Differential Geometry - Exercise 10

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June 12, 2020

1. From page 101 in the lecture notes we know that we can identify X, Y, Z with their corresponding derivations and write those as:

$$\begin{aligned} D_X &= u^1 \frac{\partial}{\partial x^1} + u^2 \frac{\partial}{\partial x^2} \\ D_Y &= v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} \\ D_Z &= w^1 \frac{\partial}{\partial x^1} + w^2 \frac{\partial}{\partial x^2} \end{aligned}$$

with $u^i, v^i, w^i \in C^\infty(\mathbb{R}^2)$. Let $u^1(x^1, x^2) = x^2$ and $w^2(x^1, x^2) = x^1$ as well as $u^2 = v^1 = v^2 = w^1 = 0$. The commutators are then

$$\begin{aligned} [X, Y] &= [X, 0] = 0 \\ [Y, Z] &= [0, Z] = 0 \\ [X, Z] &= \sum_{j=1}^{2} \sum_{i=1}^{2} \left(u^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial u^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \\ &= x^2 \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^1} \neq 0 \end{aligned}$$

Letting $Y = 0$ might be cheesy, but it is technically legal, the best kind of legal. $\square$

2. Let t be said linear map:

$$\begin{aligned} t : \mathbb{R}^{V^* \times V} &\to \mathbb{R} \\ \sum_i a_i (l_i, v_i) &\mapsto \sum_i a_i \langle l_i, v_i \rangle \end{aligned}$$

Note: on the left is a formal linear combination of basis vectors of $\mathbb{R}^{V^* \times V}$, on the right is a plain sum in $\mathbb{R}$.

(a) Ω is defined to be the linear hull of four types of elements. Since t is linear, it suffices to show that these elements vanish under t.

$$\begin{aligned} t\left((l_1 + l_2, v) - (l_1, v) - (l_2, v)\right) &= \langle l_1 + l_2, v \rangle - \langle l_1, v \rangle - \langle l_2, v \rangle \\ &= \langle 0, v \rangle = 0 \\ t\left((l, v_1 + v_2) - (l, v_1) - (l, v_2)\right) &= \langle l, v_1 + v_2 \rangle - \langle l, v_1 \rangle - \langle l, v_2 \rangle \\ &= \langle l, 0 \rangle = 0 \\ t\left((\lambda l, v) - \lambda (l, v)\right) &= \langle \lambda l, v \rangle - \lambda \langle l, v \rangle \\ &= (\lambda - \lambda) \cdot \langle l, v \rangle = 0 \\ t\left((l, \lambda v) - \lambda (l, v)\right) &= \langle l, \lambda v \rangle - \lambda \langle l, v \rangle \\ &= (\lambda - \lambda) \cdot \langle l, v \rangle = 0 \end{aligned}$$

It all essentially follows from the bilinearity of the canonical pairing. The universal property of the Tensor Product now guarantees us that the unique existence of a function $\tilde{t} =: tr$:

$$\begin{aligned} tr:\ & V^* \otimes V \to \mathbb{R} \\ & l \otimes v \mapsto \langle l, v \rangle \end{aligned}$$

(b) Just calculate:

$$\begin{aligned} tr(A) &= tr(a^i_j e_i \otimes \eta^j) && \\ &= a^i_j tr(e_i \otimes \eta^j) && \text{by linearity of } tr \\ &= a^i_j \langle e_i, \eta^j \rangle && \text{by definition of } tr \\ &= a^i_j \delta^j_i && \text{by definition of dual basis} \\ &= a^i_i && \text{by summation along the diagonal} \end{aligned}$$

□

3. Hilbert's third problem as an exercise, nice.