CHAPTER 3

Perfect Graphs

1. The Star of the Show

In this section we introduce the main character of the book—*the perfect graph.* He was "discovered" by Claude Berge, who has been his agent since the early 1960s. P.G. has appeared in such memorable works as "Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind" and "Caractérisation des graphes non orientés dont on peut orienter les arrêtes de manière à obtenir le graphe d'une relation d'ordre." Despite his seemingly assuming name, P.G. has mixed the highbrow glamorous life with an intense dedication to improving the plight of mankind. His feature role in "Perfect graphs and an application to optimizing municipal services" has won him admiration and respect around the globe. Traveling incognito, a further sign of his modesty, he has been spotted by fans disguised as a graph parfait or as a (banana) split graph in a local ice cream parlor. So, ladies and gentlemen, without further ado, the management proudly presents

THE PERFECT GRAPH

Let us recall the following parameters of an undirected graph, which were defined in Section 1.1.

$\omega(G)$, the *clique number* of G: the size of the largest complete subgraph of G.

$\chi(G)$, the *chromatic number* of G: the fewest number of colors needed to properly color the vertices of G, or equivalently, the fewest number of stable sets needed to cover the vertices of G.

$\alpha(G)$, the *stability number* of G: the size of the largest stable set of G.
$k(G)$, the *clique cover number* of G: the fewest number of complete subgraphs needed to cover the vertices of G.

The intersection of a clique and a stable set of a graph G can be at most one vertex. Thus, for any graph G,

$$\omega(G) \leq \chi(G)$$

and

$$\alpha(G) \leq k(G).$$

These equalities are dual to one another since $\alpha(G) = \omega(\bar{G})$ and $k(G) = \chi(\bar{G})$.

Let $G = (V, E)$ be an undirected graph. The main purpose of this book is to study those graphs satisfying the properties

$$(\mathrm{P}_1) \qquad \omega(G_A) = \chi(G_A) \qquad (\text{for all } A \subseteq V)$$

and

$$(\mathrm{P}_2) \qquad \alpha(G_A) = k(G_A) \qquad (\text{for all } A \subseteq V).$$

Such a graph is called *perfect*. It is clear by duality that a graph G satisfies (P_1) if and only if its complement $\bar{G}$ satisfies (P_2). A much stronger result was conjectured by Berge [1961], cultivated by Fulkerson [1969, 1971, 1972], and finally proven by Lovász [1972a], namely, that (P_1) and (P_2) are equivalent. This has become known as the Perfect Graph theorem, which will be proved in the next section along with a third equivalent condition, due to Lovász [1972b],

$$(\mathrm{P}_3) \qquad \omega(G_A)\alpha(G_A) \geq |A| \qquad (\text{for all } A \subseteq V).$$

In subsequent chapters it will be sufficient to show that a graph satisfies any (P_i) in order to conclude that it is perfect, and a perfect graph will satisfy all properties (P_i).

A fourth characterization of perfect graphs, due to Chvátal [1975], will be discussed in Section 3.3, and we shall encounter still another formulation in the chapter on superperfect graphs.

It is traditional to call a graph *χ-perfect* if it satisfies (P_1) and *α-perfect* if it satisfies (P_2). The Perfect Graph theorem then states that a graph is χ-perfect if and only if it is α-perfect. However, *the equivalence of (P_1) and (P_2) fails for uncountable graphs.* The interested reader may consult the following references on infinite perfect graphs: Hajnal and Surányi [1958], Perles [1963], and Nash-Williams [1967], Baumgartner, Malitz, and Reinhardt [1970], Trotter [1971], and Wagon [1978].

2. The Perfect Graph Theorem

In this section we shall show the equivalence of properties (P_1)–(P_3). A key to the proof is that multiplication of the vertices of a graph, as defined below, preserves each of the properties (P_i).

