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The uniqueness and existence of solutions for the 3D Helmholtz equation in a step-index waveguide with unbounded perturbation

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In this paper, we study the 3D Helmholtz equation in a step-index waveguide with unbounded perturbation, allowing the presence of guided waves. Our assumptions on the perturbed and source terms are too few. On the basis of the Green's function for the 3D homogeneous Helmholtz equation in a step-index waveguide without perturbation, we introduce a generalized (out-going) Sommerfeld–Rellich radiation condition, and then we prove the uniqueness and existence of solutions for the studied 3D Helmholtz equation satisfying our radiation condition. Copyright © 2012 John Wiley & Sons, Ltd.

Keywords: Helmholtz equation; step-index waveguide; unbounded perturbation; existence of solutions; uniqueness of solutions; Green's function; radiation condition

1. Introduction

In this paper, we consider the 3D Helmholtz equation

$$\triangle u(x_1,x_2,z)+[k^2n^2(x_1,x_2)+p(x_1,x_2,z)]u(x_1,x_2,z)=f(x_1,x_2,z),\quad (x_1,x_2,z)\in\mathbb{R}^3 \tag{1.1}$$

in a step-index waveguide with unbounded perturbation, which we will refer to as the *perturbed 3D Helmholtz equation* later, where

$$n(x_1,x_2)=\begin{cases} n_{co}, & \sqrt{x_1^2+x_2^2}<R,\\ n_{cl}, & \sqrt{x_1^2+x_2^2}\geq R.\end{cases} \tag{1.2}$$

Here, $\triangle=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial z^2}$ is the Laplacian, $k>0$, n_{co}, n_{cl}, R are positive constants with $n_{co}>n_{cl}$, and $p(x_1,x_2,z)$ is the perturbed term satisfying assumptions:

(A1) $p(x_1,x_2,z)\in L^1(\mathbb{R}^3)$ and $p(x_1,x_2,z)=0$ for $\sqrt{x_1^2+x_2^2}>\widetilde{R}$ for some positive $\widetilde{R}>0$;
(A2) $p(x_1,x_2,z)$ satisfies

$$\sup_{(\xi_1,\xi_2,\zeta)\in\mathbb{R}^3}\int_{\mathbb{R}^3}|G(x_1,x_2,z;\xi_1,\xi_2,\zeta)p(x_1,x_2,z)|dx_1dx_2dz<1, \tag{1.3}$$

where $G(x_1,x_2,z;\xi_1,\xi_2,\zeta)$ is the Green's function for the 3D homogenous Helmholtz equation in a step-index waveguide without perturbation (see section 2 for more details).

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Our work is motivated by the study of the electromagnetic waves in a cylindrical optical waveguide. For $p(x_1,x_2,z) \equiv 0$, (1.1) describes the propagation of electromagnetic waves in a step-index waveguide, where k is the wavenumber, $n(x_1,x_2)$ is the refraction index, $f(x_1,x_2,z)$ is a source of energy, R is the radius of the waveguide, and $u(x_1,x_2,z)$ is the time harmonic electromagnetic wave velocity potential. In [1], we have introduced a generalized(out-going) Sommerfeld–Rellich radiation condition, which is a collection of Sommerfeld-like conditions for all guided components of the field and for the radiative component, each of them having its own wave number, and we have studied the uniqueness and existence of solutions for the 3D Helmholtz equation in a stratified medium with unbounded perturbation, that is, the refraction index is a one-dimensional function. However, when the refraction index is cylindrically symmetric, that is, the refraction index is a two-dimensional function, it is unclear which should be the right radiation condition. In the present paper, we introduce another generalized(out-going) Sommerfeld–Rellich radiation condition, and then we consider the uniqueness and existence of solutions for the 3D Helmholtz equation in a step-index waveguide with unbounded perturbation satisfying our radiation condition.

We will use the following conventions in this paper as in [1–3]. A point in $\mathbb{R}^3$ will be described as

$$P = (x_1, x_2, z) \sim (r, \theta, z),\ P' = (\xi_1, \xi_2, \zeta) \sim (r', \theta', \zeta),$$

in different coordinate systems, respectively. There are relations

$$|P|^2 = r^2 + z^2 = x_1^2 + x_2^2 + z^2,\ x_1 = r\cos\theta,\ x_2 = r\sin\theta,$$

$$|P'|^2 = r'^2 + \zeta^2 = \xi_1^2 + \xi_2^2 + \zeta^2,\ \xi_1 = r'\cos\theta',\ \xi_2 = r'\sin\theta'.$$

The rest of the paper is organized as follows: In section 2, we recall the Green's function for the 3D homogenous Helmholtz equation in a step-index waveguide without perturbation and its asymptotic behavior, which will be useful in the rest of the paper. In section 3.1, we will give some regularity results. In section 3.2, we will prove the uniqueness of solutions for the perturbed 3D Helmholtz equation (1.1) satisfying certain out-going radiation condition to be given later. In section 3.3, we will prove the existence of solutions for the perturbed 3D Helmholtz equation (1.1) satisfying certain out-going radiation condition to be given later.

2. Preliminaries

In this section, we recall the Green's function for the 3D homogeneous Helmholtz equation in a step-index waveguide without perturbation and its asymptotic behavior, which will be useful in the rest of the paper.

2.1. *Green's function for the 3D homogeneous Helmholtz equation in a step-index waveguide without perturbation*

In this section, we recall the construction of the Green's function for the 3D homogeneous Helmholtz equation in a step-index waveguide without perturbation, which has been done in [4] and [5]. This Green's function will be used to construct the solution of the perturbed 3D Helmholtz equation (1.1) later.

