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Weak eigenfunctions of two-interval Sturm-Liouville problems together with interaction conditions

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In this study, we consider a new type boundary value problem consisting of a Sturm-Liouville equation on two disjoint intervals together with interaction conditions and with eigenvalue parameter in the boundary conditions. We suggest a special technique to reduce the considered problem into an integral equation by the use of which we define a new concept, the so-called weak eigenfunction for the considered problem. Then we construct some Hilbert spaces and define some self-adjoint compact operators in these spaces in such a way that the considered problem can be interpreted as a self-adjoint operator-pencil equation. Finally, it is shown that the spectrum is discrete and the set of weak eigenfunctions form a Riesz basis of the suitable Hilbert space. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4979615]

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

Sturm-Liouville problems are one of the most important models of various physical fields, such as quantum mechanics, electrostatics, magnetostatics, fluid mechanics, and diffusion, but more often are a result of using the method of separation of variables to solve the classical partial differential equations of physics, such as Laplace's equation, the heat equation, and the wave equation (see Refs. 10, 24, 33, and 35 and references cited therein). For example, the Schrödinger equation is, in fact, a partial differential equation, but, in the case of spherically symmetric potentials such as the Coulomb potential, the standard technique of separation of variables reduces the equation to a sequence of ordinary differential equations, one for each pair of angular momentum quantum numbers. In this way, under the assumption of spherical symmetry, Sturm-Liouville theory can be applied to the Schrödinger equation.

Two-interval Sturm-Liouville problems with supplementary interaction conditions (such conditions are known by various names including interface conditions, transmission conditions, etc.) arise naturally in solving many transfer processes in mechanics and physics, for instance, in solving the following model problem, considered[15] by the method of separation of variables. For instance, consider the equation

$$-\Delta F(x,y,z) = f \ \text{ in } \ D = D_1 \cup D_2$$

together with the Dirichlet type boundary condition

$$F(x,y,z) = 0 \ \text{ on } \ \partial D$$

and supplementary interaction conditions

$$[F] = 0 \ \text{ across } \ \mathrm{T} = \overline{D_1} \cap \overline{D_2}$$

and

$$-\left[\frac{\partial F}{\partial n}\right] = c_0\, \Delta_t\, F \ \text{ on } \ \mathrm{T}.$$

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Here D is a regular bounded domain in $\mathbb{R}^3$, a cross section T divides the domain D into two subdomains D_1 and D_2, $[F] = F_1 - F_2$ denotes the jump of F across the section T, where F_i denotes the restriction of F to D_i, $\left[\frac{\partial F}{\partial n}\right] = \frac{\partial F_1}{\partial n_1} + \frac{\partial F_2}{\partial n_2}$ denotes the jump of the normal derivatives across the section T and n_i denotes the unit normal vector on ∂D_i, which points outward from D_i, $\frac{\partial F_i}{\partial n_i}$ denotes the directional derivative of F_i in the outward normal direction, c_0 is a given positive constant, and Δ_t denotes the tangential Laplacian on T. Such type of interaction problems arise often from problems of hydraulic fracturing or from problems of electrostatics and magnetostatics. Namely, the heat transfer through an infinitely conductive layer leads to the interaction problem. In this case, the constant c_0 is the dielectric constant or the magnetic permeability. For other interesting interaction problems, we refer to Refs. 8, 9, 15, 22, 23, and 36 and references cited therein.

The literature for Sturm-Liouville problems where the spectral parameter appears in the boundary conditions is diverse (see Refs. 1, 37, 2–6, 11, 14, 26–31, and 38 and references cited therein). Walter[38] gave a Hilbert space formulation and associated a self-adjoint operator with the considered problem for such type of problems. Fulton[11] has shown that the results on the eigenfunction expansion for regular Sturm-Liouville problems are carried over to the Sturm-Liouville problems with eigenparameter dependent boundary conditions. Bobrovnitskii[7] considered oscillations of some mechanical systems and established the orthogonality relations for eigenfunctions of the corresponding Sturm-Liouville problems with the spectral parameter in the boundary conditions. The problem of completeness of eigenfunctions is not discussed in Ref. 7. Hinton[14] considered the same family of Sturm-Liouville problems as Fulton,[11] but he included the case of a singular end point. He obtained the convergence theory, uniform and absolute on compact intervals, for the eigenfunction expansions.[36] For the case where all eigenvalues of this operator are nonnegative, he proved that the class of functions where convergence takes place is the domain of the square root of the operator generated by the considered problem. Steinrück[34] considered the singular-perturbed Sturm-Liouville problem for a couple of functions. He gave an asymptotic analysis of the problem but did not consider the completeness of eigenfunctions. The general results about operator polynomials including the multidimensional case can be found in monographs by Gohberg and Krein[12] and Rodman.[32] The case where the spectral parameter appears nonlinearly in the boundary conditions is more complex. Greenberg and Babuska[13] considered Sturm-Liouville problems for second-order and fourth-order differential equations, where the eigenvalue may occur nonlinearly in the differential operator and in the boundary conditions. Among many other papers, we would like to mention only Kostyuchenko and Shkalikov,[18] who considered an operator-differential equation in connection with oscillations of an elastic semi-infinite cylinder. They reduced the problem to an operator polynomial equation of the second order $\mathcal{A}(\lambda)u = 0$ with $\mathcal{A}(\lambda) := A + \lambda B + \lambda^2 C$. Here A is bounded and self-adjoint, and B and C are compact and self-adjoint.[19]

Markus[25] considered the operator polynomials of the form $\mathcal{A}(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots + \lambda^n A_n$, where λ is a spectral parameter and $A_0, A_1, \ldots, A_n$ are linear operators acting in a Hilbert space $\mathbb{H}$. In the simplest cases, $\mathcal{A}(\lambda) = A - \lambda I$ and $\mathcal{A}(\lambda) = I - \lambda A$, we come to the usual (linear) spectral problems. Spectral problems for polynomial pencils arise naturally in diverse areas of mathematical physics (differential equations and boundary value problems, controllable systems, the theory of oscillations and waves, elasticity theory, and hydromechanics). In the fundamental works of Keldysh,[17] the concepts of associated vectors, multiplicity of an eigenvalue, and multiple completeness of the eigenvectors and associated vectors were introduced.

