Journal of Physics A:
Mathematical and Theoretical

PAPER

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To cite this article: Rakesh Kumar and Kirankumar R Hiremath 2024 *J. Phys. A: Math. Theor.* **57** 345202

IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **57** (2024) 345202 (11pp) https://doi.org/10.1088/1751-8121/ad6aaf

Mathematical analysis of bent optical waveguide eigenvalue problem

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Received 10 April 2024; revised 26 June 2024
Accepted for publication 1 August 2024
Published 12 August 2024

Abstract

This work investigates a mathematical model of the propagation of lightwaves in bent optical waveguides. This modeling leads to a non-self-adjoint eigenvalue problem for differential operator defined on the unbounded domain. Through detailed analysis, a relationship between the real and imaginary parts of the complex-valued propagation constants was constructed. Using this relation, it is found that the real and imaginary parts of the propagation constants are bounded, meaning they are limited within certain region in the complex plane. The orthogonality of these bent modes is also proved. By the asymptotic analysis of these modes, it is proved that as $r \to \infty$ the behavior of the eigenfunctions is proportional to $1/\sqrt{r}$.

Keywords: bent waveguides, non-self-adjoint operators, complex eigenvalues, eigenvalue problem, orthogonal modes, asymptotic analysis

1. Introduction

Analysis of optical wave propagation in dielectric waveguides is an active research area. The most common types of waveguides are straight and bent waveguides. A detailed mathematical analysis of the eigenvalue problem for lightwave propagation in straight waveguides was carried out in [8]. There the eigenvalue problem was set up in the appropriate function space, and it was shown that this problem is a self-adjoint eigenvalue problem. In this work, we carry out similar analysis for bent waveguides. As discussed later in this work, due to the non-self-adjointness of the bent waveguide eigenvalue problem, this analysis is more involved.

The bent waveguides were first discussed by Marcatili [12]. These waveguides were investigated by several researchers using various techniques. In [4], bent optical waveguides were

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studied by an equivalent structure of straight waveguide obtained by the conformal transformation. Treating the bent waveguide as a perturbation of a straight waveguide, Melloni expressed bent waveguide modes in the straight waveguide modes [13]. Kim and Gopinath investigated bent waveguides using the finite difference method [9], whereas Jedidi and Pierre had investigated it using finite element method [6, 7]. Recently, Han *et al* calculated bending losses for highly confined modes of optical waveguides with transformation optics, which uses form invariance property of Maxwell equations [3].

In a semi-analytical study of bent waveguide modes [5], the mode propagation constants were determined by solving the dispersion equation and power orthogonality of the modes was studied. Continuing this work, we investigated this problem as an eigenvalue problem defined on an unbounded domain [10]. It was demonstrated that the bent waveguide eigenvalue problem is a non-self-adjoint eigenvalue problem, and its eigenvalues are complex numbers. Here the imaginary part of the bent mode propagation constant is a measure of energy loss due to the bending of the waveguide. In that work, we expressed the operator involved in the eigenvalue problem into the sum of the self-adjoint and non-self-adjoint operators. Also, we studied the behavior of the eigenvalue problem with the large bent radius.

This study investigates the mathematical properties of the operator involved in the bent waveguide eigenvalue problem, the spectral properties of the eigenvalue problem, and the orthogonality of the modes. The outline of the work is as follows: In section 2, we discuss the model setting in the radial coordinate system. In section 3, we prove the boundedness of the eigenvalues of operator involved in the eigenvalue problem and derive the relation between the phase propagation and attenuation constant. In section 4, it is shown that the modes corresponding to the distinct propagation constant of the bent waveguide eigenvalue problem are orthogonal. Asymptotic analysis of the modes is explored in section 5. We conclude the work in section 6.

