Stability of essential spectra of self-adjoint subspaces under compact perturbations

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A B S T R A C T

This paper studies stability of essential spectra of self-adjoint subspaces (i.e., self-adjoint linear relations) under finite rank and compact perturbations in Hilbert spaces. Relationships between compact perturbation of closed subspaces and relatively compact perturbation of their operator parts are first established. This gives a characterization of compact perturbation in terms of difference between the operator parts of perturbed and unperturbed subspaces. It is shown that a self-adjoint subspace is still self-adjoint under either relatively bounded perturbation with relative bound less than one or relatively compact perturbation or compact perturbation with a certain additional condition. By using these results, invariance of essential spectra of self-adjoint subspaces is proved under relatively compact and compact perturbations, separately. As a special case, finite rank perturbation is discussed. The results obtained in this paper generalize the corresponding results for self-adjoint operators to self-adjoint subspaces.

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1. Introduction

Perturbation problems are one of the main topics in both pure and applied mathematics. The perturbation theory of operators (i.e., single-valued operators) has been extensively studied and many elegant results have been obtained (cf., [8,10,18]). In particular, stability of spectra of self-adjoint operators under perturbation has received lots of attention. We shall recall several most well-known results about it. If a perturbation term is a symmetric (i.e., densely-defined and Hermitian) and relatively compact operator to a self-adjoint operator, then the essential spectrum of the self-adjoint operator is invariant (see [12, Theorem 8.15] or [18, Theorem 9.9]). However, it was shown that its absolutely continuous spectrum may disappear under this perturbation even though the perturbation term is very small by H. Weyl and later generalized by von Neumann [8, Chapter 10, Theorem 2.1]. But if the perturbation term is finite rank or more generally belongs to the trace class, then its absolutely continuous spectrum is invariant [8, Chapter 10, Theorems 4.3 and 4.4]. These results have been extensively applied to study of stability of spectra of symmetric linear differen-

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tial operators and bounded Jacobi operators (i.e., second-order bounded and symmetric linear difference operators) including Schrödinger operators that have a strong physical background.

Recently, it was found that minimal and maximal operators generated by symmetric linear difference expressions are multi-valued or non-densely defined in general even though the corresponding definiteness condition is satisfied (cf., [11,14]), and similar are those generated by symmetric linear differential expressions that do not satisfy the definiteness condition [9]. So the classical perturbation theory of operators is not available in this case. Partially due to the above reason, the study of non-densely defined or multi-valued linear operators has attracted a great deal of interests in near half a century.

In 1998, Cross introduced a concept of essential spectrum of a multi-valued linear operator (also called linear relations or subspaces) in a complex normed space in terms of the nullity and deficiency of its complete closure, and showed its stability under relative compact perturbation with certain additional conditions (see [4, Theorem VII.3.2]). In 2014, Wicox gave five distinct essential spectra of linear relations in Banach spaces in terms of semi-Fredholm properties, and showed their stability under relative compact perturbation with some additional conditions and under compact perturbation, separately, in [19] (see Remarks 5.1 and 5.2 for more detailed discussions about their relations with those obtained in the present paper).

In 2009, Azizov with his coauthors introduced concepts of compact and finite rank perturbations of closed subspaces in $X \times Y$ in terms of difference between orthogonal projections of $X \times Y$ to the subspaces, where X and Y are Hilbert spaces [2]. They proved that a closed subspace is a finite rank or compact perturbation of another closed subspace if and only if the difference between their resolvents is a finite rank or compact operator in the intersection of their resolvent sets in the case that this intersection is not empty and $X = Y$ [2, Corollaries 3.4 and 4.5]. Further, they studied stability properties of spectral points of positive and negative type and type π in the non-self-adjoint case under several kinds of perturbations in the Krein spaces [3].

Minimal operators or subspaces, generated by symmetric linear differential and difference expressions, are closed Hermitian operators or subspaces, and their self-adjoint extensions are self-adjoint operators or subspaces. Their resolvents can be expressed by corresponding Green functions. In the case that the differential and difference expressions are singular, their resolvents are complicated in general, and much more complicated when their orders (or dimensions) are higher because several boundary conditions or coupled boundary conditions are involved. Moreover, the Green functions are often expressed by solutions of the systems rather than by coefficients of the systems. Therefore, in some cases it is more convenient to give out a characterization of perturbation in terms of the operators or subspaces themselves rather than their resolvents. For the operator case, concepts of relatively compact, and finite rank and trace class perturbations were given in terms of the difference between perturbed and unperturbed operators (see Definition 2.3 and Lemma 2.4 for relatively compact perturbation of operators in Section 2, and we refer to [8,12,15,18] for more detailed discussions). We shall try to give a similar characterization of compact and finite rank perturbations for closed subspaces to that for operators in the present paper.

In this paper, we focus on the stability of essential spectra of self-adjoint subspaces (i.e., self-adjoint linear relations) under perturbations in Hilbert spaces. Here, the essential spectrum of a subspace is defined as the subset of its spectrum consisting of either accumulation points or isolated eigenvalues of infinite multiplicity (see Definition 2.1, and see Remark 2.1 for detailed discussions about relations of this definition with those given in [4,19]). And the definitions of compact and finite rank perturbations of closed subspaces given in [2] are used in the present paper.

Note that the spectrum and various spectra of a self-adjoint subspace, including point, discrete, essential, continuous, singular continuous, absolutely continuous and singular spectra, can be only determined by its operator part [13, Theorems 2.1, 2.2, 3.4 and 4.1]. So it is natural to take into account perturbations of operator parts of the unperturbed and perturbed subspaces. However, we find that a summation of closed subspaces, $T = S + A$, cannot imply a similar relation of their operator parts, $T_s = S_s + A_s$, in general (see Section 3 for a detailed discussion). In addition, in dealing with the summation of subspaces, one

shall encounter another problem that does not happen for operators. If T, S and A are operators with $D(T) = D(S) \subset D(A)$ and satisfy $T = S + A$, then T can be interchanged with S or A as $S = T - A$ or $A|_{D(S)} = T - S$. However, if they are multi-valued, then this interchanging may not hold in general. This is resulted in by their multi-valued parts. See Example 3.1, in which $T = S + A$ holds, but $A|_{D(S)} \neq T - S$. These problems make the study of subspaces complicated.

The rest of this paper is organized as follows. In Section 2, some notations, basic concepts and fundamental results about subspaces are introduced. Relatively bounded and compact perturbations and finite rank perturbation of operators and closed subspaces with some properties are recalled. In Section 3, relationships among the operator parts of an unperturbed subspace, its perturbation and its perturbation term are established. Due to these relationships, relationships of compact and finite rank perturbations of closed subspaces with relatively compact and finite rank perturbations of the operator parts of the unperturbed subspace and its perturbation term are given, separately, in which characterizations of compact and finite rank perturbations are provided with relatively compact and finite rank perturbations of their operator parts, respectively. In Section 4, it is shown that a self-adjoint subspace is still self-adjoint under either relatively bounded perturbation with relative bound less than one or relatively compact perturbation or compact perturbation with an additional condition. Finally, it is proved that essential spectrum of a self-adjoint subspace is stable under either compact perturbation or relatively compact perturbation in Section 5.

2. Preliminaries

In this section, we shall first list some notations and basic concepts, including spectrum, discrete and essential spectra of subspaces, and reducing subspace. Then we recall some fundamental results about subspaces. Next, we introduce concepts of compact and finite rank perturbations for operators and closed subspaces, and list some related results.

By $\mathbf{R}$ and $\mathbf{C}$ denote the sets of the real and complex numbers, respectively, throughout this paper.

Let X be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and T a linear subspace (briefly, subspace) in the product space $X^2 := X \times X$ with the following induced inner product, still denoted by $\langle \cdot, \cdot \rangle$ without any confusion:

$$\langle (x, f), (y, g) \rangle = \langle x, y \rangle + \langle f, g \rangle, \quad (x, f), \ (y, g) \in X^2.$$

By $D(T)$ and $R(T)$ denote the domain and range of T, respectively. Its adjoint subspace is defined by

$$T^* := \{(y, g) \in X^2 : \ \langle g, x \rangle = \langle y, f \rangle \text{ for all } (x, f) \in T\}.$$

T is said to be a Hermitian subspace in X^2 if $T \subset T^*$, and said to be a self-adjoint subspace in X^2 if $T = T^*$.

