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MATHEMATICS

Conditions of Spectrum Localization for Operators not Close to Self-Adjoint Operators

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Abstract—The Keldysh theorem is generalized to an arbitrary closed operator that is not necessarily close to self-adjoint operators and has a resolvent of Schatten–von Neumann class $\mathfrak{S}_p$. Based on this theorem, conditions of spectrum localization are obtained for certain classes of non-self-adjoint differential operators.

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Assume that T_0 is a densely defined operator in a separable Hilbert space $\mathcal{H}$ with a discrete spectrum. Let $\{\lambda_k\}_{k=1}^{\infty}$ denote the eigenvalues of T_0 that are numbered in nondecreasing order of their moduli with allowance for their algebraic multiplicities and $\{f_k\}_{k=1}^{\infty}$ denote the corresponding normalized root vectors (eigenvectors and associated vectors). From now on, we assume that the system $\{f_k\}_{k=1}^{\infty}$ is complete and 0 is not an eigenvalue of T_0 (the latter restriction is insignificant).

It follows from the Keldysh theorem [1] that if the operator T_0 is positive and $T_0^{-1} \in \mathfrak{S}_p$ for some $p < \infty$, then, for any T_0-compact perturbation V, both the completeness of the system of root vectors (SRV) and the spectrum localization are preserved in the following sense: if $\{\mu_k\}_1^{\infty}$ are the eigenvalues of $T = T_0 + V$ numbered in nondecreasing order of their moduli, then

$$\forall \varepsilon > 0\ \exists K(\varepsilon) \in \mathbb{N}: |\arg \mu_k| \le \varepsilon\ \forall k \ge K(\varepsilon). \tag{1}$$

If the spectrum distribution function $N(T_0, r)$ satisfies the Tauberian condition $\exists \gamma > 0$, $R > 0$: $\dfrac{N(T_0, s)}{N(T_0, r)} < \left(\dfrac{s}{r}\right)^{\gamma}$, $\forall s > r \ge R$, then

$$N(T, r) \sim N(T_0, r), \quad r \to +\infty. \tag{2}$$

The Keldysh theorem was subsequently generalized in various directions (see [2] and the references therein).

If T_0 is not close to a self-adjoint or normal operator (that is, it cannot be represented as $T_0 = H + V$, where H is self-adjoint or normal and V is H-compact), then the spectrum of T_0 can vary greatly under the action of small perturbations (see [3, 4] and the references therein). Since the Keldysh theorem does not work for such operators, we have to use special methods for studying their spectral properties [5–7]. Meanwhile, operators that are not close to normal operators quite often arise in various branches of physics and mechanics [3, 8, 9]. Among these, there are many operators that have "correct" properties: their root vectors form a complete system and the spectrum is localized around one or several rays (even can lie entirely in a ray) [5, 6, 10, 11]. Thus, the following problem arises.

Problem. Under the assumption that an operator T_0 is not necessarily close to a normal operator and has some properties P (completeness of the SRV, basis property in a certain sense, spectrum localization, etc.), find classes of perturbations $\mathbb{V} = \mathbb{V}(P)$ for which at least some of these properties are preserved.

This paper is devoted to studying the class of perturbations preserving the asymptotics of the spectrum of an arbitrary densely defined operator with a resolvent of Schatten–von Neumann class. The main result of this study is Theorem 1, which establishes the following: If the spectrum of an operator T_0 is localized around the ray $\arg \lambda = 0$ in a more general sense than (1) and there is at least one ray along which the decay of the resolvent is fastest (with a power of -1), then, for any relatively compact perturbation, the spectrum can be localized only around the ray $\arg \lambda = 0$; under an additional regularity condition for the function $N(T_0, r)$, formula (2) is valid. In Section 2,

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we give examples of differential operators whose spectrum asymptotics can be studied easier using Theorem 1 than by traditional asymptotic methods.

1. THE MAIN RESULT

Since T_0 is not close to a self-adjoint operator, its spectrum is not necessarily localized is the sense of (1). Consequently, any angle $\{\theta_1 < \arg\lambda < \theta_2\}$ can contain infinitely many eigenvalues of T_0. Therefore, the following interpretation of the concept of spectrum localization is more natural.

