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Connection Formulas for Second Order Differential Equations with a Complex Parameter and Having an Arbitrary Number of Turning Points

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Abstract. We consider the differential equation

$$(*) \qquad -u'' + \chi(x)u = \rho^2\phi^2(x)u$$

on a finite interval I, where I contains m turning points, that is here, zeros of ϕ. Using asymptotic estimates proved by R. E. LANGER for solutions of (*) for intervals containing only one turning point we derive asymptotic estimates (for $\rho \to \infty$) for a special fundamental system of solutions of (*) in I. The results obtained are fundamental for the investigation of eigenvalue problems defined by (*) and suitable boundary conditions.

1. Introduction

The asymptotic dependence of the solutions of differential equations of the form

$$(1.1) \qquad -u'' + p(x)u' + q(x)u = \rho^2\phi^2(x)u, \quad x \in \mathrm{I} = [0, 1],$$

on the complex parameter ρ for $|\rho| \to \infty$ is well known if p and q are integrable and if $\phi \in C[0, 1]$ with $|\phi(x)| \geq c > 0$ for $x \in \mathrm{I}$. In this case, (1.1) has two linearly independent solutions of the type

$$(1.2) \qquad \phi^{-\frac{1}{2}}(x)\, e^{\pm i\rho \int_0^x |\phi(t)|dt} \left\{1 + O\left(\frac{1}{\rho}\right)\right\}$$

for $x \in \mathrm{I}$ and $\operatorname{Im}\rho \geq c_0$ or $\operatorname{Im}\rho \leq c_0$; these solutions are sometimes called Liouville-Green (LG) functions.

Since $\phi^{-\frac{1}{2}}$ is singular at zeros of ϕ it is evident that the asymptotic forms (1.2) cannot be valid in intervals containing a turning point, that is here a zero of ϕ.

The solutions of differential equations (1.1) in intervals containing just one turning point have been estimated rigorously by R. E. LANGER [14], [15] and can be expressed asymptotically by the use of Bessel functions (see also OLVER [21]). It could well seem that the estimates for the solutions of (1.1) over the entire interval I, containing m turning points, might be obtainable by dividing I into m parts, containing only one turning point, and, in the familiar way, by patching together the asymptotic forms appropriate to the neighbouring intervals; that this procedure, however, is not applicable in a straightforward way has been pointed out by R. E. LANGER [16] (see the remark at the end of the proof of Theorem 4.1).

Differential equations of the form (1.1) and of similar form have been the object of numerous investigations (cf. [2], [3], [6], [9], [11], [12], [15], [16], [17], [18], [22], [23], [24], [25]) – readers interested in a historical survey on linear turning point theory are referred to the survey article of McHUGH [19] and the book of WASOW [26].

In the case of differential equations (1.1) with several turning points asymptotic representations of the solutions which are uniformly valid in the entire interval [0,1] are often unsuitable for applications, since for this purpose one has to determine the form and the solutions of complicated comparison equations (see for example [2], [3]). On the other hand LEUNG [17], OLVER [22], [23] and HEADING [12] succeeded in proving connection formulas for the case of second order differential equations with positive parameter with an arbitrary number of turning points. In addition LEUNG [17] proved estimates on the distribution of the (positive) eigenvalues of a corresponding eigenvalue problem in $L_2(-\infty, \infty)$ (cf. also OLVER [22, Example 5]). OLVER [23], ANYANWU [2], FEDORYUK [9] and others are also investigating differential equations (1.1) in the case when ϕ is allowed to have singularities and/or where I is unbounded or a region in $\mathbf{C}$.

More recently there arose a growing interest in indefinite eigenvalue problems (cf. [20]). ATKINSON and MINGARELLI [1], BEALS [4], CURGUS and H. LANGER [5] and KAPER, KWONG, LEKKERKERKER and ZETTL [13] have studied indefinite eigenvalue problems for second order differential equations using methods from the spectral theory of selfadjoint Sturm-Liouville problems (see also the survey of MINGARELLI). Another approach has been used in [7], [8] and [10] for the investigation of nonselfadjoint indefinite eigenvalue problems associated with $n-$th order differential equations $l(y) = \lambda p(x) y$, where p is a step- function.

In order to prove estimates of the eigenvalues for a broad class of eigenvalue problems corresponding to (1.1) (see [7]) and in order to attack the associated expansion problem we need suitable estimates for fundamental systems of solutions of (1.1) being uniformly valid with respect to $|\rho| \to \infty$ in sectors $S \subset \mathbf{C}$. It turns out that a fundamental system of solutions with these properties cannot be obtained by the continuation of one solution $y_1(\cdot, \rho)$ which is recessive (in the sense of [21]) near 0 and by the continuation of a second solution $u_2(\cdot, \rho)$ which is recessive near 1, since these solutions are not linearly independent for all $\rho \in S$ – see the remarks at the end of the sections 4.2 and 4.3. In particular it is not possible to get asymptotic estimates for fundamental systems of solutions for $\rho \in S$ by a slight modification of the methods used by OLVER [22], [23] – the method used in [22] can only be used for real parameters ρ and a modification of the central connection method used in [23] would only yield estimates for the continuation of recessive solutions. In this paper we show how the difficulties, arising when we are matching the solutions appropriate for neighbouring intervals containing only one turning point, can (partially) be circumvented. For this purpose we replace the fundamental system derived by R. E. LANGER for intervals containing only one turning point x_ν by another fundamental system $w_{\nu 1}$, $w_{\nu 2}$ which is more suitable for the matching process.

In section 2 of this paper we introduce some notations and we recall the basic results of R. E. LANGER on the asymptotic behaviour of a special fundamental system $u_{\nu 1}$, $u_{\nu 2}$ of solutions of (1.1) in the neighbourhood of one turning point x_ν. In section 3 we use these estimates to derive the asymptotic form of the fundamental system $w_{\nu 1}$, $w_{\nu 2}$, and in section 4 we determine the connection matrices C_ν and the asymptotics for two pairs y_1, y_2 and u_1, u_2 of fundamental systems of solutions of (1.1) on the entire interval [0, 1]. It turns out that the estimates derived in this paper are suitable for the treatment of a broad class of eigenvalue problems – the corresponding results have been partially published in [7], [8].

2. Notations and preliminary results

We suppose that the differential equation (1.1) is given in (or transformed to) the normal form

$$-u'' + \chi(x)u = \rho^2\phi^2(x)u, \quad x \in \mathrm{I} = [0, 1], \tag{2.1}$$

and that the coefficients χ and ϕ satisfy:

Hypothesis 2.1. (i) ϕ^2 is real and has in I m zeros x_ν of order $l_\nu \in \mathbf{N}$, $1 \leqslant \nu \leqslant m$, where $0 < x_1 < x_2 < \ldots < x_m < 1$.

(ii) The function $\phi_0 : \mathrm{I} \to \mathbf{R}\backslash\{0\}$,

$$x \to \phi^2(x) \prod_{\nu=1}^{m} (x - x_\nu)^{-l_\nu} \text{ is twice continuously differentiable.}$$

(iii) χ is bounded and integrable in I.

Remark 2.2. (i) The methods of this paper can also be applied if one or both endpoints of I are turning points or/and if χ or ϕ^2 have a finite number of poles up to order two.

(ii) If I is unbounded or if Hypothesis 2.1 is only fulfilled in (0, 1), (0, 1] or [0, 1) then one has to augment Hypothesis 2.1 by an additional assumption (cf. [15, p. 41], [22, p. 675]) in order to ensure that the asymptotic integration of (2.1) is possible on the singular part of I. We will not discuss problems with singularities in this paper (see [2], [9] and [23]), problems of this type will be considered in a subsequent paper.

(iii) The assumption $l_\nu \in \mathbf{N}$ for $1 \leqslant \nu \leqslant N$ is only used for convenience and can be replaced by $l_\nu > 0$ for $1 \leqslant \nu \leqslant N$ and $\phi^2(x) \in \mathbf{R}$ for $x \in [0, 1]$ (see [15]).

Notations 2.3. For intervals containing only one turning point, R. E. LANGER has proved asymptotic estimates for a fundamental system of solutions of (2.1). For the formulation of LANGER's results we introduce the following abbreviations.

Let $\epsilon > 0$ be fixed (and sufficiently small) and let

$$D_{0,\varepsilon} = [0, x_1 - \varepsilon],$$

$$D_{\nu,\varepsilon} = [x_\nu + \varepsilon,\ x_{\nu+1} - \varepsilon] \quad \text{for} \quad 1 \leqslant \nu \leqslant m-1,$$

$$D_{m,\varepsilon} = [x_m + \varepsilon,\ 1]$$

and

$$\mathrm{I}_{\nu,\varepsilon} = D_{\nu-1,\varepsilon} \cup [x_\nu - \varepsilon,\ x_\nu + \varepsilon] \cup D_{\nu,\varepsilon}.$$

For $k \in \mathbf{Z}$ we consider the sectors

$$S_k = \left\{\rho \in \mathbf{C} \,\middle|\, \frac{k\pi}{4} \leqslant \arg\rho \leqslant \frac{(k+1)\pi}{4}\right\}, \tag{2.2}$$

$$\hat{\chi}_k = \left\{\xi \in \mathbf{C} \,\middle|\, \left(k - \frac{1}{2}\right)\pi < \arg\xi \leqslant \left(k + \frac{1}{2}\right)\pi\right\} \quad \text{and}$$

$$\tilde{\chi}_k = \left\{\xi \in \mathbf{C} \,\middle|\, \left(k - \frac{1}{2}\right)\pi \leqslant \arg\xi < \left(k + \frac{1}{2}\right)\pi\right\}.$$

We distinguish four different types of turning points; for $1 \leqslant \nu \leqslant m$

$$T_\nu := \begin{cases} I, & \text{if } l_\nu \text{ is even and } \phi^2(x)(x-x_\nu)^{-l_\nu} < 0 \text{ in } \mathrm{I}_{\nu\varepsilon}, \\ II, & \text{if } l_\nu \text{ is even and } \phi^2(x)(x-x_\nu)^{-l_\nu} > 0 \text{ in } \mathrm{I}_{\nu\varepsilon}, \\ III, & \text{if } l_\nu \text{ is odd and } \phi^2(x)(x-x_\nu)^{-l_\nu} < 0 \text{ in } \mathrm{I}_{\nu\varepsilon}, \\ IV, & \text{if } l_\nu \text{ is odd and } \phi^2(x)(x-x_\nu)^{-l_\nu} > 0 \text{ in } \mathrm{I}_{\nu\varepsilon}, \end{cases}$$

is called type of x_ν.
Further we set for $1 \leqslant \nu \leqslant m$

$$\mu_\nu = \frac{1}{2 + l_\nu}, \tag{2.3}$$

$$\sigma_\nu = \begin{cases} 1 & \text{if } \mu_\nu > \frac{1}{4}, \\ 1 - \delta_0 \quad (\text{with } \delta_0 > 0 \text{ arbitrarily small}) & \text{if } \mu_\nu = \frac{1}{4}, \\ 4\mu_\nu & \text{if } \mu_\nu < \frac{1}{4}, \end{cases} \tag{2.3}$$

$$\sigma_0 = \min\{\sigma_\nu \mid 1 \leqslant \nu \leqslant m\},$$

and for $x \in \mathrm{I}_{\nu,\varepsilon}$

$$\Phi_\nu(x) = \int_{x_\nu}^{x} \phi(t)\,\mathrm{d}t, \tag{2.4}$$

$$\psi_\nu(x) = \Phi_\nu^{\frac{1}{2}-\mu_\nu}(x)\,\phi^{-\frac{1}{2}}(x),$$

$$\xi_\nu (= \xi_\nu(x, \rho)) = \rho\Phi_\nu(x).$$

Here the choice of the root ϕ of ϕ^2 depends on the interval and the sector under consideration and has to be determined carefully. We shall use two different branches of the argument function:

(2.5) $$\text{Case A}: -\pi < \arg \phi(x) \leqslant \pi$$
$$\text{Case B}: -\pi \leqslant \arg \phi(x) < \pi .$$

Let for a moment $0 < \phi(x)$ for $x_\nu < x < x_{\nu-1} - \varepsilon$, then according to (2.4), (2.5) we have in Case A (or Case B) for $x \in \mathrm{I}_{\nu\varepsilon} \backslash \{x_\nu\}$

(2.6) $$\phi(x) = \begin{cases} |\phi(x)| & \text{for} \quad x > x_\nu \\ |\phi(x)| e^{\pm i \frac{\pi}{2} l_\nu} & \text{for} \quad x < x_\nu \end{cases}$$

$$\psi_\nu(x) = |\psi_\nu(x)| \text{ (see [15, p. 27])},$$

$$\xi_\nu(x, \rho) = \rho \int_{x_\nu}^{x} |\phi(t)| \, dt \begin{cases} 1 & \text{for} \quad x > x_\nu \\ e^{\pm i \frac{\pi}{2} l_\nu} & \text{for} \quad x < x_\nu \end{cases}$$

and

$$\psi_\nu(x) \, \xi_\nu^{\mu_\nu - \frac{1}{2}}(x, \rho) = \rho^{\mu_\nu - \frac{1}{2}} |\phi(x)|^{-\frac{1}{2}} \begin{cases} 1 & \text{for} \quad x > x_\nu \\ e^{\mp i \frac{\pi}{4} l_\nu} & \text{for} \quad x < x_\nu , \end{cases}$$

where the lower sign corresponds to Case B; similar formulas are obtained if $\phi(x) < 0$ for $x_\nu < x < x_{\nu+1} - \varepsilon$.