Further, denote

$$T(x) := \{f \in X : \ (x, f) \in T\}, \ \ T^{-1} := \{(f, x) : \ (x, f) \in T\}.$$

It is evident that $T(0) = \{0\}$ if and only if T uniquely determines a linear operator from $D(T)$ into X whose graph is T. For convenience, a linear operator in X will always be identified with a subspace in X^2 via its graph.

Let T and S be two subspaces in X^2 and $\alpha \in \mathbf{C}$. Define

$$\alpha T := \{(x, \alpha f) : (x, f) \in T\},$$
$$T + S := \{(x, f + g) : (x, f) \in T, \ (x, g) \in S\}.$$

It is evident that if T is closed, then $T - \lambda I$ is closed for any $\lambda \in \mathbf{C}$ and

$$(T - \lambda I)^* = T^* - \bar{\lambda} I. \tag{2.1}$$

On the other hand, if $T \cap S = \{(0, 0)\}$, then denote

$$T \dotplus S := \{(x + y, f + g) : (x, f) \in T, (y, g) \in S\}.$$

Further, if T and S are orthogonal; that is, $\langle (x, f), (y, g) \rangle = 0$ for all $(x, f) \in T$ and $(y, g) \in S$, then we set

$$T \oplus S := T \dotplus S.$$

The following concepts were introduced in [6,7,13].

Definition 2.1. Let T be a subspace in X^2.

(1) The set $\rho(T) := \{\lambda \in \mathbf{C} : (\lambda I - T)^{-1}$ is a bounded linear operator defined on $X\}$ is called the resolvent set of T.
(2) The set $\sigma(T) := \mathbf{C} \setminus \rho(T)$ is called the spectrum of T.
(3) The essential spectrum $\sigma_e(T)$ of T is the set of those points of $\sigma(T)$ that are either accumulation points of $\sigma(T)$ or isolated eigenvalues of infinite multiplicity.
(4) The set $\sigma_d(T) := \sigma(T) \setminus \sigma_e(T)$ is called the discrete spectrum of T.

Remark 2.1. Cross introduced a class of essential spectrum of a linear relation in a complex normed space in terms of the nullity and deficiency of its complete closure (see [4, Definition VII.1.1]). Obviously, it is quite different from the above definition. More recently, Wilcox gave five distinct essential spectra $\sigma_{ei}(T)$, $1 \leq i \leq 5$, of a subspace T in terms of semi-Fredholm properties in the case that X is a Banach space (see [19, Definition 2.2]). The second one $\sigma_e(T)$ can be defined by

$$\sigma_{e2}(T) = \{\lambda \in \mathbf{C} : R(T - \lambda I) \text{ is not closed or } n(T, \lambda) = +\infty\},$$

where $n(T, \lambda) = \dim N(T - \lambda I)$ and $N(T - \lambda I) = \{x \in X : (x, 0) \in T - \lambda I\}$. In the case that X is a Hilbert space and T is a self-adjoint subspace, it follows from (2) of Corollary 2.1 in [16] that

$$\sigma_e(T) = \sigma_{e2}(T).$$

Arens [1] introduced the following decomposition for a closed subspace T in X^2:

$$T = T_s \oplus T_\infty, \tag{2.2}$$

where

$$T_\infty := \{(0, g) \in X^2 : (0, g) \in T\}, \quad T_s := T \ominus T_\infty.$$

Then T_s is an operator. So T_s and T_∞ are called the operator and pure multi-valued parts of T, respectively. This decomposition will play an important role in our study. Now, we shall recall their fundamental properties. The following come from [1]:

$$D(T_s) = D(T), \ R(T_s) \subset T(0)^\perp, \ R(T_\infty) = T(0), \ T_\infty = \{0\} \times T(0), \tag{2.3}$$

and $D(T_s)$ is dense in $T^*(0)^\perp$.

Throughout the present paper, the resolvent set and spectrum of T_s and T_∞ mean those of T_s and T_∞ restricted to $(T(0)^\perp)^2$ and $T(0)^2$, respectively.

Lemma 2.1. *(See [13, Proposition 2.1 and Theorems 2.1, 2.2 and 3.4].) Let T be a closed Hermitian subspace in X^2. Then*

$$T_s = T \cap (T(0)^\perp)^2, \ \ T_\infty = T \cap T(0)^2, \tag{2.4}$$

T_s is a closed Hermitian operator in $T(0)^\perp$, T_∞ is a closed Hermitian subspace in $T(0)^2$, and

$$\rho(T) = \rho(T_s), \ \sigma(T) = \sigma(T_s), \ \sigma(T_\infty) = \emptyset,$$
$$\sigma_p(T) = \sigma_p(T_s), \ \sigma_e(T) = \sigma_e(T_s), \ \sigma_d(T) = \sigma_d(T_s).$$

In [5], Dijksma and Snoo introduced the concept of reducing subspace for a subspace. Let T be a subspace in X^2, X_1 a closed subspace in X and $P : X \to X_1$ an orthogonal projection. Denote

$$P^{(2)}T := \{(Px, Pf) : \ (x, f) \in T\}.$$

It is clear that $T \cap X_1^2 \subset P^{(2)}T$. If $P^{(2)}T \subset T$, then X_1 is called a reducing subspace of T. We also say that X_1 reduces T or T is reduced by X_1. In this case one has

$$T \cap X_1^2 = P^{(2)}T.$$

Lemma 2.2. *(See [5, page 26].) If T is a self-adjoint subspace in X^2, then T is reduced by $T(0)$, and T_∞ and T_s are self-adjoint subspaces in $T(0)^2$ and $(T(0)^\perp)^2$, respectively.*

Lemma 2.3. *(See [13, Theorem 2.5].) Let T be a Hermitian subspace in X^2. Then T is self-adjoint in X^2 if and only if $R(T - \lambda I) = R(T - \bar\lambda I) = X$ for some $\lambda \in \mathbf{C}$.*

Now, we present the following simple and useful result:

Proposition 2.1. *Let T be a closed Hermitian subspace in X^2. Then T is a self-adjoint subspace in X^2 if and only if T_s is a self-adjoint operator in $T(0)^\perp$.*

Proof. The necessity directly follows from Lemma 2.2. Now, we consider the sufficiency. Since T_s is a self-adjoint operator in $T(0)^\perp$, one has that $R(T_s \pm iI) = T(0)^\perp$ by Theorem 5.21 of [18] or by Lemma 2.3. So it follows from (2.2)–(2.4) that

$$R(T \pm iI) = R(T_s \pm iI) \oplus R(T_\infty) = T(0)^\perp \oplus T(0) = X,$$

which implies that T is self-adjoint in X^2 by Lemma 2.3. The proof is complete. $\square$

Next, we recall the concepts of relatively bounded and compact operators and a related result, which will be used in the sequels.

Definition 2.3. (See [18, pages 93 and 275].) Let X be a Hilbert space, and S and T operators in X. By $\|\cdot\|_S$ denote the graph norm of S, i.e., $\|x\|_S = \|x\| + \|Sx\|$, $x \in D(S)$.

(1) T is said to be S-bounded if $D(S) \subset D(T)$ and there exists a constant $c \geq 0$ such that

$$\|Tx\| \leq c\|x\|_S, \ \ x \in D(S).$$

(2) If T is S-bounded, then the infimum of all numbers $a \geq 0$ for which a constant $b \geq 0$ exists such that

$$\|Tx\| \leq a\|Sx\| + b\|x\|, \quad \forall\, x \in D(S),$$

is called the S-bound of T.

(3) T is said to be S-compact if it is compact as a mapping from $(D(S), \|\cdot\|_S)$ into X.