Definition 1. The spectrum of T_0 is said to be localized around a ray $\arg\lambda = \alpha_0$ if and only if $\forall\varepsilon > 0$ we have

$$N(T_0, r) \sim N(T_0, \alpha_0 - \varepsilon, \alpha_0 + \varepsilon, r), \quad r \to +\infty,$$

where $N(T, \eta, \zeta, r)$ and $N(T, r)$ are the numbers of eigenvalues of the operator T in the sector $\{\eta < \arg\lambda < \zeta, |\lambda| < r\}$ and the circle $\{|\lambda| < r\}$, respectively.

Theorem 1. *Let an operator T_0 satisfy the following conditions*:

(1_T) $T_0^{-1} \in \mathfrak{S}_p (p > 0)$ *and the spectrum of T_0 is localized around the ray* $\arg\lambda = 0$;

(2_T) *the spectrum of T_0 in the ray $(-\infty, 0)$ is finite and*

$$\|(T_0 + r)^{-1}\| = O(r^{-1}), \quad r \to +\infty;$$

and

(3_T) $N(T_0, r)$ *satisfies the condition*

$$\liminf_{r\to\infty} \frac{N(T_0, r)}{N(|T_0|, r)} > 0.$$

Then the following statements are true:

1. *If*

(1_V) *an operator V is bounded with respect to T_0 and its T_0-bound is zero and*

(2_V) *the spectrum of the operator $T = T_0 + V$ is localized around a ray* $\arg\lambda = \beta$, *then* $\beta = 0$.

2. *If, in addition to* (1_T)–(3_T),

$$\lim_{\substack{\varepsilon\to+0,\\ r\to+\infty}} \frac{N(T_0, r(1+\varepsilon))}{N(T_0, r)} = 1,$$

then formula (2) *is valid*.

Note that all assumptions of Theorem 1 hold if the operators T_0 and V satisfy the assumptions of the Keldysh theorem.

2. EXAMPLES

Example 1. Non-self-adjoint anharmonic oscillator. Let $H(\alpha, \theta)$ be an in $L^2(0, \infty)$ defined as

$$\begin{gathered} D(H(\alpha,\theta)) = \{y \in L^2(0,+\infty) : y, y' \in AC[0,\infty), \\ -y'' + e^{i\theta}x^{\alpha}y \in L^2(0,+\infty), y(0) = 0\}, \end{gathered} \tag{3}$$

$$H(\alpha,\theta)y = -y'' + e^{i\theta}x^{\alpha}y, \tag{4}$$

where $\theta \in (-\pi, \pi)$ and $\alpha \in (0, +\infty)$ are constants. The operator $H(\alpha, \theta)$ is usually called a non-self-adjoint anharmonic oscillator [3]. Since the cases when $\theta \in (-\pi, 0)$ and $\theta \in (0, \pi)$ are completely similar, we assume from now on that $\theta \in (0, \pi)$.

The operator $H(\alpha, \theta)$ was a subject of study for many authors (see [4, 5, 12, 13] and the references therein). In [5], it was established that for each $|\theta| \le \frac{\pi}{2}$ the spectrum of $H(\alpha, \theta)$ is discrete, all eigenvalues are simple (their algebraic multiplicities are 1) and lie in the ray $\arg z = \frac{2\theta}{2+\alpha}$, and for $\alpha > \frac{2}{3}$ the system of eigenfunctions of $H(\alpha, \theta)$ is complete in $L^2(0, \infty)$. It follows from the results of [4] that everything that has been said about the spectrum of $H(\alpha, \theta)$ remains true for all $|\theta| < \pi, \alpha > 0$: if $\{\lambda_n(\alpha, a)\}_1^{\infty}$ are the eigenvalues of $H(\alpha, \theta)$ numbered in increasing order of their moduli, we have

$$\lambda_n(\alpha, \theta) = \lambda_n(\alpha, 0)e^{2\theta i/(2+\alpha)},$$

$$\lambda_n(\alpha, 0) \sim \left(\frac{\pi}{\int_0^1 \sqrt{1 - t^{\alpha}}dt}\right)^{2\alpha/(2+\alpha)} \cdot n^{2\alpha/(2+\alpha)}, \quad n \to \infty.$$

In [13, Theorem 2], it was shown that, for $|\theta| < \frac{2\alpha\pi}{2+\alpha}$, the eigenfunctions of $H(\alpha, \theta)$ form a basis for the Abel–Lidskii summability method with an exponent $\beta \in \left(\frac{2+\alpha}{2\alpha}, \frac{\pi}{|\theta|}\right)$. The completeness of the eigenfunctions of $H(1, \theta)$ for $|\theta| < \frac{5\pi}{6}$ was also established in [13].

