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On the spectrum of second-order differential operators with complex coefficients

Dedicated to the memory of Professor Dr Friedrich Goerisch

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The main objective of this paper is to extend the pioneering work of Sims on second-order linear differential equations with a complex coefficient, in which he obtains an analogue of the Titchmarsh–Weyl theory and classification. The generalization considered exposes interesting features not visible in the special case in Sims's paper from 1957. An m-function is constructed (which is either unique or a point on a 'limit-circle'), and the relationship between its properties and the spectrum of underlying m-accretive differential operators analysed. The paper is a contribution to the study of non-self-adjoint operators; in general, the spectral theory of such operators is rather fragmentary, and further study is being driven by important physical applications, to hydrodynamics, electromagnetic theory and nuclear physics, for instance.

Keywords: Sturm–Liouville problems; spectral theory; complex theory; Titchmarsh–Weyl–Sims theory

1. Introduction

Sims (1957) obtained an extension of the Weyl limit-point, limit-circle classification for the differential equation

$$M[y] = -y'' + qy = \lambda y, \quad \lambda \in \boldsymbol{C}, \tag{1.1}$$

on an interval $[a, b)$, where q is complex valued, and the end-points a, b are, respectively, regular and singular. Under the assumption that $\operatorname{Im} q(x) \leqslant 0$ for all $x \in [a, b)$, Sims proved that for $\lambda \in \boldsymbol{C}_+$, there exists at least one solution of (1.1) that lies in the weighted space $L^2(a, b; \operatorname{Im}[\lambda - q]\,\mathrm{d}x)$; such a solution lies in $L^2(a, b)$. There are now three distinct possibilities for $\lambda \in \boldsymbol{C}_+$:

1. there is, up to constant multiples, precisely one solution of (1.1) in
$$L^2(a, b; \operatorname{Im}[\lambda - q]\,\mathrm{d}x)$$
and $L^2(a, b)$;
2. one solution in $L^2(a, b; \operatorname{Im}[\lambda - q]\,\mathrm{d}x)$, but all in $L^2(a, b)$; and

3. all in $L^2(a,b;\mathrm{Im}[\lambda-q]\,\mathrm{d}x)$.

This classification is independent of $\lambda \in \boldsymbol{C}_+$ and, indeed, if all solutions of (1.1) are in $L^2(a,b;\mathrm{Im}(\lambda-q)\,\mathrm{d}x)$ or in $L^2(a,b)$, for some λ, it remains so for all $\lambda \in \boldsymbol{C}$. At the core of the analsis of Sims (1957) is an analogue for (1.1) of the Titchmarsh–Weyl m-function, whose properties determine the self-adjoint realizations of $-(\mathrm{d}^2/\,\mathrm{d}x^2)+q$ in $L^2(0,\infty)$ when q is real and appropriate boundary conditions are prescribed at a and b. Sims made a thorough study of the 'appropriate' boundary conditions and the spectral properties of the resulting operators in the case of complex q. The extension of the theory for an interval (a,b) where both end-points are singular follows in a standard way.

We have two objectives in this paper. The first is to construct an analogue of the Sims theory to the equation

$$-(py')' + qy = \lambda wy, \tag{1.2}$$

where p and q are both complex valued, and w is a positive weight function. This is not simply a straightforward generalization of Sims (1957), for the general problem exposes problems and properties of (1.2) that are hidden in the special case considered by Sims; some of these features may also be seen in Birger & Kalyabin (1976) where a system of the form (1.2) with $p=\omega=1$ is considered (see remark 2.5 below). Second, once we have our analogue of the Titchmarsh–Weyl–Sims m-function, we are (like Sims) in a position to define natural quasi-m-accretive operators generated by

$$-\left(\frac{1}{w}\right)\left\{\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\right)+q\right\}$$

in $L^2(a,b;w\,\mathrm{d}x)$ and to investigate their spectral properties; these, of course, depend on the analogue of the three cases of Sims. Our concern, in particular, is to relate these spectral properties to those of the m-function, in a way reminiscent of that achieved for the case of real p,q by Chaudhuri & Everitt (1968). We establish the correspondence between the eigenvalues and poles of the m-function, but, unlike in the self-adjoint case considered in Chaudhuri & Everitt (1968), there is, in general, a part of the spectrum that is inaccessible from the subset of $\boldsymbol{C}$ in which the m-function is initially defined and its properties determined. However, even within this region we are able to define an m-function (definition 4.10).

2. The limit-point, limit-circle theory

Let

$$M[y] = (1/w)[-(py')'+qy], \quad \text{on } [a,b), \tag{2.1}$$

where

(i) $w>0$, $p\neq 0$ a.e. on $[a,b)$ and $w, 1/p \in L^1_{\mathrm{loc}}[a,b)$;

(ii) p,q are complex valued, $q\in L^1_{\mathrm{loc}}[a.b)$ and

$$Q = \overline{\mathrm{co}}\left\{\frac{q(x)}{w(x)} + rp(x):\ x\in[a,b),\ 0<r<\infty\right\} \neq \boldsymbol{C}, \tag{2.2}$$

where $\overline{\mathrm{co}}$ denotes the closed convex hull.

The assumptions on w, p, q ensure that a is a regular end-point of the equation $M[y] = \lambda wy$. We have in mind that b is a singular end-point, i.e. at least one of $b = \infty$ or

$$\int_a^b (w + (1/|p|) + |q|)\,\mathrm{d}x = \infty$$

holds; however, the case of regular b is included in the analysis. The conditions (i) and (ii) will be assumed hereafter without further mention.

The complement in $\boldsymbol{C}$ of the closed convex set Q has one or two connected components. For $\lambda_0 \in \boldsymbol{C}\backslash Q$, denote by $K = K(\lambda_0)$ its (unique) nearest point in Q and denote by $L = L(\lambda_0)$ the tangent to Q at K if it exists (which it does for almost all points on the boundary of Q), and otherwise any line touching Q at K. Then if the complex plane is subjected to a translation $z \mapsto z - K$ and a rotation through an appropriate angle $\eta = \eta(\lambda_0) \in (-\pi, \pi]$, the image of L coincides with the imaginary axis and the images of λ_0 and Q lie in the new negative and non-negative half-planes, respectively: in other words, for all $x \in [a, b)$ and $r \in (0, \infty)$,

$$\mathrm{Re}\left[\left\{rp(x) + \frac{q(x)}{w(x)} - K\right\}\mathrm{e}^{\mathrm{i}\eta}\right] \geqslant 0 \tag{2.3}$$

and

$$\mathrm{Re}[(\lambda_0 - K)\mathrm{e}^{\mathrm{i}\eta}] < 0.$$

For such *admissible* K, η (corresponding to some $\lambda_0 \in \boldsymbol{C}\backslash Q$), define the half-plane

$$\Lambda_{\eta,K} := \{\lambda \in \boldsymbol{C} : \ \mathrm{Re}[(\lambda - K)\mathrm{e}^{\mathrm{i}\eta}] < 0\}. \tag{2.4}$$

Note that for all $\lambda \in \Lambda_{\eta,K}$,

$$\mathrm{Re}[(\lambda - K)\mathrm{e}^{\mathrm{i}\eta}] = -\delta < 0, \tag{2.5}$$

where $\delta = \delta_{\eta,K}(\lambda)$ is the distance from λ to the boundary $\partial\Lambda_{\eta,K}$. Also $\boldsymbol{C}\backslash Q$ is the union of the half-planes $\Lambda_{\eta,K}$ over the set S of admissible values of η and K.

We shall initially establish the analogue of the Sims–Titchmarsh–Weyl theory on the half-planes $\Lambda_{\eta,K}$, but subject to the condition

$$\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}\cos\alpha\overline{\sin\alpha}] \leqslant 0, \tag{2.6}$$

for some fixed $\alpha \in \boldsymbol{C}$: the parameter α appears in the boundary condition at a satisfied by functions in the domain of the underlying operator (see § 4). Denote by $S(\alpha)$ the set $\{(\eta, K) \in S$: (2.6) is satisfied$\}$. We assume throughout that

$$Q(\alpha) := \boldsymbol{C}\backslash \cup_{S(\alpha)} \Lambda_{\eta,K} = \cap_{S(\alpha)}(\boldsymbol{C}\backslash\Lambda_{\eta,K}) \neq \emptyset. \tag{2.7}$$

The set $Q(\alpha)$ is clearly closed and convex, and $Q(\alpha) \supseteq Q$ in general: for the important special cases $\alpha = 0, \pi/2$, corresponding to the Dirichlet and Neumann problems, $Q(\alpha) = Q$. Sims (1957) assumes that $p = w = 1$ and the values of q lie in $\boldsymbol{C}_-$; thus $\eta = \pi/2$, $K = \sup_{[a,b)}[\mathrm{Im}\,q(x)]$, are admissible values, and $(\eta, K) \in S(\alpha)$ if

$$-\mathrm{Im}[\cos\alpha\overline{\sin\alpha}] = \sinh[2\,\mathrm{Im}\,\alpha] \leqslant 0,$$

the assumption made by Sims. If α is real, then (2.6) requires $|\eta| \leqslant \pi/2$ if $\alpha \in [\pi/2, \pi]$, and $|\eta| \geqslant \pi/2$ if $\alpha \in [0, \pi/2]$.

We shall prove below that the spectrum of the differential operators defined in a natural way by the problems considered lie in the set $Q(\alpha)$. This and related results can be interpreted as implying a restriction on the range of values of boundary-condition parameter, α, permitted: if α satisfies (2.6) for all η, which are such that $(\eta, K) \in S$ for some $K \in \boldsymbol{C}$, then $Q(\alpha) = Q$. However, if $\alpha \in \boldsymbol{C}$ is given, it is the set $Q(\alpha)$ and not Q, which plays the central role in general.

