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Bounds on Eigenvalues of Sturm–Liouville Problems with Discontinuous Coefficients[1])

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1. Introduction

This paper is concerned with obtaining upper and lower bounds for the eigenvalues of Sturm–Liouville problems with discontinuous coefficients. The current widespread interest in problems of this type stems from their natural appearance in diverse fields of application. Thus, for example, the propagation of harmonic waves in elastic composites with periodic structure gives rise to such Sturm–Liouville problems with quasi-periodic boundary conditions. Variational methods for these problems have been discussed by several authors (see e.g. [1–7] and the references cited therein). Finite difference and other numerical techniques have been applied [3, 8]. Vibration problems arising in geophysics also give rise to Sturm–Liouville problems with discontinuous coefficients [9, 10]. The asymptotic behavior of the eigenvalues of such a system has been considered in a recent paper [10]. The results are of major interest particularly in the context of geophysical inverse problems.

In this paper we consider the simplest prototype of such Sturm–Liouville problems ((2.1), (2.2)) arising in heat conduction in a layered composite. Our primary concern is to obtain upper and lower bounds for the eigenvalues, with emphasis on the smallest positive eigenvalue. As regards upper bounds, a natural first step is to consider approximation schemes of the classical Rayleigh–Ritz type. However, it has been shown by Nemat-Nasser and coworkers that this approach in general yields poor results, particularly when the coefficients suffer large discontinuities [1, 2], [4–7]. An alternative approximation technique, based on a modified 'new quotient' was developed in a series of papers concerned with wave propagation [1, 2], [4–6] and heat conduction [7]. Extremely accurate results were obtained for a wide variety of problems. Upper and lower bounds for eigenvalues were obtained using a Rayleigh–Ritz type approximation scheme based on the new quotient. Mathematical foundations for the numerical analysis underlying the method have been investigated in a recent paper by Babuška and Osborn [11].

Here we use somewhat different approaches to the question of eigenvalue

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estimation for such Sturm–Liouville problems. One of the key ideas is to transform the basic system (2.1), (2.2) to the Liouville normal form using the analog of the classical Liouville transformation (Section 3). The advantage of this transformation lies in the fact that it leads to a Sturm–Liouville problem with a *single* discontinuous coefficient which moreover occurs in an *undifferentiated* term. The Rayleigh–Ritz method is then formulated for the transformed problem (Section 4) to yield upper bounds. Illustrative examples discussed in Sections 6 and 7 demonstrate the accuracy of the resulting upper bounds. In Section 5 we establish two separate lower bounds for the first eigenvalue based on known results for classical Sturm–Liouville problems. These bounds are given explicitly in terms of the coefficients. The upper and lower bounds are compared with the exact eigenvalues for particular examples in Section 7 and the accuracy of the results are assessed.

2. Sturm–Liouville Problems with Discontinuous Coefficients

We consider the eigenvalue problem

$$(\kappa u')' + \lambda c u = 0, \quad 0 < x < 1, \tag{2.1}$$

$$u(0) = 0, \qquad u(1) = 0, \tag{2.2}$$

where $\kappa(x)$, $c(x)$ are positive functions, bounded on $[0, 1]$. We assume that the coefficients $\kappa(x)$, $c(x)$ have step discontinuities at a finite set of points $x_1, x_2, \ldots, x_n$ on $(0, 1)$, are continuous elsewhere and are such that the eigenvalue problem (2.1), (2.2) admits an infinite set of distinct eigenvalues $0 < \lambda_1 < \lambda_2 \cdots$. Eigenvalue problems of this type occur naturally in many areas of application; for example, in the context of heat conduction in layered composites, $\kappa(x)$ is the heat conductivity and $c(x)$ the heat capacity, both subject to possibly large discontinuities at material interfaces.

3. Liouville Transformation

For Sturm–Liouville problems with smooth coefficients, it is well known that an appropriate change of dependent and independent variables leads to a simplified problem, the Liouville normal form [12]. For problems with discontinuous coefficients, such a transformation was employed in a recent paper [10] to investigate the asymptotic behavior of the eigenvalues. In what follows, we carry out an analogous transformation for the problem (2.1), (2.2).