As was shown in [3], the resolvent norm $\|(H(\alpha, \theta) - \lambda)^{-1}\|$ exponentially increases as λ tends to infinity in the sectors $\left\{\delta \le \arg\lambda \le \frac{2\theta}{2+\alpha} - \delta\right\}$ and $\left\{\frac{2\theta}{2+\alpha} + \delta \le \arg\lambda \le \theta - \delta\right\}$. Consequently, the operator $H(\alpha, \theta)$ is spectrally unstable; therefore, the class of perturbations V for which the asymptotics of the spectrum is preserved has to be rather narrow. In [4, Theorem 2], it was shown that this class contains the operator of multiplication by a function V such that

(I) it is locally integrable on $[0,+\infty)$, admits an analytic continuation to the angle $U_\theta=\left\{-\dfrac{\theta}{2+\alpha}<\arg z<0\right\}$, and can be continuously extended to any finite point of the boundary of U_θ;

and

(II) it satisfies the estimate $V(z)=o(z^\alpha), z\to\infty$, which is uniform over $-\dfrac{\theta}{2+\alpha}\le\arg z\le 0$.

The purpose of this subsection is to show, using one class of perturbations as an example, that conditions (I) and (II) are essential for preserving the spectrum localization in the sense of (1) and how formula (1) can easily be derived using Theorem 1 if we manage to show that the spectrum is localized in the sense of Definition 1.

Let $a_0=0$ and $a_k=e^{-\theta/(2+\alpha)}(k+ik^\gamma)(k=1,2,\ldots)$, where $0<\gamma<1$. Furthermore, let $\{v_k\}$ be functions that are measurable on $[a_{k-1},a_k]$ and satisfy the following conditions:

(v1) for all $k\in\mathbb{N}$, the limits $V_k^-:=\lim\limits_{z\to a_{k-1}}v_k(z)$ and $V_k^+:=\lim\limits_{z\to a_k}v_k(z)$ exist, are finite, and $V_k^+\ne V_{k+1}^-$;

and

(v2) there exist $C>0,\delta>0$ such that, for all $k\in\mathbb{N}$, we have $\sup\limits_{[a_{k-1},a_k]}|v_k(t)|\le Ck^{-\delta}$.

Define

$$V(z)=\sum_{k=1}^{\infty}\int_{a_{k-1}}^{a_k}\frac{v_k(t)dt}{t-z},\qquad z\in\Gamma,$$

where Γ is a polygonal line with vertices at the points a_k. Under the above assumptions, the function V is holomorphic outside Γ, has angular boundary values $V_+(\zeta)$ and $V_-(\zeta)$ (from left and right, respectively) at almost each point $\zeta\in\Gamma$, and, for all $k\in\mathbb{N}$, $V_\pm\in L^2(a_{k-1},a_k)$.

Let $T=H(\alpha,\theta)+V$, where V is the operator of multiplication by the function $V(\cdot)$.

Theorem 2. *The following statements are true*:

(1) *The spectrum of the operator T is localized around the ray* $\arg\lambda=\dfrac{2\theta}{2+\alpha}$ *in the sense of Definition* 1.

(2) *For any $\varepsilon>0$, the part of the spectrum of T that lies outside the angle*

$$\left|\arg\lambda-\frac{2\theta}{2+\alpha}\right|<\varepsilon,$$

can be divided into $K_\varepsilon=\max\left\{k\in\mathbb{N}:(k+1)^\gamma-k^\gamma\ge\tan\dfrac{\varepsilon}{2}\right\}$ *series $\{\lambda_n^{(k)}\}_{n=1}^\infty$ $(k=1,2,\ldots,K_\varepsilon)$ that have the asymptotics*

$$\lambda_n^{(k)}\sim\left(\frac{\pi n}{a_{k+1}-a_k}\right)^2,\quad n\to\infty.$$

Example 2. Sturm–Liouville operator on a curve. Let γ be a curve parameterized as $z(x)=x+is(x), x\in[0,1]$, where the function s is continuously differentiable, its derivative s' is a nondecreasing function, and $s(0)=s(1)=0, s'(0)<0<s'(1)$. We introduce the notation $\alpha_0=\arctan s'(0), \alpha_1=\arctan s'(1)$. Then

$$-\frac{\pi}{2}<\alpha_0<0<\alpha_1<\frac{\pi}{2}.\tag{5}$$

Assume that a function y is absolutely continuous on the curve γ (with respect to the measure $|dz|$). The function

$$y'(z):=\lim_{\gamma\ni\zeta\to z}\frac{y(\zeta)-y(z)}{\zeta-z},$$

which is defined almost everywhere on γ, is said to be the derivative along γ. We can similarly define $y''(z)$, etc. (under the assumption that these objects exist).

