

1 Solutions in cylindrical coordinates: Bessel functions

1.1 Bessel functions

Laplace's equation in cylindrical coordinates is:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial}{\partial \phi} \left(\frac{1}{\rho} \frac{\partial \Phi}{\partial \phi} \right) + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Separate variables: Let $\Phi = R(\rho) W(\phi) Z(z)$. Then we find:

$$\frac{1}{R\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial R}{\partial \rho} \right) + \frac{1}{W\rho^2} \frac{\partial^2 W}{\partial \phi^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = 0$$

The last term is a function of z only, while the sum of the first two terms is a function of ρ and ϕ only. Thus we take each part to be a constant called k^2 . Then

$$\frac{\partial^2 Z}{\partial z^2} = k^2 Z$$

and the solutions are

$$Z = e^{\pm kz}$$

This is the appropriate solution outside of a charge distribution, say above a plane, ($\Phi \rightarrow 0$ as $z \rightarrow \pm\infty$), or inside a cylinder with grounded walls and non-zero potential on one end.

The remaining equation is:

$$\frac{1}{R\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial R}{\partial \rho} \right) + \frac{1}{W\rho^2} \frac{\partial^2 W}{\partial \phi^2} + k^2 = 0$$

Now multiply through by ρ^2 :

$$\frac{\rho}{R} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial R}{\partial \rho} \right) + k^2 \rho^2 + \frac{1}{W} \frac{\partial^2 W}{\partial \phi^2} = 0$$

Here the last term is a function of ϕ only and the first two terms are functions of ρ only. Again we often want a solution that is periodic with period 2π , so we choose a negative separation constant:

$$\frac{\partial^2 W}{\partial \phi^2} = -m^2 W \Rightarrow W = e^{\pm im\phi}$$

Finally we have the equation for the function of ρ :

$$\rho \frac{\partial}{\partial \rho} \left(\rho \frac{\partial R}{\partial \rho} \right) + k^2 \rho^2 R - m^2 R = 0$$

To see that this equation is of Sturm-Liouville form, divide through by ρ :

$$\frac{\partial}{\partial \rho} \left(\rho \frac{\partial R}{\partial \rho} \right) + k^2 \rho R - \frac{m^2}{\rho} R = 0 \quad (1)$$

Now we have a Sturm-Liouville equation with $f(\rho) = \rho$, $g(\rho) = m^2/\rho$, eigenvalue $\lambda = k^2$ and weighting function $w(\rho) = \rho$. Equation (1) is Bessel's equation. The solutions are orthogonal functions. Since $f(0) = 0$, we do not need to specify any boundary condition at $\rho = 0$ if our range is $0 \leq \rho \leq a$, as is frequently the case. (We do specify that R remain finite.) We do need a boundary condition at $\rho = a$.

It is simpler and more elegant to solve Bessel's equation if we change to the dimensionless variable $x = k\rho$. Then:

$$\begin{aligned} k \frac{\partial}{\partial k\rho} \left(k\rho \frac{\partial R}{\partial k\rho} \right) + k^2 \rho R - k \frac{m^2}{k\rho} R &= 0 \\ \frac{d}{dx} \left(x \frac{dR}{dx} \right) + xR - \frac{m^2}{x} R &= 0 \end{aligned}$$

The equation has a singular point at $x = 0$. So we look for a series solution of the Frobenius type (cf Lea Chapter 3 §3.3.2):

$$\begin{aligned} R &= x^p \sum_{n=0}^{\infty} a_n x^n \\ R' &= \sum_{n=0}^{\infty} (n+p) a_n x^{n+p-1} \\ \frac{d}{dx} \left(x \frac{dR}{dx} \right) &= \sum_{n=0}^{\infty} (n+p)^2 a_n x^{n+p-1} \end{aligned}$$

Then the equation becomes:

$$\sum_{n=0}^{\infty} (n+p)^2 a_n x^{n+p-1} + \sum_{n=0}^{\infty} a_n x^{n+p+1} - m^2 \sum_{n=0}^{\infty} a_n x^{n+p-1} = 0$$

The indicial equation is given by the coefficient of x^{p-1} :

$$p^2 - m^2 = 0 \Rightarrow p = \pm m$$

Thus one of the solutions (with $p = m$) is analytic at $x = 0$, and one (with $p = -m$) is not. To find the recursion relation, look at the $k+p-1$ power of x :

$$(k+p)^2 a_k + a_{k-2} - m^2 a_k = 0$$

and so

$$\begin{aligned} a_k &= -\frac{a_{k-2}}{(k+p)^2 - m^2} = -\frac{a_{k-2}}{k^2 + 2kp + p^2 - m^2} \\ &= -\frac{a_{k-2}}{k^2 + 2kp} = -\frac{a_{k-2}}{k(k \pm 2m)} \end{aligned}$$

Let's look first at the solution with $p = +m$. We can step down to find each a_k . If we start the series with a_0 , then k will always be even, $k = 2n$, and

$$\begin{aligned} a_{2n} &= \frac{-1}{2n(2n+2m)} \frac{-1}{(2n-2)(2n-2+2m)} a_{2n-4} \\ &= \frac{(-1)^3}{2^3 n(n-1)(n-2) 2^3 (n+m)(n+m-1)(n+m-2)} a_{2n-6} \\ &= a_0 \frac{(-1)^n}{2^n n!} \frac{1}{2^n (n+m)(n+m-1) \cdots (m+1)} \end{aligned}$$

The usual convention is to take

$$a_0 = \frac{1}{2^m \Gamma(m+1)} \quad (2)$$

Then

$$\begin{aligned} a_{2n} &= \frac{1}{2^m \Gamma(m+1)} \frac{(-1)^n}{2^n n!} \frac{1}{2^n (n+m)(n+m-1) \cdots (m+1)} \\ &= \frac{(-1)^n}{n! \Gamma(n+m+1) 2^{2n+m}} \end{aligned} \quad (3)$$

and the solution is the Bessel function:

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+m+1)} \left(\frac{x}{2}\right)^{m+2n} \quad (4)$$

The function $J_m(x)$ has only even powers if m is an even integer and only odd powers if m is an odd integer. The series converges for all values of x .