Let

$$T = \int_0^1 \kappa^{-1}(s)\, ds, \qquad t = T^{-1} \int_0^x \kappa^{-1}(s)\, ds, \qquad v(t) = u(x(t)), \tag{3.1}$$

$$f(t) = T^2 \kappa(x(t)) c(x(t)). \tag{3.2}$$

Then the eigenvalues λ_i $(i = 1, 2, \ldots)$ are the eigenvalues of the system

$$\ddot{v} + \lambda f v = 0, \quad 0 < t < 1, \tag{3.3}$$

$$v(0) = 0, \qquad v(1) = 0, \tag{3.4}$$

where the superposed dot represents differentiation with respect to t. The coefficient $f(t)$ is positive and bounded on [0, 1] and has discontinuities at the points

$$t_i = T^{-1}\left(\int_0^{x_i} \kappa^{-1}(s)\, ds\right). \tag{3.5}$$

The effect of the foregoing transformation has been to remove the discontinuous coefficient κ from its position subject to differentiation in (2.1). As we shall see in the sequel, this has a significant effect on the accuracy of eigenvalue estimates for the Sturm–Liouville problems of concern here. The points of discontinuity have also been transformed according to the relation (3.5). We shall discuss later how this feature may be used to advantage in the analysis of such problems.

4. Upper Bounds for Eigenvalues: Rayleigh Quotients and New Quotient

For the eigenvalue problem (2.1), (2.2), the Rayleigh quotient is given by

$$\lambda_R = \langle \kappa u', u' \rangle / \langle cu, u \rangle, \tag{4.1}$$

where the inner product of two functions $g(x)$, $h(x)$ is defined by

$$\langle g, h \rangle = \int_0^1 gh\, dx. \tag{4.2}$$

Upper bounds for the eigenvalues λ_i may be obtained by applying approximation schemes of the classical Rayleigh–Ritz type to the quotient in (4.1). As is discussed in [1, 2], [4–7] such an approach yields good estimates for the exact eigenvalues when κ is smooth. However, when κ admits large discontinuities, which is often the case for composites, this technique is very ineffective.

It is shown in [7] that the Rayleigh quotient (4.1) may be given a *dual* formulation which yields accurate results when c is smooth and κ is not. Thus set

$$-D\sigma = u', \qquad D = \frac{1}{\kappa}, \qquad R = \frac{1}{c}, \tag{4.3}$$

and (2.1), (2.2) yields

$$R\sigma' = \lambda u, \qquad (R\sigma')' + \lambda D\sigma = 0, \quad 0 < x < 1, \tag{4.4}$$

$$\sigma'(0) = 0, \qquad \sigma'(1) = 0. \tag{4.5}$$

The eigenvalue problem (4.4), (4.5) has the same eigenvalues as those of (2.1), (2.2) but different eigenfunctions. The corresponding dual[3]) Rayleigh quotient is

$$\bar{\lambda}_R = \langle R\sigma', \sigma' \rangle / \langle D\sigma, \sigma \rangle. \tag{4.6}$$

[3]) Clearly, the dual of the dual problem is the original problem.

The method of the 'new quotient' [1, 2], [4–7] combines the two Rayleigh quotients discussed above and yields accurate results when both κ and c are discontinuous; for the problem (2.1), (2.2), this method is described in detail in [7]. The new quotient is given by

$$\lambda_N = (2\langle -\sigma, u'\rangle - \langle D\sigma, \sigma\rangle)/\langle cu, u\rangle. \tag{4.7}$$

The major disadvantage in using (4.7) lies in the fact that this quotient gives rise to a stationary principle and not an extremum principle and thus the eigenvalue estimates are not necessarily upper bounds; it is possible however to utilize λ_N in obtaining upper and lower bounds for the eigenvalues [7]. Furthermore, when κ = constant, (4.7) yields upper bounds for the eigenvalues identical to those obtained with (4.1), while if c = constant, upper bounds identical to those obtained with (4.6) are found [7]. Babuška and Osborn [11] have recently demonstrated improved convergence for the new quotient method over that of the usual Rayleigh–Ritz technique.