The following theorem has been proved by LANGER [15, Theorem 5 and Theorem 6] for Case A and remains also valid in Case B. In the subsequent formulas the symbol B shall merely designate some computable function which is bounded for the ranges of its arguments under consideration. In a given formula different functions B will be distinguished by the use of subscripts; there is to be no presumption, however, that the same symbol in different formulas designates the same function. In order to avoid excessive repetition we shall use the symbol $\overset{(2)}{=}$ (or $\overset{(2)}{\sim}$) in place of $=$ (or $\sim$) to signify that a given formula is valid, and also the corresponding formula obtained by formal differentiation with respect to x ignoring the differentiation of all error terms B.

Let J_μ denote the Bessel function of first kind and for $1 \leqslant \nu \leqslant m$ and $x \in \mathrm{I}_{\nu\varepsilon}$ let

(2.7) $$v_{\nu j}(x, \rho) = \psi_\nu(x) \, \xi_\nu^{\mu_\nu}(x, \rho) \, J_{\mp \mu_\nu}(\xi_\nu(x, \rho)), \; j = 1, 2 .$$

Here and in the following we shall write two formulas in one by the use of double signs together with an index j; it will be understood in every such case that the upper signs are to be associated with the value $j = 1$ and the lower signs with $j = 2$.

By $u_{\nu j}(\cdot, \rho)$, $1 \leqslant \nu \leqslant m$, $1 \leqslant j \leqslant 2$, we denote the solution of (2.1) (in $\mathrm{I}_{\nu\varepsilon}$) defined by the initial conditions

(2.8) $$u_{\nu j}(x_\nu, \rho) = v_{\nu j}(x_\nu, \rho)$$
$$u'_{\nu j}(x_\nu, \rho) = v'_{\nu j}(x_\nu, \rho) ,$$

where $'$ denotes the derivative with respect to the (first) variable x. In the following $N > 0$ denotes a fixed constant.

Theorem 2.4 (Langer [15]). *For $1 \leqslant \nu \leqslant m$ the solutions $u_{\nu 1}(\cdot, \rho)$, $u_{\nu 2}(\cdot, \rho)$ satisfying the initial conditions* (2.8) *are described by the following formulas:*
For $x \in I_{\nu\varepsilon}$ and $1 \leqslant j \leqslant 2$

$$u_{\nu j}(x, \rho) \overset{(2)}{=} \psi_\nu(x)\, \xi_\nu^{\mu_\nu}(x, \rho)\, J_{\mp\mu_\nu}(\xi_\nu(x, \rho)) + \sum_{n=1}^{\infty} \frac{B_n(x, \rho)}{\rho^{4n\mu_\nu}} \tag{2.9}$$

if $|\xi_\nu(x, \rho)| \leqslant N$; otherwise we have for $k \in \mathbf{Z}$ and $\xi_\nu(x, \rho) \in \begin{cases} \hat{\chi}_k & \text{in Case } A \\ \tilde{\chi}_k & \text{in Case } B \end{cases}$

$$\begin{aligned} u_{\nu j}(x, \rho) \overset{(2)}{\sim} \psi_\nu(x)\, \xi_\nu^{\mu_\nu - \frac{1}{2}}(x, \rho) \Bigg\{ & a_{j1}^{(k)} e^{i\xi_\nu(x,\rho)} \left[1 + \sum_{n=1}^{\infty} \left(\frac{B_{n1}(x, \rho)}{\rho^{n\sigma_\nu}} + \frac{B_{n2}(x, \rho)}{\xi_\nu^n(x, \rho)} \right) \right] \\ & + a_{j2}^{(k)} e^{-i\xi_\nu(x,\rho)} \left[1 + \sum_{n=1}^{\infty} \left(\frac{B_{n3}(x, \rho)}{\rho^{n\sigma_\nu}} + \frac{B_{n4}(x, \rho)}{\xi_\nu^n(x, \rho)} \right) \right] \Bigg\}, \end{aligned} \tag{2.10}$$

where $a_{ji}^{(k)} (= a_{ji}^{(k)}(\nu))$ is defined by

$$\begin{aligned} a_{j1}^{(2p)} &= \frac{1}{\sqrt{2\pi}} e^{(2p - \frac{1}{2})(\frac{1}{2} \mp \mu_\nu)\pi i} + \sum_{n=1}^{\infty} \frac{B_n(\rho)}{\rho^{n\sigma_\nu}}, \\ a_{j2}^{(2p)} &= \frac{1}{\sqrt{2\pi}} e^{(2p + \frac{1}{2})(\frac{1}{2} \mp \mu_\nu)\pi i} + \sum_{n=1}^{\infty} \frac{B_n(\rho)}{\rho^{n\sigma_\nu}}, \\ a_{j1}^{(2p+1)} &= \frac{1}{\sqrt{2\pi}} e^{(2p + \frac{3}{2})(\frac{1}{2} \mp \mu_\nu)\pi i} + \sum_{n=1}^{\infty} \frac{B_n(\rho)}{\rho^{n\sigma_\nu}}, \\ a_{j2}^{(2p+1)} &= \frac{1}{\sqrt{2\pi}} e^{(2p + \frac{1}{2})(\frac{1}{2} \mp \mu_\nu)\pi i} + \sum_{n=1}^{\infty} \frac{B_n(\rho)}{\rho^{n\sigma_\nu}}. \end{aligned} \tag{2.11}$$

The Wronskian of $u_{\nu 1}$ and $u_{\nu 2}$ satisfies

$$W(u_{\nu 1}, u_{\nu 2}) = \frac{2}{\pi} \rho^{2\mu_\nu} \sin \pi\mu_\nu \left(1 + O\left(\frac{1}{\rho^{\sigma_\nu}} \right) \right) \tag{2.12}$$

Remark 2.5. (i) The symbol $\sim$ is used in the sense of Poincaré (see [21, p. 16]).

(ii) If $\mu_\nu = \frac{1}{4}$ we can replace the terms $\rho^{-\sigma_\nu}$ in (2.12) by $\rho^{-1} \log \rho$ (see [15, p. 37], [22, p. 676]); similarly we can modify (2.10), (2.11).

(iii) The terms $\rho^{4\mu_\nu}$ and ρ^{σ_ν} in Theorem 2.4 can be replaced by σ_0.

(iv) In this paper we use for convenience the abbreviations

$$[1] := 1 + O\left(\frac{1}{\rho^{\sigma_0}} \right) := 1 + \rho^{-\sigma_0} B(x, \rho) \tag{2.13}$$

and

$$[1;\xi]:=[1]+\frac{1}{\xi}\,O(1):=[1]+\frac{1}{\xi}\,B(x,\rho).$$

With these notations (2.10) can be written in the form

$$u_{\nu j}(x,\rho)\overset{(2)}{=}\psi_\nu(x)\,\xi_\nu^{\mu_\nu-\frac{1}{2}}(x,\rho)\,\{a_{j1}^{(k)}e^{i\xi_\nu(x,\rho)}[1;\,\xi_\nu]+a_{j2}^{(k)}e^{-i\xi_\nu(x,\rho)}[1;\xi_\nu]\}\tag{2.14}$$

for $x\in \mathrm{I}_{\nu\varepsilon}$, $1\leqslant\nu\leqslant m$, $1\leqslant j\leqslant 2$ and $\xi_\nu(x,\rho)\in\hat{\chi}_k$ (or $\tilde{\chi}_k$) and with $|\xi_\nu(x,\rho)|>N$.

3. The fundamental systems $w_{\nu1}^{T_\nu}$, $w_{\nu2}^{T_\nu}$

In this section we use Theorem 2.4 to construct for each pair $(\mathrm{I}_{\nu\varepsilon}, S_k)$ $1\leqslant\nu\leqslant m$, $k\in\mathbf{Z}$, a special fundamental system

$$w_{\nu j}^{T_\nu}:\mathrm{I}_{\nu\varepsilon}\times S_k\to\mathbf{C},\quad (x,\rho)\to w_{\nu j}^{T_\nu}(x,\rho)$$

of solutions of (2.1) – we will not indicate the dependence of $w_{\nu j}^{T_\nu}$ on the sector S_k.

For the proof of the estimates of $w_{\nu j}^{T_\nu}(x,\rho)$ for $\rho\in S_k$ we choose the argument function according to Case A resp. Case B if k is odd resp. even.

We shall formulate the estimates for $\rho\in S_k$, $-2\leqslant k\leqslant 1$, the results for the remaining sectors are easily obtained by a substitution; for example we get the formulas for the sectors S_{j+4} from the formulas for the sectors S_j by substituting therein ρ by $-\rho$. In addition it is sufficient to formulate the subsequent proof only for turning points of type II and IV, then the results for turning points of type I or III can be obtained by substitution.

Now we outline the fundamental rule for the choice of the solutions $w_{\nu1}^{T_\nu}$, $w_{\nu2}^{T_\nu}$ for $(x,\rho)\in\mathrm{I}_{\nu\varepsilon}\times S_k$. The theory of Liouville-Green (LG) approximation (cf. [21, chapter 6]) shows that for every sector S_k, $k\in\mathbf{Z}$, and for every interval $D_{\nu,\varepsilon}$, $0\leqslant\nu\leqslant m$, there are two linearly independent (LG-)solutions $\tilde{v}_{\nu1}$, $\tilde{v}_{\nu2}$ of (2.1) satisfying

$$\tilde{v}_{\nu j}(x,\rho)=\phi^{-\frac{1}{2}}(x)\exp\left\{\pm\rho\int_{x_\nu}^{x}\phi(t)\,dt\right\}[1],\qquad j=1,2,$$

for $(x,\rho)\in D_{\nu\varepsilon}\times S_k$. $\tilde{v}_{\nu j}$ is called dominant resp. subdominant (or recessive) in $D_{\nu\varepsilon}$ if $\mathrm{Re}\left(\pm\rho\int_{x_\nu}^{x}\phi(t)\,dt\right)$ is positive resp. negative for $(x,\rho)\in D_{\nu\varepsilon}\times\mathring{S}_k$. In Section 3.1 we define $w_{\nu1}^{T_\nu}$, $w_{\nu2}^{T_\nu}$ in the following way:

(i) $w_{\nu1}^{T_\nu}$ is a (*normalized*) multiple of the subdominant solution in $D_{\nu-1,\varepsilon}$, the asymptotic form of $w_{\nu1}^{T_\nu}(x,\rho)$ for $\rho\in S_k$ and $x\in\mathrm{I}_{\nu\varepsilon}\backslash D_{\nu-1,\varepsilon}$ can be derived from Theorem 2.4.

(ii) $w_{\nu2}^{T_\nu}$ is a *(normalized)* multiple of the subdominant solution in $D_{\nu\varepsilon}$ and the asymptotic form of $w_{\nu2}^{T_\nu}$ in $\mathrm{I}_{\nu\varepsilon}\backslash D_{\nu\varepsilon}$ is again derived from Theorem 2.4.

Subsequently we give a detailed proof for the representation of $w_{\nu1}^{T_\nu}(x,\rho)$, $w_{\nu2}^{T_\nu}(x,\rho)$ for $T_\nu=IV$ and $\rho\in S_k$ with $k\in\{0,1\}$. The proofs for the remaining cases are performed similarly.

