Exact analytical method for planar optical waveguides with arbitrary index profile

Article *in* Journal of the Optical Society of America A · September 1999

DOI: 10.1364/JOSAA.16.002209

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Received November 24, 1998; accepted February 11, 1999; revised manuscript received April 15, 1999

We present a novel matrix method that allows the straightforward determination of exact propagation constants as well as the field configuration in the region that is beyond the turning point at the substrate side for arbitrarily graded-index planar waveguides. © 1999 Optical Society of America [S0740-3232(99)02008-6]
OCIS codes: 130.2790, 230.7390, 260.2030, 290.3030, 350.5500.

INTRODUCTION

Many common processes for the fabrication of optical waveguides, such as ion exchange and thermal diffusion, give rise to graded-index profiles. The characteristics of the propagating modes in such waveguides are obtained by solving Maxwell's equations subject to the appropriate boundary conditions. However, exact analytical solutions are available only for a limited class of profiles. In general, one has to use either approximate or numerical methods.[1–4] There is no doubt that using numerical methods, one can obtain solutions to the desired accuracy, but a considerable amount of physical insight is lost in the use of these processes. Of the approximate methods, the WKB method is widely used because of its simplicity and the clear physical explanation available, but its solution diverges around the turning point. The variational method gives good estimates only for the lowest-order mode and requires complicated numerical integrals. The modified Airy function (MAF) method[5] and the recently developed equivalent attenuated vector (EAV) method[6] can yield extremely accurate propagation constants; however, they still show small errors in the calculations of eigenvalue and the eigenfunction.

Since a graded-index waveguide is nothing more than a large number of thin films in which the refractive index can be taken as a constant, then we can use the transfer-matrix techniques to study the characteristics of the propagation modes of the graded-index waveguides. In our method, two substantial improvements over the WKB method are proposed. First, the phase contribution of subwaves reflected from the sections of layers that is always neglected by the WKB method is taken into account. Second, we verified that the field configuration in the region beyond the turning point at the substrate side is a rigorous exponentially evanescent function. In this way, the subsequent dispersion equation can give exact propagation constants for planar optical waveguides with arbitrary refractive-index profiles in a short period of computer time.

1. TRANSFER MATRIX THEORY

We consider a planar optical waveguide with the index profile of the form

$$n^2(x) = \begin{cases} n_2^{\,2} + (n_1^{\,2} - n_2^{\,2}) f(x/d) & (x > 0) \\ n_0^{\,2} & (x < 0) \end{cases}, \quad (1)$$

where n_1, n_2, and n_0 are the refractive indices shown in Fig. 1, and $f(x/d)$ is the profile function. We assume that x_t is the turning point. To use transfer-matrix theory in determining the propagation characteristics of the waveguide, we truncate the profile at $x_s = x_t + x_c$, beyond which we assume $n(x) = n_s$. We then divide the region $(0, x_t)$ and (x_t, x_s) into l and m equal parts with width h, respectively; i.e., $x_t = lh$ and $x_c = mh$. Hence the transfer matrixes corresponding to the ith and the jth sections of layers for TE modes can be written as

$$M_i = \begin{bmatrix} \cos(k_i h) & -1/k_i \sin(k_i h) \\ k_i \sin(k_i h) & \cos(k_i h) \end{bmatrix}, \quad (i = 1, 2, \ldots, l), \quad (2)$$

$$M_j = \begin{bmatrix} \cosh(\alpha_j h) & -1/\alpha_j \sinh(\alpha_j h) \\ -\alpha_j \sinh(\alpha_j h) & \cosh(\alpha_j h) \end{bmatrix}, \quad (j = l+1, l+2, \ldots, l+m) \quad (3)$$

where

$$k_i = [k_0^{\,2} n^2(x_i) - \beta^2]^{1/2},$$
$$\alpha_j = [\beta^2 - k_0^{\,2} n^2(x_j)]^{1/2}. \quad (4)$$

β is the propagation constant along the Z direction, and $k_0 = \omega/c$ is the wave number with angular frequency ω and light velocity c in free space.

On applying the boundary conditions at $x = 0$ and $x = x_s$, we obtain

$$\begin{bmatrix} E_y(0) \\ E_y{}'(0) \end{bmatrix} = \left(\prod_{i=1}^{l} M_i \right) \left(\prod_{j=l+1}^{l+m} M_j \right) \begin{bmatrix} E_y(x_s) \\ E_y{}'(x_s) \end{bmatrix}, \quad (5)$$

where $E_y(x)$ is the y component of the electric field and the prime denotes differentiation with respect to x. As is known, the field in the cover layer $(x < 0)$ and the substrate $(x > x_s)$ decay exponentially with displacement along the x axis:

0740-3232/99/092209-04$15.00

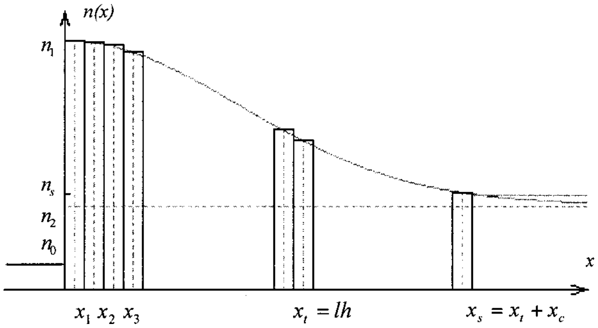

Fig. 1. Plot of planar waveguide with arbitrary index profile.

