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The kinematics behaviour of coupled pendulum using differential transformation method

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ABSTRACT

The topic of coupled oscillations is rich in physical content which is both interesting and complex. In this research work we aimed to study the kinematics behaviour of an important physical system known as coupled pendulum, in which two identical pendulums were coupled together by a spring. The Lagrangian of the system was first constructed, then the equations of motion have been derived. Analytical solutions were obtained for these equations, and in addition we apply a reliable algorithm based on an adaptation of the standard differential transformation method is presented, which is the multi-step differential transformation method to obtain numerical solutions for the considered system for some selected initial conditions. The solutions of the equations of motion were obtained by the multi-step differential transform method. Of course, this method is needed if it predicts the motion of the considered system. For this reason figurative comparisons between the multi-step differential transformation method, the standard differential transformation method and the classical fourth-order Runge–Kutta method are given. The results reveal that the proposed technique is a promising tool to predict the behaviours of the considered system in long time interval. Moreover, the motion of curves represents the initial conditions that sensitive depend and how much oscillatory motion. In conclusion, the proposed technique is a reliable method to solve the considered double pendulum problem in long time interval. We believe that the system considered in this work along with the method used is interesting for physicists, mathematicians and engineers.

Introduction

Differential equations (DEs) play a pivotal role in all aspects of science and engineering and they can be used to describe a lot of real world problems arising in physics, biology, chemistry, engineering and other fields. For example, in physics Des are very useful and effective in describing physical systems appeared in electrodynamics, modern physics, quantum mechanics, classical mechanics and many other areas, for example refer to the following important texts [1–4].

In classical mechanics, physical systems have been investigated by applying two main techniques that lead to DEs called equations on motion: Newtonian mechanics and Lagrangian mechanics. In Lagrangian mechanics the formalism of the system is based on its scalar quantities (i.e. mainly its kinetic and potential energies respectively), while Newtonian mechanics is based on vector quantities (i.e., forces acting on the system) for more details about both techniques refer to [3–7].

Upon applying the Lagrangian mechanics and obtaining the Lagrangian of the system, the Euler- Lagrange Equations (i.e. equations of motions) can be derived and obtained and these equations are DEs describing the kinematics behaviour of the system and these equations have to be solved for some specific and determined initial conditions. In simplest systems these equations of motion can be solved analytically, but in many other systems approximation methods can be used or in general simulation and numerical techniques are used, an interested reader can refer to the following references [8–10].

Of late years, many authors primarily took into consideration studying solutions of linear or nonlinear DEs via several methods, such as: Adomian decomposition method [11], homotopy perturbation method [12], homotopy analysis method [13], differential transformation method DTM [14], and variational iteration method [15]. Non– linear vibrating systems usually appear in many branches of science and engineering, and from literature many methods have been used

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Nomenclature

l	Pendulum length
m	Pendulum mass
κ	Spring stiffness
$K.E$	Kinetic energy
$P.E$	Potential energy
ϕ_1	Amplitude
ϕ_2	Amplitude
g	Acceleration gravity
ω_0	Pendulum angular velocity
t	Time variable
DEs	Differential Equations
DTM	Differential Transformation Method
Ms- DTM	Multi step- Differential transformation Method

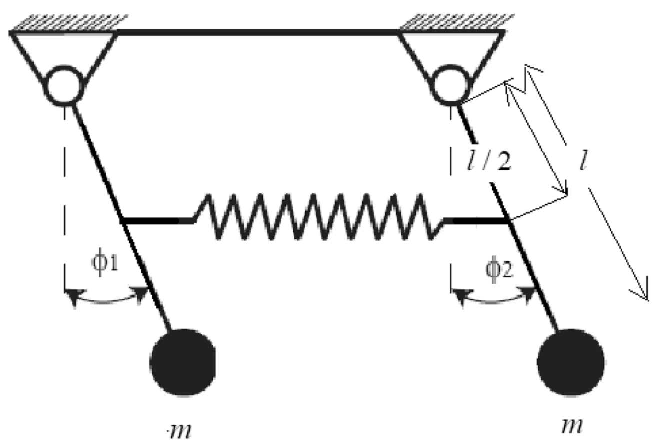

Fig. 1. Two simple pendulums coupled by a spring at midpoint.

in studying and analysing these systems analytically and numerically. For example, in [16,17] the parametrized perturbation and variational iteration methods have been used to study the non-linear vibration of Euler-Bernoulli beams [16,17], while He's energy balance and He's variational iteration methods have been applied in nonlinear vibrations and oscillations systems such as studying periodic solutions of strongly nonlinear systems and nonlinear oscillators [18,19]. In these studies one can see that these methods provided a powerful mathematical tools in studying nonlinear systems.

On the other hand, in recent years fractional calculus shows its importance in investigating many physical and engineering systems that can be described by differential equations. Many efforts have been paid in this direction, where from literature we can see that researches have used manu methods in solving physical and engineering systems using fractional calculus with accurate results, interested reader can refer to [21–26] and references therein, where many useful methods have been of great importance. As an example, the multistep reduced DTM was used to solve one dimensional fractional heat equations with time fractional derivatives, and the Klein- Gordon equation the results show that the method is a reliable technique that can be used to handle linear and nonlinear fractional partial differential equations [21,22], also a Multistep Generalized Differential Transform method was used to investigate both Rabinovich–Fabrikant model involving Caputo fractional derivative subjected to appropriate initial conditions and construct a novel robust algorithm in finding numerical approximate solutions to the following model of time–space fractional Fokker-Planck equation [23,24].

The idea of DTM was first suggested by Zhou [27] in 1986 and used in solving the linear or nonlinear ordinary DEs under initial values, which arises in electric circuit problems. This method has a major advantage on directly solving linear or nonlinear DEs since it does not require discretization, perturbation, or linearization. Contrary to the usual, Taylor series can be formulated differently with this method. This method may establish an exact solution in the polynomial form for solutions of DEs.

As known, Taylor series method necessitates symbolic computation for obtaining Taylor series solutions. However, the DTM follows an iterative algorithm for obtaining the same solutions. Moreover, this method significantly minimizes calculation time while the application of Taylor series method takes long time for higher order DEs. Despite the DTM can provide approximate solutions in form of convergent series for a wide range of DEs, it has some disadvantages: the series solution never converges in a wider region and provides slow convergent speed while the series solution constantly converges in a very small interval. The aim of this letter is the use of the multi step differential transformation Ms-DTM that has been developed in [28,29] to provide analytical approximate solutions of the equations of motion of the coupled pendulum system under some important initial conditions.

Description of the system and derivation of equations of motion

The physical system that will be studied here is illustrated and described in Fig. 1 below, where two identical simple pendulums each of length $(l)$and mass (m) coupled together with a massless spring of stiffens (κ) that connects the two strings at their mid lengths [3].

The kinetic $(K.E)$ and potential $(P.E)$ energies of the system respectively read [3]:

$$K.E = \frac{ml^2}{2}\left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right) \tag{2.1}$$

$$P.E = \frac{mgl}{2}\left(\phi_1^2 + \phi_2^2\right) + \frac{\kappa}{2}\left(\frac{l}{2}\right)^2 (\phi_2 - \phi_1)^2, \tag{2.2}$$

where the first part in (1.2) represents the non– coupling potential energy and the second term represents the coupling potential energy.

Now to construct the Lagrangian of the system we plug (2.1) and (2.2) into $L = K.E - P.E$ and as a result we have

$$L = \frac{ml^2}{2}\left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right) - \frac{mgl}{2}\left(\phi_1^2 + \phi_2^2\right) - \frac{\kappa}{2}\left(\frac{l}{2}\right)^2 (\phi_2 - \phi_1)^2 \tag{2.3}$$

The next step is deriving the Euler-Lagrange equations (the equations of motion) by substituting (2.3) into the $\dot{q}_i$ following relation [1–4] $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$. Thus for$q_1 = \phi_1$, and $q_2 = \phi_2$the CELE's respectively read:

$$\ddot{\phi}_1 + \frac{g}{l}\phi_1 + \frac{k}{4m}(\phi_1 - \phi_2) = 0, \tag{2.4}$$

$$\ddot{\phi}_2 + \frac{g}{l}\phi_2 + \frac{k}{4m}(\phi_2 - \phi_1) = 0. \tag{2.5}$$

Setting $\kappa = 0$(no coupling) then (2.4) and (2.5) reduced to two known separate independent second order linear differential equations each with angular frequency $\omega_0 = \sqrt{\frac{g}{l}}$ with very well-known analytical solution describing the oscillatory motion of the two strings refer to [5], ch. 3.