3.1. Estimates for $w_{\nu 1}^{IV}(x, \rho)$, $w_{\nu 2}^{IV}(x, \rho)$ for $(x, \rho) \in \mathrm{I}_{\nu\varepsilon} \times S_k$, $k \in \{-1, 0\}$

Let $1 \leqslant \nu \leqslant m$ and $T_\nu = IV$. Then l_ν is odd and $\frac{l_\nu + 1}{2} = 2p_\nu$ or $\frac{l_\nu + 1}{2} = 2p_\nu + 1$ with $p_\nu \in \mathbf{N}_0$. We assume in this section that $\frac{l_\nu + 1}{2} = 2p_\nu$ – the results are the same and the proofs are similar for $\frac{l_\nu + 1}{2} = 2p_\nu + 1$.

a) First we select arg according to Case A; on account of (2.6) we have in this case for $x \in D_{\nu-1,\varepsilon}$

$$-\pi < \arg \rho \leqslant 0 \Leftrightarrow \xi_\nu(x, \rho) \in \hat{\chi}_{\frac{l_\nu + 1}{2}} = \hat{\chi}_{2p_\nu}. \tag{3.1}$$

Applying Theorem 2.4 we obtain for $x \in D_{\nu-1,\varepsilon}$, $-\frac{\pi}{2} < \arg \rho \leqslant \frac{\pi}{2}$ (and especially for $\rho \in S_{-1} \cup S_0$)

$$u_{\nu j}(x, \rho) \overset{(2)}{=} \frac{1}{\sqrt{2\pi}} |\phi(x)|^{-\frac{1}{2}} \rho^{\mu_\nu - \frac{1}{2}} \times \left\{ \mathrm{e}^{\rho \int_{x_\nu}^{x} |\phi(t)| dt + i\pi\{-\frac{l_\nu}{4} + (2p_\nu - \frac{1}{2})(\frac{1}{2} \mp \mu_\nu)\}} [1] + \mathrm{e}^{-\rho \int_{x_\nu}^{x} |\phi(t)| dt + i\pi\{-\frac{l_\nu}{4} + (2p_\nu + \frac{1}{2})(\frac{1}{2} \mp \mu_\nu)\}} [1] \right\}. \tag{3.2}$$

We recall that $|\xi_\nu(x, \rho)| > N$ for $x \in D_{\nu-1,\varepsilon}$ if $|\rho| > r_0(N)$.

In contrast to (3.1) we have for $x \in D_{\nu,\varepsilon}$

$$-\frac{\pi}{2} < \arg \rho \leqslant \frac{\pi}{2} \Leftrightarrow \xi_\nu(x, \rho) \in \hat{\chi}_0. \tag{3.3}$$

Applying Theorem 2.4 for $k = 0$ and using (2.6) (for Case A) we get for $x \in D_{\nu,\varepsilon}$, $-\frac{\pi}{2} < \arg \rho \leqslant \frac{\pi}{2}$ (and especially for $\rho \in S_{-1} \cup S_0$)

$$u_{\nu j}(x, \rho) \overset{(2)}{=} \frac{1}{\sqrt{2\pi}} |\phi(x)|^{-\frac{1}{2}} \rho^{\mu_\nu - \frac{1}{2}} \times \{\mathrm{e}^{i\rho \int_{x_\nu}^{x} |\phi(t)| dt - \frac{1}{2}(\frac{1}{2} \mp \mu_\nu)\pi i} [1] + \mathrm{e}^{-i\rho \int_{x_\nu}^{x} |\phi(t)| dt + \frac{1}{2}(\frac{1}{2} \mp \mu_\nu)\pi i} [1]\}. \tag{3.4}$$

According to the theory of LG-approximation there exists a (subdominant) solution $w_{\nu 1}^{IV}(\cdot, \rho)$ satisfying

$$w_{\nu 1}^{IV}(x, \rho) \overset{(2)}{=} |\phi(x)|^{-\frac{1}{2}} \mathrm{e}^{\rho \int_{x_\nu}^{x} |\phi(t)| dt} [1] \text{ for } (x, \rho) \in D_{\nu-1,\varepsilon} \times S_{-1}. \tag{3.5}$$

Now we fix $x \in D_{\nu-1,\varepsilon}$ and use (3.2), (3.5) and Cramer's rule in order to determine for

$T_\nu = IV$ the connection coefficients $a_\nu^{T_\nu}$, $b_\nu^{T_\nu}$ with

(3.6) $$w_{\nu 1}^{T_\nu}(x,\rho) = a_\nu^{T_\nu}(\rho)\, u_{\nu 1}(x,\rho) + b_\nu^{T_\nu}(\rho)\, u_{\nu 2}(x,\rho).$$

Since $-\frac{l_\nu}{4} + p_\nu + \frac{1}{4} = \frac{1}{2} = \left(2p_\nu + \frac{1}{2}\right)\mu_\nu$, we obtain with (2.12)

(3.7) $$a_\nu^{IV}(\rho) = \frac{1}{W(u_{\nu 1}, u_{\nu 2})} \begin{vmatrix} w_{\nu 1}^{IV}(x,\rho) & u_{\nu 2}(x,\rho) \\ w_{\nu 1}^{IV\prime}(x,\rho) & u_{\nu 2}'(x,\rho) \end{vmatrix} = \frac{\sqrt{2\pi}\,\rho^{\frac{1}{2}-\mu_\nu}}{4\rho \sin \pi\mu_\nu}\, e^{-\frac{i\pi}{4} l_\nu} \times$$

$$\{e^{\rho \int_{x_\nu}^{x} |\phi(t)|dt} (\rho e^{\rho \int_{x_\nu}^{x} |\phi(t)|dt + i\pi(2p_\nu - \frac{1}{2})(\frac{1}{2}+\mu_\nu)}[1] - \rho e^{-\rho \int_{x_\nu}^{x} |\phi(t)|dt + i\pi(2p_\nu + \frac{1}{2})(\frac{1}{2}+\mu_\nu)}[1])$$

$$- \rho e^{\rho \int_{x_\nu}^{x} |\phi(t)|dt} (e^{\rho \int_{x_\nu}^{x} |\phi(t)|dt + i\pi(2p_\nu - \frac{1}{2})(\frac{1}{2}+\mu_\nu)}[1] + e^{-\rho \int_{x_\nu}^{x} |\phi(t)|dt + i\pi(2p_\nu + \frac{1}{2})(\frac{1}{2}+\mu_\nu)}[1])\}$$

$$= \frac{\sqrt{2\pi}\,\rho^{\frac{1}{2}-\mu_\nu}}{4 \sin \pi\mu_\nu}\, e^{-\frac{i\pi}{4} l_\nu} \left\{ O\left(\frac{1}{\rho}\right) - 2e^{i\pi(2p_\nu + \frac{1}{2})(\frac{1}{2}+\mu_\nu)}[1] \right\} = \frac{\sqrt{2\pi}\,\rho^{\frac{1}{2}-\mu_\nu}}{2 \sin \pi\mu_\nu}[1].$$

Similarly we get

(3.8) $$b_\nu^{IV}(\rho) = \frac{1}{W(u_{\nu 1}, u_{\nu 2})} \begin{vmatrix} u_{\nu 1}(x,\rho) & w_{\nu 1}^{IV}(x,\rho) \\ u_{\nu 1}'(x,\rho) & w_{\nu 1}^{IV\prime}(x,\rho) \end{vmatrix} = \frac{\sqrt{2\pi}\,\rho^{\frac{1}{2}-\mu_\nu}}{2 \sin \pi\mu_\nu}[1].$$

Hence we can estimate (the continuation of) $w_{\nu I}^{IV}(\cdot,\rho)$ on the interval $I_{\nu\varepsilon}$; we have

(3.9) $$w_{\nu 1}^{IV}(x,\rho) = \frac{\sqrt{2\pi}\,\rho^{\frac{1}{2}-\mu_\nu}}{2\sin\pi\mu_\nu} \{u_{\nu 1}(x,\rho)[1] + u_{\nu 2}(x,\rho)[1]\} \text{ for } (x,\rho) \in I_{\nu,\varepsilon} \times S_{-1}$$

and therefore, according to (3.9), (3.4), $\csc\alpha = \frac{1}{\sin\alpha}$ and $\sin \pi\mu_\nu = 2\sin\frac{\pi\mu_\nu}{2}\cos\frac{\pi\mu_\nu}{2}$,

(3.10) $$w_{\nu 1}^{IV}(x,\rho) = \frac{1}{2\sin\pi\mu_\nu} |\phi|^{-\frac{1}{2}}(x) \{e^{i\rho \int_{x_\nu}^{x} |\phi(t)|dt} (e^{\frac{1}{2}(\frac{1}{2}-\mu_\nu)\pi i}[1] + e^{-\frac{1}{2}(\frac{1}{2}+\mu_\nu)\pi i}[1])$$

$$+ e^{-i\rho \int_{x_\nu}^{x} |\phi(t)|dt} (e^{\frac{1}{2}(\frac{1}{2}-\mu_\nu)\pi i}[1] + e^{\frac{1}{2}(\frac{1}{2}+\mu_\nu)\pi i}[1])\}$$

$$= \frac{1}{2}\csc\frac{\pi\mu_\nu}{2} |\phi|^{-\frac{1}{2}}(x) \{e^{i\rho \int_{x_\nu}^{x} |\phi(t)|dt - \frac{i\pi}{4}}[1] + e^{-i\rho \int_{x_\nu}^{x} |\phi(t)|dt + \frac{i\pi}{4}}[1]\}$$

for $x \in D_{\nu,\varepsilon}$ and $\rho \in S_{-1}$.

b) Next we use the fact that for $T_\nu = IV$ there exists a subdominant solution $w_{\nu 2}^{T_\nu}$ with

(3.11) $$w_{\nu 2}^{IV}(x,\rho) = 2\sin\frac{\pi\mu_\nu}{2}\,\mathrm{e}^{-\mathrm{i}\rho\int_{x_\nu}^{x}|\phi(t)|dt - \mathrm{i}\frac{\pi}{4}} \quad \text{for} \quad (x,\rho)\in D_{\nu,\varepsilon}\times S_{-1}$$

and

(3.12) $$w_{\nu 2}^{IV}(\cdot,\rho) =: c_\nu^{IV}(\rho)\,u_{\nu 1}(\cdot,\rho) + d_\nu^{IV}(\rho)\,u_{\nu 2}(\cdot,\rho).$$

Again the connection coefficients can be determined from (3.4), (3.11) and (3.12) by Cramers's rule. An easy calculation (as with (3.7)) yields

(3.13) $$c_\nu^{IV}(\rho) = \frac{\sqrt{2\pi}\rho^{\frac{1}{2}-\mu_\nu}}{2\cos\frac{\pi\mu_\nu}{2}}\,\mathrm{e}^{-\mu_\nu\frac{\pi}{2}\mathrm{i}}[1]$$

and

$$d_\nu^{IV}(\rho) = -\frac{\sqrt{2\pi}\rho^{\frac{1}{2}-\mu_\nu}}{2\cos\frac{\pi\mu_\nu}{2}}\,\mathrm{e}^{\mu_\nu\frac{\pi}{2}\mathrm{i}}[1].$$

Substituting (3.13) into (3.12) we infer from (3.2) and $-\frac{l_\nu}{4}+\frac{1}{4}+p_\nu=\frac{1}{2}=\left(2p_\nu+\frac{1}{2}\right)\mu_\nu$

(3.14) $$\begin{aligned} w_{\nu 2}^{IV}(x,\rho) &\overset{(2)}{=} \frac{|\phi(x)|^{-\frac{1}{2}}}{2\cos\frac{\pi\mu_\nu}{2}}\{\mathrm{e}^{\rho\int_{x_\nu}^{x}|\phi(t)|dt}O(1) \\ &\quad + \mathrm{e}^{-\rho\int_{x_\nu}^{x}|\phi(t)|dt}(\mathrm{e}^{\mathrm{i}\pi(-\frac{\mu_\nu}{2}-\frac{l_\nu}{4}+(2p_\nu+\frac{1}{2})(\frac{1}{2}-\mu_\nu))}[1] \\ &\quad - \mathrm{e}^{\mathrm{i}\pi(\frac{\mu_\nu}{2}-\frac{l_\nu}{4}+(2p_\nu+\frac{1}{2})(\frac{1}{2}+\mu_\nu))}[1])\} \\ &= \frac{|\phi(x)|^{-\frac{1}{2}}}{2\cos\frac{\pi\mu_\nu}{2}}\{\mathrm{e}^{\mathrm{i}\pi\frac{\mu_\nu}{2}}[1]+\mathrm{e}^{-\mathrm{i}\pi\frac{\mu_\nu}{2}}[1]\}\,\mathrm{e}^{-\rho\int_{x_\nu}^{x}|\phi(t)|dt} \\ &= |\phi(x)|^{-\frac{1}{2}}\mathrm{e}^{-\rho\int_{x_\nu}^{x}|\phi(t)|dt}[1] \quad \text{for} \quad (x,\rho)\in D_{\nu-1,\varepsilon}\times S_{-1}. \end{aligned}$$

$\{w_{\nu 1}^{IV}(\cdot,\rho), w_{\nu 2}^{IV}(\cdot,\rho)\}$ is the desired fundamental system of (2.1) for $\rho\in S_{-1}$.