Let G be an undirected graph with vertex x. The graph $G \circ x$ is obtained from G by adding a new vertex x' which is connected to all the neighbors of x. We leave it to the reader to prove the elementary property

$$(G \circ x) - y = (G - y) \circ x \qquad \text{for distinct vertices } x \text{ and } y.$$

More generally, if $x_1, x_2, \ldots, x_n$ are the vertices of G and $\mathbf{h} = (h_1, h_2, \ldots, h_n)$ is a vector of non-negative integers, then $H = G \circ \mathbf{h}$ is constructed by substituting for each x_i a stable set of h_i vertices $x_i^1, \ldots, x_i^{h_i}$ and joining x_i^s with x_j^t iff x_i and x_j are adjacent in G. We say that H is obtained from G by *multiplication of vertices.*

Remark. The definition allows $h_i = 0$, in which case H includes no copy of x_i. Thus, every induced subgraph of G can be obtained by multiplication of the appropriate (0, 1)-valued vector.

Lemma 3.1 (Berge [1961]). Let H be obtained from G by multiplication of vertices.

(i) If G satisfies (P_1), then H satisfies (P_1).
(ii) If G satisfies (P_2), then H satisfies (P_2).

Proof. The lemma is true if G has only one vertex. We shall assume that (i) and (ii) are true for all graphs with fewer vertices than G. Let $H = G \circ \mathbf{h}$. If one of the coordinates of $\mathbf{h}$ equals zero, say $h_i = 0$, then H can be obtained from $G - x_i$ by multiplication of vertices. But, if G satisfies (P_1) [resp. (P_2)], then $G - x_i$ also satisfies (P_1) [resp. (P_2)]. In this case the induction hypothesis implies (i) and (ii).

Thus, we may assume that each coordinate $h_i \geq 1$, and since H can be built up from a sequence of smaller multiplications (Exercise 2), it is sufficient to prove the result for $H = G \circ x$. Let x' denote the added "copy" of x.

Assume that G satisfies (P_1). Since x and x' are nonadjacent, $\omega(G \circ x) = \omega(G)$. Let G be colored using $\omega(G)$ colors. Color x' the same color as x. This will be a coloring of $G \circ x$ in $\omega(G \circ x)$ colors. Hence, $G \circ x$ satisfies (i).

Next assume that G satisfies (P_2). We must show that $\alpha(G \circ x) = k(G \circ x)$. Let $\mathscr{K}$ be a clique cover of G with $|\mathscr{K}| = k(G) = \alpha(G)$, and let K_x be the clique of $\mathscr{K}$ containing x. There are two cases.

Case 1: *x is contained in a maximum stable set S of G, i.e., $|S| = \alpha(G)$.* In this case $S \cup \{x'\}$ is a stable set of $G \circ x$, so

$$\alpha(G \circ x) = \alpha(G) + 1.$$

Since $\mathscr{K} \cup \{\{x'\}\}$ covers $G \circ x$, we have that

$$k(G \circ x) \leq k(G) + 1 = \alpha(G) + 1 = \alpha(G \circ x) \leq k(G \circ x).$$

Thus, $\alpha(G \circ x) = k(G \circ x)$.

Case 2: *No maximum stable set of G contains x.* In this case,

$$\alpha(G \circ x) = \alpha(G).$$

Since each clique of $\mathscr{K}$ intersects a maximum stable set exactly once, this is true in particular for K_x. But x is not a member of any maximum stable set. Therefore, $D = K_x - \{x\}$ intersects each maximum stable set of G exactly once, so

$$\alpha(G_{V-D}) = \alpha(G) - 1.$$

This implies that

$$k(G_{V-D}) = \alpha(G_{V-D}) = \alpha(G) - 1 = \alpha(G \circ x) - 1.$$

Taking a clique cover of G_{V-D} of cardinality $\alpha(G \circ x) - 1$ along with the extra clique $D \cup \{x'\}$, we obtain a cover of $G \circ x$. Therefore,

$$k(G \circ x) = \alpha(G \circ x). \qquad \blacksquare$$

Lemma 3.2 (Fulkerson [1971], Lovász [1972b]). Let G be an undirected graph each of whose proper induced subgraphs satisfies (P_2), and let H be obtained from G by multiplication of vertices. If G satisfies (P_3), then H satisfies (P_3).