Let's see what the second solution looks like. With $p = -m$ the recursion relation is:

$$a_k = \frac{a_{k-2}}{k(k-2m)} \quad (5)$$

where again $k = 2n$. Now if m is an integer we will not be able to determine a_{2m} because the recursion relation blows up. One solution to this dilemma is to start the series with a_{2m} . Then we can find the succeeding a_{2n} :

$$a_{2(n+m)} = \frac{-a_{2(n-1)+2m}}{2^2 (n+m)n} = \frac{-a_{2(n-2)+2m}}{2^4 (n+m)(n+m-1)n(n-1)} = \frac{(-1)^n \Gamma(m+1)}{n! \Gamma(n+m+1) 2^{2n}} a_{2m}$$

which is the same recursion relation we had before. (Compare the equation above with equation (3). Thus we do not get a linearly independent solution

this way¹. (This dilemma does not arise if the separation constant is taken to be $-\nu^2$ with ν non-integer. In that case the second recursion relation provides a series $J_{-\nu}(x)$ that is linearly independent of the first.) Indeed we find:

$$\begin{aligned} J_{-m}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(m+1)}{n! \Gamma(n+m+1) 2^{2n}} a_{2m} x^{2(n+m)-m} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(m+1) 2^m}{n! \Gamma(n+m+1)} a_{2m} \left(\frac{x}{2}\right)^{m+2n} \end{aligned}$$

and if we choose

$$a_{2m} = \frac{(-1)^m}{\Gamma(m+1) 2^m}$$

then

$$J_{-m}(x) = (-1)^m J_m(x) \quad (6)$$

With this choice $J_\nu(x)$ is a continuous function of ν . (Notice that we can also express the series using equation (3) for the coefficients, with $m \rightarrow -m$ and $n \rightarrow k+m$, and where $a_{2n} \equiv 0$ for $n < m$.)

We still have to determine the second, linearly independent solution of the Bessel equation. We can find it by taking the limit as $\nu \rightarrow m$ of a linear combination of J_ν and $J_{-\nu}$ known as the Neumann function $N_\nu(x)$:

$$\begin{aligned} N_m(x) &= \lim_{\nu \rightarrow m} N_\nu(x) = \lim_{\nu \rightarrow m} \frac{J_\nu(x) \cos \nu\pi - J_{-\nu}(x)}{\sin \nu\pi} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{J_{m+\varepsilon}(x) \cos(m+\varepsilon)\pi - J_{-(m+\varepsilon)}(x)}{\sin(m+\varepsilon)\pi} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{J_{m+\varepsilon}(x) (\cos m\pi \cos \varepsilon\pi - \sin m\pi \sin \varepsilon\pi) - J_{-(m+\varepsilon)}(x)}{\sin m\pi \cos \varepsilon\pi + \cos m\pi \sin \varepsilon\pi} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{J_{m+\varepsilon}(x) (-1)^m \cos \varepsilon\pi - J_{-(m+\varepsilon)}(x)}{(-1)^m \sin \varepsilon\pi} \end{aligned}$$

Now we expand the functions to first order in ε . We use a Taylor series for the Bessel functions. Note that ε appears in the index, not the argument, so we have to differentiate with respect to ν .

$$\begin{aligned} N_m(x) &= \lim_{\varepsilon \rightarrow 0} \frac{J_{m+\varepsilon}(x) (-1)^m - J_{-(m+\varepsilon)}(x)}{(-1)^m \varepsilon\pi} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{(-1)^m}{\varepsilon\pi} \left[(-1)^m \left(J_m + \varepsilon \frac{dJ_\nu}{d\nu} \Big|_{\nu=m} \right) - \left(J_{-m} + \varepsilon \frac{dJ_{-\nu}}{d\nu} \Big|_{\nu=m} \right) \right] \end{aligned}$$

Using relation (6), we have:

$$N_m(x) = \frac{1}{\pi} \left[\frac{dJ_\nu}{d\nu} \Big|_{\nu=m} - (-1)^m \frac{dJ_{-\nu}}{d\nu} \Big|_{\nu=m} \right]$$

¹This happens because the two roots of the indicial equation differ by an integer: $2m$.

The derivative has a logarithmic term:

$$\begin{aligned}\frac{dJ_\nu}{d\nu} &= \frac{d}{d\nu} \left(x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n} \right) \\ &= \frac{dx^\nu}{d\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n} + x^\nu \frac{d}{d\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n}\end{aligned}$$

and

$$\frac{dx^\nu}{d\nu} = \frac{d}{d\nu} e^{\nu \ln x} = \ln x e^{\nu \ln x} = x^\nu \ln x$$

and so dJ_ν/dx has a term containing $J_\nu \ln x$. This term diverges as $x \rightarrow 0$ provided that $J_\nu(0)$ is not zero, i.e. for $\nu = 0$. The function $N_\nu(x)$ also diverges as $x \rightarrow 0$ for $\nu \neq 0$, because it contains negative powers of x . (The series for $J_{-\nu}$ starts with a term $x^{-\nu}$.) N_ν is finite as $x \rightarrow \infty$ because J_ν goes to zero sufficiently fast.