It is worth while to point out that all the connection coefficients have the form

$$\rho^{\frac{1}{2}-\mu_\nu}\cdot c[1] \quad \text{with} \quad c\neq 0;$$

this results from the fact that $w_{\nu 1}^{IV}(\cdot,\rho)$ and $w_{\nu 2}^{IV}(\cdot,\rho)$ are subdominant in the interval used for the definition of these solutions and for the calculation of the connection coefficients. If $\tilde{w}(\cdot,\rho)$ is a dominant solution of (2.1) in $D_{\nu\varepsilon}$, then we cannot determine the leading terms of the corresponding connection coefficients explicitely.

c) In order to obtain a suitable fundamental system of (2.1) for $\rho \in S_0$ we select arg according to Case B.

On account of (2.6) we have in this case for $x \in D_{\nu-1,\varepsilon}$

$$0 \leqslant \arg \rho < \pi \Leftrightarrow \xi_\nu(x, \rho) \in \tilde{\chi}_{-\frac{l_\nu+1}{2}} = \tilde{\chi}_{-2p_\nu}. \tag{3.15}$$

Applying Theorem 2.4 for $k = -2p_\nu$ we obtain with (2.6) for $x \in D_{\nu-1,\varepsilon}$, $0 \leqslant \arg \rho < \pi$ (and especially for $\rho \in S_0$)

$$\begin{aligned} u_{\nu j}(x, \rho) &\overset{(2)}{=} \frac{1}{\sqrt{2\pi}} |\phi(x)|^{-\frac{1}{2}} \rho^{\mu_\nu - \frac{1}{2}} \times \\ &\quad \{ e^{-\varrho \int_{x_\nu}^{x} |\phi(t)| dt + i\pi \{\frac{l_\nu}{4} - (2p_\nu + \frac{1}{2})(\frac{1}{2} \mp \mu_\nu)\}} [1] \\ &\quad + e^{\varrho \int_{x_\nu}^{x} |\phi(t)| dt + i\pi \{\frac{l_\nu}{4} - (2p_\nu - \frac{1}{2})(\frac{1}{2} \mp \mu_\nu)\}} [1] \}. \end{aligned} \tag{3.16}$$

Now we choose the solutions $w_{\nu 1}^{IV}$ and $w_{\nu 2}^{IV}(\cdot, \rho)$ according to the same principle that has been used in a) and b). LG-theory yields the existence of two solutions of (2.1) with

$$w_{\nu 1}^{IV}(x, \rho) \overset{(2)}{=} |\phi(x)|^{-\frac{1}{2}} e^{\varrho \int_{x_\nu}^{x} |\phi(t)| dt} [1] \quad \text{for} \quad (x, \rho) \in D_{\nu-1,\varepsilon} \times S_0 \tag{3.17}$$

and

$$w_{\nu 2}^{IV}(x, \rho) \overset{(2)}{=} 2 \sin \frac{\pi \mu_\nu}{2} |\phi(x)|^{-\frac{1}{2}} e^{i\varrho \int_{x_\nu}^{x} |\phi(t)| dt + i\frac{\pi}{4}} [1] \quad \text{for} \quad (x, \rho) \in D_{\nu,\varepsilon} \times S_0. \tag{3.18}$$

The continuations of these solutions to the interval $I_{\nu,\varepsilon}$ satisfy

$$\begin{aligned} w_{\nu 1}^{IV}(x, \rho) &= a_\nu^{IV}(\rho) u_{\nu 1}(x, \rho) + b_\nu^{IV}(\rho) u_{\nu 2}(x, \rho), \\ w_{\nu 2}^{IV}(x, \rho) &= c_\nu^{IV}(\rho) u_{\nu 1}(x, \rho) + d_\nu^{IV}(\rho) u_{\nu 2}(x, \rho) \end{aligned} \tag{3.19}$$

for $(x, \rho) \in I_{\nu,\varepsilon} \times S_0$. We use (3.4), (3.19) (or (3.16), (3.19)) in order to compute the connection coefficients a_ν^{IV}, b_ν^{IV} (or c_ν^{IV}, d_ν^{IV}) and obtain

$$\begin{aligned} a_\nu^{IV}(\rho) &= \frac{\sqrt{2\pi} \rho^{\frac{1}{2} - \mu_\nu}}{2 \sin \pi \mu_\nu} [1], \\ b_\nu^{IV}(\rho) &= \frac{\sqrt{2\pi} \rho^{\frac{1}{2} - \mu_\nu}}{2 \sin \pi \mu_\nu} [1], \\ c_\nu^{IV}(\rho) &= \frac{\sqrt{2\pi} \rho^{\frac{1}{2} - \mu_\nu}}{2 \cos \frac{\pi \mu_\nu}{2}} e^{i\frac{\pi}{2}\mu_\nu} [1], \end{aligned} \tag{3.20}$$

$$d_\nu^{IV}(\rho) = -\frac{\sqrt{2\pi}\rho^{\frac{1}{2}-\mu_\nu}}{2\cos\frac{\pi\mu_\nu}{2}} e^{-i\frac{\pi}{2}\mu_\nu}[1].$$

Inserting (3.20) into (3.19) we get from (3.4) and (3.2)

$$\begin{aligned}(3.21)\qquad w_{\nu 1}^{IV}(x,\rho) &\overset{(2)}{=} \frac{1}{2\sin\pi\mu_\nu}|\phi(x)|^{-\frac{1}{2}}\{e^{i\varrho\int_{x_\nu}^{x}|\phi(t)|dt+i\frac{\pi}{4}}(e^{i\frac{\pi}{2}\mu_\nu}[1]+e^{-i\frac{\pi}{2}\mu_\nu}[1])\\ &\quad - e^{-i\varrho\int_{x_\nu}^{x}|\phi(t)|dt}+(e^{-i\frac{\pi}{2}\mu_\nu}[1]+e^{i\frac{\pi}{2}\mu_\nu}[1])\}\\ &= \frac{1}{2}\csc\frac{\pi\mu_\nu}{2}|\phi(x)|^{-\frac{1}{2}}\{e^{i\varrho\int_{x_\nu}^{x}|\phi(t)|dt-i\frac{\pi}{4}}[1]+e^{-i\varrho\int_{x_\nu}^{x}|\phi(t)|dt+\frac{i\pi}{4}}[1])\}\end{aligned}$$

for $(x,\rho)\in D_{\nu,\varepsilon}\times S_0$ and

$$(3.22)\qquad w_{\nu 2}^{IV}(x,\rho)\overset{(2)}{=}|\phi(t)|^{-\frac{1}{2}}e^{-\varrho\int_{x_\nu}^{x}|\phi(t)|dt}[1]\quad\text{for}\quad (x,\rho)\in D_{\nu-1,\varepsilon}\times S_0.$$

In all remaining cases we can proceed similarly, it is unnecessary to record details. We summarize the results of this section in the following Theorem, where $w_{\nu 1}^{T_\nu}(\cdot,\rho)$ (resp. $w_{\nu 2}^{T_\nu}(\cdot,\rho)$) is defined on $D_{\nu-1,\varepsilon}$ (resp. $D_{\nu\varepsilon}$) by a normalized subdominant solution of (1.1) and where (3.23) and Theorem 2.4 are used to estimate the corresponding unique continuation of $w_{\nu 1}^{T_\nu}(\cdot,\rho)$ (resp. $w_{\nu 2}^{T_\nu}(\cdot,\rho)$) to the interval $I_{\nu\varepsilon}$. It is not possible to determine the connection coefficients and the estimates for the continuation of the dominant solution (see Remark 3.3 (ii)).

Theorem 3.2. *Let $1\leqslant\nu\leqslant n$ and let x_ν be a turning point of type T_ν. Then there exists a fundamental system $w_{\nu 1}^{T_\nu}(\cdot,\rho)$, $w_{\nu 2}^{T_\nu}(\cdot,\rho): I_{\nu,\varepsilon}\to\mathbf{C}$ of* (1.1) *with*

$$\begin{aligned}(3.23)\qquad w_{\nu 1}^{T_\nu}(\cdot,\rho) &= (\tilde{a}_\nu^{T_\nu}(\rho)u_{\nu 1}(\cdot,\rho)+\tilde{b}_\nu^{T_\nu}(\rho)u_{\nu 2}(\cdot,\rho))\sqrt{2\pi}\rho^{\frac{1}{2}-\mu_\nu},\\ w_{\nu 2}^{T_\nu}(\cdot,\rho) &= (\tilde{c}_\nu^{T_\nu}(\rho)u_{\nu 1}(\cdot,\rho)+\tilde{d}_\nu^{T_\nu}(\rho)u_{\nu 2}(\cdot,\rho))\sqrt{2\pi}\rho^{\frac{1}{2}-\mu_\nu}\end{aligned}$$

and which is described by the following formulas.

(i) Turning points of type II.

a) For $\rho\in S_{-2}$

$$w_{\nu 1}^{II}(x,\rho)\,|\phi(x)|^{\frac{1}{2}}\overset{(2)}{=}e^{i\varrho\int_{x_\nu}^{x}|\phi(t)|dt}[1]\quad\textit{for}\quad x\in D_{\nu-1,\varepsilon},$$

$$w_{\nu 2}^{II}(x,\rho)\,|\phi(x)|^{\frac{1}{2}}\overset{(2)}{=}\sin\pi\mu_\nu e^{-i\varrho\int_{x_\nu}^{x}|\phi(t)|dt}[1]\quad\textit{for}\quad x\in D_{\nu\varepsilon},$$

imply

$$w_{\nu 1}^{II}(x,\rho)\,|\phi(x)|^{\frac{1}{2}}\overset{(2)}{=}\csc\pi\mu_\nu e^{i\varrho\int_{x_\nu}^{x}|\phi(t)|dt}[1]\quad\textit{for}\quad x\in D_{\nu\varepsilon},$$

$$w_{\nu 2}^{II}(x,\rho)\,|\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \mathrm{e}^{-\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \text{for} \quad x\in D_{\nu-1,\varepsilon},$$

$$W(w_{\nu 1}^{II}, w_{\nu 2}^{II})(:= W(w_{\nu 1}^{II}(\cdot,\rho), w_{\nu 2}^{II}(\cdot,\rho))) = -2\mathrm{i}\rho[1],$$

$$\tilde{a}_\nu^{II}(\rho) = \frac{1}{2}\csc\pi\mu_\nu \mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}-\frac{\mu_\nu}{2})}[1],\ \tilde{b}_\nu^{II}(\rho) = \frac{1}{2}\csc\pi\mu_\nu \mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}+\frac{\mu_\nu}{2})}[1],$$

$$\tilde{c}_\nu^{II}(\rho) = \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}-\frac{\mu_\nu}{2})}[1], \textit{ and } \tilde{d}_\nu^{II}(\rho) = -\frac{1}{2}\mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}+\frac{\mu_\nu}{2})}[1].$$

b) For $\rho\in S_{-1}$

$$w_{\nu 1}^{II}(x,\rho)\,|\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \mathrm{e}^{\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \textit{for} \quad x\in D_{\nu-1,\varepsilon},$$

$$w_{\nu 2}^{II}(x,\rho)\,|\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \sin\pi\mu_\nu \mathrm{e}^{-\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \textit{for} \quad x\in D_{\nu\varepsilon},$$

imply

$$w_{\nu 1}^{II}(x,\rho)\,|\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \csc\pi\mu_\nu\{\mathrm{e}^{\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] + i\cos\pi\mu_\nu \mathrm{e}^{-\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1]\}\,\textit{for}\ x\in D_{\nu\varepsilon},$$

$$w_{\nu 2}^{II}(x,\rho)\,|\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \mathrm{e}^{-\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] + \mathrm{i}\cos\pi\mu_\nu \mathrm{e}^{\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \textit{for} \quad x\in D_{\nu-1,\varepsilon},$$