In cylindrical coordinates, the 3D homogeneous Helmholtz equation in a step-index waveguide without perturbation is

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + k^2 n^2(r) u(r,\theta,z) = 0. \tag{2.1}$$

We look for a solution for this equation in separated variables

$$u(r,\theta,z) = e^{ik\beta z} e^{im\theta} v(r)$$

with $\beta \in \mathbb{C}$ and $m \in \mathbb{Z}$. Then, $v(r)$ must satisfy the ordinary differential equation

$$v''(r) + \frac{1}{r}v'(r) + \left\{k^2 n^2(r) - k^2\beta^2 - \frac{m^2}{r^2}\right\} v(r) = 0. \tag{2.2}$$

Denote

$$d^2 = k^2(n_{co}^2 - n_{cl}^2), l = k^2(n_{co}^2 - \beta^2), q(r) = k^2[n_{co}^2 - n^2(r)]. \tag{2.3}$$

Then, this equation becomes

$$v''(r) + \frac{1}{r}v'(r) + \left\{l - q(r) - \frac{m^2}{r^2}\right\} v(r) = 0; \tag{2.4}$$

the function $q(r)$ becomes

$$q(r) = \begin{cases} 0, & r < R, \\ d^2, & r \geq R. \end{cases} \tag{2.5}$$

We will view (2.4) as an eigenvalue problem in $l \in \mathbb{C}$ and call it the associated eigenvalue problem to (2.1) (see [6] or [7]). From [4] and [5], we can obtain the Green's function $G(r,\theta,z;r',\theta',\zeta)$ of (2.1),

$$G(r,\theta,z;r',\theta',\zeta) = \frac{1}{2\pi^2\sqrt{rr'}}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}\frac{e^{i|z-\zeta|\sqrt{k^2n_{co}^2-\lambda}}}{2i\sqrt{k^2n_{co}^2-\lambda}}j_m(r,\lambda)j_m(r',\lambda)e^{im(\theta-\theta')}d\chi_m(\lambda),$$
$$0<r,r';\ 0\le\theta,\theta'\le 2\pi;\ z,\zeta\in\mathbb{R}, \tag{2.6}$$

where the function $\chi_m(\lambda)$ is identically zero for $\lambda\in(-\infty,0]$, is piecewise constant for $\lambda\in(0,d^2]$ where it has a finite number of discontinuities, and is continuous for $\lambda\in(d^2,\infty)$. Let $0<\lambda_1^m<\cdots<\lambda_{P_m}^m\le d^2$ be the points and $r_1^m,\cdots,r_{P_m}^m$ be the corresponding jumps.

Let $\lambda>d^2$,

$$j_m(r,\lambda)=\begin{cases}\sqrt{r}J_m(\sqrt{\lambda}r), & r<R,\\ \frac{\pi}{2}\sqrt{r}[\alpha_m(\lambda)J_m(\sqrt{\lambda-d^2}r)+\beta_m(\lambda)Y_m(\sqrt{\lambda-d^2}r)], & r\ge R,\end{cases} \tag{2.7}$$

with J_m, Y_m as the Bessel function of the first kind and the second kind of order m, respectively,

$$\alpha_m(\lambda)=-R[\sqrt{\lambda}Y_m(\sqrt{\lambda-d^2}R)J_m'(\sqrt{\lambda}R)-\sqrt{\lambda-d^2}Y_m'(\sqrt{\lambda-d^2}R)J_m(\sqrt{\lambda}R)],$$
$$\beta_m(\lambda)=R[\sqrt{\lambda}J_m(\sqrt{\lambda-d^2}R)J_m'(\sqrt{\lambda}R)-\sqrt{\lambda-d^2}J_m'(\sqrt{\lambda-d^2}R)J_m(\sqrt{\lambda}R)]. \tag{2.8}$$

$$d\chi_m(\lambda)=\frac{2}{\pi^2}\frac{d\lambda}{\alpha_m^2(\lambda)+\beta_m^2(\lambda)},\lambda\in(d^2,\infty). \tag{2.9}$$

Let $0<\lambda<d^2$, and let $\lambda=\lambda_j^m(j=1,2,\cdots,P_m)$ in this interval be a discontinuity point of $\chi_m(\lambda)$, the condition for discontinuity of $\chi_m(\lambda)$ at λ becomes

$$\frac{\sqrt{\lambda}J_m'(\sqrt{\lambda}R)}{J_m(\sqrt{\lambda}R)}=\frac{\sqrt{d^2-\lambda}K_m'(\sqrt{d^2-\lambda}R)}{K_m(\sqrt{d^2-\lambda}R)}, \tag{2.10}$$

with K_m as the modified Bessel function of the second kind of order m,

$$j_m(r,\lambda)=\begin{cases}\sqrt{r}J_m(\sqrt{\lambda}r), & r<R,\\ \frac{J_m(\sqrt{\lambda}R)}{K_m(\sqrt{d^2-\lambda}R)}\sqrt{r}K_m(\sqrt{d^2-\lambda}r), & r\ge R.\end{cases} \tag{2.11}$$

The jump $r_j^m(j=1,2,\cdots,P_m)$ of $\chi_m(\lambda)$ at $\lambda_j^m(j=1,2,\cdots,P_m)$ is

$$r_j^m=\frac{2\lambda_j^m(d^2-\lambda_j^m)}{d^2[\lambda_j^mR^2J_m'(\sqrt{\lambda_j^m}R)^2-m^2J_m(\sqrt{\lambda_j^m}R)^2]}. \tag{2.12}$$

Lastly, let $\lambda=d^2$, and let $\lambda=\lambda_j^m(j=1,2,\cdots,P_m)$ in this interval be a discontinuity point of $\chi_m(\lambda)$, the condition for discontinuity of $\chi_m(\lambda)$ at $\lambda=d^2$ becomes

$$\frac{dJ_m'(Rd)}{J_m(Rd)}=-\frac{|m|}{R}, \tag{2.13}$$

$$j_m(r,\lambda)=\begin{cases}\sqrt{r}J_m(rd), & r<R,\\ R^{|m|}J_m(dR)r^{1/2-|m|}, & r\ge R.\end{cases} \tag{2.14}$$

The jump $r_j^m(j=1,2,\cdots,P_m)$ of $\chi_m(\lambda)$ at $\lambda_j^m=d^2(j=1,2,\cdots,P_m)$ is

$$r_j^m=\frac{2(|m|-1)}{|m|R^2J_m(dR)^2},\ |m|\ge 2. \tag{2.15}$$

2.2. Asymptotic behavior of the Green's function

In this section, we give the following lemmas, which will be useful in the later discussion.