Belinskiy and Dauer[6] have considered the generalized solutions of a regular Sturm-Liouville problem on a finite interval with the eigenvalue parameter appearing linearly in the boundary conditions. In Ref. 39, for a non-linear eigenvalue problem similar to a linear Sturm-Liouville problem the properties of the spectrum and the eigenfunctions are analysed and the system of eigenfunctions is shown to be a Riesz basis in L_2.

In the present work, we shall investigate a Sturm-Liouville equation

$$-(pu')'(x) + q(x)u(x) = \lambda r(x)u(x) \tag{1.1}$$

on two disjoint intervals (a, c) and (c, b), together with eigenparameter dependent boundary conditions

$$p(a)\left(\frac{u'}{u}\right)(a) = \frac{\alpha_1+\lambda\alpha_1'}{\alpha_2+\lambda\alpha_2'}, \tag{1.2}$$

$$p(b)\left(\frac{u'}{u}\right)(b) = \frac{\beta_1+\lambda\beta_1'}{\beta_2+\lambda\beta_2'}, \tag{1.3}$$

and two supplementary interaction conditions,

$$u(c+0) = u(c-0), \tag{1.4}$$

$$p(c+0)u'(c+0) = \delta u(c-0) + p(c-0)u'(c-0). \tag{1.5}$$

Everywhere below we shall assume that the coefficients of the boundary value transmission problem (BVTP) (1.1)–(1.5) satisfy the following assumptions.

Assumption 1.1. i. The functions $p(x)$, $q(x)$, and $r(x)$ are positive definite, bounded, and Lebesgue integrable on (a, c) and (c, b).
ii. *The numbers $\alpha_i, \alpha_i', \beta_i, \beta_i' (i=1,2)$ are real and*

$$\theta_1 = \begin{vmatrix} \alpha_2' & \alpha_1' \\ \alpha_2 & \alpha_1 \end{vmatrix} > 0, \qquad \theta_2 = \begin{vmatrix} \beta_1' & \beta_2' \\ \beta_1 & \beta_2 \end{vmatrix} > 0.$$

iii. *$\delta \in \mathbb{R}$ and $\delta > 0$.*

II. CONSTRUCTION OF THE ADEQUATE HILBERT SPACES

At first we shall give some auxiliary terminology and facts which are needed for further consideration.

The Lebesgue space $L_2(a, b)$ is the Hilbert space of square-integrable complex-valued functions on the interval $[a, b]$, with the scalar product $\langle u, v\rangle_{L_2(\Omega)} := \int_a^b u(x)\,\overline{v}(x)dx$ and corresponding norm $\|u\|^2_{L_2(a,b)} = \langle u, u\rangle_{L_2(a,b)}$.

The Sobolev space $W_2^k(a, b)$ is the Hilbert space which consists of the elements $L_2(a, b)$ having square-integrable generalized derivatives up to kth order on the interval $[a, b]$, with the inner product $\langle u, v\rangle_{W_2^k(a,b)} := \sum_{j=0}^k \langle u^{(j)}, \overline{v}^{(j)}\rangle_{L_2(a,b)}$ and corresponding norm $\|u\|^2_{W_2^k(a,b)} = \langle u, u\rangle_{W_2^k(a,b)}$.

Definition 2.1. A bounded linear operator $\mathcal{L}:\mathbb{H}\to\mathbb{H}$ is called positive if $\langle \mathcal{L}u, u\rangle \geq 0$ for all $u\in\mathbb{H}$. If *$\mathcal{L}$ is a positive operator, we shall write $\mathcal{L}\geq 0$. The expression $\mathcal{L}\geq\mathcal{M}$ will be used to mean $\mathcal{L}-\mathcal{M}\geq 0$ (see, for example, Ref. 20).*

Theorem 2.2. *If $\mathcal{L}:\mathbb{H}\to\mathbb{H}$ is a positive bounded operator, then there is a unique $\mathcal{M}\geq 0$ such that $\mathcal{L}=\mathcal{M}^2$, which is called a positive square root of $\mathcal{L}$ and is denoted by $\mathcal{L}^{\frac{1}{2}}$ or $\sqrt{\mathcal{L}}$ (see, for example, Ref.* 20).

A family of elements $\{u_n\}$ in a Hilbert space $\mathbb{H}$ is called a basis of this space if for an arbitrary function $u\in\mathbb{H}$ there exists a unique sequence of scalars $\{a_n\}$ such that $u=\sum_{k=1}^\infty a_k u_k$. A basis $\{u_n\}$ is said to be a orthogonal basis if $\langle u_n, u_m\rangle_{\mathbb{H}}=0$ provided that $n\neq m$. A orthogonal basis $\{u_n\}$ is said to be an orthonormal basis if $\langle u_n, u_m\rangle=\delta_{nm}$, where δ_{nm} is the Kronecker delta.

Definition 2.3. A basis $\{u_n\}_{n=0,1,2,\ldots}$ in a Hilbert space $\mathbb{H}$ is called a Riesz basis in $\mathbb{H}$ if the series $\sum_{n=0}^\infty a_n u_n$ with real coefficients a_n converges in $\mathbb{H}$ if and only if $\sum_{n=0}^\infty a_n^2<\infty$ (see, for example, Ref. 12).

According to the embedding theorems for the Sobolev spaces, the functions in $W_2^2(a, b)$ are continuously differentiable on $[a, b]$, though the functions in $W_2^1(a, b)$ can only be assumed to be continuous (see Refs. 16 and 21).

Let $\Omega_1=[a, c), \Omega_2=(c, b], \Omega=\Omega_1 \cup \Omega_2$ and $u_i(x), v_i(x), p_i(x), q_i(x)$, and $r_i(x)$ $(i = 1, 2)$ are restrictions of $u(x)$, $v(x)$, $p(x)$, $q(x)$, and $r(x)$ on the interval Ω_i, respectively, $(i = 1, 2)$. For the investigation of the BVTP (1.1)–(1.5), we use the direct sum space $\oplus L_2 := L_2(\Omega_1) \oplus L_2(\Omega_2)$ with the inner-product $\langle u, v\rangle_0 := \langle u_1, v_1\rangle_{L_2(\Omega_1)} + \langle u_2, v_2\rangle_{L_2(\Omega_2)}$. Let us introduce to the consideration the set $\oplus\widetilde{W}_2^1 = \left\{u \in \oplus L_2 \mid u_i \in W_2^1(\Omega_i)(i=1,2),\ u_2(c+0) = u_1(c-0)\right\}$. We can prove easily that the linear space $\oplus\widetilde{W}_2^1$ under the inner-product $\langle u, v\rangle_1 := \langle u, v\rangle_0 + \langle u', v'\rangle_0$ is a Hilbert space.