The following result is classical in the perturbation theory of self-adjoint operators (cf., [12, Theorem 8.15] or [18, Theorem 9.9]).

Lemma 2.4. *Let T be a self-adjoint operator in X, and V a densely defined Hermitian and T-compact operator. Then $T + V$ is a self-adjoint operator and $\sigma_e(T + V) = \sigma_e(T)$.*

To end this section, we recall concepts of finite rank and compact perturbations for closed subspaces in the special case that domains and ranges of the subspaces lie in a same Hilbert space. We refer to [2] for more general definitions.

Definition 2.4. Let T and S be closed subspaces in X^2, and

$$P_T : X^2 \to T, \quad P_S : X^2 \to S$$

orthogonal projections.

(1) T is said to be a compact perturbation of S if $P_T - P_S$ is a compact operator in X.

(2) T is said to be a finite rank perturbation of S if $P_T - P_S$ is a finite rank operator in X.

The following result gives characterizations of compact and finite rank perturbations of closed subspaces in terms of resolvents.

Lemma 2.5. *(See [2, Corollaries 3.4 and 4.5].) Let T and S be closed subspaces in X^2 and $\rho(T) \cap \rho(S) \neq \emptyset$. Then*

(i) *T is a compact perturbation of S if and only if $(T - \lambda I)^{-1} - (S - \lambda I)^{-1}$ is a compact operator in X for some (and hence for all) $\lambda \in \rho(T) \cap \rho(S)$;*

(ii) *T is a finite rank perturbation of S if and only if $(T - \lambda I)^{-1} - (S - \lambda I)^{-1}$ is a finite rank operator in X for some (and hence for all) $\lambda \in \rho(T) \cap \rho(S)$.*

3. Relationships among operator parts and characterizations of compact and finite rank perturbations

In this section, we study relationships among operator parts of an unperturbed subspace, its perturbation and its perturbation term. Using them, we characterize compact and finite rank perturbations of a closed subspace in terms of difference between their operator parts.

We first study relationships between operator parts of an unperturbed subspace, its perturbation and its perturbed term. Let T, S and A be closed subspaces in X^2 with $D(T) = D(S) \subset D(A)$, and satisfy

$$T = S + A, \tag{3.1}$$

where T can be regarded as a perturbation of S by the perturbation term A.

Remark 3.1. In 1997, Hassi and Snoo considered a closed Hermitian subspace with defect numbers $(1,1)$ in [6]. They showed that its canonical self-adjoint subspace extensions can be interpreted as one-dimensional perturbations of one fixed canonical self-adjoint subspace extension. Note that the sense of the perturbation considered in [6] is different from that given in (3.1), which is a natural generalization of perturbation of operators to multi-valued operators.

It is natural to ask whether their operator parts satisfy

$$T_s = S_s + A_s. \tag{3.2}$$

This is a very interesting question itself.

The following simple fact will be repeatedly used in the sequent discussions. If S and A are Hermitian subspaces, then T, defined by (3.1), is a Hermitian subspace in X^2. But if S and A are closed, T may not be closed in general.

We first consider the following example:

Example 3.1. Let $X = l^2[1, \infty)$, $e_1 = \{e_{1n}\}_{n=1}^\infty \in X$ with $e_{11} = 1$ and $e_{1n} = 0$ for all $n \geq 2$, and $X_1 = \text{span}\{e_1\}^\perp = \{x = \{x_n\}_{n=1}^\infty \in X : x_1 = 0\}$. Define

$$S = \{(x, x + ce_1) : x \in X_1, c \in \mathbf{C}\}, \tag{3.3}$$

and define A by the following Jacobi operator:

$$(Ax)_n = a_n x_{n+1} + b_n x_n + a_{n-1} x_{n-1}, \ n \geq 1, \ x \in X, \tag{3.4}$$

with $x_0 = 0$, where $a_n \neq 0$ for $n \geq 0$ and $\{a_n, b_n\}$ is real and bounded. Then S is a closed Hermitian subspace with $D(S) = X_1$, and A is a bounded self-adjoint operator in X with $D(A) = X$ (see [14] and [17] for more detailed discussions).

Let T be defined by (3.1). Then T is a closed Hermitian subspace in X^2, and

$$T = \{(x, x + Ax + ce_1) : x \in X_1, c \in \mathbf{C}\},$$
$$D(T) = D(S) = X_1 \subset D(A), \ T(0) = S(0) = \text{span}\{e_1\}.$$

So $T(0)^\perp = S(0)^\perp = X_1$. Further, by Lemma 2.1 one has

$$T_s = T \cap (T(0)^\perp)^2 = T \cap X_1^2.$$

Thus, $(x, x + Ax + ce_1) \in T_s$ if and only if $x \in X_1$ and $x + Ax + ce_1 \in X_1$. Since $x \in D(T) = X_1$, one has that $(x, x + Ax + ce_1) \in T_s$ if and only if $Ax + ce_1 \in X_1$, which is equivalent to $(Ax)_1 + c = 0$; that is, $a_1 x_2 + c = 0$ by (3.4), which yields $c = -a_1 x_2$. Hence,

$$T_s x = x + Ax - a_1 x_2 e_1, \ \ x \in X_1. \tag{3.5}$$

In addition, we can get by (3.3) and Lemma 2.1 that

$$S_s x = x, \ \ x \in X_1. \tag{3.6}$$

It follows from (3.4)–(3.6) that

$$T_s x \neq S_s x + Ax, \ x \in X_1 \text{ with } x_2 \neq 0. \tag{3.7}$$

This means that (3.2) does not hold in general. In particular, T_s and $S_s + A_s$ have no inclusion relationships in this example. Furthermore, it follows from (3.4)–(3.6) that they satisfy

$$T_s x = S_s x + A_s x + B x, \quad x \in X_1,$$

where $Bx = -a_1 \langle e_2, x \rangle e_1$ and $e_2 = \{e_{2n}\}_{n=1}^\infty$ with $e_{22} = 1$ and $e_{2n} = 0$ for $n \neq 2$. Obviously, B is a rank one operator.

The following result gives a general relationship among T_s, S_s and A_s.

Theorem 3.1. *Let T, S and A be closed subspaces in X^2 with $D(T) = D(S) =: D \subset D(A)$, and satisfy (3.1). And let*

$$P: S(0)^\perp \to T(0)^\perp, \quad Q: A(0)^\perp \to T(0)^\perp, \tag{3.8}$$

be orthogonal projections. Then

$$T_s x = P S_s x + Q A_s x, \quad x \in D. \tag{3.9}$$

Furthermore, if S and A are Hermitian subspaces in X^2, then $P S_s$ and $Q A_s$ are Hermitian operators defined on D, respectively.

Proof. By (2.3) we have

$$R(T_s) \subset T(0)^\perp, \ R(S_s) \subset S(0)^\perp, \ R(A_s) \subset A(0)^\perp,$$
$$D(T_s) = D(T) = D(S_s) = D \subset D(A_s) = D(A). \tag{3.10}$$

It follows from (3.1) that

$$T(0) = S(0) + A(0), \tag{3.11}$$

which is a sum of sets. This implies that $S(0) \subset T(0)$ and $A(0) \subset T(0)$, and then

$$T(0)^\perp \subset S(0)^\perp, \ T(0)^\perp \subset A(0)^\perp. \tag{3.12}$$

So the projections P and Q are well defined.

Fix any $x \in D$ and let $f = T_s x$. Then $f \in T(0)^\perp$ by (3.10) and $(x, f) \in T_s \subset T$ by (2.2). So there exist $(x, g) \in S$ and $(x, h) \in A$ such that $f = g + h$ by (3.1). Further, from (2.2) and (3.10), there exist $g_s \in S(0)^\perp$, $g_\infty \in S(0)$, $h_s \in A(0)^\perp$ and $h_\infty \in A(0)$ such that

$$g = g_s + g_\infty, \ h = h_s + h_\infty, \ g_s = S_s x, \ h_s = A_s x. \tag{3.13}$$

Then

$$f = g_s + h_s + g_\infty + h_\infty$$
$$= P g_s + Q h_s + (I - P) g_s + (I - Q) h_s + g_\infty + h_\infty. \tag{3.14}$$

Note that

$$(I - P) g_s \in S(0)^\perp \ominus T(0)^\perp \subset T(0), \ (I - Q) h_s \in A(0)^\perp \ominus T(0)^\perp \subset T(0),$$

and $g_\infty, h_\infty \in T(0)$ by (3.11). Hence, it follows from (3.14) that $(I - P)g_s + (I - Q_s)h_s + g_\infty + h_\infty = 0$ and

$$f = Pg_s + Qh_s.$$

Therefore, (3.9) holds.

