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A primer on eigenvalue problems of non-self-adjoint operators

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Abstract

Non-self adjoint operators describe problems in science and engineering that lack symmetry and unitarity. They have applications in convection–diffusion processes, quantum mechanics, fluid mechanics, optics, wave-guide theory, and other fields of physics. This paper reviews some important aspects of the eigenvalue problems of non-self-adjoint differential operators and discusses the spectral properties of various non-self-adjoint differential operators. Their eigenvalues can be computed for ground and perturbed states by their spectra and pseudospectra. This work also discusses the contemporary results on the finite number of eigenvalues of non-self-adjoint operators and the implications it brings in modeling physical problems.

Keywords Non-self-adjoint operators · Pseudospectra · Complex eigenvalues · Finite number of eigenvalues

1 Introduction

Non-self adjoint operators (NSA) typically constitute differential operators. They represent a wave-type behavior where the solutions—numerical or analytic, are described

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for various physical problems by exponential and harmonic functions, which often satisfy the requirement of being infinitely differentiable in a Hilbert space $\mathscr{H}$ [1–9]. Typical examples of non-self-adjoint problems encompass the study of complex and anharmonic resonances, optical potentials, orthogonal polynomials on the unit circle, stability problems in linear systems, and spectral theory of Toeplitz operators [10–12], which under many circumstances, results in solutions with complex eigenvalues (i.e., energy and momentum in nuclear physics).

The presence of complex eigenvalues requires either complex scaling to quantify the results [13–18], the development of extensions of the non-self-adjoint operators [19–23], or transformation of the differential equations at hand into solvable forms [24–29]. Older methods for treating complex eigenvalues, which are still used to date, are to solve NSA models by deriving the asymptotics of the real part of the large eigenvalues of the operators, which are close to the self-adjoint ones [30] and ignore the right-hand side (imaginary part) of the spectrum. The evaluation of the number of eigenvalues in the complex plane and the completeness criterion has also been proposed to study NSA models [31].

Earlier studies on NSA show that in many applications, one can also come across an NSA operator with a finite number of eigenvalues. The first works on this topic were done by Gasymov et al [32], who analyzed a differential operator with a finite number of negative eigenvalues by imposing appropriate conditions on the potential function and the space on which the operator is defined. The necessity of certain restrictions on the potential term in a differential operator was also later published by [33]. Following these developments, Tunca and Bairamov's published the first study [34], which conducted the analysis of NSA operators on the infinite interval, inversely to the previous studies—which dealt with the semi-infinite interval.

A more unique and novel way to analyse eigenvalues in the complex plane was developed in the early twentyfirst century by Demuth, who constructed an analytic function corresponding to the operator such that the zeros of this function and the eigenvalues of the operator coincided [35]. Demuth discussed in his work another approach based on an operator setting, which he developed into a further study for the estimation of eigenvalues of non-compact operators [36]. However, until recently, no one has published a formula for the number of eigenvalues on the upper or lower bounds of an operator. This was also addressed by Mutlu and Arpat, who are also working on this issue of the finite number of eigenvalues [37]. Interested readers can refer to [38, 39] for basic concepts like non-self-adjoint operators, spectrum, resolvent operator, pseudo-spectra, numerical range, and the key results related to these.

The structure of this paper is as follows. In Sect. 2, we introduce the mathematical background and formalities of the non-self-adjoint operators. In Sect. 3, we perform spectral calculations on non-self-adjoint operators relevant to optics, electromagnetics, and quantum physics. One important class of differential operators is those with a finite number of eigenvalues, i.e., operators with finite discrete spectra discussed in Sect. 4. Section 5 summarizes various aspects of eigenvalue problem non-self-adjoint operators.

2 Basic definitions

This section is devoted to the basic definitions that we will use throughout this work, starting from the definition of the non-self-adjoint operator. Consider a linear operator $L : \mathcal{D}(L) \subseteq \mathbb{V} \to \mathbb{V}$ defined on the inner product space $\mathbb{V}$ with inner product $\langle \cdot, \cdot \rangle$. An operator L is said to be *non-self-adjoint* if one of the following conditions does not hold

1. Operator L is formally self-adjoint (also known as symmetric), i.e., $\langle L\psi, \phi \rangle = \langle \psi, L\phi \rangle$, $\psi, \phi \in \mathcal{D}(L)$.
2. Domain and boundary conditions corresponding to the given operator and its adjoint operator are the same.

If first condition does not satisfy, then $\langle L\psi, \phi \rangle = \langle \psi, L^{*}\phi \rangle$, i.e., $L \neq L^{*}$, where asterisks $*$ stands for adjoint and the new operator $L^{*} : \mathcal{D}(L^{*}) \to \mathbb{V}$ is called adjoint of L,, i.e., if the operator L is not self-adjoint, we could additionally get a new operator, named *adjoint operator*, from the given information [40].

The non-self-adjointness of the problem might arise by the operator, by the boundary conditions involved, or by the non-linear dependency on a parameter involved in the problem, etc [41]. In this work, we focused only on linear differential operators.

The property of non-self adjointness yields the existence of non-normalizable and non-orthogonal solutions. Non-self-adjoint operators are often constructed from unbounded forms where the following condition of boundedness is not satisfied:

$$\|L\phi\| \leq c\,\|\phi\|\,, \tag{1}$$

where L is a non-self-adjoint operator acting on the function ϕ. Equation (1) implies that the operator has no upper or lower bounds in its given form. Even if some non-self-adjoint operators generate real spectra, they may still exhibit pathological behaviour [42].

For most operators, one can obtain a spectrum of the inherent eigenvalues, which satisfies certain properties and which we here introduce with the resolvent. Consider a complex norm space $\mathbb{V} \neq 0$ and let $L : \mathcal{D}(L) \subset \mathbb{V} \to \mathbb{V}$ be a linear operator. Define a new operator $L_\lambda = L - \lambda I$, where λ can be a complex number and I is the identity operator. If inverse of L_λ exists then the *resolvent operator* $R_\lambda(L)$ of L is defined as

$$R_\lambda(L) = L_\lambda^{-1} = (L - \lambda I)^{-1}.$$

A regular value of an operator L is a complex number for which the following holds

1. The resolvent operator $R_\lambda(L)$ exists.
2. The resolvent operator $R_\lambda(L)$ is bounded.
3. The set on which the resolvent operator is defined is dense in $\mathbb{V}$, i.e., closure of $R_\lambda(L)$ $\overline{R_\lambda(L)} = \mathbb{V}$.

and the set of the *regular values* called the resolvent set $\rho(L)$. The *spectrum* $\sigma(L)$ of an operator L is the complement of its resolvent set, i.e., $\sigma(L) = \mathbb{C}\backslash\rho(L)$. The spectrum of any operator is divided into three disjoint sets, namely point spectrum (discrete spectrum) $\sigma_p(L)$, continuous spectrum $\sigma_c(L)$, and residual spectrum $\sigma_r(L)$. Each is defined based on the properties of the set of regular values. For finite or infinite dimensional space $\sigma(L) = \sigma_p(L) \cup \sigma_c(L) \cup \sigma_r(L)$ [43].

If we know the complement of the operator's resolvent, we can get the operator's numerical range, pseudospectrum, sectorial, etc., to find the spectrum of any operator. Depending on the operator's form (such as sesquilinear or coercive), various other methods for locating the spectrum exist. All of this is interconnected and aids in identifying an operator's spectral range.

