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The distribution of the eigenvalues for second order eigenvalue problems in the presence of an arbitrary number of turning points

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Dedicated to H.W. Knobloch on the occasion of his 65th birthday

We consider the differential equation

$$(1) \qquad -y'' + \chi(x)y = \rho^2\phi^2(x)y$$

on the interval $I = [0,1]$, where $[0,1]$ contains a finite number of zeros of ϕ^2, the so-called turning-points. Using asymptotic estimates proved in an earlier paper [4] of the authors for an appropriate fundamental system of solutions of (1) for $|\rho| \to \infty$ it is possible to define a class of *regular* eigenvalue problems generated by (1) and two-point boundary conditions at $0, 1$. This class is a generalization of the class of Birkhoff-regular eigenvalue problems of second order (cf. [16], p. 62) in the definite case, where ϕ^2 does not change the sign on I. Further we prove for this class of *regular* problems the formula of the asymptotic distribution of the eigenvalues. This theorem improves the results of Atkinson and Mingarelli [1], [12] for the selfadjoint case and generalizes it to nonselfadjoint equations of type (1) associated with two-point boundary conditions. Further results on indefinite eigenvalue problems can be found in [6] - [9], [11], [12] and the papers cited therein.

We suppose that the coefficients χ and ϕ satisfy:

(i) ϕ^2 is real and has in I m zeros x_ν of order $\ell_\nu \in \mathbb{N}$, $1 \le \nu \le m$ where $0 < x_1 < x_2 < \ldots < x_m < 1$.

(ii) The function $\phi_0 : I \to \mathbb{R} \setminus \{0\}, x \mapsto \phi^2(x)\prod_{\nu=1}^{m}(x - x_\nu)^{-\ell_\nu}$ is twice continuously differentiable.

(iii) χ is bounded and integrable in I.

(iv) The boundary conditions

$$(2) \qquad U_\nu(y) = U_{\nu 0}(y) + U_{\nu 1}(y) = 0 \qquad \nu = 1,2$$

are assumed to be normalized, i.e.

$$\begin{aligned} U_{\nu 0}(y) &= \alpha_\nu y^{(k_\nu)}(0) \quad + \quad \textstyle\sum_{\mu=0}^{k_\nu -1} \alpha_{\nu\mu} y^{(\mu)}(0) \\ U_{\nu 1}(y) &= \beta_\nu y^{(k_\nu)}(1) \quad + \quad \textstyle\sum_{\mu=0}^{k_\nu -1} \beta_{\nu\mu} y^{(\mu)}(1) \end{aligned}$$

with $|\alpha_\nu| + |\beta_\nu| > 0$ and $1 \geq k_1 \geq k_2 \geq 0$, where $k_1 + k_2$ is minimal with respect to all equivalent boundary conditions.

Definition: Let

$$(3) \qquad \Theta_j := \begin{vmatrix} \alpha_1 |\phi(0)|^{k_1} & \beta_1(\epsilon_j |\phi(1)|)^{k_1} \\ \alpha_2 |\phi(0)|^{k_2} & \beta_2(\epsilon_j |\phi(1)|)^{k_2} \end{vmatrix} \quad \text{where } \epsilon_1 = -1,\ \epsilon_2 = i,\ \epsilon_3 = -i.$$

The boundary conditions (2) are called *regular* if

$$(4) \qquad \begin{aligned} &\Theta_1 \neq 0 \text{ for } \phi^2(0)\phi^2(1) > 0 \\ &\text{and} \\ &\Theta_2\Theta_3 \neq 0 \text{ for } \phi^2(0)\phi^2(1) < 0. \end{aligned}$$

The eigenvalue problem (1), (2) is called *regular* if the boundary conditions are regular.

Remark 1:

(i) In the definite case, which implies $\phi^2(0)\phi^2(1) > 0$, the regularity condition $\Theta_1 \neq 0$ is identical with the well-known condition of Birkhoff ([13], p. 62).

(ii) In the case $\alpha_i, \beta_i \in \mathbb{R}$ $(i = 1,2)$ we get $\Theta_3 = \overline{\Theta}_2$,and the condition $\Theta_2 \neq 0$ is sufficient for the regularity of (1), (2) in the case $\phi^2(0)\phi^2(1) < 0$.

(iii) In the case of separated boundary conditions there holds either $\alpha_1 \neq 0 \neq \beta_2$, $\alpha_2 = \beta_1 = 0$ or $\alpha_2 \neq 0 \neq \beta_1$, $\alpha_1 = \beta_2 = 0$. This implies $\Theta_i \neq 0$ $(i = 1,2,3)$. Therefore all types of separated boundary conditions are regular.

(iv) In the case of boundary conditions of periodic (resp. antiperiodic) type: $U_i(y) = y^{(i-1)}(0) \mp y^{(i-1)}(1) = 0$, $i = 1,2$, we get $\alpha_i = \pm\beta_i = 1$ $i = 1,2$ and $k_1 = 1$, $k_2 = 0$. Therefore we obtain $-\Theta_1 = |\phi(0)| + |\phi(1)| \neq 0$, $-\Theta_{2,3} = |\phi(0)| \pm i|\phi(1)| \neq 0$, and therefore periodic (resp. antiperiodic) boundary conditions generate regular problems.

In this paper we use the estimates of [4], Section 4.2 for an appropriate fundamental system z_1, z_2 of (1) to investigate the corresponding characteristic determinant

$$\Delta = det(U_i(z_j))_{i,j=1,2}\ . \tag{5}$$

Since the zeros of Δ are the square-roots of the eigenvalues of (1), (2) we can prove the following theorem on the distribution of the eigenvalues.

Theorem: Let (1), (2) be a regular problem. Then there exist two sequences $(\lambda_k^+), (\lambda_k^-)$ of eigenvalues with the asymptotic distribution

$$\lambda_k^{\pm} = \pm\frac{k^2\pi^2}{R_{\pm}^2}[1 + O(\frac{1}{k})],\ \ k \in I\!N, \tag{6}$$

where $R_{\pm} = \int_0^1 |\phi_{\pm}(t)|dt$, $\phi_{\pm}^2(t) = max\{\pm\phi^2(t),\ 0\}$. In the case $R_+ = 0$ or $R_- = 0$ the corresponding sequence (λ_k^+) or (λ_k^-) has to be considered empty.

Remark 2:

The asymptotic formula (6) has been proved using other methods by Mingarelli and Atkinson [1], [12] for selfadjoint problems (1), (2) with Sturm-Liouville (and therefore separated) boundary conditions (2) and with $o(1)$ instead of $O(\frac{1}{k})$. Obviously (6) improves and generalizes the result of Atkinson and Mingarelli.

