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A note on the spectral singularities of non-selfadjoint matrix-valued difference operators

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ABSTRACT

Let L denote the non-selfadjoint operator generated in $\ell_2(\mathbb{N}, E)$ by the matrix difference expression $(\ell y)_n = A_{n-1}y_{n-1} + B_n y_n + A_n y_{n+1}$, $n \in \mathbb{N}$, and the boundary condition $y_0 = 0$. In this paper we investigate the Jost solution, the continuous spectrum, the eigenvalues and the spectral singularities of L.

1. Introduction

The spectral analysis of selfadjoint difference operators and Jacobi matrices has been treated by various authors in connection with the classical moment problem [1]. The spectral theory of these operators in the selfadjoint case is well known [2,3]. The spectral analysis of non-selfadjoint difference operators and Jacobi matrices has been studied in [4–9].

Study of the spectral theory of non-selfadjoint Sturm–Liouville operators (SLOs) with continuous and point spectra was initiated in [10], which proved that the spectrum of a non-selfadjoint SLO consists of the continuous spectrum, the eigenvalues and the spectral singularities. The spectral singularities are poles of the kernel of the resolvent and are also imbedded in the continuous spectrum, but they are not the eigenvalues. In [11], the effect of the spectral singularities in the spectral expansion of SLOs in terms of the principal vectors was considered.

All of the above-mentioned papers deal with difference and differential operators of scalar coefficients. Some problems of the spectral theory of selfadjoint difference and differential operators with matrix coefficients are studied in [12–16].

Let us introduce the Hilbert space $\ell_2(\mathbb{N}, E)$ consisting of all vector sequences $y = \{y_n\}$, $(y_n \in E, n \in \mathbb{N})$, such that $\sum_{n=1}^{\infty} \|y_n\|_E^2 < \infty$ with the inner product $\langle y, z\rangle = \sum_{n=1}^{\infty}(y_n, z_n)_E$, where E is m-dimensional $(m < \infty)$ Euclidean space, and $\|.\|_E$ and $(., .)_E$ denote norm and inner product in E, respectively. Further, we denote by L the operator generated in $\ell_2(\mathbb{N}, E)$ by the difference expression $(\ell y)_n = A_{n-1}y_{n-1} + B_n y_n + A_n y_{n+1}$, $n \in \mathbb{N} = \{1, 2, \ldots\}$ and the boundary condition $y_0 = 0$, where A_n, $n \in \mathbb{N} \cup \{0\}$, and B_n, $n \in \mathbb{N}$, are linear operators (matrices) acting in E. Throughout the paper we will assume that $\det A_n \neq 0$ and $A_n \neq A_n^*$, $B_n \neq B_n^*$ $(n \in \mathbb{N})$ hold. It is clear that the operator L is non-selfadjoint. Note that we can also define the operator L using the Jacobi matrix

$$J = \begin{pmatrix} B_1 & A_1 & 0 & 0 & 0 & . & . & . \\ A_1 & B_2 & A_2 & 0 & 0 & . & . & . \\ 0 & A_2 & B_3 & A_3 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix},$$

where 0 is the zero operator in E.

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In this paper we investigate the Jost solution, the continuous spectrum, the eigenvalues and the spectral singularities of L.

This paper consists of two sections. In the first section, we give the basic concepts. In the second section, we give the Jost solution of the equation $(\ell y)_n = \lambda y_n$, $n \in \mathbb{N}$, and the continuous spectrum of L, by using the Weyl compact perturbation theorem. We also investigate the eigenvalues and the spectral singularities of L. In particular, under condition (2.10), we prove that the operator L has a finite number of eigenvalues and spectral singularities with a finite multiplicity. We also give an example of a matrix difference operator having spectral singularities.

2. Jost solution and spectral properties of L

Related to the operator L, we will consider the difference equation

$$A_{n-1}y_{n-1} + B_n y_n + A_n y_{n+1} = \lambda y_n. \tag{2.1}$$

Suppose that the non-selfadjoint matrices sequences $\{A_n\}$ and $\{B_n\}$, $n \in \mathbb{N}$, satisfy

$$\sum_{n=1}^{\infty} n\left(\|I - A_n\| + \|B_n\|\right) < \infty, \tag{2.2}$$

where I denotes the identity matrix in E and $\|.\|$ denotes the matrix norm in E.