For $\kappa \neq 0$(coupling case) equations (2.4) and (2.5) can not be solved analytically, for this we are going to solve them numerically in the next section using DTM and Ms-DTM.

It is of importance to emphasize that although one can find physical description for this system in literature but it has not been studied numerically before using DTM or Ms-DTM in such details according to the authors' knowledge, but it has been studied fractionally [30].

Analytic solution of the problem

In this section we obtain the analytical solution for the system (2.4)–(2.5). The system (2.4)–(2.5) can be reformulated as a system of first-order equations as follow: letting $u_1 = \frac{d\phi_1}{dt}$ and $u_2 = \frac{d\phi_2}{dt}$, then (2.4) and (2.5) can be written as

$$\begin{cases} \dfrac{d\phi_1}{dt} = u_1, \\ \dfrac{du_1}{dt} = -\dfrac{g}{b}\phi_1 + \dfrac{k}{4m}(\phi_2 - \phi_1), \\ \dfrac{d\phi_1}{dt} = u_2, \\ \dfrac{du_2}{dt} = -\dfrac{g}{b}\phi_2 + \dfrac{k}{4m}(\phi_1 - \phi_2) \end{cases} \tag{3.1}$$

The above system can be written in the matrix form [34]

$$\dot{\boldsymbol{u}} = A\boldsymbol{u}, A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{g}{b} - \dfrac{k}{4m} & 0 & \dfrac{k}{4m} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k}{4m} & 0 & -\dfrac{g}{b} - \dfrac{k}{4m} & 0 \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} \phi_1 \\ u_1 \\ \phi_2 \\ u_2 \end{bmatrix} \tag{3.2}$$

First, find the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ -\dfrac{g}{b} - \dfrac{k}{4m} & -\lambda & \dfrac{k}{4m} & 0 \\ 0 & 0 & -\lambda & 1 \\ \dfrac{k}{4m} & 0 & -\dfrac{g}{b} - \dfrac{k}{4m} & -\lambda \end{bmatrix}$$
$$= \lambda^4 + \left(\frac{2g}{b} + \frac{k}{2m}\right)\lambda^2 + \frac{g^2}{b^2} + \frac{gk}{2bm}, \tag{3.3}$$

which means that we have four imaginary eigenvalues $\lambda_1 = -\frac{i\sqrt{g}}{\sqrt{b}}, \lambda_2 = \frac{i\sqrt{g}}{\sqrt{b}}, \lambda_3 = -\frac{i\sqrt{b^2km+2bgm^2}}{\sqrt{2}bm}$ and $\lambda_4 = \frac{i\sqrt{bk+2gm}}{\sqrt{2}\sqrt{b}\sqrt{m}}$.

Now to the eigenvectors:

Case $\lambda_1 = -\frac{i\sqrt{g}}{\sqrt{b}}$. We have

$$A - \lambda_1 I = \begin{bmatrix} \dfrac{i\sqrt{g}}{\sqrt{b}} & 1 & 0 & 0 \\ -\dfrac{g}{b} - \dfrac{k}{4m} & \dfrac{i\sqrt{g}}{\sqrt{b}} & \dfrac{k}{4m} & 0 \\ 0 & 0 & \dfrac{i\sqrt{g}}{\sqrt{b}} & 1 \\ \dfrac{k}{4m} & 0 & -\dfrac{g}{b} - \dfrac{k}{4m} & \dfrac{i\sqrt{g}}{\sqrt{b}} \end{bmatrix}, \tag{3.4}$$

hence we need to solve the following system:

$$\begin{aligned} &\frac{i\sqrt{g}v_1}{\sqrt{b}} + v_2 = 0, \\ &\left(-\frac{g}{b} - \frac{k}{4m}\right)v_1 + \frac{i\sqrt{g}v_2}{\sqrt{b}} + \frac{kv_3}{4m} = 0, \\ &\frac{i\sqrt{g}v_3}{\sqrt{b}} + v_4 = 0, \\ &\frac{kv_1}{4m} + \left(-\frac{g}{b} - \frac{k}{4m}\right)v_3 + \frac{i\sqrt{g}v_4}{\sqrt{b}} = 0. \end{aligned} \tag{3.5}$$

An eigenvector can be taken as

$$v_1 = \left(\frac{i\sqrt{b}}{\sqrt{g}}, 1, \frac{i\sqrt{b}}{\sqrt{g}}, 1\right)^T.$$

Case $\lambda_2 = \frac{i\sqrt{g}}{\sqrt{b}}$. We have

$$A - \lambda_2 I = \begin{bmatrix} -\dfrac{i\sqrt{g}}{\sqrt{b}} & 1 & 0 & 0 \\ -\dfrac{g}{b} - \dfrac{k}{4m} & -\dfrac{i\sqrt{g}}{\sqrt{b}} & \dfrac{k}{4m} & 0 \\ 0 & 0 & -\dfrac{i\sqrt{g}}{\sqrt{b}} & 1 \\ \dfrac{k}{4m} & 0 & -\dfrac{g}{b} - \dfrac{k}{4m} & -\dfrac{i\sqrt{g}}{\sqrt{b}} \end{bmatrix}, \tag{3.6}$$

hence we need to solve the following system:

$$\begin{aligned} &-\frac{i\sqrt{g}v_1}{\sqrt{b}} + v_2 = 0, \\ &\left(-\frac{g}{b} - \frac{k}{4m}\right)v_1 - \frac{i\sqrt{g}v_2}{\sqrt{b}} + \frac{kv_3}{4m} = 0, \\ &-\frac{i\sqrt{g}v_3}{\sqrt{b}} + v_4 = 0, \\ &\frac{kv_1}{4m} + \left(-\frac{g}{b} - \frac{k}{4m}\right)v_3 - \frac{i\sqrt{g}v_4}{\sqrt{b}} = 0. \end{aligned} \tag{3.7}$$

Thus, an eigenvector can be taken as

$$v_2 = \left(-\frac{i\sqrt{b}}{\sqrt{g}}, 1, -\frac{i\sqrt{b}}{\sqrt{g}}, 1\right)^T.$$

Case $\lambda_3 = -\frac{\sqrt{-bm(bk+2gm)}}{\sqrt{2}bm}$. We have

$$A - \lambda_3 I = \begin{bmatrix} \dfrac{\sqrt{-bk-2gm}}{\sqrt{2}\sqrt{b}\sqrt{m}} & 1 & 0 & 0 \\ -\dfrac{g}{b} - \dfrac{k}{4m} & \dfrac{\sqrt{-bk-2gm}}{\sqrt{2}\sqrt{b}\sqrt{m}} & \dfrac{k}{4m} & 0 \\ 0 & 0 & \dfrac{\sqrt{-bk-2gm}}{\sqrt{2}\sqrt{b}\sqrt{m}} & 1 \\ \dfrac{k}{4m} & 0 & -\dfrac{g}{b} - \dfrac{k}{4m} & \dfrac{\sqrt{-bk-2gm}}{\sqrt{2}\sqrt{b}\sqrt{m}} \end{bmatrix}, \tag{3.8}$$

It yields the following system:

$$\begin{aligned} &\frac{\sqrt{-bk-2gm}v_1}{\sqrt{2}\sqrt{b}\sqrt{m}} + v_2 = 0, \\ &\left(-\frac{g}{b} - \frac{k}{4m}\right)v_1 + \frac{\sqrt{-bk-2gm}v_2}{\sqrt{2}\sqrt{b}\sqrt{m}} + \frac{kv_3}{4m} = 0, \\ &\frac{\sqrt{-bk-2gm}v_3}{\sqrt{2}\sqrt{b}\sqrt{m}} + v_4 = 0, \\ &\frac{kv_1}{4m} + \left(-\frac{g}{b} - \frac{k}{4m}\right)v_3 + \frac{\sqrt{-bk-2gm}v_4}{\sqrt{2}\sqrt{b}\sqrt{m}} = 0. \end{aligned} \tag{3.9}$$