Further, suppose that S and A are Hermitian subspaces in X^2. Then T is a Hermitian subspace in X^2. By (2.3) and Lemma 2.1, T_s, S_s and A_s are closed Hermitian operators in $T(0)^\perp$, $S(0)^\perp$ and $A(0)^\perp$, respectively, with

$$T_s = T \cap (T(0)^\perp)^2, \ S_s = S \cap (S(0)^\perp)^2, \ A_s = A \cap (A(0)^\perp)^2,$$
$$D(S_s) = D = D(T_s) \subset T(0)^\perp, \quad D(A_s) = D(A) \subset A(0)^\perp. \tag{3.15}$$

Now, we show that PS_s is a Hermitian operator in $T(0)^\perp$. For any given $x, y \in D$, let $S_s x = f_1 + f_2$ and $S_s y = g_1 + g_2$, where $f_1, g_1 \in T(0)^\perp$ and $f_2, g_2 \in S(0)^\perp \ominus T(0)^\perp \subset T(0)$. Then $PS_s x = f_1$ and $PS_s y = g_1$. By noting that $x, y \in D \subset T(0)^\perp$, it follows that $\langle f_2, y \rangle = \langle x, g_2 \rangle = 0$. Hence, we have that

$$0 = \langle S_s x, y \rangle - \langle x, S_s y \rangle = \langle f_1, y \rangle - \langle x, g_1 \rangle = \langle PS_s x, y \rangle - \langle x, PS_s y \rangle.$$

Therefore, PS_s is a Hermitian operator. With a similar argument, one can show that QA_s is a Hermitian operator in D. This completes the proof. □

It is evident that $T(0) = S(0)$ if A is an operator in (3.1). So the following result can be directly derived from Theorem 3.1.

Corollary 3.1. *Let T and S be closed subspaces in X^2, A is a closed operator in X with $D(T) = D(S) =: D \subset D(A)$, and they satisfy (3.1). Then*

$$T_s x = S_s x + QAx, \quad x \in D,$$

where $Q : X \to S(0)^\perp$ is an orthogonal projection.

The unperturbed subspace S is often self-adjoint in X^2 in applications. Under this condition, we can get a similar result to that in Corollary 3.1, which is better than that in Theorem 3.1.

Theorem 3.2. *Let T and A be closed Hermitian subspaces in X^2, S a self-adjoint subspace in X^2 with $D(T) = D(S) =: D \subset D(A)$, and they satisfy (3.1). Then*

$$T_s x = S_s x + QA_s x, \quad x \in D, \tag{3.16}$$

and

$$A(0) \subset S(0) = T(0), \tag{3.17}$$

where $Q : A(0)^\perp \to S(0)^\perp$ is an orthogonal projection.

Proof. We first show that (3.17) holds. By the assumptions, (3.9), (3.11), (3.12) and (3.15) hold, and S_s is a self-adjoint operator in $S(0)^\perp$ by Lemma 2.2. So D is dense in $S(0)^\perp$. This yields that $\bar{D} = S(0)^\perp$. In addition, by (3.15) one has that $D \subset D(A) \subset A(0)^\perp$. Thus, $S(0)^\perp \subset A(0)^\perp$, which, together with the fact that $A(0)$ and $S(0)$ are closed, implies that $A(0) \subset S(0)$. Hence, (3.17) holds by (3.11).

Note that P, defined by (3.8), is an identity mapping from $S(0)^\perp$ onto itself in this case. Therefore, (3.16) follows. The proof is complete. $\square$

The following result gives a sufficient condition such that (3.2) holds.

Theorem 3.3. *Let T, S and A be closed Hermitian subspaces in X^2 with $D(T) = D(S) =: D \subset D(A)$, and satisfy (3.1). If $T(0)^\perp$ reduces S and A, then*

$$T_s x = S_s x + A_s x, \quad x \in D, \tag{3.18}$$

$$\hat{S}_s := S \cap (T(0)^\perp)^2 = S_s, \quad \hat{A}_s := A \cap (T(0)^\perp)^2 \subset A_s, \tag{3.19}$$

and $\hat{A}_s$ is a closed Hermitian operator in $T(0)^\perp$ with $D \subset D(\hat{A}_s)$.

Proof. By (2.3) and Lemma 2.1, T_s, S_s and A_s are closed Hermitian operators in $T(0)^\perp$, $S(0)^\perp$ and $A(0)^\perp$, respectively, and (3.11), (3.12) and (3.15) hold. From (3.12) and (3.15), we have

$$\hat{S}_s \subset S_s, \quad \hat{A}_s \subset A_s. \tag{3.20}$$

By noting that $T(0)^\perp$ is a closed subspace in X, $\hat{S}_s$ and $\hat{A}_s$ are closed Hermitian operators in $T(0)^\perp$.

We first show that the first relation in (3.19) holds. By (3.20), it suffices to show that

$$S_s \subset \hat{S}_s. \tag{3.21}$$

Fix any $(x, f) \in S_s$. Then $(x, f) \in (S(0)^\perp)^2$ and $x \in D \subset T(0)^\perp$ by (3.15). There exist $f_1 \in T(0)^\perp$ and $f_2 \in S(0)^\perp \ominus T(0)^\perp$ such that

$$(x, f) = (x, f_1) + (0, f_2), \tag{3.22}$$

where the first relation in (3.12) has been used. By the assumption that $T(0)^\perp$ reduces S, we get that $(x, f_1) \in S$, and consequently $(x, f_1) \in \hat{S}_s \subset S_s$ by (3.20). Thus, $(0, f_2) \in S_s$ by (3.22). This yields that $f_2 = 0$ because S_s is an operator. Hence, $(x, f) = (x, f_1) \in \hat{S}_s$, and so (3.21) holds. Therefore, the first relation in (3.19) holds.

Next, we show that

$$D \subset D(\hat{A}_s). \tag{3.23}$$

In fact, for any given $x \in D$, $x \in D(A) \cap T(0)^\perp$ by (3.15) and the assumption that $D \subset D(A)$. So there exists $f \in X$ such that $(x, f) \in A$. Further, there exit $f_1 \in T(0)^\perp$ and $f_2 \in T(0)$ such that $(x, f) = (x, f_1) + (0, f_2)$. By the assumption that $T(0)^\perp$ reduces A we have that $(x, f_1) \in A$ and so $(x, f_1) \in \hat{A}_s$. Thus, $x \in D(\hat{A}_s)$, and consequently (3.23) holds.

Finally, we show that (3.18) holds. We first show that

$$T_s x = \hat{S}_s x + \hat{A}_s x, \quad x \in D. \tag{3.24}$$

It follows from (3.1), (3.15) and (3.20) that

$$\hat{S}_s + \hat{A}_s \subset (S_s + A_s) \cap (T(0)^\perp)^2 \subset T \cap (T(0)^\perp)^2 = T_s. \tag{3.25}$$

By (3.15), the first relation in (3.19), and (3.23) we get that

$$D(T_s) = D = D(S_s) = D(\hat{S}_s) = D(\hat{S}_s + \hat{A}_s).$$

Hence, by the fact that $\hat{S}_s$, $\hat{A}_s$ and T_s are (single-valued) operators, (3.25) yields (3.24), which, together with (3.19) and (3.23), implies that (3.18) holds. The proof is complete. □

Remark 3.2. If T and S are self-adjoint subspaces in X^2, then T_s and S_s are self-adjoint operators in $T(0)^\perp$ and $S(0)^\perp$, respectively, by Lemma 2.2. So the various spectra of subspace S are only affected by $A_s|_{D(S)}$ by Lemma 2.1, Theorem 3.3 and Theorem 4.1 of [13] under the assumptions of Theorem 3.3.