2. Model setting

The setup of the 1-D bent waveguide segment is illustrated in figure 1. It is modeled in a cylindrical coordinate system (r, y, θ) as in previous studies [5, 10]. A bent waveguide comprised of an annular segment of a high-refractive index material sandwiched between two low-refractive index materials. The optical fields and material properties were assumed to be invariant in y-direction. This leads to decoupling of the fields, and we have transverse electric (TE) modes and transverse magnetic (TM) modes. These modes are characterized by the mode profile ψ of their respective principal field components, where $\psi(r) = \tilde{E}_y(r)$ for TE (y-component of the electric field of the mode profile), $\psi(r) = \tilde{H}_y(r)$ for TM (y-component of the magnetic field of the mode profile) [5]. This work focuses on TE modes; a similar analysis also applies to TM modes. Using (frequency domain) Maxwell equations, we get the following eigenvalue problem (EVP) for the principal mode component ψ:

$$L\psi(r) = \frac{\gamma^2}{k^2}\frac{R}{r}\psi(r), \qquad 0 < r < \infty, \tag{1}$$

where the operator L is defined as

$$L\psi(r) := \left(\frac{1}{k^2}\frac{r}{R}\psi_r\right)_r + n^2(r)\frac{r}{R}\psi.$$

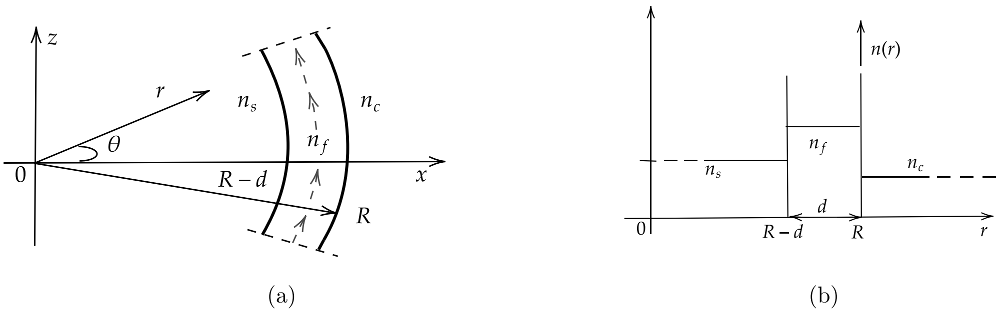

Figure 1. (a) 1-D bent waveguide model (b) constant step-index refractive index profile $n(r)$ of a bent waveguide segment in radial coordinate r with $n_f > n_s > n_c$, d is the guiding core width, and R is the bent radius.

In the above equation, the subscript r in ψ_r indicates a derivative of ψ with respect to r. The associated boundary conditions are given by

$$|\psi(r)| \to 0 \quad \text{as } r \to 0,$$
$$|\psi(r)| \to 0 \quad \text{as } r \to \infty \quad \text{and } \psi \text{ is an outgoing wave.}$$

Here γ is the propagation constant of the bent mode. In [10], it was proved that γ is a complex number with a negative imaginary part. Therefore, take $\gamma = \beta - \imath\alpha$, where $\beta > 0$ is the phase propagation constant and $\alpha \geqslant 0$ is the attenuation constant, $k = 2\pi/\lambda$ is the wave-number with wavelength λ in free space. A piece-wise constant refractive index profile $n(r)$ is given by

$$n(r) = \begin{cases} n_s & 0 < r < R - d, \\ n_f & R - d \leqslant r \leqslant R, \\ n_c & R < r < \infty, \end{cases}$$

which is illustrated in figure 1(b); n_s, n_f, and n_c are the refractive index of the substrate, the film, and the cladding medium, respectively.

The outgoing wave condition must be added to the exterior boundary condition at ∞ for the uniqueness of the solution. Note that at the material interfaces, ψ is continuous.

Thus the eigenvalue problem (1) for bent optical waveguides with the piece-wise constant refractive index profile is the weighted EVP problem with weight $\frac{R}{r}$ and with eigenvalues $\frac{\gamma^2}{k^2}$. This eigenvalue problem contains a parameter R and involves a discontinuous coefficient, defined on the semi-infinite interval $[0, \infty)$.

Note that wave number k is known fixed constant and the unknown propagation constant $\gamma = \beta - \imath\alpha$ is explicitly occurring in eigenvalues of the operator L. Therefore, in section 3, the results on the eigenvalues $\frac{\gamma^2}{k^2}$ are proved using the phase propagation constant β and α attenuation constant of the γ, instead of directly in terms of the eigenvalues $\frac{\gamma^2}{k^2}$.