Thus, an eigenvector can be taken as

$$v_3 = \left(-\frac{\sqrt{2}\sqrt{-bm(bk+2gm)}}{bk+2gm}, -1, \frac{\sqrt{2}\sqrt{-bm(bk+2gm)}}{bk+2gm}, 1\right)^T.$$

Case $\lambda_4 = \frac{\sqrt{-bm(bk+2gm)}}{\sqrt{2}bm}$. We have

$$A-\lambda_4 I=\begin{bmatrix} -\dfrac{\sqrt{-bm(bk+2gm)}}{\sqrt{2}\,bm} & 1 & 0 & 0 \\ -\dfrac{g}{b}-\dfrac{k}{4m} & -\dfrac{\sqrt{-bm(bk+2gm)}}{\sqrt{2}\,bm} & \dfrac{k}{4m} & 0 \\ 0 & 0 & -\dfrac{\sqrt{-bm(bk+2gm)}}{\sqrt{2}\,bm} & 1 \\ \dfrac{k}{4m} & 0 & -\dfrac{g}{b}-\dfrac{k}{4m} & -\dfrac{\sqrt{-bm(bk+2gm)}}{\sqrt{2}\,bm} \end{bmatrix}, \tag{3.10}$$

It yields the following system:

$$\begin{aligned} &-\frac{\sqrt{-bm(bk+2gm)}v_1}{\sqrt{2}bm}+v_2=0,\\ &\left(-\frac{g}{b}-\frac{k}{4m}\right)v_1-\frac{\sqrt{-bm(bk+2gm)}v_2}{\sqrt{2}bm}+\frac{kv_3}{4m}=0,\\ &-\frac{\sqrt{-bm(bk+2gm)}v_3}{\sqrt{2}bm}+v_4=0,\\ &\frac{kv1}{4m}+\left(-\frac{g}{b}-\frac{k}{4m}\right)v_3-\frac{\sqrt{-bm(bk+2gm)}v_4}{\sqrt{2}bm}=0. \end{aligned} \tag{3.11}$$

Thus, an eigenvector can be taken as

$$v_4=\left(\frac{\sqrt{2}\,\sqrt{-bm(bk+2gm)}}{bk+2gm},-1,-\frac{\sqrt{2}\,\sqrt{-bm(bk+2gm)}}{bk+2gm},1\right)^T.$$

We know that since v_1, v_2, v_3 are eigenvectors corresponding to distinct eigenvalues, therefore they are linearly independent, and hence the general solution to the problem can be written as [35]

$$\boldsymbol{u}(t)=C_1\begin{bmatrix}\dfrac{i\sqrt{b}}{\sqrt{g}}\\ 1\\ \dfrac{i\sqrt{b}}{\sqrt{g}}\\ 1\end{bmatrix}e^{-\frac{i\sqrt{g}}{\sqrt{b}}t}+C_2\begin{bmatrix}-\dfrac{i\sqrt{b}}{\sqrt{g}}\\ 1\\ -\dfrac{i\sqrt{b}}{\sqrt{g}}\\ 1\end{bmatrix}e^{\frac{i\sqrt{g}}{\sqrt{b}}t}+C_3\begin{bmatrix}-\dfrac{\sqrt{2}\,\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ -1\\ \dfrac{\sqrt{2}\,\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ 1\end{bmatrix}e^{-\frac{\sqrt{-bm(bk+2gm)}}{\sqrt{2}bm}t}+C_4\begin{bmatrix}\dfrac{\sqrt{2}\,\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ -1\\ -\dfrac{\sqrt{2}\,\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ 1\end{bmatrix}e^{\frac{\sqrt{-bm(bk+2gm)}}{\sqrt{2}bm}t} \tag{3.12}$$

Reading this row by row, we can rewrite this as

$$\begin{cases} \phi_1(t)=\dfrac{i\sqrt{b}\,C_1e^{-\frac{i\sqrt{g}\,t}{\sqrt{b}}}}{\sqrt{g}}-\dfrac{i\sqrt{b}\,C_2e^{\frac{i\sqrt{g}\,t}{\sqrt{b}}}}{\sqrt{g}}-\dfrac{\sqrt{2}\,C_3e^{-\frac{\sqrt{-bm(bk+2gm)}\,t}{\sqrt{2}\,bm}}\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ \qquad +\dfrac{\sqrt{2}\,C_4e^{\frac{\sqrt{-bm(bk+2gm)}\,t}{\sqrt{2}\,bm}}\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ u_1(t)=C_1e^{-\frac{i\sqrt{g}\,t}{\sqrt{b}}}+C_2e^{\frac{i\sqrt{g}\,t}{\sqrt{b}}}-C_3e^{-\frac{\sqrt{-bm(bk+2gm)}\,t}{\sqrt{2}\,bm}}-C_4e^{\frac{\sqrt{-bm(bk+2gm)}\,t}{\sqrt{2}\,bm}}\\ \phi_2(t)=\dfrac{i\sqrt{b}\,C_1e^{-\frac{i\sqrt{g}\,t}{\sqrt{b}}}}{\sqrt{g}}-\dfrac{i\sqrt{b}\,C_2e^{\frac{i\sqrt{g}\,t}{\sqrt{b}}}}{\sqrt{g}}+\dfrac{\sqrt{2}\,C_3e^{-\frac{\sqrt{-bm(bk+2gm)}\,t}{\sqrt{2}\,bm}}\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ \qquad -\dfrac{\sqrt{2}\,C_4e^{\frac{\sqrt{-bm(bk+2gm)}\,t}{\sqrt{2}\,bm}}\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ u_2(t)=C_1e^{-\frac{i\sqrt{g}\,t}{\sqrt{b}}}+C_2e^{\frac{i\sqrt{g}\,t}{\sqrt{b}}}+C_3e^{-\frac{\sqrt{-bm(bk+2gm)}\,t}{\sqrt{2}\,bm}}+C_4e^{\frac{\sqrt{-bm(bk+2gm)}\,t}{\sqrt{2}\,bm}} \end{cases} \tag{3.13}$$

To solve the system (2.4)-(2.5) under the initial conditions $\phi_1(0)=a$, $\phi_2(0)=b$, $\phi'_1(0)=c, \phi'_2(0)=d$, namely $\phi_1(0)=a, \phi_2(0)=b$, $u_1(0)=c, u_2(0)=d$, we must solve

$$C_1\begin{bmatrix}\frac{i\sqrt{b}}{\sqrt{g}}\\ 1\\ \frac{i\sqrt{b}}{\sqrt{g}}\\ 1\end{bmatrix}e^{-\frac{i\sqrt{g}}{\sqrt{b}}.0}+C_2\begin{bmatrix}-\frac{i\sqrt{b}}{\sqrt{g}}\\ 1\\ -\frac{i\sqrt{b}}{\sqrt{g}}\\ 1\end{bmatrix}e^{\frac{i\sqrt{g}}{\sqrt{b}}.0}+C_3\begin{bmatrix}-\frac{\sqrt{2}\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ -1\\ \frac{\sqrt{2}\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ 1\end{bmatrix}$$

$$e^{-\frac{\sqrt{-bm(bk+2gm)}}{\sqrt{2}bm}.0}+C_4\begin{bmatrix}\frac{\sqrt{2}\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ -1\\ -\frac{\sqrt{2}\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ 1\end{bmatrix}e^{\frac{\sqrt{-bm(bk+2gm)}}{\sqrt{2}bm}.0}=\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix} \tag{3.14}$$

That is,

$$C_1\begin{bmatrix}\frac{i\sqrt{b}}{\sqrt{g}}\\ 1\\ \frac{i\sqrt{b}}{\sqrt{g}}\\ 1\end{bmatrix}+C_2\begin{bmatrix}-\frac{i\sqrt{b}}{\sqrt{g}}\\ 1\\ -\frac{i\sqrt{b}}{\sqrt{g}}\\ 1\end{bmatrix}+C_3\begin{bmatrix}-\frac{\sqrt{2}\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ -1\\ \frac{\sqrt{2}\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ 1\end{bmatrix}$$

$$+C_4\begin{bmatrix}\frac{\sqrt{2}\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ -1\\ -\frac{\sqrt{2}\sqrt{-bm(bk+2gm)}}{bk+2gm}\\ 1\end{bmatrix}=\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}$$