In the Hilbert space $\oplus\widetilde{W}_2^1$, we introduce another inner-product by

$$\langle u, v\rangle_2 := \sum_{i=1}^{2} \int_{\Omega_i} \left\{p_i(x)u_i'(x)\overline{v_i}'(x) + q_i(x)u_i(x)\overline{v_i}(x)\right\} dx$$

and the corresponding norm $\|u\|_2^2 = \langle u, u\rangle_2$.

It is obvious that there are some positive constants M_1 and M_2, such that

$$M_1\ \|u\|_1 < \|u\|_2 < M_2\ \|u\|_1 \tag{2.1}$$

for all $u \in \oplus\widetilde{W}_2^1$.

It is easy to see that the functions in $\oplus\widetilde{W}_2^1$ are continuous on $[a, c) \cup (c, b]$, but their generalized derivatives can only be assumed to be elements of $\oplus L_2$. From the embedding theorems for Sobolev spaces (see Ref. 21) it follows that the following inequality hold:

$$|u(x_j)|^2 \le \gamma\ \|u'\|_0^2 + \frac{2}{\gamma}\ \|u\|_0^2, \tag{2.2}$$

where $j = 1, 2, 3,\ x_1 = a, x_2 = c \mp 0, x_3 = b$, and γ is any positive real number which is small enough. Moreover, for any $\xi \in \Omega$ the inequality

$$|u(\xi)| \le C(\xi)\ \|u\|_1 \tag{2.3}$$

holds where the constant $C(\xi)$ is independent of the function u, i.e., is dependent only of ξ.

III. THE ADEQUATE INTEGRAL EQUATION AND DEFINITION OF THE WEAK EIGENFUNCTIONS

The concept of a weak solution is fundamental to this work. To define this concept, let us introduce to the consideration the Hilbert space Ξ, consisting of all vector-functions $(\chi(x), \chi_1, \chi_2) \in (\oplus\widetilde{W}_2^1) \oplus \mathbb{C}^2$ with the inner product defined by

$$\langle \Gamma, \Psi\rangle_\Xi := \langle \chi, \varphi\rangle_1 + \chi_1\,\overline{\varphi_1} + \chi_2\,\overline{\varphi_2}, \tag{3.1}$$

where $\Gamma = \left(\chi, \chi_1, \chi_2\right)$ and $\Psi = \left(\varphi, \varphi_1, \varphi_2\right) \in (\oplus\widetilde{W}_2^1) \oplus \mathbb{C}^2$.

By multiplying the differential equation (1.1) by the conjugate of an arbitrary $\eta \in W_2^1(a, c) \oplus W_2^1(c, b)$ satisfying $\eta(c-0) = \eta(c+0)$ and integrating by parts over the interval $\Omega_i (i = 1, 2)$, we have

$$\int_{\Omega_i} \left(-(pu')'(x) + q(x)u(x) - \lambda r(x)u(x)\right)\,\overline{\eta}(x)dx = 0 \ \text{ for } i = 1, 2.$$

Integrating by parts and using the boundary and the jump conditions, we get

$$\begin{aligned}
&-p_2(b)u_2'(b)\overline{\eta_2}(b) + p_1(a)u_1'(a)\overline{\eta_1}(a)\\
&+\left[(p_2u_2')(c+0)\overline{\eta_2}(c+0) - (p_1u_1')(c-0)\overline{\eta_1}(c-0)\right]\\
&+\int_{\Omega_1} \left\{p_1(x)u_1'(x)\,\overline{\eta_1}'(x) + q_1(x)u_1(x)\,\overline{\eta_1}(x)\right\} dx\\
&+\int_{\Omega_2} \left\{p_2(x)u_2'(x)\,\overline{\eta_2}'(x) + q_2(x)u_2(x)\,\overline{\eta_2}(x)\right\} dx\\
&= \lambda\left\{\int_{\Omega_1} r_1(x)u_1(x)\,\overline{\eta_1}(x)dx + \int_{\Omega_2} r_2(x)u_2(x)\,\overline{\eta_2}(x)dx\right\}.
\end{aligned} \tag{3.2}$$

Let us introduce to the consideration new parameters (unknowns)

$$\kappa_1 := \alpha_1' u_1(a) - \alpha_2'(p_1 u_1')(a) \tag{3.3}$$

and

$$\kappa_2 := \beta_1' u_2(b) - \beta_2'(p_2 u_2')(b). \tag{3.4}$$

We shall consider only the case $\alpha_2' \neq 0$ and $\beta_2' \neq 0$ (the other cases are investigated similarly).

From (3.3) we get

$$(p_1 u_1')(a) = \frac{\alpha_1'}{\alpha_2'} u_1(a) - \frac{\kappa_1}{\alpha_2'}. \tag{3.5}$$

Now the boundary condition (1.2), takes the form

$$-\alpha_1 u_1(a) + \alpha_2(p_1 u_1')(a) = \lambda \kappa_1. \tag{3.6}$$

Putting (3.5) in (3.6), we see that the boundary condition (1.2) takes the form

$$-\frac{u_1(a)}{\alpha_2'} - \frac{\alpha_2}{\alpha_2'}\frac{\kappa_1}{\theta_1} = \lambda \frac{\kappa_1}{\theta_1}. \tag{3.7}$$

From (1.3) and (3.4) it follows that

$$(p_2 u_2')(b) = \frac{\beta_1'}{\beta_2'} u_2(b) - \frac{\kappa_2}{\beta_2'}. \tag{3.8}$$

Similarly, the other boundary condition (1.3) reduced to the form

$$\frac{u_2(b)}{\beta_2'} - \frac{\beta_2}{\beta_2'}\frac{\kappa_2}{\theta_2} = \lambda \frac{\kappa_2}{\theta_2}. \tag{3.9}$$

By using the jump condition (1.5) and the condition $\eta(c-0) = \eta(c+0)$, we have the next equality

$$\begin{aligned}
&(p_2 u_2')(c+0)\,\overline{\eta_2}(c+0) - (p_1 u_1')(c-0)\,\overline{\eta_1}(c-0) \\
&= \left\{(p_2 u_2')(c+0) - (p_1 u_1')(c-0)\right\} \overline{\eta_1}(c-0) \\
&= \delta u(c-0)\,\overline{\eta}(c-0).
\end{aligned}$$

After substituting (3.5) and (3.8) in the integral identity (3.2) and by using the last equality we obtain

$$\begin{aligned}
&\langle u, \eta\rangle_2 - \frac{\beta_1'}{\beta_2'} u_2(b)\overline{\eta_2}(b) + \frac{\alpha_1'}{\alpha_2'} u_1(a)\overline{\eta_1}(a) + \delta u(c-0)\overline{\eta}(c-0) \\
&+ \frac{\kappa_2}{\beta_2'}\,\overline{\eta_2}(b) - \frac{\kappa_1}{\alpha_2'}\,\overline{\eta_1}(a) = \lambda\langle ru, \eta\rangle_0.
\end{aligned} \tag{3.10}$$

Thus relation (3.2) is transformed into three relations (3.7), (3.9), and (3.10), all terms of which are defined for $u, \eta \in \oplus \widetilde{W}_2^1$.

