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SOLITONS IN NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. Using a mathematical approach accessible to graduate students of physics and engineering, we show how solitons are solutions of nonlinear Schrödinger equations. Are also given references about the history of solitons in general, their fundamental properties and how they have found applications in optics and fiber-optic communications.

(1)Introduction.

In a first approximation and we can say that a *soliton* is a solitary wave which preserves its shape and velocity when it moves, exactly as a particle does. A *soliton* is a solitary wave, i.e. a localized wave, with spectacular stability properties.[1-3] The first observation of this kind of wave was done in 1834 in water channels by the engineer John Scott Russel.[1,3,4] Only at 1895 a theory developed by Korteweg and de Vries[1,3,4] was able to explain the fascinating behavior of the *hydrodynamic soliton* observed by Russel. This amazing phenomenon was forgotten until a numerical experiment carried in 1953 by Fermi, Pasta and Ulam[3] using computers that appeared to contradict thermodynamics. Only ten years later this effect was explained by Zabusky and Kruskal[5] taking into account solitary waves that they named *solitons*. The study of Zabusky and Kruskal[5] is a landmark in the history of solitons. After this physicists noticed that solitons are solutions of nonlinear equations. Before, theoretical approaches were trying to avoid nonlinearities or to treat them as perturbations of linear theories.

The 19th.century and the first half of the 20th.century can be viewed[3] as the triumph of the *linear physics* (like Maxwell´s equations and quantum mechanics) based on a linear formalism emphasizing a superposition principle. This picture was dramatically changed after the discovery of solitons from the mathematical and physical points of view.

The body of knowledge[1] that is presently associated with the term "soliton" is enormously broad involving several significant fields with no previous contact with each other. Today, the scientific community gravitating around "*soliton equations*" (or integrable dynamical systems) includes, on the one hand, nonlinear-optics engineers, astrophysicists, theoretical biologists, oceanographers and, on the other hand, pure and applied mathematicians in algebra, geometry and functional analysis. According to Degasperis[1] the formation of the underlying concepts took place independently, in physics and in mathematics, and the discovery of solitons may be compared to the opening of the “Pandora´s box”. Strictly speaking, however, the term “soliton” indicates, in general, a peculiar solitary wave whose propagation is modeled by a nonlinear equation and whose space profile is such that the nonlinearity and the dispersion or the diffraction effects of the medium balance each other. It is a spatially localized wave with spectacular stability properties. The name soliton sounds like the name of a particle. It is a wave but moves exactly as a particle does; it is a solution of a classical field equation which simultaneously exhibits wave and quasi-particle properties.[3] These are features that one would expect from quantum systems and not from a classical one. The quantum analogue goes so far that soliton tunneling has been found.[6]

There are different kinds of solitons which are solutions of different nonlinear equations like,[3] for instance, of Kortweg-de Vries (KdV) equation, sine-Gordon equation and nonlinear Schrödinger (NLS) equation. In a recent paper, written to graduate students of physics and engineering, we have shown how to obtain the *hydrodynamic KDV solitons*. In a preceding paper[7] we have studied the existence and stability of *Gaussian solitons* in 1-dim nonlinear Schrödinger equation

In Section 1 we obtain solitons that are solutions of the 1-dim nonlinear Schrödinger equation (NLS) with no external potential. In Section 2 we study the 1-dim motion of a free particle with mass m which obeys a NSL equation. In Section 3 are analyzed the *optical solitons*[2,3,8] (spatial and temporal solitons) that are predicted by 1 and 2-dim NLS equations assuming the *Kerr nonlinearity* for the optical medium.

1) Solitons of 1-dim NLS equation.

Let us consider the 1-dim nonlinear differential equation given by[3]

$$i\partial\psi/\partial t + P(\partial^2\psi/\partial x^2) + Q\,|\psi|^2\psi = 0 \qquad (1.1),$$

where t is the time, x is the coordinate along the x-axes, P and Q are coefficients that depend on the particular problem which is being analyzed. This equation appears very similar to the Schrödinger equation (SE) if we write it as

$$i\partial\psi/\partial t = [-P(\partial^2/\partial x^2) - Q\,|\psi|^2]\psi = 0 \qquad (1.2),$$

and is *formally analogous* to the SE if $P > 0$. If $P < 0$ we take the complex conjugate of (1.2) obtaining an equation for ψ^* in which the coefficient of

$(\partial^2\psi^*/\partial x^2)$ is positive. So, without any restriction we can assume $P > 0$ in (1.1) or (1.2). Note that the complex conjugate transformation change the signs of P and Q, that is, $P \rightarrow -P$ and $Q \rightarrow -Q$ so that it does not affect the sign of the product PQ. This invariance, as will be seen, is of fundamental importance to determine the nature of the solutions of (1.1).

The potential function of the SE is here equal to the nonlinear term $-Q|\psi|^2$. As will be shown, when $Q > 0$ the ψ solution is localized, with a bell shape. Thus, the NLS equation is such that ψ generates its own potential well which, as will be seen, is a necessary condition for the existence of a solution named *spatially localized solution*. In this case the soliton is named *bright soliton*. This is a "self-trapping" phenomenon which is essential for the physics of systems obeying a NLS equation.

