Differential Geometry - Exercise 7

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1. (a) Let γ be a unit speed parametrisation of a given straight line. Obviously $\ddot{\gamma} = 0$. By Definition 3.21. we have
$$0 = \ddot{\gamma} = \kappa_n N + \kappa_g N \times \dot{\gamma}$$
Since N and $N \times \dot{\gamma}$ are orthogonal and thus linearly independent both coefficients, but especially κ_g, are zero. Thus γ is indeed geodesic. $\square$

   (b) Without loss of generality assume the plane P is a linear subspace. (Move the coordinate system so the zero-point is in the plane if necessary.) Then $\gamma \in P$ implies $\dot{\gamma} \in P$ as well as $\ddot{\gamma} \in P$. Since $N \in P$ we have $N \times \dot{\gamma} \notin P$. Again by definition 3.21.
$$\ddot{\gamma} = \kappa_n N + \kappa_g N \times \dot{\gamma} \in P$$
implies that $\kappa_g N \times \dot{\gamma} \in P$, which is only possible if $\kappa_g = 0$. $\square$

2. γ being geodesic means $\kappa_g = 0$. With Theorem 3.22. we have that γ being asympotic implies $\kappa_n = II(\dot{\gamma}, \dot{\gamma}) = 0$. Thus
$$\begin{aligned}\ddot{\gamma} &= \kappa_n N + \kappa_g N \times \dot{\gamma} \\ &= 0 \cdot N + 0 \cdot N \times \dot{\gamma} \\ &= 0\end{aligned}$$
Therefore $\gamma = \int \int 0 \; dt \; dt = x_0 + t \cdot v_0$ is a straight line. $\square$

3. 2hard4me

4. Let $I = [a, b]$ and let γ^* be an arbitrary path with the same start and end point as γ, i.e. $\gamma(a) = \gamma^*(a)$ and $\gamma(b) = \gamma^*(b)$. We write the Energy creatively as a path integral and then, because $\operatorname{grad} f$ is obviously a gradient field and path integrals of gradient fields only depend on start and endpoint, we can swap out γ with γ^*:
$$\begin{aligned}2E(\gamma) &= \int_a^b \|\dot{\gamma}\|^2 \; dt \\ &= \int_a^b \dot{\gamma} \cdot \dot{\gamma} \; dt \\ &= \int_a^b \operatorname{grad} f \cdot \dot{\gamma} \; dt \\ &= \int_a^b \operatorname{grad} f \cdot \dot{\gamma}^* \; dt \\ &\leq \int_a^b \|\operatorname{grad} f\| \cdot \|\gamma^*\| \; dt \quad \text{by Cauchy-Schwarz} \\ &= \int_a^b 1 \cdot \|\gamma^*\| \; dt \\ &= L(\gamma^*)\end{aligned}$$

Generalizing Lemma 4.5. for arbitrary intervals we get:

$$L(\gamma)^2 \leq 2(a-b)E(\gamma)$$

Combining this with the fact that $L(\gamma) = b - a$ as γ is a unit speed curve we get:

$$L(\gamma) = \frac{L(\gamma)^2}{b-a} \leq 2E \leq L(\gamma^*)$$

Thus the length of every other path is longer than the one following the gradient of f. $\square$