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Spectrum and Pseudospectrum of Non-Self-Adjoint Schrödinger Operators with Periodic Coefficients

S. V. Gal′tsev and A. I. Shafarevich

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Abstract—We consider the pseudospectrum of the non-self-adjoint operator

$$\mathfrak{D} = -h^2 \frac{d^2}{dx^2} + iV(x),$$

where $V(x)$ is a periodic entire analytic function, real on the real axis, with a real period T. In this operator, h is treated as a small parameter. We show that the pseudospectrum of this operator is the closure of its numerical image, i.e., a half-strip in $\mathbb{C}$. In this case, the pseudoeigenfunctions, i.e., the functions $\varphi(h,x)$ satisfying the condition

$$\|\mathfrak{D}\varphi - \lambda\varphi\| = O(h^N), \qquad \|\varphi\| = 1, \quad N \in \mathbb{N},$$

can be constructed explicitly. Thus, it turns out that the pseudospectrum of the operator under study is much wider than its spectrum.

Key words: *spectrum*, *pseudospectrum*, *Schrödinger operator*, *periodicity condition*, *periodic entire analytic function*, *non-self-adjoint operator*, *Riemann surface*.

1. INTRODUCTION

It is well known that the structure of the spectrum of a non-self-adjoint differential operator is, as a rule, significantly more complicated than that in the self-adjoint case; in particular, points of the spectrum can be located arbitrarily on the complex plane.

For self-adjoint problems, one of the efficient methods for studying the spectrum is the semiclassical asymptotics, which, under certain conditions, allows one to calculate the so-called spectral series, i.e., sets of numbers approximately satisfying (along with the corresponding functions) the spectral equation for the original operator. Each spectral series corresponds to an isotropic manifold in the phase space of the corresponding classical system and can be calculated using the so-called Bohr–Sommerfeld–Maslov quantization conditions; if the manifold is Lagrangian, then these conditions have the form

$$\frac{1}{2\pi h}\int_\gamma \Theta = m(\gamma) + \frac{1}{4}\mu(\gamma),$$

where γ is an arbitrary cycle on the manifold, Θ is the primitive of the symplectic structure, $m(\gamma)$ is an arbitrary integer, $\mu(\gamma)$ is the Maslov index, and $h \to 0$ is a semiclassical parameter. The quantization conditions must be satisfied for all cycles on the Lagrangian manifold; these conditions are necessary for the canonical Maslov operator determining the asymptotic eigenfunction to exist on this manifold. The fact that the original operator is self-adjoint guarantees that the spectral series is close to the spectrum of the original problem: if the spectral equation holds mod $O(h^N)$, then the h^N-neighborhood of any point of the spectral series contains a point of the spectrum.

No developed semiclassical theory describing the spectra of non-self-adjoint operators has been developed in recent years; this is related to the fact that, first, the corresponding isotropic manifolds are, as a rule, complex and, second, it is possible that there is no relationship between the spectral series and the spectrum of the original operator. In [1] and [2], the so-called pseudospectrum, i.e., a set of numbers approximately satisfying the spectral equation, was studied; in particular, it was noted that there is a difference between the pseudospectrum and the asymptotics of the point spectrum. In [3]–[6], the asymptotics of the spectrum was calculated for the Schrödinger and Orr–Sommerfeld operators defined on intervals; in particular, it was shown that the points of the spectrum tend asymptotically to a graph on the complex plane, while the pseudospectrum can fill entire domains.

Here we study the spectrum and the pseudospectrum of the operator $-h^2\partial^2/\partial x^2 + iV(x)$, where V is a periodic function real on the real axis (for example, $V = \cos x$). We show that the h^N-pseudospectrum of this operator fills a half-strip on the complex plane for any N as $h \to 0$, while the actual spectrum is $\operatorname{mod} O(h^2)$ concentrated near a one-dimensional set (graph). The edges of this graph correspond to distinct spectral series, which can be calculated using the Bohr–Sommerfeld–Maslov conditions on a complex curve (a Riemannian surface); in contrast to the self-adjoint case, distinct cycles on the same surface determine distinct series (in other words, to construct an asymptotics, it suffices to require that the quantization conditions be satisfied only on a single cycle).

