Differential Geometry - Exercise 3

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1. *Compute the evolute of the cycloid* $\gamma(t) = (t - \sin t, 1 - \cos t)$. *Show that the evolute is a congruent copy of the cycloid.*

   Recall that the evolute is defined as $\mathcal{E}(t) = \gamma(t) + \frac{1}{\kappa_s(t)}\nu_s(t)$. From the first exercise we know that
   $$\kappa_s(t) = -\frac{1}{\sqrt{8}}\frac{1}{\sqrt{1-\cos t}}.$$
   We can compute $\nu_s(t)$ as follows:
   $$\nu_s(t) = \frac{\dot\gamma(t)^\perp}{\|\dot\gamma\|} = \frac{(-\sin t, -\cos t + 1)}{\sqrt{2}\sqrt{1-\cos t}}$$
   Thus (with $s := t - \pi$):
   $$\begin{aligned}\mathcal{E}(t) &= \gamma(t) + 2(\sin t, \cos t - 1)\\ &= (t + \sin t, -1 + \cos t)\\ &= (s + \pi - \sin s, -1 - \cos s)\\ &= (s - \sin s + \pi, 1 - \cos s - 2)\\ &= \gamma(s) + (\pi, -2)\end{aligned}$$
   The evolute of the cycloid is therefore the cycloid translated by $(\pi, -2)$. □

2. *Consider the curve* $\gamma(t) = (t, \cosh t)$ *(the* catenary*). Show that the involute of* γ *with the starting point* $\gamma(0)$ *is an arc of the tractrix. (One of the parametrizations of the tractrix:* $(\log\tan\frac{s}{2} + \cos s, \sin s)$.)

   The involute of γ is defined as:
   $$\mathcal{I}_\gamma(t) = \gamma(t) - \mathcal{L}_0^t(\gamma)\frac{\dot\gamma(t)}{\|\dot\gamma(t)\|}$$
   The occurring terms are:
   $$\begin{aligned}\dot\gamma(t) &= (1, \sinh t)\\ \|\dot\gamma(t)\| &= \sqrt{1 + \sinh^2 t} = \sqrt{\cosh^2 t} = \cosh t\\ \mathcal{L}_0^t(\gamma) &= \int_0^t \cosh s\ ds = \sinh t\end{aligned}$$
   Thus:
   $$\begin{aligned}\mathcal{I}_\gamma(t) &= (t, \cosh t) - \sinh t\frac{(1, \sinh t)}{\cosh t}\\ &= \left(t - \tanh t, \cosh t - \frac{\sinh^2 t}{\cosh t}\right)\\ &= \left(t - \tanh t, \frac{\cosh^2 t}{\cosh t} - \frac{\cosh^2 t - 1}{\cosh t}\right)\\ &= \left(t - \tanh t, \frac{1}{\cosh t}\right)\end{aligned}$$

The tractrix is defined as the curve whose tangents meet the x-axis in unit distance, so we have to show

$$\mathcal{I}_\gamma(t) + 1 \cdot \frac{\dot{\mathcal{I}}_\gamma(t)}{\|\dot{\mathcal{I}}_\gamma(t)\|} = (t, 0)$$

Starting from the left side, we calculate the occurring terms:

$$\begin{aligned}
\dot{\mathcal{I}}_\gamma(t) &= \left(1 - (1 - \tanh^2 t), \frac{-\sinh t}{\cosh^2 t}\right) \\
&= \left(\tanh^2 t, \frac{-\tanh t}{\cosh t}\right) \\
\|\dot{\mathcal{I}}_\gamma(t)\| &= \sqrt{\tanh^4 t + \frac{\tanh^2 t}{\cosh^2 t}} \\
&= \tanh t \cdot \sqrt{\frac{\sinh^2 t}{\cosh^2 t} + \frac{1}{\cosh^2 t}} \\
&= \tanh t \\
\frac{\dot{\mathcal{I}}_\gamma(t)}{\|\dot{\mathcal{I}}_\gamma(t)\|} &= \left(\tanh t, -\frac{1}{\cosh t}\right)
\end{aligned}$$