In this paper, rather than use (4.7) we wish to utilize the Rayleigh quotient associated with the transformed problem (3.3), (3.4). Thus we consider

$$\lambda_{RL} = \langle \dot{v}, \dot{v}\rangle/\langle fv, v\rangle. \tag{4.8}$$

Hence, even when κ and c are discontinuous, since these quantities appear only in the denominator in (4.8) we anticipate that (4.8) should yield more accurate upper bounds for the eigenvalues than (4.1).

We may also apply a transformation analogous to (3.1), (3.2) to the dual problem (4.4), (4.5). Thus corresponding to (3.3), (3.4) we obtain

$$\ddot{w} + \lambda g w = 0, \quad 0 < t < 1, \tag{4.9}$$

$$\dot{w}(0) = 0, \qquad \dot{w}(1) = 0, \tag{4.10}$$

where

$$g = \hat{T}^2 RD = \frac{\hat{T}^2}{\kappa c}, \qquad \hat{T} = \int_0^1 c(s)\, ds, \qquad t = \hat{T}^{-1}\int_0^x c(s)\, ds. \tag{4.11}$$

The dual Rayleigh quotient associated with (4.9), (4.10) is given by

$$\bar{\lambda}_{RL} = \langle \dot{w}, \dot{w}\rangle/\langle gw, w\rangle. \tag{4.12}$$

5. Lower Bounds for λ_1

Here we wish to obtain lower bounds for the smallest positive eigenvalue λ_1 for the system (2.1), (2.2) or equivalently (3.3), (3.4). We denote these quantities by $\lambda_1(\kappa, c)$, $\lambda_1(f)$ respectively. By virtue of (2.1), (2.2),

$$\lambda_1(\kappa, c) = \langle \kappa u', u'\rangle/\langle cu, u\rangle, \tag{5.1}$$

where $u(x)$ is the eigenfunction corresponding to the eigenvalue λ_1. From (5.1) it follows that

$$\lambda_1(\kappa, c) \geq \kappa_m\langle u', u'\rangle / c_M\langle u, u\rangle, \tag{5.2}$$

where κ_m, c_M are defined by

$$\kappa_m = \inf_{(0,1)} \kappa, \qquad c_M = \sup_{(0,1)} c. \tag{5.3}$$

We are assured that $\kappa_m > 0$, $c_M > 0$ since κ, c are positive and bounded on $[0, 1]$. It follows from (5.2) that

$$\lambda_1(\kappa, c) \geq (\kappa_m/c_M) \inf (\langle h', h'\rangle / \langle h, h\rangle), \tag{5.4}$$

where the infimum on the right in (5.4) is taken over all piecewise continuously differentiable functions satisfying $h(0) = 0$, $h(1) = 0$. By the usual variational characterization of the eigenvalue $\lambda_1(1, 1)$ we may write (5.4) as

$$\lambda_1(\kappa, c) \geq (\kappa_m/c_M)\lambda_1(1, 1) \tag{5.5}$$

which is one of the lower bounds we require. Of course we have $\lambda_1(1, 1) = \pi^2$. Lower bounds of the form (5.5) are well known for classical Sturm–Liouville systems (see e.g. [13]).

The second lower bound we consider follows from an inequality due to Liapunov applied to the system (3.3), (3.4). Thus, if $f(t)$ is non-negative and integrable on $[0, 1]$, Liapunov's inequality (see e.g. [14]) reads

$$\lambda_1(f) \geq 4\Big/\int_0^1 f(t)\,dt. \tag{5.6}$$

The constant 4 in (5.6) is best possible as was shown by Hartman and Wintner [15]; see also Krein [16]. It is of interest to note that the particular function $f(t)$ given in [16] to yield equality in (5.6) has discontinuities in $(0, 1)$. Thus it might be expected that the lower bound (5.6) should be of particular interest in the analysis of Sturm–Liouville problems with discontinuous coefficients.

