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Complex Analysis and Operator Theory

Inequalities for the Eigenvalues of Non-Selfadjoint Jacobi Operators

Marcel Hansmann · Guy Katriel

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Abstract We prove Lieb-Thirring-type bounds on eigenvalues of non-selfadjoint Jacobi operators, which are nearly as strong as those proven previously for the case of selfadjoint operators by Hundertmark and Simon. We use a method based on determinants of operators and on complex function theory, extending and sharpening earlier work of Borichev, Golinskii and Kupin.

Keywords Non-selfadjoint Jacobi operators · Eigenvalues · Lieb-Thirring inequalities

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1 Introduction and Results

This paper is concerned with the study of the set of discrete eigenvalues of complex Jacobi operators $J : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$, represented by a two-sided infinite tridiagonal matrix

$$J = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_{-1} & b_0 & c_0 & & & \\ & & a_0 & b_1 & c_1 & & \\ & & & a_1 & b_2 & c_2 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

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M. Hansmann (✉) · G. Katriel
Institute of Mathematics, Technical University of Clausthal, 38678 Clausthal-Zellerfeld, Germany
e-mail: hansmann@math.tu-clausthal.de

G. Katriel
e-mail: haggaika@yahoo.com

where $\{a_k\}_{k\in\mathbb{Z}}$, $\{b_k\}_{k\in\mathbb{Z}}$ and $\{c_k\}_{k\in\mathbb{Z}}$ are bounded complex sequences. More precisely, for $u \in l^2(\mathbb{Z})$, J is defined via

$$(Ju)(k) = a_{k-1}u(k-1) + b_k u(k) + c_k u(k+1).$$

Much work has been devoted to the spectral analysis of Jacobi operators, most of it in the selfadjoint context [16], that is when $c_k = \bar{a}_k$, but also in the non-selfadjoint case, (see, e.g. [1,2,8]).

In this paper, we are interested in operators J that are compact perturbations of the free Jacobi operator J_0, which is defined as the special case with $a_k = c_k \equiv 1$ and $b_k \equiv 0$, i.e.

$$(J_0 u)(k) = u(k-1) + u(k+1).$$

So, in the following, we will always assume that $J - J_0$ is compact, or equivalently that

$$\lim_{|k|\to\infty} a_k = \lim_{|k|\to\infty} c_k = 1, \quad \text{and} \quad \lim_{|k|\to\infty} b_k = 0.$$

As is well-known, the spectrum $\sigma(J_0)$ of J_0 is equal to $[-2, 2]$ and the compactness of $J - J_0$ implies that $\sigma(J) = [-2, 2] \,\dot{\cup}\, \sigma_d(J)$, where $[-2, 2]$ is the essential spectrum of J and the discrete spectrum $\sigma_d(J) \subset \mathbb{C} \setminus [-2, 2]$ consists of a countable set of isolated eigenvalues of finite algebraic multiplicity, with possible accumulation points in $[-2, 2]$.

We define the sequence $d = \{d_k\}_{k\in\mathbb{Z}}$ as follows

$$d_k = \max\left(|a_{k-1} - 1|, |a_k - 1|, |b_k|, |c_{k-1} - 1|, |c_k - 1|\right). \tag{1}$$

Note that the compactness of $J - J_0$ is equivalent to $\{d_k\}$ converging to 0. The main results of this paper provide information on $\sigma_d(J)$, assuming the stronger condition that $d \in l^p(\mathbb{Z})$, the space of p-summable sequences (we will see below that $d \in l^p(\mathbb{Z})$ implies that $J - J_0$ is an element of the Schatten class $\mathbf{S}_p$, see Sect. 2 for relevant definitions).

Theorem 1 *Let $\tau \in (0, 1)$. If $d \in l^p(\mathbb{Z})$, where $p > 1$, then*

$$\sum_{\lambda\in\sigma_d(J)} \frac{\operatorname{dist}(\lambda, [-2, 2])^{p+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}}} \leq C(p, \tau)\|d\|_{l^p}^p. \tag{2}$$

Furthermore, if $d \in l^1(\mathbb{Z})$, then

$$\sum_{\lambda\in\sigma_d(J)} \frac{\operatorname{dist}(\lambda, [-2, 2])^{1+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}+\frac{\tau}{4}}} \leq C(\tau)\|d\|_{l^1}. \tag{3}$$

Remark 1 In the summation above, and elsewhere in this article, each eigenvalue is counted according to its algebraic multiplicity. Furthermore, the constants used in this paper are generic, i.e. the value of a constant may change from line to line. However, we will always indicate the parameters that a constant depends on.

The above estimates can be regarded as 'near-generalisations' of the following Lieb-Thirring inequalities proven by Hundertmark and Simon [11] for selfadjoint Jacobi operators (i.e. for the case that $a_k, b_k, c_k \in \mathbb{R}$ and $a_k = c_k$ for all k).

Theorem ([11], Theorem 2) *Let J be selfadjoint and suppose that $d \in l^p(\mathbb{Z})$, where $p \geq 1$. Then*

$$\sum_{\lambda \in \sigma_d(J),\ \lambda < -2} |\lambda + 2|^{p-\frac{1}{2}} + \sum_{\lambda \in \sigma_d(J),\ \lambda > 2} |\lambda - 2|^{p-\frac{1}{2}} \leq C(p)\|d\|_{l^p}^p. \tag{4}$$

To see the relation between the above result and Theorem 1, we note that in the selfadjoint case the eigenvalues of J are in $\mathbb{R} \setminus [-2, 2]$, and we have $\operatorname{dist}(\lambda, [-2, 2]) = |\lambda - 2|$ for $\lambda > 2$ and $\operatorname{dist}(\lambda, [-2, 2]) = |\lambda + 2|$ if $\lambda < -2$, so that (4) can be rewritten in the form

$$\sum_{\lambda \in \sigma_d(J)} \operatorname{dist}(\lambda, [-2, 2])^{p-\frac{1}{2}} \leq C(p)\|d\|_{l^p}^p. \tag{5}$$

One could try to generalise (5) to non-selfadjoint Jacobi operators, but we are not able to do this, and in fact we conjecture that (5) is not true in the non-selfadjoint case, due to a different behaviour of sequences of eigenvalues when converging to ± 2 or to $(-2, 2)$, respectively. To get an analogue of (4) which is valid in the non-selfadjoint case, we note that since

$$\frac{\operatorname{dist}(\lambda, [-2, 2])^p}{|\lambda^2 - 4|^{\frac{1}{2}}} \leq \frac{1}{2} \begin{cases} |\lambda - 2|^{p-\frac{1}{2}}, \ \lambda > 2 \\ |\lambda + 2|^{p-\frac{1}{2}}, \ \lambda < -2, \end{cases}$$

inequality (4) implies that in the selfadjoint case

$$\sum_{\lambda \in \sigma_d(J)} \frac{\operatorname{dist}(\lambda, [-2, 2])^p}{|\lambda^2 - 4|^{\frac{1}{2}}} \leq C(p)\|d\|_{l^p}^p. \tag{6}$$

Clearly, for $p > 1$, (6) is the same as (2) when $\tau = 0$. On the other hand, Theorem 1 requires $\tau > 0$, so that (2) (when applied to selfadjoint operators) is slightly weaker than (6), which is why we say that it is a 'near-generalisation'. The same observation applies to inequality (3) in case that $p = 1$. An interesting open question is whether (2) and (3) are valid for $\tau = 0$ in the non-selfadjoint case.

We note that the methods used by Hundertmark and Simon in [11] depend in an essential way on the selfadjointness of the operators, and so are completely different from those used here. Indeed, it is not at all obvious that results valid for selfadjoint Jacobi operators can be generalised to the non-selfadjoint case, as the distribution of eigenvalues of non-selfadjoint Jacobi operators is much less restricted than that of selfadjoint Jacobi operators. A demonstration of this fact is given in an example constructed in Sect. 6: we consider a very simple Jacobi operator for which $J - J_0$ is 0 except for two diagonal elements, and show that whereas in the selfadjoint case J can have at most two eigenvalues, in the non-selfadjoint case it can have an arbitrarily large (but finite) number of eigenvalues.