The solution is

$$\phi_1(t)=\frac{(a+c)\sqrt{g}\cos\left(\frac{\sqrt{g}\,t}{\sqrt{b}}\right)+(a-c)\sqrt{g}\cosh\left(\frac{\sqrt{-m(bk+2gm)}\,t}{\sqrt{2}\sqrt{b}m}\right)+\sqrt{b}(b+d)\sin\left(\frac{\sqrt{g}\,t}{\sqrt{b}}\right)+\frac{\sqrt{2}\sqrt{b}(b-d)\sqrt{g}m\sinh\left(\frac{\sqrt{-m(bk+2gm)}\,t}{\sqrt{2}\sqrt{b}m}\right)}{\sqrt{-m(bk+2gm)}}}{2\sqrt{g}}$$

and

$$\phi_2(t)=\frac{(a+c)\sqrt{g}\cos\left(\frac{\sqrt{g}\,t}{\sqrt{b}}\right)+(-a+c)\sqrt{g}\cosh\left(\frac{\sqrt{-m(bk+2gm)}\,t}{\sqrt{2}\sqrt{b}m}\right)+\sqrt{b}(b+d)\sin\left(\frac{\sqrt{g}\,t}{\sqrt{b}}\right)+\frac{\sqrt{2}\sqrt{b}(-b+d)\sqrt{g}m\sinh\left(\frac{\sqrt{-m(bk+2gm)}\,t}{\sqrt{2}\sqrt{b}m}\right)}{\sqrt{-m(bk+2gm)}}}{2\sqrt{g}}$$

In Section 5, the analytical solutions are given for the special choices of a, b, c, and d.

Numerical method

The DTM

The DTM is one of the semi-analytical methods that provides in the form of polynomial solutions of ordinary and partial differential equations as approximations to the exact solutions that is adequately differentiable.

The basic definitions and theorems of the DTM and its practicality for a variety of differential equations are given in [31–33].

For the reader's convenience, we review the DTM below. The differential transformation of the kth derivative of function $f(t)$ is defined as follows

$$F(k)=\frac{1}{k!}\left[\frac{d^kf(t)}{dt^k}\right]_{t=t_0}, \tag{4.1}$$

where $f(t)$ is the original function and $F(k)$ is the differential transformation of $f(t)$. The inverse differential transformation of $F(k)$ is defined as

$$f(t)=\sum_{k=0}^{\infty}F(k)(t-t_0)^k. \tag{4.2}$$

From (4.1)-(4.2), we get

$$f(t)=\sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\frac{d^kf(t)}{dt^k} \tag{4.3}$$

which refers that the idea of differential transformation is originated from Taylor series expansion, but the method does not calculate the required derivatives representatively. However, the required derivatives are evaluated by a recurrence relation which are depicted by the transformed equations corresponding to the original function.

For implementation purposes, the function $f(t)$ expressed by a finite series given in Eq. (4.2) can be written as

$$f(t)\approx\sum_{k=0}^{N}F(k)(t-t_0)^k. \tag{4.4}$$

Here N is the number of terms and is also decided by the convergence of natural frequency of the DTM.The fundamental operations of the DTM can easily be obtained and are given in Table 1.The basic steps of the DTM are as follows:

First, the differential transformation formula given by Eq. (4.1) is applied to the considered problem and then a recurrence relation is generated. Second, solving the resulting equation and using the inverse differential transformation formula (4.2), we can obtain the solution of the considered problem.

The Ms-DTM

This section is reserved for the Ms-DTM for obtaining the numerical solution of DEs. For this purpose, we consider the following nonlinear initial value problem

$$f\left(t,y,y^{'},\cdots,y^{(p)}\right)=0, y^{(p)}(0)=c_k, \text{for} k=0,1,\cdots,p-1 \tag{4.5}$$

Let's assume that we want to find the solution of the problem (4.5) on a given $[0,T]$ interval.

In the real applications, the approximate solution of the problem (4.5) via the DTM can be expressed by the following finite series

$$y(t)=\sum_{n=0}^{N}a_n t^n. \tag{4.6}$$

The multi-step approach considers the establishment of the approximate solution by a different point of view. Suppose that the interval [0, T] is divided into M subintervals $[t_0,t_1]$,$[t_1,t_2]$, $\cdots$, $[t_{M-1},t_M]$ of equal step size $h=\frac{T}{M}$, where $t_m=m.h$. The basic steps of the DTM are as follows.

First, over the first interval $[0,t_1]$, using the initial conditions ${y_1}^{(k)}(0)=c_k$, the DTM is applied to Eq. (4.5), thus the following approximate solution is obtained

$$y_1(t)=\sum_{n=0}^{K}a_{1n}t^n, t\in[0,t_1], \tag{4.7}$$

For $m\geq 2$ and at each subinterval $[t_{m-1},t_m]$, the initial conditions ${y_m}^{(k)}(t_{m-1})={y_{m-1}}^{(k)}(t_{m-1})$ are used and then the DTM is applied to Eq. (4.5) over the interval $[t_{m-1},t_m]$, where t_0 in Eq. (4.1) is altered witht_{m-1}.

The operation is repeated again and again and for the solution of $y(t)$, a sequence of approximate solutions $y_m(t), m=1,2,\cdots,M$, is generated as follows

$$y_m(t)=\sum_{n=0}^{N}a_{mn}(t-t_{m-1})^n, t\in[t_m,t_{m-1}], \tag{4.8}$$

where $N=K.M$. In actual fact, the Ms- DTM suggests the following solution

$$y(t)=\begin{cases} y_1(t), t\in[0,t_1] \\ y_2(t), t\in[t_1,t_2] \\ \vdots \\ y_m(t), t\in[t_{M-1},t_M] \end{cases} \tag{4.9}$$

The Ms-DTM has a computationally simple performance for every desired value of h. If the step size h equals to T, then it is easily observed this approach reduces to the classical DTM.

In the next section, the principal advantage of the above algorithm is that the obtained series solution converges for wide range of time is to be seen.

Simulation results and discussion

In this section, the equations of motion of the coupled pendulum system (2.4)–(2.5) are solved by an efficient, fast and high accurate method, namely the multi step differential transform method. The simulation results for the system given in Eqs. (2.4) and (2.5) are presented where certain values of the parameters m, b, and g have to be chosen ($m=1kg, b=1m$, and $g=9.8m/s^2$) in all cases. Below, the following three important cases have been considered:

Case 1: Symmetric case:

In this case, $\phi_1(0)=\phi_2(0)=A$ and $\phi'_1(0)=\phi'_2(0)=0$ are considered. In view of the multi step differential transform method, given in the previous section, and the operations of differential transform given in Table 1, applying the differential transform to the system (2.4)-(2.5), we obtain

$$\begin{cases} (k+1)(k+2)\Phi_{1,i}(k)+\frac{g}{b}\Phi_{1,i}(k)+\frac{k}{4m}\left[\Phi_{1,i}(k)-\Phi_{2,i}(k)\right]=0 \\ (k+1)(k+2)\Phi_{2,i}(k)+\frac{g}{b}\Phi_{2,i}(k)+\frac{k}{4m}\left[\Phi_{2,i}(k)-\Phi_{1,i}(k)\right]=0 \end{cases} \tag{5.1}$$

or

$$\begin{cases} \Phi_{1,i}(k)=\dfrac{-\frac{g}{b}\Phi_{1,i}(k)+\frac{k}{4m}\left[\Phi_{2,i}(k)-\Phi_{1,i}(k)\right]}{(k+1)(k+2)} \\ \Phi_{2,i}(k)=\dfrac{-\frac{g}{b}\Phi_{2,i}(k)+\frac{k}{4m}\left[\Phi_{1,i}(k)-\Phi_{2,i}(k)\right]}{(k+1)(k+2)} \end{cases}, \tag{5.2}$$

where $\Phi_1(k)$ and $\Phi_2(k)$ are the differential transforms of $\phi_1(t)$ and$\phi_2(t)$, respectively. The transformed initial conditions at $t_0=0$are

$$\begin{cases} \Phi_{1,i}(1)=\phi^{'}_{1,i}(t_i), i=1,2,\cdots,K\leq M \\ \Phi_{2,i}(1)=\phi^{'}_{2,i}(t_i), i=1,2,\cdots,K\leq M \\ \Phi_{1,i}(0)=\phi_{1,i-1}(t_i), \Phi_{1,0}(0)=A, \Phi_{1,0}(1)=0 \\ \Phi_{2,i}(0)=\phi_{2,i-1}(t_i), \Phi_{2,0}(0)=A, \Phi_{2,0}(1)=0 \end{cases} \tag{5.3}$$