Let $q\in L^1(\gamma)$. The Sturm–Liouville operator on a curve γ is an operator L_γ that acts in the space $L^2(\gamma)$ by the rule $L_\gamma y=-y''+qy$ on its domain of definition $D(L_\gamma)=\{y\in L^2(\gamma){:}y'\in AC(\gamma),-y''+qy\in L^2(\gamma),y(0)=y(1)=0\}$.

In the same way as in the case $\gamma=[0,1]$, we can prove that the operator L_γ is densely defined. Since the spectrum of L_γ is discrete [14, Subsection 2.3], it follows that the operator L_γ is closed. Using condition (5), we can easily show that all eigenvalues of L_γ, except for a finite number, lie in the angle $-2\alpha_1<\arg\lambda<-2\alpha_0$. It was shown in [7, 14] that the spectrum of L_γ is localized (in the sense of Definition 1) around the ray $\arg\lambda=0$ if and only if the function q admits a meromorphic continuation to the domain Ω bounded by the curve γ and the interval $[0,1]$ such that its poles $\{z_k\}$ can accumulate to only the interval $[0,1]$ and, in the neighborhood of each pole z_k, we have the decomposition

$$q(z)=\frac{m_k(m_k-1)}{(z-z_k)^2}+\sum_{i=0}^{m_k-1}c_{ki}(z-z_k)^{2i}+(z-z_k)^{2m_k-1}r(z),$$

where $m_k\in\mathbb{N}, c_{ki}$ are some numbers, and the function $r(z)$ is holomorphic in a neighborhood of z_k.

Let T_0 denote the operator L_γ with a potential q satisfying this criterion. Then

$$N(T_0, r) \sim \frac{\sqrt{r}}{\pi}, \quad r \to +\infty.$$

In addition, the operator T_0 satisfies all the other assumptions of Theorem 1.

Theorem 3. *The following statements are true*:

(1) *If* $-2\alpha_0 \leq \beta \leq 2\pi - 2\alpha_1$, *then*

$$\|(T_0 - r^{i\beta})^{-1}\| = O(r^{-1}), \quad r \to +\infty,$$

uniformly with respect to $\beta \in [-2\alpha_0, 2\pi - 2\alpha_1]$.

(2) *The distribution function of the spectrum of the operator* $|T_0|$ *has the asymptotics* $N(|T_0|, r) \sim \dfrac{\sqrt{r}}{\pi}, r \to +\infty$.

Let $\{a_k\}_1^\infty$ be a sequence of points in the curve γ such that the sequence $\{\mathrm{Re}\, a_k\}$ decreases and tends to zero as $k \to \infty$. Let

$$V(z) = v_k, \quad z \in \gamma_k,$$

where γ_k is the arc of the curve γ that connects the points a_k and $a_{k-1} (a_0 = 1)$ and $\{v_k\}_1^\infty$ is a bounded sequence such that $v_{k+1} \neq v_k (k = 1, 2, \ldots)$. We introduce the operator $T = T_0 + V$, where V is the operator of multiplication by the function $V(\cdot)$.

Theorem 4. *Assume that the function q can have only finitely many poles in the domain* Ω_1 *bounded by the arc* γ_1 *and the interval* $[1, a_1]$. *Then the following statements are true*:

(1) *For any* $\varepsilon > 0$, *the part of the spectrum of the operator* T *that is outside the angle* $-\varepsilon - 2\arg(1 - a_1) < \arg\lambda < -2\alpha_0$ *is finite*.

(2) *For any* $\varepsilon > 0$, *the part of the spectrum of the operator* T *that is in the angle* $-\varepsilon - 2\arg(1 - a_1) < \arg\lambda < -2\alpha_0 - \varepsilon$ *can be divided into* $K_\varepsilon = \max\{k \in \mathbb{N}: 2\arg(a_k - a_{k-1}) < -2\alpha_0 - \varepsilon\}$ *series* $\{\lambda_n^{(k)}\}_{n=1}^\infty$ $(k = 1, 2, \ldots, K_\varepsilon)$ *that have the asymptotics*

$$\lambda_n^{(k)} \sim \left(\frac{\pi n}{a_k - a_{k-1}}\right)^2, n \to \infty.$$

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