Let us look for a solution of (1.1) of the form

$$\Psi(x,t) = \phi(x,t) \exp[i\Theta(x,t)] \quad (1.3),$$

where the amplitude ϕ and the phase factor Θ are real functions. If we assume that Θ varies between 0 and 2π we can restrict the search of ϕ only to positive values. Thus, putting (1.3) in (1.1) we get, separating real and imaginary parts

$$-\phi\Theta_t + P\phi_{xx} - P\phi\,\Theta_x^2 + Q\phi^3 = 0 \quad (1.4)$$

$$\phi_t + P\phi\Theta_{xx} + 2P\phi_x\Theta_x = 0 \quad (1.5).$$

Let us look for a particular wave solution of (1.4) and (1.5) such that

$$\phi(x,t) = \phi(x-v_e t) \quad \text{and} \quad \Theta(x,t) = \Theta(x-v_p t) \quad (1.6),$$

where the envelope and the phase propagate, respectively, with velocity v_e and v_p that can assume different values. Thus, from (1.4) and (1.5) we have

$$v_p\phi\Theta_x + P\phi_{xx} - P\phi\,\Theta_x^2 + Q\phi^3 = 0 \quad (1.7)$$

$$-v_e\phi_x + P\phi\Theta_{xx} + 2P\phi_x\Theta_x = 0 \quad (1.8).$$

Multiplying (1.8) by ϕ and integrating we obtain

$$-v_e\phi^2/2 + P\phi^2\Theta_x = C \quad (1.9),$$

where C is a constant.

In order to obtain *spatially localized* solutions of the NLS equation it is necessary to assume that for $|x| \rightarrow \infty$ we have $\phi \rightarrow 0$ and $\Theta \rightarrow 0$. Consequently, from (1.9) with $\phi \neq 0$ we see that $C = 0$ and

$$\Theta_x = v_e/2P \qquad (1.10).$$

Integrating (1.10) results

$$\Theta = (v_e/2P)(x - v_p t) + C' \qquad (1.11),$$

where C´ is an integration constant that we impose to be equal to zero by an appropriate choice of the time origin. Putting (1.11) into (1.7) we obtain

$$(v_e\, v_p/2)\phi + P\phi_x - (v_e^2/4P)\, \phi + Q\, \phi^3 = 0 \qquad (1.12),$$

Multiplying (1.12) by $P\phi_x$ we get an expression that can be readily integrated resulting

$$(P^2/2)\, \phi_x^{\ 2} + V_{eff}(\phi) = 0 \qquad (1.13),$$

where $V_{eff}(\phi)$ is a "pseudo-potential" defined by

$$V_{eff}(\phi) = (PQ/4)\, \phi^4 - (v_e^2 - 2v_e v_p)\, \phi^2/8 \qquad (1.14),$$

where the constant of integration has again taken equal to zero in order to have a spatially localized solution. Since ϕ is real $\phi_x^{\ 2} \geq 0$. In this way from (1.13) we verify that the "motion of a particle" must be in a ϕ region where $V_{eff}(\phi) \leq 0$. Consequently, after a simple analysis, we see that there are two different functions $V_{eff}(\phi)$ x ϕ that are shown in Fig.1 (a) and (b) as a function of PQ: (a) PQ > 0 and (b) PQ < 0.[3]

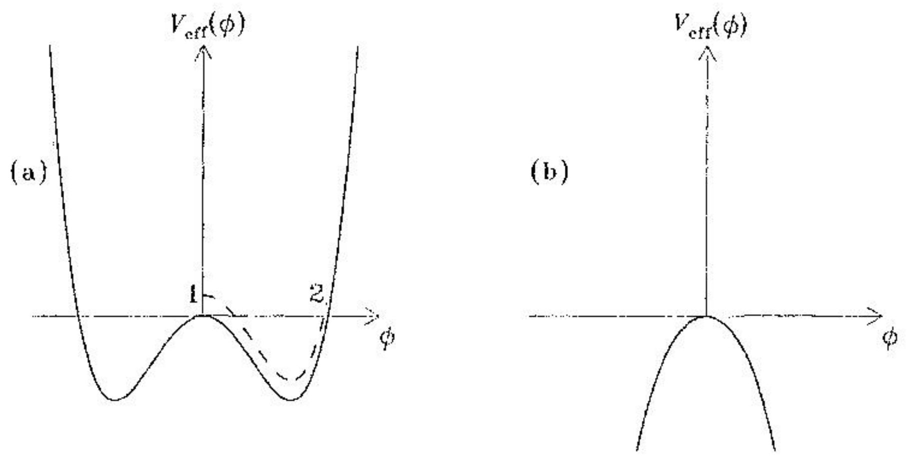

Figure 1. Shapes[3] of $V_{eff}(\phi)$ x ϕ for (a) PQ > 0 and (b) PQ < 0. The motion of a *localized soliton* evolves between the points 1 and 2. This soliton, when P > 0 and Q > 0 is known as *bright soliton*.