Two additional functions called Hankel functions are defined as linear combinations of J and N :

$$H_m^{(1)}(x) = J_m(x) + iN_m(x) \quad (7)$$

and

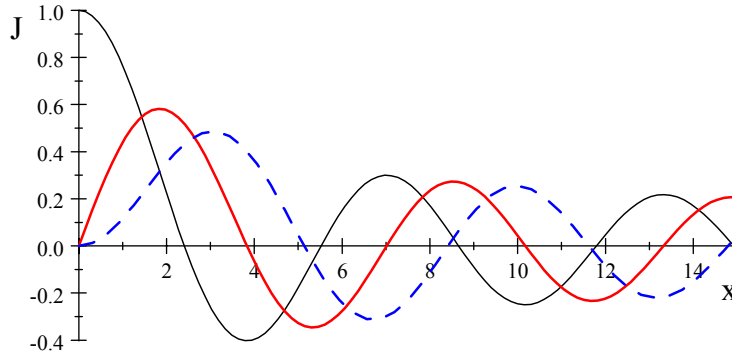
$$H_m^{(2)}(x) = J_m(x) - iN_m(x) \quad (8)$$

Compare the relation between sine, cosine, and exponential:

$$e^{\pm ix} = \cos x \pm i \sin x$$

1.2 Properties of the functions

The Bessel functions (J s) are well behaved both at the origin and as $x \rightarrow \infty$. They have infinitely many zeroes. All of them, except for J_0 , are zero at $x = 0$. The first few functions are shown in the figure.



The first three Bessel functions. J_0 , J_1 (red) and J_2

For small values of the argument, we may approximate the function with the first term in the series:

$$J_m(x) \approx \frac{1}{\Gamma(m+1)} \left(\frac{x}{2}\right)^m \quad \text{for } x \ll 1 \quad (9)$$

The Neumann functions are not well behaved at $x = 0$. N_0 has a logarithmic singularity, and for $m > 0$, N_m diverges as an inverse power of x :

$$\begin{aligned} N_0(x) &\approx \frac{2}{\pi} \ln x \quad \text{for } x \ll 1 \\ N_m(x) &\approx -\frac{(m-1)!}{\pi} \left(\frac{2}{x}\right)^m \quad \text{for } x \ll 1, \quad m > 0 \end{aligned} \quad (10)$$

For large values of the argument, both J and N oscillate: they are like damped cosine or sine functions:

$$J_m(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \quad \text{for } x \gg 1, m \quad (11)$$

$$N_m(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \quad \text{for } x \gg 1, m \quad (12)$$

and thus the Hankel functions are like complex exponentials:

$$H_m^{(1,2)} \approx \sqrt{\frac{2}{\pi x}} \exp\left[\pm i\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right] \quad \text{for } x \gg 1, m \quad (13)$$

Notice that if $m > 1$, the large argument expansions apply for $x \gg m$ rather than the usual $x \gg 1$.

1.3 Relations between the functions

As we found with the Legendre functions, we can determine a set of recursion relations that relate successive $J_m(x)$. For example (Lea §8.4.3)

$$\frac{d}{dx} \left(\frac{J_m(x)}{x^m} \right) = -\frac{J_{m+1}(x)}{x^m} \quad (14)$$

which is valid for $m \geq 0$. In particular, with $m = 0$ we obtain:

$$J_1(x) = -J'_0(x) \quad (15)$$

$$\frac{d}{dx} (x^m J_m(x)) = x^m J_{m-1}(x) \quad (16)$$

From (14) and (16) we may obtain

$$J_{m+1} + J_{m-1} = \frac{2m}{x} J_m \quad (17)$$

and similarly

$$J_{m+1} - J_{m-1} = -2 \frac{dJ_m}{dx} \quad (18)$$

The same relations hold for the N s and the H s.

1.4 Orthogonality of the J_m

Since the Bessel equation is of Sturm-Liouville form, the Bessel functions are orthogonal if we demand that they satisfy boundary conditions of the form (SL review notes eqn 2). In particular, suppose the region of interest is $\rho = 0$ to $\rho = a$, and the boundary conditions are $J_m(ka) = 0$. We do not need a boundary condition at $\rho = 0$ because the function $f(\rho) = \rho$ is zero there. Then the eigenvalues are

$$k_{mn} = \frac{\alpha_{mn}}{a}$$

where α_{mn} is the n th zero of J_m . (The zeros are tabulated in standard references such as Abramowitz and Stegun. Also programs such as Mathematica and Maple can compute them.) Then

$$\int_0^a \rho J_m(k_{mn}\rho) J_m(k_{mn'}\rho) d\rho = \frac{a^2}{2} [J'_m(k_{mn}a)]^2 \delta_{nn'} \quad (19)$$

1.5 Solving a potential problem.

Example. A cylinder of radius a and height h has its curved surface and its bottom grounded. The top surface has potential V . What is the potential inside the cylinder?