2. STATEMENT OF THE PROBLEM

Several problems naturally arising in the spectral theory of differential operators lead to the study of the spectrum of the operator

$$\mathfrak{D} = -h^2 \frac{d^2}{dx^2} + iV(x), \tag{1}$$

where $V(x)$ is a periodic entire analytic function, real on the real axis, with a real period T. In particular, Eq. (1) is the "standard" operator in the theory of hydrodynamic stability: under certain conditions, its spectrum is similar to that of the Orr–Sommerfeld operator. Another example is the spectral problem for the operator $\varepsilon\Delta + (v(x), \Delta)$ on a flat torus (here $x = (x_1, x_2) \in \mathbb{T}^2$). Let $v(x)$ be a divergenceless field of the form $v(x_1, x_2) = w(x_1)\partial/\partial x_2$. The spectral problem admits the following separation of variables: the eigenfunction $\varphi(x_1, x_2)$ has the form $e^{imx_2}\psi(x_1)$ and, moreover, ψ satisfies the spectral problem for the operator (1) with $V(x_1) = mw(x_1)$ and $\varepsilon = h^2$.

The operator $\mathfrak{D}$ can naturally be considered as an unbounded non-self-adjoint operator in $L_2(\mathbb{S}^1 = \mathbb{R}/T\mathbb{Z})$; at the same time, since the equation $(\mathfrak{D} - E)\varphi = 0$ is a linear ordinary differential equation with entire analytic coefficients, all its solutions are analytic functions and $\mathfrak{D}$ can be treated as an operator in the space $\mathscr{A}(\mathbb{C}/T\mathbb{Z})$ of periodic entire analytic functions. The goal of this paper is to compare the asymptotics as $h \to 0$ of the spectrum of the operator $\mathfrak{D}$ and the pseudospectrum of this operator in the special case $V(x) = \cos x$.

A detailed description of the asymptotics of the spectrum was given by the authors in the separate paper [7]. It turns out that this asymptotics of the spectrum can be expressed in terms of integrals of holomorphic forms over cycles on the Riemannian surface Λ determined in $\mathbb{C}^2/T\mathbb{Z}$ by the equation

$$p^2 + iV(x) = E \qquad p \in \mathbb{C}, \quad x \in \mathbb{C}^2/T\mathbb{Z}.$$

For $E \neq \pm i$, this (noncompact) surface is homeomorphic to a sphere with four holes and can is obtained as the result of gluing together two copies of the cylinder $\mathbb{C}/T\mathbb{Z}$ along the segment connecting the zeros of the function $iV(x) - E$. Namely, we have the following assertion.

Theorem 1. *Suppose that E satisfies the condition that, on the surface Λ, there exists a cycle γ such that*

$$\frac{1}{2\pi h}\int_{\gamma} p\,dx = m + \frac{\mu}{2}. \tag{2}$$

Then the operator $\mathfrak{D}$ has an eigenvalue λ such that $\lambda - E = O(h^2)$. Here $\mu = 0$ if the cycle γ becomes contractible after a hole on Λ is pasted, and $\mu = 1$ otherwise.

There are three basic cycles on the sphere with four holes. Thus, there are three different series of eigenvalues, and each of these series corresponds to a basic cycle on the surface Λ. As $h \to 0$, the eigenvalues are concentrated in the $O(h^2)$-neighborhood of a graph on the complex plane. The edges of this graph are given by the equations

$$\frac{1}{2\pi h}\int_{\gamma_j} p\,dx = 0.$$

This graph will be called the *limit spectral graph* of the operator $\mathfrak{D}$. Outside neighborhoods of the points $\pm i$, the limit spectral graph has the form shown in Fig. 1.

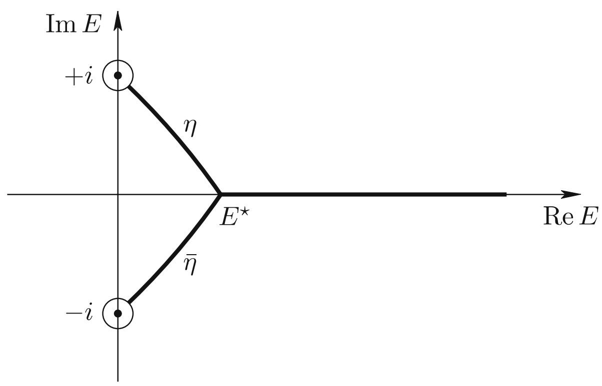

Fig. 1

In the present paper, we obtain the pseudospectrum of the operator $\mathfrak{D}$ and show that it is significantly larger than the spectrum.