We see that the "particle motion" governed by (1.13) must occurs only when PQ > 0 and between the points 1 and 2 shown in Fig.1(a). The points 1 and 2 are $\phi_1 = 0$ and $\phi_2 = \phi_o = \{(v_e^2 - 2v_e v_p)/2PQ\}^{1/2}$, respectively. In this case the amplitude ϕ_o is finite. We also verify that $v_e^2 - 2v_e v_p \geq 0$ which does not impose a sign for v_e and v_p, but shows that $v_e = v_p$ is not allowed.

From (1.13) we get

$$\partial\phi/\partial x = \{-2V_{eff}(\phi)/P^2\}^{1/2} = (A\phi^2 - B\phi^4)^{1/2} \qquad (1.15),$$

where $A = \sqrt{2}\,(2v_e v_p - v_e^2)/(8P)$ and $B = \sqrt{2}Q/4$. Integrating (1.15) remembering that $\phi = \phi(x-v_p t)$ we obtain (see Appendix A): [3,8,9]

$$\phi(x,t) = \phi_o \operatorname{sech}\{(Q/2P)^{1/2}\,\phi_o(x-v_e t)\} \qquad (1.16),$$

where $\phi_o = \{(v_e^2 - 2v_e v_p)/(2PQ)\}^{1/2}$, $PQ > 0$, $v_e^2 - 2v_e v_p \geq 0$

and assuming that for $x = t = 0$ the initial amplitude $\phi(0,0) = 0$.

Finally, using (1.3), (1.11) and (1.6) the *bright soliton* is represented by the function $\Psi(x,t)$ given by (see Appendix A)

$$\Psi(x,t) = \phi_o \operatorname{sech}\{(Q/2P)^{1/2}(x-v_e t)\,\phi_o\} \exp[i(v_e/2P)(x - v_p t)] \qquad (1.17).$$

This function $\Psi(x,t)$ can also be written as,

$$\Psi(x,t) = \phi_o \operatorname{sech}\{(x-v_e t)/\xi_e\} \exp[i(kx - \mu t)] \qquad (1.18),$$

with

$$\xi_e = (1/\phi_o)(2P/Q)^{1/2}, \quad k = v_e/2P \quad \text{and} \quad \mu = v_e v_p/2P$$

showing that $\Psi(x,t)$ is a wave packet *localized* in a region with width ξ_e which is inversely proportional to the amplitude ϕ_o. This *localization* is an effect generated by the nonlinearity of the NLS equation (1.1). In the limit of very small amplitudes (linear limit) that is, when $\phi_o \to 0$ so that $\xi_e \to \infty$ we have plane waves $\Psi(x,t) \approx \phi_o \exp[i(kx - \mu t)]$.

The intensity of a typical bright soliton as a function of $u = (x-v_e t)/\xi$ given by

$$|\psi(u)|^2 = |\phi_o|^2 \operatorname{sech}^2 u$$

is shown in Fig.2.

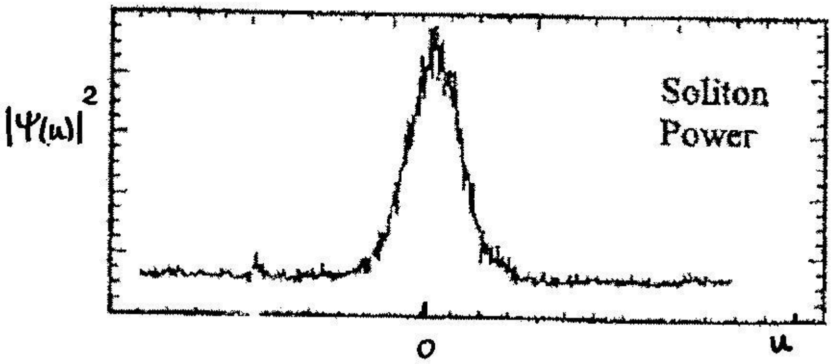

Figure 2. Bright soliton intensity (power)$|\psi(u)|^2$ as function of u in a semiconductor waveguide.[8]

1a) Grey and Dark solitons.[2,3,8,9]

When P > 0 and Q < 0 instead of (1.2) we have

$$i\partial\psi/\partial t + P(\partial^2\psi/\partial x^2) - Q|\psi|^2 = 0 \qquad (1.19).$$

Integrating this equation[8] assuming the boundary condition $\psi \rightarrow \phi_o$ when $x \rightarrow \infty$ we obtain a "soliton-like" solution $\psi(x,t)$ which is named "grey soliton" with an intensity, as a function of a parameter φ, given by

$$|\psi(x,t,\varphi)|^2 = |\phi_o|^2\{1- \cos^2\varphi\ \mathrm{sech}^2[\phi_o \cos\varphi\ (x - \phi_o \sin\varphi\ t)]\} \qquad (1.20)$$

Since the energy density of the grey soliton is not localized, strictly speaking, it is not a soliton.[3] The *grey soliton* is a "dip" in the background amplitude $|\phi_o|^2$ and its relative velocity of propagation $\phi_o \sin\varphi$ to the background depends on the angle φ. In Fig. 3, for t = 0, is shown[8] the intensity $|\psi|^2$ as function of x and of the parameter $\varphi = \phi$. For $\varphi = 0$ when the dip in the background has its maximum value we have the *dark soliton*.