Proof. Let G satisfy (P_3) and choose H to be a graph having the smallest possible number of vertices which can be obtained from G by multiplication of vertices but which fails to satisfy (P_3) itself. Thus,

$$\omega(H)\alpha(H) < |X|, \tag{1}$$

where X denotes the vertex set of H, yet (P_3) does hold for each proper induced subgraph of H.

As in the proof of the preceding lemma, we may assume that each vertex of G was multiplied by at least 1 and that some vertex u was multiplied by $h \geq 2$. Let $U = \{u^1, u^2, \ldots, u^h\}$ be the vertices of H corresponding to u. The vertex u^1 plays a distinguished role in the proof. By the minimality of H, (P_3) is satisfied by H_{X-u^1}, which gives

$$\begin{aligned} |X| - 1 = |X - u^1| &\le \omega(H_{X-u^1})\alpha(H_{X-u^1}) \qquad [\text{by } (P_3)] \\ &\le \omega(H)\alpha(H) \\ &\le |X| - 1 \qquad [\text{by } (1)]. \end{aligned}$$

Thus, equality holds throughout, and we can define

$$p = \omega(H_{X-u^1}) = \omega(H),$$
$$q = \alpha(H_{X-u^1}) = \alpha(H),$$

and

$$pq = |X| - 1. \tag{2}$$

Since H_{X-U} is obtained from $G - u$ by multiplication of vertices, Lemma 3.1 implies that H_{X-U} satisfies (P_2). Thus, H_{X-U} can be covered by a set of q complete subgraphs of H, say $K_1, K_2, \ldots, K_q$. We may assume that the K_i are pairwise disjoint and that $|K_1| \ge |K_2| \ge \cdots \ge |K_q|$. Obviously,

$$\sum_{i=1}^{q} |K_i| = |X - U| = |X| - h = pq - (h - 1) \qquad [\text{by } (2)].$$

Since $|K_i| \le p$, at most $h - 1$ of the K_i fail to contribute p to the sum. Hence,

$$|K_1| = |K_2| = \cdots = |K_{q-h+1}| = p.$$

Let H' be the subgraph of H induced by $X' = K_1 \cup \cdots \cup K_{q-h+1} \cup \{u^1\}$. Thus

$$|X'| = p(q - h + 1) + 1 < pq + 1 = |X| \qquad [\text{by } (2)], \tag{3}$$

so by the minimality of H,

$$\omega(H')\alpha(H') \ge |X'| \qquad [\text{by } (P_3)]. \tag{4}$$

But $p = \omega(H) \ge \omega(H')$, so

$$\begin{aligned} \alpha(H') &\ge |X'|/p \qquad [\text{by } (4)] \\ &> q - h + 1 \qquad [\text{by } (3)]. \end{aligned}$$

Let S' be a stable set of H' of cardinality $q - h + 2$. Certainly $u^1 \in S'$, for otherwise S' would contain two vertices of a clique (by the definition of H'). Therefore, $S = S' \cup U$ is a stable set of H with $q + 1$ vertices, contradicting the definition of q. ∎

Theorem 3.3 The Perfect Graph Theorem (Lovász [1972b]). For an undirected graph $G = (V, E)$, the following statements are equivalent:

$$(P_1) \qquad \omega(G_A) = \chi(G_A) \qquad \text{(for all } A \subseteq V\text{)},$$

$$(P_2) \qquad \alpha(G_A) = k(G_A) \qquad \text{(for all } A \subseteq V\text{)},$$

$$(P_3) \qquad \omega(G_A)\alpha(G_A) \geq |A| \qquad \text{(for all } A \subseteq V\text{)}.$$

Proof. We may assume that the theorem is true for all graphs with fewer vertices than G.