Lemma 1

Let $G(r,\theta,z;r',\theta',\zeta)$ be the Green's function mentioned in (2.6). Then, for any given r',θ',ζ, we have

$$\lim_{r\to 0}\left[j_m(r,\lambda)\frac{\partial\{\sqrt{r}G(r,\theta,z;r',\theta',\zeta)\}}{\partial r}-\frac{\partial j_m(r,\lambda)}{\partial r}\{\sqrt{r}G(r,\theta,z;r',\theta',\zeta)\}\right]=0, \tag{2.16}$$

$$\lim_{r\to \infty}\left[j_m(r,\lambda)\frac{\partial\{\sqrt{r}G(r,\theta,z;r',\theta',\zeta)\}}{\partial r}-\frac{\partial j_m(r,\lambda)}{\partial r}\{\sqrt{r}G(r,\theta,z;r',\theta',\zeta)\}\right]=0. \tag{2.17}$$

Proof

On the one hand, from (2.7), (2.11), and (2.14), we have

$$j_m(r,\lambda)=\sqrt{r}J_m(\sqrt{\lambda}r), \quad \text{as } r\to 0.$$

By using the formulas (9.1.7) and (9.1.5) from [8] (or see [9]),

$$J_m(s)\sim\frac{1}{m!}\left(\frac{s}{2}\right)^m, m\geq 0, s\to 0,$$

and

$$J_{-m}(s)=(-1)^mJ_m(s), m\in\mathbb{Z}.$$

Then, we can obtain that

$$m\geq 0, j_m(r,\lambda)\sim\sqrt{r}\frac{1}{m!}\left(\frac{\sqrt{\lambda}r}{2}\right)^m=\frac{\lambda^{m/2}}{m!2^m}r^{m+1/2}, r\to 0,$$

$$m<0, j_m(r,\lambda)\sim\sqrt{r}(-1)^m\frac{1}{(-m)!}\left(\frac{\sqrt{\lambda}r}{2}\right)^{-m}=(-1)^m\frac{\lambda^{-m/2}}{(-m)!2^{-m}}r^{-m+1/2}, r\to 0,$$

define

$$\gamma_m(\lambda)=(-1)^{(m-|m|)/2}\frac{\lambda^{|m|/2}}{2^{|m|}|m|!},$$

then,

$$j_m(r,\lambda)\sim\gamma_m(\lambda)r^{|m|+1/2}, r\to 0,$$

$$\frac{\partial j_m(r,\lambda)}{\partial r}\sim\gamma_m(\lambda)(|m|+\frac{1}{2})r^{|m|-1/2},$$

and from (2.6), we have

$$G(r,\theta,z;r',\theta',\zeta)=O(r^{|m|}), r\to 0,$$

$$\frac{\partial\{\sqrt{r}G(r,\theta,z;r',\theta',\zeta)\}}{\partial r}=O(r^{|m|-1/2}), r\to 0;$$

thus, by a straightforward computation, we can obtain (2.16) at once.

On the other hand, it remains to prove the identity (2.17). We should consider three cases: $0<\lambda<d^2, \lambda=d^2, \lambda>d^2$, in each of these intervals $j_m(r,\lambda)$ and consequently $G(r,\theta,z;r',\theta',\zeta)$, will have a different behavior.

Case 1. If $0<\lambda<d^2$, from (2.11), we have

$$j_m(r,\lambda)=\frac{J_m(\sqrt{\lambda}R)}{K_m(\sqrt{d^2-\lambda}R)}\sqrt{r}K_m(\sqrt{d^2-\lambda}r), \text{as } r\to\infty.$$

By using the formula (9.7.2) from [8] (or see [9]),

$$K_m(s) \sim \sqrt{\frac{\pi}{2s}} e^{-s}, s \to \infty.$$

Then, we can obtain that

$$j_m(r,\lambda) = O(e^{-\sqrt{d^2-\lambda}r}), r \to \infty,$$

$$\frac{\partial j_m(r,\lambda)}{\partial r} = O(e^{-\sqrt{d^2-\lambda}r}), r \to \infty,$$

$$G(r,\theta,z;r',\theta',\zeta) = O(r^{-1/2}e^{-\sqrt{d^2-\lambda}r}), r \to \infty,$$

$$\frac{\partial\{\sqrt{r}G(r,\theta,z;r',\theta',\zeta)\}}{\partial r} = O(e^{-\sqrt{d^2-\lambda}r}), r \to \infty;$$

thus, we can easily obtain (2.17) at once.

Case 2. If $\lambda = d^2$, from (2.14), we have

$$j_m(r,\lambda) = R^{|m|}J_m(dR)r^{1/2-|m|}, \text{as } r \to \infty, |m| \geq 2.$$

Then, we can obtain that

$$\frac{\partial j_m(r,\lambda)}{\partial r} = R^{|m|}J_m(dR)(\frac{1}{2} - |m|)r^{-|m|-1/2}, r \to \infty,$$

$$G(r,\theta,z;r',\theta',\zeta) = O(r^{-|m|}), r \to \infty,$$

$$\frac{\partial\{\sqrt{r}G(r,\theta,z;r',\theta',\zeta)\}}{\partial r} = O(r^{-|m|-1/2}), r \to \infty;$$

thus, we can easily obtain (2.17) at once.