The Jost solution of (2.1) is given by Serebryakov in [15].

Under condition (2.2), Eq. (2.1) has a matrix solution

$$F_n(z) = T_n e^{inz}\left[I + \sum_{m=1}^{\infty} K_{n,m} e^{imz}\right], \quad n \in \mathbb{N} \cup \{0\},$$

for $\lambda = 2\cos z$, where $z \in \bar{\mathbb{C}}_+ := \{z \in \mathbb{C}, \operatorname{Im} z \geqslant 0\}$ and T_n, $K_{n,m}$ are expressed in terms of $\{A_n\}$ and $\{B_n\}$ as

$$T_n = \prod_{p=n}^{\infty} A_n^{-1}, \quad n \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

Moreover, $K_{n,m}$ satisfies

$$\left\|K_{n,m}\right\| \leqslant c \sum_{p=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(\left\|I - A_p\right\| + \left\|B_p\right\|\right), \tag{2.4}$$

where $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$ and $c > 0$ is a constant. Due to condition (2.2), the infinite product and the series in the definition of T_n and K_{nm}, $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$, are absolutely convergent. Therefore $F_n(z)$, $n \in \mathbb{N} \cup \{0\}$, is analytic with respect to z in $\mathbb{C}_+ := \{z \in \mathbb{C}, \operatorname{Im} z > 0\}$ and continuous in $\bar{\mathbb{C}}_+$. $F_n(z)$ also satisfies

$$\begin{aligned} &F_n(z) = e^{inz}\left[I + o(1)\right], \quad z \in \bar{\mathbb{C}}_+,\ n \longrightarrow \infty, \\ &F_n(z) = T_n e^{inz}\left[I + o(1)\right], \quad z \in \bar{\mathbb{C}}_+, \qquad z \in \xi + i\tau, \quad \tau \longrightarrow \infty. \end{aligned} \tag{2.5}$$

In analogy to the Sturm–Liouville equation, the solution $F(z) := \{F_n(z)\}$, $n \in \mathbb{N} \cup \{0\}$, and the function $F_0(z) = T_0\left[I + \sum_{m=1}^{\infty} K_{0,m} e^{imz}\right]$ are called the Jost solution and the Jost function, respectively.

Let $\sigma(L)$ and $\sigma_c(L)$ denote the spectrum and the continuous spectrum of L, respectively.

Theorem 2.1. *If condition (2.2) holds, then $\sigma_c(L) = [-2, 2]$.*

Proof. Let L_1 and L_2 denote the difference operators generated in $\ell_2(\mathbb{N}, E)$ by the difference expressions $(\ell_1 y)_n = y_{n-1} + y_{n+1}$ and $(\ell_2 y)_n = (A_{n-1} - I)y_{n-1} + B_n y_n + (A_n - I)y_{n+1}$ with the boundary condition $y_0 = 0$, respectively. Note that the operators L_1 and L_2 are generated in $\ell_2(\mathbb{N}, E)$ by the following Jacobi matrices,

$$J_1 = \begin{pmatrix} I & 0 & 0 & . & . & . \\ 0 & I & 0 & . & . & . \\ 0 & 0 & I & . & . & . \\ . & . & . & . & . & . \end{pmatrix},$$

and

$$J_2 = \begin{pmatrix} B_1 & A_1 - I & 0 & 0 & 0 & . & . & . \\ A_1 - I & B_2 & A_2 - I & 0 & 0 & . & . & . \\ 0 & A_2 - I & B_3 & A_3 - I & 0 & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix},$$

respectively. It is clear that

$$L = L_1 + L_2, \qquad L_1^* = L_1,$$

and

$$\sigma(L_1) = \sigma_c(L_1) = [-2, 2],$$

[15]. It follows from (2.2) that the operator L_2 is a compact operator in $\ell_2(\mathbb{N}, E)$ [17]. By the Weyl theorem of a compact perturbation, we get

$$\sigma_c(L) = \sigma_c(L_1) = [-2, 2]$$

[18]. □

Let

$$f(z) = \det F_0(z), \quad z \in \bar{\mathbb{C}}_+, \tag{2.6}$$

where $F_0(z)$ denotes the Jost function of L. The function f is analytic in $\mathbb{C}_+$, continuous in $\bar{\mathbb{C}}_+$ and

