Differential Geometry - Exercise 12

Florian Bogner

June 25, 2020

1. Nope.
2. Thank you, lecture notes!

$$
\begin{aligned}
\nabla_X \omega(Y) - \nabla_Y \omega(X) &= D_X\langle \omega, Y\rangle - \omega(\nabla_X Y) - D_Y\langle \omega, X\rangle + \omega(\nabla_Y X) && \text{by eq. (42)} \\
&= D_X\langle \omega, Y\rangle - D_Y\langle \omega, X\rangle - \omega(\nabla_X Y + \nabla_Y X) && \text{linearity of } \omega \\
&= D_X\langle \omega, Y\rangle - D_Y\langle \omega, X\rangle - \omega([X,Y]) && \text{torsion-free} \\
&= d\omega(X,Y) && \text{Lemma 5.59}
\end{aligned}
$$

$\square$

3. Oof.
4. We will prove the equivalent statement:

$$
\begin{aligned}
D_X(g(Y,Z)) + D_Y(g(Z,X)) - D_Z(g(X,Y)) = 2g(\nabla_X Y, Z) &- g([Z,X],Y) \\
&- g([X,Y],Z) + g([Y,Z],X)
\end{aligned}
$$

For the summands of the right-hand side we have the metric property:

$$
\begin{aligned}
D_X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\
D_Y(g(Z,X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\
-D_Z(g(X,Y)) &= -g(\nabla_Z X, Y) - g(X, \nabla_Z Y)
\end{aligned}
$$

Thus, using the bilinearity and symmetry of g and we have

$$
\begin{aligned}
RHS &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \\
&= g(\nabla_Y Z - \nabla_Z Y, X) - g(\nabla_Z X - \nabla_X Z, Y) + g(\nabla_X Y + \nabla_Y X, Z) \\
&= g([Y,Z],X) - g([Z,X],Y) + g(-\nabla_X Y + \nabla_Y X + 2\nabla_X Y, Z) \\
&= g([Y,Z],X) - g([Z,X],Y) - g([X,Y],Z) + 2g(\nabla_X Y, Z)
\end{aligned}
$$

$\square$