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Existence of gap solitons in periodic discrete nonlinear Schrödinger equations ☆

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ABSTRACT

In this paper, we discuss how to use the critical point theory to study the existence of gap solitons for periodic discrete nonlinear Schrödinger equations. An open problem proposed by Professor Alexander Pankov is solved.

1. Introduction

The Schrödinger equation is one of the most generic soliton equations, and arises from a wide variety of field, such as quantum theory, weakly nonlinear dispersive water waves and nonlinear optics, see [1,12,13]. The Schrödinger equation with periodic potentials and nonlinearities has found a great deal of interest in last years because not only it is important in applications but also it provides a good model for developing mathematical methods, see [3,5,8,11,18–20]. In the past decade, localized solutions of the discrete nonlinear Schrödinger equation (DNLS) have become a topic of intense research. Much of this work concerns the standard constant coefficient cubic DNLS. For details, we can refer to Refs. [9,11] and the references therein. In the physics literature, at least, case $M=2$ is considered, see [10,16]. In this case a new phenomenon appears. While the spectral gaps of periodic operators L are typically open up. The corresponding DNLS may have standing wave solutions with frequency in such a gap.

Since the 1960s, a great number of papers have considered spatially localized standing waves for the discrete nonlinear Schrödinger equations as follows

$$i\dot{\psi}_n = -\Delta^2\psi_{n-1} + \varepsilon_n\psi_n - \sigma\chi_n|\psi_n|^2\psi_n, \quad n \in \mathbf{Z}, \tag{1.1}$$

where $\sigma = \pm 1$,

$$\Delta\psi_n = \psi_{n+1} - \psi_n$$

is the forward difference operator and the given sequences ε_n and χ_n are assumed to be M-periodic in n for a given positive integer M, i.e. $\varepsilon_{n+M} = \varepsilon_n$ and $\chi_{n+M} = \chi_n$. Such solutions are often called intrinsic localized modes or breathers, but

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in the case under consideration we prefer the name 'gap solitons' due to the obvious analogy with gap solitons in photonic crystals, see [2,6,7,10].

The gap solitons, i.e., collective nonlinear excitations which can exist in spectrum gaps forbidden for linear waves, are now studied very intensively in various physical systems [1,2,7,10,12]. In particular, mechanisms capable of producing gap solitons in chains of interacting atoms were recently proposed in Refs. [2,7,12]. The theory of gap solitons is one of the fastest developing areas of modern mathematics and has attracted much attention due to its significant nature in physical contexts, stratified internal waves, ion-acoustic wave, plasma physics [1,2,7,12].

Making use of the standing wave ansatz

$$\psi_n = u_n \exp(-i\omega t), \tag{1.2}$$

where u_n is a real valued sequence and $\omega \in \mathbf{R}$, we arrive at the equation

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = \sigma \chi_n |u_n|^2 u_n, \quad n \in \mathbf{Z}. \tag{1.3}$$

Pankov [14] recently has established the existence of nontrivial solutions in discrete periodic nonlinear Schrödinger equations with cubic nonlinearity

$$Lu_n - \omega u_n = \sigma \chi_n |u_n|^2 u_n, \quad n \in \mathbf{Z}, \tag{1.4}$$

by using the Linking Theorem. Here L is the Jacobi operator

$$Lu_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n,$$

where a_n and b_n are real valued M-periodic sequences. In [14] Pankov has pointed that superlinear behavior of nonlinearity both at 0 and ∞ seems to play a rather basic role. Therefore, Pankov has posed the following completely open problem:

Problem. The existence of discrete gap solitons of the following typical equation

$$Lu_n - \omega u_n = \sigma \chi_n u_n^3 \left(1 + c_n u_n^2\right)^{-1}, \quad n \in \mathbf{Z}, \tag{1.5}$$

where χ_n and c_n are real valued M-periodic sequences.

The aim of the present paper is to solve the above open problem.

As usual, $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \ldots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \ldots, b\}$ when $a \leqslant b$. l denotes the space of all two-side real valued sequences $u = \{u_n\}_{n \in \mathbf{Z}}$ and $l^p \subset l$, $1 < p \leqslant \infty$, the subspace of all p-summable (bounded if $p = \infty$) sequences. Endowed with the standard norm $\|\cdot\|_{l^p}$ which is defined by

$$\|u\|_{l^p} = \left(\sum_{n=-\infty}^{+\infty} |u_n|^p \right)^{\frac{1}{p}}, \quad \forall u \in l^p,$$

l^p is a Banach space (Hilbert space when $p = 2$). $*$ is the transpose sign for a vector. I denotes the identity operator.