Let θ, ϕ be the solutions of (1.2) that satisfy

$$\left.\begin{aligned} \phi(a, \lambda) &= \sin\alpha, & \theta(a, \lambda) &= \cos\alpha, \\ p\phi'(a, \lambda) &= -\cos\alpha, & p\theta'(a, \lambda) &= \sin\alpha, \end{aligned}\right\} \tag{2.8}$$

where $\alpha \in \boldsymbol{C}$. On integration by parts we have, for $a \leqslant Y < X < b$ and $u, v \in D(M)$ defined by

$$D(M) = \{y : y, py' \in \mathrm{AC}_{\mathrm{loc}}[a, b)\}, \tag{2.9}$$

that

$$\int_Y^X uM[v]w \,\mathrm{d}x = -puv'|_Y^X + \int_Y^X (pu'v' + quv)\,\mathrm{d}x, \tag{2.10}$$

$$\int_Y^X (uM[v] - vM[u])w \,\mathrm{d}x = -[u, v](X) + [u, v](Y), \tag{2.11}$$

where

$$[u, v](x) = p(x)(u(x)v'(x) - v(x)u'(x)), \tag{2.12}$$

and

$$\begin{aligned} &\int_Y^X (u\overline{M[v]} - \overline{v}M[u])w \,\mathrm{d}x \\ &= (pu'\overline{v} - \overline{p}u\overline{v}')(X) - (pu'\overline{v} - \overline{p}u\overline{v}')(Y) + \int_Y^X [(\overline{p} - p)u'\overline{v}' + (\overline{q} - q)u\overline{v}]\,\mathrm{d}x. \end{aligned} \tag{2.13}$$

Let $\psi = \theta + l\phi$ satisfy

$$\psi(X)\cos\beta + (p\psi')(X)\sin\beta = 0, \quad \beta \in \boldsymbol{C}.$$

Then

$$l \equiv l_X(\lambda, \cot\beta) = -\frac{\theta(X, \lambda)\cot\beta + p(X)\theta'(X, \lambda)}{\phi(X, \lambda)\cot\beta + p(X)\phi'(X, \lambda)}.$$

Let

$$l_X(\lambda, z) := -\frac{\theta(X, \lambda)z + p(X)\theta'(X, \lambda)}{\phi(X, \lambda)z + p(X)\phi'(X, \lambda)}, \quad z \in \boldsymbol{C}. \tag{2.14}$$

This has the inverse

$$z = z_X(\lambda, l) = -\frac{p(X)\phi'(X, \lambda)l + p(X)\theta'(X, \lambda)}{\phi(X, \lambda)l + \theta(X, \lambda)}. \tag{2.15}$$

For η satisfying (2.6), the Möbius transformation (2.14) (note that $p(\theta\phi' - \phi\theta')(X) = [\theta, \phi](X) = -1$) is such that, for $\lambda \in \Lambda_{\eta,K}$, $z \mapsto l_X(\lambda, z)$ maps the half-plane $\mathrm{Re}[z\mathrm{e}^{\mathrm{i}\eta}] \geqslant 0$ onto a closed disc $D_X(\lambda)$ in $\boldsymbol{C}$. To see this, set $\tilde{z} = z\mathrm{e}^{\mathrm{i}\eta}$ and

$$\tilde{l}_X(\lambda, \tilde{z}) = -\frac{\theta(X, \lambda)\tilde{z} + p(X)\theta'(X, \lambda)\mathrm{e}^{\mathrm{i}\eta}}{\phi(X, \lambda)\tilde{z} + p(X)\phi'(X, \lambda)\mathrm{e}^{\mathrm{i}\eta}} = l_X(\lambda, z). \tag{2.16}$$

This has critical point $\tilde{z} = -\mathrm{e}^{\mathrm{i}\eta} p(X)\phi'(X,\lambda)/\phi(X,\lambda)$, and we require this to satisfy $\mathrm{Re}[\tilde{z}] < 0$. We have

$$\mathrm{Re}[\tilde{z}] = -\,\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta} p(X)\phi'(X,\lambda)\overline{\phi}(X,\lambda)/|\phi(X,\lambda)|^2]$$

and, from (2.10),

$$\int_a^X \overline{\phi} M[\phi] w \,\mathrm{d}x = -p(X)\phi'(X,\lambda)\overline{\phi}(X,\lambda) - \cos\alpha\overline{\sin\alpha} + \int_a^X (p|\phi'|^2 + q|\phi|^2)\,\mathrm{d}x.$$

This yields

$$\begin{aligned}|\phi(X,\lambda)|^2 &\,\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta} p(X)\phi'(X,\lambda)\overline{\phi}(X,\lambda)/|\phi(X,\lambda)|^2] \\ &= -\,\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}\cos\alpha\overline{\sin\alpha}] + \mathrm{Re}\left[\int_a^X \mathrm{e}^{\mathrm{i}\eta}\left\{\frac{p}{w}|\phi'|^2 + \left(\frac{q}{w} - \lambda\right)|\phi|^2\right\} w\right]\mathrm{d}x > 0, \qquad (2.17)\end{aligned}$$

by (2.3). Thus, when (2.6) is satisfied, $z \mapsto l_X(\lambda, z)$ maps $\mathrm{Re}[z\mathrm{e}^{\mathrm{i}\eta}] \geqslant 0$ onto $D_X(\lambda)$, a closed disc with centre

$$\sigma_X(\lambda) = \tilde{l}_X(\lambda, \mathrm{e}^{-\mathrm{i}\eta}\overline{p(X)\phi'(X,\lambda)}/\overline{\phi(X,\lambda)}). \qquad (2.18)$$

Furthermore $\tilde{z} = 0$ is mapped onto a point on the circle $C_X(\lambda)$ bounding $D_X(\lambda)$, namely the point

$$\tilde{l}_X(\lambda, 0) = -\theta'(X,\lambda)/\phi'(X,\lambda), \qquad (2.19)$$

and a calculation gives, for the radius $\rho_X(\lambda)$ of $C_X(\lambda)$,

$$\begin{aligned}\rho_X(\lambda) &= (2|\,\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta} p(X)\phi'(X,\lambda)\overline{\phi}(X,\lambda)]|)^{-1} \\ &= \frac{1}{2}\left\{-\,\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}\cos\alpha\overline{\sin\alpha}] + \int_a^X \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}(p|\phi'|^2 + (q-\lambda w)|\phi|^2)]\,\mathrm{d}x\right\}^{-1}, \qquad (2.20)\end{aligned}$$

by (2.17).

The next step is to establish that the circles $C_X(\lambda)$ are nested as $X \to b$. Set $\psi_l = \theta + l\phi$ so that (2.15) gives

$$z = z_X(\lambda, l) = -p(X)\psi_l'(X,\lambda)/\psi_l(X,\lambda).$$

We have already seen that $l = l(\lambda) \in D_X(\lambda)$ if and only if $\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta} z_X(\lambda, l)] \geqslant 0$, that is

$$\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta} p(X)\psi_l'(X,\lambda)\overline{\psi}_l(X,\lambda)] \leqslant 0.$$

As in (2.17), this can be written as

$$0 \geqslant \mathrm{Re}\left[\mathrm{e}^{\mathrm{i}\eta}\left\{p(a)\psi_l'(a,\lambda)\overline{\psi}_l(a,\lambda) + \int_a^X (p|\psi_l'|^2 + (q-\lambda w)|\psi_l|^2)\,\mathrm{d}x\right\}\right].$$

On substituting (2.8), this gives that $l \in D_X(\lambda)$ if and only if

$$\begin{aligned}\int_a^X \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}\{p|\psi_l'|^2 + (q-\lambda w)|\psi_l|^2\}]\,\mathrm{d}x &\leqslant -\,\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}(\sin\alpha - l\cos\alpha)(\overline{\cos\alpha} + \overline{l}\,\overline{\sin}\alpha)] \\ &=: \mathcal{A}(\alpha, \eta; l(\lambda)), \qquad (2.21)\end{aligned}$$

say. Note that $l \in C_X(\lambda)$ if and only if equality holds in (2.21). In view of (2.3) and (2.5), the integrand on the left-hand side of (2.21) is positive, and so $D_Y(\lambda) \subset D_X(\lambda)$ if $X < Y$. Hence, the discs $D_X(\lambda)$, $a < X < b$ are nested, and as $X \to b$ they converge to a disc $D_b(\lambda)$ or a point $m(\lambda)$: these are the *limit-circle* and *limit-point* cases, respectively. The disc $D_b(\lambda)$ and point $m(\lambda)$ depend on η and K in general, but we shall only indicate this dependence explicitly when necessary for clarity.

Let

$$\psi(x,\lambda) := \theta(x,\lambda) + m(\lambda)\phi(x,\lambda), \quad \lambda \in \Lambda_{\eta,K}, \tag{2.22}$$

where $m(\lambda)$ is either a point in $D_b(\lambda)$ in the limit-circle case, or the limit-point otherwise. The nesting property and (2.21) imply that

$$\int_a^b \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}\{p|\psi'|^2 + (q-\lambda w)|\psi|^2\}]\,\mathrm{d}x \leqslant \mathcal{A}(\alpha,\eta;m(\lambda)). \tag{2.23}$$

Moreover, in the limit-point case, it follows from (2.20) that

$$\int_a^b \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}\{p|\phi'|^2 + (q-\lambda w)|\phi|^2\}]\,\mathrm{d}x = \infty, \tag{2.24}$$

whereas in the limit-circle case, the left-hand side of (2.24) is finite. Also, note that, by (2.5), a solution y of (1.2) for $\lambda \in \Lambda_{\eta,K}$ satisfies

$$\int_a^b \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}\{p|y'|^2 + (q-\lambda w)|y|^2\}]\,\mathrm{d}x < \infty, \tag{2.25}$$

if and only if

$$\int_a^b \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}\{p|y'|^2 + (q-Kw)|y|^2\}]\,\mathrm{d}x + \int_a^b |y|^2 w\,\mathrm{d}x < \infty; \tag{2.26}$$

in particular this yields

$$y \in L^2(a,b;w\,\mathrm{d}x). \tag{2.27}$$

In the limit-point case there is a unique solution of (1.2) for $\lambda \in \Lambda_{\eta,K}$ satisfying (2.26), but it may be that all solutions satisfy (2.27). We therefore have the following analogue of Sims's (1957) result. The uniqueness referred to in the theorem is only up to constant multiples.

Theorem 2.1. *For $\lambda \in \Lambda_{\eta,K}$, $(\eta,K) \in S(\alpha)$ the Weyl circles converge either to a limit-point $m(\lambda)$ or a limit-circle $C_b(\lambda)$. The following distinct cases are possible, the first two being sub-cases of the limit-point case.*

Case I: *there exists a unique solution of (1.2) satisfying (2.26), and this is the only solution satisfying (2.27).*

Case II: *there exists a unique solution of (1.2) satisfying (2.26), but all solutions of (1.2) satisfy (2.27).*

Case III: *all solutions of (1.2) satisfy (2.26) and, hence, (2.27).*

Remark 2.2. It follows by a standard argument involving the variation of parameters formula (cf. Sims (1957), § 3, theorem 2) that the classification of (1.2) in theorem 2.1 is independent of λ in the sense that

(i) if all solutions of (1.2) satisfy (2.26) for some $\lambda' \in \Lambda_{\eta,K}$ (i.e. case III), then all solutions of (1.2) satisfy (2.26) for all $\lambda \in \boldsymbol{C}$;

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(ii) if all solutions of (1.2) satisfy (2.27) for some $\lambda' \in \boldsymbol{C}$, then all solutions of (1.2) satisfy (2.27) for all $\lambda \in \boldsymbol{C}$.