The potential has no dependence on ϕ and so only eigenfunctions with $m = 0$ contribute. The potential is zero at $\rho = a$, so the solution we need is $J_0(k\rho)$ with eigenvalues chosen to make $J_0(ka) = 0$. Thus the eigenvalues are given by $k_{0n}a = \alpha_{0n}$, where α_{0n} are the zeros of the function J_0 . The remaining function of z must be zero at $z = 0$, so we choose the hyperbolic sine. Thus the potential is:

$$\Phi(\rho, z) = \sum_{n=1}^{\infty} a_n J_0(k_{0n}\rho) \sinh(k_{0n}z)$$

Now we evaluate this at $z = h$:

$$V = \Phi(\rho, h) = \sum_{n=1}^{\infty} a_n J_0(k_{0n}\rho) \sinh(k_{0n}h)$$

Next we make use of the orthogonality of the Bessel functions. Multiply both sides by $\rho J_0(k_{0r}\rho)$ and integrate from 0 to a . (Note here that the weight function $w(\rho) = \rho$. This is the first time we have seen a weight function that is not 1.) Only one term in the sum, with $n = r$, survives the integration.

$$\begin{aligned} V \int_0^a \rho J_0(k_{0r}\rho) d\rho &= \int_0^a \rho J_0(k_{0r}\rho) \sum_{n=1}^{\infty} a_n J_0(k_{0n}\rho) \sinh(k_{0n}h) d\rho \\ &= a_r \int_0^a \rho [J_0(k_{0r}\rho)]^2 d\rho \sinh(k_{0r}h) \\ &= a_r \frac{a^2}{2} [J'_0(k_{0r}a)]^2 \sinh(k_{0r}h) \end{aligned}$$

To evaluate the left hand side, we use equation (16) with $m = 1$:

$$\begin{aligned}\int_0^a \rho J_0(k\rho) d\rho &= \frac{1}{k} \int_0^a \frac{d}{dk\rho} (k\rho J_1(k\rho)) d\rho \\ &= \frac{1}{k} \rho J_1(k\rho)|_0^a = \frac{a}{k} J_1(ka)\end{aligned}$$

So

$$\begin{aligned}a_r &= \frac{Va}{k_{0r}} J_1(k_{0r}a) \frac{2}{a^2 [J'_0(k_{0r}a)]^2 \sinh(k_{0r}h)} \\ &= \frac{V}{k_{0r}a} \frac{2}{J_1(k_{0r}a) \sinh(k_{0r}h)}\end{aligned}$$

where we used the result from equation (15) that $J'_0 = -J_1$. Finally our solution is:

$$\Phi = 2V \sum_{n=1}^{\infty} \frac{J_0(\alpha_{0n}\rho/a) \sinh(\alpha_{0n}z/a)}{\alpha_{0n} J_1(\alpha_{0n}) \sinh(\alpha_{0n}h/a)}$$

The first two zeros of J_0 are: $\alpha_{01} = 2.4048$, $\alpha_{02} = 5.5201$, and thus the first two terms in the potential are:

$$\Phi = 2V \left(\frac{J_0(2.4048 \frac{\rho}{a})}{2.4048 J_1(2.4048)} \frac{\sinh(2.4048 \frac{z}{a})}{\sinh(2.4048 \frac{h}{a})} + \frac{J_0(5.5201 \frac{\rho}{a})}{5.5201 J_1(5.5201)} \frac{\sinh(5.5201 \frac{z}{a})}{\sinh(5.5201 \frac{h}{a})} + \dots \right)$$

1.6 Modified Bessel functions

Suppose we change the potential problem so that the top and bottom of the cylinder are grounded but the outer wall at $\rho = a$ has potential $V(\phi, z)$. Then we would need to choose a negative separation constant so that the solutions of the z -equation are trigonometric functions:

$$\frac{\partial^2 Z}{\partial z^2} = -k^2 Z \Rightarrow Z = a \sin kz + b \cos kz$$

At $z = 0$, $Z(z) = 0$, so we need the sine, and therefore set $b = 0$. We also need $Z(h) = 0$, so we choose the eigenvalue $k = n\pi/h$.

This change in sign of the separation constant also affects the equation for the function $R(\rho)$ because the sign of the k^2 term changes.

$$\frac{\partial}{\partial \rho} \left(\rho \frac{\partial R}{\partial \rho} \right) - k^2 \rho R - \frac{m^2}{\rho} R = 0$$

or, changing variables to $x = k\rho$:

$$\frac{\partial}{\partial x} \left(x \frac{\partial R}{\partial x} \right) - x R - \frac{m^2}{x} R = 0 \quad (20)$$

which is called the *modified Bessel equation*. The solutions to this equation are $J_m(ik\rho)$. It is usual to define the modified Bessel function $I_m(x)$ by the relation:

$$I_m(x) = \frac{1}{i^m} J_m(ix) \quad (21)$$

so that the function I_m is always real (whether or not m is an integer). Using equation (4) we can write a series expansion for I_m :

$$\begin{aligned} I_m(x) &= \frac{1}{i^m} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+m+1)} \left(\frac{ix}{2}\right)^{m+2n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n+m+1)} \left(\frac{x}{2}\right)^{m+2n} \end{aligned} \quad (22)$$

As with the J s, if m is an integer, I_{-m} is not independent of I_m : in fact:

$$I_{-m}(x) = i^m J_{-m}(ix) = i^m (-1)^m J_m(ix) = (-1)^m i^{2m} I_m(x) = I_m(x) \quad (23)$$