The *numerical range* of an operator L is defined as

$$W(L) = \{\langle L\psi, \psi\rangle : \psi \in \mathbb{V}, \|\psi\| = 1\}$$

and the numerical radius is $r(L) = \sup\{|\lambda| : \lambda \in W(L)\}$.

It was noticed that a modest perturbation in an operator might cause the spectrum of a non-self-adjoint operator to become more unstable. Studying the pseudo-spectrum, which is more stable than any operator's spectrum, is thus preferable. Put another way, the operator pseudospectrum is more significant than the spectrum. *Pseudo-spectrum* of any operator is the union of that operator's spectrum and the numbers that are almost eigenvalues of that operator. i.e. for any operator L defined on any Hilbert space $\mathbb{V}$, then

$$\sigma_\varepsilon(L) = \sigma(L) \cup \left\{z \in \rho(L) : \|zI - L\|^{-1} > \frac{1}{\varepsilon},\ \varepsilon > 0\right\}.$$

Alternatively, one can define the pseudo-spectrum as the set of all points where the resolvent norm $\|zI - L\|^{-1}$ is a large, or equivalently, set of the spectral values of a slightly perturbed operator. In other way, for any $\varepsilon' > \varepsilon$, z belongs to the spectrum of the new operator $L + B$ for some bounded operator B in $\mathbb{V}$ with norm less than or equal to ε'.

For further details about these basic concepts, please refer to [39, 44, 45].

3 Calculation of spectra of selected non-self-adjoint operators

The spectral theory helps to comprehend the eigenvalues and eigenfunctions of an operator. The spectrum of the self-adjoint operators is widely known, and its theory is well established [46, 47]. The eigenvalues (i.e., discrete spectrum) of a self-adjoint operator are always real. But this is not true for non-self-adjoint operators. The three conditions for non-selfadjointness of any eigenvalue problem are given by Wang [41].

Due to the lack of development of the general theory of non-self-adjoint operators and the uncertainty associated with each operator's behavior, such as the nature of the spectrum, eigenfunctions, etc, it becomes crucial to analyze each problem or example separately. We collected various important non-self-adjoint operators from the literature in one place, which helps the reader to understand the uncertainty in the behavior of such problems. Along with these illustrations, we will touch on various mathematical aspects related to the spectrum of non-self-adjoint operators. We shall now look at examples where we derive spectra and resolvent sets of selected operators, including theorems.

Example 1 (Convection–diffusion operator-I) The convection–diffusion operator given by (2) is an excellent example to get familiar with the behavior of the non-self-adjoint operators [48]. The nature of the eigenvalues of this operator depends on the operator's domain. This NSA operator is defined as follows:

$$L_n\psi(x) = -\psi_{xx}(x) - 2\psi_x(x), \qquad \psi \in \mathcal{L}^2[-n, n] \tag{2}$$

with the Dirichlet boundary conditions $\psi(\pm n) = 0$. For a finite domain $[-n, n]$, using the boundary conditions, the eigenvalues of the problem $-\psi_{xx}(x) - 2\psi_x(x) = \lambda\psi$, are obtained as

$$\lambda_m = 1 + \frac{\pi^2 m^2}{n^2}, \qquad \text{where } m = 1, 2, \ldots \tag{3}$$

This shows that the eigenvalues (i.e., discrete spectrum) of the NSA operator (2) is a subset of real numbers for all finite n, i.e., $\mathrm{Spec}(L_n)$ consist of real numbers. Now consider a case as $n \to \infty$, i.e., the case of the unbounded domain $(-\infty, \infty)$. From Eq. (3), it seems the resultant spectrum is real, but that is not the case. For $n \to \infty$, Eq. (2) transforms into

$$L_\infty\psi(x) = -\psi_{xx}(x) - 2\psi_x(x) \tag{4}$$

acting on $\mathcal{L}^2(-\infty, \infty)$. Applying the Fourier transformation on both sides of the equation (4), it can be shown that this limiting operator L_∞ has the complex eigenvalues, and its spectrum is the set of points in the complex plane which lie on the parabola $\{\alpha^2 + 2\imath\alpha : \alpha \in \mathbb{R}\}$.

Example 2 (Convection–diffusion operator-II) Now, we will interpret the spectrum of an operator in terms of the norm of the resolvent operator. Consider a similar example as Eq. (2) of a non-self-adjoint operator defined on the Hilbert space $\mathcal{L}^2[0, d]$ whose spectrum also depends on the interval [49]. The operator and boundary conditions are

$$Lu = u_{xx} + u_x, \qquad u(0) = u(d) = 0. \tag{5}$$

If $\lambda \in \sigma(L)$ (i.e. spectrum of L) then $\|(\lambda I - L)^{-1}\| = \infty$. For the operator defined in Eq. (5), spectrum is a subset of the negative real axis while the pseudo-spectrum of this operator is a parabola $\wedge$ in the left side of the complex plane as shown in Fig. 1.

In [49] it was shown that any point λ in the complex plane outside the parabola $\wedge$ does not belong to any of the pseudo-spectrum $\sigma_\varepsilon(L)$ with $\varepsilon < \mathrm{dist}(\lambda, \wedge)$ for any d or $\|(\lambda I - L)^{-1}\| \leq \frac{1}{\mathrm{dist}(\lambda,\wedge)}$. Here, $\mathrm{dist}(\lambda, \wedge)$ is the distance between λ and $\wedge$. For those points λ lies inside the parabola $\wedge$ will lie in the $\varepsilon-$pseudo-spectrum for a value of ε which decreases exponentially as function of d or $\|(\lambda I - L)^{-1}\|$ grows exponential as $d \to \infty$. It was also proved that

$$\sigma(L)[0, d] \not\to \sigma(L)[0, \infty) \text{ as } d \to \infty,$$

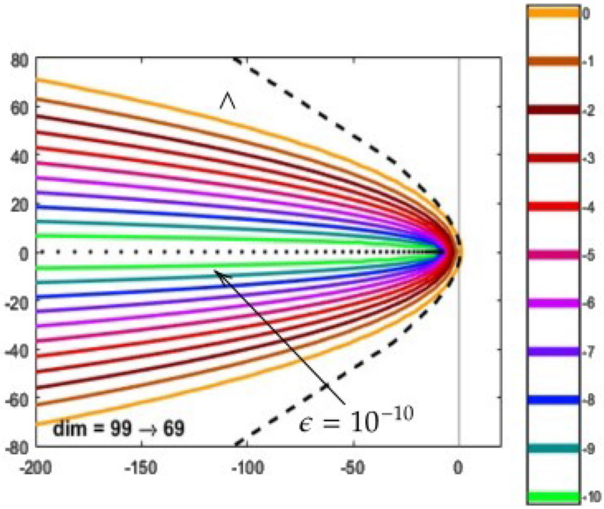

Fig. 1 Pseudo-spectrum of the convection–diffusion operator (5) for the interval $[0, d] = [0, 40]$, and number of Chebyshev points $N = 100$. Dots represents the eigenvalues of the operator, and color curves (i.e., parabolas) indicate the $\varepsilon-$pseudospectra with the different ε with base ten shown in right-hand side color strip [50]. Dashed lines indicate the numerical range or field of values of an operator (color figure online)

and

$$\sigma_\varepsilon(L)[0, d] \to \sigma_\varepsilon(L)[0, \infty) \text{ as } d \to \infty \text{ for every } \varepsilon > 0.$$

This example illustrates that the behavior of the pseudo-spectrum of the NSA operator does not alter if the interval is extended from a finite to an infinite interval, but this is not the case with the spectrum of the NSA operator.