3. GENERAL DEFINITIONS AND STATEMENT OF THE MAIN RESULT

Let a space of functions equipped with the inner product

$$\Phi = (\{\varphi(x)\}, (\,\cdot\,,\,\cdot\,))$$

be given. By $\|\cdot\|$ we denote the norm on this space generated by this inner product. In addition, we assume that an (unbounded) linear operator

$$A = A(x,\varepsilon)\colon \Phi_0 \to \Phi, \qquad \overline{\Phi_0} = \Phi,$$

depending on a parameter $\varepsilon \in (0,+\infty)$ is given.

Definition 1. A point λ belongs to the *ε-pseudospectrum* of the operator $A = A(x,\varepsilon)$ if and only if there exists a function $\varphi = \varphi(x,\varepsilon)$ belonging to the unit sphere of the space Φ_0 (i.e., $\|\varphi(\,\cdot\,,\varepsilon)\| = 1$) for any fixed $\varepsilon \in (0,+\infty)$ and having the property

$$\|A(x,\varepsilon)\varphi(x,\varepsilon) - \lambda\varphi(x,\varepsilon)\| = O(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0+0; \tag{3}$$

we denote the ε-pseudospectrum of the operator A by $\mathrm{PSP}_\varepsilon(A)$.

Remark 1. We note that the ε-pseudospectrum is independent of ε, but depends only on how the parameter ε is contained in the operator $A(x, \varepsilon)$.

Now we can formulate the main result of the present paper as follows.

Theorem 2. *For a nonconstant $V(x)$ and for any positive integer N, the h^N-pseudospectrum of the operator $-h^2\, d^2/dx^2 + iV(x)$ given on a circle is the half-strip* $[0, +\infty) + i[\min V, \max V]$.

Namely, the pseudospectrum has the form shown in Fig. 2.

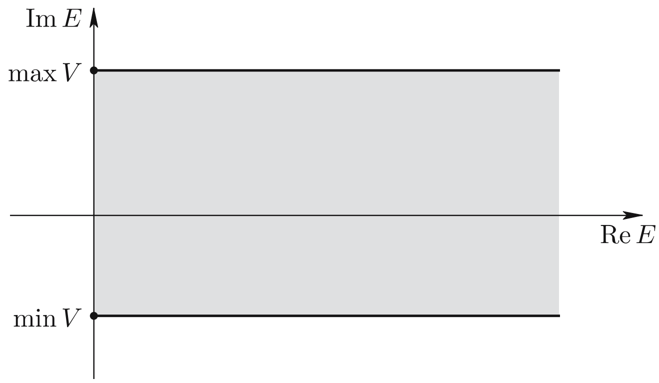

Fig. 2

Comparing this result with Theorem 1, we see how great is the difference between the spectrum and the pseudospectrum.

4. RELATIONSHIP BETWEEN THE PSEUDOSPECTRUM AND THE NUMERICAL IMAGE

Lemma 1. *The ε-pseudospectrum is closed.*

Proof. We consider an arbitrary point of the closure of the ε-pseudospectrum: $\lambda_0 \in \overline{\mathrm{PSP}_\varepsilon(A)}$. Then, by the definition of the closure, we have

$$(\forall \delta > 0)\,(\exists \lambda_\delta \in \mathrm{PSP}_\varepsilon(A))\colon |\lambda_0 - \lambda_\delta| < \delta.$$

This is also true for $\delta = \varepsilon$. Moreover, by the definition of the ε-pseudospectrum, we have

$$\exists \varphi(x, \varepsilon) \in \{\varphi(\,\cdot\,, \varepsilon) \in \Phi_0 \mid \|\varphi(\,\cdot\,, \varepsilon)\| = 1\}\colon \|A\varphi - \lambda_\delta \varphi\| = O(\varepsilon).$$

Choosing $\delta = \varepsilon$ and combining the above formulas, we obtain

$$\begin{aligned} 0 \le \|A\varphi - \lambda_0\varphi\| &= \|A\varphi - \lambda_\varepsilon\varphi + (\lambda_\varepsilon - \lambda_0)\varphi\| \\ &\le \|A\varphi - \lambda_\varepsilon\varphi\| + |\lambda_\varepsilon - \lambda_0| \cdot \|\varphi\| < O(\varepsilon) + \varepsilon \cdot 1 = O(\varepsilon). \end{aligned}$$

Hence we have $\|A\varphi - \lambda_0\varphi\| = O(\varepsilon)$; thus, λ_0 is a point of the ε-pseudospectrum. In other words, we have $\overline{\mathrm{PSP}_\varepsilon(A)} \subseteq \mathrm{PSP}_\varepsilon(A)$, which means that the ε-pseudospectrum is closed. $\square$