Our approach involves the use of some variational methods of a minimax kind. The proof is based on the Mountain Pass Lemma in combination with periodic approximations.

In what follows we consider the case $\sigma = +1$. The other case reduces to the previous one if we replace L by $-L$. Denote

$$\underline{k} = \min_{n \in \mathbf{Z}(1,M)} \chi_n, \qquad \bar{k} = \max_{n \in \mathbf{Z}(1,M)} \chi_n, \qquad \underline{l} = \min_{n \in \mathbf{Z}(1,M)} c_n, \qquad \bar{l} = \max_{n \in \mathbf{Z}(1,M)} c_n.$$

Our main results are as follows.

Theorem 1.1. *Assume that the following hypotheses are satisfied*:

(L) $a_n \neq 0$, $b_n - |a_{n-1}| - |a_n| > \omega$, *for* $n \in \mathbf{Z}$;

(F) $\chi_n > 0$, $c_n > 0$, $\bar{\lambda} < \frac{\underline{k}}{\bar{l}}$, $\underline{\lambda} + \frac{\underline{k}}{\bar{l}} > \frac{\bar{k}}{\underline{l}}$, *for* $n \in \mathbf{Z}$,

where $\underline{\lambda}$ *and* $\bar{\lambda}$ *are constants which can be referred to* (2.6).

Then Eq. (1.5) *has a nontrivial solution* $u \in l^2$.

We show that the solution u obtained in Theorem 1.1 decays exponentially at infinity.

Theorem 1.2. *Suppose that* (L) *and* (F) *are satisfied. Then the solution* u *obtained in Theorem* 1.1 *decays exponentially at infinity*:

$$|u_n| \leqslant C e^{-\gamma |n|}, \quad n \in \mathbf{Z},$$

with some constants $C > 0$ *and* $\gamma > 0$.

2. Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for Eq. (1.5) and give some lemmas which will be of fundamental importance in proving our main results. Firstly, we state some basic notations.

Let S be the set of sequences $u = (\ldots, u_{-n}, \ldots, u_{-1}, u_0, u_1, \ldots, u_n, \ldots) = \{u_n\}_{n=-\infty}^{+\infty}$, that is

$$S = \big\{\{u_n\} \bigm| u_n \in \mathbf{R},\ n \in \mathbf{Z}\big\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_n + bv_n\}_{n=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers M and m, E_m is defined as a subspace of S by

$$E_m = \{u \in S \mid u_{n+2mM} = u_n,\ \forall n \in \mathbf{Z}\}.$$

Clearly, E_m is isomorphic to $\mathbf{R}^{2mM}$. E_m can be equipped with the inner product

$$\langle u, v\rangle_m = \sum_{n=-mM}^{mM-1} \big[(L-\omega I)u_n \cdot v_n\big], \quad \forall u, v \in E_m, \tag{2.1}$$

by which the norm $\|\cdot\|_m$ can be induced by

$$\|u\|_m = \sqrt{\sum_{n=-mM}^{mM-1} \big[(L-\omega I)u_n \cdot u_n\big]}, \quad \forall u \in E_m. \tag{2.2}$$

It is obvious that $(E_m, \langle\cdot,\cdot\rangle_m)$ is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbf{R}^{2mM}$.

In what follows, l_m^r denotes the space of all real functions whose r-summable on the interval $[-mM, mM-1]$ equipped with the norm

$$\|u\|_{l_m^r} = \left(\sum_{n=-mM}^{mM-1} |u_n|^r\right)^{\frac{1}{r}}, \tag{2.3}$$

for all $u \in l_m^r$ and $r > 1$.