Proof: For the following considerations we use some notations:
For fixed $\epsilon > 0$, ϵ sufficiently small, let

$$\begin{aligned} D_{0,\epsilon} &= [0, x_1 - \epsilon], \\ D_{\nu,\epsilon} &= [x_\nu + \epsilon, x_{\nu+1} - \epsilon] \text{ for } \quad 1 \le \nu \le m-1, \\ D_{m,\epsilon} &= [x_m + \epsilon, 1] \end{aligned}$$

and

$$I_{\nu,\epsilon} = D_{\nu-1,\epsilon} \cup [x_\nu - \epsilon, x_\nu + \epsilon] \cup D_{\nu,\epsilon}.$$

For $k \in Z\!\!\!Z$ we denote by

$$S_k = \left\{\rho \in C\!\!\!\!I \,|\, \frac{k\pi}{4} \le arg\rho \le \frac{(k+1)\pi}{4}\right\}$$

sectors in the ρ-plane, and by T_ν we introduce the four different *types* of turning points:

$$T_\nu := \begin{cases} I, & \text{if } \ell_\nu \text{ is even and } \phi^2(x)(x - x_\nu)^{-\ell_\nu} < 0 \text{ in } I_{\nu,\epsilon}, \\ II, & \text{if } \ell_\nu \text{ is even and } \phi^2(x)(x - x_\nu)^{-\ell_\nu} > 0 \text{ in } I_{\nu,\epsilon}, \\ III, & \text{if } \ell_\nu \text{ is odd and } \phi^2(x)(x - x_\nu)^{-\ell_\nu} < 0 \text{ in } I_{\nu,\epsilon}, \\ IV, & \text{if } \ell_\nu \text{ is odd and } \phi^2(x)(x - x_\nu)^{-\ell_\nu} > 0 \text{ in } I_{\nu,\epsilon}. \end{cases}$$

Further we set for $1 \le \nu \le m$

$$\mu_\nu = \frac{1}{2+\ell_\nu}$$

and

$$\sigma_\nu = \begin{cases} 1 & \text{if } \mu_\nu > \frac{1}{4} \\ 1-\delta_0 \quad (\text{with } \delta_0 > 0 \text{ arbitrarily small}) & \text{if } \mu_\nu = \frac{1}{4} \\ 4\mu_\nu & \text{if } \mu_\nu < \frac{1}{4} \end{cases}$$

and

$$\sigma_0 = min\{\sigma_\nu | 1 \le \nu \le m\}.$$

We use the asymptotic estimates for a suitable fundamental system z_1, z_2 for $x \in D_{0,\epsilon}$ resp. $x \in D_{m,\epsilon}$ and $\rho \in S_k$ proved in our paper [4]; using the notation of [4], Section 4, we get:

$$(7) \qquad \begin{pmatrix} z_1(x,\rho) \\ z_2(x,\rho) \end{pmatrix} = \begin{pmatrix} n_{T_1}^{-1}(\rho) y_1(x,\rho) \\ n_{T_1}(\rho) y_2(x,\rho) \end{pmatrix} = \begin{pmatrix} w_{11}^{T_1}(x,\rho) \\ w_{12}^{T}(x,\rho) \end{pmatrix} \text{ for } (x,\rho) \in D_{0,\epsilon} \times S_k$$

$$(8) \qquad \begin{pmatrix} z_1(x,\rho) \\ z_2(x,\rho) \end{pmatrix} = \begin{pmatrix} n_{T_1}^{-1}(\rho) y_1(x,\rho) \\ n_{T_1}(\rho) y_2(x,\rho) \end{pmatrix} = \Pi_{\nu=1}^{m-1} C_\nu(T_\nu, T_{\nu+1}) \begin{pmatrix} w_{m1}^{T_m}(x,\rho) \\ w_{m2}^{T_m}(x,\rho) \end{pmatrix}$$

for $(x,\rho) \in D_{m,\epsilon} \times S_k$, where

$$n_I(\rho) = n_{IV}(\rho) = e^{\rho \int_0^{x_1} |\phi| dt} \text{ and}$$

$$n_{II}(\rho) = n_{III}(\rho) = \begin{cases} e^{i\rho \int_0^{x_1} |\phi(t)| dt} & \text{for} \quad \rho \in S_{-2} \cup S_{-1} \\ e^{-i\rho \int_0^{x_1} |\phi(t)| dt} & \text{for} \quad \rho \in S_0 \cup S_{-1}. \end{cases}$$

The functions $w_{1i}^{T_1}$ resp. $w_{mi}^{T_1}$ $(i = 1,2)$ are defined and estimated by the corresponding Liouville-Green (LG) approximation in §1 of our paper [4], and the connection matrices $C_\nu(T_\nu, T_{\nu+1})$ have been derived and estimated in §4 of the same paper. Though these estimates are not satisfactory in any case and have been completed by the discussion of a further fundamental system the quality of the estimates is sufficient for the purpose of this paper.

For the further discussion there hold the following fundamental considerations:
If $u_1(\cdot,\rho), u_2(\cdot,\rho)$ is a fundamental system of (1) with

$$(\frac{\partial}{\partial x})^{k-1} u_j(x_1,\rho) = \delta_{jk}, \ \ 1 \le j,k \le 2,$$

and $\Delta_0(\rho) = det(U_i(u_j(\cdot,\rho))_{1\le i,j\le 2}$, then Δ_0 is an entire function with zeros being the square-roots of the eigenvalues of (1), (2). Further there exists for each sector S_k a function f_k with $f_k(\rho) \neq 0$ for $\rho \in S_k$ with $|\rho| > R_0$ such that

$$\Delta_0(\rho) = f_k(\rho)\Delta(\rho), \quad \rho \in S_k.$$

It is important that Δ defined by (5), (7), (8) for $\rho \in S_k$ is dependent on k although this dependance has not been indicated by an index.

On account of $f_k(\rho) \neq 0$ it is sufficient for the proof of (6) to estimate the function Δ for each sector S_k separately.

We rewrite (8), using the functions a_i, b_i of ρ defined by

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} := \prod_{\nu=1}^{m-1} C_\nu(T_\nu, T_{\nu+1}). \tag{9}$$

In addition we set for $\alpha \in \mathbb{C}$: $[\alpha] = \alpha + O(\rho^{-\sigma_0})$.
First we consider the case

A. $\phi^2(0) < 0$, $\phi^2(1) < 0$, $\rho \in S_{-1}$.

<u>1. If $T_1 = IV$; $T_m = III$,</u>

we obtain for z_i, $i = 1, 2$ by Theorem 3.2 (i) in [4] for $(x,\rho) \in D_{0,\epsilon} \times S_{-1}$ the estimates

$$|\phi(x)|^{\frac{1}{2}} z_i(x,\rho) \overset{(2)}{=} e^{\pm\rho\int_{x_1}^{x}|\phi|dt}\,[1] \begin{cases} + & \text{for } i = 1 \\ - & \text{for } i = 2 \end{cases}$$

and for $(x,\rho) \in D_{m,\epsilon} \times S_{-1}$ we have

$$|\phi(x)|^{\frac{1}{2}} z_i(x,\rho) \overset{(2)}{=} \frac{[a_i]e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m} e^{\rho\int_{xm}^{x}|\phi|dt} + 2[b_i]sin\frac{\pi}{2}\mu_m e^{\frac{i\pi}{4}} e^{-\rho\int_{xm}^{x}|\phi|dt}; \quad i = 1, 2.$$

We shall use the symbol $\overset{(2)}{=}$ to signify that the given formula is valid and also the corresponding formula obtained by fomal differentiation with respect to x ignoring the differentiation of all error terms.