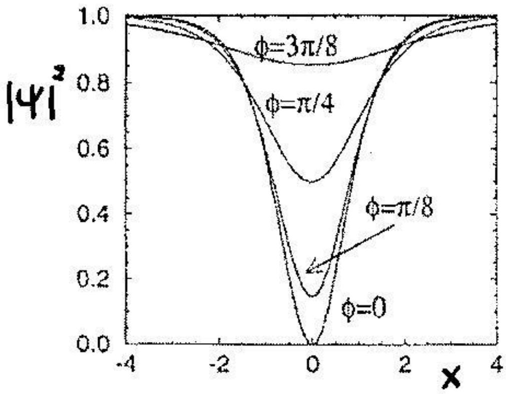

Figure 3.Normalized intensity $|\psi|^2$ of the *grey soliton* as a function of x for several[8] values of the internal phase $\varphi = \phi$; $|\psi|^2$ drops to zero at the center for the *dark soliton*.

Putting $\varphi = 0$ in (1.20) the grey soliton intensity becomes

$$|\psi(x,\varphi = 0)|^2 = |\phi_o|^2\{1- \mathrm{sech}^2(\phi_o x) \} = |\phi_o|^2\tanh^2(\phi_o x) \qquad (1.21),$$

which is equal to zero at $x = 0$, as seen in Fig.3.

2) 1-dim motion of a particle obeying a NLS equation.

Let us assume that a particle with mass m obeys a NLS equation with $g > 0$ given by

$$i\hbar(\partial\psi/\partial t) + (\hbar^2/2m)(\partial^2\psi/\partial x^2) + g|\psi|^2\psi = 0 \qquad (2.1).$$

Comparing (1.1) and (2.1) we see that $P = \hbar/2m$ and $Q = g/\hbar$. So, According to Section 1 a *localized soliton* solution of (2.1) is described by, taking into account (1.18):

$$\Psi(x,t) = \phi_o\, \mathrm{sech}\{(x-v_e t)/\xi_e\} \exp[i(kx - \mu t)] \qquad (2.2),$$

where

$$\phi_o = \{(v_e^2- 2v_e v_p) /(2PQ)\}^{1/2}= \{\hbar(v_e^2- 2v_e v_p)/m\}^{1/2}$$

$$\xi_e = (1/\phi_o)(2P/Q)^{1/2} = (1/\phi_o)(\hbar^2/2m)^{1/2},$$

$$k = v_e/2P = mv_e/\hbar$$

and

$$\mu = v_e v_p/2P = mv_e v_p/\hbar = kv_p$$

Assuming that the particle has a momentum $p = \hbar k$ we see that envelop velocity v_e obtained in (2.2) is the propagation velocity of the "pilot wave" according to the de Broglie hypothesis, that is, $v_e = \hbar k/m$. Assuming also that total energy of the soliton is E, taking $\mu = E/\hbar$ we get, using (2.2), $E/\hbar = \mu = kv_p$ which would give

$$v_p = E/\hbar k = E/mv_e \qquad (2.3).$$

3) Optical Solitons.

According to Thierry and Peyrard,[3] the optical soliton is one of the main application of solitons and an example where the idea of a theoretician of nonlinear science opened a multi-million-euro market.

We will show that the main equation governing the evolution of optical fields (electromagnetic fields) in a nonlinear medium is a NLS equation. This can be done taking into account, for instance, the Maxwell wave equation for the electric field $\mathbf{E}(\mathbf{r},t)$ associated with the wave propagating in a nonlinear optical medium with Kerr (or cubic) nonlinearity.[10] In this paper we will consider only the Kerr nonlinearity and, consequently, only *Kerr solitons*.

In order to obtain the structure of the wave in the fibre of a nonconducting ($\mathbf{j} = 0$) and nonmagnetic ($\mathbf{B} = \mu_o \mathbf{H}$) material, let us first consider the two Maxwell equations[11]

$$\mathrm{rot}\,\mathbf{E}(\mathbf{r},t) = -\,\partial\mathbf{B}(\mathbf{r},t)/\partial t \quad \text{and} \quad \mathrm{rot}\,\mathbf{H}(\mathbf{r},t)) = -\,\partial\mathbf{D}(\mathbf{r},t)/\partial t \qquad (3.1),$$

where
$$\mathbf{D} = \varepsilon_o\mathbf{E} + \mathbf{P} = \varepsilon_o\mathbf{E} + \varepsilon_o\chi(\omega)\mathbf{E} \qquad (3.2),$$

which includes only the linear part of the polarization. From (3.1) we get

$$\Delta\mathbf{E} - \mathrm{grad}(\mathrm{div}\mathbf{E}) - (1/\varepsilon_o c^2)\,\partial^2\mathbf{D}/\partial t^2 = 0 \qquad (3.3),$$

where Δ = laplacian operator. Now, taking into account the nonlinear polarization effect of the fibre it is convenient to write (3.2), separating the linear and nonlinear parts, as