Remark 2.3. Suppose that p is real and non-negative and that for some $\eta \in [-\pi/2, \pi/2]$ and $K \in \boldsymbol{C}$,

$$\theta_{K,\eta}(x) = \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}(q(x) - Kw(x))] \geqslant 0 \text{ a.e.}, \quad x \in (a,b). \tag{2.28}$$

Then the condition (2.26) in the Sims (1957) characterization of (1.2) in theorem 2.1 for $\lambda \in \Lambda_{\eta,K}$, $(\eta, K) \in S(\alpha)$, becomes

$$\cos\eta \int_a^b p|y'|^2 \,\mathrm{d}x + \int_a^b \theta_{K\eta}(x)|y(x)|^2 \,\mathrm{d}x + \int_\alpha^b |y(x)|^2 w(x) \,\mathrm{d}x < \infty. \tag{2.29}$$

In this case, remark 2.2 (i) can be extended to:

(i) if for some $\lambda' \in \boldsymbol{C}$ all the solutions of (1.2) satisfy (2.29), then for all $\lambda \in \boldsymbol{C}$, all solutions of (1.2) satisfy (2.29);

(ii) if for some $\lambda' \in \boldsymbol{C}$ all the solutions of (1.2) satisfy one of

$$\cos\eta \int_a^b p|y'|^2 \,\mathrm{d}x < \infty, \tag{2.30}$$

$$\int_a^b \theta_{K\eta}|y|^2 \,\mathrm{d}x < \infty, \tag{2.31}$$

then the same applies for all $\lambda \in \boldsymbol{C}$.

The case considered by Sims (1957) is when $\eta = \pi/2$, $K = 0$ in (2.28). This overlooks the interesting features present in (2.29) when $\eta \in (-\pi/2, \pi/2)$, namely, that the classification in theorem 2.1 involves a weighted Sobolev space as well as $L^2(a,b; w\,\mathrm{d}x)$.

Remark 2.4. We have not been able to exclude the possibility in cases II and III that there exists a solution, y, of (1.2) for $\lambda \in \Lambda_{\eta_1,K_1} \cap \Lambda_{\eta_2,K_2}$, such that

$$\int_a^b \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta_1}(p|y'|^2 + (q - K_1 w)|y|^2)] \,\mathrm{d}x + \int_a^b |y|^2 w \,\mathrm{d}x < \infty, \tag{2.32}$$

$$\int_a^b \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta_2}(p|y'|^2 + (q - K_2 w)|y|^2)] \,\mathrm{d}x + \int_a^b |y|^2 w \,\mathrm{d}x = \infty, \tag{2.33}$$

for different values of η_1, η_2 and K_1, K_2. In case I, this is not possible by remark 2.2. Thus, in cases II and III, the classification appears to depend on K, η, even under the circumstances of remark 2.3.

Remark 2.5. In Birger & Kalyabin (1976), a generalization of Weyl's limit-circles theory, which includes that of Sims, is obtained in the case of a system of the form (1.2) with $p = \omega = 1$, $\lambda = 0$ and $\mathrm{Im}[\mathrm{e}^{-\mathrm{i}\eta} q(x)] \leqslant -k < 0$. The existence of solutions that satisfy (2.26) is established, and it is shown that the analogue of case I holds when $\eta \neq \pm\pi/2$.

3. Properties of m

Throughout the rest of the paper we shall assume that $(\eta, K) \in S(\alpha)$. We denote by $m_{\eta,K}(\cdot)$ the function $m(\cdot)$ defined in § 2 on $\Lambda_{\eta,K}$ whenever there is a risk of confusion. The arguments in Titchmarsh (1958, § 2.2) and Sims (1957, theorem 3) remain valid in our problem, and give the following lemma.

Lemma 3.1. *In cases I and II, $m_{\eta,K}$ is analytic throughout $\Lambda_{\eta,K}$. In case I, the function defined by*

$$m(\lambda) = m_{\eta,K}(\lambda), \quad \lambda \in \Lambda_{\eta,K}, \tag{3.1}$$

is well defined on each of the possible two connected components of $\boldsymbol{C}\backslash Q(\alpha) = \cup_{S(\alpha)} \Lambda_{\eta,K}$, (see (2.7)); the restriction to a connected component is analytic on that set.

In case III, given $m_0 \in C_b(\lambda_0), \lambda_0 \in \Lambda_{\eta,K}$, there exists a function $m_{\eta,K}$ that is analytic in $\Lambda_{\eta,K}$ and $m_{\eta,K}(\lambda_0) = m_0$; moreover, a function $m_{\eta,K}$ can be found such that $m_{\eta,K}(\lambda) \in C_b(\lambda)$ for all $\lambda \in \Lambda_{\eta,K}$.

Proof. The only part not covered by the argument in Sims (1957, theorem 3) is that pertaining to (3.1) on $\boldsymbol{C}\backslash Q(\alpha)$ in case I. We need only show that $m_{\eta_1,K_1}(\lambda) = m_{\eta_2,K_2}(\lambda)$ if $\lambda \in \Lambda_{\eta_1,K_1} \cap \Lambda_{\eta_2,K_2}$. Since in case I, the function in (2.22) (now denoted by $\psi_{\eta,K}(\cdot, \lambda)$ for $\lambda \in \Lambda_{\eta,K}$) is the unique solution of (1.2) in $L^2(a, b; w\,\mathrm{d}x)$ it follows that

$$\psi_{\eta_1,K_1}(x, \lambda) = K(\lambda)\psi_{\eta_2,K_2}(x, \lambda),$$

for some $K(\lambda)$. On substituting the initial conditions (2.8), we obtain $m_{\eta_1,K_1}(\lambda) = m_{\eta_2,K_2}(\lambda)$.

In case I, if $\boldsymbol{C}\backslash Q(\alpha)$ has two connected components C_1, C_2 say, and $m^{(1)}, m^{(2)}$ are the m-functions defined on C_1 and C_2, respectively, by lemma 3.1, we define m on $\boldsymbol{C}\backslash Q(\alpha)$ by

$$m(\lambda) = \begin{cases} m^{(1)}, & \lambda \in C_1, \\ m^{(2)}, & \lambda \in C_2. \end{cases}$$

■

Remark 3.2. Let $\alpha \in \{0, \pi\}$ in (2.21). Then $l \in D_X(\lambda)$ implies that $\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta} l] \geqslant 0$. Thus $z \mapsto l_X(\lambda, z)$ maps the half-plane $\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta} z] \geqslant 0$ into itself and, in particular, $m(\cdot)$ possesses an analogue of the Nevanlinna property enjoyed by the Titchmarsh–Weyl function in the formally symmetric case. If $\alpha = \pi/2$, then $l \in D_X(\lambda)$ implies that $\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta} \bar{l}] \leqslant 0$.

The argument in Titchmarsh (1962, lemma 2.3) requires only a slight modification to give the following important lemma.

Lemma 3.3. *Let $\lambda, \lambda' \in \Lambda_{\eta,K}$ and*

$$\psi(\cdot, \lambda) = \theta(\cdot, \lambda) + m(\lambda)\phi(\cdot, \lambda),$$

where $m(\lambda)$ is either the limit-point or an arbitrary point in $D_b(\lambda)$ in the limit-circle case. Then

$$\lim_{X\to b} [\psi(\cdot, \lambda), \psi(\cdot, \lambda')](X) \equiv \lim_{X\to b} \{p(X)[\psi(X, \lambda)\psi'(X, \lambda') - \psi'(X, \lambda)\psi(X, \lambda')]\} = 0. \tag{3.2}$$

In case I, (3.2) continues to hold for all $\lambda, \lambda' \in \boldsymbol{C}\backslash Q(\alpha)$.

Proof. The starting point is the observation that if $\mathrm{Re}[z e^{\mathrm{i}\eta}] \geqslant 0$ and, hence, $l_X(\lambda, z)$ in (2.14) lies on the disc $D_X(\lambda)$, then, with $\psi_X = \theta + l_X \phi$,

$$z\psi_X(X,\lambda) + p\psi'_X(X,\lambda) = 0$$

and, similarly, for λ'. Then

$$[\psi_X(\cdot,\lambda), \psi_X(\cdot,\lambda')](X) = 0$$

and the argument proceeds as in Titchmarsh (1962). ∎

Lemma 3.3 and (2.10) yield the following corollary.

Corollary 3.4. *For all* $\lambda, \lambda' \in \Lambda_{\eta,K}$

$$(\lambda' - \lambda)\int_a^b \psi(x,\lambda)\psi(x,\lambda')w(x)\,\mathrm{d}x = m(\lambda) - m(\lambda'); \tag{3.3}$$

this holds for all $\lambda, \lambda' \in \boldsymbol{C}\backslash Q$ *in case I. It follows that in cases II and III, for a fixed* $\lambda' \in \Lambda_{\eta,K}$,

$$m(\lambda) = \frac{m(\lambda') - (\lambda - \lambda')\displaystyle\int_a^b \theta(x,\lambda)\psi(x,\lambda')w(x)\,\mathrm{d}x}{1 + (\lambda - \lambda')\displaystyle\int_a^b \phi(x,\lambda)\psi(x,\lambda')w(x)\,\mathrm{d}x}, \tag{3.4}$$

defines $m(\lambda)$ *as a meromorphic function in* $\boldsymbol{C}$*; it has a pole at* λ, *if and only if*

$$1 + (\lambda - \lambda')\int_a^b \phi(x,\lambda)\psi(x,\lambda')w(x)\,\mathrm{d}x = 0. \tag{3.5}$$

Proof. The identity (3.3) follows easily from (2.11) and lemma 3.3. In cases II and III, $\theta(\cdot,\lambda), \phi(\cdot,\lambda) \in L^2(a,b,w\,\mathrm{d}x)$, and (3.4) is derived from (3.3) on writing $\psi(\cdot,\lambda) = \theta(\cdot,\lambda) + m(\lambda)\phi(\cdot,\lambda)$. ∎

Theorem 3.5. *Suppose that (1.2) is in case I. Define*

$$Q_c := \overline{\mathrm{co}}\left\{\frac{q(x)}{w(x)} + rp(x) :\ x \in [c,b),\ r \in (0,\infty)\right\}, \tag{3.6}$$

$$Q_b := \cap_{c\in(a,b)} Q_c, \qquad Q_b(\alpha) = \cap_{c\in(a,b)} Q_c(\alpha), \tag{3.7}$$

where $Q_c(\alpha)$ *is the set* $Q(\alpha)$ *defined in (2.7) when the underlying interval is* $[c,b)$ *rather than* $[a,b)$. *Then* $m(\lambda)$ *is defined throughout* $\boldsymbol{C}\backslash Q(\alpha)$, *and has a meromorphic extension to* $\boldsymbol{C}\backslash Q_b(\alpha)$, *with poles only in* $Q(\alpha)\backslash Q_b(\alpha)$.