Definition 2. The *numerical image* of the operator A is defined to be the set

$$\{(A\psi, \psi) \mid \psi \in \Phi_0,\ \|\psi\| = 1\}.$$

Lemma 2. *The ε-pseudospectrum of the operator A is contained in the closure of its numerical image.*

Proof. Indeed, we assume that E is a point of the ε-pseudospectrum; then there exists a function ψ, $\|\psi\| = 1$, such that $A\psi = E\psi + O(\varepsilon)$. We scalarly multiply this relation by ψ and obtain

$$(A\psi, \psi) = (E\psi, \psi) + (O(\varepsilon), \psi).$$

Using the relations $(\psi, \psi) = \|\psi\| = 1$ and $(O(\varepsilon), \psi) = O(\varepsilon)$, we see that

$$(A\psi, \psi) = E + O(\varepsilon).$$

In other words, $E \in \{(A\psi, \psi) \mid \psi \in \Psi_0,\ \|\psi\| = 1\} + O(\varepsilon)$. We let ε tend to zero and obtain the desired result:

$$E \in \overline{\{(A\psi, \psi) \mid \psi \in \Psi_0,\ \|\psi\| = 1\}}. \quad \square$$

Corollary 1. *Suppose that $\varepsilon(h)$ is a continuous invertible function of the parameter h and $\varepsilon(0+0) = 0+0$. Then the $\varepsilon(h)$-pseudospectrum of the operator $\mathfrak{D}$ is contained in the half-strip $[0, +\infty) + i[\min V, \max V]$.*

Proof. In our case,

$$\Phi = L_2(\mathbb{S}^1), \qquad \Phi_0 = W_2^2(\mathbb{S}^1), \qquad A(x, \varepsilon) = D_{\mathbb{S}^1} = -h(\varepsilon)^2 \frac{d^2}{dx^2} + iV(x).$$

The inner product on the space $L_2(\mathbb{S}^1)$ is given by the formula

$$(u, v) = \int_{\mathbb{S}^1} u\bar{v}\, dx.$$

We write the expression $(D_{\mathbb{S}^1}\psi, \psi)$ in more detail:

$$(D_{\mathbb{S}^1}\psi, \psi) = (-h^2\psi'' + iV\psi, \psi) = -h^2(\psi'', \psi) + i(V\psi, \psi).$$

Taking into account the relation $(\psi'', \psi) = -(\psi', \psi')$ (since ψ is a periodic function), we obtain

$$(D_{\mathbb{S}^1}\psi, \psi) = h^2(\psi', \psi') + i(V\psi, \psi).$$

In the right-hand side, the first term is real and nonnegative, while the second term is imaginary and can easily be estimated (according to the mean-value theorem, since the function

$$\psi(x)\overline{\psi(x)} = |\psi(x)|^2 \geq 0$$

does not change sign) as follows:

$$\begin{aligned}(V\psi, \psi) &= \int_{\mathbb{S}^1} V(x)\psi(x)\overline{\psi(x)}\, dx = V(x)|_{x=x_0\in\mathbb{S}^1} \int_{\mathbb{S}^1} \psi(x)\overline{\psi(x)}\, dx \\ &\in \left[\min_{x\in\mathbb{S}^1} V(x), \max_{x\in\mathbb{S}^1} V(x)\right] \cdot \|\psi\| = [\min V, \max V].\end{aligned}$$

Thus, we see that

$$\{(D_{\mathbb{S}^1}\psi, \psi) \mid \psi \in W_2^2(\mathbb{S}^1),\ \|\psi\| = 1\} \subseteq [0, +\infty) + i[\min V, \max V],$$

and hence

$$\mathrm{PSP}_{h^N}\left(-h^2\frac{d^2}{dx^2} + iV(x)\right) \in \overline{[0, +\infty) + i[\min V, \max V]} = [0, +\infty) + i[\min V, \max V]. \quad \square$$

Remark 2. For an analytic $V(x) \not\equiv \text{const}$, the numerical image of the operator $\mathfrak{D}$ is the half-strip $[0, +\infty) + i(\min V, \max V)$. Indeed, in the estimate of the integral $(V\psi, \psi)$, the equality to $\min V$ is attained only for $V|_{\operatorname{supp}\psi} = \min V$. Since ψ is continuous and not equal to zero at a certain point, this means that V is constant on some interval. Then the analytic function V must be constant. Namely, the value of $\min V$ is not attained in the estimate of the integral $(V\psi, \psi)$ for $V \not\equiv \text{const}$. A similar argument remains valid for $\max V$. In this case, one can approach $\min V$ (and $\max V$) as closely as possible if for ψ one takes functions concentrated near the point $\arg\min V$ (respectively, near the point $\arg\max V$).