$$W(w_{\nu 1}^{II}, w_{\nu 2}^{II}) = -2\mathrm{i}\rho[1],$$

$$\tilde{a}_\nu^{II}(\rho) = \frac{1}{2}\csc\pi\mu_\nu e^{\mathrm{i}\pi(\frac{1}{4}-\frac{\mu_\nu}{2})}[1] \textit{ and } \tilde{b}_\nu^{II}(\rho) = \frac{1}{2}\csc\pi\mu_\nu \mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}+\frac{\mu_\nu}{2})}[1],$$

$$\tilde{c}_\nu^{II}(\rho) = \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}-\frac{\mu_\nu}{2})}[1] \textit{ and } \tilde{d}_\nu^{II}(\rho) = -\frac{1}{2}\mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}+\frac{\mu_\nu}{2})}[1].$$

c) For $\rho\in S_0$

$$w_{\nu 1}^{II}(x,\rho)\,|\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \mathrm{e}^{-\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \textit{for} \quad x\in D_{\nu-1,\varepsilon},$$

$$w_{\nu 2}^{II}(x,\rho)\,|\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \sin\pi\mu_\nu \mathrm{e}^{\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \textit{for} \quad x\in D_{\nu\varepsilon}$$

imply

$$w_{\nu 1}^{II}(x,\rho)\,|\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \csc\pi\mu_\nu\{\mathrm{e}^{-\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] - \mathrm{i}\cos\pi\mu_\nu \mathrm{e}^{\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1]\}\,\textit{for}\ x\in D_{\nu\varepsilon},$$

$$w_{\nu 2}^{II}(x,\rho)\,|\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \mathrm{e}^{\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] - \mathrm{i}\cos\pi\mu_\nu \mathrm{e}^{-\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \textit{for} \quad x\in D_{\nu-1,\varepsilon},$$

$$W(w_{\nu 1}^{II}, w_{\nu 2}^{II}) = 2\mathrm{i}\rho[1],$$

$$\tilde{a}_\nu^{II}(\rho) = \frac{1}{2}\csc \pi\mu_\nu \mathrm{e}^{\mathrm{i}\pi(-\frac{1}{4}+\frac{\mu_\nu}{2})}[1],\ \tilde{b}_\nu^{II}(\rho) = \frac{1}{2}\csc \pi\mu_\nu \mathrm{e}^{\mathrm{i}\pi(-\frac{1}{4}-\frac{\mu_\nu}{2})}[1],$$

$$\tilde{c}_\nu^{II}(\rho) = \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi(-\frac{1}{4}+\frac{\mu_\nu}{2})}[1] \quad \textit{and} \quad \tilde{d}_\nu^{II}(\rho) = -\frac{1}{2}\mathrm{e}^{\mathrm{i}\pi(-\frac{1}{4}-\frac{\mu_\nu}{2})}[1].$$

d) For $\rho \in S_1$

$$w_{\nu 1}^{II}(x, \rho)\ |\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \mathrm{e}^{-\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \textit{for} \quad x \in D_{\nu-1,\varepsilon},$$

$$w_{\nu 2}^{II}(x, \rho)\ |\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \sin \pi\mu_\nu \mathrm{e}^{\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \textit{for} \quad x \in D_{\nu\varepsilon}$$

imply

$$w_{\nu 1}^{II}(x, \rho)\ |\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \csc \pi\mu_\nu \mathrm{e}^{-\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \textit{for} \quad x \in D_{\nu\varepsilon},$$

$$w_{\nu 2}^{II}(x, \rho)\ |\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \mathrm{e}^{\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \textit{for} \quad x \in D_{\nu-1,\varepsilon},$$

$$W(w_{\nu 1}^{II}, w_{\nu 2}^{II}) = 2\mathrm{i}\rho[1],$$

$$\tilde{a}_\nu^{II}(\rho) = \frac{1}{2}\csc \pi\mu_\nu e^{\mathrm{i}\pi(-\frac{1}{4}+\frac{\mu_\nu}{2})}[1],\ \tilde{b}_\nu^{II}(\rho) = \frac{1}{2}\csc \pi\mu_\nu \mathrm{e}^{\mathrm{i}\pi(-\frac{1}{4}-\frac{\mu_\nu}{2})}[1],$$

$$\tilde{c}_\nu^{II}(\rho) = \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi(-\frac{1}{4}+\frac{\mu_\nu}{2})}[1] \quad \textit{and} \quad \tilde{d}_\nu^{II}(\rho) = -\frac{1}{2}\mathrm{e}^{\mathrm{i}\pi(-\frac{1}{4}-\frac{\mu_\nu}{2})}[1].$$

(ii) Turning points of type IV.

a) For $\rho \in S_{-2}$

$$w_{\nu 1}^{IV}(x, \rho)\ |\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \mathrm{e}^{\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \textit{for} \quad x \in D_{\nu-1,\varepsilon},$$

$$w_{\nu 2}^{IV}(x, \rho)\ |\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} 2\sin\frac{\pi\mu_\nu}{2}\mathrm{e}^{-\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t-\mathrm{i}\frac{\pi}{4}}[1] \quad \textit{for} \quad x \in D_{\nu\varepsilon}$$

imply

$$w_{\nu 1}^{IV}(x, \rho)\ |\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \frac{1}{2}\csc\frac{\pi\mu_\nu}{2}\mathrm{e}^{\mathrm{i}\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t-\mathrm{i}\frac{\pi}{4}}[1] \quad \textit{for} \quad x \in D_{\nu\varepsilon},$$

$$w_{\nu 2}^{IV}(x, \rho)\ |\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \mathrm{e}^{-\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] - \mathrm{i}\mathrm{e}^{\varrho\int_{x_\nu}^{x}|\phi(t)|\mathrm{d}t}[1] \quad \textit{for} \quad x \in D_{\nu-1,\varepsilon},$$

$$W(w_{\nu 2}^{IV}, w_{\nu 2}^{IV}) = -2\rho[1],$$

$$\tilde{a}_\nu^{IV}(\rho)=\frac{1}{2}\csc\pi\mu_\nu[1],\ \tilde{b}_\nu^{IV}(\rho)=\frac{1}{2}\csc\pi\mu_\nu[1].$$

$$\tilde{c}_\nu^{IV}(\rho)=\frac{1}{2}\sec\frac{\pi\mu_\nu}{2}e^{-i\frac{\pi}{2}\mu_\nu}[1] \quad and \quad \tilde{d}_\nu^{IV}(\rho)=-\frac{1}{2}\sec\frac{\pi\mu_\nu}{2}e^{i\frac{\pi}{2}\mu_\nu}[1].$$

b) For $\rho\in S_{-1}$

$$w_{\nu 1}^{IV}(x,\rho)\,|\phi(x)|^{\frac{1}{2}}\overset{(2)}{=}e^{\varrho\int_{x_\nu}^{x}|\phi(t)|dt}[1] \quad for \quad x\in D_{\nu-1,\varepsilon},$$

$$w_{\nu 2}^{IV}(x,\rho)\,|\phi(x)|^{\frac{1}{2}}\overset{(2)}{=}2\sin\frac{\pi\mu_\nu}{2}e^{-i\varrho\int_{x_\nu}^{x}|\phi(t)|dt-i\frac{\pi}{4}}[1] \quad for \quad x\in D_{\nu\varepsilon}$$

imply

$$w_{\nu 1}^{IV}(x,\rho)\,|\phi(x)|^{\frac{1}{2}}\overset{(2)}{=}\frac{1}{2}\csc\frac{\pi\mu_\nu}{2}\{e^{i\varrho\int_{x_\nu}^{x}|\phi(t)|dt-i\frac{\pi}{4}}[1]+e^{-i\varrho\int_{x_\nu}^{x}|\phi(t)|dt+i\frac{\pi}{4}}[1]\} \quad for \quad x\in D_{\nu\varepsilon},$$

$$w_{\nu 2}^{IV}(x,\rho)\,|\phi(x)|^{\frac{1}{2}}\overset{(2)}{=}e^{-\varrho\int_{x_\nu}^{x}|\phi(t)|dt}[1] \quad for \quad x\in D_{\nu-1,\varepsilon},$$

$$W(w_{\nu 2}^{IV},w_{\nu 2}^{IV})=-2\rho[1],$$

$$\tilde{a}_\nu^{IV}(\rho)=\frac{1}{2}\csc\pi\mu_\nu[1],\quad \tilde{b}_\nu^{IV}(\rho)=\frac{1}{2}\csc\pi\mu_\nu[1],$$

$$\tilde{c}_\nu^{IV}(\rho)=\frac{1}{2}\sec\frac{\pi\mu_\nu}{2}e^{-i\pi\frac{\mu_\nu}{2}}[1] \quad and \quad \tilde{d}_\nu^{IV}(\rho)=-\frac{1}{2}\sec\frac{\mu\pi_\nu}{2}e^{i\pi\frac{\mu_\nu}{2}}[1].$$

c) For $\rho\in S_0$

$$w_{\nu 1}^{IV}(x,\rho)\,|\phi(x)|^{\frac{1}{2}}\overset{(2)}{=}e^{\varrho\int_{x_\nu}^{x}|\phi(t)|dt}[1] \quad for \quad x\in D_{\nu-1,\varepsilon},$$

$$w_{\nu 2}^{IV}(x,\rho)\,|\phi(x)|^{\frac{1}{2}}\overset{(2)}{=}2\sin\frac{\pi\mu_\nu}{2}e^{i\varrho\int_{x_\nu}^{x}|\phi(t)|dt+i\frac{\pi}{4}}[1] \quad for \quad x\in D_{\nu\varepsilon}$$

imply

$$w_{\nu 1}^{IV}(x,\rho)\,|\phi(x)|^{\frac{1}{2}}\overset{(2)}{=}\frac{1}{2}\csc\frac{\pi\mu_\nu}{2}\{e^{i\varrho\int_{x_\nu}^{x}|\phi(t)|dt-i\frac{\pi}{4}}[1]+e^{-i\varrho\int_{x_\nu}^{x}|\phi(t)|dt+i\frac{\pi}{4}}[1]\} \quad for \quad x\in D_{\nu\varepsilon},$$

$$w_{\nu 2}^{IV}(x,\rho)\,|\phi(x)|^{\frac{1}{2}}\overset{(2)}{=}e^{-\varrho\int_{x_\nu}^{x}|\phi(t)|dt}[1] \quad for \quad x\in D_{\nu-1,\varepsilon},$$

$$W(w_{\nu 1}^{IV},w_{\nu 2}^{IV})=-2\rho[1],$$

$$\tilde{a}_\nu^{IV}(\rho)=\frac{1}{2}\csc\pi\mu_\nu[1],\quad \tilde{b}_\nu^{IV}(\rho)=\frac{1}{2}\csc\pi\mu_\nu[1],$$

$$\tilde{c}_\nu^{IV}(\rho)=\frac{1}{2}\sec\frac{\pi\mu_\nu}{2}e^{i\frac{\pi}{2}\mu_\nu}[1] \ and \ \tilde{d}_\nu^{IV}(\rho)=-\frac{1}{2}\sec\frac{\mu\pi_\nu}{2}e^{-i\frac{\pi}{2}\mu_\nu}[1].$$

d) For $\rho \in S_1$

$$w_{\nu 1}^{IV}(x, \rho)\, |\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} e^{\rho \int_{x_\nu}^{x} |\phi(t)| dt}[1] \quad \textit{for} \quad x \in D_{\nu-1,\varepsilon},$$

$$w_{\nu 2}^{IV}(x, \rho)\, |\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} 2\sin\frac{\pi\mu_\nu}{2}\, e^{i\rho \int_{x_\nu}^{x} |\phi(t)| dt + i\frac{\pi}{4}}[1] \quad \textit{for} \quad x \in D_{\nu\varepsilon},$$

imply

$$w_{\nu 1}^{IV}(x, \rho)\, |\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \frac{1}{2}\csc\frac{\pi\mu_\nu}{2}\, e^{-i\rho \int_{x_\nu}^{x} |\phi(t)| dt + i\frac{\pi}{4}}[1] \quad \textit{for} \quad x \in D_{\nu\varepsilon},$$

$$w_{\nu 2}^{IV}(x, \rho)\, |\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} e^{-\rho \int_{x_\nu}^{x} |\phi(t)| dt}[1] + i e^{\rho \int_{x_\nu}^{x} |\phi(t)| dt}[1] \quad \textit{for} \quad x \in D_{\nu-1,\varepsilon},$$

$$W(w_{\nu 1}^{IV}, w_{\nu 2}^{IV}) = -2\rho[1],$$

$$\tilde{a}_\nu^{IV}(\rho) = \frac{1}{2}\csc \pi\mu_\nu[1], \quad \tilde{b}_\nu^{IV}(\rho) = \frac{1}{2}\csc \pi\mu_\nu[1],$$

$$\tilde{c}_\nu^{IV}(x, \rho) = \frac{1}{2}\sec\frac{\pi\mu_\nu}{2}\, e^{i\frac{\pi}{2}\mu_\nu}[1] \quad \textit{and} \quad \tilde{d}_\nu^{IV}(\rho) = -\frac{1}{2}\sec\frac{\pi\mu_\nu}{2}\, e^{-i\frac{\pi}{2}\mu_\nu}[1].$$

(iii) Let x_ν *be a turning point of type I (or III). Then the estimates for* $w_{\nu 1}^{I}(x, \rho)$, $w_{\nu 2}^{I}(x, \rho)$ *(or* $w_{\nu 1}^{III}(x, \rho)$, $w_{\nu 2}^{III}(x, \rho)$*) for* $(x, \rho) \in \mathrm{I}_{\nu\varepsilon} \times S_j (j \in \mathbf{Z})$ *are obtained from the corresponding estimates for* $w_{\nu 1}^{II}(x, \rho)$, $w_{\nu 2}^{II}(x, \rho)$ *(or* $w_{\nu 1}^{IV}(x, \rho)$, $w_{\nu 2}^{II}(x, \rho)$*) for* $(x, \rho) \in \mathrm{I}_{\nu\varepsilon} \times S_{j+2}$ *by substituting therein* ρ *by* $i\rho$.