Proof. Let $m_c(\cdot)$ denote the limit-point in the problem on $[c,b)$ with c now replacing a in the initial conditions (2.8); it is defined and analytic throughout each of the possible two connected components of $\boldsymbol{C}\backslash Q_c(\alpha)$, by lemma 3.1. Also, $\psi_c := \theta_c + m_c\phi_c$ can be uniquely extended to $[a,b)$ with $\psi_c(x,\cdot)$ and $p\psi'_c(x,\cdot)$ analytic in $\boldsymbol{C}\backslash Q_c(\alpha)$ for fixed x. Since we are in case I, there exists $K(\lambda)$ such that

$$\psi(x,\lambda) = K(\lambda)\psi_c(x,\lambda).$$

On substituting (2.8), we obtain

$$m(\lambda) = \frac{\sin\alpha\psi_c(a,\lambda) - \cos\alpha p\psi'_c(a,\lambda)}{\cos\alpha\psi_c(a,\lambda) + \sin\alpha p\psi'_c(a,\lambda)}. \tag{3.8}$$

This defines $m(\lambda)$ as a meromorphic function in $\boldsymbol{C}\backslash Q_c(\alpha)$ with isolated poles at the zeros of the denominator in (3.8). In the case $b = \infty$, Q_b appears in Glazman (1965, § 35). ∎

4. Operator realizations of M

For $\lambda \in \Lambda_{\eta,K}$, $(\eta, K) \in S(\alpha)$ define

$$G(x,y;\lambda) = \begin{cases} -\phi(x,\lambda)\psi(y,\lambda), & a<x<y<b, \\ -\psi(x,\lambda)\phi(y,\lambda), & a<y<x<b, \end{cases} \tag{4.1}$$

where ϕ, ψ are the solutions of (1.2) in (2.8) and (2.22). Recall that m, and hence ψ, depends on (η, K) in general, but for simplicity of notation we suppress this dependency. In case I, however, lemma 3.1 shows that m is properly defined throughout $\boldsymbol{C}\backslash Q(\alpha)$. In cases II and III, we know from theorem 3.5 that $m(\cdot)$ can be continued as a meromorphic function throughout $\boldsymbol{C}$ (but apparently still depends on η and K). For $\lambda \in \Lambda_{\eta,K}$ and $f \in L^2(a,b;w\,\mathrm{d}x)$ define

$$R_\lambda f(x) := \int_a^b G(x,y;\lambda)f(y)w(y)\,\mathrm{d}x. \tag{4.2}$$

It is readily verified that $p(R_\lambda f)' \in \mathrm{AC_{loc}}[a,b)$ and from

$$[\phi,\psi](x) = [\phi,\psi](a) = 1 \quad (x \in (a,b))$$

(see (2.11) and (2.12)), that for a.e. $x \in (a,b)$

$$(M-\lambda)R_\lambda f(x) = f(x). \tag{4.3}$$

Also, for any $\lambda' \in \boldsymbol{C}$

$$[R_\lambda f, \phi(\cdot,\lambda')](a) = -[\phi(\cdot,\lambda),\phi(\cdot,\lambda')](a)\int_a^b \psi f w\,\mathrm{d}x = 0. \tag{4.4}$$

Moreover, if f is supported away from b, then, by lemma 3.3, for any $\lambda, \lambda' \in \Lambda_{\eta,K}$,

$$\begin{aligned}[R_\lambda f, \psi(\cdot,\lambda')](b) &:= \lim_{X\to b}[R_\lambda f, \psi(\cdot,\lambda')](X) \\ &= -\lim_{X\to b}\left\{[\psi(\cdot,\lambda),\psi(\cdot,\lambda')](X)\int_a^X \phi f w\,\mathrm{d}x\right\} \\ &= 0. \end{aligned} \tag{4.5}$$

In cases II and III, (4.5) holds for all $f \in L^2(a,b,;w\,\mathrm{d}x)$, since then the integral on the right-hand side remains bounded as $X \to b$ and $\lim_{X\to b}[\psi(\cdot,\lambda),\psi(\cdot,\lambda')](X)$ is zero by (3.2). In case I, (4.5) continues to be true for all $\lambda, \lambda' \in \boldsymbol{C}\backslash Q(\alpha)$.

Before proceeding to define the realizations of M that are natural to the problem, we need the following theorem, which provides our basic tool. In the theorem, $\|\cdot\|$ denotes the $L^2(a,b;w\,\mathrm{d}x)$ norm.

Theorem 4.1. *Let $f \in L^2(a,b;w\,\mathrm{d}x)$ and $\lambda \in \Lambda_{\eta,K}$, $(\eta,K) \in S(\alpha)$. Then, in every case, with $\Phi \equiv R_\lambda f$, and $\delta = \mathrm{dist}(\lambda, \partial\Lambda_{\eta,K})$,*

$$\begin{aligned}&\int_a^b \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}(p|\Phi'|^2 + (q-Kw)|\Phi|^2)]\,\mathrm{d}x \\ &\qquad + (\mathrm{Re}[(K-\lambda)\mathrm{e}^{\mathrm{i}\eta}] - \epsilon)\int_a^b |\Phi|^2 w\,\mathrm{d}x \leqslant \frac{1}{4\epsilon}\int_a^b |f|^2 w\,\mathrm{d}x, \end{aligned} \tag{4.6}$$

for any $\epsilon > 0$. In particular, R_λ is bounded and

$$\| R_\lambda f \| \leqslant (1/\delta) \| f \| . \tag{4.7}$$

Proof. Let $f_X = \chi_{(a,X)} f$ and $\Phi_X = R_\lambda f_X$. Then, by (2.10) and (4.3),

$$\begin{aligned}
\int_a^X (p|\Phi_X'|^2 + (q-\lambda w)|\Phi_X|^2)\,\mathrm{d}x
&= p\overline{\Phi_X}\Phi_X' \,|_a^X + \int_a^X \overline{\Phi}_X f w\,\mathrm{d}x \\
&= p(X)\overline{\psi(X)}\psi'(X)\left|\int_a^X \phi f w\,\mathrm{d}x\right|^2 \\
&\quad - p(a)\overline{\phi}(a)\phi'(a)\left|\int_a^X \psi f w\,\mathrm{d}x\right|^2 + \int_a^X \overline{\Phi_X} f w\,\mathrm{d}x \\
&= \left\{\int_a^X (p|\psi'|^2 + (q-\lambda w)|\psi|^2)\,\mathrm{d}x\right. \\
&\quad \left. + (\overline{\cos\alpha + m\sin\alpha})(\sin\alpha - m\cos\alpha)\right\}\left|\int_a^X \phi f w\,\mathrm{d}x\right|^2 \\
&\quad + \overline{\sin\alpha}\cos\alpha\left|\int_a^X \psi f w\,\mathrm{d}x\right|^2 + \int_a^X \overline{\Phi}_X f w\,\mathrm{d}x,
\end{aligned}$$

from (2.10) again, and (2.8). Hence, by (2.21) and (2.23),

$$\begin{aligned}
&\int_a^X \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}(p|\Phi_X'|^2 + (q-\lambda w)|\Phi_X|^2)]\,\mathrm{d}x \\
&\quad = \int_a^X \{\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}(p|\psi'|^2 + (q-\lambda w)|\psi|^2)]\,\mathrm{d}x - \mathcal{A}(\alpha,\eta;m(\lambda))\}\left|\int_a^X \phi f w\,\mathrm{d}x\right|^2 \\
&\qquad + \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}\overline{\sin\alpha}\cos\alpha]\left|\int_a^X \psi f w\,\mathrm{d}x\right|^2 + \mathrm{Re}\left[\mathrm{e}^{\mathrm{i}\eta}\int_a^X \overline{\Phi}_X f w\right]\mathrm{d}x \\
&\quad \leqslant \int_a^X |\Phi_X|\,|f|w\,\mathrm{d}x \leqslant \epsilon\int_a^X |\Phi_X|^2 w\,\mathrm{d}x + \frac{1}{4\epsilon}\int_a^X |f_X|^2 w\,\mathrm{d}x,
\end{aligned}$$

whence

$$\begin{aligned}
\int_a^b \mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}(p|\Phi_X'|^2 + (q-Kw)|\Phi_X|^2)\,\mathrm{d}x] + (\mathrm{Re}[\mathrm{e}^{\mathrm{i}\eta}(K-\lambda)] - \epsilon)\int_a^b |\Phi_X|^2 w\,\mathrm{d}x \\
\leqslant \frac{1}{4\epsilon}\int_a^b |f_X|^2 w\,\mathrm{d}x.
\end{aligned}$$

As $X \to b$, $\Phi_X(x) \to \Phi(x)$ and (4.6) follows by Fatou's lemma. We also obtain from (4.6), (2.3), (2.5) and (2.10) that

$$(\delta - \epsilon)\int_a^b |\Phi|^2 w\,\mathrm{d}x \leqslant \frac{1}{4\epsilon}\int_a^b |f|^2 w\,\mathrm{d}x.$$

The choice $\epsilon = \delta/2$ yields (4.7). ■

Theorem 4.1 enables us to establish (4.5) for all $f \in L^2(a,b;w\,\mathrm{d}x)$ in case I (and hence in all cases).

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Lemma 4.2. *For* $\lambda, \lambda' \in \Lambda_{\eta,K}$, $(\eta, K) \in S(\alpha)$ *and* $f \in L^2(a, b; w\,\mathrm{d}x)$,

$$[R_\lambda f, \psi(\cdot, \lambda')](b) = 0.$$

Proof. Let $f_c = \chi_{[a,c]} f$, so that, as $c \to b$, we have

$$f_c \to f, \qquad R_\lambda f_c \to R_\lambda f, \quad \text{in } L^2(a, b; w\,\mathrm{d}x), \tag{4.8}$$

$$[R_\lambda f_c, \psi(\cdot, \lambda')](a) \to [R_\lambda f, \psi(\cdot, \lambda')](a), \tag{4.9}$$

since

$$(R_\lambda f_c)(a) = -\phi(a, \lambda) \int_a^b \psi(y, \lambda) f_c(y) w\,\mathrm{d}y \to (R_\lambda f)(a),$$

$$[p(R_\lambda f_c)'](a) = -p\phi'(a, \lambda) \int_a^b \psi(y, \lambda) f_c(y) w(y)\,\mathrm{d}y \to [p(R_\lambda f)'](a),$$

and, by (4.5),

$$[R_\lambda f_c, \psi(\cdot, \lambda')](b) = 0. \tag{4.10}$$

Hence, by (2.11),

$$\begin{aligned}
&[R_\lambda f, \psi(\cdot, \lambda')](X) \\
&\qquad = [R_\lambda(f - f_c), \psi(\cdot, \lambda')](a) + [R_\lambda f_c, \psi(\cdot, \lambda')](X) \\
&\qquad\qquad + \int_a^X \{(\lambda - \lambda')\psi(x, \lambda') R_\lambda [f - f_c](x) + \psi(x, \lambda')[f - f_c](x)\} w(x)\,\mathrm{d}x \\
&\qquad \to [R_\lambda(f - f_c), \psi(\cdot, \lambda')](a) + \int_a^b \{(\lambda - \lambda')\psi(x, \lambda') R_\lambda [f - f_c](x) \\
&\qquad\qquad + \psi(x, \lambda')[f - f_c](x)\} w(x)\,\mathrm{d}x,
\end{aligned}$$

as $X \to b$, by (4.10),

$$\to 0$$

by (4.8) and (4.9). ∎

Remark 4.3. In cases II and III, R_λ is obviously Hilbert–Schmidt for any $\lambda \in \Lambda_{\eta,K}$, $(\eta, K) \in S(\alpha)$.