Case 3. If $\lambda > d^2$, from (2.7), we have

$$j_m(r,\lambda) = \frac{\pi}{2}\sqrt{r}[\alpha_m(\lambda)J_m(\sqrt{\lambda - d^2}r) + \beta_m(\lambda)Y_m(\sqrt{\lambda - d^2}r)], \text{as } r \to \infty.$$

By using the formulas (9.2.1) and (9.2.2) from [8] (or see [9]),

$$J_m(s) = \sqrt{\frac{2}{\pi s}}\left[\cos(s - m\frac{\pi}{2} - \frac{\pi}{4}) + O(s^{-1})\right], s \to \infty,$$

$$Y_m(s) = \sqrt{\frac{2}{\pi s}}\left[\sin(s - m\frac{\pi}{2} - \frac{\pi}{4}) + O(s^{-1})\right], s \to \infty.$$

Then, by a straightforward computation, we can obtain (2.17) at once. It finishes the proof. □

Denote

$$\widetilde{G}(\lambda,\theta,z;r',\theta',\zeta) = \int_0^{+\infty} j_m(\rho,\lambda)\sqrt{\rho}G(\rho,\theta,z;r',\theta',\zeta)d\rho,$$

then we can obtain our second lemma as follows:

Lemma 2

Let $\widetilde{G}(\lambda,\theta,z;r',\theta',\zeta)$ previously given. Then, for any given r',θ',ζ, we have

$$\begin{cases} \lim\limits_{|z|\to\infty}\left[\frac{\partial\widetilde{G}(\lambda,\theta,z;r',\theta',\zeta)}{\partial|z|} - i\sqrt{k^2n_{co}^2 - \lambda}\widetilde{G}(\lambda,\theta,z;r',\theta',\zeta)\right] = 0, & \lambda \leq k^2n_{co}^2 \text{ and } d\chi_m(\lambda) \neq 0, \\ \lim\limits_{|z|\to\infty} \widetilde{G}(\lambda,\theta,z;r',\theta',\zeta) = 0, & \lambda > k^2n_{co}^2. \end{cases} \tag{2.18}$$

Proof
From (2.7), (2.11), and (2.14) and the Green's function $G(r,\theta,z;r',\theta',\zeta)$ defined by (2.6), we can obtain that

$$\begin{cases} \lim\limits_{|z|\to\infty}\left[\frac{\partial G(r,\theta,z;r',\theta',\zeta)}{\partial |z|} - i\sqrt{k^2n_{co}^2-\lambda}G(r,\theta,z;r',\theta',\zeta)\right]=0, & \lambda\le k^2n_{co}^2 \text{ and } d\chi_m(\lambda)\neq 0,\\ \lim\limits_{|z|\to\infty} G(r,\theta,z;r',\theta',\zeta)=0, & \lambda>k^2n_{co}^2. \end{cases}$$

Then, from Fubini–Tonelli's theorem, we can easily obtain (2.18) at once. It finishes the proof. □

Lemma 3
Let $(r,\theta,z),(r',\theta',\zeta)\in\mathbb{R}^3$, and $P-P'=(r\cos\theta-r'\cos\theta', r\sin\theta-r'\sin\theta', z-\zeta)$ with $|P-P'|<1$. There exists a positive constant C_1 independent on $r,\theta,z,r',\theta',\zeta$ such that

$$|G(r,\theta,z;r',\theta',\zeta)-\frac{1}{4\pi|P-P'|}|\le C_1. \tag{2.19}$$

Proof
The proof is analogous to part of the proof of Lemma 2.19 in [10] and, hence, is omitted. □

3. The uniqueness and existence of solutions for the perturbed 3D Helmholtz equation

In this section, we will study the uniqueness and existence of solutions for the perturbed 3D Helmholtz equation (1.1). Before studying them, we should give some regularity results.

3.1. Regularity results

In this section, we recall the global regularity of weak solutions for the 3D Helmholtz equation; the results in this section can be found in [11] and [10] in a more general text.

Let $\mu(x_1,x_2,z):\mathbb{R}^3\to\mathbb{R}$ be a positive function. We will denote by $L^2(\mu)$ the weighted space consisting of all the complex valued measurable functions $u(x_1,x_2,z)$, $(x_1,x_2,z)\in\mathbb{R}^3$, such that

$$\mu^{\frac{1}{2}}(x_1,x_2,z)u(x_1,x_2,z)\in L^2(\mathbb{R}^3),$$

equipped with the norm

$$\|u(x_1,x_2,z)\|^2_{L^2(\mu)}=\int_{\mathbb{R}^3}|u(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz.$$

$L^2(\mu)$ is commonly used when dealing with solutions of Helmholtz equation (see [11] and [12]). In 2D Helmholtz equation, we refer to [13], which has proved that the spectrum-based solution derived in [14] belongs to $L^2(\mu)$.

In a similar way, we define the weighted Sobolev spaces $H^1(\mu)$ and $H^2(\mu)$. The norms in $H^1(\mu)$ and $H^2(\mu)$ are given respectively by

$$\begin{aligned}\|u(x_1,x_2,z)\|^2_{H^1(\mu)}=&\int_{\mathbb{R}^3}|u(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz\\ &+\int_{\mathbb{R}^3}|\nabla u(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz,\end{aligned}$$

and

$$\begin{aligned}\|u(x_1,x_2,z)\|^2_{H^2(\mu)}=&\int_{\mathbb{R}^3}|u(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz\\ &+\int_{\mathbb{R}^3}|\nabla u(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz\\ &+\int_{\mathbb{R}^3}|\nabla^2 u(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz.\end{aligned}$$

Lemma 4
Let $u(x_1,x_2,z)\in H^1_{loc}(\mathbb{R}^3)$ be a weak solution of

$$\triangle u(x_1,x_2,z)+k^2n^2(x_1,x_2,z)u(x_1,x_2,z)=f(x_1,x_2,z),\ (x_1,x_2,z)\in\mathbb{R}^3, \tag{3.1}$$

with $n(x_1,x_2,z) \in L^\infty(\mathbb{R}^3)$, $f(x_1,x_2,z) \in L^2(\mu)$, and the positive weight function $\mu(x_1,x_2,z)$ satisfies the following assumptions:

$$\mu(x_1,x_2,z) \in C^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3),$$
$$|\bigtriangledown \mu(x_1,x_2,z)| \le C_2\mu(x_1,x_2,z), |\bigtriangledown^2 \mu(x_1,x_2,z)| \le C_3\mu(x_1,x_2,z), \text{ in } \mathbb{R}^3. \tag{3.2}$$