In addition the estimates for $\rho \in S_{j\pm 4}$ *are obtained from the estimates for* S_j *by substituting therein* ρ *by* $-\rho$.

Remark 3.3. (i) On account of Theorem 2.4 all the estimates stated in Theorem 3.2 for $w_{\nu 1}^{T_\nu}(x, \rho)$ and $w_{\nu 2}^{T_\nu}(x, \rho)$ with $(x, \rho) \in D_{\nu-1,\varepsilon} \times S_j$ or $(x, \rho) \in D_{\nu\varepsilon} \times S_j$ are also valid for $(x, \rho) \in \mathrm{I}_{\nu\varepsilon} \times S_j$ with $|\xi_\nu(x, \rho)| \geqslant N > 0$, if we substitute therein all terms [1] by $[1; \xi_\nu] = [1] + \dfrac{1}{\xi_\nu(x, \rho)} O(1)$. Combining (2.9), (3.23) and the estimates of Theorem 3.2 for the coefficients $\tilde{a}_\nu^{T_\nu}(\rho)$, $\tilde{b}_\nu^{T_\nu}(\rho)$, $\tilde{c}_\nu^{T_\nu}(\rho)$ and $\tilde{d}_\nu^{T_\nu}(\rho)$ we obtain estimates for $(x, \rho) \in \mathrm{I}_{\nu\varepsilon} \times S_j$ with $|\xi_\nu(x, \rho)| < N$.

(ii) It is not allowed to interpret the results of Theorem 3.2 in reversed order since there are infinitely many solutions satisfying similar asymptotic formulas. For example for arbitrary $c \in \mathbf{C} \backslash \{0\}$ and $\rho \in S_{-2}$

$$(w_{\nu 1}^{II}(x, \rho) + c\, w_{\nu 2}^{II}(x, \rho))|\phi(x)|^{\frac{1}{2}} \overset{(2)}{=} \csc \pi\mu_\nu e^{i\rho \int_{x_\nu}^{x} |\phi(t)| dt}[1] \quad \text{for} \quad x \in D_{\nu\varepsilon}.$$

This means that $w(x, \rho) = w_{\nu 1}^{II}(x, \rho) + c w_{\nu 2}^{II}(x, \rho)$ behaves for $x \in D_{\nu\varepsilon}$ like $w_{\nu 1}^{II}(x, \rho)$ but – according to Theorem 3.2 a) – $w(x, \rho)$ behaves for $x \in D_{\nu-1,\varepsilon}$ asymptotically like $c w_{\nu 2}^{II}(x, \rho)$.

4. Connection matrices and the solutions y_1, y_2, u_1, u_2

In this section we use Theorem 3.2 in order to determine the connection matrices $C_\nu(T_\nu, T_{\nu+1})$ with

$$\begin{pmatrix} w_{\nu 1}^{T_\nu} \\ w_{\nu 2}^{T_\nu} \end{pmatrix} = C_\nu(T_\nu, T_{\nu+1}) \begin{pmatrix} w_{\nu+1,1}^{T_{\nu+1}} \\ w_{\nu+1,2}^{T_{\nu+1}} \end{pmatrix}, \quad 1 \leqslant \nu \leqslant m-1. \tag{4.1}$$

For this purpose we introduce some abbreviations. For $1 \leqslant \nu \leqslant m$ and $\alpha, \beta, \gamma, z \in \mathbf{C}$ we set

$$F_\nu^I = F_\nu^{II} = \begin{pmatrix} \csc \pi\mu_\nu & 0 \\ 0 & \sin \pi\mu_\nu \end{pmatrix},$$

$$F_\nu^{III} = F_\nu^{IV} = \begin{bmatrix} \frac{1}{2}\csc\frac{\pi\mu_\nu}{2} & 0 \\ 0 & 2\sin\frac{\pi\mu_\nu}{2} \end{bmatrix},$$

$$E_\nu(z) = \begin{bmatrix} e^{z \int_{x_\nu}^{x_{\nu+1}} |\phi(t)| dt} & 0 \\ 0 & e^{-z \int_{x_\nu}^{x_{\nu+1}} |\phi(t)| dt} \end{bmatrix}$$

and

$$H_\nu(\rho; \alpha, \beta, \gamma) =$$

$$\begin{bmatrix} \left([1] + [\alpha] e^{-2\varrho \int_{x_\nu}^{x_{\nu+1}} |\phi(t)| dt}\right) & \left(e^{-2\varrho \int_{x_\nu+\varepsilon}^{x_{\nu+1}} |\phi(t)| dt} O\left(\frac{1}{\rho^{\sigma_0}}\right) + [\beta] e^{-2\rho \int_{x_\nu}^{x_{\nu+1}} |\phi(t)| dt}\right) \\ e^{2\varrho \int_{x_{\nu+1}-\varepsilon}^{x_{\nu+1}} |\phi(t)| dt} O\left(\frac{1}{\rho^{\sigma_0}}\right) + [\gamma] & [1] \end{bmatrix}.$$

Here $\varepsilon > 0$ is the number fixed in 2.3 and $H_\nu(\rho; \alpha, \beta, \gamma)$ is only determined in first approximation, which means that

$$(\alpha, \beta, \gamma) = (\alpha_1, \beta_1, \gamma_1) \text{ does not imply } H_\nu(\rho; \alpha, \beta, \gamma) = H_\nu(\rho; \alpha_1, \beta_1, \gamma_1).$$

Theorem 4.1. *The connection matrices satisfy the following identities.*

(i) For $\rho \in S_{-2}$ and $1 \leqslant \nu \leqslant m$

$$C_\nu(I, I) \quad = F_\nu^I E_\nu(\rho) H_\nu(\rho; \cos \pi\mu_\nu \cos \pi\mu_{\nu+1}, -\mathrm{i}\cos \pi\mu_\nu, \mathrm{i}\cos \pi\mu_{\nu+1}),$$

$$C_\nu(I, IV) \quad = F_\nu^I E_\nu(\rho) H_\nu(\rho; \cos\pi\mu_\nu, -\mathrm{i}\cos\pi\mu_\nu, \mathrm{i})$$

$$C_\nu(T_\nu, T_{\nu+1}) = F_\nu^{T_\nu} E_\nu(\mathrm{i}\rho) H_\nu(\mathrm{i}\rho; 0, 0, 0) \begin{cases} 1 & \textit{for } T_\nu = II \textit{ and } T_{\nu+1} \in \{II, III\}, \\ \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} & \textit{for } T_\nu = IV \textit{ and } T_{\nu+1} \in \{II, III\}, \end{cases}$$

$$C_\nu(III, I) \quad = F_\nu^{III} E_\nu(\rho) H_\nu(\rho; \cos\pi\mu_{\nu+1}, -\mathrm{i}, \mathrm{i}\cos\pi\mu_{\nu+1}) \mathrm{e}^{\mathrm{i}\frac{\pi}{4}},$$

$$C_\nu(III, IV) = F_\nu^{III} E_\nu(\rho) H_\nu(\rho; 1, -\mathrm{i}, \mathrm{i}) \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}.$$

(ii) For $\rho \in S_{-1}$ and $1 \leqslant \nu \leqslant m$

$$C_\nu(T_\nu, T_{\nu+1}) = F_\nu^{T_\nu} E_\nu(\rho) H_\nu(\rho; 0, 0, 0) \begin{cases} 1 & \textit{for } T_\nu = I \textit{ and } T_{\nu+1} \in \{I, IV\}, \\ \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} & \textit{for } T_\nu = III \textit{ and } T_{\nu+1} \in \{I, IV\}, \end{cases}$$

$$C_\nu(II, II) \quad = F_\nu^{II} E_\nu(\mathrm{i}\rho) H_\nu(\mathrm{i}\rho; \cos\pi\mu_\nu \cos\pi\mu_{\nu+1}, \mathrm{i}\cos\pi\mu_\nu, -\mathrm{i}\cos\pi\mu_{\nu+1}),$$

$$C_\nu(II, III) \quad = F_\nu^{II} E_\nu(\mathrm{i}\rho) H_\nu(\mathrm{i}\rho; \cos\pi\mu_\nu, \mathrm{i}\cos\pi\mu_\nu, -\mathrm{i}),$$

$$C_\nu(IV, II) \quad = F_\nu^{IV} E_\nu(\mathrm{i}\rho) H_\nu(\mathrm{i}\rho; \cos\pi\mu_{\nu+1}, \mathrm{i}, -\mathrm{i}\cos\pi\mu_{\nu+1}) \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}},$$

$$C_\nu(IV, III) \quad = F_\nu^{IV} E_\nu(\mathrm{i}\rho) H_\nu(\mathrm{i}\rho; 1, \mathrm{i}, -\mathrm{i}) \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}.$$

(iii) For $\rho \in S_0$ and $1 \leqslant \nu \leqslant m$

$$C_\nu(T_\nu, T_{\nu+1}) = F_\nu^{T_\nu} E_\nu(\rho) H_\nu(\rho, 0, 0, 0) \begin{cases} 1 & \textit{for } T_\nu = I \textit{ and } T_{\nu+1} \in \{I, IV\}, \\ \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} & \textit{for } T_\nu = III \textit{ and } T_{\nu+1} \in \{I, IV\}, \end{cases}$$

$$C_\nu(II, II) \quad = F_\nu^{II} E_\nu(-\mathrm{i}\rho) H_\nu(-\mathrm{i}\rho; \cos\pi\mu_\nu \cos\pi\mu_{\nu+1}, -\mathrm{i}\cos\pi\mu_\nu, \mathrm{i}\cos\pi\mu_{\nu+1}),$$

$$C_\nu(II, III) \quad = F_\nu^{II} E_\nu(-\mathrm{i}\rho) H_\nu(-\mathrm{i}\rho; \cos\pi\mu_\nu, -\mathrm{i}\cos\pi\mu_\nu, \mathrm{i}),$$

$$C_\nu(IV, II) \quad = F_\nu^{IV} E_\nu(-\mathrm{i}\rho) H_\nu(-\mathrm{i}\rho; \cos\pi\mu_{\nu+1}, -\mathrm{i}, \mathrm{i}\cos\pi\mu_{\nu+1}) \mathrm{e}^{\mathrm{i}\frac{\pi}{4}},$$

$$C_\nu(IV, III) \quad = F_\nu^{IV} E_\nu(-\mathrm{i}\rho) H_\nu(-\mathrm{i}\rho; 1, -\mathrm{i}, \mathrm{i}) \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}.$$

(iv) For $\rho \in S_{-3}$ and $1 \leqslant \nu \leqslant m$

$$C_\nu(I, I) \quad = F_\nu^I E_\nu(-\rho) H_\nu(-\rho; \cos\pi\mu_\nu \cos\pi\mu_{\nu+1}, \mathrm{i}\cos\pi\mu_\nu, -\mathrm{i}\cos\pi\mu_{\nu+1}),$$

$$C_\nu(I, IV) \quad = F_\nu^I E_\nu(-\rho) H_\nu(-\rho; \cos\pi\mu_\nu, \mathrm{i}\cos\pi\mu_\nu, -\mathrm{i})$$

$$C_\nu(T_\nu, T_{\nu+1}) = F_\nu^{T_\nu} E_\nu(\mathrm{i}\rho) H_\nu(\mathrm{i}\rho, 0, 0, 0) \begin{cases} 1 & \textit{for } T_\nu = II \textit{ and } T_{\nu+1} \in \{II, III\}, \\ \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} & \textit{for } T_\nu = IV \textit{ and } T_{\nu+1} \in \{II, III\}, \end{cases}$$

$$C_\nu(III, I) \quad = F_\nu^{III} E_\nu(-\rho) H_\nu(-\rho; \cos\pi\mu_{\nu+1}, \mathrm{i}, -\mathrm{i}\cos\pi\mu_{\nu+1}) \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}},$$

$$C_\nu(III, IV) = F_\nu^{III} E_\nu(-\rho) H_\nu(-\rho; 1, \mathrm{i}, -\mathrm{i}) \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}.$$

The connection matrices for S_{j+4} are obtained from the connection matrices for S_j by substituting therein ρ by $-\rho$.