In view of theorem 4.1 and preceding remarks, it is natural to define the following operators. Let $\lambda' \in \Lambda_{\eta,K}$, $(\eta, K) \in S(\alpha)$, be fixed and set

$$\left.\begin{aligned}
D(\tilde{M}) := \{u : u, pu' \in \mathrm{AC}_{\mathrm{loc}}[a, b), u, Mu \in L^2(a, b; w\,\mathrm{d}x),& \\
[u, \phi(\cdot, \lambda')](a) = 0 \text{ and } [u, \psi(\cdot, \lambda')](b) = 0\},& \\
\tilde{M}u := Mu, \quad u \in D(\tilde{M}).&
\end{aligned}\right\} \tag{4.11}$$

The dependence, or otherwise, of $D(\tilde{M})$ on λ' is made clear in the following theorem.

Theorem 4.4. *In case I*

$$\begin{aligned}
D(\tilde{M}) = D_1 := \{u : u, pu' \in \mathrm{AC}_{\mathrm{loc}}[a, b), u, Mu \in L^2(a, b; w\,\mathrm{d}x),& \\
(\cos\alpha)u(a) + (\sin\alpha)p(a)u'(a) = 0\}.&
\end{aligned} \tag{4.12}$$

In cases II and III, D_1 *is the direct sum*

$$D_1 = D(\tilde{M}) \dot{+} [\phi(\cdot, \lambda')], \tag{4.13}$$

where $[\cdot]$ *indicates the linear span.*

Proof. Clearly $D(\tilde{M}) \subset D_1$: note that the boundary condition at a in (4.12) can be written as $[u, \phi(\cdot, \lambda')](a) = 0$. Let $u \in D_1$, and for $\lambda' \in \Lambda_{\eta,K}$ set $v = R_{\lambda'}[(M - \lambda')u]$. Then $(M - \lambda')v = (M - \lambda')u$ and $[v - u, \phi(\cdot, \lambda')](a) = 0$. It follows that $v - u = K_1\phi(\cdot, \lambda')$ for some constant K_1. In case I, this implies that $K = 0$ since $v \in D(\tilde{M})$ and $\phi(\cdot, \lambda') \notin L^2(a, b; w\,\mathrm{d}x)$. The decomposition (4.13) also follows since the right-hand side of (4.13) is obviously in D_1 in cases II and III. ∎

In the next theorem, J stands for the conjugation operator $u \mapsto \overline{u}$. An operator T is J-symmetric if $JTJ \subset T^*$ and J-self-adjoint if $JTJ = T^*$ (see Edmunds & Evans 1987, § III.5). Also, T is m-accretive if $\operatorname{Re}\lambda < 0$ implies that $\lambda \in \rho(T)$, the resolvent set of T, and $\| (T - \lambda I)^{-1} \| \leqslant |\operatorname{Re}\lambda|^{-1}$. If, for some $K \in \boldsymbol{C}$ and $\eta \in (-\pi, \pi)$, $\mathrm{e}^{\mathrm{i}\eta}(T - K)$ is m-accretive, we shall say that T is quasi-m-accretive; note that this is slightly different to the standard notion that does not involve the rotation $\mathrm{e}^{\mathrm{i}\eta}$ (cf. Edmunds & Evans 1987, § III).

Let $\sigma(\tilde{M})$ denote the spectrum of $\tilde{M}$. We define the essential spectrum, $\sigma_{\mathrm{e}}(\tilde{M})$, of $\tilde{M}$ to be the complement in $\boldsymbol{C}$ of the set

$$\Delta(\tilde{M}) = \{\lambda : (\tilde{M} - \lambda I) \text{ is a Fredholm operator and } \operatorname{ind}(\tilde{M} - \lambda I) = 0\}.$$

Recall that a Fredholm operator A is one with closed range, finite nullity $\operatorname{nul} A$ and finite deficiency $\operatorname{def} A$, and $\operatorname{ind} A = \operatorname{nul} A - \operatorname{def} A$. Thus, any $\lambda \in \sigma(\tilde{M}) \backslash \sigma_{\mathrm{e}}(\tilde{M})$ is an eigenvalue of finite (geometric) multiplicity.

Theorem 4.5. *The operators defined in (4.11) for any $\lambda' \in \Lambda_{\eta,K}$, $(\eta, K) \in S(\alpha)$ (or (4.12) in case I), are J-self-adjoint and quasi-m-accretive, and $\sigma(\tilde{M}) \subseteq \boldsymbol{C}\backslash\Lambda_{\eta,K}$. For any $\lambda \in \Lambda_{\eta,K}$, $(\tilde{M} - \lambda)^{-1} = R_\lambda$.*

In case I, $\sigma(\tilde{M}) \subseteq Q(\alpha)$ and $\sigma_{\mathrm{e}}(\tilde{M}) \subseteq Q_b(\alpha)$, where $Q_b(\alpha)$ is defined in (3.7): in $Q(\alpha)\backslash Q_b(\alpha)$, $\sigma(\tilde{M})$ consists only of eigenvalues of finite geometric multiplicity.

In cases II and III, R_λ is compact for any $\lambda \in \rho(\tilde{M})$ and $\sigma(\tilde{M})$ consists only of isolated eigenvalues (in $\boldsymbol{C}\backslash\Lambda_{\eta,K}$) having finite algebraic multiplicity.

Proof. From $JMJ = M^+$, the Lagrange adjoint of M, it follows that M is J-symmetric. Since $(\tilde{M} - \lambda)^{-1} = R_\lambda$ and $\Lambda_{\eta,K} \subseteq \rho(\tilde{M})$ are established in theorem 4.1 and the preceding remarks, it follows that $\tilde{M}$ is quasi-m-accretive, and hence also J-self-adjoint by theorem III 6.7 in Edmunds & Evans (1987).

In case I, theorem 4.1 holds for any $\lambda \in \boldsymbol{C}\backslash Q(\alpha)$ and, hence, $\sigma(\tilde{M}) \subseteq Q(\alpha)$. Also, by the 'decomposition principle' (see Edmunds & Evans (1987), theorem IX 9.3 and remark IX 9.8), $\sigma_{\mathrm{e}}(\tilde{M}) \subseteq Q_b(\alpha)$.

The compactness of R_λ for $\lambda \in \Lambda_{\eta,K}$ in cases II and III is noted in remark 4.3, and the rest of the theorem follows. ∎

Remark 4.6. The argument in Glazman (1965, theorem 35.29) can be used to prove that in case I of theorem 4.5, either $\sigma(\tilde{M})\backslash Q_b(\alpha)$ consists of isolated points of finite algebraic multiplicity and with no limit-point outside $Q_b(\alpha)$, or else each point of at least one of the (possible two) connected components of $Q(\alpha)\backslash Q_b(\alpha)$ is an eigenvalue. We now prove that the latter is not possible.

Theorem 4.7. *Let (1.2) be in case I. Then $\sigma(\tilde{M}) \subseteq Q(\alpha)$, $\sigma_{\mathrm{e}}(\tilde{M}) \subseteq Q_b(\alpha)$ and in $Q(\alpha)\backslash Q_b(\alpha)$, $\sigma(\tilde{M})$ consists only of isolated eigenvalues of finite algebraic multiplicity, these points being the poles of the meromorphic extension of m defined in theorem 3.5.*

Proof. Let $\lambda \in Q(\alpha)\backslash Q_b(\alpha)$ be such that the meromorphic extension of m in theorem 3.5 is regular at λ, and for $c \in (a,b)$, let $\psi(\cdot,\lambda) = K(\lambda)\psi_c(\cdot,\lambda)$ in the notation of the proof of theorem 3.5. Then $\psi(\cdot,\lambda) = \theta(\cdot,\lambda) + m(\lambda)\phi(\cdot,\lambda) \in L^2(a,b;w\,\mathrm{d}x)$ and the operator R^c_λ defined by

$$R^c_\lambda f(x) := -\psi_c(x,\lambda)\int_c^x \phi(y,\lambda)f(y)w(y)\,\mathrm{d}y - \phi(x,\lambda)\int_x^b \psi_c(y,\lambda)f(y)w(y)\,\mathrm{d}y,$$

is bounded on $L^2(c,b;w\,\mathrm{d}x)$ for c sufficiently close to b (so that $\lambda \notin Q_c(\alpha)$), by theorem 4.1 applied to $[c,b)$. Moreover, (4.3) and (4.4) are satisfied by R_λ, now defined for this $\lambda \in Q(\alpha)\backslash Q_b(\alpha)$, and, hence, if we can prove that R_λ is bounded on $L^2(a,b;w\,\mathrm{d}x)$, it will follow that $\lambda \in \rho(\tilde{M})$, whence the theorem in view of remark 4.6. But, for any $f \in L^2(a,b;w\,\mathrm{d}x)$, it is readily verified that

$$\| R_\lambda f \| \leqslant \text{const.}\{\| \phi \|_{(a,c)} \| \psi \| + \| R^c_\lambda \|\} \| f \| .$$

Hence, $\lambda \in \rho(\tilde{M})$. In lemma 4.12 below, we shall prove that m is analytic on $\rho(\tilde{M})$, hence, any pole of m in $Q(\alpha)\backslash Q_b(\alpha)$ lies in $\sigma(\tilde{M})$. The theorem is therefore proved. ■

Remark 4.8. Suppose that case I holds. In the notation of Edmunds & Evans (1987, § IX.1), our essential spectrum σ_e is $\sigma_{\mathrm{e}4}$. However, since the operator $\tilde{M}$ is J-self-adjoint, by theorem 4.5, all the essential spectra $\sigma_{\mathrm{e}k}(\tilde{M})$, $k = 1,2,3,4$ defined in Edmunds & Evans (1987, § IX.1) coincide (by Edmunds & Evans (1987), § IX.1.6). Furthermore, for any α, $\tilde{M}$ is a two-dimensional extension of the closed minimal operator generated by M on

$$D_0 = \{u : u, pu' \in \mathrm{AC_{loc}}[a,b), u, Mu, \in L^2(a,b,w\,\mathrm{d}x), u(a) = p(a)u'(a) = 0\}$$

(cf. Edmunds & Evans (1987), theorem III 10.13 and lemma IX 9.2). It therefore follows from Edmunds & Evans (1987, IX.1, 4.2) that the essential spectrum $\sigma_\mathrm{e}(\tilde{M})$ is independent of α. Thus, in theorem 4.7, $\sigma_\mathrm{e}(\tilde{M}) \subseteq Q_b$, since $Q_b(0) = Q_b$.