Consider the following inner product space (in radial coordinate):

$$\mathbb{V}_{r,R} = \left\{ \psi \in \mathcal{L}^2_w(0,\infty) : \psi_r \in \mathcal{L}^2_w(0,\infty) \right\},$$

with the inner product given as

$$\langle \psi, \phi \rangle_{\mathbb{V}_{r,R}} = \int_0^\infty \psi(r)\,\phi^*(r)\,\frac{R}{r}\mathrm{d}r. \tag{2}$$

The corresponding norm in this inner product space is defined as:

$$\|\psi(r)\|^2_{\mathbb{V}_{r,R}} = \int_0^\infty |\psi(r)|^2 \frac{R}{r}\mathrm{d}r,$$

where r in the suffix of the inner product notation and the norm notation indicates the use of the radial coordinate system. Here the radial derivative is defined in the weak sense with respect to the inner product (2).

The governing eigenvalue problem (1) for the bent waveguide model is non-self-adjoint, with its adjoint problem

$$\begin{aligned} L^*\psi(r) &= \frac{\gamma^{*2}}{k^2}\frac{R}{r}\psi, \qquad 0 < r < \infty, \\ L^*\psi(r) &:= \frac{r}{Rk^2}\psi_{rr} + \frac{1}{Rk^2}\psi_r + n^2(r)\frac{r}{R}\psi + \left(\frac{1}{rRk^2}\psi - \frac{2}{Rk^2}\psi_r\right), \end{aligned} \tag{3}$$

and has complex eigenvalues [10]. According to the model mentioned above, the propagation constants γ_j are located in the fourth quadrant of the complex plane, i.e. $\gamma_j = \beta_j - \imath\alpha_j$, $\beta_j > 0$ and $\alpha_j \geqslant 0$.

In section 3, we will prove that both the real part β_j and the imaginary part α_j of the propagation constant γ_j are bounded. This means that propagation constants γ_j are contained inside a bounded region in the fourth quadrant in the complex plane.

3. Boundedness of propagation constants

In this section, we demonstrate the boundedness of eigenvalues of the operator L defined in (1). We begin by discussing the relationship between the real part β (phase propagation constant) and the imaginary part α (attenuation constant) of γ and boundedness of these constants. First, we will show that the phase propagation constant β is proportional to $\frac{1}{\alpha}\exp(\alpha R\pi)$, where α is the attenuation constant and R is a fixed bent radius.

3.1. *Relationship between phase propagation and attenuation constant*

To prove this relationship between β and α, multiply (1) by ψ^* and integrating both sides, one obtains

$$\frac{1}{Rk^2} r\psi_r\psi^*|_0^\infty - \int_0^\infty \frac{1}{Rk^2} r\psi_r\psi_r^*\mathrm{d}r + \int_0^\infty n^2(r)\frac{r}{R}\psi\,\psi^*\mathrm{d}r = \frac{\gamma^2}{k^2}\int_0^\infty \frac{R}{r}\psi\,\psi^*\mathrm{d}r.$$

At $r = 0$, the modal solution $\psi(r)$ and its derivative $\psi_r(r)$ are represented in terms of Bessel functions of the first kind [5]. The Bessel function of the first kind $J_{\gamma R}(\cdot)$ is zero at $r = 0$ for $\gamma R \neq 0$. Here γR is non-zero. So, the term $\frac{1}{Rk^2} r\psi_r\psi^*$ is zero at $r = 0$. Therefore, the above equation can be written as

$$\lim_{r\to\infty} \frac{1}{Rk^2} r\psi_r\psi^* - \int_0^\infty \frac{1}{Rk^2} r|\psi_r|^2\mathrm{d}r + \int_0^\infty n^2(r)\frac{r}{R}|\psi|^2\mathrm{d}r = \frac{\beta^2 - \alpha^2 - 2\imath\alpha\beta}{k^2}\int_0^\infty \frac{R}{r}|\psi|^2\mathrm{d}r. \tag{4}$$