We denote

$$A_V = \{a \in \mathbb{R} \mid \exists x \in \mathbb{S}^1 \colon V(x) = a,\ V'(x) \neq 0\}.$$

The inclusion $A_V \subseteq (\min V, \max V)$ is obvious; moreover, the set $(\min V, \max V) \setminus A_V$ is finite. Indeed, if this set were infinite, the analytic function V' would have infinitely many zeros on the compact set $\mathbb{S}^1$. Then we would have

$$V' \equiv 0, \quad V = \text{const}, \qquad [\min V, \max V] = \{\,\cdot\,\},$$

but a single-point set cannot contain an infinite subset; thus, A_V is the interval $(\min V, \max V)$ with finitely many points deleted (or $A_V = \varnothing$ for $V = \text{const}$). The case of a constant V is trivial and has already been considered here. In what follows, we assume that $V \not\equiv \text{const}$.

In Fig. 3, the set $(0, +\infty) + iA_V$ is distinguished by gray color.

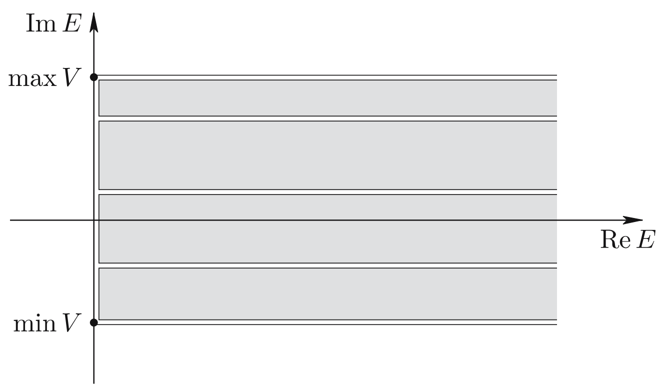

Fig. 3

5. EXPLICIT CONSTRUCTION OF ALMOST EIGENFUNCTIONS CORRESPONDING TO POINTS OF THE HALF-STRIP

For any positive integer N, we show that the h^N-pseudospectrum contains the set $(0, +\infty) + iA_V$, i.e., a set of finitely many half-strips whose closure (in the case of a nonconstant V) is the strip $[0, +\infty) + i[\min V, \max V]$.

Lemma 3. *For any $E \in (0, +\infty) + iA_V$ and for any positive integer N, there exists a function ψ_N whose norm $\|\psi_N\| = 1$ satisfies relation* (3) *with $\varepsilon = h^N$.*

Proof. We choose arbitrary $E \in (0, +\infty) + iA_V$ and $N \in \mathbb{N}$. We construct the function ψ_N as follows:

$$\psi_N(x) = e^{iS(x)/h} \sum_{\ell=0}^{N-2} h^\ell \chi_\ell(x). \tag{4}$$

We do not write the subscript N for the functions S and χ_ℓ in order to make the formulas less cumbersome; in addition, we note that h^0 is assumed to be equal to one. Substituting representation (4) into Eq. (3), we obtain the following condition:

$$\begin{aligned}((S')^2 + iV(x) - E)e^{iS(x)/h} \sum_{\ell=0}^{N-1} h^\ell \chi_\ell(x) - 2iS' e^{iS(x)/h} \sum_{\ell=0}^{N-1} h^{\ell+1} \chi'_\ell(x) \\ - iS'' e^{iS(x)/h} \sum_{\ell=0}^{N-1} h^{\ell+1} \chi_\ell(x) - e^{iS(x)/h} \sum_{\ell=0}^{N-1} h^{\ell+2} \chi''_\ell(x) = O(h^N).\end{aligned}$$

This condition can be written in powers of h; then we have the system

$$\begin{cases} ((S')^2 + iV(x) - E)e^{iS(x)/h}\chi_0 = O(h^N), \\ ((S')^2 + iV(x) - E)e^{iS(x)/h}\chi_1 - (2iS'\chi'_0 - iS''\chi_0)e^{iS(x)/h} = O(h^{N-1}), \\ ((S')^2 + iV(x) - E)e^{iS(x)/h}\chi_{\ell+1} - (2iS'\chi'_\ell - iS''\chi_\ell + \chi''_{\ell-1})e^{iS(x)/h} = O(h^{N-\ell-1}), \\ -(2iS'\chi'_{N-2} - iS''\chi_{N-2} + \chi''_{N-3})e^{iS(x)/h} = O(h), \\ \chi''_{N-2}e^{iS(x)/h} = O(1). \end{cases} \tag{5}$$