$(P_1) \Rightarrow (P_3)$. Suppose we can color G_A in $\omega(G_A)$ colors. Since there are at most $\alpha(G_A)$ vertices of a given color it follows that $\omega(G_A)\alpha(G_A) \geq |A|$.

$(P_3) \Rightarrow (P_1)$. Let $G = (V, E)$ satisfy (P_3); then by induction each proper induced subgraph of G satisfies (P_1)–(P_3). It is sufficient to show that $\omega(G) = \chi(G)$.

If we had a stable set S of G such that $\omega(G_{V-S}) < \omega(G)$, we could then paint S orange and paint G_{V-S} in $\omega(G) - 1$ other colors, and we would have $\omega(G) = \chi(G)$.

Suppose G_{V-S} has an $\omega(G)$-clique $K(S)$ for every stable set S of G. Let $\mathscr{S}$ be the collection of all stable sets of G, and keep in mind that $S \cap K(S) = \emptyset$. For each $x_i \in V$, let h_i denote the number of cliques $K(S)$ which contain x_i. Let $H = (X, F)$ be obtained from G by multiplying each x_i by h_i. On the one hand, by Lemma 3.2,

$$\omega(H)\alpha(H) \geq |X|.$$

On the other hand, using some simple counting arguments we can easily show that

$$|X| = \sum_{x_i \in V} h_i$$

$$= \sum_{S \in \mathscr{S}} |K(S)| = \omega(G)|\mathscr{S}|, \tag{5}$$

$$\omega(H) \leq \omega(G), \tag{6}$$

$$\alpha(H) = \max_{T \in \mathscr{S}} \sum_{X_i \in T} h_i \tag{7}$$

$$= \max_{T \in \mathscr{S}} \left[\sum_{S \in \mathscr{S}} |T \cap K(S)| \right] \tag{8}$$

$$\leq |\mathscr{S}| - 1, \tag{9}$$

which together imply that

$$\omega(H)\alpha(H) \leq \omega(G)(|\mathscr{S}| - 1) < |X|,$$

a contradiction.*

$(P_2) \Leftrightarrow (P_3)$. By what we have already proved, we have the following implications:

$$\begin{aligned} G \text{ satisfies } (P_2) &\Leftrightarrow \bar{G} \text{ satisfies } (P_1) \\ &\Leftrightarrow \bar{G} \text{ satisfies } (P_3) \Leftrightarrow G \text{ satisfies } (P_3). \end{aligned}$$

▮

Corollary 3.4. **A graph G is perfect if and only if its complement $\bar{G}$ is perfect.**

Corollary 3.5. **A graph G is perfect if and only if every graph H obtained from G by multiplication of vertices is perfect.**

Historical note. The equivalence of (P_1) and (P_2) was almost proved by Fulkerson. He heard the news of the success of Lovász, who was not aware of Fulkerson's work at that time, from a postcard sent by Berge. Fulkerson immediately returned to his previous results on pluperfection and, within a few hours, obtained his own proof. Such are the joys and sorrows of research. His consolation, to our benefit, was that in the process of his investigations, Fulkerson invented and developed the notion of antiblocking pairs of polyhedra, an idea which has become an important topic in the rapidly growing field of polyhedral combinatorics.†

Briefly, and in our terminology, Fulkerson had proved the following:

Let $\mathscr{M}(G)$ be the collection of all graphs H which can be constructed from a graph G by multiplication of vertices. Then, H satisfies (P_1) for all $H \in \mathscr{M}(G)$ if and only if H satisfies (P_2) for all $H \in \mathscr{M}(G)$.

* Equations (5)–(9) are justified as follows:

(5) Consider the incidence matrix whose rows are indexed by the vertices $x_1, x_2, \ldots, x_n$ and whose columns correspond to the cliques $K(S)$ for $S \in \mathscr{S}$. Then, h_i equals the number of nonzeros in row i, and $|K(S)|$ equals the number of nonzeros in its corresponding column, which is by definition equal to $\omega(G)$.