$$E_y(x) = \begin{cases} A_0 \exp(P_0 x) & (x < 0) \\ A_s \exp[-P_s(x - x_s)] & (x > x_s) \end{cases}, \tag{6}$$

where P_0 and P_s are the attenuated coefficients, i.e.,

$$P_0 = (\beta^2 - k_0{}^2 n_0{}^2)^{1/2},$$
$$P_s = (\beta^2 - k_0{}^2 n_s{}^2)^{1/2}. \tag{7}$$

A_0 and A_s are the coefficients to be determined.

The use of Eq. (6) in Eq. (5) yields

$$(-P_0 \quad 1)\left(\prod_{i=1}^{l} M_i\right)\left(\prod_{j=l+1}^{l+m} M_j\right)\begin{pmatrix} 1 \\ -P_s \end{pmatrix} = 0. \tag{8}$$

Eq. (8) can be reduced by simple algebra manipulations to

$$(-P_0 \quad 1)\left(\prod_{i=1}^{l} M_i\right)\begin{pmatrix} 1 \\ -P_t \end{pmatrix} = 0, \tag{9}$$

where

$$P_t = P_{l+1}, \tag{10}$$

$$P_j = \alpha_j \frac{\sinh(\alpha_j h) + \dfrac{P_{j+1}}{\alpha_j}\cosh(\alpha_j h)}{\cosh(\alpha_j h) + \dfrac{P_{j+1}}{\alpha_j}\sinh(\alpha_j h)},$$
$$(j = l+1, l+2, \ldots, l+m),$$
$$P_{l+m+1} = P_s. \tag{11}$$

From the derivation of Eqs. (9)–(11), it is confirmed that the field solution in the region beyond the turning point at the substrate side is an exponentially evanescent function:

$$E_y(x) = A_t \exp[-P_t(x - x_t)] \qquad (x > x_t), \tag{12}$$

where P_t is an attenuated coefficient and A_t is another coefficient to be determined. P_t can be characterized by an equivalent refractive index n_{eq} and is defined by

$$P_t = (\beta^2 - k_0{}^2 n_{\mathrm{eq}}^2)^{1/2}. \tag{13}$$

For the guiding region $(0, x_t)$, when we use a treatment analogous to that in Eq. (8), Eq. (9) can be written as

$$P_0 + P_1 = 0, \tag{14}$$

where

$$P_i = k_i \tan\left[\tan^{-1}\left(\frac{P_{i+1}}{k_i}\right) - k_i h\right], \qquad (i = 1, 2, 3, \ldots, l),$$
$$P_{l+1} = P_t. \tag{15}$$

To obtain an appropriate form that can give clear physical insight, here we set

$$\Phi_i = \tan^{-1}\left(\frac{P_i}{k_i}\right), \tag{16}$$

which, by use of Eq. (15), becomes

$$\Phi_i = m_i \pi + \tan^{-1}\left(\frac{P_{i+1}}{k_i}\right) - k_i h,$$
$$= m_i \pi + \tan^{-1}\left(\frac{k_{i+1}}{k_i}\tan\Phi_{i+1}\right) - k_i h$$
$$(m_i = 0, 1, 2, \ldots; \qquad i = 1, 2, \ldots, l-1). \tag{17}$$

Rearranging Eq. (17) yields

$$k_i h + \left[\Phi_{i+1} - \tan^{-1}\left(\frac{k_{i+1}}{k_i}\tan\Phi_{i+1}\right)\right]$$
$$= m_i \pi + (\Phi_{i+1} - \Phi_i). \tag{18}$$

The solution for $i = l$ is

$$k_l h = m_l \pi + \tan^{-1}\left(\frac{P_t}{k_l}\right) - \Phi_l. \tag{19}$$

Summing all the indexes i, we have

$$\sum_{i=1}^{l} k_i h + \sum_{i=1}^{l-1}\left[\Phi_{i+1} - \tan^{-1}\left(\frac{k_{i+1}}{k_i}\tan\Phi_{i+1}\right)\right]$$
$$= m\pi + \tan^{-1}\left(\frac{P_t}{k_l}\right) - \Phi_1. \tag{20}$$

Combining Eqs. (14) and (16), we obtain

$$\Phi_1 = -\tan^{-1}\left(\frac{P_0}{k_1}\right). \tag{21}$$

Substituting Eq. (21) into Eq. (20) and letting $l \to \infty$ ($h \to 0$), we obtain

$$\int_0^{x_t} k(x)\mathrm{d}x + \Phi(r) = m\pi + \tan^{-1}\left(\frac{P_0}{k_1}\right) + \tan^{-1}\left(\frac{P_t}{k_l}\right)$$
$$(m = 0, 1, 2, \ldots), \tag{22}$$

where

$$\Phi(r) = \sum_{i=1}^{l-1}\left[\Phi_{i+1} - \tan^{-1}\left(\frac{k_{i+1}}{k_i}\tan\Phi_{i+1}\right)\right]. \tag{23}$$

It is clear that subscripts i and $i+1$ indicate the neighboring sections of layers in the guiding region. If we equate $k_{i+1} = k_i$ in Eq. (23), which means we have neglected the refractive-index difference of the neighboring sections of layers, we obtain $\Phi(r) = 0$; thus $\Phi(r)$ can be interpreted as the phase contribution of the subwaves reflected from every interface between the two neighboring sections of layers. This phase contribution is always neglected by the WKB method.