To satisfy system (5), it suffices to require (we must only show that there exists a solution, but we need not find all the solutions) that

$$\begin{cases} ((S')^2 + iV(x) - E)e^{iS(x)/h} = O(h^N), \\ \chi_\ell(x)e^{iS(x)/h} = O(1), \\ (-2iS'\chi'_0 - iS''\chi_0)e^{iS(x)/h} = O(h^{N-1}), \\ (-2iS'\chi'_\ell - iS''\chi_\ell)e^{iS(x)/h} = \chi''_{\ell-1}e^{iS(x)/h} + O(h^{N-\ell-1}), \\ \ell \in \{1, \dots, N-1\}. \end{cases} \tag{6}$$

To simplify the calculations, we additionally narrow down the class of admissible functions ψ_N; namely, we seek these functions among those asymptotically (as $h \to 0$) concentrated at a single point, i.e., among the functions such that

$$\psi_N(x) \xrightarrow{h \to 0} 0 \qquad \text{for} \quad x \neq x_0$$

for some $x_0 \in \mathbb{S}^1$. Then $e^{iS(x)/h}$ tends to zero as $h \to 0$ for $x \neq x_0$ faster than any power h^k, i.e.,

$$\operatorname{Im} S(\mathbb{S}^1 \setminus \{x_0\}) \subseteq \mathbb{R}_+, \qquad \operatorname{Im} S(x_0) = 0.$$

Thus, we obtain $\operatorname{Im}(S'(x_0)) = 0$ and, to satisfy the first condition in system (6), we choose x_0 and S so that

$$\begin{cases} \operatorname{Re} E = (S'(x_0))^2, \\ \operatorname{Im} E = V(x_0). \end{cases}$$

This means that, for a given E, we choose a point $x_0 \in \mathbb{S}^1$ at which V attains the value $\operatorname{Im} E$ (and at which $V'(x_0) \neq 0$; this fact will be used later). Such a point x_0 exists on the circle according to the definition of the set A_V and the fact that $\operatorname{Im} E \in A_V$. Moreover, we have also obtained a condition on the function S.

Now we construct the function S as a power series near the point x_0:

$$S(x) = \sum_{k \in \mathbb{Z}_+} S^{(k)}(x_0) \frac{(x - x_0)^k}{k!}.$$

Since the desired function $\psi_N(x)$ is determined by condition (3) up to a nonzero multiplicative factor, it is natural to choose $S(x_0)$ arbitrarily from $\mathbb{C}$ under the assumption that $e^{iS(x_0)/h} \neq 0$. To be definite, we set $S(x_0) = 0$. To obtain ψ_N with unit norm, we simply divide ψ_N by its norm at the end of the construction procedure. The condition on $S'(x_0)$ has already been obtained: $S'(x_0) = \pm\sqrt{\operatorname{Re} E}$.

We represent the function $V(x)$ as a power series near the point x_0 (this can be done, since $V(x)$ is an analytic function):

$$V(x) = \sum_{k\in\mathbb{Z}_+} V^{(k)}(x_0)\frac{(x-x_0)^k}{k!};$$

moreover, as was previously shown, $\operatorname{Im} E = V(x_0)$. Then the condition

$$(S'(x))^2 + iV(x) - E \equiv 0$$

can be written as an (infinite) system for $S^k(x_0)$, $k \in \mathbb{N}$:

$$\begin{cases} S^{(0)}(x_0) = 0, \\ S^{(1)}(x_0) = \pm\sqrt{\operatorname{Re} E}, \\ iV^{k-1}(x_0) + \displaystyle\sum_{\substack{j,m\in\mathbb{N}\\ j+m=k+1}} \frac{S^{(j)}(x_0)S^{(m)}(x_0)}{(j-1)!(m-1)!} = 0, \quad k \in \mathbb{N}\setminus\{1\}. \end{cases}$$