$$\mathbf{D} = \varepsilon_o\mathbf{E} + \mathbf{P} = \varepsilon_o\mathbf{E} + \varepsilon_o\chi^{(1)}\mathbf{E} + \varepsilon_o\chi^{(3)}|\mathbf{E}|^2\mathbf{E} = \mathbf{D}_\ell + \varepsilon_o\chi^{(3)}|\mathbf{E}|^2\mathbf{E} \qquad (3.4),$$

where $\mathbf{D}_\ell = \varepsilon_o\mathbf{E} + \varepsilon_o\chi^{(1)}\mathbf{E}$. Note[3] that as a change in the sign of $\mathbf{E}$ must reverse the polarization, the tensor $\chi^{(2)}$ must vanish, so the first nonlinear term of (3.2)-(3.4) is the third order term $\chi^{(3)}$ which is of order ε^2.

In what follows it will be assumed to simplify the calculations that the electric field $\mathbf{E}$ is *linearly polarized.* It is also important to remark that we are not taking into account the decrease of the soliton intensity along the optical fibre.[3,8]

3.1) ***Spatial solitons.***

Let us consider the case of a *monochromatic* electric field *linearly polarized* propagating in an infinite fibre with a diameter much larger than the wavelength of the light. So, we only have to consider one component of the electric field[3]

$$E(x,y,z) = \phi(x,y,z)\exp[i(k_o z - \omega_o t)] + \text{c.c.} \qquad (3.5),$$

where $\phi(x,y,z)$ describes the structure of the field that propagates along the z-axis. In this way, with $\mathrm{div}(\mathbf{E}) = 0$, from (3.3)-(3.5) results

$$\Delta E - (1/\varepsilon_o c^2)\,\partial^2 D_\ell/\partial t^2 = (\chi^{(3)}/c^2)\,\partial^2(|E|^2E)/\partial t^2 \qquad (3.6),$$

where for a monochromatic wave $D_\ell = \varepsilon(\omega_o)E$. Taking into account that

$$\Delta E = (\Delta\phi) \exp[i(k_o z - \omega_o t)] + (2ik_o\partial\phi/\partial z - k_o^2\phi) \exp[i(k_o z - \omega_o t)]$$

and using the dispersion relation $k_o^2 = \omega_o^2\varepsilon(\omega_o)/\varepsilon_o c^2$ the eq.(3.6) becomes

$$2ik_o(\partial\phi/\partial z) + (\Delta\phi) + (\omega_o^2\chi^{(3)}/c^2)\,|\phi|^2\phi = 0 \qquad (3.7),$$

showing that (3.7) belongs to the family of NLS equations of the form

$$i(\partial\phi/\partial z) + P(\Delta\phi) + Q\,|\phi|^2\phi = 0 \qquad (3.8),$$

where $P = (1/2k_o)$, $Q = (\omega_o^2\chi^{(3)}/2k_o c^2)$ and the laplacian operator Δ acts in a *D* dimension space.[3] For a light beam in a nonlinear medium the variation of ϕ with space in transverse direction (x,y) is much slower than the space variation of the exponential factor of (3.5).[3] While the exponential factor varies over a length of a micron or below (which is order of the light wavelength) the variation of ϕ occurs over a length of the order of the diameter of the beam, such as millimeters (usual transverse dimensions of optical fibres). In other words, the envelope ϕ changes slowly while propagating, i.e. $|\partial^2\phi/\partial z^2| << |k_o\partial\phi/\partial z|$. In these conditions, in (3.8) the term $\partial^2\phi/\partial z^2$ will be neglected. In this way (3.8) becomes, putting $\Delta = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, since $P > 0$ and that Q can be positive or negative, because $\chi^{(3)}$ can be positive or negative,

$$i(\partial\phi/\partial z) + P(\partial^2/\partial x^2 + \partial^2/\partial y^2)\phi \pm |Q|\,|\phi|^2\phi = 0 \qquad (3.9),$$

where the signs $\pm$ correspond to *bright* and *grey* solitons, respectively. Since $P > 0$ and we have *bright solitons* only when $PQ > 0$, that is, when $Q = (\omega_o^2\chi^{(3)}/2k_o c^2) > 0$. This condition is satisfied for $\chi^{(3)} > 0$, as occurs with dielectric materials.

The standard NLS equation has the time variable t in place of z. Of course one can use $z = ct/n_o$, where $n_o = n(\omega_o)$ is the refraction index of the fibre and write (3.9) in terms of t. However, in optics it is common to use z as the propagation variable.[8]

It is important to remark that the soliton solutions of (3.9) are named *spatial solitons*: they are generated while propagating in the medium when nonlinear effect balance the diffraction.[2,3,8]

1-dim planar waveguide

The 1-dim motion occurs when the nonlinear medium has a form of a planar waveguide. So, the optical field is confined in one of the transverse direction, say the y axis. In this case the beam will spread only along the x direction. In these conditions (3.9) becomes written as[8]

$$i(\partial\phi/\partial z) + P(\partial^2\phi/\partial x^2) \pm |Q|\,|\phi|^2\phi \;=\; 0 \qquad (3.10)$$

which is the 1-dim NLS equation analyzed in Section 1. The solitons of (3.10), or mono-dimensional solitons, are stable and often referred as (1+1)D solitons, meaning that they are limited in one dimension (x or t) and propagate along one (z).