To prove Theorem 1 we develop and sharpen ideas of Borichev et al. [3]. Using determinants of Schatten class operators, we define a function $g(\lambda)$ whose zeros coincide with the eigenvalues of J, and then we study these zeros by applying complex function theory. Other applications of this approach can be found in [4,5]. In [3], Theorem 2.3, the authors used this approach to show that for $p > 1$ and $\tau > 0$,

$$\sum_{\lambda \in \sigma_d(J)} \frac{\operatorname{dist}(\lambda, [-2, 2])^{p+1+\tau}}{|\lambda^2 - 4|} \leq C(\tau, \|J - J_0\|, p)\|J - J_0\|_{\mathbf{S}_p}^p, \tag{7}$$

where $\|.\|_{\mathbf{S}_p}$ denotes the pth Schatten norm. We note that the Schatten norm $\|J - J_0\|_{\mathbf{S}_p}$ is equivalent to $\|d\|_{l^p}$, as is shown in Lemma 8 below. Inequality (7) was originally derived for Jacobi operators on $l^2(\mathbb{N})$ but it carries over to the whole-line case. The authors of [3] also derived a more refined estimate in case $p = 1$, similar to (3), but here their proof seems to use special properties of the half-line operator and is thus not directly transferable to the whole-line setting.

We remark that inequality (7) is an easy consequence of Theorem 1 since

$$\frac{\operatorname{dist}(\lambda, [-2, 2])^{p+1+\tau}}{|\lambda^2 - 4|} \leq \frac{\operatorname{dist}(\lambda, [-2, 2])^{p+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}}},$$

as a direct calculation shows. Theorem 1 improves upon (7) in another respect since the constants on the right-hand side of (2) and (3) are independent of J. To get rid of this dependence, we develop, in Theorem 4 below, a variant of the complex function result used in [3]. The proof of Theorem 1 further depends on a more subtle estimate on the function $g(\lambda)$ (mentioned above) than the one used in [3], exploiting the structure of the Jacobi operators in an essential way.

In relation to Theorem 1, it is interesting to discuss another approach to generalising inequality (4) to non-selfadjoint operators, developed by Golinskii and Kupin [10]. For $\theta \in [0, \frac{\pi}{2})$ we define the following sectors in the complex plane:

$$\Omega_\theta^\pm = \{\lambda : 2 \mp \operatorname{Re}(\lambda) < \tan(\theta)|\operatorname{Im}\lambda|\}.$$

Theorem ([10], Theorem 1.5) *Let $\theta \in [0, \frac{\pi}{2})$. Then for $p \geq \frac{3}{2}$*

$$\sum_{\lambda \in \sigma_d(J) \cap \Omega_\theta^+} |\lambda - 2|^{p-\frac{1}{2}} + \sum_{\lambda \in \sigma_d(J) \cap \Omega_\theta^-} |\lambda + 2|^{p-\frac{1}{2}} \leq C(p, \theta)\|d\|_{l^p}^p, \tag{8}$$

where $C(p, \theta) = C(p)(1 + 2\tan(\theta))^p$.

Clearly, this theorem, when restricted to the selfadjoint case, gives (4). This is not a coincidence, but due to the fact that its proof is obtained by a reduction to the case of selfadjoint operators and employing (4). A drawback of this result is that the sum is not over all eigenvalues since it excludes a diamond-shaped region around the interval $[-2, 2]$ (thus avoiding sequences of eigenvalues converging to some point in $(-2, 2)$).

However, we shall show that by a suitable integration, the inequalities (8) can be used to derive an inequality where the sum is over all the eigenvalues:

Theorem 2 *Let $\tau \in (0, 1)$. If $d \in l^p(\mathbb{Z})$, where $p \geq \frac{3}{2}$, then*

$$\sum_{\lambda \in \sigma_d(J)} \frac{\operatorname{dist}(\lambda, [-2, 2])^{p+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}+\tau}} \leq C(p, \tau)\|d\|_{l^p}^p. \tag{9}$$

We emphasise that the proof of Theorem 2 does not involve any complex analysis and is thus completely different from the proof of Theorem 1. Let us note the similarities, and the differences, between these two results: inequality (9) is in fact somewhat stronger than (2), because of the τ in the denominator of (9). However, while Theorem 2 requires the condition $p \geq \frac{3}{2}$, Theorem 1 is valid for $p \geq 1$, just like the corresponding inequality (4) in the selfadjoint case.

We can thus conclude that the approach based on complex analysis provides very satisfying results for non-selfadjoint Jacobi operators, which are almost as strong as those obtained in the selfadjoint case by specialised methods relying on the selfadjointness of the operators.

Finally, let us give a short overview of the contents of this paper: in Sect. 2, we gather information about Schatten classes and infinite determinants. In Sect. 3 we present some complex analysis results that are used in the proof of Theorem 1, which is provided in Sect. 4. In Sect. 5 we are concerned with the proof of Theorem 2. Finally, in Sect. 6 we construct the simple example mentioned above, showing the great difference in the behaviour of eigenvalues of Jacobi operators in the non-selfadjoint case as compared to the selfadjoint case.

2 Preliminaries

For a Hilbert space $\mathcal{H}$ let $\mathbf{C}(\mathcal{H})$ and $\mathbf{B}(\mathcal{H})$ denote the classes of closed and of bounded linear operators on $\mathcal{H}$, respectively. We denote the ideal of all compact operators on $\mathcal{H}$ by $\mathbf{S}_\infty$ and the Schatten class operators by $\mathbf{S}_p$, $p > 0$, i.e. a compact operator $C \in \mathbf{S}_p$ if

$$\|C\|_{\mathbf{S}_p}^p = \sum_{n=1}^{\infty} \mu_n(C)^p < \infty,$$

where $\mu_n(C)$ denotes the nth singular value of C.

Schatten class operators obey the following Hölder's inequality: let p, p_1, p_2 be positive numbers with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. If $C_i \in \mathbf{S}_{p_i} (i = 1, 2)$, then the operator

$C = C_1 C_2 \in \mathbf{S}_p$ and

$$\|C\|_{\mathbf{S}_p} \leq \|C_1\|_{\mathbf{S}_{p_1}} \|C_2\|_{\mathbf{S}_{p_2}}, \tag{10}$$

(see, e.g. [15], Theorem 2.8).

Let $C \in \mathbf{S}_n$, where $n \in \mathbb{N}$. Then one can define the (regularized) determinant

$$\det_n(I - C) = \prod_{\lambda \in \sigma(C)} \left[(1 - \lambda) \exp\left(\sum_{j=1}^{n-1} \frac{\lambda^j}{j} \right) \right],$$

having the following properties (see, e.g. [6,9,15]):

1. $I - C$ is invertible if and only if $\det_n(I - C) \neq 0$.
2. $\det_n(I) = 1$.
3. For $A, B \in \mathbf{B}(\mathcal{H})$ with $AB, BA \in \mathbf{S}_n$:

$$\det_n(I - AB) = \det_n(I - BA). \tag{11}$$

4. If $C(\lambda) \in \mathbf{S}_n$ depends holomorphically on $\lambda \in \Omega$, where $\Omega \subset \mathbb{C}$ is open, then $\det_n(I - C(\lambda))$ is holomorphic on Ω.
5. If $C \in \mathbf{S}_p$ for some $p > 0$, then $C \in \mathbf{S}_{\lceil p \rceil}$, where

$$\lceil p \rceil = \min\{n \in \mathbb{N} : n \geq p\},$$

and the following inequality holds,

$$|\det_{\lceil p \rceil}(I - C)| \leq \exp\left(\Gamma_p \|C\|_{\mathbf{S}_p}^p \right), \tag{12}$$

where Γ_p is some positive constant (see, [6, p. 1106]). We remark that $\Gamma_p = \frac{1}{p}$ for $p \leq 1$, $\Gamma_2 = \frac{1}{2}$ and $\Gamma_p \leq e(2 + \log p)$ when $p \geq 3$ is an integer (see, [14]).