Now, we are ready to give out relationships between compact perturbation of a closed Hermitian subspace and relatively compact perturbation of the operator parts of the subspace and its perturbation term, one of which is a characterization of compact perturbation of the subspaces in terms of their operator parts under the assumptions of Theorem 3.3.

Theorem 3.4. Let T, S and A be closed Hermitian subspaces in X^2 with $D(T) = D(S) =: D \subset D(A)$, and satisfy (3.1). Assume that $\rho(T) \cap \rho(S) \neq \emptyset$.

  (i) If A_s is an S_s-compact operator, then T is a compact perturbation of S.
 (ii) If T is a compact perturbation of S, $T(0)^\perp$ reduces S and A, and T_s is a bounded operator in D, then A_s is an S_s-compact operator.

Proof. By the assumptions, (3.15) holds. For convenience, denote

$$B := (T - \lambda I)^{-1} - (S - \lambda I)^{-1}. \tag{3.26}$$

(i) Suppose that A_s is an S_s-compact operator. In order to show that T is a compact perturbation of S, it suffices to show that B is a compact operator for all $\lambda \in \rho(T) \cap \rho(S)$ by Lemma 2.5. Fix any $\lambda \in \rho(T) \cap \rho(S)$ and any bounded sequence $\{f_n\}_{n=1}^\infty \subset X$. We want to show that $\{Bf_n\}$ has a convergent subsequence.

Set

$$x_n = (T - \lambda I)^{-1} f_n, \quad y_n = (S - \lambda I)^{-1} f_n, \quad n \geq 1.$$

Then $\{x_n\}$ and $\{y_n\}$ are bounded sequences in D,

$$Bf_n = x_n - y_n, \quad n \geq 1, \tag{3.27}$$

and

$$(x_n, f_n + \lambda x_n) \in T, \quad (y_n, f_n + \lambda y_n) \in S, \quad n \geq 1. \tag{3.28}$$

By noting that $y_n \in D \subset S(0)^\perp$ by (3.15), it follows from (2.2) that there exist $g_n \in S(0)^\perp$ and $h_n \in S(0)$ such that

$$f_n = g_n + h_n,$$

and

$$(y_n, g_n + \lambda y_n) \in S_s, \quad (0, h_n) \in S_\infty.$$

Since $\{f_n\}$ is bounded, $\{g_n\}$ is bounded, and consequently $\{g_n + \lambda y_n\}$ is bounded. Thus, by the assumption that A_s is an S_s-compact operator we have that $\{A_s y_n\}$ has a convergent subsequence. For simplicity, denote $u_n = A_s y_n$ for $n \geq 1$. It is evident that

$$(y_n, f_n + \lambda y_n + u_n) \in S + A_s \subset T,$$

which, together with the first relation in (3.28), implies that $(y_n - x_n, \lambda(y_n - x_n) + u_n) \in T$. This yields that $(y_n - x_n, u_n) \in T - \lambda I$, and then

$$y_n - x_n = (T - \lambda I)^{-1} u_n.$$

Hence, $\{y_n - x_n\}$, namely, $\{Bf_n\}$ by (3.27), has a convergent subsequence since $(T - \lambda I)^{-1}$ is a bounded operator on X. Therefore, B is a compact operator in X, and then T is a compact perturbation of S.

(ii) Now, suppose that T is a compact perturbation of S, $T(0)^\perp$ reduces S and A, and T_s is a bounded operator. Then B is a compact operator in X by Lemma 2.5.

Since all the assumptions of Theorem 3.3 hold, T_s, S_s, A_s, $\hat{S}_s$, and $\hat{A}_s$ satisfy (3.18) and (3.19), where $\hat{A}_s$ and $\hat{S}_s$ are defined by (3.19). Note that $\rho(T) \cap \rho(S) = \rho(T_s) \cap \rho(S_s)$ by Lemma 2.1. So it can be easily verified by (3.18) that for any $\lambda \in \rho(T) \cap \rho(S)$,

$$(S_s - \lambda I)^{-1} g = (T_s - \lambda I)^{-1} g + (T_s - \lambda I)^{-1} A_s (S_s - \lambda I)^{-1} g, \quad g \in T(0)^\perp. \tag{3.29}$$

In addition, it follows from (2.2) that

$$(T - \lambda I)^{-1} = (T_s - \lambda I)^{-1} \oplus T_\infty^{-1}, \quad (S - \lambda I)^{-1} = (S_s - \lambda I)^{-1} \oplus S_\infty^{-1}. \tag{3.30}$$

For any given $f \in X$, there exist $g \in T(0)^\perp$ and $h \in T(0)$ such that $f = g + h$. By (3.30) one has

$$(T_s - \lambda I)^{-1} g = (T - \lambda I)^{-1} f, \quad (S_s - \lambda I)^{-1} g = (S - \lambda I)^{-1} f, \tag{3.31}$$

where the first relation in (3.19) and the assumption that $T(0)^\perp$ reduces S have been used for the above second relation. So, it follows from (3.26), (3.29) and (3.31) that

$$Bf = (T_s - \lambda I)^{-1} g - (S_s - \lambda I)^{-1} g = -(T_s - \lambda I)^{-1} A_s (S_s - \lambda I)^{-1} g. \tag{3.32}$$

Fix any bounded sequence $\{(y_n, S_s y_n)\}_{n=1}^\infty \subset S_s$. Set $g_n = (S_s - \lambda I) y_n$. Then $g_n \in T(0)^\perp$ by the first relation in (3.19), $\{g_n\}$ is a bounded sequence, and $y_n = (S_s - \lambda I)^{-1} g_n$. By (3.32) we get that

$$Bg_n = -(T_s - \lambda I)^{-1} A_s y_n, \quad n \geq 1,$$

and then

$$A_s y_n = -(T_s - \lambda I) B g_n, \quad n \geq 1. \tag{3.33}$$

Since B is compact, there exists a subsequence $\{g_{n_k}\}$ such that $\{Bg_{n_k}\}$ is convergent. Thus, $\{(T_s - \lambda I) B g_{n_k}\}$ is convergent by the assumption that T_s is bounded, and consequently so is $\{A_s y_{n_k}\}$ by (3.33). Therefore, A_s is an S_s-compact operator. The whole proof is complete. $\square$

Remark 3.3. By Theorem 3.4, one can see that the relatively compact perturbation of operator parts of closed Hermitian subspaces is stronger than the compact perturbation, in general.

If S is a self-adjoint subspace in X^2, then we can give the following results:

Theorem 3.5. *Let T and A be closed Hermitian subspaces in X^2, S a self-adjoint subspace in X^2 with $D(T) = D(S) =: D \subset D(A)$, and they satisfy (3.1). Assume that $\rho(T) \cap \rho(S) \neq \emptyset$.*

(i) *If QA_s is an S_s-compact operator, then T is a compact perturbation of S.*

(ii) *If T is a compact perturbation of S and T_s is a bounded operator in D, then QA_s is an S_s-compact operator.*

Here Q is specified in Theorem 3.2.

Proof. The proof of assertion (ii) is similar to that of (ii) of Theorem 3.4, where (3.18) is replaced by (3.16). So we omit its details.