We now proceed to analyse the connections between the spectrum of $\tilde{M}$ and the singularities of extensions of the $m(\cdot)$ function as is done for the Sturm–Liouville problem in Chaudhuri & Everitt (1968). An important observation for this analysis is the following lemma. In it, $(\cdot,\cdot)$ denotes the $L^2(a,b;w\,\mathrm{d}x)$ inner product.

Lemma 4.9. *For all* $\lambda, \lambda' \in \varLambda_{\eta,K}$, $(\eta,K) \in S(\alpha)$,

$$m(\lambda) = m(\lambda') - (\lambda - \lambda')\int_a^b \psi^2(x,\lambda')w(x)\,\mathrm{d}x - (\lambda-\lambda')^2(R_\lambda\psi(\cdot,\lambda'),\overline{\psi}(\cdot,\lambda')), \tag{4.14}$$

$$m(\lambda) = [\psi(\cdot,\lambda),\theta(\cdot,\lambda')](a), \tag{4.15}$$

and

$$\psi(\cdot,\lambda) = \psi(\cdot,\lambda') + (\lambda-\lambda')R_\lambda\psi(\cdot,\lambda'). \tag{4.16}$$

Proof. The identity (4.14) is an immediate consequence of (3.3) and (4.16), and (4.15) follows from (2.8) and (2.22). To prove (4.16), set $u = \psi(\cdot,\lambda) - \psi(\cdot,\lambda')$. Then $u \in D(\tilde{M})$ by lemma 3.3 and since

$$[\psi(\cdot,\lambda),\phi(\cdot,\lambda')](a) - [\psi(\cdot,\lambda'),\phi(\cdot,\lambda')](a) = 0. \tag{4.17}$$

Also $(\tilde{M}-\lambda)u = (\lambda-\lambda')\psi(\cdot,\lambda')$. This yields $u = (\lambda-\lambda')R_\lambda\psi(\cdot,\lambda')$ and (4.16) is established. The lemma is therefore proved. ■

Motivated by (4.15) and (4.16) in lemma 4.9, we have

Definition 4.10. For $\lambda' \in \Lambda_{\eta,K}$, $(\eta, K) \in S(\alpha)$ and $R_\lambda = (\tilde{M} - \lambda)^{-1}$, we define m on $\rho(\tilde{M})$ by

$$m(\lambda) = [\Psi(\cdot, \lambda), \theta(\cdot, \lambda')](a), \tag{4.18}$$

where

$$\Psi(\cdot, \lambda) = \psi(\cdot, \lambda') + (\lambda - \lambda')R_\lambda \psi(\cdot, \lambda'). \tag{4.19}$$

Remark 4.11. In cases II and III, the points $m(\lambda')$ on the limit-circle for $\lambda' \in \Lambda_{\eta,K}$ seem to depend on η, K (see remark 2.4) and, hence, so does the extension to $\rho(\tilde{M})$ in definition 4.10. This is not so in case I, in view of lemma 3.1.

Lemma 4.12. *Let $\lambda' \in \Lambda_{\eta,K}$, $(\eta, K) \in S(\alpha)$, and define m by (4.18) on $\rho(\tilde{M})$, where $R_\lambda = (\tilde{M} - \lambda)^{-1}$. Then, in (4.19),*

$$\Psi(\cdot, \lambda) = \theta(\cdot, \lambda) + m(\lambda)\phi(\cdot, \lambda). \tag{4.20}$$

Also (3.3) and (4.14) hold for all $\lambda \in \rho(\tilde{M})$. Hence, m is analytic on $\rho(\tilde{M})$, and in cases II and III, (4.18) and (3.4) define the same meromorphic extension of m, while in case I, (4.18) defines the same meromorphic extension to $\boldsymbol{C}\backslash Q_b(\alpha)$ as that described in theorem 3.5.

Proof. Since

$$(M - \lambda)\Psi(\cdot, \lambda) = [(\lambda' - \lambda) + (\lambda - \lambda')]\psi(\cdot, \lambda') = 0,$$

we have

$$\Psi(\cdot, \lambda) = A\theta(\cdot, \lambda) + B\phi(\cdot, \lambda),$$

for some constants A and B. On using (2.8) and (4.11), it is readily verified that

$$\begin{aligned} A &= -A[\theta(\cdot, \lambda), \phi(\cdot, \lambda')](a) \\ &= -[\Psi(\cdot, \lambda), \phi(\cdot, \lambda')](a) \\ &= -[\psi(\cdot, \lambda'), \phi(\cdot, \lambda')](a) - (\lambda - \lambda')[R_\lambda \psi(\cdot, \lambda'), \phi(\cdot, \lambda')](a) \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} B &= B[\phi(\cdot, \lambda), \theta(\cdot, \lambda')](a) \\ &= [\Psi(\cdot, \lambda), \theta(\cdot, \lambda')](a) \\ &= m(\lambda), \end{aligned}$$

whence (4.20). Also, from (4.19),

$$\begin{aligned} (\lambda - \lambda')^2 (R_\lambda \psi(\cdot, \lambda'), \overline{\psi}(\cdot, \lambda')) &+ (\lambda - \lambda') \int_a^b \psi^2(x, \lambda') w(x) \, \mathrm{d}x \\ &= (\lambda - \lambda') \int_a^b \Psi(x, \lambda)\psi(x, \lambda') w(x) \, \mathrm{d}x \\ &= -\int_a^b \{\Psi(x, \lambda) M\psi(x, \lambda') - \psi(x, \lambda') M\Psi(x, \lambda)\} w \, \mathrm{d}x \\ &= [\Psi(\cdot, \lambda), \psi(\cdot, \lambda')](b) - [\Psi(\cdot, \lambda), \psi(\cdot, \lambda')](a), \end{aligned}$$

by (2.11),

$$= -[\Psi(\cdot,\lambda), \psi(\cdot,\lambda')](a),$$

by (4.19), and, since $\lambda \in \rho(\tilde{M})$,

$$\begin{aligned} &= -m(\lambda) - m(\lambda')[\Psi(\cdot,\lambda), \phi(\cdot,\lambda')](a) \\ &= m(\lambda') - m(\lambda), \end{aligned}$$

on account of (4.18) and again using $\lambda \in \rho(\tilde{M})$. The lemma is therefore proved. ∎

We now define, for $\lambda \in \rho(\tilde{M})$ and $f \in L^2(a,b;w\,\mathrm{d}x)$,

$$\tilde{G}(x,y;\lambda) = \begin{cases} -\phi(x,\lambda)\Psi(y,\lambda), & a<x<y<b, \\ -\Psi(x,\lambda)\phi(y,\lambda), & a<y<x<b, \end{cases} \tag{4.21}$$

$$\tilde{R}_\lambda f(x) := \int_a^b \tilde{G}(x,y;\lambda) f(x) w(x)\,\mathrm{d}y, \tag{4.22}$$

where Ψ is defined in (4.20) and m in definition 4.10. Thus, for $\lambda \in \boldsymbol{C}\backslash Q(\alpha)$, ($\lambda \in \Lambda_{\eta,K}$, $(\eta,K) \in S(\alpha)$, in cases II and III), we have $R_\lambda = \tilde{R}_\lambda$. We can say more, for (4.3), (4.4) and (4.5) hold for $\tilde{R}_\lambda$, whenever $m(\lambda)$ is defined, and thus $\tilde{R}_\lambda = R_\lambda$ for every λ that is such that $\tilde{R}_\lambda$ is bounded. This is true for every λ at which m is regular in cases II and III. From (4.18) and lemma 4.12 we know that in cases II and III, λ is a pole of $m(\lambda)$ if and only if λ is an eigenvalue of $\tilde{M}$; this is also true in case I for $\lambda \notin Q_b(\alpha)$.

Theorem 4.13. *In cases II and III, λ_0 is a pole of m of order s if and only if λ_0 is an eigenvalue of $\tilde{M}$ of algebraic multiplicity s.*

Proof. For any $f \in L^2(a,b;w\,\mathrm{d}x)$, $R_\lambda f(x)$ has a pole of order s at λ_0 with residue

$$\left\{\frac{1}{(s-1)!}\frac{\partial^{s-1}}{\partial\lambda^{s-1}}\left[(\lambda-\lambda_0)^s m(\lambda)\int_a^b \phi(x,\lambda)\phi(y,\lambda)f(y)w(y)\,\mathrm{d}y\right]\right\}_{\lambda=\lambda_0}.$$

This is of the form

$$\sum_{j=0}^{s-1}\frac{\partial^j}{\partial\lambda^j}\phi(x,\lambda_0)c_j(\lambda_0,f), \tag{4.23}$$

where the coefficients $c_j(\lambda_0,f)$ are linear combinations of

$$\int_a^b \frac{\partial^j}{\partial\lambda^j}\phi(y,\lambda_0)f(y)w(y)\,\mathrm{d}y, \quad j=0,1,\dots,s-1. \tag{4.24}$$

From $(M-\lambda)\phi(\cdot,\lambda)=0$, it follows that for $j=0,1,\dots,s-1$,

$$(M-\lambda_0)\phi_j = j\phi_{j-1}, \tag{4.25}$$

$$(M-\lambda_0)^{j+1}\phi_j = 0, \tag{4.26}$$

where

$$\phi_j = \frac{\partial^j}{\partial\lambda^j}\phi(\cdot,\lambda_0), \quad j=0,s-1. \tag{4.27}$$

It follows inductively from (4.25), on using the variation of parameters, that

$$\phi_j \in L^2(a,b;w\,\mathrm{d}x), \quad j=0,1,\dots,s-1. \tag{4.28}$$

Let Γ_{λ_0} be a positively orientated small circle enclosing λ_0 but excluding the other eigenvalues of $\tilde{M}$. We have

$$\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{\lambda_0}} R_\lambda\,\mathrm{d}\lambda = P_{\Gamma_{\lambda_0}}, \tag{4.29}$$

where $P_{\Gamma_{\lambda_0}}$ is a bounded operator of finite rank given by (4.23): its range is spanned by ϕ_j, $j=0,1,\dots,s-1$. The identity (4.26) readily implies that the functions in (4.27) are linearly independent. Thus P_{λ_0} is of rank s, and s is the algebraic multiplicity of λ_0. The functions in (4.27) span the algebraic eigenspace of $\tilde{M}$ at λ_0 and are the generalized eigenfunctions corresponding to λ_0: they satisfy

$$(\tilde{M}-\lambda_0)^{j+1}\phi_j \neq 0, \qquad (\tilde{M}-\lambda_0)^j\phi_j = 0, \quad j=0,1,\dots,s-1; \tag{4.30}$$

see Kato (1976, § III.4) and McLeod (1961). In case I, we expect theorem 4.13 to remain true for $\lambda_0 \in Q(\alpha)\backslash Q_b(\alpha)$, but we have been unable to prove (4.28) in this case. ■