For $A, B \in \mathbf{B}(\mathcal{H})$ with $B - A \in \mathbf{S}_p$, the $\lceil p \rceil$-regularized perturbation determinant of A by $B - A$ is a well defined holomorphic function on $\rho(A) = \mathbb{C} \setminus \sigma(A)$, given by

$$d(\lambda) = \det_{\lceil p \rceil}(I - (\lambda - A)^{-1}(B - A)).$$

Furthermore, $\lambda_0 \in \rho(A)$ is an eigenvalue of B of algebraic multiplicity k_0 if and only if λ_0 is a zero of $d(\cdot)$ of the same multiplicity.

3 Complex Analysis

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The following result is due to Borichev et al. [3].

Theorem 3 *Let $h : \mathbb{D} \to \mathbb{C}$ be holomorphic with $h(0) = 1$, and suppose that*

$$\log |h(z)| \leq K \frac{1}{(1 - |z|)^{\alpha}} \prod_{j=1}^{N} \frac{1}{|z - \xi_j|^{\beta_j}},$$

where $|\xi_j| = 1$ ($1 \leq j \leq N$), and the exponents α, β_j are nonnegative. Let $\tau > 0$. Then the zeros of h satisfy the inequality

$$\sum_{z \in \mathbb{D}, h(z)=0} (1 - |z|)^{\alpha+1+\tau} \prod_{j=1}^{N} |z - \xi_j|^{(\beta_j - 1+\tau)_+} \leq C(\alpha, \{\beta_j\}, \{\xi_j\}, \tau)K,$$

where $x_+ = \max(x, 0)$, and each zero of h is counted according to its multiplicity.

For the holomorphic function h that we will consider below, there will be some additional information on the speed of convergence of $\log |h(z)| \to 0$ as $z \to 0$. The following modification of the above theorem takes this into account.

Theorem 4 *Let $h : \mathbb{D} \to \mathbb{C}$ be holomorphic with $h(0) = 1$, and suppose that*

$$\log |h(z)| \leq K \frac{|z|^{\gamma}}{(1 - |z|)^{\alpha}} \prod_{j=1}^{N} \frac{1}{|z - \xi_j|^{\beta_j}}, \tag{13}$$

where $|\xi_j| = 1$ ($1 \leq j \leq N$), and the exponents α, β_j and γ are nonnegative. Let $0 < \tau < 1$. Then the zeros of h satisfy the inequality

$$\sum_{z \in \mathbb{D}, h(z)=0} \frac{(1 - |z|)^{\alpha+1+\tau}}{|z|^{(\gamma-1+\tau)_+}} \prod_{j=1}^{N} |z - \xi_j|^{(\beta_j - 1+\tau)_+} \leq C(\alpha, \{\beta_j\}, \gamma, \{\xi_j\}, \tau)K, \tag{14}$$

where each zero is counted according to its multiplicity.

We note that the results of the previous theorem differ from the results obtained in Theorem 3 both in the hypothesis, which requires $\log |h(z)|$ to vanish at 0 at a specified rate, and requires $\tau < 1$, and in its conclusion, which includes the $\frac{1}{|z|}$ term.

Proof of Theorem 4 In view of Theorem 3, we only need to consider the case $\gamma > 1 - \tau$. Set $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. Using the boundedness of $\frac{1}{|z|}$ on $\mathbb{D} \setminus \mathbb{D}_{\frac{1}{2}}$, we obtain from Theorem 3 that

$$\sum_{z \in \mathbb{D} \setminus \mathbb{D}_{\frac{1}{2}},\, h(z)=0} \frac{(1 - |z|)^{\alpha+1+\tau}}{|z|^{\gamma-1+\tau}} \prod_{j=1}^{N} |z - \xi_j|^{(\beta_j - 1+\tau)_+} \leq C(\alpha, \{\beta_j\}, \gamma, \{\xi_j\}, \tau)K.$$

Hence, the proof is completed by showing that

$$\sum_{z\in\mathbb{D}_{\frac{1}{2}},\, h(z)=0} \frac{1}{|z|^{\gamma-1+\tau}} \leq C(\alpha, \{\beta_j\}, \gamma, \{\xi_j\}, \tau)K. \tag{15}$$

Let $N_h(\mathbb{D}_r)$ denote the number of zeros of h in $\mathbb{D}_r$ (multiplicities taken into account). Then we can rewrite the sum in (15) as follows:

$$\begin{aligned}\sum_{z\in\mathbb{D}_{\frac{1}{2}},\, h(z)=0} \frac{1}{|z|^{\gamma-1+\tau}} &= (\gamma-1+\tau) \sum_{z\in\mathbb{D}_{\frac{1}{2}},\, h(z)=0} \int_0^{\frac{1}{|z|}} \mathrm{d}t\ t^{\gamma-2+\tau} \\ &= (\gamma-1+\tau)\left[\int_0^2 \mathrm{d}t\ t^{\gamma-2+\tau} N_h\left(\mathbb{D}_{\frac{1}{2}}\right) + \int_2^\infty \mathrm{d}t\ t^{\gamma-2+\tau} N_h(\mathbb{D}_{t^{-1}})\right].\end{aligned} \tag{16}$$

To estimate the last two integrals the following lemma is used.

Lemma 5 *Assume* (13)*. Then for* $r \in (0, \frac{1}{2}]$ *we have*

$$N_h(\mathbb{D}_r) \leq C(\alpha, \{\beta_j\}, \{\xi_j\})Kr^\gamma.$$

Proof of Lemma 5 Let $0 < r < s < 1$. From Jensen's identity (see, e.g. [13], Theorem 15.18) and assumption (13) we obtain

$$\begin{aligned} N_h(\mathbb{D}_r) &= \frac{1}{\log(\frac{s}{r})} \sum_{z\in\mathbb{D}_r, h(z)=0} \log\left(\frac{s}{r}\right) \leq \frac{1}{\log(\frac{s}{r})} \sum_{z\in\mathbb{D}_r, h(z)=0} \log\left(\frac{s}{|z|}\right) \\ &\leq \frac{1}{\log(\frac{s}{r})} \sum_{z\in\mathbb{D}_s, h(z)=0} \log\left(\frac{s}{|z|}\right) = \frac{1}{\log(\frac{s}{r})} \frac{1}{2\pi} \int_0^{2\pi} \log|h(se^{i\theta})|\mathrm{d}\theta \\ &\leq \frac{1}{\log(\frac{s}{r})} \frac{Ks^\gamma}{(1-s)^\alpha} \frac{1}{2\pi} \int_0^{2\pi} \prod_{j=1}^N \frac{1}{|se^{i\theta}-\xi_j|^{\beta_j}} \mathrm{d}\theta. \end{aligned}$$

Choosing $s = \frac{3}{2}r$ (i.e. $s \leq \frac{3}{4}$) concludes the proof of the lemma. □

Returning to (16), we can use Lemma 5 and the fact that $\gamma > 1-\tau$ to conclude

$$\int_0^2 \mathrm{d}t\ t^{\gamma-2+\tau} N_h(\mathbb{D}_{\frac{1}{2}}) \leq C(\alpha, \{\beta_j\}, \gamma, \{\xi_j\}, \tau)K.$$

Similarly, using that $\tau < 1$, Lemma 5 implies that

$$\int_2^\infty \mathrm{d}t\ t^{\gamma-2+\tau} N_h(\mathbb{D}_{t^{-1}}) \leq C(\alpha, \{\beta_j\}, \{\xi_j\})K \int_2^\infty \mathrm{d}t\ t^{-2+\tau}$$
$$\leq C(\alpha, \{\beta_j\}, \gamma, \{\xi_j\}, \tau)K.$$

This concludes the proof of Theorem 4. □

We now translate the result of Theorem 4 into a result about holomorphic functions on $\mathbb{C} \setminus [-2, 2]$, which is the one we will use below.