6. Illustrative Examples

To assess the accuracy of the upper and lower bounds discussed in Sections 4 and 5, we now consider some specific examples. The case of heat conduction through a composite composed of two identical homogeneous outer layers enclosing an inner homogeneous layer has been treated in [7]. Thus the coefficients $\kappa(x)$, $c(x)$ in (2.1) are piecewise constant and are given by

$$c_1, \kappa_1 \quad \text{in } 0 \leq x < (1 - b)/2,\ (1 + b)/2 < x \leq 1, \tag{6.1}$$

$$c_2, \kappa_2 \quad \text{in } (1 - b)/2 \leq x \leq (1 + b)/2. \tag{6.2}$$

The eigenvalue problem (2.1), (2.2) with coefficients given by (6.1), (6.2) has been solved exactly in [7]. Solutions to Eqn. (2.1) were obtained in each region, satisfying the boundary condition (2.2) and then matched by ensuring continuity of u and $\kappa u'$ at the interfaces. In this way, transcendental equations governing the exact eigenvalues were found and solved numerically for various combinations of the constants c_β, κ_β, $\beta = 1, 2$. Some of the results obtained for λ_1 will be presented in Section 7.

It is convenient to introduce the following notation:

$$\gamma = \frac{\kappa_2}{\kappa_1}, \qquad \theta = \frac{c_2}{c_1}, \qquad n_1 = 1 - b, \qquad n_2 = b, \\ \bar{\kappa} = n_1\kappa_1 + n_2\kappa_2 = 1, \qquad \bar{c} = n_1 c_1 + n_2 c_2 = 1, \tag{6.3}$$

and so

$$c_1 = (n_1 + n_2\theta)^{-1}, \qquad c_2 = \theta(n_1 + n_2\theta)^{-1}; \qquad \kappa_1 = (n_1 + n_2\gamma)^{-1}, \\ \kappa_2 = \gamma(n_1 + n_2\gamma)^{-1}. \tag{6.4}$$

The corresponding dimensionless eigenvalue is then denoted by ν and given by

$$\nu \equiv (\lambda\bar{c}/\bar{\kappa})^{1/2} = \lambda^{1/2}. \tag{6.5}$$

For given values of the geometric parameters n_1, n_2, the effect of the material discontinuities on ν is conveniently analyzed through consideration of the dependence of ν on the dimensionless material parameters γ and θ. For continuous conductivities $\gamma = 1$ while for continuous capacities $\theta = 1$.

Application of the Rayleigh–Ritz method based on the Rayleigh quotients (4.1), (4.6) to the above example problem has been carried out in [7]. (See also Section 7 of the present paper). As might be anticipated, continuous test functions satisfying the boundary conditions (2.2) do not yield very accurate results, the underlying reason being that the exact eigenfunctions must be such that u' is discontinuous in such a way that $\kappa u'$ is continuous. Improved test functions satisfying these continuity properties were also utilized leading to more accurate bounds. Approximate techniques based on the new quotient (4.7) have the advantage of having such continuity properties naturally built into the method. Thus extremely accurate results are obtained requiring much less computational effort than the use of the improved functions.

The new upper bounds to be considered in the present paper will be obtained by employing a direct Rayleigh–Ritz approximate scheme based on the quotients (4.8), (4.12) associated with the Liouville normal form (3.3). Continuous test functions of the form

$$v = \sum_{n=1}^{M} v_n \varphi_n(t), \tag{6.6}$$

are used, where

$$\varphi_n(t) = \sin n\pi t. \tag{6.7}$$

For the purposes of illustration, in this paper we shall confine our attention to results for the smallest eigenvalue λ_1, or equivalently, ν_1. It will be seen in the next section that extremely accurate upper bounds are obtained with minimal computational effort using a three-term approximation (6.6).

