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Integral Equations and Operator Theory

Open Problems

The Growth Order of a Solution of a Singular Non-Self-Adjoint Differential Equation, as a Function of the Spectral Parameter

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Abstract. For solutions of regular initial value problems which depend analytically on a spectral parameter it is usually easy to calculate the growth order of the solutions as functions of the parameter. The situation is less clear when the equation is singular in the spatial variable.

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In 2003, Benilov et al. [1] proposed a model for the stability of the process used for coating fluorescent tubes. This led to a differential equation eigenproblem for which any reasonable operator realization is highly non-self-adjoint, yet for suitable parameter regimes the eigenvalues are all real. The proof of this fact is due to Weir: see [3], where a full description of the problem and further references may be found.

A generalization of this work [2] considers the differential equation

$$\mathrm{i}\epsilon \frac{d}{dx}\left(f(x)\frac{du}{dx}\right) + \mathrm{i}\frac{du}{dx} = \lambda u,$$

in which $f(x) > 0$ on $(0, \pi)$, $f'(0) = 1$, and f has (say) a $C^2(\mathbb{R})$ 2π-periodic odd extension to the whole real line. The parameter ϵ is real with $0 < \epsilon < 2$.

It is known that there exists, for each $\lambda \in \mathbb{C}$, a unique solution of the differential equation, say $\psi(x, \lambda)$, with the properties

$$\lim_{x\to 0} \psi(x, \lambda) = 1,$$
$$\psi(-x, \lambda) = \psi(x, -\lambda) = \overline{\psi(x, \overline{\lambda})}.$$

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Open problem: for fixed $x \in (0, \pi]$, what is the growth order of $\psi(x, \lambda)$ as an analytic function of λ?

The conjecture is that it should be 1/2, which can be proved in many special cases, e.g. when f is linear near $x = 0$.

A similar problem arises in consideration of Schrödinger equations with PT-symmetric potentials on the real line. Here one has two solutions

$$\psi_{\mathrm{left}}(x, \lambda) \in L^2(-\infty, 0) \quad \text{and} \quad \psi_{\mathrm{right}}(x, \lambda) \in L^2(0, \infty),$$

Jost solutions, for instance, if the potential lies in $L^1(\mathbb{R})$. Normalizing these solutions in some appropriate way, what can one say about their growth orders? In this case it seems that some results are available in the self-adoint case in the inverse spectral theory literature, in particular. These results all make special assumptions on the class of potential under consideration; the decay conditions are usually essential but the self-adjointness is usually not.

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