(6) At most one "copy" of any vertex of G could be in a clique of H.

(7) If a maximum stable set of H contains some of the "copies" of x_i, then it will contain all of the "copies."

(8) Restrict attention to those rows of the matrix pertinent to (5) which belong to elements of T.

(9) $|T \cap K(S)| \leq 1$ and $|T \cap K(T)| = 0$.

† Polyhedral combinatorics deals with the interplay between concepts from combinatorics and mathematical programming.

Clearly, this result together with Lemma 3.1 would give a proof of the equivalence of (P_1) and (P_2) for G.

3. p-Critical and Partitionable Graphs*

An undirected graph G is called *p-critical* if it is minimally imperfect, that is, G is *not* perfect but every proper induced subgraph of G *is* a perfect graph. Such a graph, in particular, satisfies the inequalities

$$\alpha(G-x) = k(G-x) \qquad \text{and} \qquad \omega(G-x) = \chi(G-x)$$

for all vertices x, where $G - x$ denotes the resulting graph after deleting x. The following properties of p-critical graphs are easy consequences of the Perfect Graph theorem.

Theorem 3.6. If G is a p-critical graph on n vertices, then

$$n = \alpha(G)\omega(G) + 1,$$

and for all vertices x of G,

$$\alpha(G) = k(G-x) \qquad \text{and} \qquad \omega(G) = \chi(G-x).$$

Proof. By Theorem 3.3, since G is p-critical we have $n > \alpha(G)\omega(G)$ and $n - 1 \leq \alpha(G-x)\omega(G-x)$ for all vertices x. Thus,

$$n - 1 \leq \alpha(G-x)\omega(G-x) \leq \alpha(G)\omega(G) < n.$$

Hence, $n - 1 = \alpha(G)\omega(G)$, $\alpha(G) = \alpha(G-x) = k(G-x)$, and

$$\omega(G) = \omega(G-x) = \chi(G-x). \qquad \blacksquare$$

Let $\alpha, \omega \geq 2$ be arbitrary integers. An undirected graph G on n vertices is called *(α, ω)-partitionable* if $n = \alpha\omega + 1$ and for all vertices x of G

$$\alpha = k(G-x), \qquad \omega = \chi(G-x).$$

We have shown in Theorem 3.6 that every p-critical graph is (α, ω)-partitionable with $\alpha = \alpha(G)$ and $\omega = \omega(G)$. A more general result holds.

Remark 3.7. After removing any vertex x of an (α, ω)-partitionable graph, the remaining graph has $\alpha\omega$ vertices, chromatic number ω, and clique cover number α. So an ω-coloring of $G - x$ will partition the vertices into ω stable sets, one of which must be at least of size α. Similarly, a minimum clique

* Sections 3.3–3.5 were written jointly with Mark Buckingham.

cover of $G - x$ will partition the vertices into α cliques, one of which must be at least of size ω.

Theorem 3.8. If G is an (α, ω)-partitionable graph, then $\alpha = \alpha(G)$ and $\omega = \omega(G)$.

Proof. Let $G = (V, E)$ be (α, ω)-partitionable. By Remark 3.7, $\alpha \leq \alpha(G)$ and $\omega \leq \omega(G)$. Conversely, take a maximum stable set S of G and let $y \in V - S$. Then S is also a maximum stable set of $G - y$, so

$$\alpha(G) = |S| = \alpha(G - y) \leq k(G - y) = \alpha.$$

Thus, $\alpha(G) \leq \alpha$. Similarly, $\omega(G) \leq \omega$. Therefore, $\alpha = \alpha(G)$ and $\omega = \omega(G)$. ▌

Theorem 3.8 shows that the integers α and ω for a partitionable graph are unique. Therefore, we shall simply use the term partitionable graph and assume that $\alpha = \alpha(G)$ and $\omega = \omega(G)$. The class of p-critical graphs is properly contained in the class of partitionable graphs which, in turn, is properly contained in the class of imperfect graphs (Exercise 10).