Corollary 6 *Let* $g : \mathbb{C} \setminus [-2, 2] \to \mathbb{C}$ *be holomorphic with* $\lim_{|\lambda|\to\infty} g(\lambda) = 1$. *For* $\alpha, \beta \geq 0$ *suppose that*

$$\log |g(\lambda)| \leq \frac{K_0}{\operatorname{dist}(\lambda, [-2, 2])^\alpha |\lambda^2 - 4|^\beta}. \tag{17}$$

Let $0 < \tau < 1$ *and set*

$$\begin{aligned} \eta_1 &= \alpha + 1 + \tau, \\ \eta_2 &= (2\beta + \alpha - 1 + \tau)_+. \end{aligned} \tag{18}$$

Then,

$$\sum_{\lambda \in \mathbb{C}\setminus[-2,2], g(\lambda)=0} \frac{\operatorname{dist}(\lambda, [-2, 2])^{\eta_1}}{|\lambda^2 - 4|^{\frac{\eta_1-\eta_2}{2}}} \leq C(\alpha, \beta, \tau)K_0, \tag{19}$$

where each zero of g is counted according to its multiplicity.

Proof We note that $z \mapsto z + z^{-1}$ maps $\mathbb{D} \setminus \{0\}$ conformally onto $\mathbb{C} \setminus [-2, 2]$. So we can define

$$h(z) = g(z + z^{-1}), \quad z \in \mathbb{D} \setminus \{0\}.$$

Setting $h(0) = 1$, h is holomorphic on $\mathbb{D}$ and

$$\log |h(z)| \leq \frac{K_0}{\operatorname{dist}(\lambda, [-2, 2])^\alpha |\lambda^2 - 4|^\beta}, \tag{20}$$

where $\lambda = z + z^{-1}$. To express the right-hand side of the last equation in terms of z, the following lemma is needed. Its proof is given below.

Lemma 7 *For* $\lambda = z + z^{-1}$ *with* $z \in \mathbb{D}$, *we have*

$$\frac{1}{2} \frac{|z^2 - 1|(1 - |z|)}{|z|} \leq \operatorname{dist}(\lambda, [-2, 2]) \leq \frac{1 + \sqrt{2}}{2} \frac{|z^2 - 1|(1 - |z|)}{|z|}. \tag{21}$$

In addition to the last lemma, a direct calculation shows that

$$|\lambda^2 - 4| = \left|\frac{z^2-1}{z}\right|^2. \tag{22}$$

Using (21) and (22), we can estimate (20) as follows

$$\log|h(z)| \le \frac{2^\alpha K_0 |z|^{2\beta+\alpha}}{(1-|z|)^\alpha |z^2-1|^{2\beta+\alpha}}.$$

Hence, Theorem 4 implies that for $0 < \tau < 1$ and η_1, η_2 as defined in (18),

$$\sum_{z\in\mathbb{D}, h(z)=0} (1-|z|)^{\eta_1} \left|\frac{z^2-1}{z}\right|^{\eta_2} \le C(\alpha,\beta,\tau)K_0.$$

By (21) and (22),

$$(1-|z|)^{\eta_1} \left|\frac{z^2-1}{z}\right|^{\eta_2} \ge \left(\frac{2}{1+\sqrt{2}}\right)^{\eta_1} \frac{\operatorname{dist}(\lambda,[-2,2])^{\eta_1}}{|\lambda^2-4|^{\frac{\eta_1-\eta_2}{2}}}.$$

This concludes the proof of Corollary 6. □

Proof of Lemma 7 Let $z \in \mathbb{D}$ and set $\lambda = z + z^{-1}$. We define $G_1 = \{z : \operatorname{Re}(\lambda) \le -2\}$, $G_2 = \{z : \operatorname{Re}(\lambda) \ge 2\}$ and $G_3 = \{z : |\operatorname{Re}(\lambda)| < 2\}$. Then

$$\operatorname{dist}(\lambda,[-2,2]) = \begin{cases} |\lambda+2| = \frac{|1+z|^2}{|z|}, & z \in G_1 \\ |\lambda-2| = \frac{|1-z|^2}{|z|}, & z \in G_2 \\ |\operatorname{Im}\lambda| = |\operatorname{Im} z|\frac{1-|z|^2}{|z|^2}, & z \in G_3. \end{cases} \tag{23}$$

We first show that for $z \in G_3$ the following holds

$$\frac{1}{\sqrt{2}} \frac{|z^2-1|(1-|z|)}{|z|} \le \operatorname{dist}(\lambda,[-2,2]) \le \frac{1+\sqrt{2}}{2} \frac{|z^2-1|(1-|z|)}{|z|}. \tag{24}$$

By (23), this is equivalent to

$$\frac{1}{\sqrt{2}} \le |\operatorname{Im} z| \frac{1+|z|}{|z||z^2-1|} \le \frac{1+\sqrt{2}}{2}. \tag{25}$$

Switching to polar coordinates we see that $re^{i\theta} \in G_3$ if $\cos^2(\theta) < \frac{4r^2}{(1+r^2)^2}$ and (25) can be rewritten as

$$\frac{1}{\sqrt{2}} \le \frac{(1+r)\sqrt{1-\cos^2(\theta)}}{\sqrt{(1+r^2)^2 - 4r^2\cos^2(\theta)}} \le \frac{1+\sqrt{2}}{2}. \tag{26}$$

For $x = \cos^2(\theta)$ and fixed r we define

$$f(x) = \frac{1-x}{(1+r^2)^2 - 4r^2 x}, \quad 0 \leq x < \frac{4r^2}{(1+r^2)^2}.$$

It is easy to see that f is monotonically decreasing. We thus obtain

$$\frac{1}{1+6r^2+r^4} = f\left(\frac{4r^2}{(1+r^2)^2}\right) \leq f(x) \leq f(0) = \frac{1}{(1+r^2)^2}.$$

The last chain of inequalities implies the validity of (26) and (25) since

$$\sup_{r\in[0,1]} \frac{1+r}{1+r^2} = \frac{1+\sqrt{2}}{2} \quad \text{and} \quad \inf_{r\in[0,1]} \frac{1+r}{\sqrt{1+6r^2+r^4}} = \frac{1}{\sqrt{2}}.$$

Next, we show that the following holds for $z \in G_1 \cup G_2$,

$$\frac{1}{2}\frac{|z^2-1|(1-|z|)}{|z|} \leq \operatorname{dist}(\lambda, [-2,2]) \leq \frac{1+\sqrt{2}}{2}\frac{|z^2-1|(1-|z|)}{|z|}.$$

By symmetry, it is sufficient to show it for $z \in G_1$. In this case, by (23), the last inequality is equivalent to

$$\frac{1}{2} \leq \frac{|z+1|}{|z-1|(1-|z|)} \leq \frac{1+\sqrt{2}}{2}. \tag{27}$$

Switching to polar coordinates , we have to show that

$$\frac{1}{2} \leq \frac{1}{1-r}\sqrt{\frac{r^2+1+2r\cos(\theta)}{r^2+1-2r\cos(\theta)}} \leq \frac{1+\sqrt{2}}{2}, \tag{28}$$

where $\cos(\theta) \leq \frac{-2r}{1+r^2}$. For $y = \cos(\theta)$ and fixed r we define

$$\tilde{f}(y) = \frac{r^2+1+2ry}{r^2+1-2ry}, \quad -1 \leq y \leq \frac{-2r}{1+r^2}. \tag{29}$$

A short calculation shows that $\tilde{f}$ is monotonically increasing and we obtain that

$$\left(\frac{1-r}{1+r}\right)^2 = \tilde{f}(-1) \leq \tilde{f}(y) \leq \tilde{f}\left(\frac{-2r}{1+r^2}\right) = \frac{(1-r^2)^2}{1+6r^2+r^4}. \tag{30}$$