For $r \to \infty$, modal solution $\psi(r)$ and its derivative $\psi_r(r)$ are represented by the Hankel functions of the second kind. Later in section 5, we will analytically show that the asymptotic behavior of ψ is as given in [5]. The asymptotic expansions for the Hankel functions of the second kind for $r \to \infty$ are

$$\psi \sim A_c \sqrt{\frac{2}{\pi n_c k r}} \exp -\imath \left(n_c k r - \gamma R \frac{\pi}{2} - \frac{\pi}{4} \right),$$
$$\psi_r \sim \left(-\frac{1}{2r} - \imath n_c k \right) \psi,$$

where A_c is an arbitrary constant [5]. Using the above expressions, we get

$$r \psi_r \psi^* \sim |A_c|^2 \frac{2}{\pi n_c k} \left(-\frac{1}{2r} - \imath n_c k \right) \exp(\alpha R \pi),$$
$$\lim_{r \to \infty} \frac{1}{R k^2} r \psi_r \psi^* = -|A_c|^2 \frac{2\imath}{\pi R k^2} \exp(\alpha R \pi).$$

Thus from equation (4), one gets

$$-|A_c|^2 \frac{2\imath}{\pi R k^2} \exp(\alpha R \pi) - \int_0^\infty \frac{1}{R k^2} r |\psi_r|^2 \mathrm{d}r + \int_0^\infty n^2(r) \frac{r}{R} |\psi|^2 \mathrm{d}r = \frac{\beta^2 - \alpha^2 - 2\imath \alpha \beta}{k^2} \int_0^\infty |\psi|^2 \frac{R}{r} \mathrm{d}r.$$

The second and third terms on the LHS of the above equation are real. Therefore, on equating the imaginary parts in the above equation, assuming the mode is normalized i.e. $\int_0^\infty |\psi|^2 \frac{R}{r} \mathrm{d}r = \|\psi\|_r = 1$, we get

$$\beta = |A_c|^2 \frac{1}{\alpha R \pi} \exp(\alpha R \pi), \tag{5}$$

where R is a fixed bent radius. This equation expresses the phase propagation constant β in terms of the attenuation constant α.

Nevertheless, equation (5) is useful to find bounds of α. We will employ the above expression equation (5), not only to express β in terms of α, but also to express α in terms of β in the subsequent results, as necessary.

Next, to prove the behavior of phase propagation constant β with respect to attenuation constant α. For this, we first construct a new operator B, using operator $Q = \dfrac{L - L^*}{2}$. From (1) and (3), we get

$$Q\psi(r) := -\frac{2\imath \alpha \beta}{k^2} \frac{R}{r} \psi, \qquad 0 < r < \infty, \tag{6}$$
$$Q\psi(r) := \frac{1}{R k^2} \psi_r - \frac{1}{2 r R k^2} \psi.$$

The operator Q is not self-adjoint due to $Q^* = \dfrac{(L - L^*)^*}{2} = \dfrac{L^* - L}{2} = -Q$, i.e. $Q^* \neq Q$.

Let us now construct a new operator B on multiplying operator Q with $-\frac{1}{2\imath}$,

$$B\psi(r) := \frac{\alpha \beta}{k^2} \frac{R}{r} \psi, \qquad 0 < r < \infty, \tag{7}$$
$$B\psi(r) := -\frac{1}{2\imath R k^2} \psi_r + \frac{1}{4\imath r R k^2} \psi,$$

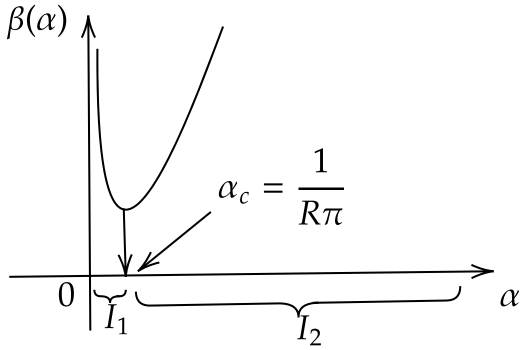

Figure 2. Behavior of the real part β (phase propagation constant) with respect to imaginary part α (attenuation constant) of the propagation constant $\gamma = \beta - \imath\alpha$. At $\alpha_c = \frac{1}{R\pi}$, β changes its behavior.