Before considering the lower bounds (5.5) and (5.6) for our example problem, it is of interest to discuss some properties of the Liouville transformation (3.1), (3.2) in this case. For convenience we let $b = \frac{1}{2}$ $(n_1 = n_2 = \frac{1}{2})$ and so the jumps in κ, c occur at $x_1 = \frac{1}{4}$ and $x_2 = \frac{3}{4}$. From (3.1), (6.1), and (6.2), we obtain

$$T = (\gamma + 1)^2/4\gamma, \tag{6.8}$$

and thus from (3.5), it follows that the points of discontinuity for $f(t)$ are given by

$$t_1 = \frac{1}{2} - \frac{1}{2(\gamma + 1)}, \qquad t_2 = \frac{1}{2} + \frac{1}{2(\gamma + 1)}. \tag{6.9}$$

We observe that t_β $(\beta = 1, 2)$ are independent of θ. The magnitude of the discontinuity in f is given by

$$T^2(\kappa_2 c_2 - \kappa_1 c_1) = (\gamma + 1)^3(\gamma\theta - 1)/4\gamma^2(1 + \theta). \tag{6.10}$$

For increasing relative jumps in the conductivity κ, that is, for increasing values of γ, we see from (6.9) that the effect of the transformation is to move the location of discontinuities closer together in the t coordinate system. Furthermore the magnitude of the discontinuity increases with γ. This suggests that for the problem (2.1), (2.2) in the case of large γ, it may be advantageous to consider the transformed problem (3.3), (3.4) as governing the oscillations of a vibrating string with mass density concentrated at its center point. Using the dual formulation (4.9), (4.10) analogous interpretations are possible in the case of large values of θ. Such analogies and their implications for the basic problem (2.1), (2.2) will be discussed in detail elsewhere.

We now return to the lower bounds (5.5), (5.6). For convenience, it is again assumed that $n_1 = n_2 = 1/2$. From (6.4) and the definitions of κ_m, c_M in (5.3) we obtain (for $\theta \geq 1$, $\gamma \geq 1$)

$$\kappa_m = 2/(1 + \gamma), \qquad c_M = 2/(1 + \theta^{-1}), \tag{6.11}$$

and so the lower bound (5.5) yields

$$\nu_1 \geq \pi[(1 + \theta^{-1})/(1 + \gamma)]^{1/2}. \tag{6.12}$$

By virtue of the definition of $f(t)$ in (3.2), and using (6.1)–(6.4), (6.8) and (6.9), we obtain from (5.6) the second lower bound

$$\nu_1 \geq 4\gamma^{1/2}/(\gamma + 1). \tag{6.13}$$

The striking feature of the bound in (6.13) is its independence of the parameter θ. This aspect and the comparative merit of (6.12) and (6.13) will be discussed in conjunction with numerical results to be given in the next section.

7. Numerical Results and Discussion

Here we present some numerical results for the first eigenvalue ν_1 for the example problem described in the previous section. In Table 1 we have listed the exact eigenvalues ν_1 for various combinations of γ and θ, together with four alternative upper bounds calculated from a three-term Rayleigh–Ritz type approximation based on the quotients (4.1), (4.6), (4.8) and (4.12). The results for the exact eigenvalues and the first two of the upper bounds are taken from [7]. The improved accuracy resulting from using the quotients (4.8), (4.12) based on the Liouville normal form is evident.

The lower bounds furnished by (6.12), (6.13) are also given in Table 1. When either $\gamma = 1$ (continuous κ) or $\theta = 1$ (continuous c), we see that (6.12) yields the better lower bound. For other values of γ and θ, (6.13) gives a sharper result.

It is clear that when $\gamma = \theta = 1$, all estimates (except (6.13)) in Table 1 coincide with the exact value π. We also note that when $\gamma = 1$, the results of (4.1) and (4.8) coincide, and, similarly, when $\theta = 1$, the results of (4.6) and (4.12) become identical. Finally, when $\gamma = \theta$, the results of (4.1) and (4.6) coincide, as well as the results of (4.8) and (4.12).

We examine these features of the lower bounds in more detail. We write $\nu_1 \equiv \nu_1(\gamma, \theta)$ to emphasize the dependence of ν_1 on the material parameters γ and θ. From the standard Sturm comparison theory, it can be shown that for fixed γ, $\nu_1(\gamma, \theta)$ is monotonic decreasing in θ, and for fixed θ, $\nu_1(\gamma, \theta)$ is monotonic decreasing in γ.