(29) and (30) imply the validity of (28) and (27) since

$$\inf_{r\in[0,1]} \frac{1}{1+r} = \frac{1}{2} \quad \text{and} \quad \sup_{r\in[0,1]} \frac{1+r}{\sqrt{1+6r^2+r^4}} \leq \frac{1+\sqrt{2}}{2}. \tag{31}$$

This concludes the proof of Lemma 7. □

4 Proof of Theorem 1

We denote the standard basis of $l^2(\mathbb{Z})$ by $\{\delta_k\}_{k\in\mathbb{Z}}$, i.e. $\delta_k(k) = 1$ and $\delta_k(j) = 0$ for $j \neq k$. Let the diagonal operator $D \in \mathbf{B}(l^2(\mathbb{Z}))$ be defined via $D\delta_k = d_k\delta_k$, where the sequence $d = \{d_k\}$ is as defined in (1), i.e.

$$d_k = \max\left(|a_{k-1} - 1|, |a_k - 1|, |b_k|, |c_{k-1} - 1|, |c_k - 1|\right).$$

Furthermore, the operator $U \in \mathbf{B}(l^2(\mathbb{Z}))$ is given by

$$U\delta_k = u_k^-\delta_{k-1} + u_k^0\delta_k + u_k^+\delta_{k+1},$$

where

$$u_k^- = \frac{c_{k-1} - 1}{\sqrt{d_{k-1}d_k}}, \quad u_k^0 = \frac{b_k}{d_k}, \quad u_k^+ = \frac{a_k - 1}{\sqrt{d_{k+1}d_k}}.$$

Here we use the convention $\frac{0}{0} = 1$. It is then easily checked that

$$J - J_0 = D^{\frac{1}{2}}UD^{\frac{1}{2}}. \tag{32}$$

Moreover, the definition of $\{d_k\}$ implies

$$|u_k^-| \leq 1, \quad |u_k^0| \leq 1, \quad |u_k^+| \leq 1,$$

showing that $\|U\| \leq 3$.

The following lemma is a variation on Theorem 2.4 of [12].

Lemma 8 *Let $p \geq 1$ and $d \in l^p(\mathbb{Z})$. Then $J - J_0 \in \mathbf{S}_p$ and*

$$6^{-\frac{1}{p}}\|d\|_{l^p} \leq \|J - J_0\|_{\mathbf{S}_p} \leq 3\|d\|_{l^p}. \tag{33}$$

Proof From (32) we obtain

$$\begin{aligned}\|J - J_0\|_{\mathbf{S}_p} &= \|D^{\frac{1}{2}}UD^{\frac{1}{2}}\|_{\mathbf{S}_p} \leq \|D^{\frac{1}{2}}\|_{\mathbf{S}_{2p}}\|UD^{\frac{1}{2}}\|_{\mathbf{S}_{2p}} \\ &\leq \|U\|\|D^{\frac{1}{2}}\|^2_{\mathbf{S}_{2p}} \leq 3\|D^{\frac{1}{2}}\|^2_{\mathbf{S}_{2p}} = 3\|d\|_{l^p}.\end{aligned}$$

In the first estimate we applied Hölder's inequality for Schatten norms, see (10). The last equality is valid since the diagonal operator $D^{\frac{1}{2}}$ is selfadjoint and nonnegative with eigenvalues $d_k^{\frac{1}{2}}$.

For the inequality in the other direction, we use that (see [12], Lemma 2.3(iii))

$$\sum_{k\in\mathbb{Z}}[|a_k-1|^p+|c_k-1|^p+|b_k|^p]\le 3\|J-J_0\|^p_{\mathbf{S}^p}.$$

Since

$$\begin{aligned}\|d\|^p_{l^p}&=\sum_{k\in\mathbb{Z}}\max\left(|a_{k-1}-1|^p,|a_k-1|^p,|b_k|^p,|c_{k-1}-1|^p,|c_k-1|^p\right)\\&\le 2\sum_{k\in\mathbb{Z}}[|a_k-1|^p+|c_k-1|^p+|b_k|^p],\end{aligned}$$

the result follows. □

In the sequel, we suppose that $d\in l^p(\mathbb{Z})$ for some fixed $p\ge 1$. Using the results of Sect. 2, we can define the holomorphic function $g:\mathbb{C}\setminus[-2,2]\to\mathbb{C}$ via

$$g(\lambda)=\det_{\lceil p\rceil}(I-(\lambda-J_0)^{-1}(J-J_0)), \tag{34}$$

and the zeros of g coincide with the eigenvalues of J in $\mathbb{C}\setminus[-2,2]$, where multiplicity is taken into account. We further note that $\lim_{|\lambda|\to\infty}g(\lambda)=1$.

For $\lambda\in\mathbb{C}\setminus[-2,2]$ we define

$$G(\lambda)=D^{\frac12}(\lambda-J_0)^{-1}D^{\frac12}. \tag{35}$$

We will see below that $G(\lambda)\in\mathbf{S}_p$. Hence, (11) and (32) allow to derive the following alternative representation of g.

$$\begin{aligned}g(\lambda)&=\det_{\lceil p\rceil}(I-(\lambda-J_0)^{-1}D^{\frac12}UD^{\frac12})\\&=\det_{\lceil p\rceil}(I-G(\lambda)U).\end{aligned} \tag{36}$$

From (12) we further obtain that

$$\log|g(\lambda)|\le\Gamma_p\|G(\lambda)U\|^p_{\mathbf{S}_p}\le\Gamma_p3^p\|G(\lambda)\|^p_{\mathbf{S}_p}. \tag{37}$$

The following lemma provides some information on the Schatten norm of $G(\lambda)$.

Lemma 9 *Let $d\in l^p(\mathbb{Z})$, where $p\ge 1$. Then $G(\lambda)\in\mathbf{S}_p$ and for $p>1$,*

$$\|G(\lambda)\|^p_{\mathbf{S}_p}\le\frac{C(p)\|d\|^p_{l^p}}{\operatorname{dist}(\lambda,[-2,2])^{p-1}|\lambda^2-4|^{\frac12}}. \tag{38}$$

Furthermore, for every $0<\varepsilon<1$,

$$\|G(\lambda)\|_{\mathbf{S}_1}\le\frac{C(\varepsilon)\|d\|_{l^1}}{\operatorname{dist}(\lambda,[-2,2])^{\varepsilon}|\lambda^2-4|^{\frac{(1-\varepsilon)}{2}}}. \tag{39}$$

The proof of Lemma 9 will be given below. First, let us continue with the proof of Theorem 1. Fix $\tau \in (0, 1)$. We consider the case $p > 1$ first. From (37) and (38) we obtain that

$$\log |g(\lambda)| \leq \frac{C(p)\|d\|_{l^p}^p}{\operatorname{dist}(\lambda, [-2, 2])^{p-1}|\lambda^2 - 4|^{\frac{1}{2}}}.$$

Hence, we can apply Corollary 6 with $\alpha = p - 1$ and $\beta = \frac{1}{2}$ (i.e. $\eta_1 = p + \tau$ and $\eta_2 = p - 1 + \tau$) to obtain

$$\sum_{\lambda \in \mathbb{C}\setminus[-2,2], g(\lambda)=0} \frac{\operatorname{dist}(\lambda, [-2, 2])^{p+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}}} \leq C(p, \tau)\|d\|_{l^p}^p. \tag{40}$$

Noting that the eigenvalues of J coincide with the zeros of g concludes the proof in case that $p > 1$. In case $p = 1$, we obtain from (37) and (39) that for $0 < \varepsilon < 1$,

$$\log |g(z)| \leq \frac{C(\varepsilon)\|d\|_{l^1}}{\operatorname{dist}(\lambda, [-2, 2])^{\varepsilon}|\lambda^2 - 4|^{\frac{(1-\varepsilon)}{2}}}.$$

As above, an application of Corollary 6 shows that for every $\tilde{\tau} \in (0, 1)$

$$\sum_{\lambda \in \sigma_d(J)} \frac{\operatorname{dist}(\lambda, [-2, 2])^{1+\varepsilon+\tilde{\tau}}}{|\lambda^2 - 4|^{\frac{(1+\varepsilon)}{2}}} \leq C(\tilde{\tau}, \varepsilon)\|d\|_{l^1}. \tag{41}$$

Choosing $\varepsilon = \tilde{\tau} = \frac{\tau}{2}$ concludes the proof of Theorem 1.