2-dim waveguide

In this case according to (3.9) we have

$$i(\partial\phi/\partial z) + P(\partial^2/\partial x^2 + \partial^2/\partial x^2)\phi \pm |Q|\,|\phi|^2\phi \;=\; 0 \qquad (3.11).$$

Solving (3.11) one verify that the 2-dim beam propagation is more dramatic than in 1-dim case since there appear many unstable solutions.[3] The (2+1)D spatial solitons are unstable, so any small perturbation (due to noise, for instance) can cause the soliton to diffract as a field in a linear medium or to collapse, thus damaging the material.[2,3] This can be seen, for instance, solving this equation taking into account the cylindrical symmetry of the beam.[3] In this case the ***2-dim NLS*** equation (3.11) can be written as

$$i(\partial\phi/\partial z) + (P/r)\{\partial/\partial r[r\,(\partial\phi\,/\partial r)]\} \pm |Q|\,|\phi|^2\phi \;=\; 0 \qquad (3.12).$$

In order to investigate the origin of these instabilities and to obtain stable (2+1)D spatial solitons more general forms of NLS equations were proposed like, for instance,[3]

$$i(\partial\phi/\partial z) + P(\Delta\phi) + Q\,|\phi|^{2\sigma}\phi \;=\; 0 \qquad (3.13),$$

where the nonlinearity is controlled by a parameter σ.

In Appendix B is shown the equation (3.11) written in a compact form as is usually done in optics.

Many detailed descriptions of experiments about generation, stability and properties of optical solitons can be found, for instance, in the books “Physics of Solitons”[3] and “Optical Solitons.”[8] and also in reference 2.

The first experiment[2] on spatial solitons was reported in 1974 by Ashkin and Bjorkholm in a cell with sodium vapor. About 1985 this field was revisited in experiments at the Limoges University in carbon disulphide. After these experiments spatial solitons have been demonstrated in glass, semiconductors and polymers. During the last ten years several experiments have been reported on solitons in nematic liquid crystals.

3.2) Temporal solitons: propagation of a pulse of light in optical fibres.

Now let us consider the propagation of an electric field in a dispersive nonlinear optical fibre. In these conditions solitons are created when the linear dispersion effect and the nonlinear Kerr effect balance each other.[2,3,8] These solitons are called *temporal solitons*.

As is shown, for instance, by Thierry and Peyard[3] the amplitude of an electric field $\mathbf{E}(\mathbf{r},\omega)$ with a given frequency ω propagates in a nonlinear fibre with constant amplitude. So it cannot be used to transfer information along a fibre. It is only possible through *wave packets* which combine several modes with frequencies $\omega = \omega(k)$ centered around a reference frequency $\omega_o = \omega(k_o)$. As we know, a wave packet which propagates along a z direction is represented by[11]

$$E(z,t) = \psi(z,t) \exp[i(k_o z - \omega_o t)] \qquad (3.13),$$

where $\psi(z,t)$ is the shape of the envelop of the wave packet centered at the point $z = v_g t$. The propagation velocity of the envelope is v_g, called *group velocity*. It is given by $v_g = [d\omega(k)/dk]_{k=ko}$, where $\omega(k) = ck/n(k)$ is the dispersion relation and n(k) the refraction index of the material expressed as a function of k.

Our goal now is to determine how the wave packet evolves along the z axis of the optical fibre. To do this we must solve (3.6) assuming that[3]

$$E(z,t) \sim \phi(z,t) \exp[i(k_o z - \omega_o t)] \qquad (3.14).$$

noting that

$$k^2 = \omega^2/c^2_{medium} = \omega^2\varepsilon(\omega)/\varepsilon_o c^2 ,$$

$$\partial k/\partial\omega = \omega\varepsilon(\omega)/\varepsilon_o k c^2 + (\omega^2/2k\varepsilon_o c^2)[\partial\varepsilon(\omega)/\partial\omega] \qquad (3.15).$$

and that

$$v_g = 1/(\partial k/\partial\omega)_{\omega o}$$

Performing the calculation assuming that $\omega \approx \omega_o$, up to a second order approximation, we have[3]

$$i(\partial\phi/\partial\xi) - P(\partial^2\phi/\partial\tau^2) + Q|\phi|^2\phi = 0 \qquad (3.16),$$

where the amplitude $\phi = \phi(\xi,\tau)$, $\xi = z$, $\tau = t - z/v_g$,

$$P = (1/2)(\partial^2 k/\partial\omega^2)_{\omega o} \quad \text{and} \quad Q = \omega_o^2\chi^{(3)}/2k_o c^2.$$

Showing that (3.16) is a 1-dim NLS equation as a function of the variables

$\xi = z$ and $\tau = t - z/v_g$, in which the role of the time and space have been switched with respect with the usual 1-dim NLS equation. According to Section 1 this equation has the following soliton solution

$$\phi(\xi,\tau) = \phi_o \,\mathrm{sech}\{(Q/2P)^{1/2}\phi_o\xi\} \exp(iQ\phi_o^{2}\tau/2) \qquad (3.17).$$