i.e, $B = -\frac{1}{2\imath}Q$. Using the properties of B, we will prove behavior of phase propagation constant β with respect to attenuation constant α. From the definition of B, we have

$$B^* = \frac{1}{2\imath}Q^* = -\frac{1}{2\imath}Q = B.$$

It follows that B is a self-adjoint operator on $\mathbb{V}_{r,R}$. We now show that operator B is a positive operator. Since, eigenvalues of operator B are of the type $\frac{\alpha\beta}{k^2}$. For finite bent radius R, both $\alpha > 0$, and $\beta > 0$. This concludes that, for a finite bent radius R, eigenvalues of an operator B are positive, i.e. $(\frac{\alpha\beta}{k^2} > 0)$. This means that operator B is a positive operator. We can also conclude this using

$$\frac{\alpha\beta}{k^2} = |A_c|^2 \frac{1}{Rk^2\pi} \exp\left(\alpha R\pi\right), \text{(by equation (5))}.$$

3.2. *Behavior of phase propagation constant w.r.t. attenuation constant*

Now, we will show that β and α are inversely proportional to each other. For a given bent waveguide configuration, consider the distinct bent mode propagation constants $\gamma_i = \beta_i - \imath\alpha_i$ and $\gamma_j = \beta_j - \imath\alpha_j$. We will show that if $\alpha_i > \alpha_j$ then $\beta_i < \beta_j$ and if $\alpha_i < \alpha_j$ then $\beta_i > \beta_j$. This indicates that for a given bent waveguide configuration, if the phase propagation constant increases, then the attenuation constant decreases and vice-versa.

Since, operator B (7) is a self-adjoint operator. Therefore, its eigenvalues $\alpha\beta/k^2$ can be arranged in increasing order as $\cdots < \frac{\alpha_i\beta_i}{k^2} < \frac{\alpha_j\beta_j}{k^2} < \cdots$ [1, 2]. From this, one gets

$$\frac{\alpha_j}{\alpha_i} > \frac{\beta_i}{\beta_j}. \tag{8}$$

For fixed R, the function $\beta(\alpha) = \frac{1}{\alpha R\pi}\exp(\alpha R\pi)$ in equation (5) has minima at $\alpha_c = \frac{1}{R\pi}$ shown in figure 2 (here without loss of the generality, we are assuming the constant $A_c = 1$). The function $\beta(\alpha)$ has distinct behavior on intervals $I_1 = (0, \alpha_c)$ and $I_2 = (\alpha_c, \infty)$. On the interval I_1, when α decreases, β increases. Whereas on the interval I_2, when α decreases, β also decreases.

Consider the distinct bent mode propagation constants $\gamma_i = \beta_i - \imath\alpha_i$ and $\gamma_j = \beta_j - \imath\alpha_j$. Suppose both of these lie in the interval I_2 and without loss of the generality assume $\alpha_i > \alpha_j$. i.e. $\frac{\alpha_j}{\alpha_i} < 1$. Then, from inequality (8), we get $\frac{\beta_i}{\beta_j} < 1$, i.e. $\beta_i < \beta_j$, implying β is decreasing for increasing α, which is not possible in interval I_2. Therefore, the only valid interval is I_1, where if $\alpha_i > \alpha_j$ then $\beta_i < \beta_j$ or if $\alpha_i < \alpha_j$ then $\beta_i > \beta_j$.

3.3. Boundedness of propagation constants

The boundedness of propagation constants $\gamma_j = \beta_j - \imath\alpha_j$, implies the eigenvalues of the operator L defined in (1) are bounded.

To prove this, in the section 3.2, we found that α_j are bounded and lie in $(0, \alpha_c)$. Also, for finite R, α can not be zero [5, 10]. From equation (5), β_j are also bounded. Therefore, the propagation constants γ_j are bounded, and hence eigenvalues of the operator L are bounded.

For the straight waveguide eigenvalue problem, modes corresponding to distinct eigenvalues are orthogonal [8]. So, our next interest is to know whether modes or eigenfunctions corresponding to distinct eigenvalues for the bent waveguide problem are orthogonal.