Moreover, l_m^∞ denotes the space of all bounded real functions on the interval $[-mM, mM-1]$ endowed with the norm

$$\|u\|_{l_m^\infty} = \max_{n\in\mathbf{Z}(-mM, mM-1)} |u_n|, \quad \forall u \in l_m^\infty.$$

For all $u \in E_m$, we define the functional J_m on E_m as follows:

$$J_m(u) := \frac{1}{2}\sum_{n=-mM}^{mM-1}\left[(L-\omega I)u_n \cdot u_n - \frac{\chi_n}{c_n}\left(u_n^2 - \frac{1}{c_n}\ln\big(1+c_n u_n^2\big)\right)\right]. \tag{2.4}$$

Clearly, $J_m \in C^1(E_m, \mathbf{R})$ and for any $u = \{u_n\}_{n\in\mathbf{Z}} \in E_m$, by using $u_0 = u_{2mM}$, $u_1 = u_{2mM+1}$, we can compute the partial derivative as

$$\frac{\partial J_m(u)}{\partial u_n} = Lu_n - \omega u_n - \chi_n u_n^3\big(1+c_n u_n^2\big)^{-1}, \quad \forall n \in \mathbf{Z}(-mM, mM-1).$$

Thus, u is a critical point of J_m on E_m if and only if

$$Lu_n - \omega u_n = \chi_n u_n^3\big(1+c_n u_n^2\big)^{-1}, \quad \forall n \in \mathbf{Z}(-mM, mM-1).$$

Due to the periodicity of $u = \{u_n\}_{n\in\mathbf{Z}} \in E_m$, we reduce the existence of periodic solutions of Eq. (1.5) to the existence of critical points of J_m on E_m. That is, the functional J_m is just the variational framework of Eq. (1.5).

For all $u \in E_m$ and $mM > 1$, $J_m(u)$ is rewritten as

$$J_m(u) = \frac{1}{2}(P_m u, u) - \frac{1}{2}\sum_{n=-mM}^{mM-1}\left[\frac{\chi_n}{c_n}\left(u_n^2 - \frac{1}{c_n}\ln\big(1+c_n u_n^2\big)\right)\right], \tag{2.5}$$

where

$$u = (u_{-mM}, \ldots, u_{-1}, u_0, u_1, \ldots, u_{mM-1})^*,$$

$$P_m = \begin{pmatrix} b_{-mM}-\omega & a_{-mM} & 0 & \cdots & 0 & a_{-mM-1} \\ a_{-mM} & b_{-mM+1}-\omega & a_{-mM+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_{mM-2}-\omega & a_{mM-2} \\ a_{mM-1} & 0 & 0 & \cdots & a_{mM-2} & b_{mM-1}-\omega \end{pmatrix}_{2mM\times 2mM},$$

and $(\cdot,\cdot)$ is the usual product in $\mathbf{R}^{2mM}$.

Remark 2.1. For the case $mM = 1$, P_m has a different form, namely,

$$P_m = \begin{pmatrix} b_{-1}-\omega & a_{-1}+a_0 \\ a_{-1}+a_0 & b_0-\omega \end{pmatrix}.$$

However, in this case, it is easy to complete the proof of Theorem 1.1.

We define

$$\underline{\lambda} = \min_{n\in\mathbf{Z}(1,M)} \big(b_n - \omega - |a_{n-1}| - |a_n|\big) > 0, \qquad \bar{\lambda} = \max_{n\in\mathbf{Z}(1,M)} \big(b_n - \omega + |a_{n-1}| + |a_n|\big). \tag{2.6}$$

By the Gerschgörin Theorem, all eigenvalues of the matrix P_m satisfy $\underline{\lambda} \leqslant \lambda_i \leqslant \bar{\lambda}$, $i \in \mathbf{Z}[-mM, mM-1]$. Thus,

$$\underline{\lambda}\|u\|^2_{l^2_m} \leqslant \|u\|^2_m \leqslant \bar{\lambda}\|u\|^2_{l^2_m}. \tag{2.7}$$

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to be satisfying the Palais–Smale condition (P.S. condition for short) if any sequence $\{u^{(k)}\} \subset E$ for which $\{J(u^{(k)})\}$ is bounded and $J'(u^{(k)}) \to 0$ $(k \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 *(Mountain Pass Lemma). (See [4,15].) Let E be a real Banach space and $J \in C^1(E, \mathbf{R})$ satisfies the P.S. condition. If $J(0) = 0$ and*

(J_1) *there exist constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho} \geqslant \alpha$, and*
(J_2) *there exists $e \in E \setminus B_\rho$ such that $J(e) \leqslant 0$, then J possesses a critical value $c \geqslant \alpha$ given by*

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} J\big(g(s)\big), \tag{2.8}$$

where

$$\Gamma = \big\{g \in C\big([0,1], E\big) \bigm| g(0) = 0,\ g(1) = e\big\}. \tag{2.9}$$

Lemma 2.2. *For any $x > 0$, the following inequalities hold*:

$$\frac{2x}{2+x} < \ln(1+x) < x, \tag{2.10}$$

$$\frac{2x}{2+x} < \ln(1+x) < \sqrt{x}. \tag{2.11}$$

Lemma 2.3. *Assume that (L) and (F) are satisfied. Then J satisfies the P.S. condition.*

Proof. Let $\{u^{(k)}\} \subset E_m$ be such that $\{J_m(u^{(k)})\}$ is bounded and $J'_m(u^{(k)}) \to 0$ as $k \to \infty$. Then there exists a positive constant K such that $|J_m(u^{(k)})| \leqslant K$. We shall prove $\{u^{(k)}\}$ is bounded.