After solving Eq. (3.10) and using the Eq.(3.11) the solution for the system (2.4)-(2.5) can be obtained for any each time step. For this case, the following initial conditions were considered:$\phi_1(0)=\phi_2(0)=5$ and $\phi'_1(0)=\phi'_2(0)=0$. Thus, we obtain the following solutions for the time step 0.25:

$$\phi_1(t)=\begin{cases} \phi_{1,0}(t)=5.-24.5t^2+20.0083t^4-6.53606t^6+1.14381t^8-0.124548t^{10}t\in[0,0.25) \\ \phi_{1,0.25}(t)=3.54533-11.0372(t-0.25)-17.3721(t-0.25)^2+18.0275(t-0.25)^3 \\ +14.1872(t-0.25)^4-8.83346(t-0.25)^5-4.63449(t-0.25)^6 \\ +2.06114(t-0.25)^7 0.811036(t-0.25)^8-0.280544(t-0.25)^9-0.0883129(t-0.25)^{10}t\in[0.25,0.5) \\ \phi_{1,0.5}(t)=0.0277436-15.6522(t-0.5)-0.135943(t-0.5)^2+25.5653(t-0.5)^3 \\ +0.11102(t-0.5)^4-12.527(t-0.5)^5-0.0362667(t-0.5)^6+2.92297(t-0.5)^7 \\ +0.00634667(t-0.5)^8-0.397848(t-0.5)^9-0.000691082(t-0.5)^{10}t\in[0.5,0.75) \\ \phi_{1,0.75}(t)=-3.50599-11.1597(t-0.75)+17.1793(t-0.75)^2+18.2275(t-0.75)^3 \\ -14.0298(t-0.75)^4-8.93149(t-0.75)^5+4.58306(t-0.75)^6+ \\ 2.08401(t-0.75)^7-0.802036(t-0.75)^8-0.283657(t-0.75)^9+0.0873328(t-0.75)^{10}t\in[0.75,1) \\ \vdots \\ \phi_{1,4}(t)=4.99507+0.694584(t-4)-24.4759(t-4)^2-1.13449(t-4)^3 \\ +19.9886(t-4)^4+0.555899(t-4)^5-6.52962(t-4)^6-0.12971(t-4)^7 \\ +1.14268(t-4)^8+0.0176549(t-4)^9-0.124425(t-4)^{10}t\in[4,4.25) \\ \phi_{1,4.25}(t)=3.69829-10.5338(t-4.25)-18.1216(t-4.25)^2+17.2053(t-4.25)^3+14.7993(t-4.25)^4 \\ -8.43059(t-4.25)^5-4.83445(t-4.25)^6+1.96714(t-4.25)^7 \\ +0.846028(t-4.25)^8-0.267749(t-4.25)^9-0.0921231(t-4.25)^{10}t\in[4.25,4.5) \\ \phi_{1,4.5}(t)=0.24959-15.633(t-4.5)-1.22299(t-4.5)^2+25.5338(t-4.5)^3 \\ +0.998774(t-4.5)^4-12.5116(t-4.5)^5-0.326266(t-4.5)^6+2.91937(t-4.5)^7 \\ +0.0570966(t-4.5)^8-0.397359(t-4.5)^9-0.00621718(t-4.5)^{10}t\in[4.5,4.75) \\ \phi_{1,4.75}(t)=-3.34434-11.6358(t-4.75)+16.3873(t-4.75)^2+19.0051(t-4.75)^3 \\ -13.3829(t-4.75)^4-9.31248(t-4.75)^5+4.37176(t-4.75)^6+2.17291(t-4.75)^7 \\ -0.765058(t-4.75)^8-0.295758(t-4.75)^9+0.0833063(t-4.75)^{10}t\in[4.75,5) \end{cases} \tag{5.4}$$

$$\phi_2(t)=\begin{cases} \phi_{1,0}(t)=5.-24.5t^2+20.0083t^4-6.53606t^6+1.14381t^8-0.124548t^{10}t\in[0,0.25)\\ \phi_{1,0.25}(t)=3.54533-11.0372(t-0.25)-17.3721(t-0.25)^2+18.0275(t-0.25)^3\\ +14.1872(t-0.25)^4-8.83346(t-0.25)^5-4.63449(t-0.25)^6\\ +2.06114(t-0.25)^7 0.811036(t-0.25)^8-0.280544(t-0.25)^9-0.0883129(t-0.25)^{10}t\in[0.25,0.5)\\ \phi_{1,0.5}(t)=0.0277436-15.6522(t-0.5)-0.135943(t-0.5)^2+25.5653(t-0.5)^3\\ +0.11102(t-0.5)^4-12.527(t-0.5)^5-0.0362667(t-0.5)^6+2.92297(t-0.5)^7\\ +0.00634667(t-0.5)^8-0.397848(t-0.5)^9-0.000691082(t-0.5)^{10}t\in[0.5,0.75)\\ \phi_{1,0.75}(t)=-3.50599-11.1597(t-0.75)+17.1793(t-0.75)^2+18.2275(t-0.75)^3\\ -14.0298(t-0.75)^4-8.93149(t-0.75)^5+4.58306(t-0.75)^6+\\ 2.08401(t-0.75)^7-0.802036(t-0.75)^8-0.283657(t-0.75)^9+0.0873328(t-0.75)^{10}t\in[0.75,1)\\ \vdots\\ \phi_{1,4}(t)=4.99507+0.694584(t-4)-24.4759(t-4)^2-1.13449(t-4)^3\\ +19.9886(t-4)^4+0.555899(t-4)^5-6.52962(t-4)^6-0.12971(t-4)^7\\ +1.14268(t-4)^8+0.0176549(t-4)^9-0.124425(t-4)^{10}t\in[4,4.25)\\ \phi_{1,4.25}(t)=3.69829-10.5338(t-4.25)-18.1216(t-4.25)^2+17.2053(t-4.25)^3+14.7993(t-4.25)^4\\ -8.43059(t-4.25)^5-4.83445(t-4.25)^6+1.96714(t-4.25)^7\\ +0.846028(t-4.25)^8-0.267749(t-4.25)^9-0.0921231(t-4.25)^{10}t\in[4.25,4.5)\\ \phi_{1,4.5}(t)=0.24959-15.633(t-4.5)-1.22299(t-4.5)^2+25.5338(t-4.5)^3\\ +0.998774(t-4.5)^4-12.5116(t-4.5)^5-0.326266(t-4.5)^6+2.91937(t-4.5)^7\\ +0.0570966(t-4.5)^8-0.397359(t-4.5)^9-0.00621718(t-4.5)^{10}t\in[4.5,4.75)\\ \phi_{1,4.75}(t)=-3.34434-11.6358(t-4.75)+16.3873(t-4.75)^2+19.0051(t-4.75)^3\\ -13.3829(t-4.75)^4-9.31248(t-4.75)^5+4.37176(t-4.75)^6+2.17291(t-4.75)^7\\ -0.765058(t-4.75)^8-0.295758(t-4.75)^9+0.0833063(t-4.75)^{10}t\in[4.75,5) \end{cases} \tag{5.5}$$

On the other hand, for $a=5, b=5, c=0$, and $d=0$, according to section 3 and the considered parameters, the analytical solutions are $\phi_1(t)=5\cos\left(\frac{\sqrt{g}t}{\sqrt{b}}\right)$ and $\phi_2(t)=5\cos\left(\frac{\sqrt{g}t}{\sqrt{b}}\right)$

The behaviour of ϕ_1 and ϕ_2 against time are presented with $k=10, 50$, and $100(N/m)$ in Figs. 2–4, respectively. From these three figures it is obvious that the motion of both masses is in phase with simple harmonic motion behaviour with a frequency of 0.50 unit, the spring constant k has no effect on the frequency of the two masses.