2. NUMERICAL RESULTS AND COMPARISON

To illustrate the reliability of our method, we present here some typical results for the propagation constants obtained by the present theory, the MAF method, the EAV method, the WKB approximation, and the exact numerical results. For simplicity we have introduced the normalized frequency

$$V = k_0 d(n_1{}^2 - n_2{}^2)^{1/2}, \tag{24}$$

and the normalized mode index

$$b = \frac{n_e{}^2 - n_2{}^2}{n_1{}^2 - n_2{}^2}, \qquad n_e = \beta/k_0. \tag{25}$$

In our analysis we consider three general classes of profiles:

$$f(x/d) = \begin{cases} \exp(-x/d) & \text{exponential} \\ \exp(-x^2/d^2) & \text{Gaussian} \\ \operatorname{erfc}(-x/d) & \text{complimentary error} \end{cases}, \tag{26}$$

where d is the effective depth of diffusion.

Calculations were carried out with $n_2 = 2.177$, $n_0 = 1.0$, and $n_1{}^2 - n_2{}^2 = 0.187$. The value of d has been varied to produce results as a function of V. The calculated values of b are given in Tables 1–3, with the region $(0, x_t)$ divided into several hundred sections of layers. Meanwhile, for comparison with the MAF method and the improved MAF method, we also performed calculations for the examples presented in previous papers,[5,6] for which the index profiles are the exponential function and the parabolic function. The latter one is

$$n^2(x) = \begin{cases} n_0{}^2 & (x < 0) \\ n_1{}^2 - (n_1{}^2 - n_2{}^2)(x/d)^2 & (0 < x < d). \\ n_2{}^2 & (x > d) \end{cases} \tag{27}$$

The comparison is shown in Table 4.

Calculations of the exponentially attenuated coefficient and equivalent index beyond the turning point are carried at $V = 4.0$, as shown in Table 5.

We verified that the present method has a convergence behavior on the step number in the effective-index calculation, and the evidence is given in Table 6.

Table 1. Exponential Profiles

	b				
V	Exact	Present	MAF	EAV	WKB
1.5	0.03501	0.03501	0.03502	0.03513	0.03783
2.0	0.10495	0.10495	0.10503	0.10457	0.10861
2.5	1.17144	0.17144	0.17152	0.17084	0.17531
3.0	0.22919	0.22919	0.22926	0.22848	0.23308
3.5	0.27865	0.27865	0.27872	0.27783	0.28249
4.0	0.32117	0.32117	0.32124	0.32045	0.32493

Table 2. Gaussian Profiles

	b			
V	Exact	Present	EAV	WKB
2.0	0.0817	0.0817	0.0814	0.0451
3.0	0.2750	0.2750	0.2741	0.2537
4.0	0.4133	0.4133	0.4124	0.4007

Table 3. Complementary Error Profiles

	b			
V	Exact	Present	EAV	WKB
3.0	0.0675	0.0675	0.0672	0.0574
4.0	0.1694	0.1694	0.1644	0.1650

Table 4. Comparison at $V = 4.0$

	b				
Profile	Exact	Present	IMAF	MAF	WKB
Exponential	0.32117	0.32117	0.32116	0.32150	0.32493
Parabolic	0.32617	0.32617	0.32703	0.31691	0.30790

Table 5. Exponentially Attenuated Coefficient and Equivalent Index beyond the Turning Point

Profiles	$x_c = x_s - x_t$ (μm)	P_t (μm^{-1})	n_{eq}	$n(x_t)$
Exponential	3.9419	1.2460	2.1871	2.1907
Gaussian	4.1243	1.6807	2.1881	2.1947
Complementary Error	4.0949	1.2704	2.1805	2.1843

3. CONCLUSION

We have developed a novel method for obtaining the modal characteristics of a planar optical waveguide with arbitrary index profile. Dispersion equations are presented in a simple and clear manner. For the first time we have also proved that the field solution in the region beyond the turning point at the substrate side is an exponentially evanescent function. The test calculations

Table 6. Calculating Results

Step Number	10^2	10^3	10^4	10^5	10^6	10^7
b	0.32116	0.32117	0.32117	0.32117	0.32117	0.32117
P_t (μm^{-1})			1.2460			
n_{eq}			2.1871			
CPU time (s)	0.4	0.5	0.7	2.9	27.0	273.5

show that the present method is reliable for analyzing various types of practical waveguides.

ACKNOWLEDGMENT

The authors acknowledge support received from Municipal Grants Committee in Shanghai, China.

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