Plugging these estimates into $U_j (j = 1, 2)$ we get for $i, j = 1, 2$

$$(10)\ U_j(z_i) = \alpha_j(\pm\rho)^{k_j}|\phi(0)|^{k_j-\frac{1}{2}}e^{\mp\rho\int_0^{x_1}|\phi|dt}[1] + \beta_j\rho^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}\frac{[a_i]e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}e^{\rho\int_{x_m}^1|\phi|dt}$$
$$+ \beta_j(-\rho)^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}[b_i]2sin\frac{\pi}{2}\mu_m e^{\frac{i\pi}{4}}e^{-\rho\int_{x_m}^1|\phi|dt}.$$

Using (10) for the estimation of the characteristic determinant Δ we can split $\Delta(\rho)$ for $|\rho| > R_0$ into the sum of a term $r(\rho)$ which does not influence the leading term of the zeros of Δ and Δ_0 (see [10]) and the *dominant* part $\Delta(\rho) - r(\rho) =: \Delta_d(\rho)$, here

$$\Delta_d(\rho) = \begin{vmatrix} \beta_1\rho^{k_1}|\phi(1)|^{k_1-\frac{1}{2}} & \alpha_1(-\rho)^{k_1}|\phi(0)|^{k_1-\frac{1}{2}} \\ \beta_2\rho^{k_2}|\phi(1)|^{k_2-\frac{1}{2}} & \alpha_2(-\rho)^{k_2}|\phi(0)|^{k_2-\frac{1}{2}} \end{vmatrix} \frac{[a_1]e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}e^{\rho(\int_0^{x_1}+\int_{x_m}^1)|\phi|dt}$$

$$(11)\qquad = -(-\rho)^{k_1+k_2}|\phi(0)\phi(1)|^{-\frac{1}{2}}\frac{[a_1]e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}\Theta_1 e^{\rho(\int_0^{x_1}+\int_{x_m}^1)|\phi|dt}.$$

A similar formula can be proved for the sector S_0; combining both results we get the dominant terms of $\Delta_0(\rho)$ for $\rho \in S_{-1} \cup S_0$. The asymptotic distribution of the zeros of Δ_0 in $S_{-1} \cup S_0$ is therefore asymptotically determined by that of a_1 , where a_1 is the function defined by (9).

Therefore we have to discuss in (9) the asymptotic behaviour of $\prod_{\nu=1}^{m-1} C_\nu(T_\nu, T_{\nu+1})$, where the connection matrices are dependent on the respective type of the turning point x_ν . We confine our discussion to the sector S_{-1}, for S_0 we get an equivalent result for the dominant terms of $\Delta_0(\rho)$.
First we discuss the case

1a. $\mathbf{T_1 = IV,\ T_2 = \cdots = T_{m-1} = II,\ T_m = III}$.

By Theorem 4.1 in [4] we obtain for $C_\nu(II, II)$ $(\nu = 2, \ldots, m-2)$ and for $\rho \in S_{-1}$ the asymptotic estimates

$$C_\nu(II,II) = \begin{pmatrix} \frac{[1]}{sin\pi\mu_\nu}e^{i\rho\int_{x_\nu}^{x_{\nu+1}}|\phi|dt} + \frac{[cos\pi\mu_\nu]}{sin\pi\mu_\nu}cos\pi\mu_{\nu+1}e^{-i\rho\int_{x_\nu}^{x_{\nu+1}}|\phi|dt} & \{e^{-2i\rho\int_{x_\nu+\epsilon}^{x_{\nu+1}}|\phi|dt}O(\rho^{-\sigma_0}) + \frac{[i]cos\pi\mu_\nu}{sin\pi\mu_\nu}e^{-2i\rho\int_{x_\nu}^{x_{\nu+1}}|\phi|dt}\}e^{i\rho\int_{x_\nu}^{x_{\nu+1}}|\phi|dt} \\ \{e^{2i\rho\int_{x_{\nu+1}-\epsilon}^{x_{\nu+1}}|\phi|dt}O(\rho^{-\sigma_0}) - [i]sin\pi\mu_\nu cos\pi\mu_{\nu+1}\}e^{-i\rho\int_{x_\nu}^{x_{\nu+1}}|\phi|dt} & [sin\pi\mu_\nu]e^{-i\rho\int_{x_\nu}^{x_{\nu+1}}|\phi|dt} \end{pmatrix} \quad (12)$$

$$C_1(IV,II) = \begin{pmatrix} \frac{[1]}{2sin\frac{\pi}{2}\mu_1}e^{i\rho\int_{x_1}^{x_2}|\phi|dt} + \frac{[cos\pi\mu_2]}{2sin\frac{\pi}{2}\mu_1}e^{-i\rho\int_{x_1}^{x_2}|\phi|dt} & \{e^{-2i\rho\int_{x_1+\epsilon}^{x_2}|\phi|dt}O(\rho^{-\sigma_0}) + \frac{[i]}{2sin\frac{\pi}{2}\mu_1}e^{-2i\rho\int_{x_1}^{x_2}|\phi|dt}\}e^{i\rho\int_{x_1}^{x_2}|\phi|dt} \\ \{e^{2i\rho\int_{x_2-\epsilon}^{x_2}|\phi|dt}O(\rho^{-\sigma_0}) - [i]2cos\pi\mu_2 sin\frac{\pi}{2}\mu_1\}e^{-i\rho\int_{x_1}^{x_2}|\phi|dt} & 2[sin\frac{\pi}{2}\mu_1]e^{-i\rho\int_{x_1}^{x_2}|\phi|dt} \end{pmatrix} e^{-\frac{i\pi}{4}} \tag{13}$$

and

$$C_{m-1}(II,III) = \begin{pmatrix} \frac{[1]}{sin\pi\mu_{m-1}}e^{i\rho\int_{x_{m-1}}^{x_m}|\phi|dt} + \frac{[cos\pi\mu_{m-1}]}{sin\pi\mu_{m-1}}e^{-i\rho\int_{x_{m-1}}^{x_m}|\phi|dt} & \{e^{-2i\rho\int_{x_{m-1}}^{x_m}|\phi|dt}O(\rho^{-\sigma_0}) + \frac{[i]cos\pi\mu_{m-1}}{sin\pi\mu_{m-1}}e^{-2i\rho\int_{x_\nu}^{x_\nu+1}|\phi|dt}\}e^{i\rho\int_{x_{m-1}}^{x_m}|\phi|dt} \\ \{e^{2i\rho\int_{x_m-\epsilon}^{x_m}|\phi|dt}O(\rho^{-\sigma_0}) - [i]sin\pi\mu_{m-1}cos\pi\mu_{m-1}\}e^{-i\rho\int_{x_{m-1}}^{x_m}|\phi|dt} & [sin\pi\mu_{m-1}]e^{-i\rho\int_{x_{m-1}}^{x_m}|\phi|dt}. \end{pmatrix}. \tag{14}$$

Multiplying these $m-1$ matrices we recognize that a_1 and therefore Δ_0 and Δ can be represented for $\rho \in S_{-1}$ (and similarly for $\rho \in S_0$) as an asymptotic exponential sum:

$$\Delta(\rho) = f_{-1}^{-1}(\rho)\Delta_0(\rho) = h(\rho)\sum_{\nu=1}^{p}\hat{c}_\nu e^{i\rho\vartheta_\nu}[1] \tag{15}$$

with an appropriate number p, a non-vanishing function h, $\hat{c}_\nu \in \mathbb{C}$ and with $\vartheta_1 < \vartheta_2 < \ldots \vartheta_p$.
The distribution of the zeros of (15) is described by the following Lemma:

Lemma 1: Let $p \in \mathbb{N}\setminus\{1\}$, $\vartheta_1 < \vartheta_2 < \ldots < \vartheta_p$, $\epsilon_j : S_{-1} \cup S_0 \to \mathbb{C}$ be functions with $lim_{|\rho|\to\infty}\epsilon_j(\rho) = 0$, $\hat{c}_j \in \mathbb{C}$, for $1 \le j \le p$ with $\hat{c}_1 \neq 0 \neq \hat{c}_p$ and let $F : S_0 \cup S_{-1} \to \mathbb{C}$, $\rho \mapsto \sum_{j=1}^{p}(\hat{c}_j + \epsilon_j(\rho))e^{i\vartheta_j\rho}$ be holomorphic. Then the zeros ρ_k of F fulfill the estimates

$$\rho_k = \frac{2k\pi}{\vartheta_p - \vartheta_1}[1 + O(\frac{1}{k})],\ k \in \mathbb{N}. \tag{16}$$

Remark 3:

(i) (16) is an immediate consequence of Langer [10], Theorem 7 or Tamarkin [14], p. 26.

(ii) In the case of an asymptotic exponential sum F of a more special shape we can improve (16), compare Langer [10] and the estimates for the eigenvalues proved by Langer [9] and Dorodnicyn [3] for special eigenvalue problems (1), (2).

We use Lemma 1 to estimate the zeros of Δ, Δ_0 resp. a_1 and have to determine the coefficients

$$c_1 = h(\rho)\hat{c}_1, \quad c_p = h(\rho)\hat{c}_p$$

and ϑ_1, ϑ_p in (15) for $\rho \in S_{-1}$ and similarly for each sector S_k $(k = -2, -1, 0, 1)$. For the determination of c_1, ϑ_1 we have to take in consideration all terms in $\prod_{\nu=1}^{m-1} C_\nu(T_\nu, T_{\nu+1})$ having a factor $e^{-i\rho\int_{x_1}^{x_m}|\phi|dt}$. Therefore we have to consider the following *relevant partial matrices* in (12), (13), (14).

$$(17)\quad C_{\nu,1}(II, II) = \begin{pmatrix} \frac{cos\pi\mu_\nu cos\pi\mu_{\nu+1}}{sin\pi\mu_\nu} & i\frac{cos\pi\mu_\nu}{sin\pi\mu_\nu} \\ -isin\pi\mu_\nu cos\pi\mu_{\nu+1} & sin\pi\mu_\nu \end{pmatrix} e^{-i\rho\int_{x_\nu}^{x_{\nu+1}}|\phi|dt}, \nu = 2, \cdots, m-2.$$

$$(18)\qquad C_{1,1}(IV, II) = \begin{pmatrix} \frac{cos\pi\mu_2}{2sin\frac{\pi}{2}\mu_1} & \frac{i}{2sin\frac{\pi}{2}\mu_1} \\ -i2sin\frac{\pi}{2}\mu_1 cos\pi\mu_2 & 2sin\frac{\pi}{2}\mu_1 \end{pmatrix} e^{-\frac{i\pi}{4}} e^{-i\rho\int_{x_1}^{x_2}|\phi|dt}$$

and

$$(19)\qquad C_{m-1,1}(II, III) = \begin{pmatrix} \frac{cos\pi\mu_{m-1}}{sin\pi\mu_{m-1}} & i\frac{cos\pi\mu_{m-1}}{sin\pi\mu_{m-1}} \\ -isin\pi\mu_{m-1} & sin\pi\mu_{m-1} \end{pmatrix} e^{-i\rho\int_{x_{m-1}}^{x_m}|\phi|dt}.$$

Multipliying these matrices we get

$$\prod_{\nu=1}^{m-1} C_{\nu,1}(T_\nu, T_{\nu+1})$$

$$(20)\qquad = e^{-\frac{i\pi}{4}} \prod_{\nu=2}^{m-1}(sin\pi\mu_\nu)^{-1} \begin{pmatrix} \frac{1}{2sin\frac{\pi}{2}\mu_1} & \frac{i}{2sin\frac{\pi}{2}\mu_1} \\ -i2sin\frac{\pi}{2}\mu_1 & 2sin\frac{\pi}{2}\mu_1 \end{pmatrix} e^{-i\rho\int_{x_1}^{x_m}|\phi|dt}.$$

This yields $\vartheta_1 = -\int_{x_1}^{x_m}|\phi|dt = -R_+$, and comparing (9), (11) and (20) we get

$$(21)\qquad c_1 = h(\rho)\hat{c}_1 = -(-\rho)^{k_1+k_2}|\phi(0)\phi(1)|^{-\frac{1}{2}} \frac{e^{\rho R_-}}{4sin\frac{\pi}{2}\mu_1 sin\frac{\pi}{2}\mu_m \prod_{\nu=2}^{m-1} sin\pi\mu_\nu}\Theta_1.$$

In order to determine c_p and ϑ_p we have to consider all terms in $\Pi_{\nu=1}^{m-1} C_\nu(T_\nu, T_{\nu+1})$ possessing a factor $e^{i\rho\int_{x_1}^{x_m}|\phi|dt}$; from (12), (13), (14) we get as the corresponding relevant partial matrices

(22) $$C_{\nu,p}(II,II) = \begin{pmatrix} \frac{1}{sin\pi\mu_\nu} & 0 \\ 0 & 0 \end{pmatrix} e^{i\rho\int_{x_\nu}^{x_{\nu+1}}|\phi|dt}, \nu = 2,\dots,m-2,$$

(23) $$C_{1,p}(IV,II) = \begin{pmatrix} \frac{1}{2sin\frac{\pi}{2}\mu_1} & 0 \\ 0 & 0 \end{pmatrix} e^{i\rho\int_0^{x_1}|\phi|dt},$$

(24) $$C_{m-1,p}(II,III) = \begin{pmatrix} \frac{1}{sin\pi\mu_{m-1}} & 0 \\ 0 & 0 \end{pmatrix} e^{i\rho\int_{x_{m-1}}^{x_m}|\phi|dt}.$$

An elementary calculation implies $\vartheta_p = \int_{x_1}^{x_m}|\phi|dt = R_+$, $c_p = c_1$, and the application of Lemma 1 yields the assertion of the Theorem for $\rho_k \in S_{-1} \cup S_0$.

Now we discuss the case

1b. $\mathbf{T_1 = IV}$, $\mathbf{T_2 = T_{m-1} = II}$, $\mathbf{T_m = III}$, and two $\mathbf{T_i \neq II}$ for $\mathbf{i \in \{3,\dots,m-2\}}$.

We consider the combination $C_\nu(T_\nu,T_{\nu+1}) = C_\nu(II,II)$ for $\nu = 2,\dots,k-2$, $C_{k-1}(T_{k-1},T_k) = C_{k-1}(II,III)$, $C_k(T_k,T_{k+1}) = C_k(III,IV)$ $C_{k+1}(T_k,T_{k+1}) = C_{k+1}(IV,II)$, $C_\nu(T_\nu,T_{\nu+1}) = C_\nu(II,II)$ for $\nu = k+2,\dots,m-1$.