Here, C_2 and C_3 are positive constants. Then,

$$\int_{\mathbb{R}^3} |\bigtriangledown u(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz \le \tfrac{1}{2}\int_{\mathbb{R}^3} |f(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz +$$
$$\left(2C_3 + k^2n_*^2 + \tfrac{1}{2}\right)\int_{\mathbb{R}^3} |u(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz, \tag{3.3}$$

where $n_* = \|n(x_1,x_2,z)\|_{L^\infty(\mathbb{R}^3)}$.

Proof
The proof is analogous to the proof of Lemma 4.1 in [10] and, hence, is omitted. □

Lemma 5
Let $u(x_1,x_2,z) \in H^1(\mu)$ be a weak solution of (3.1) with $n(x_1,x_2,z) \in L^\infty(\mathbb{R}^3)$, $f(x_1,x_2,z) \in L^2(\mu)$, and the positive weight function $\mu(x_1,x_2,z)$ satisfies the assumptions (3.2). Then,

$$\int_{\mathbb{R}^3} |\bigtriangledown^2 u(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz \le 2\int_{\mathbb{R}^3} |f(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz$$
$$+2k^2n_*^4\int_{\mathbb{R}^3} |u(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz$$
$$+4C_3\int_{\mathbb{R}^3} |\bigtriangledown u(x_1,x_2,z)|^2\mu(x_1,x_2,z)dx_1dx_2dz, \tag{3.4}$$

where $n_* = \|n(x_1,x_2,z)\|_{L^\infty(\mathbb{R}^3)}$.

Proof
The proof is analogous to the proof of Lemma 4.3 in [10] and, hence, is omitted. □

3.2. Uniqueness of solutions for the perturbed 3D Helmholtz equation

We will give the generalized Sommerfeld–Rellich radiation condition, which we will refer to as the *out-going radiation condition*. First, we impose the condition that

$$u \in C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3). \tag{3.5}$$

Second, we support that for all $m \in \mathbb{Z}, z \in \mathbb{R}$, the following equality holds:

$$\lim_{r\to\infty}\left[j_m(r,\lambda)\frac{\partial\{\sqrt{r}u_m(r,z)\}}{\partial r} - \frac{\partial j_m(r,\lambda)}{\partial r}\{\sqrt{r}u_m(r,z)\}\right] = 0, \tag{3.6}$$

with the function $u_m(r,z)$ being the Fourier coefficients from the Fourier series

$$u(r,\theta,z) = \sum_{m\in\mathbb{Z}} e^{im\theta}u_m(r,z).$$

We denote by $U_m(\lambda,z)$ the transform of the function $r \to \sqrt{r}u_m(r,z)$ given by

$$U_m(\lambda,z) = \int_0^\infty j_m(\rho,\lambda)\sqrt{\rho}u_m(\rho,z)d\rho.$$

The third requirement is

$$\begin{cases} \lim\limits_{|z|\to\infty}\left[\frac{\partial U_m(\lambda,z)}{\partial|z|} - i\sqrt{k^2n_{co}^2-\lambda}U_m(\lambda,z)\right] = 0, & \lambda \le k^2n_{co}^2 \text{ and } d\chi_m(\lambda) \ne 0, \\ \lim\limits_{|z|\to\infty} U_m(\lambda,z) = 0, & \lambda > k^2n_{co}^2. \end{cases} \tag{3.7}$$

These conditions are physically motivated, we refer to [4], [5], and references therein for a more detailed description.

Lemma 6
Let $u(x_1,x_2,z) \in L^2(\mu)$ satisfy

$$\bigtriangleup u(x_1,x_2,z) + [k^2n^2(x_1,x_2) + p(x_1,x_2,z)]u(x_1,x_2,z) = 0 \tag{3.8}$$

in $\mathbb{R}^3$, where $n(x_1,x_2)$ is given by (1.2). Then, for $\lambda > k^2n_{co}^2$,

$$\lim_{|z|\to\infty} u(x_1,x_2,z)e^{-\sqrt{\lambda-k^2n_{co}^2}|z|} = \lim_{|z|\to\infty} u_z(x_1,x_2,z)e^{-\sqrt{\lambda-k^2n_{co}^2}|z|} = 0. \tag{3.9}$$

Proof

On the one hand, because $u(x_1,x_2,z)$ is a solution of (3.8), from Lemmas 4 and 5, we can obtain that $|\bigtriangledown u(x_1,x_2,z)|^2\mu(x_1,x_2,z)$ and $|\bigtriangledown^2 u(x_1,x_2,z)|^2\mu(x_1,x_2,z)$ are integrable in $\mathbb{R}^3$. Thus, it easily follows that the function

$$\Phi(x_1,x_2,z) = u(x_1,x_2,z)\mu^{\frac{1}{2}}(x_1,x_2,z)$$

belongs to the Sobolev space $W^{2,2}(\mathbb{R}^3)$. From the Sobolev imbedding theorem (see Theorem in [15]), we can obtain that $\Phi(x_1,x_2,z) \in L^\infty(\mathbb{R}^3)$, and hence, the first limit in (3.9) follows at once.