Proof. The connection matrices listed in Theorem 4.1 are all determined in the same way, using Theorem 3.2 and Cramer's rule. Therefore it is sufficient to give a detailed proof for one example:

Let $\rho \in S_{-1}$, $1 \leqslant v \leqslant m-1$, $T(x_v) = IV$ and $T(x_{v+1}) = III$. For the elements of the matrix

$$C_v(IV, III) =: \begin{pmatrix} a(\rho) & b(\rho) \\ c(\rho) & d(\rho) \end{pmatrix}$$

we infer from (4.1) and Theorem 3.2 by Cramer's rule and with $x := x_v + \varepsilon$

$$a(\rho) = \frac{1}{W(w^{III}_{v+1,1}, w^{III}_{v+1,2})} \begin{vmatrix} w^{IV}_{v1}(x_v+\varepsilon, \rho) & w^{III}_{v+1,2}(x_v+\varepsilon, \rho) \\ w^{IV'}_{v1}(x_v+\varepsilon, \rho) & w^{III'}_{v+1,2}(x_v+\varepsilon, \rho) \end{vmatrix} = \frac{i}{4}\csc\frac{\pi\mu_v}{2} \times$$

$$\begin{vmatrix} (e^{i\varrho \int_{x_v}^{x_v+\varepsilon} |\phi(t)|dt - i\frac{\pi}{4}}[1] + e^{-i\varrho \int_{x_v}^{x_v+\varepsilon} |\phi(t)|dt + i\frac{\pi}{4}}[1]) & (ie^{i\varrho \int_{x_{v+1}}^{x_v+\varepsilon} |\phi(t)|dt}[1] + e^{-i\varrho \int_{x_{v+1}}^{x_v+\varepsilon} |\phi(t)|dt}[1]) \\ (ie^{i\varrho \int_{x_v}^{x_v+\varepsilon} |\phi(t)|dt - i\frac{\pi}{4}}[1] - ie^{-i\varrho \int_{x_v}^{x_v+\varepsilon} |\phi(t)|dt + i\frac{\pi}{4}}[1]) & (-e^{i\varrho \int_{x_{v+1}}^{x_v+\varepsilon} |\phi(t)|dt}[1] - ie^{-i\varrho \int_{x_{v+1}}^{x_v+\varepsilon} |\phi|dt}[1]) \end{vmatrix}$$

$$= \frac{1}{2}\csc\frac{\pi\mu_v}{2} e^{i\varrho \int_{x_v}^{x_{v+1}} |\phi(t)|dt - i\frac{\pi}{4}} \{[1] + [1] e^{-2i\varrho \int_{x_v}^{x_{v+1}} |\phi(t)|dt}\}$$

and

$$b(\rho) = -\frac{[1]}{2i\rho} \begin{vmatrix} w^{III}_{v+1,1}(x_v+\varepsilon, \rho) & w^{IV}_{v,1}(x_v+\varepsilon, \rho) \\ w^{III'}_{v+1,1}(x_v+\varepsilon, \rho) & w^{IV'}_{v1}(x_v+\varepsilon, \rho) \end{vmatrix} = \frac{i}{4}\csc\frac{\pi\mu_v}{2} \times$$

$$\begin{vmatrix} e^{i\varrho \int_{x_{v+1}}^{x_v+\varepsilon} |\phi(t)|dt}[1] & (e^{i\varrho \int_{x_v}^{x_v+\varepsilon} |\phi(t)|dt - i\frac{\pi}{4}}[1] + e^{-i\varrho \int_{x_v}^{x_v+\varepsilon} |\phi(t)|dt + i\frac{\pi}{4}}[1]) \\ ie^{i\varrho \int_{x_{v+1}}^{x_v+\varepsilon} |\phi(t)|dt}[1] & (ie^{i\varrho \int_{x_v}^{x_v+\varepsilon} |\phi(t)|dt - i\frac{\pi}{4}}[1] - ie^{-i\varrho \int_{x_v}^{x_v+\varepsilon} |\phi(t)|dt + i\frac{\pi}{4}}[1]) \end{vmatrix}$$

$$= e^{i\varrho \int_{x_v}^{x_{v+1}} |\phi(t)|dt - i\frac{\pi}{4}} \left\{ O\left(\frac{1}{\rho^{\sigma_0}}\right) e^{-2i\varrho \int_{x_v+\varepsilon}^{x_{v+1}} |\phi(t)|dt} \right.$$

$$\left. + [i] e^{-2i\varrho \int_{x_v}^{x_{v+1}} |\phi(t)|dt} \right\} \frac{1}{2}\csc\frac{\pi\mu_v}{2}.$$

Similarly we obtain with $x := x_{v+1} - \varepsilon$

$$c(\rho) = i\sin\frac{\pi\mu_v}{2} \begin{vmatrix} e^{-i\varrho \int_{x_v}^{x_{v+1}-\varepsilon} |\phi(t)|dt - \frac{i\pi}{4}}[1] & (ie^{i\varrho \int_{x_{v+1}}^{x_{v+1}-\varepsilon} |\phi(t)|dt}[1] + e^{-i\varrho \int_{x_{v+1}}^{x_{v+1}-\varepsilon} |\phi(t)|dt}[1]) \\ -ie^{-i\varrho \int_{x_v}^{x_{v+1}-\varepsilon} |\phi(t)|dt - i\frac{\pi}{4}}[1] & (-e^{i\varrho \int_{x_{v+1}}^{x_{v+1}-\varepsilon} |\phi(t)|dt}[1]) - ie^{-i\varrho \int_{x_{v+1}}^{x_{v+1}-\varepsilon} |\phi(t)|dt}[1]) \end{vmatrix}$$

$$= 2\sin\frac{\pi\mu_v}{2} e^{-i\varrho \int_{x_v}^{x_{v+1}} |\phi(t)|dt - \frac{i\pi}{4}} \left\{ O\left(\frac{1}{\rho^{\sigma_0}}\right) e^{2i\varrho \int_{x_{v+1}-\varepsilon}^{x_{v+1}} |\phi(t)|dt} + [-i] \right\}$$

and

$$d(\rho)=\mathrm{i}\sin\frac{\pi\mu_\nu}{2}\begin{vmatrix} \mathrm{e}^{\mathrm{i}\rho\int_{x_{\nu+1}}^{x_{\nu+1}-\varepsilon}|\phi(t)|\mathrm{d}t}[1] & \mathrm{e}^{-\mathrm{i}\rho\int_{x_\nu}^{x_{\nu+1}-\varepsilon}|\phi(t)|\mathrm{d}t-\mathrm{i}\frac{\pi}{4}}[1] \\ \mathrm{i}\mathrm{e}^{\mathrm{i}\rho\int_{x_{\nu+1}}^{x_{\nu+1}-\varepsilon}|\phi(t)|\mathrm{d}t}[1] & -\mathrm{i}\mathrm{e}^{-\mathrm{i}\rho\int_{x_\nu}^{x_{\nu+1}-\varepsilon}|\phi(t)|\mathrm{d}t-\mathrm{i}\frac{\pi}{4}}[1] \end{vmatrix}$$

$$=2\sin\frac{\pi\mu_\nu}{2}\mathrm{e}^{-\mathrm{i}\rho\int_{x_\nu}^{x_\nu+1}|\phi(t)|\mathrm{d}t-\mathrm{i}\frac{\pi}{4}}[1].$$

This yields the formula for $C_\nu(IV, III)$ for $\rho\in S_{-1}$; the values of x have been selected so that the modulus of the exponential term in $b(\rho)$ and $c(\rho)$ which are multiplied by $O\left(\frac{1}{\rho^{\sigma_0}}\right)$ becomes as small as possible. All remaining connection matrices have been calculated similarly. We note that the asymptotic representation of the connection matrices corresponding to $C_\nu(T_\nu, T_{\nu+1})$ cannot be found if we substitute the functions $w_{\nu j}^{T_\nu}$ and $w_{\nu+1,j}^{T_{\nu+1}}$ in (4.1) by the functions $u_{\nu j}$ and $u_{\nu+1,j}$ defined in Theorem 2.4; in this case the connection coefficients for example for the sector S_0 are of the form

$$[c]+O\left(\frac{1}{\rho}\right)\mathrm{e}^{\rho d} \quad \text{with} \quad c, \quad d\in\mathbf{C}$$

and with $\Re(\rho d)>0$ for $0<\arg\rho\leqslant\frac{\pi}{4}$ and $\Re(\rho d)=0$ for $\arg\rho=0$.

This means that the connection matrices can only be determined for positive (or negative) values of the parameter ρ and not for sectors in the ρ-plane – cf. [22] for the discussion of problems with a positive parameter. On the other hand we need estimates of the connection matrices and of fundamental systems of solutions of (2.1) which are valid in sectors of the ρ-plane for the investigation of the corresponding nonselfadjoint eigenvalue problems.

4.2. The fundamental system y_1, y_2

Now we are able to define a normalized fundamental system of solutions of (2.1) which can be estimated on the whole interval [0, 1]. We set

$$n_I(\rho)=n_{IV}(\rho)=\mathrm{e}^{\rho\int_0^{x_1}|\phi(t)|\mathrm{d}t},$$

$$n_{II}(\rho)=n_{III}(\rho)=\begin{cases}\mathrm{e}^{\mathrm{i}\rho\int_0^{x_1}|\phi(t)|\mathrm{d}t} & \text{for} \quad \rho\in S_{-2}\cup S_{-1},\\ \mathrm{e}^{-\mathrm{i}\rho\int_0^{x_1}|\phi(t)|\mathrm{d}t} & \text{for} \quad \rho\in S_0\cup S_1.\end{cases}$$

Let $k\in\{-2, -1, 0, 1\}$ be fixed and $\rho\in S_k$; then we denote by $y_1(\cdot, \rho)$ and $y_2(\cdot, \rho)$ the continuation of the functions $n_{T_1}(\rho)\,w_{11}^{T_1}(\cdot, \rho)$ and $\frac{1}{n_{T_1}(\rho)}w_{12}^{T_1}(\cdot, \rho)$ to the whole interval

[0, 1]. Since $W(y_1(\cdot,\rho), y_2(\cdot,\rho)) = W(w_{11}^{T_1}(\cdot,\rho), w_{12}^{T_1}(\cdot,\rho)) \neq 0$ for $\rho \in S_k$ with $|\rho| > r_0$, $\{y_1(\cdot,\rho), y_2(\cdot,\rho)\}$ is (for every sector S_k) a fundamental system of (2.1). According to Theorem 4.1.

$$\begin{pmatrix} y_1(x,\rho) \\ y_2(x,\rho) \end{pmatrix} = \begin{pmatrix} n_{T_1}(\rho) & 0 \\ 0 & (n_{T_1}(\rho))^{-1} \end{pmatrix} \prod_{v=1}^{j} C_v(T_v, T_{v+1}) \begin{pmatrix} w_{j+1,1}^{T_{j+1}}(x,\rho) \\ w_{j+1,2}^{T_{j+1}}(x,\rho) \end{pmatrix} \tag{4.2}$$

for $(x,\rho) \in \mathrm{I}_{j+1,\varepsilon} \times S_k$, $0 \leqslant j \leqslant m-1$. Combining Theorem 3.2, Theorem 4.1 and (4.2) we get asymptotic estimates of $y_1(x,\rho)$ and $y_2(x,\rho)$ for $\rho \in S_k$ and $x \in [0, 1]$. It turns out that the resulting estimates are satisfactory for y_1 but rather unsatisfactory for y_2 (see (4.4), (4.5)); for the formulation of these estimates we introduce the following notations. For $x \in [0, 1]$ let

$$R_+(x) = \int_0^x \sqrt{\max\{0, \phi^2(t)\}}\, dt, \quad R_-(x) = \int_0^x \sqrt{\max\{0, -\phi^2(t)\}}\, dt,$$