Eigenvalues of the operator (5) are given by the following theorem:

Theorem 1 ([49]) *The spectrum of an operator L defined in Eq.* (5) *is a discrete set*

$$\sigma(L) = \left\{\lambda_n : \lambda_n = -\frac{1}{4} - \frac{\pi^2 n^2}{d^2}, n = 1, 2, 3, \ldots\right\}.$$

The eigenvalues are lying on the negative real axis in the interval $(-\infty, -\frac{1}{4})$ and the corresponding eigenfunctions are $u_n(x) = e^{-\frac{x}{2}} \sin\left(\frac{n\pi x}{d}\right), n = 1, 2, 3, \ldots$. For this operator, both the functions $e^{\alpha_+ x}$ and $e^{\alpha_- x}$ with $\alpha_\pm = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\lambda}$ are decreasing functions due to the choices of the λ for which both α_+ and α_- lie in the left half of the complex plane. Generally, it happens when the real part of the eigenvalues lies in set $S = \{\alpha \in \mathbb{C} : -1 \leq \Re(\alpha) \leq 0\}$. The corresponding image of this set in the λ plane is given by $\wedge = \{\lambda \in \mathbb{C} : \lambda = \alpha^2 + \alpha, -1 \leq \Re(\alpha) \leq 0\}$.

The critical parabola (dashed curve in Fig. 1) that bounds this set $\wedge$ is the image of the boundary of set S under the same function is

$$P = \{\lambda \in \mathbb{C} : \lambda = \alpha^2 + \alpha, \Re(\alpha) = 0\}.$$

The above result gives an idea about the spectrum of the operator L. The next result discusses the pseudo-spectrum of the operator L in Eq. (5) (Fig. 2).

Theorem 2 ([49]) *For each $\varepsilon \geq 0$, the $\varepsilon-$pseudo-spectrum of the operator L satisfies*

$$\sigma_\varepsilon(L) \subset \wedge + \delta_\varepsilon.$$

where $\delta_\varepsilon = \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon\}$ is a closed disk. Also, for $\lambda \notin \wedge$, the resolvent norm

$$\|(\lambda I - L)^{-1}\| \leq \frac{1}{dist(\lambda, \wedge)},$$

and numerical range $W(L)$ is subset of $\wedge$.

The above discussion was for the finite interval $[0, d]$. In the limiting case when $d \to \infty$ or for the infinite interval $[0, \infty)$,

$$L_\infty u = u_{xx} + u_x, \tag{6}$$
$$u(0) = \lim_{d\to\infty} u(d) = 0, \tag{7}$$

defined in $\mathcal{L}^2[0, \infty)$.

Theorem 3 ([49]) *The spectrum $\sigma_\infty(L_\infty)$ of an operator in Eq. (6) is the set P defined above and for $\varepsilon \geq 0$, the $\varepsilon-$pseudo-spectrum is*

$$\sigma_\varepsilon(L_\infty) = \wedge + \delta_\varepsilon.$$

In another way, if $\lambda \notin P$ the norm of the resolvent

$$\|(\lambda I - L_\infty)^{-1}\| = \frac{1}{dist(\lambda, \wedge)}.$$

Also, for the case where operator L_∞ defined on $(-\infty, \infty)$, then the above result is the same only set $\wedge$ change by set P.

Theorem 4 ([49]) *For each $\lambda \in \mathbb{C}$,*

$$\|(\lambda I - L)^{-1}\| \to \|(\lambda I - L_\infty)^{-1}\| \text{ as } d \to \infty.$$

The above examples highlight the behavior of the spectrum of an operator for changing the interval from finite to infinite.

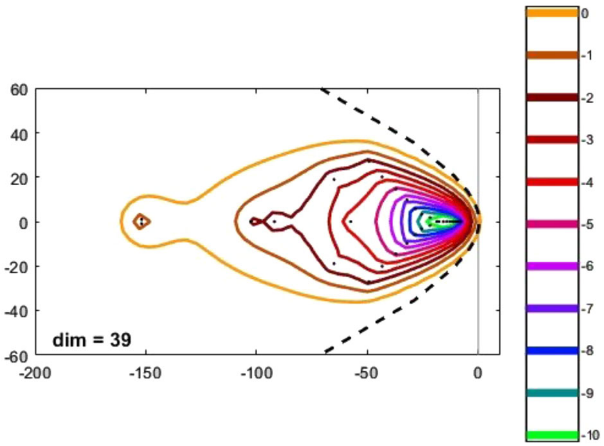

Fig. 2 Pseudo-spectrum of the convection–diffusion operator Eq. (5) including the complex eigenvalues for number of Chebyshev points $N = 40$. Dots represent the eigenvalues of the operator, and color curves indicate the $\varepsilon-$pseudospectra with the different ε with base ten shown in right-hand side color strip [50]. Dashed lines indicate the numerical range or field of values of an operator (color figure online)

Example 3 (Swanson hamiltonian model) Consider a non-self-adjoint operator defined as

$$L := -a\frac{d^2}{dx^2} + x^2 - \gamma\left(\frac{1}{2} + x\frac{d}{dx}\right), \quad \gamma \in \mathbb{R}\setminus\{0\}, \quad |\gamma| \le 1, \tag{8}$$

on $\mathcal{L}^2(\mathbb{R})$. This operator is self -adjoint if and only if $\gamma = 0$ [42], which then makes it a Schrödinger operator. This shows that the real spectrum results only when the parameter γ attains specific values. An interesting point is that if we take the modulus of γ sufficiently small, then the operator has a real spectrum despite being a non-self-adjoint operator. We can also look at this operator as the small perturbation of the operator

$$L = -a\frac{d^2}{dx^2} + x^2 - \left(\frac{1}{2} + x\frac{d}{dx}\right).$$

Interestingly, the potential term in the operator,

$$V = -\gamma\left(\frac{1}{2} + x\frac{d}{dx}\right) \tag{9}$$

includes γ as the real parameter, which acts as a measure of the degree of non-Hermiticity of the resulting Swanson model Hamiltonian Eq. (8).

Here the domain $\mathscr{D}(L)$ is:

$$\mathscr{D} := \{\Psi(x) \in W^{1,2}(\mathbb{R}) : x^2\Psi(x) \in \mathcal{L}^2(\mathbb{R})\}, \tag{10}$$

where $W^{1,2}(\mathbb{R})$ denotes the usual Sobolev space of functions on $\mathcal{L}^2(\mathbb{R})$ whose weak first and second derivatives belong to $\mathcal{L}^2(\mathbb{R})$. This domain contains the set $\mathscr{S}$ of functions f such that $e^{(\gamma x^2)} f(x) \in \mathcal{L}^2(\mathbb{R})$ [42]:

$$\tilde{\mathscr{S}} := \{\Psi(x) \in \mathscr{D} : e^{\gamma x^2}\Psi(x) \in \mathcal{L}^2(\mathbb{R})\} \tag{11}$$

which is dense and contains the set of all C^∞ functions with compact support [42]. Because the range of H, $\mathscr{R}(H)$ is closed, and V is relatively bounded with respect to $\mathscr{R}(H)$ with the relative bound smaller than 1, then H is closed. Hence, the non-self-adjoint Hamiltonian in (8) with extension V (9) yields a meaningful spectrum and discrete eigenvalues, provided that $|\gamma|$ is sufficiently small [42]. When $|\gamma|$ increases, the eigenvalues move further toward the right of the imaginary axis.