If otherwise, we assume that $\|u^{(k)}\|_m \to +\infty$ as $k \to \infty$. So $\|u^{(k)}\|_{l^2_m} \to +\infty$ as $k \to \infty$. Therefore, $\|u^{(k)}\|_{l^\infty_m} \to +\infty$ as $k \to \infty$. By (2.7) and (2.11),

$$\begin{aligned} J_m\big(u^{(k)}\big) &\leqslant \frac{\bar{\lambda}}{2}\big\|u^{(k)}\big\|^2_{l^2_m} - \frac{1}{2}\sum_{n=-mM}^{mM-1} \frac{\chi_n}{c_n}\big(u^{(k)}_n\big)^2 + \frac{1}{2}\sum_{n=-mM}^{mM-1} \frac{\chi_n}{c_n^2}\ln\big(1 + c_n\big(u^{(k)}_n\big)^2\big) \\ &\leqslant \frac{1}{2}\sum_{n=-mM}^{mM-1}\left[\bar{\lambda}\big(u^{(k)}_n\big)^2 - \frac{k}{\bar{\bar{l}}}\big(u^{(k)}_n\big)^2 + \frac{\chi_n}{c_n^{\frac{3}{2}}}\big|u^{(k)}_n\big|\right] \\ &\leqslant \frac{1}{2}\sum_{n=-mM}^{mM-1}\left[\left(\bar{\lambda} - \frac{k}{\bar{\bar{l}}}\right)\big(u^{(k)}_n\big)^2 + \frac{\bar{k}}{\underline{l}^{\frac{3}{2}}}\big|u^{(k)}_n\big|\right]. \end{aligned}$$

Since $\bar{\lambda} < \frac{\underline{k}}{\bar{l}}$, we get $J_m(u^{(k)}) \to -\infty$ as $k \to \infty$. This contradicts the fact that $|J_m(u^{(k)})| \leqslant K$. Thus the P.S. condition is verified. $\square$

3. Proof of the main results

In this section, we firstly prove the existence of a nontrivial $2mM$-periodic solution to Eq. (1.5). Actually, we show that the functional J_m possesses a nontrivial critical point. Secondly, we get uniform estimates of $u^{(m)}$ in E_m, l_m^2 and l_m^4 which are independent of $m \in \mathbf{N}$. Thirdly, $\{u^{(m)}\}_{m\in\mathbf{N}}$ passes to a nontrivial solution $u \in l^2$ as $m \to \infty$ and we complete the proof of Theorem 1.1. Finally, we prove Theorem 1.2.

3.1. Existence of periodic solutions

For our setting, clearly $J_m(0) = 0$ and J_m satisfies the P.S. condition. Hence, it suffices to prove that J_m satisfies the conditions (J_1) and (J_2). By (2.7) and (2.10), we have

$$\begin{aligned}
J_m(u) &\geqslant \frac{\underline{\lambda}}{2}\|u\|_{l_m^2}^2 - \frac{1}{2}\sum_{n=-mM}^{mM-1}\frac{\chi_n}{c_n}u_n^2 + \frac{1}{2}\sum_{n=-mM}^{mM-1}\frac{\chi_n}{c_n^2}\ln\bigl(1 + c_n u_n^2\bigr) \\
&\geqslant \frac{1}{2}\sum_{n=-mM}^{mM-1}\left(\underline{\lambda}u_n^2 - \frac{\bar{k}}{\underline{l}}u_n^2 + \frac{\chi_n}{c_n}\cdot\frac{2u_n^2}{2 + c_n u_n^2}\right).
\end{aligned}$$

Since $\underline{\lambda} + \frac{\underline{k}}{\bar{l}} > \frac{\bar{k}}{\underline{l}}$, there exists a constant $\rho > 0$ such that $\underline{\lambda} + \frac{2\underline{k}\cdot\underline{\lambda}}{\bar{l}(2\underline{\lambda}+\bar{l}\rho^2)} > \frac{\bar{k}}{\underline{l}}$. Therefore, for any $u \in E_m$ and $\|u\|_m \leqslant \rho$, we have $\|u\|_{l_m^\infty} \leqslant \|u\|_{l_m^2} \leqslant \frac{1}{\sqrt{\underline{\lambda}}}\|u\|_m \leqslant \frac{1}{\sqrt{\underline{\lambda}}}\rho$.