Now we are ready to introduce a new concept for the two-interval Sturm-Liouville problems of type (1.1)–(1.5), the so-called weak solution of this problem.

Definition 3.1. The element $\Gamma = \left(u(x),\ \kappa_1,\ \kappa_2\right) \in \Xi := (\oplus \widetilde{W}_2^1) \oplus \mathbb{C}^2$ *is said to be a weak solution of the BVTP* (1.1)–(1.5) *if equations* (3.7), (3.9), *and* (3.10) *are satisfied for any* $\eta \in \oplus \widetilde{W}_2^1$.

Let us introduce to the consideration some linear forms in $\eta \in \oplus \widetilde{W}_2^1$ given by the following equalities:

$$\tau_0(u, \eta) := -\frac{\beta_1'}{\beta_2'} u_2(b)\overline{\eta_2}(b) + \frac{\alpha_1'}{\alpha_2'} u_1(a)\overline{\eta_1}(a) + \delta u(c-0)\overline{\eta}(c-0), \tag{3.11}$$

$$\tau_1(u, \eta) := \langle ru, \eta\rangle_0 := \sum_{i=1}^{2} \int_{\Omega_i} r_i(x) u_i(x)\,\overline{\eta_i}(x)dx, \tag{3.12}$$

$$\tau_2(\kappa_1,\eta) := -\frac{\kappa_1}{\alpha_2'}\,\overline{\eta_1}(a), \tag{3.13}$$

$$\tau_3(\kappa_2,\eta) := \frac{\kappa_2}{\beta_2'}\,\overline{\eta_2}(b), \tag{3.14}$$

where $u \in \oplus\widetilde{W}_2^1$, $\kappa_1, \kappa_2 \in \mathbb{C}$.

Theorem 3.2. (i) *$\tau_0(u,\eta)$ and $\tau_1(u,\eta)$ are linear functionals in $\eta \in \oplus\widetilde{W}_2^1$ for any given $u \in \oplus\widetilde{W}_2^1$,*

(ii) *$\tau_2(\kappa_1,\eta)$ and $\tau_3(\kappa_2,\eta)$ are linear functionals in η for any given κ_1, $\kappa_2 \in \mathbb{C}$, respectively.*

Proof. From (3.11), (3.13), and (3.14), it follows immediately that

$$|\tau_0(u,\eta)| \le C_1\left\{|u_2(b)|\,|\eta_2(b)| + |u_1(a)|\,|\eta_1(a)| + |u(c-0)|\,|\eta(c-0)|\right\}, \tag{3.15}$$

$$|\tau_2(\kappa_1,\eta)| \le C_2|\kappa_1|\,|\eta_1(a)|, \tag{3.16}$$

and

$$|\tau_3(\kappa_2,\eta)| \le C_3|\kappa_2|\,|\eta_2(b)|, \tag{3.17}$$

respectively, so the linear forms $\tau_i(.\,,\ \eta) : \oplus\widetilde{W}_2^1 \to \mathbb{C}$ for $i = 0, 1, 2$ are continuous with respect to the second argument, which completes the proof.

Remark 3.3. Here, and below, the symbols C_n, for $n = 1, 2, \ldots$, are used to denote different constants which do not depend on the functions under consideration and whose exact values are not important for the proof.

By using the interpolation inequalities (2.2) and (2.3), we have the following inequalities:

$$|\tau_0(u,\eta)| \ \le\ C_4\ \|u\|_1\ \|\eta\|_1,$$
$$|\tau_2(\kappa_1,\eta)| \ \le\ C_5\ |\kappa_1|\ |\overline{\eta_1}(a)|\ \le\ C_6\ |\kappa_1|\ \|\eta\|_1,$$

and, similarly

$$|\tau_3(\kappa_2,\eta)| \ \le\ C_7\ |\kappa_2|\ \|\eta\|_1.$$

Now by using the well-known Schwarz inequality, we get

$$\begin{aligned}|\tau_1(u,\eta)| &\le \int_{\Omega_1} |r_1(x)|\,|u_1(x)|\,|\,\overline{\eta_1}(x)|dx + \int_{\Omega_2} |r_2(x)|\,|u_2(x)|\,|\overline{\eta_2}(x)|dx\\ &\le C_8\left\{\int_{\Omega_1} |u_1(x)|\,|\overline{\eta_1}(x)|dx + \int_{\Omega_2} |u_2(x)|\,|\overline{\eta_2}(x)|dx\right\}\\ &= C_8\left\{||u_1||\ ||\eta_1|| + ||u_2||\ ||\eta_2||\right\} \le\ C_9\ \|u\|_1\ \|\eta\|_1.\end{aligned} \tag{3.18}$$

The proof is complete. □

From (2.1) follows immediately the next corollary.

Corollary 3.4. The following inequalities hold:

$$|\tau_0(u,\eta)| \le C_{10}||u||_2\ ||\eta||_2, \tag{3.19}$$

$$|\tau_1(u,\eta)| \le C_{11}||u||_2\ ||\eta||_2, \tag{3.20}$$

$$|\tau_2(\kappa_1,\eta)| \le C_{12}|\kappa_1|\ ||\eta||_2, \tag{3.21}$$

$$|\tau_3(\kappa_2,\eta)| \le C_{13}|\kappa_2|\ ||\eta||_2. \tag{3.22}$$

The well-known Riesz representation theorem (see, for example, Ref. 20) shows that the following representations are valid for some bounded operators $S_k (k = 0, 1, 2, 3)$.