5. Examples

(*a*) *The sets Q and $Q(\alpha)$*

Suppose that $[a,b)=[1,\infty)$ and the coefficients are of the form

$$p(x)=|p(x)|\mathrm{e}^{\mathrm{i}\phi}, \qquad q(x)=q_1x^{b_1}+\mathrm{i}q_2x^{b_2}, \qquad w(x)=x^\omega, \tag{5.1}$$

where ϕ, q_1, q_2, b_1, b_2 and w are real constants. Then $q(x)/w(x)$, $x\in[1,\infty)$, lie on the curve

$$C := \{z\in \boldsymbol{C}:\ z=q_1x^{b_1-\omega}+\mathrm{i}q_2x^{b_2-\omega},\ x\in[1,\infty)\}. \tag{5.2}$$

The determination of the sets Q and $Q(\alpha)$ is a straightforward exercise. As an illustration, we consider the case $\phi\in[-\pi/2,\pi/2]$, $q_1<0$, $q_2\leqslant 0$, $b_2>b_1>\omega$ in figures 1, 2 and 3. The arrows indicate addition by $rp(x)$, $0<r<\infty$, to the point $q(x)/w(x)$ on C, and the other shading in each figure is the fill-in required to produce the closed convex set Q. We set $z_0=q_1+\mathrm{i}q_2$, $\tan\theta_0$ is the gradient of the tangent to C at z_0, and z_1 the point on C where the gradient is $\tan\phi$ when $\phi\geqslant\theta_0$ and $z_1=z_0$ if $\phi<\theta_0$.

The admissible values of η (for an appropriate K) and the sets $Q(\alpha)$ for real values of the boundary value parameters $\alpha\in(-\pi,\pi]$ are as follows (recall that $Q(\alpha)$ is defined in (2.7), where the admissible values of η must now satisfy $\sin 2\alpha\cos\eta\leqslant 0$).

Figure 1: $0<\eta\leqslant\pi/2-\phi<\pi/2$;

$$Q(\alpha)=\begin{cases} Q, & \text{if } \alpha\in[-\pi/2,0]\cup[\pi/2,\pi],\\ \boldsymbol{C}, & \text{if } \alpha\in(-\pi,-\pi/2)\cup(0,\pi/2).\end{cases}$$

Figure 2: $0<\eta\leqslant\pi/2-\phi<\pi$;

$$Q(\alpha)=\begin{cases} Q, & \text{if } \alpha\in\{-\pi/2,0,\pi/2,\pi\},\\ Q\cup\{z:\phi<\arg(z-z_0)\leqslant 0\}, & \text{if } \alpha\in(-\pi/2,0)\cup(\pi/2,\pi),\\ \{z:-\pi\leqslant\arg(z-z_0)\leqslant\phi\}, & \text{if } \alpha\in(-\pi,-\pi/2)\cup(0,\pi/2).\end{cases}$$

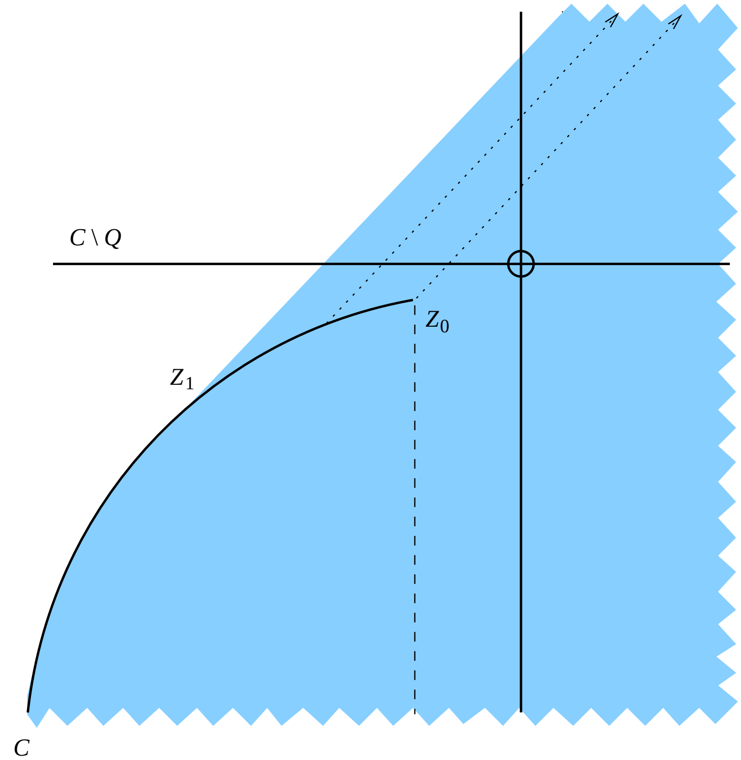

Figure 1. $0 < \phi < \pi/2$.

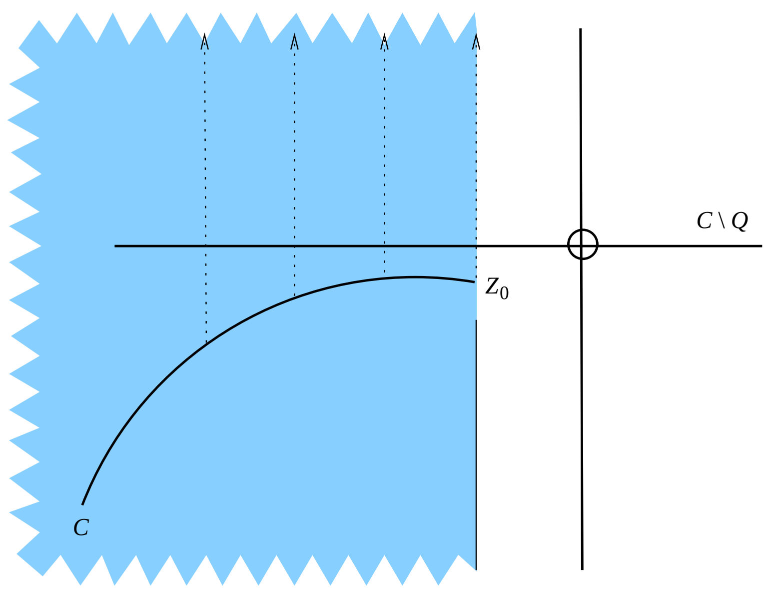

Figure 2. $-\pi/2 < \phi \leqslant 0$.

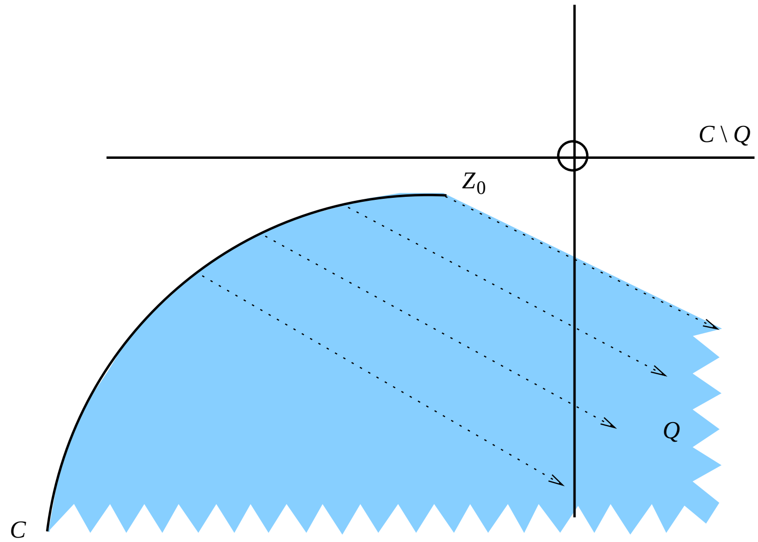

Figure 3. $\phi = \pi/2$.

Figure 3: $\eta = \pi$;

$$Q(\alpha) = \begin{cases} Q, & \text{if } \alpha \in [-\pi, -\pi/2] \cup [0, \pi/2], \\ \boldsymbol{C}, & \text{if } \alpha \in (-\pi/2, 0) \cup (\pi/2, \pi). \end{cases}$$

(*b*) *The classification of equation (1.2)*

In this section we analyse the Sims classification of (1.2) when the coefficients are

$$p(x) = p_1 x^{a_1} + \mathrm{i} p_2 x^{a_2}, \qquad q(x) = q_1 x^{b_1} + \mathrm{i} q_2 x^{b_2}, \qquad w(x) = x^{\omega}, \tag{5.3}$$

where p_j, q_j, a_j, b_j $(j = 1, 2)$ and ω are real, and $x \in [1, \infty)$. We write $A = \max(a_1, a_2)$ and $B = \max(b_1, b_2, \omega)$. Our results follow from an analysis of the asymptotic behaviour of linearly independent solutions of (1.2) at infinity as given by the Liouville–Green formulae (Eastham 1989). A general description covering all cases is far too complicated and hardly helpful. Instead, we provide a prescription for determining the classification. In each specific case the details are straightforward, though tedious.

(i) *The case $A - B < 2$*

In this case, linearly independent solutions $y_\pm$ exist that are such that, as $x \to \infty$,

$$y_\pm(x) \sim [p(x)s(x)]^{-1/4} \exp\left(\pm \int_1^x \mathrm{Re}[(s/p)^{1/2}]\,\mathrm{d}t\right), \tag{5.4}$$

$$p(x)y'_\pm(x) \sim [p(x)s(x)]^{1/4} \exp\left(\pm \int_1^x \mathrm{Re}[(s/p)^{1/2}]\,\mathrm{d}t\right), \tag{5.5}$$

where $s(x) = q(x) - \lambda w(x)$ (see Eastham (1989), p. 58). We use the notation $f(x) \sim g(x)$ to mean that $f(x)/g(x) \to 1$ as $x \to \infty$, and $f(x) \asymp g(x)$ if $|f(x)/g(x)|$ is bounded above and below by positive constants. Note that, for $z = r\mathrm{e}^{\mathrm{i}\theta} \in \boldsymbol{C}$, $0 \leqslant \theta < 2\pi$, $r > 0$, we define the nth root of z to be the complex number $r^{1/n}\mathrm{e}^{\mathrm{i}\theta/n}$.