The second independent solution is usually chosen to be:

$$K_m(x) = \frac{\pi}{2} i^{m+1} H_m^{(1)}(ix) \quad (24)$$

Then these functions have the limiting forms:

$$I_m(x) \approx \frac{1}{\Gamma(m+1)} \left(\frac{x}{2}\right)^m \quad \text{for } x \ll 1 \quad (25)$$

and

$$K_0(x) \approx -0.5772 - \ln \frac{x}{2} \quad \text{for } x \ll 1 \quad (26)$$

$$K_m(x) \approx \frac{\Gamma(m)}{2} \left(\frac{2}{x}\right)^m \quad \text{for } m > 0, x \ll 1 \quad (27)$$

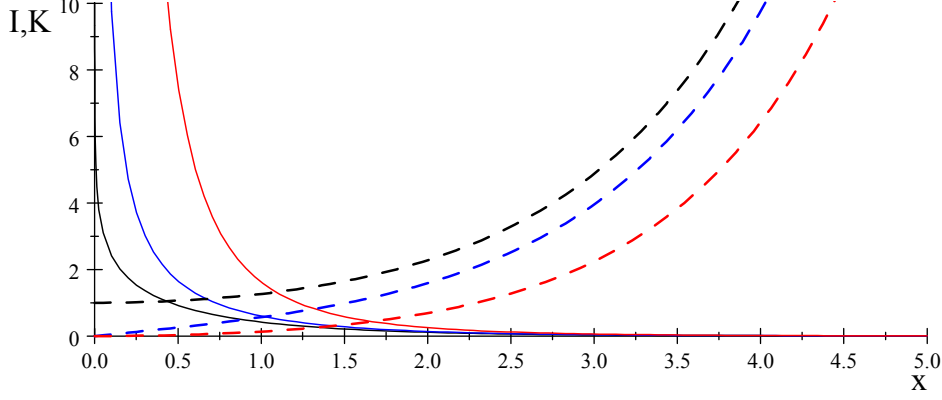
At large x , $x \gg 1, m$, the asymptotic forms are:

$$I_m(x) \approx \frac{1}{\sqrt{2\pi x}} e^x \quad (28)$$

and

$$K_m(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \quad (29)$$

(cf Lea Chapter 3 Example 3.9) These functions, like the real exponentials, do not have multiple zeros and are not orthogonal functions. Note that the I s are well behaved at the origin but diverge at infinity. For the K s, the reverse is true. They diverge at the origin but are well behaved at infinity (See figure below).



Modified Bessel functions. K solid I dashed Black $m = 0$, Blue 1, Red 2

The recursion relations satisfied by the modified Bessel functions are similar to, but not identical to, the relations satisfied by the J s. For the I s, again we can start with the series:

$$\begin{aligned} \frac{d}{dx} \left(\frac{I_m}{x^m} \right) &= \frac{1}{2^m} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n+m+1)} \left(\frac{x}{2} \right)^{2n} \\ &= \frac{1}{2^m} \sum_{n=1}^{\infty} \frac{n}{n! \Gamma(n+m+1)} \left(\frac{x}{2} \right)^{2n-1} \end{aligned}$$

Now let $k = n - 1$:

$$\begin{aligned} \frac{d}{dx} \left(\frac{I_m}{x^m} \right) &= \frac{1}{2^m} \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k+m+2)} \left(\frac{x}{2} \right)^{2k+1} \\ &= \frac{1}{x^m} \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k+m+2)} \left(\frac{x}{2} \right)^{2k+m+1} \\ \frac{d}{dx} \left(\frac{I_m}{x^m} \right) &= \frac{I_{m+1}(x)}{x^m} \end{aligned} \tag{30}$$

and similarly

$$\frac{d}{dx} (x^m I_m) = x^m I_{m-1} \tag{31}$$

Expanding out and combining, we get:

$$\begin{aligned} 2I'_m &= I_{m+1} + I_{m-1} \\ \frac{2m}{x} I_m &= I_{m-1} - I_{m+1} \end{aligned} \tag{32}$$

For the K s, the relations are:

$$\frac{d}{dx} (x^m K_m) = -x^m K_{m-1}; \quad \frac{d}{dx} \left(\frac{K_m(x)}{x^m} \right) = -\frac{K_{m+1}(x)}{x^m}$$

and consequently:

$$\begin{aligned} K_{m-1} - K_{m+1} &= -\frac{2m}{x} K_m \\ K_{m-1} + K_{m+1} &= -2K'_m \end{aligned} \quad (33)$$

1.7 Combining functions

When solving a physics problem, we start with a partial differential equation and a set of boundary conditions. Separation of variables produces a set of *coupled* ordinary differential equations in the various coordinates. The standard solution method ("Orthogonal" notes §2) requires that we choose the separation constants by fitting the zero boundary conditions first. In a standard 3-dimensional problem, once we have chosen the two separation constants we have no more freedom and the third function is determined.

When solving Laplace's equation in cylindrical coordinates, the functions couple as follows:

Zero boundary conditions in ρ : The eigenfunctions are of the form:

$$J_m \left(\alpha_{mn} \frac{\rho}{a} \right) \left(A_{mn} \sinh \alpha_{mn} \frac{z}{a} + B_{mn} \cosh \alpha_{mn} \frac{z}{a} \right) e^{\pm im\phi}$$

The set of functions $J_m \left(\alpha_{mn} \frac{\rho}{a} \right) e^{\pm im\phi}$ form a complete orthogonal set on the surfaces $z = \text{constant}$ that bound the region.