On the other hand, it remains to prove the second limit in (3.9). A straightforward computation shows that $\Phi(x_1,x_2,z)$ satisfies the equation

$$\triangle\Phi(x_1,x_2,z) + b(x_1,x_2,z)\cdot\bigtriangledown\Phi(x_1,x_2,z) + c(x_1,x_2,z)\Phi(x_1,x_2,z) = 0$$

in $\mathbb{R}^3$, where

$$b(x_1,x_2,z) = (-\mu^{-1}\mu_{x_1}, -\mu^{-1}\mu_{x_2}, -\mu^{-1}\mu_z) = -\mu^{-1}\cdot\bigtriangledown\mu,$$

$$\begin{aligned} c(x_1,x_2,z) = {} & k^2n^2(x_1,x_2) + p(x_1,x_2,z) + \frac{1}{4}\mu^{-2}\mu_{x_1} + \frac{1}{4}\mu^{-2}\mu_{x_2} + \frac{1}{4}\mu^{-2}\mu_z + \\ & \frac{1}{2}\mu^{-2}\mu_{x_1}^2 + \frac{1}{2}\mu^{-2}\mu_{x_2}^2 + \frac{1}{2}\mu^{-2}\mu_z^2 - \frac{1}{2}\mu^{-1}\mu_{x_1x_1} - \frac{1}{2}\mu^{-1}\mu_{x_2x_2} - \frac{1}{2}\mu^{-1}\mu_{zz} \\ = {} & k^2n^2(x_1,x_2) + p(x_1,x_2,z) + \frac{1}{4}\mu^{-2}(\mu_{x_1}+\mu_{x_2}+\mu_z) + \frac{1}{2}\mu^{-2}(\mu_{x_1}^2+\mu_{x_2}^2+\mu_z^2) - \frac{1}{2}\mu^{-1}\triangle\mu \end{aligned}$$

are functions of (x_1,x_2,z).

Because $\Phi(x_1,x_2,z) \in W^{2,2}(\mathbb{R}^3)$, and from Theorem 8.10 in [16], we can obtain that $\Phi(x_1,x_2,z) \in W^{3,2}(H_+)$, where $H_+ = \{(x_1,x_2,z) \in \mathbb{R}^3 | |z| \geq h\}$, where $h > 0$ is a constant. Again, from the Sobolev imbedding theorem, we can obtain that $|\bigtriangledown\Phi(x_1,x_2,z)| \in L^\infty(H_+)$, and hence, the second limit in (3.9) follows. It finishes the proof. □

Let $u(x_1,x_2,z) = u(r,\theta,z)$ be the solution of the perturbed 3D Helmholtz equation (1.1). We define

$$\widetilde{u}(\lambda,\theta,z) = \int_0^{+\infty} j_m(\rho,\lambda)\sqrt{\rho}u(\rho,\theta,z)d\rho,$$

$$U(r,\theta,z) = \frac{1}{\pi\sqrt{r}}\int_{-\infty}^{+\infty} j_m(r,\lambda)\widetilde{u}(\lambda,\theta,z)d\chi_m(\lambda).$$

Then, we have the lemma as follows:

Lemma 7

Let $u(x_1,x_2,z) = u(r,\theta,z)$ be a weak solution of (3.8) and $U(r,\theta,z)$ defined previously. Then, $U(r,\theta,z)$ is a weak solution of

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + k^2n^2(r)u(r,\theta,z) = -\psi(r,\theta,z), \tag{3.10}$$

where

$$\psi(r,\theta,z) = \frac{1}{\pi\sqrt{r}}\int_{-\infty}^{+\infty} j_m(r,\lambda)\int_0^{+\infty} j_m(\rho,\lambda)\sqrt{\rho}p(\rho,\theta,z)u(\rho,\theta,z)d\rho d\chi_m(\lambda). \tag{3.11}$$

Proof

The proof is analogous to the proof of Lemma 6 in [1] and, hence, is omitted. □

Lemma 8
Let $(r', \theta', \zeta) \in \mathbb{R}^3$ be fixed and R' large enough such that $(r', \theta', \zeta) \in \Omega_{R'}$. Let $u(r, \theta, z)$ be a solution of (3.8). Then, we have the following identity:

$$\frac{1}{\pi\sqrt{r}}\int_{-\infty}^{+\infty} j_m(r,\lambda)\int_0^{R'} j_m(\rho,\lambda)\sqrt{\rho}u(\rho,\theta,z)d\rho d\chi_m(\lambda) + \int_{\Omega_{R'}} G\psi r dr d\theta dz = \int_{\partial\Omega_{R'}}\left(u\frac{\partial G}{\partial \nu} - G\frac{\partial u}{\partial \nu}\right)ds, \tag{3.12}$$

where $\Omega_{R'} = \{(r,\theta,z)|(r\cos\theta)^2 + (r\sin\theta)^2 + z^2 \le (R')^2\}$, $\psi(r,\theta,z)$ is given by (3.11) and ν is the outward normal of $\Omega_{R'}$.

Proof
It can be easily verified that $\triangle G + k^2n^2(r)G$ has a singularity for $(\theta, z) \equiv (\theta', \zeta)$. So from Lemma 7, we can obtain that

$$\begin{aligned}
&\int_{\Omega_R'\backslash\Omega_\varepsilon}(u\triangle G - G\triangle u) r dr d\theta dz \\
&\quad = \int_{\Omega_R'\backslash\Omega_\varepsilon}[u(\triangle G + k^2n^2(r)G) - G(\triangle u + k^2n^2(r)u)] r dr d\theta dz \\
&\quad = \int_{\Omega_R'\backslash\Omega_\varepsilon}[u(\triangle G + k^2n^2(r)G) + G\psi] r dr d\theta dz,
\end{aligned}$$

where $\Omega_\varepsilon = \{(r,\theta,z) \in \mathbb{R}^3|(r\cos\theta - r'\cos\theta')^2 + (r\sin\theta - r'\sin\theta')^2 + (z-\zeta)^2 < \varepsilon^2\}$.