Case 2: Anti- symmetric case

In this case, $\phi_1(0)=-\phi_2(0)=A$ and $\phi'_1(0)=\phi'_2(0)=0$. The transformed initial conditions that correspond to this case are as follows:

$$\begin{cases} \Phi_{1,i}(1)=\phi'_{1,i}(t_i), i=1,2,\cdots,K\le M\\ \Phi_{2,i}(1)=\phi'_{2,i}(t_i), i=1,2,\cdots,K\le M\\ \Phi_{1,i}(0)=\phi_{1,i-1}(t_i), \Phi_{1,0}(0)=A, \Phi_{1,0}(1)=0\\ \Phi_{2,i}(0)=\phi_{2,i-1}(t_i), \Phi_{2,0}(0)=-A, \Phi_{2,0}(1)=0 \end{cases}. \tag{5.6}$$

The following initial conditions were taken:$\phi_1(0)=-\phi_2(0)=5$ and $\phi'_1(0)=\phi'_2(0)=0$. For $a=5, b=-5, c=0$, and $d=0$, according to section 3 and the considered parameters, the analytical solutions are $\phi_1(t)=5\cosh\left(\frac{\sqrt{-bm(bk+2gm)}\,t}{\sqrt{2}bm}\right)$ and$\phi_2(t)=-5\cosh\left(\frac{\sqrt{-bm(bk+2gm)}\,t}{\sqrt{2}bm}\right)$

The behaviour of ϕ_1 and ϕ_2 against time are presented with $k=10$, 50, and $100(N/m)$ in Figs. 5–7, respectively. In this case (anti- symmetric case) the two masses are moving in a simple harmonic motion but out of phase to each other. In this case, the frequency of the two masses is directly proportional to the spring constant k and increases with increasing k. It was approximately 0.61, 1.0, and 1.25 unit for k = 10, 50, and 100 N/m respectively.

Case 3: Beats case

In this case, $\phi_1(0)=A$, while $\phi_2(0)=0$, and $\phi'_1(0)=\phi'_2(0)=0$. The transformed initial conditions at $t_0=0$are

Table 1
Operations of DTM.

Original function	Transformed function
$f(t)=u(t)\pm v(t)$	$F(k)=U(k)\pm V(k)$
$f(t)=\alpha u(t)$	$F(k)=\alpha U(k)$
$f(t)=u(t)v(t)$	$F(k)=\sum_{l=0}^{k}U(l)V(k-l)$
$f(t)=\frac{du}{dt}$	$F(k)=(k+1)U(k)$
$f(t)=\frac{d^m u}{dt^m}$	$F(k)=(k+1)(k+2)\cdots(k+m)U(k+m)$
$f(t)=\int_{t_0}^{t}u(t)dt$	$F(k)=\frac{U(k-1)}{k}, k\ge1$
$f(t)=t^m$	$F(k)=\begin{cases}1, k=m\\0, k\ne m\end{cases}$
$f(t)=\exp(\lambda t)$	$F(k)=\frac{\lambda^k}{k!}$
$f(t)=\sin(\omega t+\alpha)$	$F(k)=\frac{\omega^k}{k!}\sin\left(\frac{\pi k}{2}+\alpha\right)$
$f(t)=\cos(\omega t+\alpha)$	$F(k)=\frac{\omega^k}{k!}\cos\left(\frac{\pi k}{2}+\alpha\right)$
$h(t)=\frac{f(t)}{g(t)}$	$H(k)=\frac{1}{G(0)}\left[F(k)-\sum_{m=0}^{k-1}H(m)G(k-m)\right]$
$f(t)=[g(t)]^b$	$F(k)=\begin{cases}G(0), k=0\\ \sum_{m=0}^{k-1}\frac{(b+1)m-k}{kG(0)}G(m)F(k-m), & k\ge1\end{cases}$

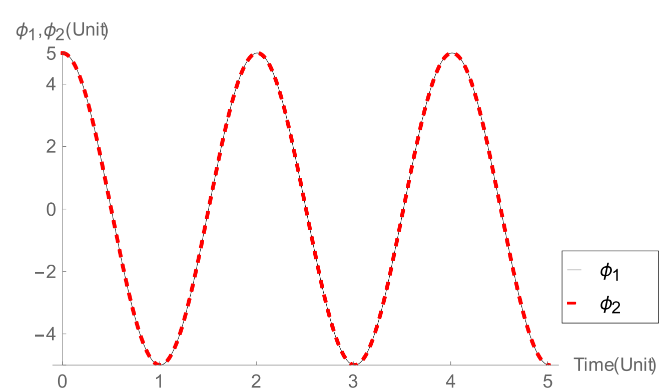

Fig. 2. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k=10$.

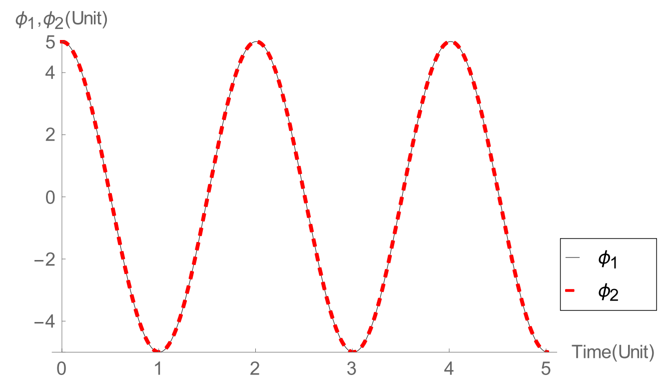

Fig. 3. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 50$.

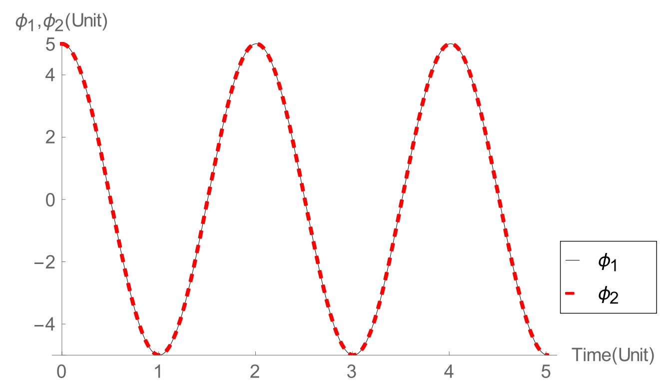

Fig. 4. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 100$.

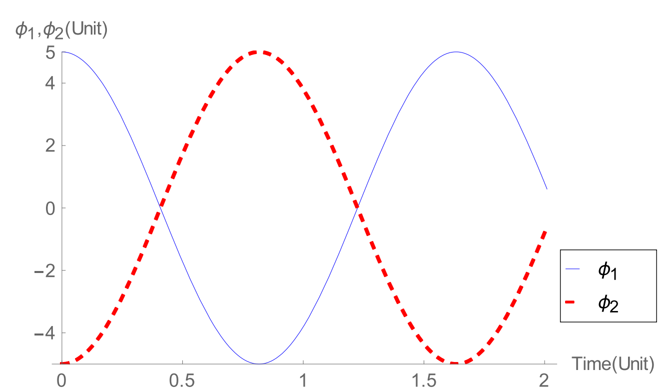

Fig. 5. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 10$.

$$\begin{cases} \Phi_{1,i}(1) = \phi'_{1,i}(t_i), i = 1, 2, \cdots, K \le M \\ \Phi_{2,i}(1) = \phi'_{2,i}(t_i), i = 1, 2, \cdots, K \le M \\ \Phi_{1,i}(0) = \phi_{1,i-1}(t_i), \Phi_{1,0}(0) = A, \Phi_{1,0}(1) = 0 \\ \Phi_{2,i}(0) = \phi_{2,i-1}(t_i), \Phi_{2,0}(0) = 0, \Phi_{2,0}(1) = 0 \end{cases} \tag{5.7}$$