Corollary 3.5. There are bounded linear operators $S_0, S_1 : \oplus\widetilde{W}_2^1 \to \oplus\widetilde{W}_2^1$ *and* $S_2, S_3 : \mathbb{C} \to \oplus\widetilde{W}_2^1$ *such that*

$$\langle S_0 u, \eta\rangle_2 = \tau_0(u, \eta) \qquad \text{for all } u, \eta \in \oplus\widetilde{W}_2^1, \tag{3.23}$$

$$\langle S_1 u, \eta\rangle_2 = \tau_1(u, \eta) \qquad \text{for all } u, \eta \in \oplus\widetilde{W}_2^1, \tag{3.24}$$

$$\langle S_2 \kappa_1, \eta\rangle_2 = \tau_2(\kappa_1, \eta) \qquad \text{for all } \kappa_1 \in \mathbb{C}, \eta \in \oplus\widetilde{W}_2^1, \tag{3.25}$$

$$\langle S_3 \kappa_2, \eta\rangle_2 = \tau_3(\kappa_2, \eta) \qquad \text{for all } \kappa_2 \in \mathbb{C}, \eta \in \oplus\widetilde{W}_2^1. \tag{3.26}$$

Lemma 3.6. The operator $S_0 : \oplus\widetilde{W}_2^1 \to \oplus\widetilde{W}_2^1$ *is self-adjoint and compact.*

Proof. Let the sequence (u_k) converge weakly in $\oplus\widetilde{W}_2^1$ to an element $u \in \oplus\widetilde{W}_2^1$. Then the sequence $(S_0 u_k)$ converges weakly to $(S_0 u)$, since S_0 is bounded. Since each weakly convergent sequence is bounded, we have that the sequence (u_k) is bounded, so there is a constant $c_1 > 0$ such that $||u_k||_2 \le c_1$ for all $k = 1, 2, \ldots$. It is easy to see that the embedding operator $J : \oplus\widetilde{W}_2^1 \hookrightarrow \oplus L_2$ is compact; consequently the sequences (u_k) and $(S_0 u_k)$ converge strongly to u and $S_0 u$ in $\oplus L_2$, respectively.

Moreover, since for each finite interval $[\alpha, \beta]$ the embedding operator $J : W_2^1(\alpha, \beta) \hookrightarrow C[\alpha, \beta]$ is compact, we have that the sequences (u_k) and $(S_0 u_k)$ converge in $C(\Omega_1) \oplus C(\Omega_2)$. Consequently for each $c \in \Omega$ the sequences $(u_k(c))$ and $(S_0 u_k)(c)$ converge to $u(c)$ and $(S_0 u)(c)$, respectively.

Then by using Corollary 3.5, we get that there is a constant $C_{15} > 0$ such that

$$\begin{aligned}
&\| S_0 u_k - S_0 u_m\|_2^2 = \langle S_0(u_k - u_m), S_0(u_k - u_m)\rangle_2 \\
&= \tau_0(u_k - u_m\ ,\ S_0(u_k - u_m)) \\
&\le C_{14}\{\, |(u_k(b) - u_m(b))| \cdot |(S_0(u_k - u_m))(b)| \\
&+ |(u_k(a) - u_m(a))| \cdot |(S_0(u_k - u_m))(a)| \\
&+ |(u_k(c-0) - u_m(c-0))| \cdot |(S_0(u_k - u_m))(c-0)|\,\}.
\end{aligned}$$

Therefore $(S_0 u_k)$ is the Cauchy sequence in the Hilbert space $\oplus\widetilde{W}_2^1$. Thus S_0 is compact in $\oplus\widetilde{W}_2^1$.

Now, let $u, \eta \in \oplus\widetilde{W}_2^1$ be arbitrary functions. Then by using (3.11) and (3.23), we have immediately that

$$\langle u, S_0\eta\rangle_2 = \overline{\langle S_0\eta, u\rangle_2} = \overline{\tau_0(\eta, u)} = \langle \tau_0 u, \eta\rangle_2 = \langle S_0 u, \eta\rangle_2,$$

so the operator S_0 is self-adjoint in the Hilbert space $\oplus\widetilde{W}_2^1$. The proof is complete. □

Lemma 3.7. The operator S_1 *is self-adjoint, compact, and positive in the Hilbert space* $\oplus\widetilde{W}_2^1$.

Proof. Let the sequence (u_k) be any weakly convergent sequence in $\oplus\widetilde{W}_2^1$. By using Corollary 3.5 and Cauchy-Schwarz inequality from (3.12), we have

$$\begin{aligned}
&\| S_1 u_k - S_1 u_m\|_2^2 = \langle S_1(u_k - u_m), S_1(u_k - u_m)\rangle_2 \\
&= \tau_1(u_k - u_m\ ,\ S_1(u_k - u_m)) \\
&\le \sum_{i=1}^{2} \left| \int_{\Omega_i} r_i(u_k(x) - u_m(x)) \cdot \overline{S_1(u_k - u_m)} dx \right| \\
&\le C_{15}\ \|u_k - u_m\|_0 \cdot \|S_1(u_k - u_m)\|_2.
\end{aligned}$$

Consequently

$$\|S_1 u_k - S_1 u_m\|_2 \le C_{16}\ \|u_k - u_m\|_0.$$

The embedding theorems for Sobolev spaces (see Ref. 21) imply that the sequence (u_k) converges strongly in $\oplus L_2$, from which it follows immediately that the sequence $(S_1 u_k)$ converges in the Hilbert space $\oplus \widetilde{W}_2^1$. That is, S_1 is compact in $\oplus \widetilde{W}_2^1$.

Since $r_1(x)$ and $r_2(x)$ are positive definite functions, the positivity of S_1 is obvious. The self-adjointness of S_1 follows immediately from (3.12) and (3.24). The proof is complete. □

Lemma 3.8. The operators $S_2, S_3 : \mathbb{C} \to \oplus \widetilde{W}_2^1$ are compact.

Proof. It is easy to verify that the adjoint operators of S_2 and S_3 are defined on whole $\oplus \widetilde{W}_2^1$ with equalities $S_2^*(u) = -\frac{1}{\alpha_2'}\, u(a)$ and $S_3^*(u) = \frac{1}{\beta_2'}\, u(b)$, respectively. From this representation, it follows that the operators S_2^* and S_3^* from $\oplus \widetilde{W}_2^1$ to $\mathbb{C}$ are bounded, i.e.,

$$|S_k^* u| \leq C_{17}\, \|u\|_2$$

for $k = 2, 3$. Consequently the operators S_2^* and S_3^* are bounded linear operators with a finite dimensional range and therefore are compact.