Example 4 (Airy operator) Consider the Airy operator

$$L_{0,h} = -h^2 \frac{d^2}{dx^2} + \imath x, \tag{12}$$

defined on the space $\mathcal{L}^2(-1, 1)$ with the Dirichlet's boundary conditions $\psi(\pm 1) = 0$, h is a real parameter and have the complex potential $\imath x$ [51]. If $h = 1$, then

$$L_{0,1} = -\frac{d^2}{dx^2} + \imath x, \tag{13}$$

and its adjoint operator is

$$L^*_{0,1} = -\frac{d^2}{dx^2} - \imath x,$$

which shows that $L_{0,1} \neq L^*_{0,1}$. Therefore, operator $L_{0,1}$ is non-self-adjoint operator. The operator $L_{0,1}$ has real eigenvalues whose limiting point is infinity. But, for the case where $h \to 0$, computer simulations (Matlab) show that eigenvalues are complex with positive real part [51], i.e., $z = \beta + \imath\alpha$, $\beta > 0$ and spectrum $\sigma(L_{0,1})$ is subset of union

$$\{z = \beta + \imath\alpha : \alpha = -\sqrt{3\beta} + 1, \text{ and } \alpha = \sqrt{3\beta} - 1, \beta \in (0, 5.77]\} \cup \left[\frac{1}{\sqrt{3}}, \infty\right).$$

Consider the perturbation for zeroth order derivative term in (12), and the resultant perturbed operator is

$$L_{\delta,h} = \begin{cases} -h^2 \frac{d^2}{dx^2} + \imath(x + \delta), \ x \leq 0 \\ -h^2 \frac{d^2}{dx^2} + \imath(x - \delta), \ x > 0 \end{cases} \tag{14}$$

where both $h > 0$ and $\delta > 0$. For fixed $\delta > 0$, when $h \to 0$, spectrum of this operator shows a bifurcation and decoupled at $x = 0$. This operator behave like a two

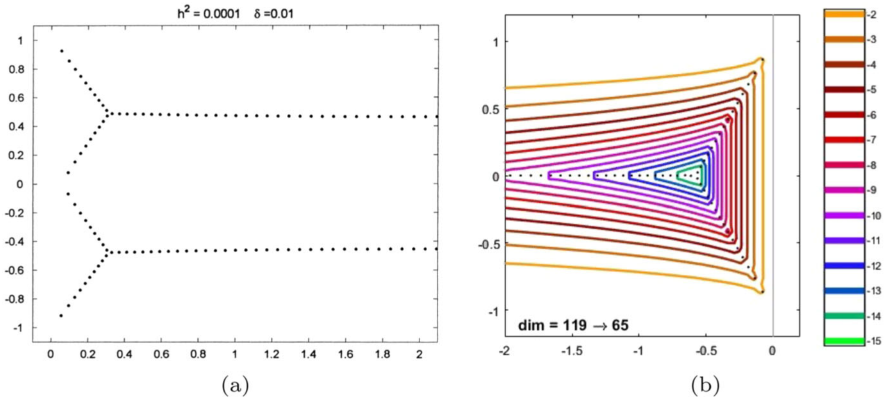

Fig. 3 Left-hand side figure shows the eigenvalues of the perturbed Airy operator in Eq. (14) with $h^2 = 0.0001$ and $\delta = 0.01$. Right-hand side figure shows the pseudo-spectrum of the Airy operator in Eq. (13) for the number of Chebyshev points $N = 120$. Dots represent the eigenvalues of the operator, and color curves indicate the $\varepsilon-$pseudospectra with the different ε with base ten shown in right-hand side color strip: [50] (color figure online)

operators one on $\mathcal{L}^2(-1, 0)$ and other is on $\mathcal{L}^2(0, 1)$ and spectrum $\sigma(L_{\delta,h})$ looks like two $Y-$shaped curves instead one $Y-$shaped curves shown in Fig. 3a.

Redparth [51] provided later an explanation of the origins of the double Y-shaped spectrum, which are attributed to the imaginary potential term in Eq. (14). Also, it was proved that both the limits on the parameters are different

$$\lim_{\delta\to 0}\lim_{h\to 0}\sigma(L_{\delta,h}) \neq \lim_{h\to 0}\lim_{\delta\to 0}\sigma(L_{\delta,h}).$$

The work we described above was based on the perturbation of a non-self-adjoint operator. In some cases, we may create a non-self-adjoint operator by perturbing the self-adjoint operator. Lenhoff's work is an excellent example of this kind of study, where weak perturbations of the self-adjoint operators are studied [52]. Using the eigenvalues of closely related self-adjoint operators as a starting point, researchers discovered an imbedding approach for computing the location of any non-self-adjoint operator's eigenvalues. Consider a compact operator T and its adjoint operator T^*. The *s−number* is defined as the eigenvalues of the compact operator $(T^*T)^{\frac{1}{2}}$. The operator T considered above is said to be of finite order if, for some finite real p, $\sum_{i=1}^{\infty} s_i^p < \infty$. Then, the infimum of these p is called the *order of an operator* T, i.e., $p(T)$.

Theorem 5 ([52]) *Let $L = P + Q$, where P is self-adjoint operator with a discrete spectrum, and Q is a $P-$completely continuous (i.e compact) operator such that*

$$p((P - \lambda_0 I)^{-1} Q (P - \lambda_0 I)^{-1}) < \infty,$$

where λ_0 is a regular value of P. Then the spectrum of L contains λ normal eigenvalues, i.e., isolated eigenvalues of finite multiplicity with a unique limit point at infinity. For *any* $\theta > 0$, *all of them, with the possible exception of a finite number, lie in the sectors*

$$-\theta < \arg\lambda < \theta, \ \textit{and}\ \pi - \theta < \arg\lambda < \pi + \theta.$$

The system of eigenvectors of the operator L is complete in Hilbert space $\mathcal{H}$.

Now define an operator $L_\varepsilon = P + \varepsilon Q$, which is the perturbation of the operator P with ε as a perturbation parameter. Therefore,

$$\text{If } \varepsilon = 0, L_\varepsilon = P, \text{ and if } \varepsilon = 1, L_\varepsilon = L.$$

For $\varepsilon = 0$, operator L_ε became operator P, which is the self-adjoint operator, and their eigenvalues can be located on the real axis. When ε varies from 0 to 1, we can find the location of the eigenvalues of the operator L_ε. Thus, by considering the eigenvalues of the operator L as the initial points, one can find the eigenvalues of the operator P.

Even relatively small perturbations in the mathematical setup might produce large-scale changes in the spectrum of an operator. One can ask, is it possible to design a perturbation that has no impact on the spectrum of an operator? In the next section, we discuss a continuous deformation operator constructed so that there is no impact on the spectrum of the operator due to the deformation. Valeev [53] investigated finding the location of the spectrum of the non-self-adjoint operator that connects the operator L by the continuous deformation as

$$L(s) = T(s)LT^{-1}(s).$$

Here L is a closed operator which is defined on the dense domain $\mathcal{D}(\mathcal{L})$ in separable Hilbert space H, and $T(s)$ is the family of the bounded operator which continuously depends on the real parameter such that inverse of $T(s)$ is bounded for every s.

Example 5 (Minimal differential operator) Now we discuss the behaviour of the non-self-adjoint operator resulting from the perturbation of the self-adjoint operator. In the following example, we will consider a non-self-adjoint operator and construct a self-adjoint operator from it. The relationship between the eigenvalues of the non-self-adjoint operator and the obtained self-adjoint operator will be discussed.