Thus,

$$\begin{aligned}
J_m(u) &\geqslant \frac{1}{2}\sum_{n=-mM}^{mM-1}\left(\underline{\lambda} - \frac{\bar{k}}{\underline{l}} + \frac{\underline{k}}{\bar{l}}\cdot\frac{2}{2 + \frac{\bar{l}}{\underline{\lambda}}\rho^2}\right)u_n^2 \\
&\geqslant \frac{1}{2\bar{\lambda}}\left(\underline{\lambda} - \frac{\bar{k}}{\underline{l}} + \frac{\underline{k}}{\bar{l}}\cdot\frac{2\underline{\lambda}}{2\underline{\lambda} + \bar{l}\rho^2}\right)\|u\|_m^2.
\end{aligned}$$

Taking $\alpha = \frac{1}{2\bar{\lambda}}[\underline{\lambda} - \frac{\bar{k}}{\underline{l}} + \frac{2\underline{k}\cdot\underline{\lambda}}{\bar{l}(2\underline{\lambda}+\bar{l}\rho^2)}]\rho^2$, we obtain

$$J_m(u)|_{\partial B_\rho} \geqslant \alpha > 0,$$

which implies that J_m satisfies the condition (J_1) of the Mountain Pass Lemma.

Next, we shall verify the condition (J_2).

By (2.7) and (2.11), we get

$$\begin{aligned}
J_m(u) &\leqslant \frac{\bar{\lambda}}{2}\|u\|_{l_m^2}^2 - \frac{1}{2}\sum_{n=-mM}^{mM-1}\frac{\chi_n}{c_n}u_n^2 + \frac{1}{2}\sum_{n=-mM}^{mM-1}\frac{\chi_n}{c_n^2}\ln\bigl(1 + c_n u_n^2\bigr) \\
&\leqslant \frac{1}{2}\sum_{n=-mM}^{mM-1}\left(\bar{\lambda}u_n^2 - \frac{\underline{k}}{\bar{l}}u_n^2 + \frac{\chi_n}{c_n^{\frac{3}{2}}}|u_n|\right) \\
&\leqslant \frac{1}{2}\sum_{n=-mM}^{mM-1}\left[\left(\bar{\lambda} - \frac{\underline{k}}{\bar{l}}\right)u_n^2 + \frac{\bar{k}}{\underline{l}^{\frac{3}{2}}}|u_n|\right].
\end{aligned}$$

Since $\bar{\lambda} < \frac{\underline{k}}{\bar{l}}$ and $\|u\|_m \leqslant \sqrt{\bar{\lambda}}\|u\|_{l_m^2} \leqslant 2mM\sqrt{\bar{\lambda}}\|u\|_{l_m^\infty}$, we can choose $\|u\|_m > \rho$ large enough to ensure that $J_m(u) \leqslant 0$.

All the assumptions of the Mountain Pass Lemma have been verified. Consequently, J_m possesses a critical value c_m given by (2.8) and (2.9) with $E = E_m$ and $\Gamma = \Gamma_m$, where $\Gamma_m = \{g_m \in C([0,1], E_m) \mid g_m(0) = 0,\ g_m(1) = e_m,\ e_m \in E_m \setminus B_\rho\}$. We denote $u^{(m)}$ the corresponding critical point of J_m on E_m. It is obvious that $\|u^{(m)}\|_m \neq 0$ since $c_m > 0$.

3.2. Uniform estimates for $\|u^{(m)}\|_m$

The next step in the proof is to obtain m independent estimates for c_m and $u^{(m)}$.