The following initial conditions were taken: $\phi_1(0) = 5, \phi_2(0) = 0$ and $\phi'_1(0) = \phi'_2(0) = 0$. $a = 5, b = 0, c = 0$, and $d = 0$, according to section 3 and the considered parameters, the analytical solutions are $\phi_1(t) = \frac{5}{2}\left(\cos\left(\frac{\sqrt{g}t}{\sqrt{b}}\right) + \cosh\left(\frac{\sqrt{-bm(bk+2gm)}t}{\sqrt{2}bm}\right)\right)$ and $\phi_2(t) = \frac{5}{2}\left(\cos\left(\frac{\sqrt{g}t}{\sqrt{b}}\right) - \cosh\left(\frac{\sqrt{-bm(bk+2gm)}t}{\sqrt{2}bm}\right)\right)$

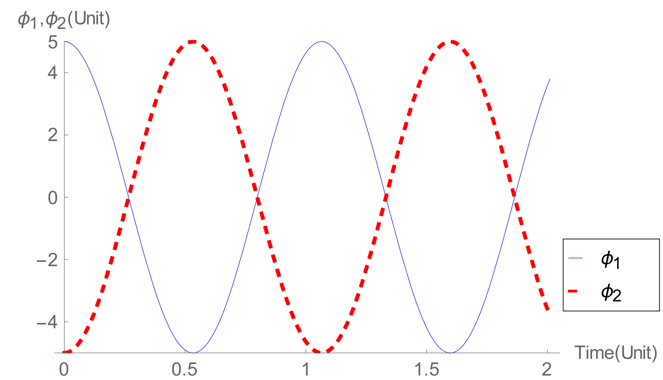

Fig. 6. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 50$.

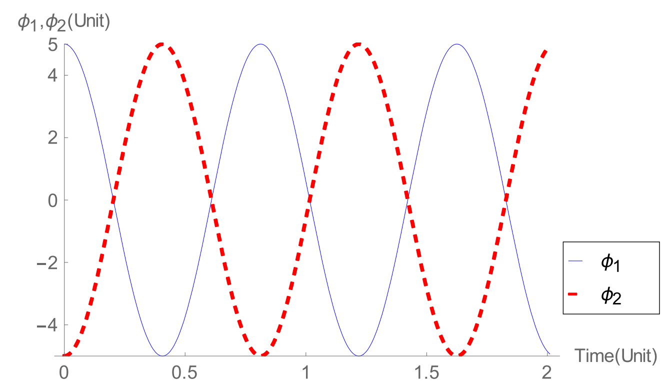

Fig. 7. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 100$.

Figs. 8–10 show the behaviour of ϕ_1 and ϕ_2 against time with $k = 10$, 50, and $100(N/m)$, respectively.

From Figs. 8–10 one can notice that the two masses are still have periodic motion but with changing amplitude. As k increases the two masses became more out of phase in motion giving a rise to beats phenomenon.

Furthermore, in Fig. 11 below, we investigate a special case where $\phi_1(0) = \phi_2(0) = 0$, while $\phi'_1(0) = 5$ and $\phi'_2(0) = 0$ when $k = 10$. $a = 0, b = 0, c = 5$, and $d = 0$, according to section 3 and the considered parameters, the analytical solutions are $\phi_1(t) = \frac{5}{2}\sqrt{b}\left(\frac{\sin\left(\frac{\sqrt{g}t}{\sqrt{b}}\right)}{\sqrt{g}} + \frac{\sqrt{2}m\sinh\left(\frac{\sqrt{-m(bk+2gm)}t}{\sqrt{2}\sqrt{b}m}\right)}{\sqrt{-m(bk+2gm)}}\right)$

and $\phi_2(t) = \frac{5}{2}\sqrt{b}\left(\frac{\sin\left(\frac{\sqrt{g}t}{\sqrt{b}}\right)}{\sqrt{g}} - \frac{\sqrt{2}m\sinh\left(\frac{\sqrt{-m(bk+2gm)}t}{\sqrt{2}\sqrt{b}m}\right)}{\sqrt{-m(bk+2gm)}}\right)$ Moreover, the transformed initial conditions that correspond to this case are as follows:

In Figs. 12–14, the Ms-DTM results are compared with obtained result from the DTM for $k = 10$ and all cases studied above respectively (symmetric, anti- symmetric, beats and the special case). From these curves, it can be said that Ms-DTM is more accurate than DTM in the long time interval as expected (Fig. 15).

Conclusion

Two pendulums coupled at mid length with a spring is studied. The Lagrangian of the system was first constructed and as a result the

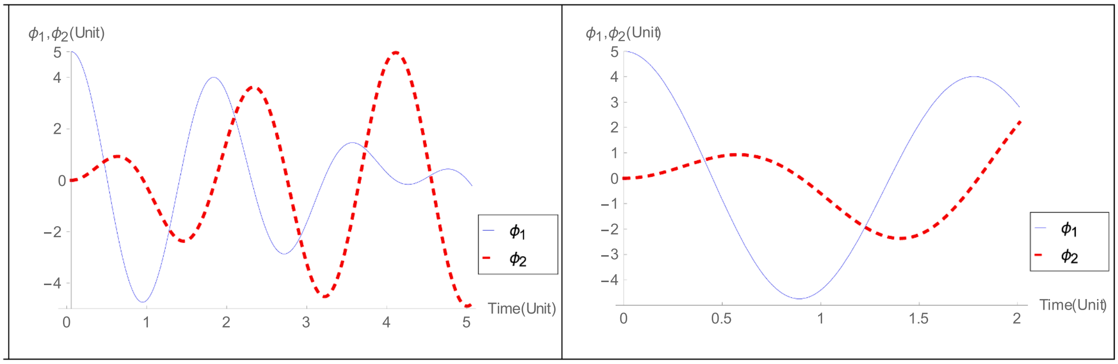

Fig. 8. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 10$.

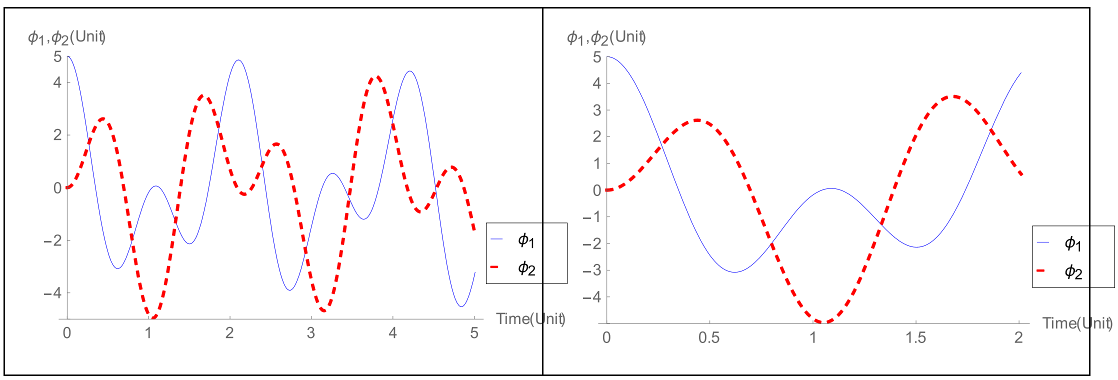

Fig. 9. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 50$.

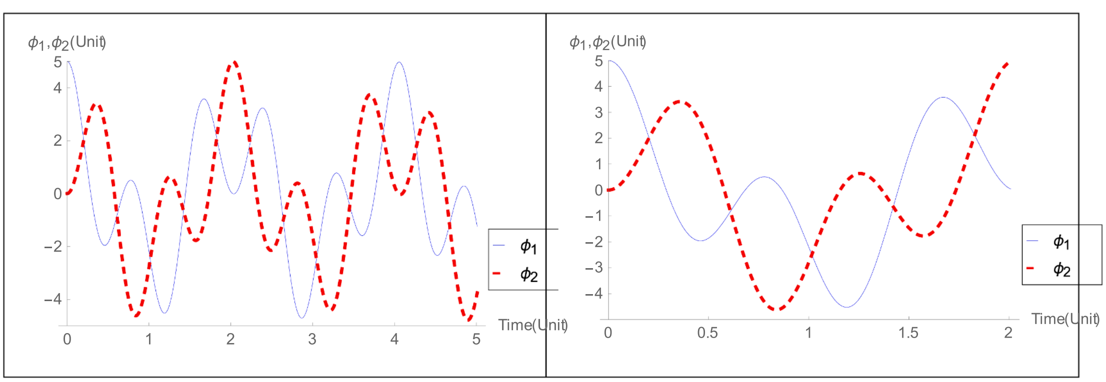

Fig. 10. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 100$.