Lemma 3.9. If G is a partitionable graph on n vertices, then the following conditions hold:

(i) G contains a set of n maximum cliques $K_1, K_2, \ldots, K_n$ that cover each vertex of G exactly $\omega(G)$ times;
(ii) G contains a set of n maximum stable sets $S_1, S_2, \ldots, S_n$ that cover each vertex of G exactly $\alpha(G)$ times; and
(iii) $K_i \cap S_j = \varnothing$ if and only if $i = j$.

Proof. Choose a maximum clique K of G and, for each $x \in K$, choose a minimum clique cover $\mathscr{K}_x$ of $G - x$. By Remark 3.7, all of the members of $\mathscr{K}_x$ must be cliques of size ω. Finally, let $\mathbf{A}$ be the $n \times n$ matrix whose first row is the characteristic vector of K and whose subsequent rows are the characteristic vectors of each of the cliques in $\mathscr{K}_x$ for all $x \in K$. (Note that the number of rows is $1 + \alpha\omega = n$.)

Each vertex $y \notin K$ is covered once by $\mathscr{K}_x$ for all $x \in K$. Each vertex $z \in K$ is covered once by K and once by $\mathscr{K}_x$ for all $z \neq x \in K$. Therefore, every vector is covered ω times. For each row $\mathbf{a}_i$ of $\mathbf{A}$ we let K_i be the clique whose characteristic vector is $\mathbf{a}_i$. We may express (i) by the matrix equation $\mathbf{1A} = \omega\mathbf{1}$, where $\mathbf{1}$ is the row vector containing all ones. Condition (i) will be satisfied once we show that the K_i are distinct.

For each i, pick a vertex $v \in K_i$ and let $\mathscr{S}$ denote a minimum stable set covering (coloring) of $G - v$. By Remark 3.7 and an easy counting exercise, there must be some stable set $S_i \in \mathscr{S}$ such that $K_i \cap S_i = \varnothing$. Let $\mathbf{b}_i$ be the characteristic vector of S_i, and let $\mathbf{B}$ denote the $n \times n$ matrix having rows $\mathbf{b}_i$ for $i = 1, \ldots, n$. Since $\mathbf{1} \cdot \mathbf{b}_i^{\mathrm{T}} = \alpha$, we have

$$\mathbf{1AB}^{\mathrm{T}} = \omega\mathbf{1B}^{\mathrm{T}} = \omega\alpha\mathbf{1} = (n - 1)\mathbf{1}.$$

But $\mathbf{a}_i \cdot \mathbf{b}_i^{\mathrm{T}} = 0$, so $\mathbf{AB}^{\mathrm{T}} = \mathbf{J} - \mathbf{I}$, where $\mathbf{J}$ is the matrix containing all ones and $\mathbf{I}$ is the identity matrix. This proves (iii).

Finally, both $\mathbf{A}$ and $\mathbf{B}$ are nonsingular matrices since $\mathbf{J} - \mathbf{I}$ is nonsingular. Thus, the K_i are distinct and the S_i are distinct. Furthermore,

$$\begin{aligned}\mathbf{1B} &= \mathbf{1BA}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}})^{-1} = \mathbf{1}(\mathbf{J} - \mathbf{I})(\mathbf{A}^{\mathrm{T}})^{-1} = (n - 1)\mathbf{1}(\mathbf{A}^{\mathrm{T}})^{-1} \\ &= [(n - 1)/\omega]\mathbf{1} = \alpha\mathbf{1},\end{aligned}$$

which proves (ii). ∎

The next result shows that *all* the maximum cliques and stable sets of G are among those in Lemma 3.9.

Lemma 3.10. A partitionable graph G contains exactly n maximum cliques and n maximum stable sets.