Lemma 3.2.1. *Suppose that (L) and (F) are satisfied. Then there exists a constant $\tilde{C}$ independent of m such that*

$$\|u^{(m)}\|_m \leqslant \tilde{C}, \quad \forall m \in \mathbf{N}.$$

Proof. Let $e \in E_1 \setminus \{0\}$ such that $e_{-M} = e_M = 0$ and $J_1(e) \leqslant 0$. Define $e^{(m)} = \{e_n^{(m)}\}$ satisfying

$$e_n^{(m)} = \begin{cases} e_n, & n \in \mathbf{Z}(-M, M), \\ 0, & M < |n| \leqslant mM. \end{cases}$$

Then by the Mountain Pass Lemma, $e^{(m)} \in E_m \setminus \{0\}$ and $J_m(e^{(m)}) = J_1(e) \leqslant 0$. Note also that $g_m(s) = se_m \in \Gamma_m$ for any $m \in \mathbf{N}$ and $J_m(g_m(s)) = J_1(g_1(s)) = J_1(se)$. Therefore, by (2.8),

$$0 < \alpha \leqslant c_m \leqslant \max_{0\leqslant s\leqslant 1} J_m\big(g_m(s)\big) = \max_{0\leqslant s\leqslant 1} J_1\big(g_1(s)\big) = \max_{0\leqslant s\leqslant 1} J_1(se) = d_1, \tag{3.1}$$

where d_1 is a constant independent of m.

Since $J_m'(u^{(m)}) = 0$, we have

$$\begin{aligned} c_m &= J_m\big(u^{(m)}\big) - \frac{1}{2}\big\langle J_m'\big(u^{(m)}\big), u^{(m)}\big\rangle_m \\ &= \frac{1}{2}\sum_{n=-mM}^{mM-1} \chi_n\left[\frac{1}{c_n^2}\ln\big(1 + c_n\big(u_n^{(m)}\big)^2\big) - \frac{(u_n^{(m)})^2}{c_n(1 + c_n(u_n^{(m)})^2)}\right]. \end{aligned}$$

Let

$$f(x) = \frac{1}{c_n^2}\ln\big(1 + c_n x^2\big) - \frac{x^2}{c_n(1 + c_n x^2)}, \quad x \geqslant 0.$$

Then $f(x) \geqslant 0$.

We choose $\eta_i > 0$ $(i = 1, 2, \ldots, M)$ such that

$$\frac{1}{2}\underline{k}\left[\frac{1}{c_i^2}\ln\big(1 + c_i\eta_i^2\big) - \frac{\eta_i^2}{c_i(1 + c_i\eta_i^2)}\right] = d_1, \quad i = 1, 2, \ldots, M.$$

Take

$$\eta = \max\{\eta_1, \eta_2, \ldots, \eta_M\}.$$

By $c_m \leqslant d_1$ and the monotone increasing property of $f(x)$ with respect to x, we have

$$\|u^{(m)}\|_{l_m^\infty} \leqslant \eta.$$

Thus, by (2.10),

$$\begin{aligned} c_m &\geqslant \frac{\underline{k}}{2}\sum_{n=-mM}^{mM-1}\left[\frac{2(u_n^{(m)})^2}{c_n(2 + c_n(u_n^{(m)})^2)} - \frac{(u_n^{(m)})^2}{c_n(1 + c_n(u_n^{(m)})^2)}\right] \\ &= \frac{\underline{k}}{2}\sum_{n=-mM}^{mM-1}\frac{(u_n^{(m)})^4}{(2 + c_n(u_n^{(m)})^2)(1 + c_n(u_n^{(m)})^2)} \\ &\geqslant \frac{\underline{k}}{2(2 + \bar{l}\eta^2)(1 + \bar{l}\eta^2)}\|u^{(m)}\|_{l_m^4}^4. \end{aligned}$$

This together with (3.1) yields

$$\|u^{(m)}\|_{l_m^4} \leqslant \sqrt[4]{\frac{2(2 + \bar{l}\eta^2)(1 + \bar{l}\eta^2)d_1}{\underline{k}}} \equiv d. \tag{3.2}$$

Since $J_m'(u_n^{(m)}) = 0$, we have

$$\begin{aligned} \big\langle J_m'\big(u^{(m)}\big), u^{(m)}\big\rangle_m &= \big(Lu^{(m)} - \omega u^{(m)}, u^{(m)}\big) - \sum_{n=-mM}^{mM-1}\frac{\chi_n}{c_n}\left[\big(u_n^{(m)}\big)^2 - \frac{(u_n^{(m)})^2}{1 + c_n(u_n^{(m)})^2}\right] \\ &= \big(Lu^{(m)} - \omega u^{(m)}, u^{(m)}\big) - \sum_{n=-mM}^{mM-1}\frac{\chi_n(u_n^{(m)})^4}{1 + c_n(u_n^{(m)})^2}, \end{aligned}$$