Consider the minimal differential operator defined on space $\mathcal{L}^2(0,\infty)$

$$L_0(b) = y_{xx} - bx^\alpha y, \qquad 0 \le x < \infty, \tag{15}$$

with boundary conditions $y(0) = 0$, where $\alpha > 0$, and $-\pi < \beta = \arg(b) < \pi$ [53]. The main cause behind the non-self-adjointness of the operator L_0 is the complex coefficient b. From this, one can also show the dependence of the eigenvalues on b. Also, the numerical range of the operator L_0 is a subset of the central sector S_b of an angle β, (= $\{z \in \mathbb{C} : \theta_1 \le \arg z \le \theta_2,$ where $\theta_2 - \theta_1 = \beta\}$) which lies either in the upper half or lower half of the complex plane. This gives an idea about the spectrum and regular values of the operator in the complex plane.

To discuss the connection between the non-self-adjoint operator's eigenvalues and the self-adjoint operator's eigenvalues, let us construct a self-adjoint operator corresponding to the minimal differential operator (15). To investigate the effect of the complex number b on the non-self-adjointness of the operator L_0 in Eq. (15), define a new operator by taking the modulus of b as

$$L_0(|b|) = y_{xx} - |b|x^\alpha y, \quad 0 \leq x < \infty. \tag{16}$$

The relation of the eigenvalues of the operators $L_0(b)$ and $L_0(|b|)$ is expressed in the next theorem.

Theorem 6 ([53]) *Let $\lambda_k(b)$ and $\lambda_k(|b|)$ be the eigenvalue of the operators $L_0(b)$ and $L_0(|b|)$ respectively. Then*

$$\lambda_k(b) = \exp\left(\frac{2\imath\beta}{2+\alpha}\right)\lambda_k(|b|). \tag{17}$$

Example 6 (Orr-Sommerfeld model operator) One more interesting operator in this direction is the Orr-Sommerfeld operator [54]

$$L = -\frac{1}{\imath\alpha R}\frac{d^2}{dx^2} + x.$$

The domain of operator L is $\mathcal{D}(L) = \{\psi \in \mathcal{L}^2[-1,1] : \psi$ has absolutely continuous first-order derivative, $\psi_{xx} \in \mathcal{L}^2[-1,1], \psi(\pm 1) = 0\}$. The corresponding eigenvalue problem is

$$L\psi = -\frac{1}{\imath\alpha R}\psi_{xx} + x\psi = \lambda\psi, \quad \psi \in \mathcal{L}^2[-1,1] \tag{18}$$

with the boundary conditions $\psi(\pm 1) = 0$. It is a non-normal operator whose eigenfunctions are non-orthogonal [54].

The operator L is closed and domain $\mathcal{D}(L)$ is dense in $\mathcal{L}^2[-1,1]$. It can be shown that spectrum $\sigma(L)$ is countable, $Y-$shaped consists of three branches, symmetric about the imaginary axis, and eigenvalues are highly sensitive at the intersection of the branches to the perturbations. For the case where $|\lambda|$ is very large, we can neglect the term x in Eq. (18), and their eigenvalues satisfy

$$\lambda_n \approx -\frac{\imath n^2\pi^2}{4R^2} \text{ as } n \to \infty.$$

For $\psi \in \mathcal{D}(L)$, one gets

$$\langle L\psi, \psi\rangle = -\frac{\imath}{R}\int_{-1}^{1}|\psi|^2 dx + \int_{-1}^{1} x|\psi|^2 dx.$$

The first term and second terms in the R.H.S. of the above equation are clearly imaginary and real, respectively. Also, the second term is bounded by 1 in absolute value for $\|\psi\| = 1$. Therefore, we can conclude that

Theorem 7 ([54]) *For all* $R > 0$, *the numerical range* $Num(L)$ *is subset of the set*

$$\{z : Im(z) \leq 0, |Re(z)| \leq 1\}.$$

Refer to [54] for discussions regarding finding its numerical range, $\varepsilon-$pseudo-spectra, and the spectrum of the operator. Using the transformation $\gamma^3 = \imath R$ and $w = \gamma(x - \lambda)$ for Eq. (18), one gets

$$\psi_{ww} - w\psi = 0, \tag{19}$$

which is an Airy equation whose solution can be written as the linear combination of two of the three Airy functions such as $Ai(w)$, $Ai(\beta w)$ and $Ai(\beta^2 w)$ where $\beta = e^{\imath \frac{2\pi}{3}}$. Similarly, to get familiar about the numerical range, $\varepsilon-$pseudo-spectra one can also read the work [49].

Example 7 (The harmonic oscillator) In work [55], it was proved that the pseudo-spectra of a non-self-adjoint harmonic and anharmonic oscillator operator is non-trivial. Consider the operator

$$L\psi = -\psi_{xx} + cx^2\psi, \quad \psi \in \mathcal{L}^2(-\infty, \infty), \tag{20}$$

where c is a complex number with positive real and imaginary parts. The operator L has a closed sectorial form with the domain-independent of c [55]. It was also proved that operator L is not normal by showing that there does not exist any other bounded invertible operator S such that SLS^{-1} is normal and the set of the eigenfunctions of the operator L does not form the Riesz basis.

Using an analytic continuation argument, the spectrum of operator L found depends on c

$$Spec(L) = \{c^{\frac{1}{2}}(2n - 1) : n = 1, 2, 3, \dots\}.$$

The resolvent operator $(L - zI)^{-1}$ is compact for all complex $z \in \mathbb{C} \setminus Spec(L)$. In terms of the resolvent bound, it was found that there does not exist any number a such that

$$\|(L - zI)^{-1}\| \leq a \,\mathrm{dist}(z, Spec(L))^{-1}, \quad \forall\, z \notin Spec(L)$$

hold.

Theorem 8 ([55]) *If* $0 < \theta < \arg c$, *then*

$$\lim_{r\to\infty} \|(L - re^{\imath\theta} I)^{-1}\| = \infty,$$

and if $\arg c < \theta < 2\pi$, *then*

$$\lim_{r\to\infty} \|(L - re^{\imath\theta} I)^{-1}\| = 0.$$

The above results also hold for one other non-self-adjoint operator

$$K\psi = -\psi_{xx} + cx^2\psi + V(x)\psi,$$

defined on $\mathcal{L}^2(-\infty, \infty)$ with the same complex number c as before and bounded complex potential function $V(x)$ satisfy

$$\lim_{|x|\to\infty} V(x) = 0.$$

The result discussed above can also be checked with the numerical simulation based on the Chebyshev spectral collocation method rather than the finite differences method. It was found that operator L is invariant w.r.t the reflection about the origin. Computations show that the above results are not always the case. For this, construct the associated matrix $A = \{A_{r,s}\}$ (corresponding to the operator L) given by

$$A_{r,s} = \begin{cases} -m^2, & \text{if } |r - s| = 1, \\ c\left(\frac{r}{m}\right)^2 + 2m^2, & \text{if } r = s, \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

where m is the number of the points uniformly distributed in the unit interval, and let's say b is the length of the interval. See [55] for the details of the numerical setup (Fig. 4).