The envelope would keep a permanent profile and move at the group velocity v_g, in agreement with its definition. Observing the envelope passing through any section of the fibre we would always observe the same function, but shifted by the amount z/v_g depending on the point of observation. As said above, such pulses are called *temporal solitons*. The pulse does not change while propagating due to two contrary effects that balance each other: the linear dispersion and the nonlinear Kerr effect.[2,3,8]

Note that since for dielectric material $Q = \omega_o^{2}\chi^{(3)}/2k_oc^2 > 0$, since $\chi^{(3)}$ is positive, the product $P(\omega)Q > 0$ only when $P(\omega) > 0$. This occurs only in a frequency region of *anomalous dispersion*, that is, when $\partial(1/v_g)/\partial\omega < 0$. Only for this region we have *bright solitons*. For a region of *normal dispersion* we have *grey solitons*.

The NLS equation for an optical fibre was proposed in 1973 by two theoreticians, A. Hasegawa and F.Tappert.[2,3,8] Also in 1973 R. Boullogh made the first mathematical report of the existence of temporal solitons. However, the first experimental checks were only made in 1980 by L. Mollenauer suggesting that solitons could exist in optical fibres.[2] In1987, P. Emplit et al. made the first experimental observation of the propagation of a dark soliton in an optical fibre. In 1988, L. Mollenauer et al. transmitted solitons pulses over 4000 km. In 1991, a Bell Labs research team transmitted solitons over more than 14000 km.[2] Since then, the fiber solitons have been studied extensively and have even found applications in the field of fiber-optic communications.[8]

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Appendix A. Jacobi elliptic functions.

Let us consider the nonlinear differential equation for $V(x)$

$$d^2V/dx^2 = 2V(K-V^2) \qquad (A.1),$$

where K is a constant. This equation can be solved by multiplying it by $2(dV/dx)$ and integrating over x obtaining

$$(dV/dx)^2 = 2KV^2-V^4 + C \qquad (A.2),$$

where C is a constant of integration. Using the boundary conditions $V(x) \to 0$ and $dV(x)/dx \to 0$ when $|x| \to \infty$, C is found to be 0. Assuming also that $V(0) = a$ and $(dV/dx)_{x=0} = 0$ we get, using (A.2), that $K = a^2/2$. In this way (A.2) becomes

$$dx = dV/(a^2V^2 - V^4)^{1/2} \qquad \text{(A.3)}.$$

Integrating (A.2) taking into account the Jacobi elliptic functions[12] we get

$$V(x) = a\,\mathrm{sech}(ax) \qquad \text{(A.4)}.$$

Appendix B. Compact NLS equation for optical spatial solitons.

Usually in optics[2,8] the NLS equations are written in a compact form as will be seen in what follows. According to (3.3) and (3.4) the nonlinear polarization $\mathbf{P}_{NL}(\mathbf{r},t)$ to a Kerr medium is given by

$$\mathbf{P}_{NL}(\mathbf{r},t) \approx \varepsilon_o \varepsilon_{NL} \mathbf{E}(\mathbf{r},t) = \varepsilon_o \chi^{(3)} |\mathbf{E}|^2 \mathbf{E} = (\varepsilon_o \chi^{(3)} I)\mathbf{E} \qquad \text{(B.1)},$$

where $\varepsilon_{NL} = \chi^{(3)} I$ and I is the field intensity $I = |\mathbf{E}|^2$**.** The Fourier transform of the dielectric constant $\varepsilon^*(\omega)$ is written as

$$\varepsilon^*(\omega) = 1 + \chi^{(1)}(\omega) + \varepsilon_{NL} = 1 + \chi^{(1)}(\omega) + \chi^{(3)} I \qquad \text{(B.2)}.$$

The dielectric constant can be used to define[11] the refractive index $n^*(\omega)$ and the *absorption coefficient* $\alpha^*(\omega)$. Due to the nonlinear effect both $n^*(\omega)$ and $\alpha^*(\omega)$ become intensity dependent because of ε_{NL}. It is customary to introduce [8]

$$n^*(\omega) = n(\omega) + n_2(\omega) I \qquad \text{and} \qquad \alpha^*(\omega) = \alpha(\omega) + \alpha_2(\omega) I \qquad \text{(B.3)},$$

where $n(\omega)$ and $\alpha(\omega)$ are related to the real and imaginary parts of the linear parameter $\chi^{(1)}(\omega)$, respectively. Analogously, $n_2(\omega)$ and $\alpha_2(\omega)$ are related to the real and imaginary parts of $\chi^{(3)}(\omega)$, respectively.[8]

At this point we believe that is important to remember[11] that the wavenumber that is defined as $k(\omega) = \omega/v = (\omega/c)\sqrt{\varepsilon(\omega)\mu(\omega)}$ can be written as $k = \beta + i\alpha/2$.Thus, assuming that an electric field propagates in the Kerr medium with frequency ω_o we define $n_o = n(\omega_o)$, $\varepsilon_o = \varepsilon(\omega_o)$, $\lambda_o = 2\pi c/\omega_o$, $k_o = 2\pi/\lambda_o$ and $\beta_o = 2\pi n_o/\lambda_o = k_o n_o$.