From the aforementioned formula and the second Green's formula, we can obtain that

$$\int_{\partial(\Omega_R'\backslash\Omega_\varepsilon)}\left(u\frac{\triangle G}{\partial\nu} - G\frac{\triangle u}{\partial\nu}\right)ds = \int_{\Omega_R'\backslash\Omega_\varepsilon}[u(\triangle G + k^2n^2(r)G) + G\psi] r dr d\theta dz. \tag{3.13}$$

Thus, by taking the limit as $\varepsilon \to 0^+$ in the aforementioned identity (3.13), we can easily obtain (3.12) at once. It finishes the proof. □

So, we can obtain our first result as follows:

Theorem 1 (Uniqueness of solutions)
Let $p(x_1, x_2, z)$ satisfy the assumptions (A1) and (A2). There exists at most one bounded solution of the perturbed 3D Helmholtz equation (1.1) satisfying the out-going radiation condition (3.5), (3.6), and (3.7).

Proof
We assume that $u^1(x_1,x_2,z)$ and $u^2(x_1,x_2,z)$ are two bounded solutions of the perturbed 3D Helmholtz equation (1.1) satisfying the out-going radiation condition(3.5), (3.6), and (3.7) and set $u(x_1,x_2,z) = u^1(x_1,x_2,z) - u^2(x_1,x_2,z)$. It is clear that $u(x_1,x_2,z)$ is a bounded solution of equation (3.8) and satisfies the out-going radiation condition (3.5), (3.6), and (3.7).

From (3.12), we have

$$\begin{aligned}
&\frac{1}{\pi\sqrt{r}}\int_{-\infty}^{+\infty} j_m(r,\lambda)\int_0^{R'} j_m(\rho,\lambda)\sqrt{\rho}u(\rho,\theta,z)d\rho d\chi_m(\lambda) + \int_{\Omega_{R'}} G\psi r dr d\theta dz \\
&= \int_{\partial\Omega_{R'}}\left(u\frac{\partial G}{\partial\nu} - G\frac{\partial u}{\partial\nu}\right)ds \\
&= \int_{\partial\Omega_{R'}}\left[u\left(\frac{\partial G}{\partial\nu} - i\sqrt{k^2n_{co}^2 - \lambda}G\right) - G\left(\frac{\partial u}{\partial\nu} - i\sqrt{k^2n_{co}^2 - \lambda}u\right)\right]ds.
\end{aligned} \tag{3.14}$$

From triangular and Cauchy–Schwartz inequalities, we obtain that the right hand of (3.14) is as follows:

$$\begin{aligned}
&\int_{\partial\Omega_{R'}}\left[u\left(\frac{\partial G}{\partial\nu} - i\sqrt{k^2n_{co}^2 - \lambda}G\right) - G\left(\frac{\partial u}{\partial\nu} - i\sqrt{k^2n_{co}^2 - \lambda}u\right)\right]ds \\
&\le \left(\int_{\partial\Omega_{R'}}|u|^2 ds\right)^{\frac{1}{2}}\left(\int_{\partial\Omega_{R'}}\left|\frac{\partial G}{\partial\nu} - i\sqrt{k^2n_{co}^2 - \lambda}G\right|^2 ds\right)^{\frac{1}{2}} + \\
&\left(\int_{\partial\Omega_{R'}}|G|^2 ds\right)^{\frac{1}{2}}\left(\int_{\partial\Omega_{R'}}\left|\frac{\partial u}{\partial\nu} - i\sqrt{k^2n_{co}^2 - \lambda}u\right|^2 ds\right)^{\frac{1}{2}} \\
&= I_1 + I_2.
\end{aligned}$$

From (2.6), and because $j_m(r,\lambda)$ is bounded, we can easily obtain that

$$I_1 \to 0, \quad I_2 \to 0, \qquad \text{as } R' \to \infty.$$

Thus, it follows that the right hand of (3.14) vanishes as $R' \to \infty$.

From [6], [7], and Fubini–Tonelli's theorem, we obtain that the left hand of (3.14) is as follows:

$$\frac{1}{\pi\sqrt{r}} \int_{-\infty}^{+\infty} j_m(r,\lambda) \int_0^{R'} j_m(\rho,\lambda)\sqrt{\rho}u(\rho,\theta,z)d\rho d\chi_m(\lambda) + \int_{\Omega_{R'}} G\psi r dr d\theta dz$$
$$\to \quad u(r',\theta',\zeta) + \int_{\mathbb{R}^3} G(r,\theta,z;r',\theta',\zeta)p(r,\theta,z)u(r,\theta,z) r dr d\theta dz, \quad \text{as } R' \to \infty.$$

Thus, by taking the limit for $R' \to \infty$ in (3.14), we can obtain that

$$u(r',\theta',\zeta) + \int_{\mathbb{R}^3} G(r,\theta,z;r',\theta',\zeta)p(r,\theta,z)u(r,\theta,z) r dr d\theta dz = 0. \tag{3.15}$$

Because $u(r,\theta,z)$ is bounded and by setting $M = \sup_{(r,\theta,z)\in\mathbb{R}^3} |u(r,\theta,z)|$, from the aforementioned formula (3.15), we have that

$$M \le M \sup_{(r',\theta',\zeta)\in\mathbb{R}^3} \int_{\mathbb{R}^3} |G(r,\theta,z;r',\theta',\zeta)p(r,\theta,z)| r dr d\theta dz. \tag{3.16}$$

From (1.3) and the above inequality (3.16), we can obtain that $M = 0$, that is, $u^1(x_1,x_2,z) = u^2(x_1,x_2,z)$. It finishes the proof. □

3.3. Existence of solutions for the perturbed 3D Helmholtz equation

Before studying the existence of solutions for the perturbed 3D Helmholtz equation (1.1), we should give two lemmas.