4.1 Proof of Lemma 9

We define the Fourier transform $\mathcal{F} : l^2(\mathbb{Z}) \to L^2(0, 2\pi)$ by

$$(\mathcal{F}f)(\theta) = \frac{1}{\sqrt{2\pi}} \sum_k e^{ik\theta} f_k$$

and note that for $f \in l^2(\mathbb{Z})$ and $\theta \in [0, 2\pi)$:

$$(\mathcal{F}J_0 f)(\theta) = 2\cos(\theta)(\mathcal{F}f)(\theta).$$

Consequently, for $\lambda \in \mathbb{C} \setminus [2, 2]$,

$$(\lambda - J_0)^{-1} = \mathcal{F}^{-1} M_{v_\lambda} \mathcal{F},$$

where $M_{v_\lambda} \in \mathbf{B}(L^2(0, 2\pi))$ is the operator of multiplication with the bounded function

$$v_\lambda(\theta) = (\lambda - 2\cos(\theta))^{-1}, \quad \theta \in [0, 2\pi). \tag{42}$$

Since

$$v_\lambda = |v_\lambda|^{\frac{1}{2}} \cdot \frac{v_\lambda}{|v_\lambda|} \cdot |v_\lambda|^{\frac{1}{2}},$$

we can define the unitary operator $T = \mathcal{F}^{-1} M_{\frac{v_\lambda}{|v_\lambda|}} \mathcal{F}$, to obtain the following identity

$$(\lambda - J_0)^{-1} = \mathcal{F}^{-1} M_{|v_\lambda|^{\frac{1}{2}}} \mathcal{F} T \mathcal{F}^{-1} M_{|v_\lambda|^{\frac{1}{2}}} \mathcal{F}.$$

With definition (35) and Hölder's inequality, see (10), we obtain

$$\begin{aligned}
\|G(\lambda)\|_{\mathbf{S}_p}^p &= \|D^{\frac{1}{2}} \mathcal{F}^{-1} M_{|v_\lambda|^{\frac{1}{2}}} \mathcal{F} T \mathcal{F}^{-1} M_{|v_\lambda|^{\frac{1}{2}}} \mathcal{F} D^{\frac{1}{2}} \|_{\mathbf{S}_p}^p \\
&\leq \|D^{\frac{1}{2}} \mathcal{F}^{-1} M_{|v_\lambda|^{\frac{1}{2}}} \mathcal{F}\|_{\mathbf{S}_{2p}}^p \|T \mathcal{F}^{-1} M_{|v_\lambda|^{\frac{1}{2}}} \mathcal{F} D^{\frac{1}{2}} \|_{\mathbf{S}_{2p}}^p \\
&\leq \|D^{\frac{1}{2}} \mathcal{F}^{-1} M_{|v_\lambda|^{\frac{1}{2}}} \mathcal{F}\|_{\mathbf{S}_{2p}}^p \|\mathcal{F}^{-1} M_{|v_\lambda|^{\frac{1}{2}}} \mathcal{F} D^{\frac{1}{2}} \|_{\mathbf{S}_{2p}}^p \\
&= \|D^{\frac{1}{2}} \mathcal{F}^{-1} M_{|v_\lambda|^{\frac{1}{2}}} \mathcal{F}\|_{\mathbf{S}_{2p}}^{2p}. \qquad (43)
\end{aligned}$$

For the last identity we used the selfadjointness of $D^{\frac{1}{2}}$ and $\mathcal{F}^{-1} M_{|v_\lambda|^{\frac{1}{2}}} \mathcal{F}$, and the fact that the Schatten norm is invariant under taking the adjoint. To derive an estimate on the Schatten norm of $D^{\frac{1}{2}} \mathcal{F}^{-1} M_{|v_\lambda|^{\frac{1}{2}}} \mathcal{F}$, we will use the following lemma. Here, as above, we denote the diagonal operator corresponding to a sequence $k = \{k_m\} \in l^\infty(\mathbb{Z})$ by K, i.e. $K\delta_m = k_m \delta_m$.

Lemma 10 *Let $q \geq 2$. Suppose that $k = \{k_m\} \in l^q(\mathbb{Z})$ and $v \in L^q(0, 2\pi)$. Then the following holds,*

$$\|K \mathcal{F}^{-1} M_v \mathcal{F}\|_{\mathbf{S}_q} \leq (2\pi)^{-1/q} \|k\|_{l^q} \|v\|_{L^q}. \qquad (44)$$

For operators on $L^2(\mathbb{R}^d)$, this is a well-known result, we refer to Theorem 4.1 in Simon [15]. Since the proofs in the discrete and continuous settings are completely analogous, we only provide a sketch.

Sketch of proof of Lemma 10 We note that $K\mathcal{F}^{-1} M_v \mathcal{F}$ is an integral operator on $l^2(\mathbb{Z})$ with kernel $(2\pi)^{-\frac{1}{2}} k_m (\mathcal{F}^{-1} v)_{m-n}$ where $m, n \in \mathbb{Z}$. We thus obtain for the Hilbert-Schmidt norm

$$\begin{aligned}
\|K \mathcal{F}^{-1} M_v \mathcal{F}\|_{\mathbf{S}_2}^2 &= (2\pi)^{-1} \sum_{m,n} |k_m (\mathcal{F}^{-1} v)_{m-n}|^2 \\
&= (2\pi)^{-1} \|k\|_{l^2}^2 \|\mathcal{F}^{-1} v\|_{l^2}^2 = (2\pi)^{-1} \|k\|_{l^2}^2 \|v\|_{L^2}^2.
\end{aligned}$$

Clearly, for the operator norm we have

$$\|K \mathcal{F}^{-1} M_v \mathcal{F}\| \leq \|K\| \|\mathcal{F}^{-1} M_v \mathcal{F}\| = \|K\| \|M_v\| = \|k\|_{l^\infty} \|v\|_{L^\infty}.$$

The general result now follows by complex interpolation. For details, see the proof of Theorem 4.1 in [15]. □

We return to the proof of Lemma 9. With $d^{\frac{1}{2}} = \{d_k^{\frac{1}{2}}\}$ the previous lemma and estimate (43) imply

$$\|G(\lambda)\|_{\mathbf{S}_p}^p \leq (2\pi)^{-1} \|d^{\frac{1}{2}}\|_{l^{2p}}^{2p} \| |v_\lambda|^{\frac{1}{2}} \|_{L^{2p}}^{2p} = (2\pi)^{-1} \|d\|_{l^p}^p \|v_\lambda\|_{L^p}^p.$$

The proof of Lemma 9 is completed by an application of the following result.

Lemma 11 *Let* $\lambda \in \mathbb{C} \setminus [-2, 2]$ *and let* $v_\lambda : [0, 2\pi) \to \mathbb{C}$ *be defined by* (42)*. Then for* $p > 1$,

$$\|v_\lambda\|_{L^p}^p \leq \frac{C(p)}{\mathrm{dist}(\lambda, [-2, 2])^{p-1} |\lambda^2 - 4|^{\frac{1}{2}}}. \tag{45}$$

Furthermore, for every $0 < \varepsilon < 1$,

$$\|v_\lambda\|_{L^1} \leq \frac{C(\varepsilon)}{\mathrm{dist}(\lambda, [-2, 2])^{\varepsilon} |\lambda^2 - 4|^{\frac{(1-\varepsilon)}{2}}}. \tag{46}$$

Proof of Lemma 11 Let us first show that (46) is an immediate consequence of (45): for $r > 1$ Hölder's inequality and (45) imply (remember that $L^2 = L^2(0, 2\pi)$)

$$\begin{aligned} \|v_\lambda\|_{L^1} &= \|v_\lambda \cdot 1\|_{L^1} \leq \|v_\lambda\|_{L^r} \|1\|_{L^{r/(r-1)}} \\ &\leq \frac{C(r)}{\mathrm{dist}(\lambda, [-2, 2])^{1-\frac{1}{r}} |\lambda^2 - 4|^{\frac{1}{2r}}}. \end{aligned}$$

Choosing $r = \frac{1}{1-\varepsilon}$, where $0 < \varepsilon < 1$, implies the validity of (46). It remains to show (45).