Table 1
The smallest eigenvalue ν_1 ($n_1 = n_2 = \frac{1}{2}$, $M = 3$)

γ \ θ		1	10	100
1	$\bar{\nu}_{RL}$	3.142	2.560	2.505
	ν_{RL}	3.142	2.530	2.444
	$\bar{\nu}_{R}$	3.142	2.691	3.621
	ν_{R}	3.142	2.530	2.444
	Exact ν_1	3.142	2.529	2.443
	(6.13)	2.000	2.000	2.000
	(6.12)	3.142	2.330	2.233
10	$\bar{\nu}_{RL}$	1.455	1.256	1.246
	ν_{RL}	1.472	1.256	1.223
	$\bar{\nu}_{R}$	1.455	1.328	1.807
	ν_{R}	1.547	1.328	1.294
	Exact ν_1	1.454	1.225	1.190
	(6.13)	1.150	1.150	1.150
	(6.12)	1.340	.993	.952
100	$\bar{\nu}_{RL}$	.484	.421	.419
	ν_{RL}	.496	.429	.419
	$\bar{\nu}_{R}$	.484	.446	.608
	ν_{R}	.717	.622	.608
	Exact ν_1	.484	.410	.399
	(6.13)	.396	.396	.396
	(6.12)	.442	.328	.314

Table 2
The smallest eigenvalue ν_1 ($n_1 = n_2 = \frac{1}{2}$, $M = 3$)

	Exact	Upper Bounds				Lower Bounds	
	ν_1	ν_R	$\bar{\nu}_R$	ν_{RL}	$\bar{\nu}_{RL}$	(6.13)	(6.12)
$\gamma = 3$ $\theta = 100$	1.902	1.955	2.863	1.912	1.977	1.732	1.579
$\gamma = 100$ $\theta = 3$	.435	.655	.447	.452	.437	.396	.361
$\gamma = 100$ $\theta = 100$	.399	.608	.608	.419	.419	.396	.314

The lower bound (6.12) also has these monotonicity properties. The lower bound (6.13) is independent of θ; it is, however, monotonic decreasing in γ. We see from the table that for fixed γ, the lower bound (6.13) yields sharper results with increasing values of θ. (The exact values of ν_1 are relatively insensitive to variations in θ for fixed γ. This is not the situation for the case when θ is fixed.)

In Table 2, we present in summary form the upper and lower bounds obtained for some representative pairs (γ, θ). Extremely high accuracy is obtained on using the best results from the alternative available bounds.

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Added in Proof

Since this paper was completed, sharper lower bounds than those presented in Section 5 have been obtained. For these, and a discussion of the relation with eigenvalue optimization problems (Krein [16]), see authors' paper: "Variational Methods for Eigenvalue Problems in Composites," Proceedings of the IUTAM Symposium on *Variational Methods in the Mechanics of Solids*, Northwestern University, Evanston, Illinois, Sept. 1978, Pergamon Press (in press).

Abstract

This paper is concerned with obtaining upper and lower bounds for the eigenvalues of Sturm-Liouville problems with discontinuous coefficients. Such problems occur naturally in many areas of composite material mechanics.

The problem is first transformed by using an analog of the classical Liouville transformation. Upper bounds are obtained by application of a Rayleigh-Ritz technique to the transformed problem. Explicit lower bounds in terms of the coefficients are established. Numerical examples illustrate the accuracy of the results.

Résumé

Dans cet article les bornes supérieures et inférieures sont déterminées pour les valeurs caractéristiques des problèmes de Sturm-Liouville avec des coefficients discontinus. De tels problèmes se trouvent naturellement dans la mécanique des materiaux composites.

Après avoir transformé ce problème en utilisant un analogue de la transformation classique de Liouville, les bornes supérieures sont obtenues par l'application d'une technique de Rayleigh-Ritz au problème transformé. Les bornes inférieurs sont determinées en fonction des coefficients sous une forme explicite. Quelques exemples numériques montrent l'exactitude des résultats.

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