Differential Geometry - Exercise 9

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1. Let $(U_i, \phi_i)_{i\in I}$ be an enumeration of the smooth structure of M and similarly $(V_j, \psi_j)_{j\in J}$ be an enumeration of the smooth structure of N. Lots of properties held by components are also true for the products:

   - $(U_i \times V_j)_{(i,j)\in I\times J}$ is a covering of $M \times N$ as are the components.
   - $\phi_i \times \psi_j : M \times N \to R^m \times R^n : (p, q) \mapsto \phi_i(p) \times \psi_j(q)$ is a homeomorphism for every $i \in I, j \in J$ as are the components.
   - The transition maps $(\phi_{i_1} \times \psi_{j_1}) \circ (\phi_{i_2} \times \psi_{j_2})^{-1} = (\phi_{i_1} \circ \phi_{i_2}^{-1}) \times (\psi_{j_1} \circ \psi_{j_2}^{-1})$ are diffeomorphisms as the component maps are diffeomorphisms.

   Therefore $(U_i \times V_j, \phi_i \times \psi_j)_{(i,j)\in I\times J}$ is a smooth Atlas on $M \times N$. It has a unique maximal atlas as the lecture notes tell us on p.94. This is the desired natural smooth structure. □

2. Let
$$U_1 \subset \mathbb{R}P^2 = \{[x, y, z]_\sim \mid x, y, z \in \mathbb{R}, z \neq 0\}$$
   and
$$\begin{aligned} \phi_1 : U_1 &\to \mathbb{R}^2 \\ [x, y, z]_\sim &\mapsto (\frac{x}{z}, \frac{y}{z}) \end{aligned}$$
   We have the inverse as
$$\begin{aligned} \phi_1^{-1} : \mathbb{R}^2 &\to U_1 \\ (x, y) &\mapsto [x, y, 1]_\sim \end{aligned}$$
   Note that these definitions do not depend on the representation (x, y, z) of $[x, y, z]_\sim$.
$$\phi_1([x, y, z]_\sim) = \phi_1([\lambda x, \lambda y, \lambda z]_\sim) = (\frac{\lambda x}{\lambda z}, \frac{\lambda y}{\lambda z}) = (\frac{x}{z}, \frac{y}{z})$$
   Therefore and because $z \neq 0$ they are well defined. Both ϕ_1 and ϕ_1^{-1} are obviously continuous, thus ϕ_1 is indeed a homeomorphism.

   We define U_2, U_3 and ϕ_2, ϕ_3 analogously swapping the roles of z with x and y respectively. Since for $[x, y, z]_\sim \in \mathbb{R}P^2$ we have one of x, y, z not equal to 0, thus it is in one of U_i. Therefore we have an Atlas.

   Finally, are the transition maps smooth? Lets consider
$$U_2 \cap U_3 = \{[x, y, z]_\sim \mid x, y, z \in \mathbb{R}, x \neq 0, y \neq 0\}$$
   Then we have the transition map
$$\begin{aligned} \phi_3 \circ \phi_2^{-1} : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (y, z) &\mapsto \phi_3([1, y, z]_\sim) = (\frac{1}{y}, \frac{z}{y}) \end{aligned}$$
   This is obviously smooth for $y \neq 0$. Its inverse is smooth for $x \neq 0$. This also holds for the other two transition maps analogously. Therefore we have a *smooth* Atlas. □

3. The smooth structure generated by id equals the set of all diffeomorphisms on open subsets of $\mathbb{R}$. However, ϕ is not a diffeomorphism, as its inverse is not differentiable at 0. Thus the smooth structure generated by ϕ, which of course contains ϕ cannot be equal to the one generated by id.

   $F : p \mapsto p^{\frac{1}{3}}$ is a diffeomorphism from the smooth structure of id to the smooth structure of ϕ as

$$\begin{aligned} \phi \circ F \circ id^{-1}(p) &= (p^{\frac{1}{3}})^3 = p \\ id \circ F^{-1} \circ phi^{-1}(p) &= (p^3)^{\frac{1}{3}} = p \end{aligned}$$

   Both of these are as smooth as a babys butt. □

4. We identify vector fields with derivations as is allowed by Theorem 5.24

   (a) Antisymmetry:

$$\begin{aligned} [X, Y] &= f \mapsto D_X D_Y f - D_Y D_X f \\ &= f \mapsto -(D_Y D_X f - D_X D_Y f) \\ &= -[Y, X] \end{aligned}$$

   (b) Leibnitz property:

$$\begin{aligned} [X, gY] &= f \mapsto D_X(g \cdot D_Y f) - g \cdot D_Y D_X f \\ &= f \mapsto D_X g \cdot D_Y f + g \cdot D_X D_Y f - g \cdot D_X D_Y f \\ &= D_X g \cdot Y + g \cdot [X, Y] \end{aligned}$$