Zero boundary conditions in z : The eigenfunctions are of the form:

$$\left(A_{mn} I_m \left(\frac{n\pi\rho}{h} \right) + B_{mn} K_m \left(\frac{n\pi\rho}{h} \right) \right) \sin \left(\frac{n\pi z}{h} \right) e^{\pm im\phi}$$

The set of functions $\sin \left(\frac{n\pi z}{h} \right) e^{\pm im\phi}$ form a complete orthogonal set on the boundary surface $\rho = \text{constant}$.

Thus in solutions of Laplace's equation, J s in ρ always couple with the hyperbolic sines and hyperbolic cosines (or real exponentials) in z , while the I s and K s in ρ always couple with the sines and cosines (or complex exponentials) in z .

Example: Suppose the potential on the curved wall of the cylinder is $V(a, \phi, z) = V \sin \phi$. The top and bottom are grounded.

The solution is of the form

$$\Phi(\rho, \phi, z) = \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{\infty} \sin \left(\frac{n\pi z}{h} \right) e^{im\phi} \left(A_{mn} I_m \left(\frac{n\pi\rho}{h} \right) + B_{mn} K_m \left(\frac{n\pi\rho}{h} \right) \right)$$

The solution must be finite on axis at $\rho = 0$, so $B_{mn} = 0$. Now we evaluate the potential at $\rho = a$

$$\Phi(a, \phi, z) = \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{h}\right) e^{im\phi} A_{mn} I_m\left(\frac{n\pi a}{h}\right) = V \sin \phi$$

We make use of orthogonality by multiplying by $\sin p\pi z/h$ and integrating from 0 to h .

$$\begin{aligned} \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{\infty} \int_0^h \sin \frac{p\pi z}{h} \sin\left(\frac{n\pi z}{h}\right) dz e^{im\phi} A_{mn} I_m\left(\frac{n\pi a}{h}\right) &= V \sin \phi \int_0^h \sin \frac{p\pi z}{h} dz \\ \sum_{m=-\infty}^{+\infty} \frac{h}{2} e^{im\phi} A_{mp} I_m\left(\frac{p\pi a}{h}\right) &= V \sin \phi \frac{h}{p\pi} \cos \frac{p\pi z}{h} \Big|_0^h \\ &= V \frac{h}{p\pi} \sin \phi [(-1)^p - 1] \end{aligned}$$

The result is non-zero only for odd p .

Next we make use of the orthogonality of the $e^{im\phi}$

$$\begin{aligned} \sum_{m=-\infty}^{+\infty} \frac{h}{2} \int_0^{2\pi} e^{im\phi} e^{-im'\phi} d\phi A_{mp} I_m\left(\frac{p\pi a}{h}\right) &= 2V \frac{h}{p\pi} \int_0^{2\pi} \sin \phi e^{-im'\phi} d\phi \quad p \text{ odd} \\ &= 0 \quad p \text{ even} \end{aligned}$$

To do the integral on the right hand side, express the sine in terms of exponentials.

$$\begin{aligned} \int_0^{2\pi} \sin \phi e^{-im'\phi} d\phi &= \int_0^{2\pi} \frac{e^{i\phi} - e^{-i\phi}}{2i} e^{-im'\phi} d\phi \\ &= \frac{2\pi}{2i} (\delta_{m'1} - \delta_{m',-1}) \end{aligned}$$

Only one term with $m = m'$ survives the integration on the left hand side. Thus

$$\begin{aligned} \frac{h}{2} 2\pi A_{m'p} I_{m'}\left(\frac{p\pi a}{h}\right) &= 2V \frac{h}{p\pi} \frac{2\pi}{2i} (\delta_{m'1} - \delta_{m',-1}) \\ A_{m'p} &= \frac{4V}{p\pi} \frac{1}{2i} \frac{(\delta_{m'1} - \delta_{m',-1})}{I_{m'}\left(\frac{p\pi a}{h}\right)} \end{aligned}$$

$$\Phi(\rho, \phi, z) = \frac{4V}{\pi} \sum_{n=1, \text{ odd}}^{\infty} \sin\left(\frac{n\pi z}{h}\right) \frac{1}{2in} \left[e^{i\phi} \frac{I_1\left(\frac{n\pi\rho}{h}\right)}{I_1\left(\frac{n\pi a}{h}\right)} - e^{-i\phi} \frac{I_{-1}\left(\frac{n\pi\rho}{h}\right)}{I_{-1}\left(\frac{n\pi a}{h}\right)} \right]$$

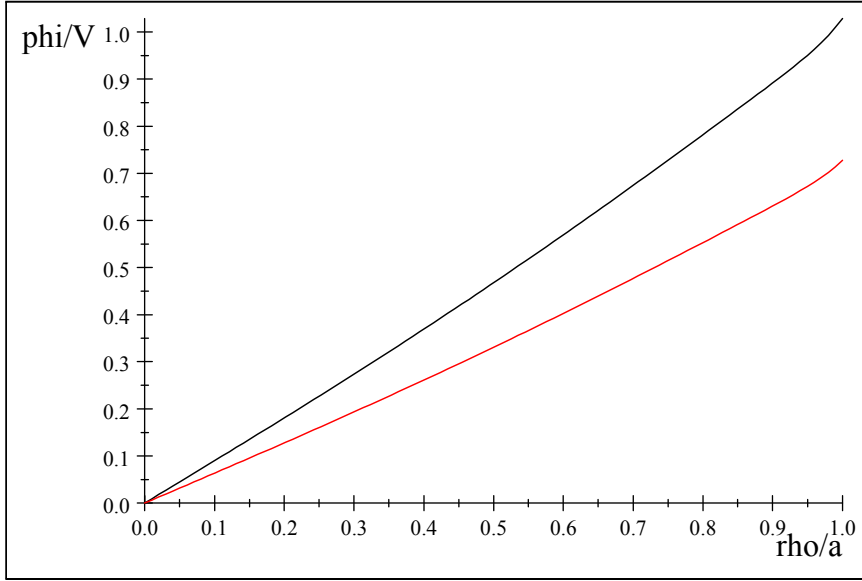
But $I_1 = I_{-1}$, (eqn 23), so

$$\Phi(\rho, \phi, z) = \frac{4V}{\pi} \sum_{n=1, \text{ odd}}^{\infty} \sin\left(\frac{n\pi z}{h}\right) \frac{\sin \phi}{n} \frac{I_1\left(\frac{n\pi\rho}{h}\right)}{I_1\left(\frac{n\pi a}{h}\right)}$$