4. Orthogonality of the bent modes

In the context of a self-adjoint eigenvalue problem, it is established that eigenfunctions corresponding to distinct eigenvalues are orthogonal [1]. However, for a non-self-adjoint problem, eigenfunctions corresponding to different eigenvalues may or may not be orthogonal [11]. Thus, our next objective is also highlight that whether eigenfunctions corresponding to distinct eigenvalues of the non-self-adjoint operator L also exhibit orthogonality. We investigate it using the adjoint operator L^* given by (3).

Consider the EVP (1)

$$L\psi := \left(\frac{1}{k^2}\frac{r}{R}\psi_r\right)_r + n^2(r)\frac{r}{R}\psi = \frac{\gamma^2}{k^2}\frac{R}{r}\psi. \tag{9}$$

Assume that ϕ is an eigenfunction of this problem corresponding to the eigenvalue δ^2/k^2. The adjoint eigenvalue problem for ϕ is

$$L^*\phi = \left(\frac{r}{Rk^2}\phi_r\right)_r + n^2(r)\frac{r}{R}\phi + \left(\frac{\phi}{rRk^2} - \frac{2}{Rk^2}\phi_r\right) = \frac{\delta^{*2}}{k^2}\frac{R}{r}\phi.$$

On taking the complex conjugate $'*'$ both the sides, one gets

$$\left(\frac{r}{Rk^2}\phi_r^*\right)_r + n^2(r)\frac{r}{R}\phi^* + \left(\frac{\phi^*}{rRk^2} - \frac{2}{Rk^2}\phi_r^*\right) = \frac{\delta^2}{k^2}\frac{R}{r}\phi^*, \tag{10}$$

where $\gamma = \beta - \imath\alpha \neq \delta = \beta' - \imath\alpha'$, i.e. either $\beta \neq \beta'$or $\alpha \neq \alpha'$. On multiplying equation (9) with ϕ^* and equation (10) with ψ and then integrating both the sides, we get

$$\int_0^\infty \left(\frac{1}{k^2}\frac{r}{R}\psi_r\right)_r \phi^* \mathrm{d}r + \int_0^\infty n^2(r)\frac{r}{R}\psi\phi^* \mathrm{d}r = \int_0^\infty \frac{\gamma^2}{k^2}\frac{R}{r}\psi\phi^* \mathrm{d}r, \tag{11}$$

and

$$\int_0^\infty \left(\frac{1}{k^2}\frac{r}{R}\phi^*{}_r\right)_r \psi\,\mathrm{d}r + \int_0^\infty n^2(r)\,\frac{r}{R}\phi^*\psi\,\mathrm{d}r + \int_0^\infty \frac{1}{rRk^2}\psi\,\phi^*\mathrm{d}r - \int_0^\infty \frac{2}{Rk^2}\psi\phi_r{}^*\mathrm{d}r = \int_0^\infty \frac{\delta^2}{k^2}\frac{R}{r}\phi^*\psi\,\mathrm{d}r. \tag{12}$$

Integrate the first term in equation (11) by part, and using the fact that the boundary term is zero [5], we get

$$\int_0^\infty \left(\frac{1}{k^2}\frac{r}{R}\phi^*{}_r\right)_r \psi\,\mathrm{d}r + \int_0^\infty n^2(r)\,\frac{r}{R}\psi\phi^*\mathrm{d}r = \int_0^\infty \frac{\gamma^2}{k^2}\frac{R}{r}\psi\phi^*\mathrm{d}r.$$

Subtracting the above equation from equation (12), we get

$$-2\int_0^\infty \psi\left(\frac{1}{Rk^2}\phi_r{}^* - \frac{1}{2rRk^2}\phi^*\right)\mathrm{d}r = \frac{\delta^2-\gamma^2}{k^2}\int_0^\infty \psi\,\phi^*\frac{R}{r}\mathrm{d}r.$$

Using the conjugate of equation (6) gives

$$-\frac{4\imath\alpha'\beta'}{k^2}\int_0^\infty \psi\,\phi^*\frac{R}{r}\mathrm{d}r = \frac{\delta^2-\gamma^2}{k^2}\int_0^\infty \psi\,\phi^*\frac{R}{r}\mathrm{d}r.$$