Let us analyze two different ways to write the NLS equations in compact forms.

(B.1) Let us assume that propagating field along the z-axis is given by the $E(\mathbf{r},t) = A(\mathbf{r})\exp(i\,\beta_o z)$. Using the same approach adopted in Section (3.1) we get the following equation for the amplitude A(x,y,z):[8]

$$2i\,\beta_o(\partial A/\partial z) + (\partial^2/\partial x^2 + \partial^2/\partial y^2)A + 2\,\beta_o\,k_o n_2(I)A = 0 \qquad \text{(B.4)},$$

where $n_2(I) = n_2(\omega_o)I$. In the absence of the nonlinear effects (B.4) reduces to the well-known paraxial equation used extensively in the context of scalar diffraction theory.

Assuming that the widths of the optical fibre along the x and y axes are equal to w_o it is useful to introduce the scaled dimensionless variables

$$x = x/w_o\ ,\ \ y = y/w_o\ ,\ \ z = z/L_d,\ \ L_d = \beta_o w_o^2 \ \text{ and }\ u = A(k_o n_2 L_d)^{1/2} \qquad \text{(B.5)},$$

where L_d is the *diffraction length* (or Rayleigh range). In terms of these dimensionless variables Eq.(B.4) takes the form of a standard (2+1)-dimensional NLS equation:[8]

$$i(\partial u/\partial z) + (\partial^2/\partial x^2 + \partial^2/\partial y^2)u \pm |u|^2 u = 0 \qquad \text{(B.6)},$$

where the choice of the sign depends on the sign of the nonlinear parameter $n_2 = n_2(\omega_o)$.

(B.2) Let us study only the simplest 1-dim case and write[2]

$$E(x,z,t) = A_m\, a(x,z)\exp[i\,(\beta_o z - \omega_o)t], \qquad \text{(B.7)}$$

where A_m is the maximum amplitude of the field and a(x,z) is a dimensionless normalized function (so that is maximum value is 1) that represent the shape of the field among the x-axis and that propagates along the z-axis. Now for this field we have to solve the Helmholtz equation:

$$\Delta E + k_o n_2(I)\,E = 0 \qquad \text{(B.8)}.$$

Considering that $|\partial^2 a/\partial z^2| << |k_o\,\partial a/\partial z|$ we verify that (B.8) becomes:

$$\partial^2 a/\partial x^2 + 2ik_o n_o(\partial a/\partial z) + k_o^2[n^2(I) - n_o^2]a = 0 \qquad \text{(B.9)}$$

Taking into account that the nonlinear effects are always much smaller than the linear one: $[n^2(I) - n_o^2] = [n(I) - n_o][n(I) + n_o] = n_2 I\,(2n_o + n_2 I) \approx 2n_o n_2 I$. With this approximation (B.9) becomes

$$(1/2k_o n_o)\,(\partial^2 a/\partial x^2) + i(\partial a/\partial z) + (k_o n_o n_2|A_m|^2/2)\,|a|^2 a = 0 \qquad \text{(B.10)}.$$

Let us define the dimensionless variables $\xi = x/w_o$, where w_o is the width of fibre along the x-axis and $\zeta = z/L_d$ where $L_d = k_o n_o w_o^2 = \beta_o w_o^2$ is the *diffraction length* or Rayleigh length. In addition, putting $N = L_d/L_{nl}$ where $L_{nl} = (k_o n_o n_2 |A_m|^2/2)$ is the *self-focusing length* the Eq.(B.10) becomes,

$$(1/2)\,(\partial^2 a/\partial\xi^2) + i(\partial a/\partial\zeta) \pm N^2|a|^2 a = 0 \qquad \text{(B.11)},$$

where the choice $\pm$ depends on the sign of the parameter $n_2 = n_2(\omega_o)$.

a) $N >> 1 \rightarrow$ *nonlinear* effects (self-focusing effects) are much larger than the *linear effects* (diffraction effects*)*.The field will tend to focus.

b) $N << 1 \rightarrow$ *linear effects* are much larger than *nonlinear effects*. The field will diffract.

c) $N \approx 1 \rightarrow$ the linear and nonlinear effects balance each other and we have to solve (B.11).

For $N = 1$ and signs $\pm$ we verify[2,8] that the solutions of (B.11) are the *bright soliton* and *dark soliton*, respectively,

$$a_b(\xi, \zeta) = \mathrm{sech}(\xi)\exp(i\zeta/2) \quad \text{and} \quad a_d(\xi, \zeta) = \tanh(\xi)\exp(-i\zeta)$$

For $N = 2$ and + it is still possible to obtain the solution in a closed form:[2]

$$a(\xi, \zeta) = \{4[\cosh(3\xi) + 3e^{4i\zeta}\cosh(\xi)]e^{i\zeta/2}\}/\{\cosh(4\xi)+4\cosh(2\xi)+3\cosh(4\xi)\},$$

with a shape that changes during the propagation along z with period $\zeta=\pi/2$.

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