Differential Geometry - Exercise 11

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1. (a) Lemma 5.55 with $\alpha := \beta := \omega$ and $r := q$ yields

$$\omega \wedge \omega = (-1)^{q^2} \omega \wedge \omega = -\omega \wedge \omega$$

Therefore $\omega \wedge \omega = 0$. □

(b) Let e_1, e_2, e_3, e_4 be a basis of $\Omega^1(\mathbb{R}^4)$. Let $\omega := (e_1 \wedge e_2) + (e_3 \wedge e_4)$. Then

$$\omega \wedge \omega = 2(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \neq 0$$

2. *Disclaimer:* This is probably a false solution as I actually disprove what is to be proven. I still want to show my thought process though as I cannot find an error in my reasoning.

Step 1: What is dx? Definition 5.58 of d tells us dx is the differential of x. The differential as defined in 2.28 is

$$dx_p(v) = \frac{d(x \circ \gamma)}{dt}(t_0)$$

with $\gamma(t_0) = p$ and $\dot{\gamma}(t_0) = v$. Let us set $t_0 = 0$ and $\gamma(t) = p + t \cdot v$. Then

$$dx_p(v) = \frac{d(x(p + t \cdot v))}{dt}(0) = x(v)$$

So dx is the functional that reads the x-coordinate of a vector. dy and dz are analogue.

Step 2: What is $dx \wedge dy$? Example 5.54 tells us

$$(dx \wedge dy)(v, w) = dx(v)dy(w) - dx(w)dy(v)$$

If we write v and w in the xyz-basis, we can write this bilinear form in matrix style.

$$(dx \wedge dy)(v, w) = v^T \cdot \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot w$$

The other combinations are analogue.

Step 3: What is ω? For every point $p = (x, y, z)$ there is a bilinear form ω_p which we can again write in matrix form.

$$\omega_p(v, w) = v^T \cdot \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} \cdot w$$

Let us call that matrix B. (Note that for every bilinear form σ on $\mathbb{R}^3$ there exits a point p such that $\sigma = \omega_p$.) We can write the pullback under a rotation A as

$$A^*\omega_p(v, w) = \omega_p(Av, Aw) = (Av)^T \cdot B \cdot Aw = v^T \cdot A^T BA \cdot w$$

We thus have to show that $A^T BA = B$ (or equivalently $A^{-1}BA = B$). But pick your favorite rotation matrix and you will see this is not the case.[1]

[1] `https://www.wolframalpha.com/input/?i=%28%280%2C0%2C1%29%2C%281%2C0%2C0%29%2C%280%2C1%2C0%29%29%5E-1+*+%28%280%2C+z%2C-y%29%2C%28-z%2C0%2Cx%29%2C%28y%2C-x%2C0%29%29+*+%28%280%2C0%2C1%29%2C%281%2C0%2C0%29%2C%280%2C1%2C0%29%29`

3. Because mixing postfix and prefix notation is extremely messy and in music sharps and flats are prefixed anyway, I redefine them as $\flat X := X^\flat$ and $\sharp\omega := \omega^\sharp$. This allows us to skip brackets altogether. Screw you, Marcel Berger.

   (a) • Gradient:

$$\begin{aligned}
\sharp df &= \sharp(\frac{\partial f}{\partial x_1}dx^1 + \frac{\partial f}{\partial x_2}dx^2 + \frac{\partial f}{\partial x_3}dx^3) \\
&= \frac{\partial f}{\partial x_1}\partial_1 + \frac{\partial f}{\partial x_2}\partial_2 + \frac{\partial f}{\partial x_3}\partial_3 \\
&= (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}) \\
&= \mathrm{grad} f
\end{aligned}$$

   • Divergence:

$$\begin{aligned}
d\flat X &= *d*\flat(a\ \partial_1 + b\ \partial_2 + c\ \partial_3) \\
&= *d*(a\ dx^1 + b\ dx^2 + c\ dx^3) \\
&= *d(a(dx^2 \wedge dx^3) + b(dx^3 \wedge dx^1) + c(dx^1 \wedge dx^2)) \\
&= *(da \wedge dx^2 \wedge dx^3 + a \wedge d(dx^2 \wedge dx^3) + ...) \\
&= *((\frac{\partial a}{\partial x_1}dx^1 + \frac{\partial a}{\partial x_2}dx^2 + \frac{\partial a}{\partial x_3}dx^3) \wedge dx^2 \wedge dx^3 + 0 + ...) \\
&= *(\frac{\partial a}{\partial x_1}dx^1 \wedge dx^2 \wedge dx^3 + ...) \\
&= \frac{\partial a}{\partial x_1} + \frac{\partial b}{\partial x_2} + \frac{\partial c}{\partial x_3} = \mathrm{div} X
\end{aligned}$$

   • Curl:

$$\begin{aligned}
\sharp*d\flat X &= \sharp*d(a\ dx^1 + b\ dx^2 + c\ dx^3) \\
&= \sharp*(da \wedge dx^1 + a \wedge ddx^1 + ...) \\
&= \sharp*((\frac{\partial a}{\partial x_1}dx^1 + \frac{\partial a}{\partial x_2}dx^2 + \frac{\partial a}{\partial x_3}dx^3) \wedge dx^1 + 0 + ...) \\
&= \sharp*(\frac{\partial a}{\partial x_3}dx^3 \wedge dx^1 - \frac{\partial a}{\partial x_2}dx^1 \wedge dx^2 + ...) \\
&= \sharp(\frac{\partial a}{\partial x_3}dx^2 - \frac{\partial a}{\partial x_2}dx^3 + ...) \\
&= \frac{\partial a}{\partial x_3}\partial_2 - \frac{\partial a}{\partial x_2}\partial_3 + \frac{\partial b}{\partial x_1}\partial_3 - \frac{\partial b}{\partial x_3}\partial_1 + \frac{\partial c}{\partial x_2}\partial_1 - \frac{\partial c}{\partial x_1}\partial_2 \\
&= (\frac{\partial c}{\partial x_2} - \frac{\partial b}{\partial x_3}, \frac{\partial a}{\partial x_3} - \frac{\partial c}{\partial x_1}, \frac{\partial b}{\partial x_1} - \frac{\partial a}{\partial x_2}) \\
&= \mathrm{curl} X
\end{aligned}$$

(b)

$$
\begin{aligned}
\mathrm{curl}(\mathrm{grad} f) &= \mathrm{curl}(\sharp df) \\
&= \sharp{*}d\flat\sharp df \\
&= \sharp{*}ddf \\
&= \sharp{*}0 \\
&= 0 \\
\mathrm{div}(\mathrm{curl} f) &= \mathrm{div}(\sharp{*}d\flat X) \\
&= {*}d{*}\flat\sharp{*}d\flat X \\
&= {*}d{*}{*}d\flat X \\
&= {*}dd\flat X \\
&= {*}0 \\
&= 0
\end{aligned}
$$

$\square$

4. Nope.