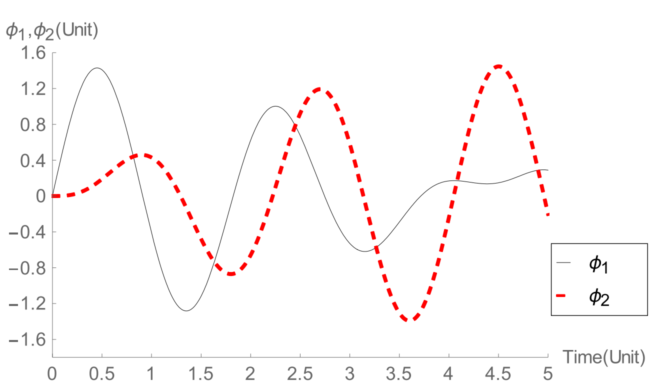

Fig. 11. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 10$.

equations of motion was derived. The system was studied analytically and after that a numerical solution was obtained using DTM, Ms-DTM and RK4 methods. It was shown that Ms-DTM is more accurate than DTM in the long time interval by virtue of extending the applicable domain to overcome the drawback of DTM solution, and in close agreement with RK4. The various initial conditions are also considered in order to illustrate the behavior of the considered system. It is observed that the system is sensitive to the change in the initial conditions, and that this causes a different behaviour. We demonstrate that Ms-DTM is a suitable and effective method for studying the dynamics and solutions of the considered problem.

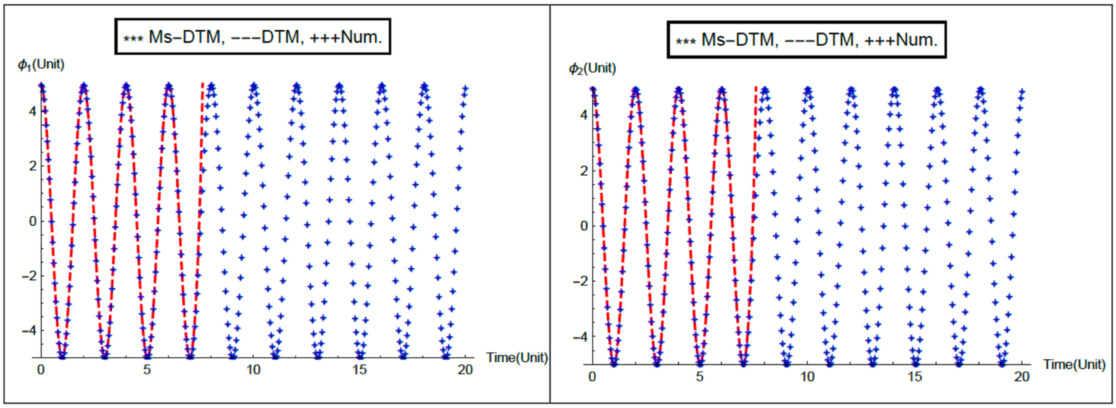

Fig. 12. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 10$.

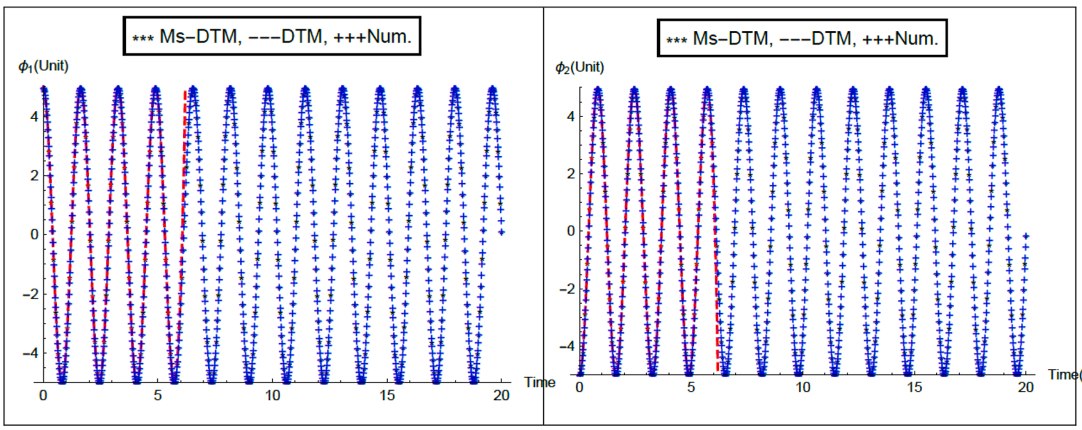

Fig. 13. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 10$.

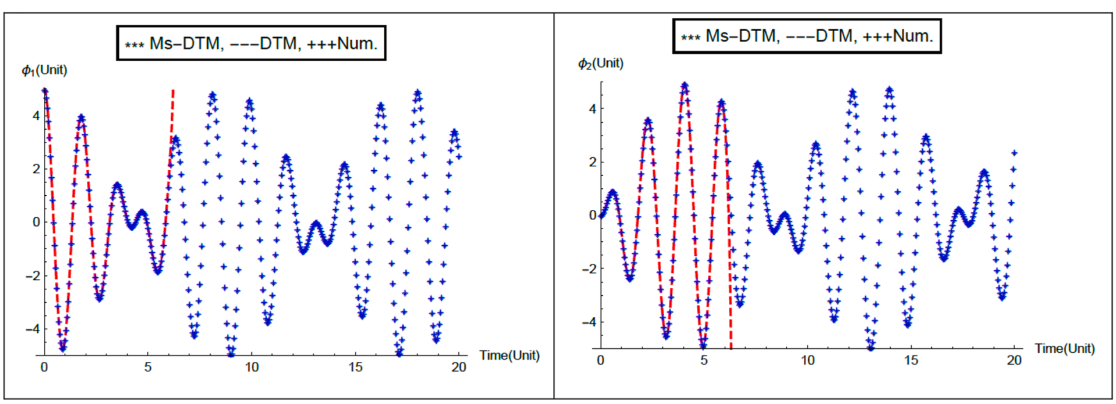

Fig. 14. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 10$.

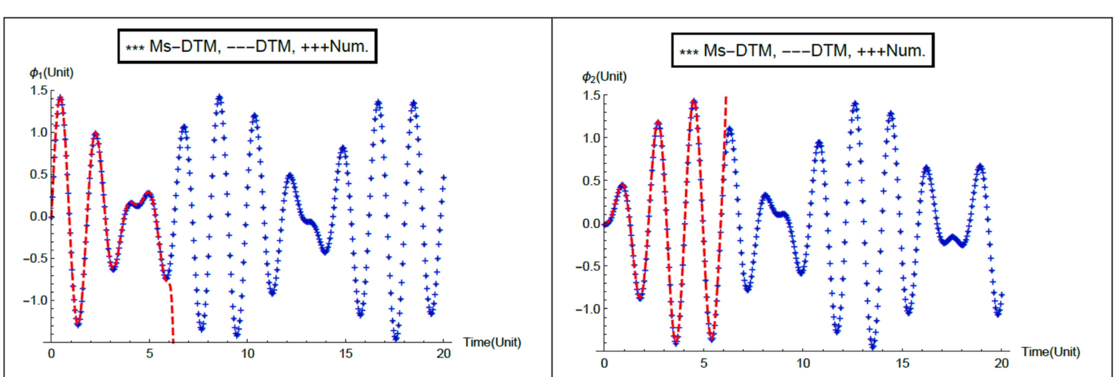

Fig. 15. The dynamical behaviour of ϕ_1 and ϕ_2 against time for $k = 10$.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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