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be the matrices whose rows are the characteristic vectors of the cliques and stable sets, respectively, satisfying $\mathbf{AB}^{\mathrm{T}} = \mathbf{J} - \mathbf{I}$ as specified in Lemma 3.9. Suppose that $\mathbf{c}$ is the characteristic vector of some maximum clique of G. We will show that $\mathbf{c}$ is a row of $\mathbf{A}$.

We first observe that $\mathbf{A}^{-1} = \omega^{-1}\mathbf{J} - \mathbf{B}^{\mathrm{T}}$ since

$$\mathbf{A}(\omega^{-1}\mathbf{J} - \mathbf{B}^{\mathrm{T}}) = \omega^{-1}\mathbf{AJ} - \mathbf{AB}^{\mathrm{T}} = \mathbf{J} - \mathbf{AB}^{\mathrm{T}} = \mathbf{I}.$$

A solution $\mathbf{t}$ to the equation $\mathbf{tA} = \mathbf{c}$ will satisfy

$$\mathbf{t} = \mathbf{cA}^{-1} = \omega^{-1}\mathbf{cJ} - \mathbf{cB}^{\mathrm{T}} = \omega^{-1}(\omega\mathbf{1}) - \mathbf{cB}^{\mathrm{T}} = \mathbf{1} - \mathbf{cB}^{\mathrm{T}}.$$

Therefore, $\mathbf{t}$ is a (0, 1)-valued vector. Also,

$$\mathbf{t} \cdot \mathbf{1}^{\mathrm{T}} = (\mathbf{1} - \mathbf{cB}^{\mathrm{T}}) \cdot \mathbf{1}^{\mathrm{T}} = n - \alpha\mathbf{c} \cdot \mathbf{1}^{\mathrm{T}} = n - \alpha\omega = 1.$$

Therefore, $\mathbf{t}$ is a unit vector. This implies that $\mathbf{c}$ is a row of $\mathbf{A}$.

Similarly, every characteristic vector of a maximum stable set is a row of $\mathbf{B}$. ∎

Theorem 3.11. Let G be an undirected graph on n vertices, and let $\alpha = \alpha(G)$ and $\omega = \omega(G)$. Then G is partitionable if and only if the following conditions hold:

(i) $n = \alpha\omega + 1$;
(ii) G has exactly n maximum cliques and n maximum stable sets;

(iii) every vertex of G is contained in exactly ω maximum cliques and in exactly α maximum stable sets;

(iv) each maximum clique intersects all but one maximum stable set and vice versa.

Proof. ($\Rightarrow$) This implication follows from Lemmas 3.9 and 3.10.

($\Leftarrow$) Following our previous notation, conditions (ii)–(iv) imply that

$$\mathbf{AJ} = \mathbf{JA} = \omega\mathbf{J}, \qquad \mathbf{BJ} = \mathbf{JB} = \alpha\mathbf{J}, \qquad \mathbf{AB}^{\mathrm{T}} = \mathbf{J} - \mathbf{I},$$

where $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices whose rows are the characteristic vectors of the maximum cliques and maximum stable sets, respectively. Let x_i be a vertex of G and let $\mathbf{h}_i^{\mathrm{T}}$ be its corresponding column in $\mathbf{A}$. Since

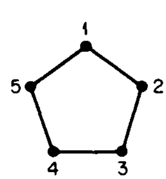
$$\begin{aligned}\mathbf{A}^{\mathrm{T}}\mathbf{B} &= \mathbf{B}^{-1}\mathbf{B}\mathbf{A}^{\mathrm{T}}\mathbf{B} = \mathbf{B}^{-1}(\mathbf{J} - \mathbf{I})\mathbf{B} = \mathbf{B}^{-1}(\alpha\mathbf{J} - \mathbf{B})\\ &= \alpha\alpha^{-1}\mathbf{J} - \mathbf{I} = \mathbf{J} - \mathbf{I},\end{aligned}$$