Lemma 9
Let $\Upsilon(r',\theta',\zeta)$ be a complex valued function satisfying the assumption (A1). Then, the function

$$w(r,\theta,z) = \int_{\mathbb{R}^3} G(r,\theta,z;r',\theta',\zeta)\Upsilon(r',\theta',\zeta) r' dr' d\theta' d\zeta$$

satisfies

$$\lim_{r\to\infty} \left[j_m(r,\lambda)\frac{\partial\{\sqrt{r}w(r,\theta,z)\}}{\partial r} - \frac{\partial j_m(r,\lambda)}{\partial r}\{\sqrt{r}w(r,\theta,z)\}\right] = 0.$$

Proof
From Lemma 1 and the assumption (A1), we can easily obtain Lemma 9 at once. □

Lemma 10
Let $\Upsilon(r',\theta',\zeta)$ be a complex valued function satisfying the assumption (A1). Then, the function

$$\widetilde{w}(\lambda,\theta,z) = \int_{\mathbb{R}^3} \widetilde{G}(\lambda,\theta,z;r',\theta',\zeta)\Upsilon(r',\theta',\zeta) r' dr' d\theta' d\zeta$$

satisfies

$$\begin{cases} \lim\limits_{|z|\to\infty} \left[\frac{\partial \widetilde{w}(\lambda,\theta,z)}{\partial |z|} - i\sqrt{k^2 n_{co}^2 - \lambda}\,\widetilde{w}(\lambda,\theta,z)\right] = 0, & \lambda \le k^2 n_{co}^2 \text{ and } d\chi_m(\lambda) \ne 0, \\ \lim\limits_{|z|\to\infty} \widetilde{w}(\lambda,\theta,z) = 0, & \lambda > k^2 n_{co}^2. \end{cases}$$

Proof
From Lemma 2 and the assumption (A1), we can easily obtain Lemma 10 at once. □

The second result of this paper concerns the existence of solutions for the perturbed 3D Helmholtz equation (1.1) satisfying the out-going radiation condition (3.5), (3.6), and (3.7). Then, we obtain our result as follows:

Theorem 2 (Existence of solutions)
Let $f(x_1,x_2,z) \in L^2(\mathbb{R}^3)$ and $p(x_1,x_2,z) \in L^2(\mathbb{R}^3)$ satisfy the assumptions (A1) and (A2). Then, there exists a uniqueness bounded solution of the perturbed 3D Helmholtz equation (1.1) satisfying the out-going radiation condition (3.5), (3.6), and (3.7).

In particular, such a solution is the only bounded solution of the following integral equation:

$$u(x_1,x_2,z) = \int_{\mathbb{R}^3} G(x_1,x_2,z;\xi_1,\xi_2,\zeta)[f(\xi_1,\xi_2,\zeta) - p(\xi_1,\xi_2,\zeta)u(\xi_1,\xi_2,\zeta)]d\xi_1 d\xi_2 d\zeta \tag{3.17}$$

Proof
First, we prove that $u(x_1,x_2,z)$ is bounded and then show that it satisfies the out-going radiation condition (3.5), (3.6), and (3.7).
On the one hand, we write

$$\begin{aligned}\int_{\mathbb{R}^3} G(x_1,x_2,z;\xi_1,\xi_2,\zeta)f(\xi_1,\xi_2,\zeta)d\xi_1 d\xi_2 d\zeta &= \int_{B_1(x_1,x_2,z)} G(x_1,x_2,z;\xi_1,\xi_2,\zeta)f(\xi_1,\xi_2,\zeta)d\xi_1 d\xi_2 d\zeta \\ &\quad + \int_{\mathbb{R}^3\setminus B_1(x_1,x_2,z)} G(x_1,x_2,z;\xi_1,\xi_2,\zeta)f(\xi_1,\xi_2,\zeta)d\xi_1 d\xi_2 d\zeta \\ &= J_1 + J_2,\end{aligned} \tag{3.18}$$

where $B_1(x_1,x_2,z) = \{(x_1,x_2,z) \in \mathbb{R}^3 | (x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (z-\zeta)^2 \le 1$.
From Lemma 3 and (2.6), we have that $|J_1|$ is estimated (up to multiplicative constant) by

$$\int_{B_1(x_1,x_2,z)} \frac{|f(\xi_1,\xi_2,\zeta)|}{|P-P'|} d\xi_1 d\xi_2 d\zeta.$$

From Hölder inequality, we estimated $|J_1|^2$ by

$$\begin{aligned}&\left(\int_{B_1(x_1,x_2,z)} \frac{|f(\xi_1,\xi_2,\zeta)|}{|P-P'|} d\xi_1 d\xi_2 d\zeta\right)^2 \\ &\le \int_{B_1(x_1,x_2,z)} |f(\xi_1,\xi_2,\zeta)|^2 d\xi_1 d\xi_2 d\zeta \cdot \int_{B_1(x_1,x_2,z)} \frac{1}{|P-P'|^2} d\xi_1 d\xi_2 d\zeta \\ &= 4\pi \int_{B_1(x_1,x_2,z)} |f(\xi_1,\xi_2,\zeta)|^2 d\xi_1 d\xi_2 d\zeta,\end{aligned}$$

because $f(x_1,x_2,z) \in L^2(\mathbb{R}^3)$; thus, the first integral J_1 on the right hand of (3.18) is bounded.
From Lemmas 1, 2, and 3, we know that $G(x_1,x_2,z;\xi_1,\xi_2,\zeta)$ is bounded outside $B_1(x_1,x_2,z)$; thus, from the assumption (A1) on the $f(x_1,x_2,z)$, we can obtain that the second integral J_2 on the right hand of (3.18) is bounded.
So, we prove that $\int_{\mathbb{R}^3} G(x_1,x_2,z;\xi_1,\xi_2,\zeta)f(\xi_1,\xi_2,\zeta)d\xi_1 d\xi_2 d\zeta$ is bounded.
From (1.3) and a contraction mapping theorem, then we conclude that $u(x_1,x_2,z)$ is bounded at once.
On the other hand, it remains to prove that $u(x_1,x_2,z)$ satisfies the radiation condition (3.5), (3.6), and (3.7). Because $u(x_1,x_2,z)$ is bounded, and from the assumptions on $f(x_1,x_2,z)$ and $p(x_1,x_2,z)$ by using Lemmas 9 and 10, we can obtain the conclusion at once. It finishes the proof. □

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