Now, we show that assertion (i) holds. Suppose that QA_s is an S_s-compact operator. Since all the assumptions of Theorem 3.2 are satisfied, (3.15), (3.16) and (3.17) hold. Fix any $\lambda \in \rho(T) \cap \rho(S)$. Then $\lambda \in \rho(T_s) \cap \rho(S_s)$ by Lemma 2.1. For any given $f \in X$, set $x = (T - \lambda I)^{-1}f$ and $y = (S - \lambda I)^{-1}f$. Then $(x, f + \lambda x) \in T$ and $(y, f + \lambda y) \in S$. There exist $g \in T(0)^{\perp}$ and $h \in T(0)$ such that $f = g + h$. So we get that

$$(x, f + \lambda x) = (x, g + \lambda x) + (0, h), \ (y, f + \lambda y) = (y, g + \lambda y) + (0, h).$$

Note that $x \in D \subset T(0)^{\perp}$ and $S(0) = T(0)$ by (3.17). Hence, $(x, g + \lambda x) \in T_s$ and $(y, g + \lambda y) \in S_s$ by (3.15), which implies that

$$x = (T_s - \lambda I)^{-1}g, \ \ y = (S_s - \lambda I)^{-1}g.$$

This shows that

$$Bf = (T_s - \lambda I)^{-1}g - (S_s - \lambda I)^{-1}g, \tag{3.34}$$

where B is defined by (3.26). On the other hand, it follows from (3.16) that

$$(S_s - \lambda I)^{-1}g' = (T_s - \lambda I)^{-1}g' + (T_s - \lambda I)^{-1}QA_s(S_s - \lambda I)^{-1}g', \ \ \forall g' \in T(0)^{\perp},$$

which, together with (3.34), implies that

$$Bf = -(T_s - \lambda I)^{-1}QA_s(S_s - \lambda I)^{-1}g. \tag{3.35}$$

Now, fix any bounded sequence $\{f_n\}_{n=1}^{\infty} \subset X$. We shall show that $\{Bf_n\}$ has a convergent subsequence. There exist $g_n \in T(0)^{\perp}$ and $h_n \in T(0)$ such that

$$f_n = g_n + h_n, \ \ n \geq 1.$$

Then $\{g_n\}$ is bounded. By (3.35) one has that

$$Bf_n = -(T_s - \lambda I)^{-1}QA_s(S_s - \lambda I)^{-1}g_n. \tag{3.36}$$

Set $y_n = (S_s - \lambda I)^{-1}g_n$ for $n \geq 1$. Then we get from (3.36) that

$$Bf_n = -(T_s - \lambda I)^{-1}QA_s y_n, \ \ n \geq 1, \tag{3.37}$$

and $\{(y_n, g_n + \lambda y_n)\} \subset S_s$ is bounded by the fact that $(S_s - \lambda I)^{-1}$ is a bounded operator. Thus, $\{QA_s y_n\}$ has a convergent subsequence by the assumption. And consequently, $\{Bf_n\}$ has a convergent subsequence. This means that B is a compact operator. Therefore, T is a compact perturbation of S by Lemma 2.5. The proof is complete. $\square$

We shall remark that the results of Theorem 3.5 cannot be directly derived from Theorem 3.4. It is evident that the results of Theorem 3.5 are better than those of Theorem 3.4 in the case that S is a self-adjoint subspace in X^2.

To end this section, we give relationships between finite rank perturbation of a closed Hermitian subspace and finite rank perturbation of the operator parts of the subspace and its perturbation term, one of which is a characterization of finite rank perturbation of the subspace in terms of their operator parts.

Theorem 3.6. *Let T, S and A satisfy the assumptions of Theorem 3.4.*

(i) *If A_s is a finite rank operator in D, then T is a finite rank perturbation of S.*
(ii) *If T is a finite rank perturbation of S and $T(0)^\perp$ reduces S and A, then A_s is a finite rank operator in D.*

Proof. With a similar argument to that used in the proof of Theorem 3.4, one can easily show Theorem 3.6 by Lemmas 2.1 and 2.5, Theorem 3.3 and (3.32). So we omit its details. □

Note that the assumption that T_s is a bounded operator in D in (ii) of Theorem 3.4 has been removed in (ii) of Theorem 3.6 because that if B is finite rank, then $(T_s - \lambda I)B$ is finite rank for every linear operator T_s.

Theorem 3.7. *Let T, S and A satisfy the assumptions of Theorem 3.5. Then QA_s is a finite rank operator in D if and only if T is a finite rank perturbation of S, where Q is specified in Theorem 3.2.*

Proof. With a similar argument to that used in the proof of Theorem 3.5, one can easily show Theorem 3.7 by Lemmas 2.1 and 2.5, Theorem 3.2 and (3.35). So we omit its details. □

Remark 3.4. The finite rank property of QA_s is equivalent to the finite rank perturbation of the self-adjoint subspace S under the assumptions of Theorem 3.7. It is evident that if A_s is finite rank in $D = D(S)$, then so is QA_s. Its converse may not hold in general. But, in the case that $S(0) \ominus A(0)$ is finite-dimensional, then the converse is true because $A_s = QA_s + (I - Q)A_s$, while $I - Q : A(0)^\perp \to A(0)^\perp \ominus S(0)^\perp$ is finite rank by the fact that $A(0)^\perp \ominus S(0)^\perp = S(0) \ominus A(0)$.

Remark 3.5. A trace class perturbation is a special compact perturbation and a finite rank perturbation is a simple case of trace class perturbation. We shall study the trace class perturbation of closed subspaces in detail in our forthcoming paper.

4. Self-adjoint subspaces under compact perturbations

In this section, we show that a self-adjoint subspace is still self-adjoint under relatively bounded and relatively compact perturbations as well as compact perturbation.

The following result is a generalization of the well-known Kato–Rellich theorem for self-adjoint operators (cf., [10, Theorem 10.12] or [12, Theorem 8.5] or [18, Theorem 5.28]) to self-adjoint subspaces.

Theorem 4.1. *Let S be a self-adjoint subspace in X^2 and A a closed Hermitian subspace in X^2 with $D(S) \subset D(A)$ and $A_s x \in S(0)^\perp$ for every $x \in D(S)$. If A_s is S_s-bounded with S_s-bound less than 1, then $S + A$ is a self-adjoint subspace in X^2.*

Proof. We shall show that $S + A$ is self-adjoint in X^2 by Lemma 2.3.

It follows from Lemmas 2.1 and 2.2 that S_s is a self-adjoint operator in $S(0)^\perp$, and A_s is a closed Hermitian operator in $A(0)^\perp$ with $D(S_s) = D(S) \subset D(A) = D(A_s)$. Then $D(S)$ is dense in $S(0)^\perp$, and

consequently $A_s|_{D(S)}$ is densely defined in $S(0)^\perp$. Since A_s is S_s-bounded with S_s-bound less than 1, $S_s + A_s$ is a self-adjoint operator in $S(0)^\perp$ by the Kato–Rellich theorem [10, Theorem 10.12]. Hence, by Lemma 2.3 one has

$$R(S_s + A_s \pm iI) = S(0)^\perp. \tag{4.1}$$

Fix any $f \in X$. There exist $f_1 \in S(0)^\perp$ and $f_2 \in S(0)$ such that $f = f_1 + f_2$. Further, by (4.1), there exists $x \in D(S_s)$ such that

$$f_1 = S_s x + A_s x \pm ix. \tag{4.2}$$

Note that $(x, S_s x) \in S_s \subset S$, $(x, A_s x) \in A_s \subset A$, and $(0, f_2) \in S_\infty \subset S$, which implies that $(x, S_s x + A_s x + f_2) \in S + A$. This, together with (4.2), yields that $(x, f) = (x, S_s x + A_s x + f_2 \pm ix) \in S + A \pm iI$. This means that $f \in R(S + A \pm iI)$. Hence, $R(S + A \pm iI) = X$, and consequently $S + A$ is a self-adjoint subspace in X^2 by Lemma 2.3. This completes the proof. $\quad\square$

If the condition that $S + A$ is closed is added to the assumptions of Theorem 4.1, then the condition that A_s is S_s-bounded with S_s-bound less than 1 can be weakened as follows. We shall point out again that $S + A$ may not be closed if S and A are closed.

