer n. We also use the scalar product

$$\langle f|g\rangle = \int_0^{2\pi} f^*(x)g(x)\,dx.$$

Then

$$\begin{aligned} a_{mn} &= \langle\varphi_m|A\varphi_n\rangle \\ &= \left\langle \frac{1}{\sqrt{2\pi}}e^{imx} \middle| \frac{d}{dx}\left[\frac{1}{\sqrt{2\pi}}e^{inx}\right]\right\rangle \\ &= \frac{1}{2\pi}\langle e^{imx}|ine^{inx}\rangle \\ &= \frac{in}{2\pi}\langle e^{imx}|e^{inx}\rangle \\ &= in\langle\varphi_m|\varphi_n\rangle \\ &= in\delta_{mn}, \end{aligned}$$

where, as before, δ_{mn} is the Kronecker delta. It follows that the matrix A of the operator A in this basis is a *diagonal* matrix, meaning that the only nonzero entries are on the main diagonal, on which the row index equals the column index. That is,

$$a_{mn} = \begin{cases} in & m = n \\ 0 & m \neq n \end{cases}.$$

$\square$

Basis Expansion of Adjoint

Now we wish to be able to express the matrix B of the adjoint $A^\dagger$ of an operator A in terms of its matrix A, with respect to a given basis $\{\varphi_n\}$. Because the adjoint of $A^\dagger$ is $(A^\dagger)^\dagger = A$, we have

$$b_{mn} = \langle\varphi_m|A^\dagger\varphi_n\rangle = \langle A\varphi_m|\varphi_n\rangle = \langle\varphi_n|A\varphi_m\rangle^* = a_{nm}^*.$$

That is, $\mathsf{B} = \mathsf{A}^\dagger$, meaning that the matrix of the adjoint is the adjoint of the matrix, also known as the *Hermitian transpose* (transpose and complex conjugate).

Example Let $A = \frac{d}{dx}$ be the operator from the previous example. Then the matrix of the adjoint $A^\dagger$ is $\mathsf{A}^\dagger$, which has entries

$$[\mathsf{A}^\dagger]_{mn} = \begin{cases} -in & m = n \\ 0 & m \neq n \end{cases}.$$

That is, $\mathsf{A}^\dagger = -\mathsf{A}$, from which it follows that $A^\dagger = -A$. That is, differentiation in the space $C_p[0, 2\pi]$ is anti-Hermitian. $\square$