Then by virtue of the well-known theorem of functional analysis (see, for example, Ref. 20), the operators S_2 and S_3 are also compact. The proof is complete. □

IV. OPERATOR TREATMENT OF THE PROBLEM

Observe that from the integral identities (3.7), (3.9), and (3.10), it follows that the following equalities hold:

$$\langle u + S_0 u + S_2 \kappa_1 + S_3 \kappa_2\ ,\ \eta \rangle_2 = \lambda \langle S_1 u, \eta \rangle_2, \tag{4.1}$$

$$S_2^* u - \frac{\alpha_2}{\alpha_2'} \frac{\kappa_1}{\theta_1} = \lambda \frac{\kappa_1}{\theta_1}, \tag{4.2}$$

$$S_3^* u - \frac{\beta_2}{\beta_2'} \frac{\kappa_2}{\theta_2} = \lambda \frac{\kappa_2}{\theta_2}. \tag{4.3}$$

The arbitrariness of $\eta \in \oplus \widetilde{W}_2^1$ implies

$$u + S_0 u + S_2 \kappa_1 + S_3 \kappa_2 = \lambda S_1 u. \tag{4.4}$$

Let us define two operators $\prod$ and $\coprod$ in the Hilbert space Ξ by the equalities

$$\prod (u, \kappa_1, \kappa_2) = \left(u + S_0 u + S_2 \kappa_1 + S_3 \kappa_2\ ,\ S_2^* u - \frac{\alpha_2}{\alpha_2'} \frac{\kappa_1}{\theta_1}\ ,\ S_3^* u - \frac{\beta_2}{\beta_2'} \frac{\kappa_2}{\theta_2} \right) \tag{4.5}$$

and

$$\coprod (u, \kappa_1, \kappa_2) = \left(S_1 u\ ,\ \frac{\kappa_1}{\theta_1}\ ,\ \frac{\kappa_2}{\theta_2} \right), \tag{4.6}$$

where κ_1, κ_2 are defined in (3.3) and (3.4) and consider the operator-pencil equation

$$\mathcal{L}(\lambda)\, \Gamma = 0\,, \tag{4.7}$$

with $\mathcal{L}(\lambda) = \prod - \lambda\ \coprod$.

Remark 4.1. It is important to note that if $\lambda \in \mathbb{C}$ be any eigenvalue with the weak eigenfunction $\widetilde{u}(x, \lambda) := \left(u(x, \lambda)\ ,\ \kappa_1\ ,\ \kappa_2 \right)$, then the operator polynomial equation

$$\mathcal{L}(\lambda)\, \widetilde{u}(., \lambda) = 0 \tag{4.8}$$

holds in the Hilbert space Ξ.

Theorem 4.2. *The operator polynomial $\mathcal{L}(-\lambda_0)$ is positive definite for sufficiently large positive values of λ_0.*

Proof. By using (3.23)–(3.26), we get

$$\begin{aligned}\langle \mathcal{L}(-\lambda_0)\Gamma,\Gamma\rangle_{\Xi} &= \langle u(x),u(x)\rangle_2 + \langle S_0u(x),u(x)\rangle_2 + \langle S_2\kappa_1,u(x)\rangle_2\\ &\quad + \langle S_3\kappa_2,u(x)\rangle_2 + (S_2^*u(x))\overline{\kappa_1} + (S_3^*u(x))\overline{\kappa_2} - \frac{\alpha_2}{\alpha_2'\,\theta_1}|\kappa_1|^2\\ &\quad - \frac{\beta_2}{\beta_2'\,\theta_2}|\kappa_2|^2 + \lambda_0\left\{\langle S_1u(x),u(x)\rangle_2 + \frac{1}{\theta_1}|\kappa_1|^2 + \frac{1}{\theta_2}|\kappa_2|^2\right\}. \end{aligned}\tag{4.9}$$

Below we will use the following functionals:

$$P(u) := \langle pu',u'\rangle_0,\ \ Q(u) := \langle qu,u\rangle_0,\ \ R(u) := \langle ru,u\rangle_0. \tag{4.10}$$

From the well-known embedding theorems for Sobolev spaces (see, for example, Ref. 12), it follows that there exists a constant $C_{jk} > 0$ such that, the inequalities

$$|u(x_j)|^2 \le C_{j1}\varepsilon_j P(u) + \frac{C_{j2}}{\varepsilon_j}Q(u), \qquad u\in\oplus\widetilde{W}_2^1\ (j=1,2,3) \tag{4.11}$$

hold for sufficiently small positive ε_j, where $x_1=a, x_2=c\mp 0, x_3=b$.

By using (4.10) and inequality (4.11), we have the following estimates:

$$\begin{aligned}\langle S_0u(x),u(x)\rangle_2 &= -\frac{\beta_1'}{\beta_2'}\,|u_2(b)|^2 + \frac{\alpha_1'}{\alpha_2'}\,|u_1(a)|^2 + \delta|u(c-0)|^2\\ &\ge -\left(\left|\frac{\alpha_1'}{\alpha_2'}\right|C_{11}\varepsilon_1 + |\delta|C_{21}\varepsilon_2 + \left|\frac{\beta_1'}{\beta_2'}\right|C_{31}\varepsilon_3\right)P(u)\\ &\quad -\left(\left|\frac{\alpha_1'}{\alpha_2'}\right|\frac{C_{12}}{\varepsilon_1} + |\delta|\frac{C_{22}}{\varepsilon_2} + \left|\frac{\beta_1'}{\beta_2'}\right|\frac{C_{32}}{\varepsilon_3}\right)Q(u).\end{aligned}\tag{4.12}$$