and obtain, using the Cauchy–Schwarz inequality, that

$$\underline{\lambda}\|u^{(m)}\|_{l_m^2}^2 \leqslant \sum_{n=-mM}^{mM-1} \frac{\chi_n(u_n^{(m)})^4}{1+c_n(u_n^{(m)})^2} \leqslant \bar{k} \sum_{n=-mM}^{mM-1} \big(u_n^{(m)}\big)^3 u_n^{(m)}$$
$$\leqslant \bar{k}\|u^{(m)}\|_{l_m^6}^3 \|u^{(m)}\|_{l_m^2} \leqslant \bar{k}\|u^{(m)}\|_{l_m^4}^3 \|u^{(m)}\|_{l_m^2}.$$

Hence, by (3.2),

$$\|u^{(m)}\|_{l_m^2} \leqslant \underline{\lambda}^{-1}\bar{k}d^3. \tag{3.3}$$

Therefore,

$$\|u^{(m)}\|_m \leqslant \bar{\lambda}\underline{\lambda}^{-1}\bar{k}d^3 \equiv \tilde{C}. \qquad \square \tag{3.4}$$

3.3. Limit process for $\{u_m\}_{m\in\mathbf{N}}$

Next, we shall give the existence of a nontrivial solution $u \in l^2$ in Theorem 1.1.

Consider the sequence $u^{(m)} = \{u_n^{(m)}\}$ of $2mM$-periodic solutions found in Section 3.1.

For a sequence $u = \{u_n\} \in l$ we set

$$R_m u_n = \begin{cases} u_n, & n \in \mathbf{Z}(-mM, mM-1), \\ 0, & n \notin \mathbf{Z}(-mM, mM-1). \end{cases}$$

Firstly, we claim that for any $m \in \mathbf{N}$, there exist constants $\xi > 0$, n_m and n'_m such that

$$\big|u_{n_m}^{(n'_m)}\big| \geqslant \xi. \tag{3.5}$$

Indeed, if not, then $u^{(m)} \to 0$ in l^∞. Hence, $v^{(m)} = R_m u^{(m)} \to 0$ in l^∞. By (3.3), $\|v^{(m)}\|_{l^2} = \|u^{(m)}\|_{l_m^2}$ is bounded. Now the following simple inequality

$$\|v^{(m)}\|_{l^p}^p \leqslant \|v^{(m)}\|_{l^\infty}^{p-2} \|v^{(m)}\|_{l^2}^2,$$

where $p > 2$ shows that $v^{(m)} \to 0$ in all l^p, $p > 2$. By (2.10), we have

$$\begin{aligned}
c_m &= J_m\big(u^{(m)}\big) - \frac{1}{2}\big\langle J'_m\big(u^{(m)}\big), u^{(m)}\big\rangle_m \\
&= \frac{1}{2}\sum_{n=-mM}^{mM-1} \chi_n \left[\frac{1}{c_n^2}\ln\big(1+c_n\big(u_n^{(m)}\big)^2\big) - \frac{(u_n^{(m)})^2}{c_n(1+c_n(u_n^{(m)})^2)}\right] \\
&\leqslant \frac{\bar{k}}{2}\sum_{n=-mM}^{mM-1} \left[\frac{(u_n^{(m)})^2}{c_n} - \frac{(u_n^{(m)})^2}{c_n(1+c_n(u_n^{(m)})^2)}\right] \\
&= \frac{\bar{k}}{2}\sum_{n=-mM}^{mM-1} \frac{(u_n^{(m)})^4}{1+c_n(u_n^{(m)})^2} \\
&\leqslant \frac{\bar{k}}{2}\|v^{(m)}\|_{l^4}^4 \to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

However, this contradicts (3.1) and we obtain (3.5).

Due to periodicity of coefficients, $\{u_{n_m+M}^{(n'_m)}\}$ is also a solution of Eq. (1.5). Hence, making such shifts, we can assume that $0 \leqslant n_m \leqslant M-1$ in (3.5). Moreover, passing to a subsequence of ms, we can even assume that $n_m = n_0$ is independent of m.