Suppose that for some $\Lambda_{\eta,K}$, $(\eta, K) \in S(\alpha)$, and $\lambda \in \Lambda_{\eta,K}$, as $x \to \infty$,

$$\operatorname{Re}\left[\left(\frac{s(x)}{p(x)}\right)^{1/2}\right] = Dx^{\tau}\left(1 + O\left(\frac{1}{x^{\epsilon}}\right)\right), \quad D \neq 0, \quad \epsilon > 0, \quad D, \tau \in \boldsymbol{R}, \tag{5.6}$$

and

$$|p(x)s(x)| \asymp x^{\gamma}, \quad \gamma \in \boldsymbol{R}. \tag{5.7}$$

In each of the following cases, at least one of the solutions y_+ and y_- is not in $L^2(1, \infty; w\,\mathrm{d}x)$, and, hence, (1.2) is in case I:

1. $\tau > -1$;
2. $\tau = -1$ and $2|D| + \omega - \gamma/2 + 1 \geqslant 0$;
3. $\tau < -1$ and $\omega - \gamma/2 + 1 \geqslant 0$.

In all other cases when $A - B < 2$, and (5.6) and (5.7) hold, we are either in case II or case III: on setting

$$W_{\pm}(x) := \operatorname{Re}[\mathrm{e}^{\mathrm{i}\eta}(p(x)|y_{\pm}'(x)|^2 + s(x)|y_{\pm}(x)^2|)], \tag{5.8}$$

we have that case III prevails if W_+ and W_- are both integrable (which can be verified using (5.4) and (5.5)) and case II otherwise.

(ii) *The case $A - B = 2$*

In this case, the equation (1.2) is asymptotically of Euler type. Here the results of Eastham (1989, p. 75) give, with $c = \frac{1}{4}(\sqrt{17} - 1)$,

$$|y_+| \asymp x^{2(A-1)c}, \qquad |py_+'| \asymp x^{2(A-1)((1/2)+c)}$$

and

$$|y_-| \asymp x^{-2(A-1)((1/2)+c)}, \qquad |py_-'| \asymp x^{-2(A-1)c}.$$

At least one of the solutions y_+, y_- is not in $L^2(1, \infty; w\,\mathrm{d}x)$, and, hence, (1.2) is in case I, in each of the following cases.

1. $A > 1$ and $\omega + 4(A - 1)c + 1 \geqslant 0$.
2. $A = 1$ and $\omega \geqslant -1$.
3. $A < 1$ and $\omega - 4(A - 1)(\frac{1}{2} + c) + 1 \geqslant 0$.

In all other cases when $A - B = 2$, we are in case III when W_+ and W_- defined in (5.8) are both integrable, and case II otherwise.

(iii) *The case* $A - B > 2$

Here the relevant analysis is that in Eastham (1989, p. 78). It follows that

$$|y_+| \asymp 1, \qquad |y_+'| \asymp x^{(B-A)/2},$$
$$|y_-| \asymp x^{-(A+B)/2}, \qquad |y_-'| \asymp x^{-A}.$$

At least one of the solutions y_+, y_- is not in $L^2(1,\infty; w\,\mathrm{d}x)$, and hence, (1.2) is in case I if $\omega - \min\{0, A+B\} \geqslant -1$. If $\omega - \min\{0, A+B\} < -1$, (1.2) is in case III if $W_\pm$ are both integrable and case II otherwise.

The case $p = w = 1$ is covered in detail in Edmunds & Evans (1987, theorem III, 10.28); this includes the original example of Sims (1957, p. 257) establishing the existence of case II.

(*c*) *The spectra*

Finally, we investigate the spectra of the operators $\tilde{M}$ generated in $L^2(0,\infty)$ by expressions M of the form

$$M[y] = -y'' + cx^\beta y, \quad 0 \leqslant x < \infty, \tag{5.9}$$

where $\beta > 0$ and $c \in \boldsymbol{C}$ with $\arg c \in [0, \pi]$; the case $\arg c \in (\pi, 2\pi)$ is similar.

If $\arg c \neq \pi$, we have

$$Q = \{z : 0 \leqslant \arg z \leqslant \arg c\}, \qquad Q_\infty = \emptyset. \tag{5.10}$$

Suppose that

$$\operatorname{Im}[\overline{\sin\alpha}\cos\alpha] \geqslant 0. \tag{5.11}$$

Then, (2.6) is satisfied for $\eta = -\pi/2$ and, for any $K > 0$, $(-\pi/2, K) \in S(\alpha)$. Consequently,

$$Q(\alpha) \subseteq \boldsymbol{C}\backslash\Lambda_{-\pi/2,K} = \overline{\boldsymbol{C}_+}, \tag{5.12}$$

and, similarly,

$$Q_\infty(\alpha) = \emptyset \tag{5.13}$$

(see (3.7)). Also, it follows from § 5 *b* (i) (item 1) that case I holds. Hence, by theorem 4.7 and remark 4.8, for $\arg c \neq \pi$, the operator realization $\tilde{M}$ of M defined in (4.12) has the empty essential spectrum $\sigma_{\mathrm{e}}(\tilde{M})$. Such a result is given in Glazman (1965, theorem 30) for the analogous problem on $(-\infty,\infty)$.

If $\arg c = \pi$, we have

$$Q = Q_\infty = \boldsymbol{R}, \tag{5.14}$$

and, if (5.11) is satisfied,

$$Q(\alpha) \subseteq (\boldsymbol{C}\backslash\Lambda_{-\pi/2,K}) \cap (\boldsymbol{C}\backslash\Lambda_{\pi/2,K}) = \boldsymbol{R},$$

and hence,

$$Q(\alpha) = Q_\infty(\alpha) = \boldsymbol{R}. \tag{5.15}$$

For $\lambda = \mathrm{i}$ and $\eta = \pm\pi/2$, we now have $|W_\pm| = |y_\pm|^2$ and in § 5 *b* (i),

$$y_\pm(x) \asymp \begin{cases} x^{-\beta/4}, & \text{if } \beta > 2, \\ x^{-(1/2)\mp 1/(2|c|^{1/2})}, & \text{if } \beta = 2, \\ x^{-\beta/4}\exp\left[\mp\dfrac{x^{1-\beta/2}}{|c|^{1/2}(2-\beta)}\right], & \text{if } \beta < 2. \end{cases}$$

It follows that case I holds if $\beta \leqslant 2$ and case III if $\beta > 2$; note that case III is now the Weyl limit-circle case since M is formally symmetric. Hence, if $\arg c = \pi$, by theorem 4.5,

$$\sigma_{\mathrm{e}}(\tilde{M}) \begin{cases} = \emptyset, & \text{if } \beta > 2, \\ \subseteq \boldsymbol{R}, & \text{if } \beta \leqslant 2. \end{cases} \tag{5.16}$$

If α is real, (5.11) is satisfied. In this case, when $\beta \leqslant 2$, M is in the Weyl limit-point case at ∞ (so that $\tilde{M}$ is self-adjoint), and $\sigma_{\mathrm{e}}(\tilde{M}) = \boldsymbol{R}$ (see Titchmarsh 1962, theorem V.5.10).

In case I, the identity (5.17) below (which holds for (1.2) in general) is often useful and reinforces remark 4.8. Denote the functions θ, ϕ in (2.8) by θ_α, ϕ_α, respectively, and the corresponding m-function by m_α. Since $\alpha = 0, \pi/2$ satisfy (2.6) for any η, we have $Q(0) = Q(\pi/2) = Q$. Also, for $\lambda \notin Q(\alpha)$, there exist $K \neq 0$ such that

$$\theta_\alpha(x,\lambda) + m_\alpha(\lambda)\phi_\alpha(x,\lambda) = K[\theta_{\pi/2}(x,\lambda) + m_{\pi/2}(\lambda)\phi_{\pi/2}(x,\lambda)].$$

On substituting (2.8), we have

$$m_\alpha(\lambda) = \frac{m_{\pi/2}(\lambda)\sin\alpha - \cos\alpha}{m_{\pi/2}(\lambda)\cos\alpha + \sin\alpha}. \tag{5.17}$$

Hence, if $m_{\pi/2}$ is meromorphic in $\boldsymbol{C}$, the same is true of m_α, for any α.

An important special case of (5.9) is the expression for the harmonic oscillator

$$M[y] = -y'' + cx^2 y, \quad 0 \leqslant x < \infty.$$

On setting $x = z/(\sqrt{2}c^{1/4})$, the equation $(M-\lambda)[y] = 0$ becomes

$$-y'' + \tfrac{1}{4}z^2 y = \mu y, \tag{5.18}$$

where the prime now denotes differentiation with respect to z along the ray with argument $\frac{1}{4}\arg c$, and $\mu = \lambda/(2\sqrt{c})$. From Whittaker & Watson (1915, p. 341), for $0 \leqslant \arg c < \pi$, the unique solution of (5.18) in $L^2(0,\infty)$ is the parabolic cylinder function $D_{\mu-1/2}(z)$. It follows from (2.8) and the fact that our function ψ in (2.22) must be a constant multiple of $D_{\mu-1/2}(z)$ that

$$m_{\pi/2}(\lambda) = \frac{D_{\mu-1/2}(0)}{D'_{\mu-1/2}(0)},$$

and this gives

$$m_{\pi/2}(\lambda) = -\frac{1}{2c^{1/4}}\frac{\Gamma(1/4 - (\lambda/4\sqrt{c}))}{\Gamma(3/4 - (\lambda/4\sqrt{c}))}. \tag{5.19}$$

This is meromorphic with poles at

$$\lambda_n = (4n+1)\sqrt{c}, \quad n = 0, 1, 2, \ldots.$$

When $\arg c = \pi$, $Q = Q_\infty = \boldsymbol{R}$, and for $\alpha = \pi/2$ there are m-functions defined in $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$:

$$m^{(1)}_{\pi/2} = -\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{2|c|^{1/4}}\frac{\Gamma(1/4 - (\lambda/4\sqrt{c}))}{\Gamma(3/4 - (\lambda/4\sqrt{c}))}, \quad (\lambda \in \boldsymbol{C}_+),$$

$$m^{(2)}_{\pi/2} = -\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{2|c|^{1/4}}\frac{\Gamma(1/4 + (\lambda/4\sqrt{c}))}{\Gamma(3/4 + (\lambda/4\sqrt{c}))}, \quad (\lambda \in \boldsymbol{C}_-);$$

$\boldsymbol{C}_+$ and $\boldsymbol{C}_-$ are the connected components C_1 and C_2 referred to in lemma 3.1 and the following comment. These functions are not analytic continuations of each other and the self-adjoint operator $\tilde{M}$, with $\alpha = \pi/2$, has $\sigma_{\mathrm{e}}(\tilde{M}) = \boldsymbol{R}$: this is, therefore, true for all values of α by remark 4.8. Criteria on q for $\sigma_{\mathrm{e}}(\tilde{M}) \supseteq [0, \infty)$ in the case $p = w = 1$ are given in Kamimura (1979); see also Glazman (1965, Ch. VII).

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