Theorem 4.2. *Let S be a self-adjoint subspace in X^2, and A and $S + A$ closed Hermitian subspaces in X^2 with $D(S) \subset D(A)$. If QA_s is S_s-bounded with S_s-bound less than 1, then $S + A$ is a self-adjoint subspace in X^2, where Q is specified in Theorem 3.2.*

Proof. Let T be defined by (3.1). By Theorem 3.2, T_s, S_s and A_s satisfy (3.16) and (3.17) holds. Further, S_s is a self-adjoint operator in $S(0)^\perp$ by Lemma 2.2, and QA_s is a Hermitian operator in $D(S) \subset D(A)$ by Theorem 3.1. So $(QA_s)|_{D(S)}$ is densely defined and Hermitian in $S(0)^\perp$, where the fact that $D(S)$ is dense in $S(0)^\perp$ has been used. Since QA_s is S_s-bounded with S_s-bound less than 1 by the assumption, it follows from (3.16) that T_s is a self-adjoint operator in $S(0)^\perp$ by the Kato–Rellich theorem [10, Theorem 10.12]. Note that $T(0)^\perp = S(0)^\perp$ by (3.17). Therefore, T, namely, $S + A$, is a self-adjoint subspace in X^2 by Proposition 2.1. This completes the proof. $\quad\square$

By Theorem 9.7 of [18], if an operator U is relatively compact to another operator V, then U is V-bounded with V-bound zero. So the following result can be derived from Theorem 4.1.

Theorem 4.3. *Let S be a self-adjoint subspace in X^2 and A a closed Hermitian subspace in X^2 with $D(S) \subset D(A)$ and $A_s x \in S(0)^\perp$ for every $x \in D(S)$. If A_s is S_s-compact, then $S + A$ is a self-adjoint subspace in X^2.*

Since a finite rank or more general trace class operator is compact, the following result can be directly derived from Theorem 4.3.

Corollary 4.1. *Let S be a self-adjoint subspace in X^2 and A a closed Hermitian subspace in X^2 with $D(S) \subset D(A)$. If A_s is finite rank or belongs to the trace class in $D(S)$, then $S + A$ is a self-adjoint subspace in X^2.*

The following result is a direct consequence of Theorem 4.2.

Theorem 4.4. *Let S be a self-adjoint subspace in X^2, and A and $S + A$ closed Hermitian subspaces in X^2 with $D(S) \subset D(A)$. If QA_s is S_s-compact, then $S + A$ is a self-adjoint subspace in X^2, where Q is specified in Theorem 3.2.*

The following result can be directly derived from Theorem 4.4.

Corollary 4.2. *Let S be a self-adjoint subspace in X^2, and A and $S + A$ closed Hermitian subspaces in X^2 with $D(S) \subset D(A)$. If QA_s is finite rank or belongs to the trace class in $D(S)$, then $S + A$ is a self-adjoint subspace in X^2, where Q is specified in Theorem 3.2.*

The following result is a direct consequence of Theorems 3.5 and 4.4.

Corollary 4.3. *Let S be a self-adjoint subspace in X^2, and A and $S + A$ closed Hermitian subspaces in X^2 with $D(S) \subset D(A)$. Assume that $\rho(T) \cap \rho(S) \neq \emptyset$. If $S + A$ is a compact perturbation of S and $(S + A)_s$ is a bounded operator, then $S + A$ is a self-adjoint subspace in X^2.*

If we strengthen the assumption about the resolvent sets of S and $S + A$ in Corollary 4.3, the assumption that $(S + A)_s$ is a bounded operator can be removed as follows.

Theorem 4.5. *Let S be a self-adjoint subspace in X^2 and T a closed Hermitian subspace in X^2. Assume that $\rho(S) \cap \rho(T) \cap \mathbf{R} \neq \emptyset$. If T is a compact perturbation of S, then T is a self-adjoint subspace X^2.*

Proof. Fix any $\lambda \in \rho(S) \cap \rho(T) \cap \mathbf{R}$. By the definition of self-adjoint subspace, it can be easily verified that a subspace C is self-adjoint (resp., closed Hermitian) in X^2 if and only if C^{-1} is self-adjoint (resp., closed Hermitian) in X^2. Hence, $(S - \lambda I)^{-1}$ is a bounded self-adjoint operator on X and $(T - \lambda I)^{-1}$ is a bounded and closed Hermitian operator on X. Let B be defined by (3.26). Then B is a compact Hermitian operator defined on X and

$$(T - \lambda I)^{-1} = (S - \lambda I)^{-1} + B,$$

which yields that $(T - \lambda I)^{-1}$ is a self-adjoint operator on X by Lemma 2.4. Thus, $T - \lambda I$, and then T by (2.1), is a self-adjoint subspace in X. This completes the proof. $\square$

The following result directly follows from Theorem 4.5.

Corollary 4.4. *Let S be a self-adjoint subspace in X^2 and T a closed Hermitian subspace in X^2. Assume that $\rho(S) \cap \rho(T) \cap \mathbf{R} \neq \emptyset$. If T is a finite rank perturbation of S or $(T - \lambda I)^{-1} - (S - \lambda I)^{-1}$ belongs to the trace class in X for some $\lambda \in \rho(S) \cap \rho(T) \cap \mathbf{R}$, then T is a self-adjoint subspace in X^2.*

Remark 4.1. The finite rank perturbation is of special interest in the existing literature. So we specially point out this condition in the above results.

5. Stability of essential spectra of self-adjoint subspaces under perturbations

In this section, we study stability of essential spectra of self-adjoint subspaces under compact and relatively compact perturbations.

Theorem 5.1. *Let T and S be self-adjoint subspaces in X^2. If T is a compact perturbation of S, then*

$$\sigma_e(T) = \sigma_e(S), \tag{5.1}$$

and consequently S has a pure discrete spectrum if and only if so does T.

Proof. By Definition 2.4, T is a compact perturbation of S if and only if S is a compact perturbation of T. So it suffices to show that

$$\sigma_e(S) \subset \sigma_e(T) \tag{5.2}$$

because its inverse inclusion can be obtained by interchanging S and T.

Fix any $\lambda \in \sigma_e(S)$. Then, by Theorem 3.7 of [13], there exists a sequence $\{(x_n, f_n)\} \subset S$ satisfying that

$$x_n \overset{w}{\to} 0, \quad \liminf_{n\to\infty} \|x_n\| > 0, \quad f_n - \lambda x_n \to 0 \text{ as } n \to \infty. \tag{5.3}$$

Since T and S are self-adjoint subspaces in X^2, $\mathbf{C} \setminus \mathbf{R} \subset \rho(T) \cap \rho(S)$ by Theorem 2.5 of [13], and then $\rho(T) \cap \rho(S) \neq \emptyset$. Take any $\mu \in \rho(T) \cap \rho(S)$. Then B is a compact operator defined on X by Lemma 2.5, where B is defined by (3.26) with λ replaced by μ. Further, noting that $(x_n, f_n - \mu x_n) \in S - \mu I$, we have that

$$x_n = (S - \mu I)^{-1}(f_n - \mu x_n). \tag{5.4}$$

Set

$$y_n = (T - \mu I)^{-1}(f_n - \mu x_n). \tag{5.5}$$

Then $(y_n, f_n - \mu x_n) \in T - \mu I$, which implies that $(y_n, f_n + \mu(y_n - x_n)) \in T$. Now, we want to show that

$$y_n \overset{w}{\to} 0, \quad \liminf_{n\to\infty} \|y_n\| > 0, \quad f_n + \mu(y_n - x_n) - \lambda y_n \to 0 \text{ as } n \to \infty. \tag{5.6}$$

If it is true, then $\lambda \in \sigma_e(T)$ again by Theorem 3.7 of [13], and consequently, (5.2) holds. It follows from (5.4) and (5.5) that

$$y_n - x_n = B(f_n - \mu x_n). \tag{5.7}$$

In addition, by (5.3) we get that

$$f_n - \mu x_n = f_n - \lambda x_n + (\lambda - \mu)x_n \overset{w}{\to} 0,$$

which, together with (5.7), the compactness of B and Theorem 6.3 of [18], yields that

$$y_n - x_n \to 0 \text{ as } n \to \infty. \tag{5.8}$$

This implies that the first relation in (5.6) holds, and $\liminf_{n\to\infty} \|y_n\| = \liminf_{n\to\infty} \|x_n\| > 0$. Moreover, we have

$$f_n + \mu(y_n - x_n) - \lambda y_n = f_n - \lambda x_n + (\mu - \lambda)(y_n - x_n),$$

which, together with the third relation in (5.3) and (5.8), implies that the third relation in (5.6) holds. Hence, (5.6) has been shown. Therefore, (5.1) holds.