By using Corollary 3.5 and applying the well-known Young inequality, we have

$$\begin{aligned}\langle S_2\kappa_1,u(x)\rangle_2 + (S_2^*u(x))\overline{\kappa_1} &= -\frac{2}{\alpha_2'}\ Re(\kappa_1\,\overline{u_1}(a))\\ &\ge -\frac{1}{|\alpha_2'|\,\gamma_1}\,|u_1(a)|^2 - \frac{\gamma_1}{|\alpha_2'|}|\kappa_1|^2\\ &\ge -\frac{1}{|\alpha_2'|\,\gamma_1}\left\{C_{11}\varepsilon_1P(u) + \frac{C_{12}}{\varepsilon_1}Q(u)\right\}\\ &\quad -\frac{\gamma_1}{|\alpha_2'|}|\kappa_1|^2\end{aligned}\tag{4.13}$$

and

$$\begin{aligned}\langle S_3\kappa_2,u(x)\rangle_2 + (S_3^*u(x))\overline{\kappa_2} &= \frac{2}{\beta_2'}\ Re(\kappa_2\,\overline{u_2}(b))\\ &\ge -\frac{1}{|\beta_2'|\,\gamma_2}\,|u_2(b)|^2 - \frac{\gamma_2}{|\beta_2'|}|\kappa_2|^2\\ &\ge -\frac{1}{|\beta_2'|\,\gamma_2}\left\{C_{31}\varepsilon_3P(u) + \frac{C_{32}}{\varepsilon_3}Q(u)\right\}\\ &\quad -\frac{\gamma_2}{|\beta_2'|}|\kappa_2|^2\end{aligned}\tag{4.14}$$

for arbitrary $\gamma_1,\gamma_2 > 0$.

Since the functions $q(x)$ and $r(x)$ are positive and bounded, there exists a constant $M > 0$ such that

$$\langle S_1u(x),u(x)\rangle_2 = \tau_1(u,u) \ge MQ(u). \tag{4.15}$$

Taking in view the equality

$$\|u\|_2^2 = P(u) + Q(u) \tag{4.16}$$

and substituting (4.12)–(4.16) in (4.9), we have

$$\langle \mathcal{L}(-\lambda_0)\Gamma, \Gamma\rangle_\Xi \ \geq \ \daleth_1 P(u) \ + \ \daleth_2(\lambda_0)Q(u) \ + \ \daleth_3(\lambda_0)|\kappa_1|^2 + \daleth_4(\lambda_0)|\kappa_2|^2, \tag{4.17}$$

where

$$\daleth_1 := 1 \ - \ \left(\left|\frac{\alpha_1'}{\alpha_2'}\right| + \frac{1}{\gamma_1\,|\alpha_2'|}\right) C_{11}\varepsilon_1 - |\delta|\, C_{21}\,\varepsilon_2 - \left(\left|\frac{\beta_1'}{\beta_2'}\right| + \frac{1}{\gamma_2\,|\beta_2'|}\right) C_{31}\varepsilon_3, \tag{4.18}$$

$$\daleth_2(\lambda_0) := 1 - \left(\left|\frac{\alpha_1'}{\alpha_2'}\right| + \frac{1}{\gamma_1\,|\alpha_2'|}\right)\frac{C_{12}}{\varepsilon_1} - |\delta|\,\frac{C_{22}}{\varepsilon_2} - \left(\left|\frac{\beta_1'}{\beta_2'}\right| + \frac{1}{\gamma_2\,|\beta_2'|}\right)\frac{C_{32}}{\varepsilon_3} + \lambda_0 M, \tag{4.19}$$

$$\daleth_3(\lambda_0) := -\frac{\alpha_2}{\alpha_2'\theta_1} - \frac{\gamma_1}{|\alpha_2'|} + \frac{\lambda_0}{\theta_1}, \tag{4.20}$$

$$\daleth_4(\lambda_0) := -\frac{\beta_2}{\beta_2'\,\theta_2} \ - \frac{\gamma_2}{|\beta_2'|} + \frac{\lambda_0}{\theta_2}. \tag{4.21}$$

Since $\theta_1 > 0,\ \theta_2 > 0$, it is possible to choose the arbitrary positive parameters $\gamma_1, \gamma_2, \varepsilon_1, \varepsilon_2$ and ε_3 so small and the positive parameter λ_0 so large that the inequalities $\daleth_1 > 0,\ \daleth_2(\lambda_0) > 0,\ \daleth_3(\lambda_0) > 0$, and $\daleth_4(\lambda_0) > 0$ hold. Now denoting

$$\daleth(\lambda_0) := \min\left(\daleth_1\ ,\ \daleth_2(\lambda_0)\ ,\ \daleth_3(\lambda_0)\ ,\ \daleth_4(\lambda_0)\right) \tag{4.22}$$

we have

$$\langle \mathcal{L}(-\,\lambda_0)\Gamma, \Gamma\rangle_\Xi \geq \daleth(\lambda_0)\ \|\Gamma\|_\Xi^2 \tag{4.23}$$

for all $\Gamma \in \Xi$ and hence the quadratic form $\langle \mathcal{L}(-\,\lambda_0)\Gamma, \Gamma\rangle_\Xi$ is positive definite for sufficiently large positive values of λ_0. Thus the operator pencil $\mathcal{L}(-\lambda_0)$ is positive definite for sufficiently large $\lambda_0 \ > \ 0$. The proof is complete. □

V. DISCRETENESS OF THE SPECTRUM AND COMPLETENESS OF THE WEAK EIGENFUNCTIONS

In this section, we shall prove that the system of weak eigenfunctions of the BVTP (1.1)–(1.5) forms a Riesz basis of the Hilbert space Ξ.

Theorem 5.1. *The operators* $\prod$ *and* $\coprod$ *are self-adjoint in the Hilbert space* Ξ.

Proof. Let $\Gamma_1 = (u(.), \kappa_1, \kappa_2)$ and $\Gamma_2 = (v(.), \kappa_1', \kappa_2')$ be any two elements of Ξ. Then making use of the representations $S_2^* u = -\frac{1}{\alpha_2'}\ u_1(a),\ S_3^* u = \frac{1}{\beta_2'}\ u_2(b)$ from (3.23)–(3.26), (4.5), and (4.6) we have

$$\langle \prod \Gamma_1, \Gamma_2\rangle_\Xi = \langle u + S_0 u + S_2\kappa_1 + S_3\kappa_2\ ,\ v\rangle_2 \ + \ \left(S_2^* u - \frac{\alpha_2}{\alpha_2'}\frac{\kappa_1}{\theta_1}\right)\overline{\kappa_1'} + \left(S_3^* u - \frac{\beta_2}{\beta_2'}\frac{\kappa_2}{\theta_2}\right)\overline{\kappa_2'}, \tag{5.1}$$

$$\langle \Gamma_1, \prod \Gamma_2\rangle_\Xi = \overline{\langle \prod \Gamma_2, \Gamma_1\rangle_\Xi} = \overline{\langle v + S_0 v + S_2\kappa_1' + S_3\kappa_2'\ ,\ u\rangle_2} + \overline{\left(S_2^* v - \frac{\alpha_2}{\alpha_2'}\frac{\kappa_1'}{\theta_1}\right)}\ \overline{\kappa_1} + \overline{\left(S_3^* v - \frac{\beta_2}{\beta_2'}\frac{\kappa_2'}{\theta_2}\right)}\ \overline{\kappa_2}. \tag{5.2}$$