Although this system is infinite, we need only a finite part of this system. Indeed, to satisfy the first equation in system (6), we need not have the exact relation $(S')^2 + iV - E \equiv 0$, but it suffices to require that the following conditions be satisfied: $\operatorname{Re}(iS'(x_0)) = 0$ (this already holds according to the second equation in the system), $\operatorname{Re}(iS''(x_0)) < 0$ (this condition must be added to the system), and $(S')^2 + iV - E = O((x-x_0)^{2N})$. Let us show this:

$$\begin{aligned} e^{iS(x)/h}O((x-x_0)^{2N}) &= e^{\operatorname{Re}(iS(x)/h)}e^{i\operatorname{Im}(\dots)}O((x-x_0)^{2N}) \\ &= e^{\operatorname{Re}(iS''(x_0))(x-x_0)^2/h+O((x-x_0)^3)}O((x-x_0)^{2N}). \end{aligned} \tag{7}$$

Now the required property follows from the obvious estimate (in the norm of L_2)

$$e^{\lambda(x-x_0)^2/h}(x-x_0)^k = e^{\lambda(x-x_0\sqrt{h})^2}(x-x_0\sqrt{h})^k \cdot h^{k/2} = O(h^{k/2+1/4})$$

for $\operatorname{Re}\lambda < 0$. Thus, we have

$$e^{iS(x)/h}O((x-x_0)^{2N}) = O(h^N),$$

and hence, for the first relation in system (6) to be satisfied, it is required that

$$(S')^2 + iV - E = O((x-x_0)^{2N});$$

hence we need only the following finite part of the system written above for $S^{(k)}(x_0)$ (with a small addition in the form of an inequality):

$$\begin{cases} \operatorname{Re}(iS''(x_0)) < 0, \\ S^{(0)}(x_0) = 0, \\ S^{(1)}(x_0) = \pm\sqrt{\operatorname{Re} E}, \\ iV^{k-1}(x_0) + \displaystyle\sum_{\substack{j,m\in\mathbb{N}\\ j+m=k+1}} \frac{S^{(j)}(x_0)S^{(m)}(x_0)}{(j-1)!(m-1)!} = 0, \quad k \in \mathbb{N}\cap[2, 2N-1]. \end{cases} \tag{8}$$

The last equation in this system can be written as

$$\frac{2}{(k-1)!}S^{(1)}(x_0)S^{(k)}(x_0) = -iV^{(k-1)}(x_0) - \sum_{\substack{j,m\in\mathbb{N}\setminus\{1\}\\ j+m=k+1}} \frac{S^{(j)}(x_0)S^{(m)}(x_0)}{(j-1)!(m-1)!};$$

hence (since $S^{(1)}(x_0) = \pm\sqrt{\operatorname{Re} E} \neq 0$) this system allows us to find all $S^{(k)}$ uniquely (for a fixed sign of $S'(x_0)$); thus, we obtain S. It remains only to satisfy the additional requirement $\operatorname{Re}(iS''(x_0)) < 0$. From the last equation of the system for $k = 2$, we have $S''(x_0) = -iV'(x_0)/(2S'(x_0))$; hence the additional requirement can be satisfied by an appropriate choice of the sign of $S'(x_0)$ (which has been arbitrary until now) if only $V'(x_0) \neq 0$. Precisely for this, it was required to introduce the set A_V and to choose x_0 so as to satisfy the relation $V'(x_0) \neq 0$. Thus, we have determined the sign of $S'(x_0)$:

$$S'(x_0) = -\operatorname{sign}(V'(x_0))\sqrt{\operatorname{Re} E}.$$

Then, determining $S^{(k)}$ for $k < 2N$ from system (8) and setting $S^{(k)}(x_0) = 0$ for $k \geq 2N$, we uniquely construct the function S in the form of a polynomial in $(x - x_0)$. In this case, the first and second equations of system (6) are satisfied.

Now let us construct the function $\chi_\ell(x)$. For the third and fourth equations in system (6) to be satisfied, it suffices (by the construction of the function S) to require that

$$\begin{cases} -2iS'\chi_0' - iS''\chi_0 = 0, \\ -2iS'\chi_\ell' - iS''\chi_\ell = \chi_{\ell-1}'', \quad \ell \in \{1, \dots, N-1\} \end{cases}$$

in a neighborhood $U'(x_0)$ of the point x_0. Since we have $S'(x_0) \neq 0$ by construction, we can choose the neighborhood $U'(x_0)$ so that S' be bounded away from zero in this neighborhood. Then we obtain χ_0 from the relations:

$$2S'\chi_0' + S''\chi_0 = 0 \iff \chi_0 = \operatorname{const}(S')^{-1/2}.$$

The multiplicative constant in the formula for χ_0 can be an arbitrary nonzero number. To be definite, we assume that it is equal to one:

$$\chi_0 = (S')^{-1/2}.$$

Next, from the equation

$$2S'\chi_1' + S''\chi_1 = i\chi_0'' = i\frac{3S'' - 2S'S'''}{4(S')^{5/2}},$$

we obtain χ_1, and so on. Since S' is bounded away from zero on $U'(x_0)$, the equation

$$\chi_\ell' = \frac{-S''\chi_\ell + i\chi_{\ell-1}''}{2S'}$$

has a solution in the neighborhood $U'(x_0)$, and this solution is smooth.