The final assertion can be directly derived from (5.1) and (4) of Definition 2.1. The proof is complete. □

Remark 5.1. In [19, Theorem 5.3], Wilcox showed that the five essential spectra $\sigma_{ei}(S), 1 \leq i \leq 5$, given in [19] are stable under compact perturbation in the resolvent sense, in which X is a Banach space, and T

and S are closed subspaces in X^2. By (i) of Lemma 2.5, the compact perturbation in the resolvent sense is equivalent to that in the sense of (1) of Definition 2.4. So the result of [19, Theorem 5.3] is more general than Theorem 5.1 by Remark 2.1. We shall remark that its proof is different from that of Theorem 5.1.

By Theorems 4.5 and 5.1 one can easily get the following result:

Corollary 5.1. *Let S be a self-adjoint subspace in X^2 and T a closed Hermitian subspace in X^2. Assume that $\rho(S) \cap \rho(T) \cap \mathbf{R} \neq \emptyset$. If T is a compact perturbation of S, then the results of Theorem 5.1 hold.*

The following result is a direct consequence of Corollary 5.1.

Corollary 5.2. *Let S be a self-adjoint subspace in X^2 and T a closed Hermitian subspace in X^2. Assume that $\rho(S) \cap \rho(T) \cap \mathbf{R} \neq \emptyset$. If T is a finite rank perturbation of S or $(T - \lambda I)^{-1} - (S - \lambda I)^{-1}$ belongs to the trace class in X for some $\lambda \in \rho(S) \cap \rho(T) \cap \mathbf{R}$, then the results of Theorem 5.1 hold.*

The following result can be directly derived from Theorem 5.1.

Corollary 5.3. *Let T and S be self-adjoint subspaces in X^2. If T is a finite rank perturbation of S or $(T - \lambda I)^{-1} - (S - \lambda I)^{-1}$ belongs to the trace class in X for some $\lambda \in \rho(S) \cap \rho(T)$, then the results of Theorem 5.1 hold.*

Theorem 5.2. *Let S be a self-adjoint subspace in X^2 and A a closed Hermitian subspace in X^2 with $D(S) \subset D(A)$ and $A_s x \in S(0)^{\perp}$ for every $x \in D(S)$. If A_s is an S_s-compact operator, then*

$$\sigma_e(S) = \sigma_e(S + A), \tag{5.9}$$

and consequently S has a pure discrete spectrum if and only if so does $S + A$.

Proof. Let T be defined by (3.1). By Theorem 4.3, T is a self-adjoint subspace in X^2. Thus, $\mathbf{C} \setminus \mathbf{R} \subset \rho(T) \cap \rho(S)$ by Theorem 2.5 of [13], and consequently $\rho(T) \cap \rho(S) \neq \emptyset$. Further, it follows from Theorem 3.4 that T is a compact perturbation of S. Therefore, (5.9) holds by Theorem 5.1. This completes the proof. □

Remark 5.2. In [19, Theorem 4.4], Wilcox showed that the four essential spectra $\sigma_{ei}(S)$, $1 \le i \le 4$, are stable if X is a Banach space, subspace $S \subset X^2$ is closed, subspace $A \subset X^2$ is S-compact with S-bound less than one, $\overline{D(S)} \subset D(A)$, and $A(0) \subset S(0)$. In the case that X is a Hilbert space, S is self-adjoint, and A is closed and Hermitian, it can be verified that A is S-compact in the sense given in [19] if and only if A_s is an S_s-compact operator; and A is S-bounded with S-bound a in the sense given in [19] if and only if A_s is an S_s-bounded operator with S_s-bound a (see Definition 2.3). By [18, Theorem 9.7], A_s is an S_s-bounded operator with S_s-bound zero if A_s is S_s-compact. Obviously, the assumptions of Theorem 5.2 are weaker than those of [19, Theorem 4.4] in this case.

The following result directly follows from Theorem 5.2.

Corollary 5.4. *Let S be a self-adjoint subspace in X^2 and A a closed Hermitian subspace in X^2 with $D(S) \subset D(A)$ and $A_s x \in S(0)^{\perp}$ for every $x \in D(S)$. If A_s is finite rank or belongs to the trace class in $D(S)$, then the results of Theorem 5.2 hold.*

Theorem 5.3. *Let S be a self-adjoint subspace in X^2, and A and $S + A$ closed Hermitian subspaces in X^2 with $D(S) \subset D(A)$. If QA_s is S_s-compact, then the results of Theorem 5.2 hold, where Q is specified in Theorem 3.2.*

Proof. By Theorem 4.4, $T = S + A$ is a self-adjoint subspace in X^2, and by Theorem 3.2, T_s, S_s and A_s satisfy (3.16) and (3.17) holds. So T_s and S_s are self-adjoint operators in $S(0)^\perp = T(0)^\perp$ by Lemma 2.2 and (3.17). Further, QA_s is a Hermitian operator in $D(S)$ by Theorem 3.1. By noting that $D(S)$ is dense in $S(0)^\perp$, $QA_s|_{D(S)}$ is a densely defined and Hermitian operator in $S(0)^\perp$. Therefore, $\sigma_e(T_s) = \sigma_e(S_s)$ by Lemma 2.4, (3.16) and the assumption that QA_s is S_s-compact. This yields the results of Theorem 5.3 by Lemma 2.1, and then the proof is complete.   □

By Theorem 5.3 one can easily get the following result:

Corollary 5.5. *Let S be a self-adjoint subspace in X^2, and A and $S + A$ closed Hermitian subspaces in X^2 with $D(S) \subset D(A)$. If QA_s is finite rank or belongs to the trace class in $D(S)$, then the results of Theorem 5.2 hold, where Q is specified in Theorem 3.2.*

Finally, we shall give several remarks on the assumptions and results of Theorems 5.1–5.3 and Corollaries 5.1–5.5.

Remark 5.3. Theorems 5.1 and 5.2 extend Theorems 8.12 and 8.15 of [12] for self-adjoint operators to self-adjoint subspaces, respectively.

Remark 5.4. In the case that it is known that a subspace and its perturbation are both self-adjoint, and their resolvents can be explicitly expressed, then Theorem 5.1 and Corollary 5.3 are applicable. In the case that it is known that the resolvent sets of the unperturbed self-adjoint subspace and its perturbation both contain a real value, then Corollaries 5.1 and 5.2 are applicable. Instead, if it is more easy to get the related information of the operator part of the perturbed term, then Theorems 5.2 and 5.3 and Corollaries 5.4 and 5.5 are more applicable.

Remark 5.5. As we have mentioned in the first section, it is very important for us to study spectral properties of multi-valued or non-densely defined Hermitian operators because a minimal operator, generated by a symmetric linear difference or differential expression that does not satisfy the corresponding definiteness condition, may be multi-valued and non-densely defined and so may be its self-adjoint extensions (cf., [9, 11,14]). The results given in this section are available in this case.

Remark 5.6. By Theorems 5.1 and 5.2, the essential spectrum $\sigma_e(S)$ of a self-adjoint subspace S is invariant if the perturbation is compact or the operator part A_s of the perturbed term A is S_s-compact. If the perturbation or A_s is finite rank or more generally belongs to the trace class, then $\sigma_e(S)$ is invariant by Corollaries 5.2–5.5. We shall further study invariance of the absolutely continuous spectrum of S under this perturbation in our forthcoming paper. In addition, we shall apply these results to study dependence of absolutely continuous spectra on regular endpoints and boundary conditions and invariance of essential and absolutely continuous spectra under perturbation for singular linear Hamiltonian systems in our other forthcoming papers, including that the systems are in the limit point and middle limit cases at singular endpoints.

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