$V_{T_v}(x) =$ Number of turning points of type T_v in $(0, x)$, $1 \leqslant v \leqslant 4$,

$$K_\pm(x) = \left(\prod_{T_v \in \{III, IV\}}^{x_v < x} \frac{1}{2} \csc \frac{\pi\mu_v}{2} \right) \left(\prod_{T_v \in \{I, II\}}^{x_v < x} \csc \pi\mu_v \right) e^{\pm i \frac{\pi}{4}(V_{III}(x) - V_{IV}(x))}.$$

Further we denote by $E_k(x,\rho)$, $-2 \leqslant k \leqslant 1$, $(x,\rho) \in [0, 1] \times S_k$, an exponential sum of the form

$$E_k(x,\rho) = [1] + \sum_{n=1}^{v(x)} e^{\varrho\alpha_k\beta_{kn}(x)} [b_{kn}(x)], \tag{4.3}$$

where $\alpha_{-2} = \alpha_1 = -1$, $\alpha_0 = -\alpha_{-1} = i$, $\beta_{kv(x)}(x) \neq 0$, $0 < \delta \leqslant \beta_{k1}(x) < \beta_{k2}(x) < \ldots < \beta_{kv(x)}(x) \leqslant 2\max\{R_+(1), R_-(1)\}$ and where the functions v and b_{kn} are constant in every interval $D_{j\varepsilon}$, $0 \leqslant j \leqslant m$. Using these abbreviations, we easily infer from (4.2), Theorem 3.2 and Theorem 4.1 for $\rho \in S_k$ and $x \in \bigcup_{v=1}^{m} D_{v\varepsilon}$

$$y_1(x,\rho) \overset{(2)}{=} \frac{1}{\sqrt{|\phi(x)|}} e^{\varrho(R_-(x) \pm iR_+(x))} K_\pm(x) E_k(x,\rho) \tag{4.4}$$

and

$$y_2(x,\rho) \overset{(2)}{=} \frac{1}{\sqrt{|\phi(x)|}} e^{\varrho(R)_-(x) \pm iR_+(x) - \delta_1(\varepsilon) - \delta_2(\varepsilon)i)} O\left(\frac{1}{\rho^{\sigma_0}}\right), \tag{4.5}$$

where $\delta_1(\varepsilon) + \delta_2(\varepsilon) > 0$, $\delta_1(\varepsilon), \delta_2(\varepsilon) \geqslant 0$ and where the upper or lower sign in (4.4) and (4.5) has to be used if $k \in \{-2, -1\}$ or $\{0, 1\}$ respectively.

For $x \in \bigcup_{v=1}^{m} (x_v - \varepsilon, x_v + \varepsilon)$ and $\rho \in S_k$ we can estimate $y_1(x,\rho)$ and $y_2(x,\rho)$ using (4.2), Theorem 4.1 and Theorem 3.2.

Since $y_2(x,\rho) = (n_{T_1}(\rho))^{-1} w_{12}^{T_1}(x,\rho)$ for $(x,\rho) \in [0, x_1 - \varepsilon] \times S_k$, Theorem 3.2 yields an estimate of $y_2(x,\rho)$ for $(x,\rho) \in [0, x_1 - \varepsilon] \times S_k$ which is much more precise than (4.5). In addition it turns out that the estimates obtained for y_1, y_2 are precise enough to determine the asymptotic behaviour of the eigenvalues of a large class of regular eigenvalue problems defined by (2.1) and two-point boundary conditions; these results have been partially published in [7]. For the proof of expansion theorems with respect to the eigenfunctions of these eigenvalue problems we cannot use the estimates (4.5) for $y_2(x, \rho)$; for this reason we derive asymptotic estimates for another fundamental system of solutions of (2.1) which are more adequate for the proof of expansion theorems in [8].

4.3. The fundamental systems u_1, u_2 and y_1, u_2

Let $k \in \{-2, -1, 0, 1\}$ be fixed, $\rho \in S_k$ and let $w_{m1}^{T_m}$, $w_{m2}^{T_m}$ be the fundamental system of solutions of (2.1) defined in section 3. Then we set for $x \in [0, 1]$ and $\rho \in S_k$

$$\begin{pmatrix} u_1(x, \rho) \\ u_2(x, \rho) \end{pmatrix} = \begin{pmatrix} \tilde{n}_{T_m}(\rho) & 0 \\ 0 & \tilde{n}_{T_m}(\rho)^{-1} \end{pmatrix} (F_m^{T_m})^{-1} e^{i\frac{\pi}{4}\nu(k, T_m)} \begin{pmatrix} w_{m1}^{T_m}(x, \rho) \\ w_{m2}^{T_m}(x, \rho) \end{pmatrix}$$

where

$$\tilde{n}_I(\rho) = \tilde{n}_{III}(\rho) = e^{-\varrho \int_{x_m}^{1} |\phi(t)| dt} \quad \text{for} \quad \operatorname{Re} \rho \geqslant 0,$$

$$\tilde{n}_{II}(\rho) = \tilde{n}_{IV}(\rho) = \begin{cases} e^{i\varrho \int_{x_m}^{1} |\phi(t)| dt} & \text{for} \quad \rho \in S_0 \cup S_1, \\ e^{-i\varrho \int_{x_m}^{1} |\phi(t)| dt} & \text{for} \quad \rho \in S_{-2} \cup S_{-1}, \end{cases}$$

and

$$\nu(k, T_m) = \begin{cases} 0 & \text{for} \quad T_m \in \{I, II\}, \ -2 \leqslant k \leqslant 1, \\ -1 & \text{for} \quad T_m = III, \ k \in \{-2, -1\} \text{ and } T_m = IV, \ k \in \{0, 1\}, \\ 1 & \text{for} \quad T_m = III, \ k \in \{0, 1\} \text{ and } T_m = IV, \ k \in \{-2, -1\}. \end{cases}$$

From (4.1) we infer for $1 \leqslant \nu \leqslant m-1$, $\rho \in S_k$ and $x \in I_{\nu\varepsilon}$

$$(4.6) \qquad \begin{pmatrix} u_1(x, \rho) \\ u_2(x, \rho) \end{pmatrix} = \begin{pmatrix} \tilde{n}_{T_m}(\rho) & 0 \\ 0 & \tilde{n}_{T_m}(\rho)^{-1} \end{pmatrix}$$
$$\times (F_m^{T_m})^{-1} e^{i\frac{\pi}{4}\nu(k, T_m)} C_{m-1}(T_{m-1}, T_m)^{-1} \dots C_\nu(T_\nu, T_{\nu+1})^{-1} \begin{pmatrix} w_{\nu1}^{T_\nu}(x, \rho) \\ w_{\nu2}^{T_\nu}(x, \rho) \end{pmatrix}.$$

On account of Theorem 4.1, $C_\nu(T_\nu, T_{\nu+1})$ is the product of two regular diagonal matrices and of a matrix of the form $H_\nu(z; \alpha, \beta, \gamma)$. We show how $C_\nu(T_\nu, T_{\nu+1})^{-1}$ is easily obtained from Theorem 4.1; for this purpose we denote by $\check{H}_\nu(z; \alpha, \beta, \gamma)$ a matrix of the form

$$\tilde{H}_\nu(z;\alpha,\beta,\gamma)=\begin{bmatrix}[1]+[\alpha]e^{-2z\int_{x_\nu}^{x_{\nu+1}}|\phi(t)|dt} & e^{2z\int_{x_{\nu+1}-\varepsilon}^{x_{\nu+1}}|\phi(t)|dt}O\left(\frac{1}{\rho^{\sigma_0}}\right)+[\beta] \\ e^{-2z\int_{x_\nu+\varepsilon}^{x_{\nu+1}}|\phi(t)|dt}O\left(\frac{1}{\rho^{\sigma_0}}\right)+[\gamma]e^{-2z\int_{x_\nu}^{x_{\nu+1}}|\phi(t)|dt} & [1]\end{bmatrix}$$

for $z, \alpha, \beta, \gamma \in \mathbf{C}$. Since $\det H_\nu(z;\alpha,\beta,\gamma)=[1]$ for $\alpha, \beta, \gamma \in \mathbf{C}$ and $\operatorname{Re} z>0$, we conclude that $H_\nu(z;\alpha,\beta,\gamma)^{-1}$ is a matrix of the form $\tilde{H}_\nu(z;\alpha,-\gamma,-\beta)$ for $\operatorname{Re} z>0$. Now we can use Theorem 3.2 and Theorem 4.1 in order to determine the asymptotic form of $u_1(x,\rho)$, $u_2(x,\rho)$ for $x\in[0,1]$ and $\rho\in S_k$. It turns out that the estimates obtained for $u_2(x,\rho)$ are satisfactory and that the estimates for $u_1(x,\rho)$ are unsatisfactory. Using the abbreviations

$$\tilde{R}_+(x)=R_+(1)-R_+(x),\quad \tilde{R}_-(x)=R_-(1)-R_-(x)$$

and

$$\tilde{K}_\pm(x)=\left(\prod_{\substack{x<x_\nu\\ T_\nu\in\{III,IV\}}}2\sin\frac{\pi\mu_\nu}{2}\right)^{-1}\left(\prod_{\substack{x<x_\nu\\ T_\nu\in\{I,II\}}}\sin\pi\mu_\nu\right)^{-1}e^{\pm i\frac{\pi}{4}(V_{III}(x)-V_{III}(1)-V_{IV}(x)+V_{IV}(1))},$$

we get as with (4.4)

$$u_2(x,\rho)\overset{(2)}{=}\frac{1}{\sqrt{|\phi(x)|}}e^{\rho(\tilde{R}_-(x)\pm i\tilde{R}_+(x))}\tilde{K}_\pm(x)\tilde{E}_k(x,\rho) \tag{4.7}$$

for $x\in\bigcup_{\nu=0}^{m}D_{\nu\varepsilon}$ and $\rho\in S_k$, $-2\leqslant k\leqslant 1$, where the upper or lower sign in (4.7) has to be used if $k\in\{-2,-1\}$ or $\{0,1\}$ respectively and where $\tilde{E}_k(x,\rho)$ is an exponential sum of the form (4.3).

The estimates obtained for $u_1(x,\rho)$ are unsatisfactory and will not be used in the sequel. From (4.4), (4.6) and Theorem 3.2 we infer for $\rho\in S_k$, $-2\leqslant k\leqslant 1$,

$$W(y_1(\cdot,\rho),u_2(\cdot,\rho))=\begin{vmatrix}y_1(1,\rho) & u_2(1,\rho)\\ y_1'(1,\rho) & u_2'(1,\rho)\end{vmatrix}$$

$$=2\rho\alpha K_\pm(1)E_k(1,\rho)\tilde{n}_{T_m}(\rho)^{-1}e^{i\frac{\pi}{4}\nu(k,T_m)}(F_m^{T_m})_{1,1},$$

where $\alpha\in\{1,-1,i,-i\}$ and where $E_k(1,\rho)$ is an asymptotic exponential sum of the form (4.3); in addition, $(A)_{i,k}$ denotes the element in the i-th row and k-th column of the matrix A. Hence $W(y_1(\cdot,\rho),u_2(\cdot,\rho))\neq 0$ for $\rho\in S_k\backslash C_k$ and $|\rho|>r_0$, where C_k is the countable set of zeros of $E_k(1,\rho)$. For $\rho\in S_k\backslash C_k$ and $|\rho|>r_0$, $-2\leqslant k\leqslant 1$, $\{y_1(\cdot,\rho), u_2(\cdot,\rho)\}$ is a fundamental system of solutions of (2.1) satisfying the estimates (4.2), (4.4) and (4.6), (4.7) which are adequate for the application of the contour-integration-method and the proof of expansion theorems corresponding to eigenvalue problems defined by (2.1) and two-point boundary conditions in [8]; for the proof of theorems on the pointwise convergence of the corresponding expansions we need the complete information on

$y_1(\cdot, \rho)$, $u_2(\cdot, \rho)$ contained in Theorem 2.4, Theorem 3.2 and Section 4. Notice that the solutions $y_1(\cdot, \rho)$, $u_2(\cdot, \rho)$ cannot be used to determine the asymptotic behaviour of the eigenvalues of these problems; for details see [7], [8].

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