The function ψ_N thus constructed has only one drawback; it is not periodic. But this drawback can easily be removed. We multiply this function by a smooth truncation function equal to one in a certain neighborhood of the point x_0 and to zero outside another neighborhood of this point. Now we can easily redefine ψ_N so that it becomes a periodic function; we simply replace it by the sum

$$\sum_{k\in\mathbb{Z}} \psi_N(x + kT)$$

(this sum is finite at each point x). It only remains to divide ψ_N by its nonzero (since ψ_N is continuous and $\psi_N \not\equiv 0$) norm in $L_2(\mathbb{S}^1)$. The proof is complete. $\square$

Theorem 3. *For a nonconstant $V(x)$ and for any positive integer N, the h^N-pseudospectrum of the operator $-h^2\,d^2/dx^2 + iV(x)$ given on the circle is the half-strip $[0, +\infty) + i[\min V, \max V]$.*

Proof. We have already noted that the closure of the set $(0, +\infty) + iA_V$ is the half-strip

$$[0, +\infty) + i[\min V, \max V]$$

(for a nonconstant V). Hence, to prove the theorem, it suffices to combine the results of Lemmas 1 and 3 and of Corollary 1. □

In [8], an exponential estimate was announced for the rate of growth of the resolvent of a similar operator in the half-strip of the numerical image (with boundary conditions on an interval instead of the periodicity condition).

BIBLIOGRAPHY

1. S. Yu. Dobrokhotov, V. N. Kolokoltsov, and V. Martinez Olive, "Quasimodes of the diffusion operator $-\varepsilon\Delta + v(x)\cdot\nabla$, corresponding to asymptotically stable limit cycles of the field v," *Sobretiro de Socieded Matematica Mexicana*, **11** (1994), 81–89.
2. S. Yu. Dobrokhotov, V. N. Kolokoltsov, and V. Martinez Olive, "Asymptotically stable invariant tori of the vector field $V(x)$ and quasimodes of the diffusion operator," *Mat. Zametki* [*Math. Notes*], **58** (1995), no. 2, 880–884.
3. S. A. Stepin, "Non-self-adjoint singular perturbations and the spectral properties of the Orr–Sommerfeld problem," *Mat. Sb.* [*Math. USSR-Sb.*], **188** (1997), 129–146.
4. A. A. Shkalikov, "On the limit behavior of the spectrum for large values of the parameter of a model problem," *Mat. Zametki* [*Math. Notes*], **62** (1997), no. 6, 950–953.
5. S. A. Stepin and A. A. Arzhanov, "On localization of the spectrum in a problem of the singular perturbation theory," *Uspekhi Mat. Nauk* [*Russian Math. Surveys*], **57** (2002), no. 3, 161–162.
6. S. N. Tumanov and A. A. Shkalikov, "On the limit behaviour of the spectrum of a model problem for the Orr–Sommerfeld equation with Poiseuille profile," *Izv. Ross. Akad. Nauk Ser. Mat.* [*Russian Acad. Sci. Izv. Math.*], **66** (2002), no. 4, 177–204.
7. S. V. Gal′tsev and A. I. Shafarevich, "Asymptotics of the discrete spectrum of a non-self-adjoint periodic operator," in: *Proceedings of the XXVIIth Conference of Young Reserchers*, Isd. Mekh.-Mat. Fak. Moskov. Gos. Univ., Moscow, 2005, pp. 18–22.
8. A. A. Schkalikov (A. A. Shkalikov), *Spectral portraits and the resolvent growth of a model problem associated with the Orr–Sommerfeld equation*, `http://arXiv.org/math. FA/0306342v1`, 2003.

S. V. Gal′tsev
M. V. Lomonosov Moscow State University
E-mail: `galtsev@mccme.ru`

A. I. Shafarevich
M. V. Lomonosov Moscow State University
E-mail: `shafar@mech.math.msu.su`