Rearranging this expression, we get

$$\left(\delta^2-\gamma^2+4\imath\alpha'\beta'\right)\langle\psi,\phi\rangle_{\mathbb{V}_{r,R}} = 0.$$

Putting the valued of δ and γ in this equation, we get

$$\left(\left(\beta'^2-\beta^2\right)+\left(\alpha'^2-\alpha^2\right)+2\imath\alpha'\beta'+2\imath\alpha\beta\right)\langle\psi,\phi\rangle_{\mathbb{V}_{r,R}} = \left(\bar{\delta}^2-\gamma^2\right)\langle\psi,\phi\rangle_{\mathbb{V}_{r,R}} = 0.$$

Further investigation of the orthogonality of eigenfunctions depends on the product of two terms in LHS in the above equation. Suppose that eigenfunctions are not orthogonal in the sense $\langle\psi,\phi\rangle \neq 0$. In this case $(\bar{\delta}^2-\gamma^2) = 0$i.e. $(\beta'^2-\beta^2)+(\alpha'^2-\alpha^2)+2\imath\alpha'\beta'+2\imath\alpha\beta = 0$. Equating the real and the imaginary parts, one gets

$$\left(\beta'^2-\beta^2\right)+\left(\alpha'^2-\alpha^2\right) = 0 \text{ and } \alpha'\beta'+\alpha\beta = 0.$$

From [10], (for forward propagating modes) both $\beta,\beta' > 0$. Therefore, from the equation $\alpha'\beta'+\alpha\beta = 0$, one gets $\alpha' = -\dfrac{\alpha\beta}{\beta'}$ and the another equation gives either $\beta' = \beta$ or $\beta' = \alpha$. If $\beta' = \alpha$ then $\alpha' = -\beta$ which is not possible because $\alpha' \geqslant 0$. Implies, $\beta' = \beta$, and then $\alpha' = -\alpha$. Again, this is not possible. This means the term $\bar{\delta}^2-\gamma^2$ is non-zero. So, $\langle\psi,\phi\rangle_{\mathbb{V}_{r,R}} = 0$. Therefore, eigenfunctions of the eigenvalue problem (9) corresponding to distinct eigenvalues are orthogonal, i.e. bent waveguide modes correspond to distinct eigenvalues are orthogonal.

This indicates that for a non-self-adjoint operator, eigenfunctions corresponding to distinct eigenvalues can be orthogonal even if their eigenvalues are complex. In the next section, we discuss the asymptotic behavior of these modes.

5. Asymptotic behavior of modes

Using the asymptotic form of the Hankel function, the semi-analytical study showed that the asymptotic behavior of the eigenfunctions is of the type $\frac{1}{\sqrt{r}}$ [5]. In this section, we will prove this mathematically directly from the eigenvalue problem without referring to the solution to the problem. Rewrite the eigenvalue problem (1) as

$$r^2\psi_{rr} + r\psi_r + \left(n^2(r)k^2r^2 - \gamma^2R^2\right)\psi = 0.$$

As $r \to \infty$, the refractive index $n(r)$ in the cladding region ($R < r < \infty$) is n_c. Therefore, in the cladding region

$$r^2\psi_{rr} + r\psi_r + \left(n_c^2k^2r^2 - \gamma^2R^2\right)\psi = 0.$$

The solution's leading behavior can be found by substituting $\psi(r) \sim e^{S(r)}$ as $r \to \infty$. This gives $\psi_r \sim S_re^{S(r)}, \psi_{rr} \sim (S_{rr} + (S_r)^2)e^{S(r)}$. Putting it in the above equation,

$$r^2\left(S_{rr} + S_r^2\right) + rS_r + \left(n_c^2k^2r^2 - \gamma^2R^2\right) \sim 0. \tag{13}$$

Assume $S(r)$ is growing algebraically, and $|S_{rr}| \ll |S_r^2|$ as $r \to \infty$. This notation means that when $r \to \infty$, the function $|S_{rr}|$ is negligible as compared to $|S_r^2|$, i.e.