Starting with (10), (11) we get in a similar way to case 1a

$$-\vartheta_1 = \vartheta_p = (\int_{x_1}^{x_k} + \int_{x_{k+1}}^{x_m})|\phi|dt = R_+.$$

When we determine c_1 we observe that in the product $\prod_{\nu=1}^{k-1} C_\nu(T_\nu,T_{\nu+1})$ the relevant partial matrix

$$e^{-\frac{i\pi}{4}} \prod_{\nu=2}^{k-1}(sin\pi\mu_\nu)^{-1} \begin{pmatrix} \frac{1}{2sin\frac{\pi}{2}\mu_1} & \frac{i}{2sin\frac{\pi}{2}\mu_1} \\ -2isin\frac{\pi}{2}\mu_1 & 2sin\frac{\pi}{2}\mu_1 \end{pmatrix} e^{-i\rho\int_{x_1}^{x_k}|\phi|dt}$$

has to be considered. In the product $\prod_{\nu=k+1}^{m-1} C_\nu(T_\nu,T_{\nu+1})$ only the matrix

$$\prod_{\nu=k+2}^{m-1}(sin\pi\mu_\nu)^{-1} \begin{pmatrix} \frac{1}{2sin\frac{\pi}{2}\mu_{k+1}} & \frac{i}{2sin\frac{\pi}{2}\mu_{k+1}} \\ -2isin\frac{\pi}{2}\mu_{k+1} & 2sin\frac{\pi}{2}\mu_{k+1} \end{pmatrix} e^{-i\rho\int_{x_{k+1}}^{x_m}|\phi|dt}$$

is relevant. Using the relevant partial matrix

$$\begin{pmatrix} \frac{1}{2sin\frac{\pi}{2}\mu_k} & 0 \\ 0 & 0 \end{pmatrix} e^{\rho\int_{x_k}^{x_{k+1}}|\phi|dt}$$

for $C_k(III, IV)$ we get for the product $\prod_{\nu=1}^{m-1} C_\nu(T_\nu, T_{\nu+1})$ the relevant matrix

$$\frac{e^{-\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_k 2sin\frac{\pi}{2}\mu_{k+1}} \prod_{\substack{\nu=2 \\ \nu\neq k,k+1}}^{m-1} (sin\pi\mu_\nu)^{-1} \begin{pmatrix} \frac{1}{2sin\frac{\pi}{2}\mu_1} & \frac{i}{2sin\frac{\pi}{2}\mu_1} \\ -2isin\frac{\pi}{2}\mu_1 & 2sin\frac{\pi}{2}\mu_1 \end{pmatrix} e^{\rho(-iR_+ + \int_{x_k}^{x_{k+1}}|\phi|dt)}.$$

Analogous condiderations as in the case $T_i = II,\ i = 2, \ldots, m-1$ yield for c_1 the result

$$c_1 = \frac{-|\phi(0)\phi(1)|^{-\frac{1}{2}}(-\rho)^{k_1+k_2}e^{\rho R_-}}{16sin\frac{\pi}{2}\mu_1 sin\frac{\pi}{2}\mu_k sin\frac{\pi}{2}\mu_{k+1} sin\frac{\pi}{2}\mu_m \prod_{\substack{\nu=2 \\ \nu\neq k,k+1}}^{m-1} sin\pi\mu_\nu}\Theta_1.$$

The coefficient c_p is determined – neglecting exponential terms and nonvanishing constant factors – by the relevant matrices $\begin{pmatrix} \frac{1}{sin\pi\mu_\nu} & 0 \\ 0 & 0 \end{pmatrix}$ for $C_\nu(II, II)$, $\nu = 2, \ldots, k-1,\ k+2, \ldots, m-1$ resp. by $\begin{pmatrix} \frac{1}{2sin\frac{\pi}{2}\mu_\nu} & 0 \\ 0 & 0 \end{pmatrix}$ for $C_k(III, IV)$, and $C_{k+1}(IV, II)$. This yields $c_p = c_1$. Using the assumption $\Theta_1 \neq 0$ we derive from Lemma 1 the assertion of the theorem for the sequence (λ_k^+).

Now let us consider:

1c. The general case $\phi^2(0) < 0$, $\phi^2(1) < 0$, $\rho \in S_{-1}; T_1 = IV, T_m = III$.

Each combination of turning points of type T_ν, $T_\nu \neq II$ generates for the determination of c_1 a relevant partial matrix of the type

$$\prod_{j=n_1}^{n_2} \tau_j \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tau_j \neq 0.$$

At the appearance of turning points of type II there holds:

- At the beginning in the combination $IV, II, \ldots, II, III$, a partial matrix

$$\left(\prod_{j=2}^{n_1} \tau_j\right) \begin{pmatrix} \frac{1}{2sin\frac{\pi}{2}\mu_1} & \frac{i}{2sin\frac{\pi}{2}\mu_1} \\ -i2sin\frac{\pi}{2}\mu_1 & 2sin\frac{\pi}{2}\mu_1 \end{pmatrix}; \quad \tau_j \neq 0,$$

- at the midst in the combination $III, IV, II, \dots, II, III$, or $I, IV, II, \dots, II, III$, a partial matrix of the shape

$$\left(\prod_{j=n_1}^{n_2} \tau_j\right)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{cos\pi\mu_{n_1}}{sin\pi\mu_{n_1}} & i\frac{cos\pi\mu_{n_1}}{sin\pi\mu_{n_1}} \\ -isin\pi\mu_{n_1} & sin\pi\mu_{n_1} \end{pmatrix} = \tau\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} = \tau\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}; \quad \tau_j, \tau \neq 0,$$

- at the end in the combination $III, IV, II, \dots, II, III$ or $I, IV, II, \dots, II, III$, a partial matrix of the shape

$$\left(\prod_{j=n_1}^{m-1} \tau_j\right)\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}, \quad \tau_j \neq 0$$

is relevant for the determination of c_1. Therefore each admissible combination of turning points of type $T_1 = IV, T_m = III$ with arbitrary T_i $(i = 2, \dots, m-1)$, generates for the determination of c_1- neglecting non-vanishing factors - a relevant partial matrix of the shape

$$\tau\begin{pmatrix} \frac{1}{2sin\frac{\pi}{2}\mu_1} & \frac{i}{2sin\frac{\pi}{2}\mu_1} \\ -i2sin\frac{\pi}{2}\mu_1 & 2sin\frac{\pi}{2}\mu_1 \end{pmatrix}; \quad \tau \neq 0.$$

Therefore we have for all cases

$$c_1 = \rho^{k_1+k_2}\Theta_1\tilde{\tau}_1 e^{\rho R_-}, \ c_p = \rho^{k_1+k_2}\Theta_1\tilde{\tau}_2 e^{\rho R_-} \ \text{ with } \ \tilde{\tau}_1, \tilde{\tau} \in \mathbb{C}\setminus\{0\}.$$

Hence the regularity condition $\Theta_1 \neq 0$ is sufficient for the assertion of the theorem.

1d. The combination of turning points I, …, I, IV, …, III.