Subtracting (5.2) from (5.1) and realizing that

$$\langle S_i\kappa_j, v\rangle_2 \ = \ \kappa_j\ \overline{(S_i^* v)} \ \text{ and } \ S_i^* u\overline{\kappa_j'} \ = \ \langle u, S_i\kappa_j'\rangle_\Xi$$

by the definition of the adjoint operator, we see that

$$\langle \prod \Gamma_1, \Gamma_2 \rangle_{\Xi} - \langle \Gamma_1, \prod \Gamma_2 \rangle_{\Xi} = 0, \tag{5.3}$$

so the operator $\prod$ is self-adjoint in the Hilbert space Ξ.

Similarly we have

$$\langle \coprod \Gamma_1, \Gamma_2 \rangle_{\Xi} = \langle S_1 u\ ,\ v \rangle_2\ +\ \frac{\kappa_1\ \overline{\kappa_1'}}{\theta_1^2} + \frac{\kappa_2\ \overline{\kappa_2'}}{\theta_2^2}, \tag{5.4}$$

$$\begin{aligned} \langle \Gamma_1, \coprod \Gamma_2 \rangle_{\Xi} &= \overline{\langle \coprod \Gamma_2, \Gamma_1 \rangle_{\Xi}} = \overline{\langle S_1 v\ ,\ u \rangle_2}\ +\ \overline{\frac{\kappa_1'\ \overline{\kappa_1}}{\theta_1^2}} + \overline{\frac{\kappa_2'\ \overline{\kappa_2}}{\theta_2^2}} \\ &= \langle S_1 u\ ,\ v \rangle_2\ +\ \frac{\kappa_1\ \overline{\kappa_1'}}{\theta_1^2} + \frac{\kappa_2\ \overline{\kappa_2'}}{\theta_2^2}. \end{aligned} \tag{5.5}$$

Consequently,

$$\langle \coprod \Gamma_1, \Gamma_2 \rangle_{\Xi} = \langle \Gamma_1, \coprod \Gamma_2 \rangle_{\Xi} \tag{5.6}$$

so the operator $\coprod$ is also compact in the Hilbert space Ξ, which completes the proof. □

Corollary 5.2. The operator $\mathcal{L}(-\lambda) = \prod + \lambda\ \coprod$ *is self-adjoint for each real* λ.

Since the operator $\mathcal{L}(-\lambda_0)$ is positive definite and self-adjoint for sufficiently large $\lambda_0 > 0$, there exists a positive square root of this operator for sufficiently large $\lambda_0 > 0$. Since $\prod + \lambda_0\ \coprod$ is positive definite, the operator $(\prod + \lambda_0\ \coprod)^{\frac{1}{2}}$ is invertible. Therefore we can introduce to the consideration a new operator $\mathrm{S}(\lambda_0)$ defined by

$$\mathrm{S}(\lambda_0) := \left((\prod + \lambda_0\ \coprod)^{\frac{1}{2}} \right)^{-1} \coprod \left((\prod + \lambda_0\ \coprod)^{\frac{1}{2}} \right)^{-1} \tag{5.7}$$

in the Hilbert space Ξ.

Theorem 5.3. *The operator* $\mathrm{S}(\lambda_0)$ *is positive, self-adjoint, and compact in the Hilbert space* Ξ *for sufficiently large* $\lambda_0 > 0$.

Proof. From representation (4.6), we have

$$\begin{aligned} \langle \coprod \Gamma, \Gamma \rangle_{\Xi} &= \langle S_1 u, u \rangle_2\ + \frac{1}{\theta_1} |\kappa_1|^2 + \frac{1}{\theta_2} |\kappa_2|^2 \\ &= R(u)\ + \frac{1}{\theta_1} |\kappa_1|^2\ +\ \frac{1}{\theta_2} |\kappa_2|^2 \geq 0 \end{aligned} \tag{5.8}$$

for all $\Gamma = (u(x), \kappa_1, \kappa_2) \in \Xi$, so $\coprod$ is positive in Ξ. On the other hand, by virtue of (4.17) and Lemma 5.1 the operator $\prod + \lambda_0\ \coprod$ is self-adjoint and positive definite, and the operator $\left((\prod + \lambda_0\ \coprod)^{\frac{1}{2}} \right)^{-1}$ is positive, self-adjoint, and bounded. Moreover, since $\coprod$ is compact, then by virtue of the standard theorems for compact operators, the operator $\mathrm{S}(\lambda_0)$ is a positive, self-adjoint, and compact operator in the Hilbert space Ξ. □

Lemma 5.4. Let λ_n *be any eigenvalue with corresponding eigenfunction* Γ_n *of the BVTP* (1.1)–(1.5). *Then* $\eta_n = \frac{1}{(\lambda_0 + \lambda_n)}$ *is the eigenvalue of the operator* $\mathrm{S}(\lambda_0)$ *with a corresponding weak eigenfunction* $\Psi_n = (\prod + \lambda_0\ \coprod)^{\frac{1}{2}} \Gamma_n$.

Proof. The proof of this lemma follows from (5.7). □

Corollary 5.5. For sufficiently large λ_0*, the eigenvalues of the operator* $S(\lambda_0)$ *form a real discrete set with the only point of accumulation at zero and the set of eigenfunctions* $\{\Psi_k\}$ *forms an orthogonal basis of* Ξ.

Proof. The proof of this corollary follows from the well-known theorems for compact self-adjoint operators in the Hilbert space (see, for example, Ref. 20). □

Now, by using the well-known fact that every bounded invertible operator transforms any orthonormal basis of a Hilbert space Ξ into a Riesz basis of Ξ (see, for example, Ref. 12), we get the following important result.

Theorem 5.6. *The eigenvalues of the BVTP* (1.1)–(1.5) *form a real discrete set* $\{\lambda_n\}$ *with* $\lambda_n \to +\infty$. *The system of the weak eigenfunctions of the BVTP* (1.1)–(1.5) *forms a Riesz basis of* Ξ.

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