$$\lim_{r\to\infty}\frac{|S_{rr}|}{|S_r^2|} = 0.$$

Also, $|\gamma|^2R^2 \ll n_c^2k^2r^2$ as $r \to \infty$. Hence, the above equation leads to the asymptotic equation

$$r^2S_r^2 + rS_r + n_c^2k^2r^2 \sim 0,$$

which is a quadratic equation in S_r. Therefore,

$$S_r \sim -\frac{1}{2r} \pm \imath\sqrt{n_c^2k^2 - \frac{1}{4r^2}}.$$

Since we are dealing with $r \to \infty$, $S_r \sim \pm\imath n_ck$, which implies that $S \sim \pm\imath n_ckr = S_0$, where S_0 is the zeroth-order approximation of the solution. Also, the condition $|S_{rr}| \ll |S_r^2|$ as $r \to \infty$ is satisfied. So, the result for S_0 is consistent with the assumptions.

For the next term in the asymptotic expansion of ψ, let $S(r) = \pm\imath n_ckr + S_1(r)$, and $|S_1(r)| \ll |\imath n_ckr|$. From equation (14), one gets

$$r^2(S_1)_r^2 \pm 2\imath n_ckr^2(S_1)_r + r^2(S_1)_{rr} \pm \imath n_ckr + r(S_1)_r - \gamma^2R^2 \sim 0.$$

As, $|S_1(r)| \ll |\pm\imath n_ckr|$ and $|(S_1)_r| \ll |\pm\imath n_ck|, |(S_1)_{rr}| \ll 0$. $|r^2(S_1)_r^2|, |r(S_1)_r| \ll |\pm 2\imath n_ckr^2(S_1)_r|$ and $|-\gamma^2R^2| \ll |\pm\imath n_ckr|$. Therefore,

$$\pm 2\imath n_ckr^2(S_1)_r \pm \imath n_ckr \sim 0,$$
$$\implies S_1(r) \sim -\frac{1}{2}\ln r.$$

The assumption $|S_1(r)| \ll |\pm\imath n_c kr|$ is consistent and the term $S_1(r)$ is not tending to zero as $r\to\infty$, Therefore, $S_1(r) = -\frac{1}{2}\ln r + S_2(r)$. So,

$$S(r) \sim S_0(r) + S_1(r) \sim \pm\imath n_c kr - \frac{1}{2}\ln r + S_2(r).$$
$$\implies \psi(r) \sim e^{S(r)} \sim \frac{1}{\sqrt{r}} e^{\pm\imath n_c kr + S_2(r)}.$$

This shows the asymptotic behavior of the eigenfunction (i.e. the mode of a bent waveguide) ψ is proportional to $\dfrac{1}{\sqrt{r}}$, which clearly illustrates the decaying behavior of the waves in the cladding region. This result is consistent with the earlier findings of semi-analytic studies of the bent waveguides [5].

6. Conclusions

Non-self-adjoint operators play a central role in modeling physical systems involving losses. In optics-photonics, it was well known that lightwave propagation in dielectric bent waveguides is a lossy phenomenon. The extent of these losses depends on the bending radius of the waveguide. In this work, we investigated the bent waveguide model, which was mathematically formulated as a non-self-adjoint eigenvalue problem. In applications, its eigenfunctions are known as waveguide modes (also called 'normal modes'), characterized by scalars called propagation constants. These propagation constants are related to the eigenvalues of the non-self-adjoint operator. Earlier studies showed these eigenvalues are complex, but further insights were available.

In this work, we first showed that the complex-valued propagation constants are bounded and obtained the relationship between their real and imaginary parts. It was also shown that despite having complex eigenvalues, eigenfunctions corresponding to distinct eigenvalues are orthogonal. It will be interesting to see if such an orthogonality of eigenfunctions is also generally valid for any non-self-adjoint eigenvalue problem. For waveguides, the asymptotic behavior of the eigenfunctions (i.e. bent modes) dictates the distribution of electromagnetic energy in the radial directions. In this work, we showed mathematically that the asymptotic behavior of the eigenfunctions is proportional to $\dfrac{1}{\sqrt{r}}$. This information helps to define the appropriate function space and the subsequent mathematical analysis of the wave propagation.

Data availability statement

No new data were created or analysed in this study.

Acknowledgment

Rakesh Kumar acknowledges the support of the CSIR fellowship for his PhD work.

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