The estimates of z_i $(i = 1, 2)$ for turning points of type IV and I and $\rho \in S_{-1}$ are the same in $D_{0,\epsilon}$. Therefore the statements concerning a_1 and Δ do not change. In the product $\prod_{\nu=1}^{m-1} C_\nu(T_\nu, T_{\nu+1})$ we now obtain the factor $\prod_{\nu=1}^{n_1} C_\nu(I, I)$, with $n_1 \geq 1$, where the relevant partial matrix corresponding to $C_\nu(I, I)$ is given by

$$\frac{1}{sin\pi\mu_\nu}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\tau_\nu \ \text{ with } \ \tau_\nu \neq 0.$$

Obviously each relevant partial matrix corresponding to $C_\nu(I, I)$ contributes the factor $\frac{1}{sin\pi\mu_\nu}e^{\rho\int_{x_\nu}^{x_{\nu+1}}|\phi(t)|dt}$ to c_1 and c_p; therefore we infer from 1c that for all cases

$$c_1 = \rho^{k_1+k_2}\Theta_1\tilde{\tau}_1 e^{\rho R_-}, \ c_p = \rho^{k_1+k_2}\Theta_1\tilde{\tau}_2 e^{\rho R_-} \ \text{ with } \ \tau_1, \tau_2 \in \mathbb{C}\setminus\{0\}.$$

Hence the regularity condition $\Theta_1 \neq 0$ is sufficient for the assertion of the theorem.

1e. The combination of turning points of the type I,...,I,IV,...,III,I,...,I.

In the case of a combination of turning points of type I at the end we have to substitute the factor $2sin\frac{\pi}{2}\mu_m$ by $sin\pi\mu_m$ in the estimates (11). For the determination of a_1 we have to take into account that there appears a factor $\prod_{\nu=n_2}^{m-1} C_\nu(I,I)$ at the end of the product (9) involving a relevant partial matrix of the type

$$\tau\left(\prod_{\nu=n_2}^{m-1}\frac{1}{sin\pi\mu_\nu}\right)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \tau\neq 0.$$

Therefore we have to replace the corresponding factors of the representations of c_1 and c_p by $\prod_{\nu=n_2}^{m-1}\frac{1}{sin\pi\mu_\nu}$.

B. $\phi^2(0)<0, \phi^2(1)>0$ and $\rho\in S_{-1}$

1. Let $T_1 = IV$ or $T_1 = I$ and $T_m = IV$

Using Theorem 3.2 in [4] we get in this case

$$|\phi(x)|^{\frac{1}{2}}z_i(x,\rho)\overset{(2)}{=} e^{\pm\rho\int_{x_1}^{x}|\phi|dt}[1]\begin{cases} + & \text{for } i=1\\ - & \text{for } i=2\end{cases} \quad \text{for } (x,\rho)\in D_{0,\epsilon}\times S_{-1}$$

and

$$|\phi(x)|^{\frac{1}{2}}z_i(x,\rho)\overset{(2)}{=}\frac{[a_i]e^{-\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}e^{i\rho\int_{x_m}^{x}|\phi|dt}+\left\{\frac{[a_i]e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}+2[b_i]e^{-\frac{i\pi}{4}}sin\frac{\pi}{2}\mu_m\right\}e^{-i\rho\int_{x_m}^{1}|\phi|dt}$$
for $i=1,2,\quad (x,\rho)\in D_{m,\epsilon}\times S_{-1}$.

These estimates yield

$$\begin{aligned}U_j(z_i) &= \alpha_j(\pm\rho)^{k_j}|\phi(0)|^{k_j-\frac{1}{2}}e^{\mp\rho\int_0^{x_1}|\phi|dt}[1]+\beta_j(i\rho)^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}\frac{[a_i]e^{-\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}e^{i\rho\int_{x_m}^{1}|\phi|dt}+\\ &+\beta_j(-i\rho)^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}\left\{\frac{[a_i]e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}+2[b_i]e^{-\frac{i\pi}{4}}sin\frac{\pi}{2}\mu_m\right\}e^{-i\rho\int_{x_m}^{1}|\phi|dt},\ i,j=1,2.\end{aligned}$$

For $1\le\nu\le m,\ j\in\{1,p\}$ and $\rho\in S_{-1}$ let

$$\begin{pmatrix}a_{1j} & b_{1j}\\ a_{2j} & b_{2j}\end{pmatrix}:=\prod_{\nu=1}^{m}\hat{C}_{\nu,j}(T_\nu,T_{\nu+1}), \tag{25}$$

where $\hat{C}_{\nu,j}(T_\nu,T_{\nu+1})=C_{\nu,j}(T_\nu,T_{\nu+1})e^{\gamma_{\nu j}\rho\int_{x_\nu}^{x_{\nu+1}}|\phi|dt}$

$$\text{with } \gamma_{\nu j}=\begin{cases}-1 & \text{for} & \nu\in\{I,III\} & \text{and} & j\in\{1,p\}\\ i & \text{for} & \nu\in\{II,IV\} & \text{and} & j=1\\ -i & \text{for} & \nu\in\{II,IV\} & \text{and} & j=p.\end{cases}$$

Proceeding as for case A we infer that Δ_d can be represented as an exponential sum

$$\Delta_d(\rho) = \sum_{k=1}^{p} c_k e^{i\rho\vartheta_k}[1]$$

with

$$-\vartheta_1 = \vartheta_p = R_+$$

and

$$\begin{aligned} c_1 &= -|\phi(0)\phi(1)|^{-\frac{1}{2}}(-\rho)^{k_1+k_2}\left\{\tfrac{[a_{1,1}]e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m} + 2[b_{1,1}]e^{-\frac{i\pi}{4}}sin\tfrac{\pi}{2}\mu_m\right\}\Theta_2 e^{\rho R_-} \\ c_p &= -|\phi(0)\phi(1)|^{-\frac{1}{2}}(-\rho)^{k_1+k_2}\tfrac{a_{1,p}e^{-\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}\Theta_3 e^{\rho R_-}. \end{aligned}$$

Since the relevant partial matrix for $C_{m-1}(T_{m-1}, IV)$ is of the type $\tau\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $(\tau \neq 0)$ we obtain for all admissable combinations of turning points $b_{1,1} = 0$ and $a_{1,1} \neq 0$. Analogously we deduce $a_{1,p} \neq 0$. The regularity condition $\Theta_2\Theta_3 \neq 0$ implies therefore the assertion of the theorem.

2. Let $T_1 = I$ or IV, $T_m = II$

From Theorem 3.2 in [4] we get in this case

$$|\phi(x)|^{\frac{1}{2}} z_i(x,\rho) \overset{(2)}{=} e^{\pm\rho\int_{x_1}^{x}|\phi|dt}[1] \begin{cases} + & \text{for } i = 1 \\ - & \text{for } i = 2 \end{cases}, (x,\rho) \in D_{0,\epsilon} \times S_{-1}$$

$$|\phi(x)|^{\frac{1}{2}} z_i(x,\rho) \overset{(2)}{=} \tfrac{[a_i]}{sin\pi\mu_m} e^{i\rho\int_{x_m}^{x}|\phi|dt} + \left\{\tfrac{i[a_i]cos\pi\mu_m}{sin\pi\mu_m} + [b_i]sin\pi\mu_m\right\} e^{-i\rho\int_{x_m}^{x}|\phi|dt}$$

for $i = 1, 2,\ (x,\rho) \in D_{m,\epsilon} \times S_{-1}$.