The eigenvalues in the complex plane closest to the origin were plotted using the specific values $b = 16$, $c = 1 + 3\imath$, and $m = 20$. It was noticed that these eigenvalues lie along the line $\{z : \arg(z) = \frac{1}{2}\arg(c)\}$. The plots for the eigenvalues, pseudo-spectra, and eigenfunctions can see in [48, 55]. Additionally, it is found that even a perturbation of order $\varepsilon = 10^{-4}$ in the entries of A has an impact on the distribution of eigenvalues that are located at some distance from the origin.

Example 8 (Schrödinger's operator) The work in [55] discussed the Schrödinger's operator defined on $\mathcal{L}^2(-\infty, \infty)$ with the locally bounded potential function V which is asymptotically given by the complex multiple of the positive power of the variable x

$$L\psi(x) = -\psi_{xx} + V(x)\psi(x), \quad \forall\, \psi \in C_c^\infty(0, \infty), \tag{22}$$

where $C_c^\infty(0, \infty)$ is subset of $\mathcal{L}^2(-\infty, \infty)$ which contains the smooth functions with the compact support in $(0, \infty)$ and for some $n \in \mathbb{Z}^+$

$$\lim_{x\to\infty} (V(x) - cx^n) = 0.$$

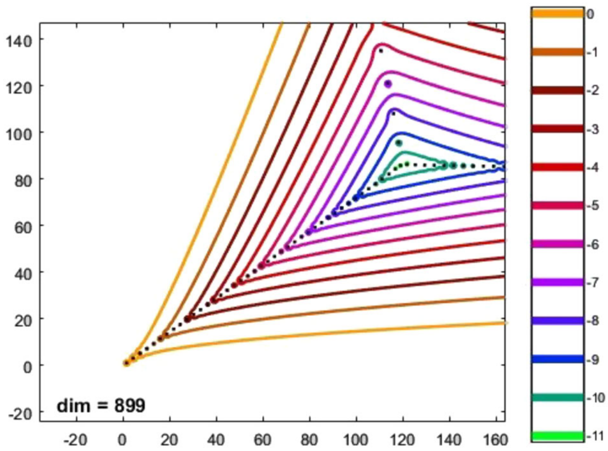

Fig. 4 Pseudo-spectrum of the harmonic oscillator operator Eq. (20) near origin for $c = 1 + 3\imath$, the interval $[-9, 9]$, and number of Chebyshev point $N = 900$. Dots represents the eigenvalues of the operator, and color curves indicate the $\varepsilon-$pseudospectra with the different ε with base ten shown in right-hand side color strip [50] (color figure online)

It was proved that

$$\lim_{|z|\to\infty} \|(L - zI)^{-1}\| = \infty,$$

inside the sector $S = \{z : 0 < \arg(z) < \arg(c)\}$.

Consider a non-negative self-adjoint operator defined in $\mathcal{L}^2(\mathbb{R})$

$$L\psi(x) = -\psi_{xx} + kx_+\psi(x) + \delta_0\psi(x), \tag{23}$$

where constant $k > 0$ and δ_0 is delta function at origin. For $c > 0$, by dilation transformation, the operator L is equivalent to the operator

$$L_c\psi(x) = -c^{-2}\psi_{xx} + kcx_+\psi(x) + c^{-1}\delta_0\psi(x), \tag{24}$$

where $x_+ = \max(x, 0)$. It is interesting to know that for some $c \in \mathbb{C}$ with $\mathrm{Re}(c) > 0$ and $\mathrm{Im}(c) > 0$, the operator L_c has the complex eigenvalues which are independent of c [55]. The real part $\mathrm{Re}(c)$ is close to the eigenvalues of the operator K given by

$$K\psi(x) = -\psi_{xx} + kx\psi(x), \quad \psi \in \mathcal{L}^2(-\infty, \infty),$$

with the Dirichlet's boundary conditions. With the same method as above for the operator $c^2 L_c$, we found that pseudo-spectra of the operator L_c is the sector $\{z : -2\arg(c) < \arg(z) < \arg(c)\}$, which means that under the small perturbation for the operator L, resonance of the operator L_c which are not close to the origin are unstable.

Example 9 (Bent optical waveguide operator) While discussing the non-self-adjoint operator (15), we mentioned that the non-self-adjointness of an operator depends on different factors involved in the governing equation. An example of the parameter dependency of the non-self-adjointness is the 1-D bent waveguide eigenvalue problem [56] given by

$$L\psi := \frac{d}{dr}\left(\frac{1}{k^2}\frac{r}{R}\frac{d\psi}{dr}\right) + n^2(r)\frac{r}{R}\psi = \frac{\gamma^2}{k^2}\frac{R}{r}\psi. \tag{25}$$

Here the eigenfunction ψ is the principal component of the mode (either a component of the electric field for transverse electric modes or a component of the magnetic field for transverse magnetic modes), $r \in [0, \infty)$ is the radial coordinate, $n(r)$ is a piecewise constant refractive index profile, k is the free space wave vector, R is a parameter representing the radius of the bent waveguide and γ is the propagation constant. Equation (25) is a weighted eigenvalue problem for the operator L, where $\psi \in \mathbb{V} = \{\psi : \psi, \frac{r}{R}\psi_r \in \mathcal{L}^2[0, \infty)\}$ is the eigenfunction corresponding to eigenvalue $\dfrac{\gamma^2}{k^2}$, and $\dfrac{R}{r}$ is the weight function. Here, one is interested in guided mode solutions of Eq. (25), which mathematically represent bounded functions in $\mathbb{V}$ with outgoing wave property.

The formal adjoint L^* of the operator L (in the standard weighted inner product defined on $\mathcal{L}^2[0, \infty)$) is given by

$$L^* = \frac{d}{dr}\left(\frac{1}{k^2}\frac{r}{R}\frac{d^2}{dr^2}\right) + n^2(r)\frac{r}{R} + \left(\frac{1}{rRk^2} - \frac{2}{Rk^2}\frac{d}{dr}\right). \tag{26}$$

As $L^* \neq L$, the operator L is not self-adjoint. Going further, it was shown that the bent waveguide eigenvalue problem for the guided modes defined by Eq. (25) with the appropriate boundary conditions ($|\psi| \to 0$ as $r \to 0$, $|\psi| \to 0$ as $r \to \infty$ and out-going wave condition) has complex eigenvalues [56].

In this example, the bend radius R acts as a parameter that influences the non-self-adjointness of the problem (25). To investigate the behaviour of L as $R \to \infty$, using the transformation $r = R\exp(x/R)$, the eigenvalue problem (25) was transformed to

$$\frac{1}{k^2}(\psi(x))_{xx} + N_R^2(x)\psi(x) = \frac{\gamma^2}{k^2}\psi(x), \quad -\infty < x < \infty, \tag{27}$$

where $N_R(x)$ denotes the transformed refractive index profile given by

$$N_R(x) = \begin{cases} n_s \exp\left(\frac{x}{R}\right) & -\infty < x < R\ln\left(\frac{R-d}{R}\right), \\ n_f \exp\left(\frac{x}{R}\right) & R\ln\left(\frac{R-d}{R}\right) \le x \le 0, \\ n_c \exp\left(\frac{x}{R}\right) & 0 < x < \infty. \end{cases} \tag{28}$$

Here, n_s, n_f, and n_c are the constant refractive index of the substrate, the core (interior medium), and the cladding medium, respectively. The constant d is the core width of the bent waveguide. As $R \to \infty$, (27) becomes

$$\frac{1}{k^2}(\psi_\infty)_{xx} + N_\infty^2(x)\psi_\infty(x) = \frac{\gamma_\infty^2}{k^2}\psi_\infty \quad -\infty < x < \infty, \tag{29}$$

where

$$\psi_\infty(x) = \lim_{R\to\infty} \psi(x) \tag{30}$$

and

$$N_\infty(x) := \lim_{R\to\infty} N_R(x) = \begin{cases} n_s & -\infty < x < -d, \\ n_f & -d \le x \le 0, \\ n_c & 0 < x < \infty. \end{cases}$$