Next, we extract a subsequence, still denote by $u_n^{(m)}$, such that

$$u_n^{(m)} \to u_n, \quad m \to \infty, \ \forall n \in \mathbf{Z}.$$

Inequality (3.5) implies that $|u_{n_0}| \geqslant \xi$ and, hence, $u = \{u_n\}$ is a nonzero sequence. Moreover,

$$\begin{aligned}
Lu_n - \omega u_n - \chi_n|u_n|^3\big(1+c_n u_n^2\big)^{-1} &= \lim_{m\to\infty}\big[Lu_n^{(m)} - \omega u_n^{(m)} - \chi_n\big|u_n^{(m)}\big|^3\big(1+c_n\big(u_n^{(m)}\big)^2\big)^{-1}\big] \\
&= \lim_{m\to\infty} 0 = 0.
\end{aligned}$$

So $u = \{u_n\}$ is a solution of Eq. (1.5).

Finally, for any fixed $D \in \mathbf{Z}$ and m large enough, we have that

$$\sum_{n=-D}^{D} \left|u_n^{(m)}\right|^2 \leqslant \underline{\lambda}^{-1} \left\|u^{(m)}\right\|_m^2 \leqslant \underline{\lambda}^{-1} \tilde{C}^2.$$

Since $\underline{\lambda}^{-1} \tilde{C}^2$ is a constant independent of m, passing to the limit, we have that

$$\sum_{n=-D}^{D} |u_n|^2 \leqslant \underline{\lambda}^{-1} \tilde{C}^2.$$

Due to arbitrariness of D, $u \in l^2$. The proof of Theorem 1.1 is complete.

3.4. Proof of Theorem 1.2

Next, we shall complete the proof of Theorem 1.2. To complete the proof of Theorem 1.2, it remains to show that the solution obtained in Theorem 1.1 decays exponentially fast.

Take

$$\underline{\tilde{\lambda}} = \min_{n \in \mathbf{Z}(1,M)} \big(b_n - |a_{n-1}| - |a_n|\big), \qquad \bar{\tilde{\lambda}} = \max_{n \in \mathbf{Z}(1,M)} \big(b_n + |a_{n-1}| + |a_n|\big).$$

The operator L is a bounded and self-adjoint operator in l^2. By the Jacobi operators theory [17], it is easy to know that the spectrum $\sigma(L) \subset [\underline{\tilde{\lambda}}, \bar{\tilde{\lambda}}]$, and the frequency ω belongs to a spectral gap $(-\infty, \underline{\tilde{\lambda}})$.

Let

$$v_n = -\chi_n u_n^2 \big(1 + c_n u_n^2\big)^{-1}, \quad n \in \mathbf{Z}.$$

Then

$$\tilde{L} u_n - \omega u_n = 0, \tag{3.6}$$

where

$$\tilde{L} u_n = L u_n + v_n u_n.$$

Since $u \in l^2$, we know that $\lim_{|n| \to \infty} v_n = 0$, the multiplication by v_n is compact operator in l^2. Hence,

$$\sigma_{ess}(\tilde{L}) = \sigma_{ess}(L),$$

where σ_{ess} stands for the essential spectrum. Now (3.6) means that $u = \{u_n\}$ is an eigenfunction that corresponds to the eigenvalue of finite multiplicity $\omega \notin \sigma_{ess}(\tilde{L})$ of the operator $\tilde{L}$. Therefore, the result follows from the standard theorem on exponential decay for such eigenfunctions, see, for example [17].

Remark 3.1. As an application of Theorems 1.1 and 1.2, finally, we give an example to illustrate our results.

For all $n \in \mathbf{Z}$, assume that

$$u_{n+1} + u_{n-1} + \left[4 - \sin^2\left(\frac{n\pi}{3}\right)\right] u_n = \left[16 + \frac{1}{16} \cos^2\left(\frac{n\pi}{3}\right)\right] u_n^3 \big(1 + 2u_n^2\big)^{-1}. \tag{3.7}$$

We have

$$a_n \equiv 1, \qquad b_n = -\left[2 + \sin^2\left(\frac{n\pi}{3}\right)\right], \qquad \omega = -6, \qquad \chi_n = 16 + \frac{1}{16} \cos^2\left(\frac{n\pi}{3}\right), \qquad c_n \equiv 2, \qquad M = 3.$$

Then

$$\underline{\lambda} = \frac{5}{4}, \qquad \bar{\lambda} = 6.$$

It is easy to verify all the assumptions of Theorems 1.1 and 1.2 are satisfied. Thus, Eq. (3.7) has a nontrivial solution $u \in l^2$ and moreover, the solution u decays exponentially at infinity.

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