Therefore:

$$\mathcal{I}_\gamma(t) + 1 \cdot \frac{\dot{\mathcal{I}}_\gamma(t)}{\|\dot{\mathcal{I}}_\gamma(t)\|} = \left(t - \tanh t + \tanh t, \frac{1}{\cosh t} - \frac{1}{\cosh t}\right) = (t, 0)$$

Thus, the involute of the catenary is indeed the tractrix. $\square$

3. *Let γ be a curve with monotonic and nowhere vanishing curvature. Show that involutes of the evolute γ are curves parallel to γ.*

   We have the evolute and the involute of the evolute as:

$$\begin{aligned}
\mathcal{E}_\gamma &= \gamma + \frac{1}{\kappa_s}\nu_s \\
\mathcal{I}_{\mathcal{E}_\gamma, t_0} &= \mathcal{E}_\gamma - \mathcal{L}_{t_0}^t(\mathcal{E}_\gamma)\frac{\dot{\mathcal{E}}_\gamma}{\|\dot{\mathcal{E}}_\gamma\|}
\end{aligned}$$

   Lets calculate the occurring terms (using $\dot{\nu}_s = -\kappa_s\dot{\gamma}$):

$$\begin{aligned}
\dot{\mathcal{E}}_\gamma &= \dot{\gamma} + \frac{\dot{\nu}_s(t)\kappa_s - \nu_s\dot{\kappa}_s}{\kappa_s^2} \\
&= \dot{\gamma} - \frac{\kappa_s^2\dot{\gamma}}{\kappa_s^2} - \frac{\dot{\kappa}_s\nu_s}{\kappa_s^2} \\
&= -\frac{\dot{\kappa}_s\nu_s}{\kappa_s^2} \\
\|\dot{\mathcal{E}}_\gamma\| &= \left|\frac{\dot{\kappa}_s}{\kappa_s^2}\right| \cdot \|\nu_s\| = \left|\frac{\dot{\kappa}_s}{\kappa_s^2}\right|
\end{aligned}$$

   Without loss of generality assume that $\kappa_s = \kappa > 0$ and $\dot{\kappa} > 0$. Mirror and/or reverse the curve if necessary. This is valid because of the monotonic and nowhere vanishing curvature.

$$
\begin{aligned}
\mathcal{L}_{t_0}^t(\mathcal{E}_\gamma) &= \int_{t_0}^t \frac{\dot{\kappa}}{\kappa^2} dt = -\frac{1}{\kappa}\Big|_{t_0}^t = \frac{1}{\kappa(t_0)} - \frac{1}{\kappa(t)} \\
\mathcal{I}_{\mathcal{E}_\gamma, t_0} &= \gamma + \frac{1}{\kappa}\nu_s - \left(\frac{1}{\kappa(t_0)} - \frac{1}{\kappa(t)}\right)\nu_s \\
&= \gamma + \frac{1}{\kappa(t_0)}\nu_s
\end{aligned}
$$

Therefore the involute of the evolute is indeed a parallel curve with the distance being the radius of the osculation circle in the starting point of the evolute. □

4. *Prove the Tait-Kneser theorem: the osculating circles of a curve with a monotonic non-vanishing curvature are disjoint and nested. (*Hint: *two circles are disjoint and nested if and only if the distance between their centers is smaller than the difference of their radii.)*

   Consider the two osculating circles at t_1 and t_2. Their centers are connected by the evolute of γ, whose length has been calculated in the previous example. This length is of course longer than the distance between the centers. Thus we have:

$$
\|c_1 - c_2\| < \mathcal{L}_{t_1}^{t_2}(\mathcal{E}_\gamma) = \left| \frac{1}{\kappa(t_1)} - \frac{1}{\kappa(t_2)} \right| = |r_1 - r_2|
$$

□

5. Oof.

6. Big oof.