Let $\lambda = z + z^{-1}$, where $z \in \mathbb{D}$. By $dm(.)$ we denote normalized Lebesgue measure on $\mathbb{T} = \partial\mathbb{D}$. Then

$$\begin{aligned} \|v_\lambda\|_{L^p}^p &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{|\lambda - 2\cos(\theta)|^p} \\ &= \int_{\mathbb{T}} \frac{\mathrm{d}m(w)}{|w + \overline{w} - z - z^{-1}|^p} = |z|^p \int_{\mathbb{T}} \frac{\mathrm{d}m(w)}{|w - z|^p |w - \overline{z}|^p}. \end{aligned} \tag{47}$$

Let $\mathbb{T}^+ = \{w \in \mathbb{T} : \mathrm{Im}(w) > 0\}$ and $\mathbb{T}^- = \mathbb{T} \setminus \mathbb{T}^+$. In the following, let us suppose that $\mathrm{Im}(z) \geq 0$ (the other case can be handled similarly). Then for $w \in \mathbb{T}^+$ we have

$$|w - \overline{z}| \geq \min(|1 - \overline{z}|, |1 + \overline{z}|) \geq \frac{1}{2}|1 - z^2|.$$

Similarly, for $w \in \mathbb{T}^-$ we obtain

$$|w - z| \geq \min(|1 - z|, |1 + z|) \geq \frac{1}{2}|1 - z^2|.$$

From (47) we can thus deduce that

$$\|v_\lambda\|_{L^p}^p \leq \frac{2^p |z|^p}{|1 - z^2|^p} \left(\int_{\mathbb{T}^+} \frac{\mathrm{d}m(w)}{|w - z|^p} + \int_{\mathbb{T}^-} \frac{\mathrm{d}m(w)}{|w - \overline{z}|^p} \right). \tag{48}$$

We use the estimate (see, e.g. [7, p. 78]), for $\mu \in \mathbb{D}$,

$$\int_{\mathbb{T}} \frac{\mathrm{d}m(w)}{|w - \mu|^p} = O((1 - |\mu|)^{1-p}), \quad |\mu| \to 1.$$

Applying this estimate to (48), and using (21) and (22), we obtain

$$\|v_\lambda\|_{L^p}^p \leq \frac{C(p)|z|^p}{|1 - z^2|^p (1 - |z|)^{p-1}} \leq \frac{C(p)}{\mathrm{dist}(\lambda, [-2, 2])^{p-1} |\lambda^2 - 4|^{1/2}}.$$

□

5 Proof of Theorem 2

Let $p \geq \frac{3}{2}$ and $\tau \in (0, 1)$. From (8) we know that for $\theta \in [0, \frac{\pi}{2})$

$$\sum_{\lambda \in \sigma_d(J) \cap \Omega_\theta^+} |\lambda - 2|^{p - \frac{1}{2}} \leq C(p)(1 + 2\tan(\theta))^p \|d\|_{l^p}^p, \tag{49}$$

where $\Omega_\theta^+ = \{\lambda : 2 - \mathrm{Re}(\lambda) < \tan(\theta)|\,\mathrm{Im}\,\lambda|\}$. We define

$$\Psi_1 = \{\lambda : \mathrm{Re}(\lambda) > 0,\ 2 - \mathrm{Re}(\lambda) < |\,\mathrm{Im}(\lambda)|\} \subset \Omega_{\pi/4}^+.$$

An easy calculation shows that for $\lambda \in \Psi_1$ we have

$$|\lambda - 2|^{p - \frac{1}{2}} \geq C(\tau) \frac{\mathrm{dist}(\lambda, [-2, 2])^{p+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}+\tau}}, \tag{50}$$

so (49) implies that

$$\sum_{\lambda \in \sigma_d(J) \cap \Psi_1} \frac{\mathrm{dist}(\lambda, [-2, 2])^{p+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}+\tau}} \leq C(p, \tau) \|d\|_{l^p}^p. \tag{51}$$

Let $\Psi_2 = \{\lambda : \mathrm{Re}(\lambda) > 0\} \setminus \Psi_1$ and set $x = \tan(\theta) \in [0, \infty)$. From (49) we obtain

$$\sum_{\lambda \in \sigma_d(J) \cap \Psi_2,\ \frac{2-\mathrm{Re}(\lambda)}{|\operatorname{Im}\lambda|} < x} |\lambda - 2|^{p-\frac{1}{2}} \leq C(p)(1+2x)^p \|d\|_{l^p}^p. \tag{52}$$

We multiply both sides of (52) with $x^{-p-1-\tau}$ and integrate with respect to $x \in [1, \infty)$. For the left-hand side we obtain

$$\begin{aligned}
&\int_1^\infty \mathrm{d}x\, x^{-p-1-\tau} \sum_{\lambda \in \sigma_d(J) \cap \Psi_2,\ \frac{2-\mathrm{Re}(\lambda)}{|\operatorname{Im}\lambda|} < x} |\lambda - 2|^{p-\frac{1}{2}} \\
&= \sum_{\lambda \in \sigma_d(J) \cap \Psi_2} |\lambda - 2|^{p-\frac{1}{2}} \int_{\max(1, \frac{2-\mathrm{Re}(\lambda)}{|\operatorname{Im}\lambda|})}^\infty \mathrm{d}x\, x^{-p-1-\tau} \\
&= C(p,\tau) \sum_{\lambda \in \sigma_d(J) \cap \Psi_2} |\lambda - 2|^{p-\frac{1}{2}} \left(\frac{|\operatorname{Im}\lambda|}{2 - \mathrm{Re}(\lambda)} \right)^{p+\tau} \\
&= C(p,\tau) \sum_{\lambda \in \sigma_d(J) \cap \Psi_2} |\lambda - 2|^{p-\frac{1}{2}} \left(\frac{\mathrm{dist}(\lambda, [-2,2])}{2 - \mathrm{Re}(\lambda)} \right)^{p+\tau} \\
&\geq C(p,\tau) \sum_{\lambda \in \sigma_d(J) \cap \Psi_2} \frac{\mathrm{dist}(\lambda, [-2,2])^{p+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}+\tau}}.
\end{aligned}$$

Similarly, for the right-hand side of (52) we obtain,

$$C(p)\|d\|_{l^p}^p \int_1^\infty \mathrm{d}x\, x^{-p-1-\tau}(1+2x)^p \leq C(p,\tau)\|d\|_{l^p}^p.$$

We have thus shown that

$$\sum_{\lambda \in \sigma_d(J) \cap \Psi_2} \frac{\mathrm{dist}(\lambda, [-2,2])^{p+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}+\tau}} \leq C(p,\tau)\|d\|_{l^p}^p. \tag{53}$$

Noting that Ψ_1 and Ψ_2 are disjoint with $\Psi_1 \cup \Psi_2 = \{\lambda : \mathrm{Re}(\lambda) > 0\}$ we conclude from (51) and (53) that

$$\sum_{\lambda \in \sigma_d(J), \mathrm{Re}(\lambda) > 0} \frac{\mathrm{dist}(\lambda, [-2,2])^{p+\tau}}{|\lambda^2 - 4|^{\frac{1}{2}+\tau}} \leq C(p,\tau)\|d\|_{l^p}^p.$$

Finally, starting with the estimate

$$\sum_{\lambda\in\sigma_d(J)\cap\Omega_\theta^-} |\lambda+2|^{p-\frac{1}{2}} \leq C(p)(1+2\tan(\theta))^p \|d\|_{l^p}^p,$$

which follows from (8), we can show in exactly the same manner as above that

$$\sum_{\lambda\in\sigma_d(J),\mathrm{Re}(\lambda)\leq 0} \frac{\mathrm{dist}(\lambda,[-2,2])^{p+\tau}}{|\lambda^2-4|^{\frac{1}{2}+\tau}} \leq C(p,\tau)\|d\|_{l^p}^p.$$

This concludes the proof of Theorem 2.