The dimensions are correct, and the $1/n$ gives us reasonable convergence. The ratio

$$\frac{I_1\left(\frac{n\pi\rho}{h}\right)}{I_1\left(\frac{n\pi a}{h}\right)} \leq 1 \quad \text{for } \rho \leq a.$$

The potential at $\rho = 0$ is zero, as expected. The $m = 1$ "source" (the potential on the surface) generates an $m = 1$ response. Again we can look at the first few terms (here 10). Let's take $h/a = 2$ and $\phi = \pi/4$ (red) and $\Phi = \pi/2$ (black)



1.8 Continuous set of eigenvalues: the Fourier Bessel Transform

In Lea Chapter 7 we approached the Fourier transform by letting the length of the domain in a Fourier series problem become infinite. The orthogonality relation for the exponential functions:

$$\frac{1}{2L} \int_{-L}^L \exp\left(i\frac{n\pi x}{L}\right) \exp\left(-i\frac{m\pi x}{L}\right) dx = \delta_{mn}$$

becomes

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-ik'x} dx = \delta(k - k')$$

That is, the Kronecker delta becomes a delta function, and the countable set of eigenvalues $n\pi/L$ becomes a continuous set of values $-\infty < k < \infty$.

The same thing happens with Bessel functions. With a finite domain in ρ , say $0 \leq \rho \leq a$, we can determine a countable set of eigenvalues from the set of

zeros of the Bessel functions J_m . If our domain in ρ becomes infinite, then we cannot determine the eigenvalues, and instead we have a continuous set. The orthogonality relation:

$$\int_0^a J_m\left(\alpha_{mn}\frac{\rho}{a}\right) J_m\left(\alpha_{mk}\frac{\rho}{a}\right) \rho d\rho = \frac{a^2}{2} [J'_m(\alpha_{mn})]^2 \delta_{nk}$$

becomes:

$$\int_0^\infty \rho J_m(k\rho) J_m(k'\rho) d\rho = \frac{\delta(k-k')}{k} \quad (34)$$

(The proof of this relation is in Lea Appendix 7.) Then the solution to the physics problem is determined as an integral over k . For example, a solution of Laplace's equation may be written as:

$$\Phi(\rho, \phi, z) = \sum_m e^{im\phi} \int_0^\infty A(k) f(kz) J_m(k\rho) dk$$

where $f(kz)$ depends on the boundary conditions in z . It will be a combination of the exponentials e^{-kz} and e^{+kz} .

Example: Suppose the potential is $V(\rho) = V_0 \left(\frac{a}{\rho}\right) \sin \frac{\rho}{a}$ on a plane at $z = 0$, and we want to find the potential for $z > 0$. Then the appropriate function of z is e^{-kz} , chosen so that $\Phi \rightarrow 0$ as $z \rightarrow \infty$ (a long way from the plane). We also need the J_m , which remain finite at $\rho = 0$. Then the solution is of the form:

$$\Phi(\rho, \phi, z) = \sum_m e^{im\phi} \int_0^\infty A_m(k) e^{-kz} J_m(k\rho) dk$$

Evaluating Φ on the plane at $z = 0$, we get:

$$V_0 \left(\frac{a}{\rho}\right) \sin \frac{\rho}{a} = \Phi(\rho, \phi, 0) = \sum_m e^{im\phi} \int_0^\infty A_m(k) J_m(k\rho) dk$$

Now we can make use of the orthogonality of the $e^{im\phi}$. Multiply both sides by $e^{-im'\phi}$ and integrate over the range 0 to 2π . On the LHS, only the term with $m = 0$ survives the integration, and on the RHS only the term with $m = m'$ survives.

$$\begin{aligned} \int_0^{2\pi} V_0 \left(\frac{a}{\rho}\right) \sin \frac{\rho}{a} e^{-im'\phi} d\phi &= \int_0^{2\pi} \sum_m e^{im\phi} e^{-im'\phi} d\phi \int_0^\infty A_m(k) J_m(k\rho) dk \\ 2\pi V_0 \left(\frac{a}{\rho}\right) \sin \frac{\rho}{a} \delta_{m'0} &= 2\pi \int_0^\infty A_{m'}(k) J_{m'}(k\rho) dk \end{aligned}$$

—a Fourier Bessel transform. Next² multiply both sides by $\rho J_m(k'\rho)$, integrate from 0 to ∞ in ρ , and use equation (34) to get:

$$\begin{aligned} V_0 \int_0^\infty \left(\frac{a}{\rho}\right) \sin \frac{\rho}{a} J_m(k'\rho) \rho d\rho \delta_{m'0} &= \int_0^\infty \int_0^\infty A_m(k) J_m(k\rho) J_m(k'\rho) \rho d\rho dk \\ &= \int_0^\infty A_m(k) \frac{\delta(k-k')}{k} dk = \frac{A_m(k')}{k'} \end{aligned}$$

²We also drop the primes on the m' for convenience.

which determines the coefficient $A_m(k')$ in terms of the known potential on the plane. Only A_0 is non-zero, as expected from the azimuthal symmetry.