Using the resulting estimates

$$\begin{aligned} U_j(z_i) &= \alpha_j\rho^{k_j}|\phi(0)|^{k_j-\frac{1}{2}}e^{\mp\rho\int_0^{x_1}|\phi|dt}[1] + \beta_j(i\rho)^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}\tfrac{[a_i]}{sin\pi\mu_m}e^{i\rho\int_{x_m}^{1}|\phi|dt} + \\ &+\beta_j(-i\rho)^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}\left\{\tfrac{i[a_i]cos\pi\mu_m}{sin\pi\mu_m} + [b_i]sin\pi\mu_m\right\}e^{-i\rho\int_{x_m}^{1}|\phi|dt} \end{aligned}$$

we conclude that

$$\Delta_d(\rho) = \sum_{k=1}^{p} c_k e^{i\rho\vartheta_k}[1]$$

with

$$-\vartheta_1 = \vartheta_p = R_+$$

and

$$c_1 = |\phi(0)\phi(1)|^{-\frac{1}{2}}(-\rho)^{k_1+k_2+1}\left\{\frac{i[a_{1,1}]cos\pi\mu_m}{sin\pi\mu_m} + [b_{1,1}]sin\pi\mu_m\right\}\Theta_2 e^{\rho R_-}$$
$$c_p = |\phi(0)\phi(1)|^{-\frac{1}{2}}(-\rho)^{k_1+k_2+1}\frac{[a_{1,p}]}{sin\pi\mu_m}\Theta_3 e^{\rho R_-}.$$

For all admissable combinations of turning points we obtain $a_{1,1} = \tau cos\pi\mu_m$, $b_{1,1} = i\tau$, $a_{1,p} = \tilde{\tau}$ where $\tau, \tilde{\tau} \in \mathbb{C}\setminus\{0\}$. The condition $\Theta_2\Theta_3 \neq 0$ implies therefore the assertion of the theorem.

C. $\phi^2(0) > 0$, $\phi^2(1) < 0$ and $\rho \in S_{-1}$

1. $T_1 = II$, $T_m = III$

By Theorem 3.2 in [4] we have:

$$|\phi(x)|^{\frac{1}{2}} z_1(x,\rho) \overset{(2)}{=} e^{i\rho\int_{x_1}^{x}|\phi|dt}[1]$$
$$|\phi(x)|^{\frac{1}{2}} z_2(x,\rho) \overset{(2)}{=} e^{-i\rho\int_{x_1}^{x}|\phi|dt}[1] + icos\pi\mu_1 e^{i\rho\int_{x_1}^{x}|\phi|dt}[1] \quad \text{for } (x,\rho) \in D_{0,\epsilon} \times S_{-1}$$

and

$$|\phi(x)|^{\frac{1}{2}} z_i(x,\rho) \overset{(2)}{=} \frac{[a_i]e^{\frac{i\pi}{4}}}{sin\pi\mu_m} e^{\rho\int_{x_m}^{x}|\phi|dt} + 2[b_i]sin\tfrac{\pi}{2}\mu_m e^{-\rho\int_{x_m}^{x}|\phi|dt}$$
for $i = 1, 2$ and $(x,\rho) \in D_{m,\epsilon} \times S_{-1}$.

Using the resulting estimates

$$U_j(z_1) = \alpha_j(i\rho)^{k_j}|\phi(0)|^{k_j-\frac{1}{2}}e^{-i\rho\int_0^{x_1}|\phi|dt}[1] + \beta_j\rho^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}\frac{[a_1]e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}e^{\rho\int_{x_m}^{1}|\phi|dt} +$$
$$+\beta_j(-i\rho)^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}[b_1]2sin\tfrac{\pi}{2}\mu_m e^{\frac{i\pi}{4}}e^{-\rho\int_{x_m}^{1}|\phi|dt}$$

and

$$U_j(z_2) = \alpha_j(-i\rho)^{k_j}|\phi(0)|^{k_j-\frac{1}{2}}e^{i\rho\int_0^{x_1}|\phi|dt}[1] + [i]\alpha_j(i\rho)^{k_j}|\phi(0)|^{k_j-\frac{1}{2}}cos\ \pi\mu_1 e^{-i\rho\int_0^{x_1}|\phi|dt} +$$
$$+\beta_j\rho^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}\frac{[a_2]e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}e^{\rho\int_{x_m}^{1}|\phi|dt} + \beta_j(-\rho)^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}[b_2]sin\tfrac{\pi}{2}\mu_m e^{\frac{i\pi}{4}}e^{-\rho\int_{x_m}^{1}|\phi|dt}$$

we get in the corresponding exponential sum

$$-\vartheta_1 = \vartheta_p = R_+$$

and

$$(26)\qquad \begin{aligned} c_1 &= |\phi(0)\phi(1)|^{-\frac{1}{2}}(i\rho)^{k_1+k_2}\frac{e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m} + (-[a_{2,1}] + i[a_{1,1}]cos\pi\mu_1)\Theta_3 e^{\rho R_-}, \\ c_p &= |\phi(0)\phi(1)|^{-\frac{1}{2}}(-\rho)^{k_1+k_2}\frac{[a_{1,p}]e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}\Theta_2 e^{\rho R_-}. \end{aligned}$$

In the case $T_i = II$ for $i = 1, \dots, m-1$ and $T_m = III$, we conclude for all admissable combinations of turning points that

$$a_{1,1} = \tau \tfrac{cos\pi\mu_1}{sin\pi\mu_1}$$
$$a_{2,1} = \tau(-i) sin\pi\mu_1 \quad (\tau \neq 0).$$

This yields

$$-a_{2,1} + ia_{1,1} cos\pi\mu_1 = \frac{2\tau i}{sin\pi\mu_1} \neq 0.$$

Together with an analogous conclusion for c_p, we get the assertion of the theorem.

2. Let $\mathbf{T_1 = II}$, $\mathbf{T_m = I}$.

In this case there is at least one relevant partial matrix $C_{\nu,j}(T_\nu, T_{\nu+1}) = \tau \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for some $\nu \in \{2, \dots, m-2\}$. Consequently for all combinations of turning points we derive that of $a_{2,1} = 0$ in (26) and

$$c_1 = \tau a_{1,1}\Theta_2, \; c_p = \tilde{\tau} a_{1,p}\Theta_3 \text{ with } \tau\tilde{\tau} \neq 0 \text{ where } \frac{\tilde{\tau}}{\tau} \text{ is a constant.}$$

This proves the assertion.

3. Let $\mathbf{T_1 = III}$, $\mathbf{T_m = I}$ or $\mathbf{III}$

In this case (26) has the shape

$$c_1 = |\phi(0)\phi(1)|^{-\frac{1}{2}}(i\rho)^{k_1+k_2} \tfrac{e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}(-[a_{2,1}] + i[a_{1,1}])\Theta_3 e^{\rho R_-}$$
$$c_p = |\phi(0)\phi(1)|^{-\frac{1}{2}}(i\rho)^{k_1+k_2} \tfrac{[a_{1,p}]e^{\frac{i\pi}{4}}}{2sin\frac{\pi}{2}\mu_m}\Theta_2 e^{\rho R_-}.$$

For each combination of turning points we get, as for 2., $a_{2,1} = 0$. This implies the assertion of the theorem since $a_{1,1}a_{1,p} \neq 0$.