6 On a Difference Between the Behaviour of Eigenvalues of Selfadjoint and Non-Selfadjoint Jacobi Operators

In this section we construct a simple example which demonstrates a sharp contrast between selfadjoint and non-selfadjoint Jacobi operators.

We consider a Jacobi operator J with $a_k = c_k = 1$ for all k, $b_k = 0$ for $k \notin \{0, N\}$ and $b_0, b_N \neq 0$, where $N \geq 1$. In other words we allow the diagonal elements to differ from 0 in only two places. Note that J is a rank two perturbation of the free Jacobi operator J_0. In the selfadjoint case, that is when b_0, b_N are real numbers, such a rank two perturbation can have at most two eigenvalues. In contrast to this, we will show that in the non-selfadjoint case the following holds true.

Proposition 12 *J can have an arbitrarily large (but finite) number of eigenvalues.*

More precisely: fixing N, the operator J can have up to $2N$ eigenvalues, so that by increasing N we can make the number of eigenvalues arbitrarily large. This result is based on the fact that the eigenvalues of J can be characterized explicitly as the following lemma shows.

Lemma 13 *Let J be defined as above. Then the discrete spectrum of J coincides with the set*

$$\left\{ w + w^{-1} : |w| < 1, \;\; b_0 b_N (w^{2N}-1)w^2 = [w^2 + (b_0+b_N)w - 1](w^2-1) \right\}. \tag{54}$$

Proof We search for eigenvectors of J of the form

$$u(k) = \alpha w^{|k|} + \beta w^{|k-N|}, \tag{55}$$

where, in order to have $u \in l^2(\mathbb{Z})$, we must assume $0 < |w| < 1$. Applying J we obtain

$$(Ju)(k) = (w + w^{-1})u(k), \qquad k \notin \{0, N\}, \tag{56}$$
$$(Ju)(0) = \alpha(2w + b_0) + \beta[w + w^{-1} + b_0]w^N, \tag{57}$$
$$(Ju)(N) = \alpha w^N[w + w^{-1} + b_N] + \beta(2w + b_N). \tag{58}$$

From (56) we see that if u is an eigenvector of J, then the corresponding eigenvalue is $\lambda = w + w^{-1}$. Therefore from (57) and (58) we see that for u to be an eigenvector it is necessary and sufficient that

$$[(w - w^{-1}) + b_0]\alpha + b_0 w^N \beta = 0,$$
$$b_N w^N \alpha + [(w - w^{-1}) + b_N]\beta = 0,$$

which has a nontrivial solution (α, β) iff the determinant of the coefficient matrix is 0, that is

$$[w^2 - 1 + (b_0 + b_N)w](1 - w^2) - b_0 b_N w^2[1 - w^{2N}] = 0,$$

which is equivalent to the equation in (54). An additional argument, which we will skip here, shows that *any* eigenvector of J is of the form (55), so that we have found all eigenvalues. □

To make computations simpler in constructing our example, we will now take

$$b_0 = \epsilon^{-1}i, \quad b_N = -\epsilon^{-1}i, \qquad \epsilon > 0, \tag{59}$$

so that the equation determining the eigenvalues becomes

$$(w^{2N} - 1)w^2 = \epsilon^2(w^2 - 1)^2. \tag{60}$$

We will prove the following lemma.

Lemma 14 *For $\epsilon \neq 0$ sufficiently small,* (60) *has precisely $2N$ solutions satisfying $|w| < 1$.*

Thus, when we define b_0, b_N by (59) and take $\epsilon \neq 0$ sufficiently small, the corresponding operator J has $2N$ eigenvalues. As an example, Fig. 1 shows the eigenvalues of J when $N = 100$, $\varepsilon = \frac{1}{2}$, calculated by numerically solving (60). There are 200 eigenvalues, but many of them are very close to ± 2, and so cannot be distinguished in the plot.

Proof of Lemma 14 Setting $z = w^2$, $\mu = \epsilon^2$ we need to show that the equation

$$(z^N - 1)z = \mu(z - 1)^2 \tag{61}$$

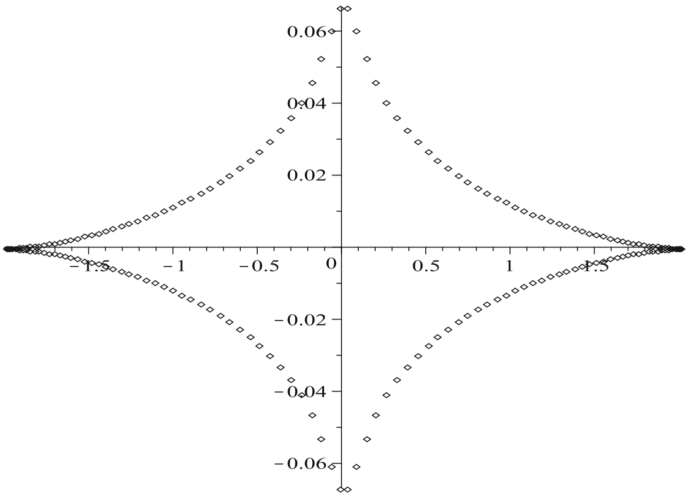

Fig. 1 Eigenvalues of J when $a_k = c_k = 1$, $b_0 = 2i$, $b_{100} = -2i$, $b_k = 0$ for $k \notin \{1, 100\}$

has N solutions satisfying $|z| < 1$ when $\mu > 0$ is sufficiently small. Note first that when $\mu = 0$, (61) has $N + 1$ solutions: $z_0(0) = 0$ and N roots of unity

$$z_k(0) = e^{\frac{2\pi k}{N} i}, \quad k = 1, \ldots, N.$$

Since all these roots are simple, the implicit function theorem implies that for $\mu \neq 0$ small each of these roots is perturbed to a root $z_k(\mu)$, depending analytically on μ. Clearly $|z_0(\mu)| < 1$ for μ small. On the other hand, since $z = 1$ is a solution of (61) for any μ, we have $z_N(\mu) \equiv 1$. It remains to show that for $k = 1, \ldots, N - 1$, $z_k(\mu)$ enter the unit disk for $\mu > 0$ small, which we do by showing that

$$\frac{\mathrm{d}}{\mathrm{d}\mu} |z_k(\mu)|^2 \Big|_{\mu=0} < 0.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}\mu} |z_k(\mu)|^2 \Big|_{\mu=0} = z_k'(0)\overline{z_k(0)} + z_k(0)\overline{z_k'(0)} = 2\,\mathrm{Re}\left(z_k'(0) e^{-\frac{2\pi k}{N} i}\right).$$

To compute $z_k'(0)$ we differentiate the relation

$$[(z_k(\mu))^N - 1] z_k(\mu) = \mu (z_k(\mu) - 1)^2,$$

with respect to μ, and then set $\mu = 0$, obtaining

$$N(z_k(0))^N z_k'(0) + [(z_k(0))^N - 1] z_k'(0) = (z_k(0) - 1)^2,$$

so that

$$z_k'(0) = \frac{1}{N}(e^{\frac{2\pi k}{N}i} - 1)^2,$$

which gives

$$\mathrm{Re}\left(z_k'(0)e^{-\frac{2\pi k}{N}i}\right) = \frac{2}{N}\left(\cos\left(\frac{2\pi k}{N}\right) - 1\right) < 0$$

for $k = 1, \ldots, N - 1$, which is what we needed to prove. □

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