On the RHS, let $\rho/a = x$. We get (dropping the primes on k)

$$V_0 a^2 \int_0^\infty \sin x J_0(xka) dx = \frac{A_0(k)}{k}$$

This integral is GR 6.671#7. So

$$A(k) = V_0 a^2 k \begin{cases} 0 & \text{if } 0 < 1 < ka \\ \frac{1}{\sqrt{1-(ka)^2}} & \text{if } 0 < ka < 1 \end{cases}$$

So finally we have

$$\begin{aligned} \Phi(\rho, z) &= V_0 a^2 \int_0^{1/a} \frac{k J_0(k\rho)}{\sqrt{1-(ka)^2}} e^{-kz} dk \\ &= V_0 \int_0^1 \frac{x J_0\left(x \frac{\rho}{a}\right)}{\sqrt{1-x^2}} e^{-xz/a} dx \end{aligned}$$

For $z = 0$ we have

$$\Phi(\rho, 0) = V_0 \int_0^1 \frac{x J_0\left(x \frac{\rho}{a}\right)}{\sqrt{1-x^2}} dx = V_0 \frac{1}{\rho/a} \sin \frac{\rho}{a} \quad \text{GR 6.554\#2}$$

which gives us back the potential on the plane.

The charge density on the plane is given by

$$\vec{E} \cdot \hat{z} \Big|_{z=0} = \frac{\sigma}{\varepsilon_0}$$

So we need

$$\begin{aligned} \frac{\partial \Phi}{\partial z} \Big|_{z=0} &= -\frac{V_0}{a} \int_0^1 \frac{x^2 J_0\left(x \frac{\rho}{a}\right)}{\sqrt{1-x^2}} dx \\ &= -\frac{V_0}{a} \int_0^1 \frac{x^2}{\sqrt{1-x^2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{[n!]^2} \left(\frac{x \rho}{2a}\right)^{2n} dx \\ &= -\frac{V_0}{a} \sum_{n=0}^{\infty} \frac{(-1)^n}{[n!]^2} \left(\frac{\rho}{2a}\right)^{2n} \int_0^1 \frac{x^{2(n+1)}}{\sqrt{1-x^2}} dx \end{aligned}$$

Let $x = \sin \theta$, $dx = \cos \theta d\theta$ Then the integral is

$$I_n = \int_0^{\pi/2} \frac{\sin^{2(n+1)} \theta}{\cos \theta} \cos \theta d\theta = \int_0^{\pi/2} \sin^{2(n+1)} \theta d\theta = \frac{(2n+1)!!}{(2n+2)!!} \frac{\pi}{2} \quad \text{GR 3.621\#3, Lea Ch2 P29d}$$

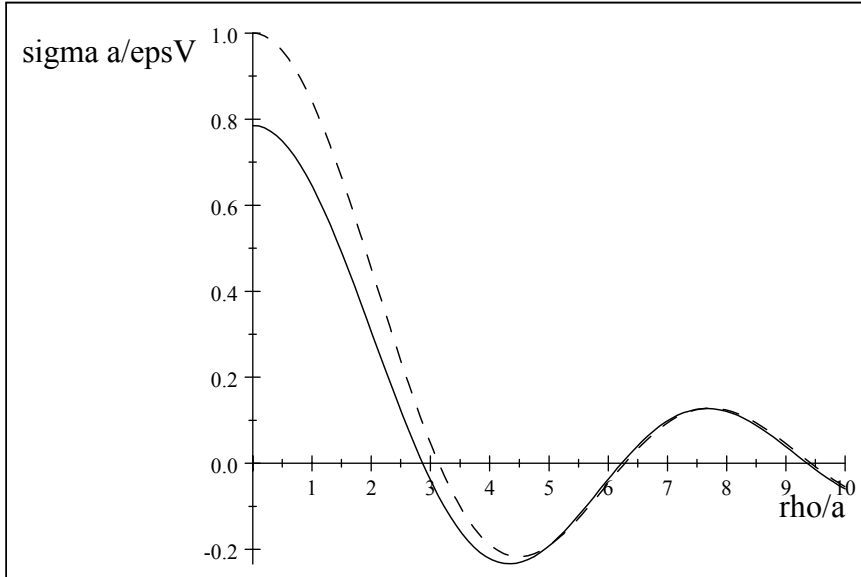
Thus

$$\begin{aligned}
\left. \frac{\partial \Phi}{\partial z} \right|_{z=0} &= -\frac{V_0}{a} \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{[n!]^2} \left(\frac{\rho}{2a} \right)^{2n} \frac{(2n+1)!!}{2^{n+1}(n+1)!} \\
&= -\frac{V_0}{a} \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\rho}{2a} \right)^{2n} \frac{(2n+1)!!}{2^n(n+1)!}
\end{aligned}$$

and so the charge density on the plane is

$$\sigma = \varepsilon_0 E_z = \varepsilon_0 \frac{V_0}{a} \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\rho^2}{8a^2} \right)^n \frac{(2n+1)!!}{(n+1)!}$$

The plot shows the series up to $n = 40$.



Convince yourself that this agrees with the field line diagram. (The dashed line is the potential.)