$$f(z) = f(z + 2\pi).$$

Let us define the semi-strip $P_0 = \{z : z = \xi + i\tau, 0 \leqslant \xi \leqslant 2\pi, \tau > 0\}$ and $P = P_0 \cup [0, 2\pi]$. We will denote the set of all eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From the definition of the eigenvalues and the spectral singularities of non-selfadjoint operators [11], we have

$$\sigma_d(L) = \{\lambda : \lambda = 2\cos z, z \in P_0, f(z) = 0\}, \tag{2.7}$$

$$\sigma_{ss}(L) = \{\lambda : \lambda = 2\cos z, z \in [0, 2\pi], f(z) = 0\} \setminus \{0\}. \tag{2.8}$$

Theorem 2.2. *Under condition (2.2),*

(i) *the set of eigenvalues of L is bounded, countable and its limit points lie in $[-2, 2]$;*
(ii) *$\sigma_{ss}(L) \subset [-2, 2]$ and $\mu\{\sigma_{ss}(L)\} = 0$, where $\mu\{\sigma_{ss}(L)\}$ denotes the linear Lebesgue measure of $\sigma_{ss}(L)$.*

Proof. It follows from (2.7) and (2.8) that in order to investigate the quantitative properties of the eigenvalues and the spectral singularities of L, we need to discuss the quantitative properties of the zeros of f in P.

Using (2.5) and (2.6), we find

$$f(z) = \det T_0 [1 + o(1)], \quad z \in P_0, \qquad z = \xi + i\tau, \quad \tau \longrightarrow \infty, \tag{2.9}$$

where $\det T_0 \neq 0$ by (2.2) and (2.3). Now, (2.9) shows the boundedness of the zeros of f in P_0. Since f is a 2π-periodic function and is analytic in $\mathbb{C}_+$, we obtain that f has at most a countable number of zeros in P_0. By the uniqueness of analytic functions, we find that the limit points of zeros of f in P_0 can lie only in $[0, 2\pi]$. From (2.8), we have $\sigma_{ss}(L) \subset [-2, 2]$. Using the boundary uniqueness theorem of analytic functions, we get $\mu\{\sigma_{ss}(L)\} = 0$ [19]. □

Definition 2.1. The multiplicity of a zero of f in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of L.

Theorem 2.3. *If, for some $\varepsilon > 0$,*

$$\sup_{n \in \mathbb{N}} \left\{ e^{\varepsilon n} \left(\|I - A_n\| + \|B_n\| \right) \right\} < \infty \tag{2.10}$$

holds, then the operator L has a finite number of eigenvalues of finite multiplicity and spectral singularities.

Proof. From (2.4) and (2.10), we find that

$$\|K_{0m}\| \leqslant c e^{-\frac{\varepsilon}{4} m}, \quad m \in \mathbb{N}, \tag{2.11}$$

where $c > 0$ is a constant. By (2.6) and (2.11), we observe that the function f has an analytic continuation to the half-plane $\operatorname{Im} z > -\frac{\varepsilon}{4}$. Since f is a 2π periodic function, the limit points of its zeros in P cannot lie in $[0, 2\pi]$. Using Theorem 2.2, we find that the bounded sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ have no limit points; i.e. the sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ have a finite number of elements. From analyticity of f in $\operatorname{Im} z > -\frac{\varepsilon}{4}$, we get that all zeros of f in P have a finite multiplicity. Consequently, all eigenvalues and spectral singularities of L have a finite multiplicity. □

Now we give an example of a non-selfadjoint matrix difference operator having a spectral singularity.

Example 2.1. Let L_0 denote the operator generated in $\ell_2(\mathbb{N}, E)$ by the difference expression $(\ell_0 y)_n = y_{n-1} + y_{n+1}$, $n \in \mathbb{N}$, and the boundary condition $y_1 - e^{i\frac{2\pi}{3}} y_0 = 0$. It is clear that $F(z) = \left\{ e^{inz} I \right\}$, $n \in \mathbb{N} \cup \{0\}$, and $G(z) = (e^{iz} - e^{i\frac{2\pi}{3}}) I$ are the Jost solution and the Jost function of L_0, respectively. Using Theorem 2.1 and Eqs. (2.6)–(2.8), we get

$$\sigma_c(L_0) = [-2, 2], \qquad \sigma_d(L_0) = \varnothing, \qquad \sigma_{ss}(L) = \{-1\}.$$

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