This $N_\infty(x)$ represents the refractive index profile of the 1D straight waveguide with a core width d. Then one can interpret ψ_∞ as a mode of the corresponding straight waveguide with refractive index profile $N_\infty(x)$, and γ_∞ is the mode propagation constant of the mode of the corresponding straight waveguide. Denote it by β_s. Thus for the limiting case $R \to \infty$, the eigenvalue problem (27) becomes

$$L_\infty\psi_\infty := \frac{1}{k^2}(\psi_\infty)_{xx} + N_\infty^2(x)\psi_\infty = \frac{\beta_s^2}{k^2}\psi_\infty \quad -\infty < x < \infty$$

It has been shown in [57] that this is a self-adjoint eigenvalue problem with real eigenvalues $\frac{\beta_s^2}{k^2}$ with $\beta_s \in \mathbb{R}\backslash\{0\}$. Thus as $R \to \infty$, the non-self-adjoint eigenvalue problem (25) of bent waveguides changes into a self-adjoint eigenvalue problem. This implies β_s is real, which agrees with the theory of straight waveguides [58]. β_s represents the phase constant of a mode of the corresponding straight waveguide.

Example 10 (Bloch–Torrey operator) In 1−D case [59], the Bloch–Torrey operator, also called the complex Airy operator, is defined as

$$L_g\psi = -\psi_{xx}(x) + \imath g x\psi(x), \;\; g \in \mathbb{R}, \psi \in \mathcal{L}^2(\Omega).$$

The Bloch–Torrey operator is used in modeling electromagnetism, focusing on spin-bearing particles in nuclear magnetic resonance experiments. The eigenvalue problem $L_g\psi = \lambda\psi$ has different spectrum behavior depending on the Ω and the boundary conditions [59, 60]. Based on that, cases are divided as

1. If $\Omega = \mathbb{R}$, then the spectrum is empty.
2. If $\Omega = (a, b)$, $a, b \in \mathbb{R}$, and boundary condition are either Dirichlet, Neumann, or Robin, then the spectrum is discrete.
3. If $\Omega = \mathbb{R}^+$, and boundary condition are either Dirichlet, Neumann, or Robin, then the spectrum is discrete.

4. If $\Omega = \mathbb{R} \setminus \{0\}$, and transmission boundary condition at the origin, then the spectrum is discrete.

These four domains for the Bloch–Torrey operator have the given properties since the potential $igx\psi(x)$ in the Bloch–Torrey operator is not necessarily a periodic, unbounded, and changing sign. Therefore, based on the type of potential $\psi(x)$ we find in a Bloch–Torrey equation, we can obtain either of the four cases of the spectra mentioned above.

Example 11 (Schrödinger operator with a potential which is sum of harmonic oscillator and imaginary cubic oscillator) Define an operator $L : \mathcal{D}(L) \subset \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$

$$L\psi = -\psi_{xx} + x^2\psi + \imath x^3\psi, \tag{31}$$

with domain $\mathcal{D}(L) = \{\psi \in W^{2,2}(\mathbb{R}) : \psi, L\psi \in \mathcal{L}^2(\mathbb{R})\}$ [61]. This is a Schrödinger operator with a potential which is the sum of the harmonic oscillator and imaginary cubic oscillator. Operator L has compact resolvent due to which spectrum of L consists of discrete eigenvalues. Also, eigenvalues are positive real numbers and are algebraically simple. The set of the eigenfunctions of L form a complete set in Hilbert space $\mathcal{L}^2(\mathbb{R})$, even if they do not form a Schauder or countable basis. In case of the pseudospectra of this operator L, for any $\delta > 0$, there exists constants $c_1, c_2 > 0$ such that

$$\left\{\lambda \in \mathbb{C} : |\lambda| > c_1, |\arg \lambda| < \frac{\pi}{2} - \delta, |\lambda| \geq c_2 \left(\log \frac{1}{\varepsilon}\right)^{\frac{6}{5}}\right\} \subset \sigma_\varepsilon(L).$$

The plots for the pseudospectra of the above operator can be seen in Fig. 5. For the proof of the aforementioned results for this particular example, we refer the reader to check [61], which was discussed based on scaling techniques, and the operator is treated using semi-classical methods. For all the operators corresponding to the odd power of the imaginary oscillator, a reader can check [62].

All the non-self-adjoint operator examples presented above illustrate how the behaviour of the spectrum could change depending on the parameters present in the problems and demonstrates how difficult it is to deal with non-self-adjoint operators. Moreover, it is challenging to make predictions without analysing the operator.

4 Non-self-adjoint operators with finite discrete spectra

The non-self-adjoint operators discussed so far have infinitely many eigenvalues. NSA operators with a finite number of eigenvalues constitute a very important class. Typically, these are the operators defined on an unbounded domain. However, here we discuss NSA operators with a finite number of eigenvalues. Naimark was one of the first researchers to investigate conditions under which a Schrödinger differential operator has the finite number of eigenvalues as mentioned in [32] where the operator

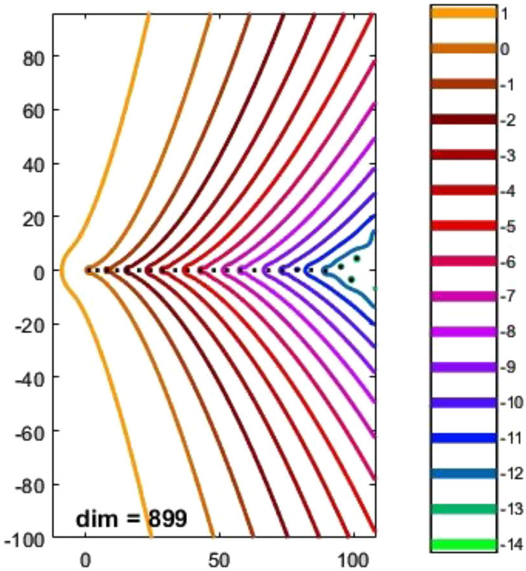

Fig. 5 Pseudo-spectrum of the imaginary cubic oscillator operator in Eq. (31) with $N = 900$ number of Chebyshev points. Dots represent the eigenvalues of the operator, and color curves indicate the ε−pseudospectra with the different ε with base ten shown in the right-hand side color strip. For more details about the plot, see [61, 63]. Plots were made using [50] (color figure online)

considered was

$$L(y) = -y_{xx} + q(x)y = \lambda y, \qquad 0 \leq x < \infty, \tag{32}$$

with the boundary condition $y(0) = 0$, and functions $y(x) \in \mathcal{L}^2(0, \infty)$. The condition for the requirement of a finite number of eigenvalues of the above operator was related to the potential function $q(x)$ as

$$\int_0^\infty |q(x)|e^{\varepsilon x} < \infty, \qquad \varepsilon > 0.$$

As per the above constraint, the operator L has a finite number of eigenvalues, and each has a finite multiplicity. In another study, Naimark proposed an alternative condition that ensures

$$\sup_x |q(x)|e^{\varepsilon x} < \infty, \tag{33}$$

is sufficient for the finite number of eigenvalues of the operator L. After that, Levin modified this condition [33] and introduced a new condition for a finite number of eigenvalues of the operator L as

$$|q(x)| < e^{-\gamma(x)},$$

where the function $\gamma(x)$ satisfy $x\gamma_x(x) \to \infty$ as $x \to \infty$ and $\int_0^\infty \gamma(x)x^{-2}dx = \infty$.