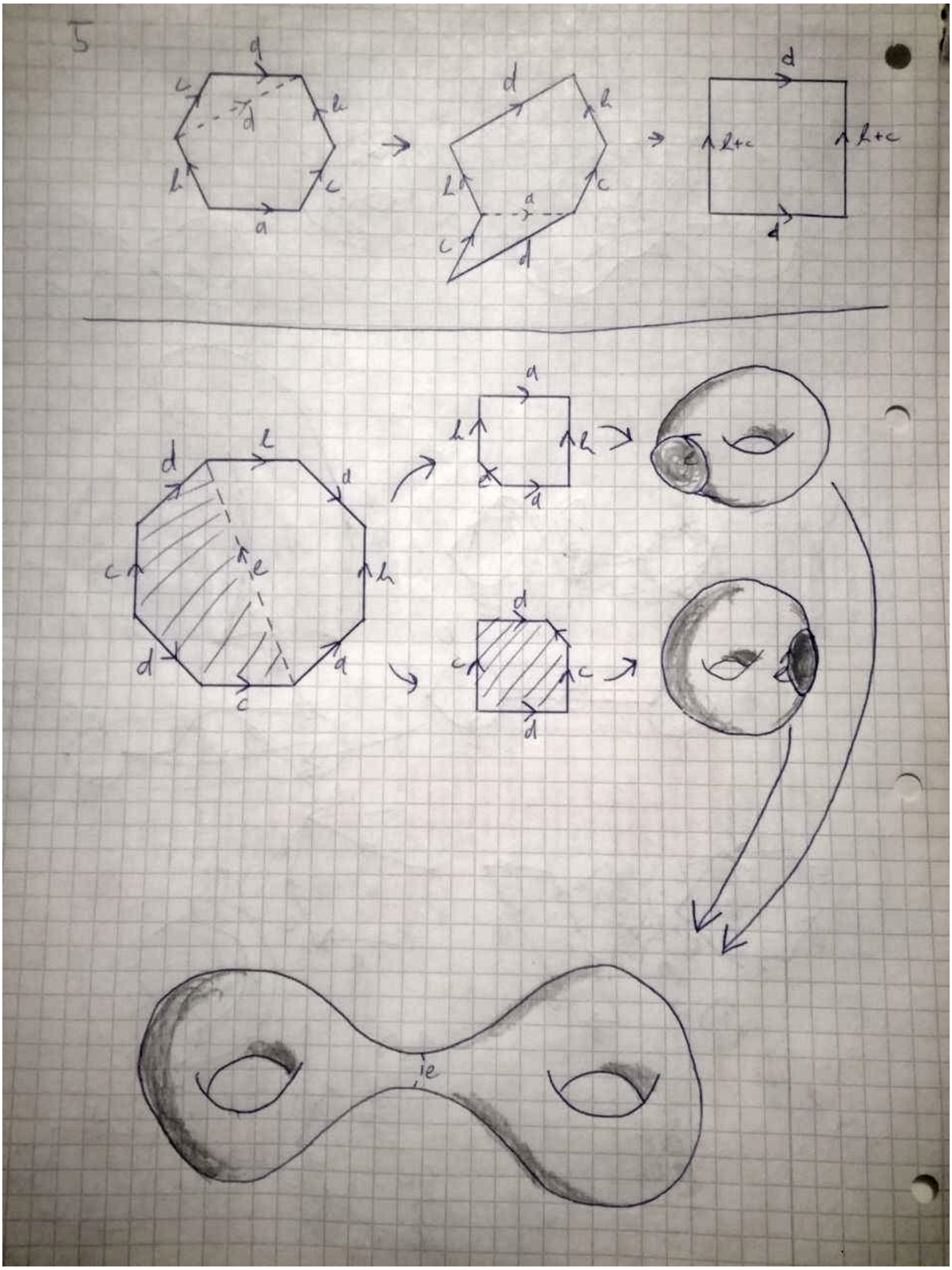
   The second identity follows from the first and a).

   (c) Jacobi identity:

$$\begin{aligned} [X, [Y, Z]] &= D_X D_{[Y,Z]} - D_{[Y,Z]} D_X \\ &= D_X D_Y D_Z - D_X D_Z D_Y - D_Y D_Z D_X + D_Z D_Y D_X \\ [Z, [X, Y]] &= D_Z D_{[X,Y]} - D_{[X,Y]} D_Z \\ &= D_Z D_X D_Y - D_Z D_Y D_X - D_X D_Y D_Z + D_Y D_X D_Z \\ [Y, [Z, X]] &= D_Y D_{[Z,X]} - D_{[Z,X]} D_Y \\ &= D_Y D_Z D_X - D_Y D_X D_Z - D_Z D_X D_Y + D_X D_Z D_Y \end{aligned}$$

   Notice each permutation of X, Y and Z appears exactly twice, once with each sign. Thus the sum is zero. □

5. The left one is a doughnut. The right one is a two-holed doughnut. See image on page 3.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