D. $\boldsymbol{\phi^2(0) > 0}$, $\boldsymbol{\phi^2(1) > 0}$, $\boldsymbol{\rho \in S_{-1}}$

1. Let $\mathbf{T_1 = II}$, $\mathbf{T_m = II}$,

By Theorem 3.2 (i) in [4] we get for $(x, \rho) \in D_{0,\epsilon} \times S_{-1}$:

$$|\phi(x)|^{\frac{1}{2}} z_1(x,\rho) \overset{(2)}{=} e^{i\rho\int_{x_1}^x |\phi| dt}[1]$$
$$|\phi(x)|^{\frac{1}{2}} z_2(x,\rho) \overset{(2)}{=} e^{-i\rho\int_{x_1}^x |\phi| dt}[1] + icos\pi\mu_1 e^{i\rho\int_{x_1}^x |\phi| dt}[1]$$

and for $(x, \rho) \in D_{m,\epsilon} \times S_{-1}$ for $i = 1, 2$

$$|\phi(x)|^{\frac{1}{2}} z_i(x,\rho) \overset{(2)}{=} \tfrac{[a_i]}{sin\pi\mu_m} e^{i\rho\int_{x_m}^x |\phi| dt} + \left\{\tfrac{i[a_i]cos\pi\mu_m}{sin\pi\mu_m} + [b_i] sin\pi\mu_m\right\} e^{-i\rho\int_{x_m}^x |\phi| dt}.$$

Plugging these estimates into U_j we obtain:

$$\begin{aligned} U_j(z_1) &= \alpha_j(i\rho)^{k_j}|\phi(0)|^{k_j-\frac{1}{2}}e^{-i\rho\int_0^{x_1}|\phi|dt} + \beta_j(i\rho)^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}\frac{[a_1]}{sin\pi\mu_m}e^{i\rho\int_{x_m}^1|\phi|dt} + \\ &+\beta_j(-i\rho)^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}\left(i\frac{[a_1]cos\pi\mu_m}{sin\pi\mu_m} + [b_1]sin\pi\mu_m\right)e^{-i\rho\int_{x_m}^1|\phi|dt} \\ U_j(z_2) &= \alpha_j(i\rho)^{k_j}|\phi(0)|^{k_j-\frac{1}{2}}\left\{(-1)^{k_j}e^{i\rho\int_0^{x_1}|\phi|dt}[1] + [i]cos\pi\mu_1 e^{-i\rho\int_0^{x_1}|\phi|dt}\right\} + \\ &+\beta_j(i\rho)^{k_j}|\phi(1)|^{k_j-\frac{1}{2}}\left\{\frac{[a_2]}{sin\pi\mu_m}e^{i\rho\int_{x_m}^1|\phi|dt} + (-1)^{k_j}\left(\frac{i[a_2]cos\pi\mu_m}{sin\pi\mu_m} + [b_2]sin\pi\mu_m\right)e^{-i\rho\int_{x_m}^1|\phi|dt}\right\}. \end{aligned}$$

We derive for the main terms $c_1e^{i\rho\vartheta_1}$ resp. $c_pe^{i\rho\vartheta_p}$ in the resulting exponential sum

$$\begin{aligned} c_1 &= |\phi(0)\phi(1)|^{-\frac{1}{2}}(i\rho)^{k_1+k_2}\left\{\frac{i[a_{2,1}]cos\pi\mu_m}{sin\pi\mu_m} + [b_{2,1}]sin\pi\mu_m + \frac{[a_{1,1}]cos\pi\mu_1\ cos\pi\mu_m}{sin\pi\mu_m} - \right. \\ &\left. -i[b_{1,1}]cos\pi\mu_1\ sin\pi\mu_m\ \right\}\Theta_1e^{\rho R_-} \end{aligned}$$

and

$$c_p = -|\phi(0)\phi(1)|^{-\frac{1}{2}}(i\rho)^{k_1+k_2}\frac{[a_{1,p}]}{sin\pi\mu_m}\Theta_1e^{\rho R_-}.$$

The coefficients $a_{i,1}$, $b_{i,1}$, $i = 1,2$ respectively $a_{1,p}$ are determined by (25) and the corresponding considerations for all admissable combinations of turning points in the case $A : \phi^2(0) < 0,\ \phi^2(1) < 0$. In any case $a_{1,p}$ and the contents of the brackets in the expressions for c_1 are non-vanishing for $\rho \in S_{-1},\ |\rho| > R_0$. Therefore the condition $\Theta_1 \neq 0$ for the regularity of (2) is sufficient for the assertion of the theorem concerning the sequence (λ_k^+) of eigenvalues.

The corresponding estimates for $\Delta_0(\rho)$ and $\Delta(\rho)$ for $\rho \in S_k, k \in \{-2,0,1\}$ and the estimate for the sequence (λ_k^-) are derived similarly (see [5] for further details).

Final Remarks:

(i) The regularity conditions are independent of the sector $S_k, -2 \le k \le 1$.

(ii) Using the method of this paper we can also investigate problems with turning points in 0 or (and) 1. For the discussion we use the representation of the fundamental system u_{1j} resp. u_{mj} $(j = 1,2)$ (see [4]) and the corresponding initial conditions at 0 or 1; in addition we must use a modified regularity condition depending on the type of the turning point in 0 or (and) 1 .

(iii) We can also apply the methods used in [4] and in this paper in the presence of poles, of χ or ϕ^2 up to order 2 in x_ν, $1 \leq \nu \leq m$ (see [2] and R.E. Langers papers cited therein). In this case one has to develop connection formulas for a fundamental system $w_{\nu j}$ $(j = 1,2)$ in $D_{\nu,\epsilon} \times S_k$ analogous to those in our paper [4]. These problems will be studied in another paper and will permit the treatment of the classical equations of second order with weak singularities (for example the Bessel-, Legendre-, and hypergeometric functions).

(iv) The explicit character of the estimates of Theorem 3.2 in [4] also allow the investigation of eigenvalue problems with boundary conditions at more than two points:

$$U_\nu(y) = U_{\nu 0}(y) + \textstyle\sum_{j=1}^{k} U_{\nu a_k}(y) + U_{\nu 1}(y) = 0, \quad 1 \leq \nu \leq 2, \qquad (27)$$
$$\text{where} \quad 0 < a_1 < \ldots < a_k < 1,$$

$$U_{\nu a_j}(y) = \gamma_{j\nu} y^{(k_\nu)}(a_j) + \sum_{\mu=0}^{k_\nu - 1} \gamma_{\nu j \mu} y^{(\mu)}(a_j), \quad 1 \leq j \leq k,$$

and where $U_{\nu 0}(y)$ and $U_{\nu 1}(y)$ are defined as in (2). The boundary conditions (27) are called regular if the corresponding boundary condition (2) are regular and in this case the eigenvalues of (1), (27) satisfy the estimates (6) of the theorem. In the same manner we can investigate multipoint - integral - boundary value problems. For details we refer to the extensive literature for the corresponding definite case.

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