Levin then investigated the discreteness and finiteness of the set of negative eigenvalues for Schrödinger's equation [32]. The operator was same as in Eq. (32) however with a modified boundary condition

$$y_x(0) - hy(0) = 0, \tag{34}$$

where y belong to Hilbert space $\mathscr{H}$ and the potential function $q(x)$ is strongly measurable operator function in $\mathscr{H}$. In the boundary condition, h is a fixed, completely continuous operator.

If the potential function $q(x)$ and h in the boundary condition Eq. (34) are real, then the condition for the finite number of eigenvalues on the operator in Eq. (32) is $\int_0^\infty x|q(x)|dx < \infty$. But when either $q(x)$ or h are not self-conjugate, leading to non-self-adjointness of the operator L, then this condition is not sufficient for the finite number of eigenvalues. For this case, the main result of Levin on the discreteness of the spectrum for the above operator is given by the following theorem:

Theorem 9 (Discreteness of the negative eigenvalues [32]) *If*

1. *For almost all x, the operator function $q(x)$ is completely continuous in $\mathscr{H}$,*
2. *the operator h is completely continuous,*
3. $\int_0^\infty \|q(x)\|^2\, dx < \infty$,

then the negative spectrum of the operator L in Eq. (32) *with the condition Eq.* (34) *is discrete.*

The next result in the same work discussed the condition on the finiteness of the number of the negative eigenvalues for the operator L.

Theorem 10 (Finiteness of the number of negative eigenvalues [32]) *The operator L defined in Eq.* (32) *with the boundary condition $y(0) = 0$ has finite negative spectrum if the potential function satisfies $\int_0^\infty (1 + x)\, \|q(x)\|\, dx < \infty$.*

Neither the Theorem 9 nor the Theorem 10 provides any information regarding the whole spectrum. The potential function $q(x)$ must be of a specified sort that meets some constraint for these findings to be valid.

Pavlov [33] improved some of the limitations of the aforementioned findings. This study and the references therein are significant in this approach. For the operator and boundary condition in Eqs. (32) and (34), he focused on the non-self-adjointness situation. To accommodate both finite and infinite numbers of eigenvalues of the operator L, the condition in Eq. (33) was modified. The new common condition introduced corresponding to operator L was

$$\sup_{0\le x<\infty} |q(x)|e^{\varepsilon x^\beta} < \infty.$$

In the above condition, the number of eigenvalues will vary depending on the values of real β. If $\beta \in (0, \frac{1}{2})$, then the number of the eigenvalues will be infinite, and if $\beta \ge \frac{1}{2}$, then the number of the eigenvalues will be finite for the operator.

The aforementioned findings have improved somewhat, although certain questions were not addressed, e.g., the above analysis was carried out for the operator was defined on the semi-infinite interval $[0, \infty)$. It is equally interesting to know how such analysis is carried over for the operator defined on the infinite interval.

This difficulty of the infinite interval was taken up by Tunca and Bairamov [34]. In their work, the operator considered was the same as in Eq. (32) on the infinite interval:

$$L(y) = -y_{xx} + q(x)y = \lambda y, \qquad -\infty < x < \infty, \tag{35}$$

where $y \in \mathcal{L}^2(-\infty, \infty)$. The main condition for the finite number of eigenvalues of the operator L in the above equation was

$$\sup_{-\infty<x<\infty} |q(x)|e^{\varepsilon\sqrt{|x|}} < \infty, \ \varepsilon > 0. \tag{36}$$

After a decade gap, Demuth published a paper in 2013, comparing two approaches to the distribution of eigenvalues of the closed operators, with a specific emphasis on the non-self-adjoint operator [35]. The non-self-adjoint operators considered in this work were the perturbations of the self-adjoint operators. The first approach was based on the complex analysis concept of holomorphic function. In this approach, a holomorphic function was constructed for the linear operator. The zeros of this holomorphic function coincide with the eigenvalues of the linear operator. The other approach discussed in this work is the operator theoretic method, based on the idea of the numerical range. However, no conclusion was reached on this study's certainty of the finite number of eigenvalues of the operators.

Using the number of eigenvalues estimated for the linear operator on Banach space, Demuth produced his next study [36]. In this study, non-compact operators on the finite-dimensional space or of finite rank were examined. If the operator is specified on an infinite dimensional space, this does not fulfill the prerequisites for the condition. Using the ideas of the zeros of an analytic function, the approach remained the same.

Frank and Safronov followed a similar approach for the Schrödinger operator defined on odd-dimensional space [64]. They identified a bound on the number of eigenvalues in the scenario when the potential function decays exponentially to infinity. Among the most important findings in this study were the limits on the number of eigenvalues.

Theorem 11 ([64]) *Consider the operator L defined in Eq.* (32) *with the Dirichlet boundary condition at zero. Let N be the number of eigenvalues of the operator L counting with the algebraic multiplicities. Then N satisfies the following inequality*

$$N \leq \frac{1}{\varepsilon^2}\left(\int_0^\infty e^{\varepsilon x}|q(x)|dx\right)^2, \ \varepsilon > 0. \tag{37}$$

This theorem was proved using trace formulas, in which we build an analytic function whose zeros coincide with the eigenvalues of the corresponding operator. The upper bound on the number of eigenvalues N was found by the bound of the zeros of the analytic function. This is the first condition in the literature that provides the upper bound on the number of eigenvalues for the operator in Eq. (37).

The most contemporary work in this area that we found was published by Mutlu and Arpat, where the operator considered was the same form as in Eq. (35) [37]. Additionally, it was assumed that the potential function $q(x)$ is non-self-adjoint and completely continuous in a separable Hilbert space $\mathscr{H}$ for each x. An operator-specific Jost function was constructed corresponding to the operator in the same manner as the complex analysis approach. Based on the properties of the Jost function, the discrete spectrum and the finite number of the eigenvalues of the operator were examined.

5 Conclusion

While the spectral theory of self-adjoint operators is a well-established domain, the same can not be said for the spectral theory of non-self-adjoint operators. One of the principal objectives of this review article is to present to the readers the multifaceted world of non-self-adjoint operators and give a glimpse of ongoing research ideas in the spectral analysis of non-self-adjoint operators.

As we have seen in this review article, there are non-self-adjoint operators that have real eigenvalues on the finite domain, but these eigenvalues become complex on the infinite domain. Quite often, some parametric perturbations of self-adjoint operators produce non-self-adjoint operators having complex eigenvalues. Due to the sensitivity of eigenvalues towards perturbation, the pseudo-spectrum of non-self-adjoint operators is generally investigated. Eigtool is a powerful aid in this analysis. There are many physical phenomena that lead to non-self-adjoint models with a finite number of eigenvalues. This work explored several important contemporary results in this direction.

To get further insights, researchers explore the construction of projections and the development of spectral expansions of non-self-adjoint operators. We will take up these advanced topics in the future. While no review can be possibly exhaustive, we hope that this timely primer on eigenvalue problems of non-self-adjoint operators will attract fresh minds to this field, and thereby paving the way to address various challenges in this field.

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Declarations

Conflict of interest The authors declare that they do not have any conflict of interest.

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