we obtain $\mathbf{h}_i\mathbf{B} = \mathbf{1} - \mathbf{e}_i$, where $\mathbf{e}_i$ is the ith unit vector. Thus, $\mathbf{h}_i$ designates ω rows of $\mathbf{B}$ (i.e., stable sets of G) which cover $G - x_i$. Thus, $\chi(G - x_i) \leq \omega$. By a similar argument, $k(G - x_i) \leq \alpha$ for all x_i. But since $n - 1 = \alpha\omega$, we must have $\chi(G - x_i) = \omega$ and $k(G - x_i) = \alpha$. Therefore, G is partitionable. ∎

Corollary 3.12 (Padberg [1974]). If G is a p-critical graph, then conditions (i)–(iv) of Theorem 3.11 hold.

Padberg's investigation of the facial structure of polyhedra associated with (0, 1)-valued matrices first led him to a proof of Corollary 3.12. (We shall discuss some of Padberg's work in Section 3.5.) The proof presented here, using only elementary linear algebra, is due to Bland, Huang, and Trotter [1979]. Additional results on p-critical graphs can be found in Section 3.6.

The only p-critical graphs known are the chordless cycles of odd length and their complements. Figures 3.1 and 3.2 illustrate the conditions of Theorem 3.11 for the graphs C_5 and $\bar{C}_7$.



$K_1 = \{1, 2\}, K_2 = \{2, 3\}, K_3 = \{3, 4\}, K_4 = \{4, 5\}, K_5 = \{5, 1\}$

$S_1 = \{3, 5\}, S_2 = \{1, 4\}, S_3 = \{2, 5\}, S_4 = \{1, 3\}, S_5 = \{2, 4\}$

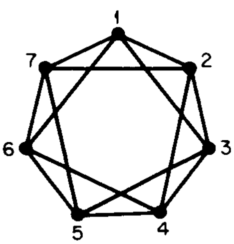
$\{1\} \in K_1, K_5, S_2, S_4$; $\{2\} \in K_1, K_2, S_3, S_5$;

$\{3\} \in K_2, K_3, S_1, S_4$; $\{4\} \in K_3, K_4, S_2, S_5$;

$\{5\} \in K_4, K_5, S_1, S_3$

$n = 5$, $\alpha(G) = 2$, $\omega(G) = 2$

Figure 3.1. The graph C_5 and its maximum clique and stable set structure as specified in Theorem 3.11.



$n = 7$ $\alpha(G) = 2$ $\omega(G) = 3$

$$K_1 = \{1, 2, 3\},\ K_2 = \{2, 3, 4\},\ K_3 = \{3, 4, 5\},\ K_4 = \{4, 5, 6\}$$
$$K_5 = \{5, 6, 7\},\ K_6 = \{6, 7, 1\},\ K_7 = \{7, 1, 2\}$$
$$S_1 = \{4, 7\},\ S_2 = \{1, 5\},\ S_3 = \{2, 6\},\ S_4 = \{3, 7\}$$
$$S_5 = \{1, 4\},\ S_6 = \{2, 5\},\ S_7 = \{3, 6\}$$
$$\{1\} \in K_1, K_6, K_7, S_2, S_5;\quad \{2\} \in K_1, K_2, K_7, S_3, S_6;$$
$$\{3\} \in K_1, K_2, K_3, S_4, S_7;\quad \{4\} \in K_2, K_3, K_4, S_1, S_5;$$
$$\{5\} \in K_3, K_4, K_5, S_2, S_6;\quad \{6\} \in K_4, K_5, K_6, S_3, S_7;$$
$$\{7\} \in K_5, K_6, K_7, S_1, S_4$$

Figure 3.2. The graph $\bar{C}_7$ and its maximum clique and stable set structure as specified in Theorem 3.11.

4. A Polyhedral Characterization of Perfect Graphs

Let **A** be an $m \times n$ matrix. We consider the two polyhedra

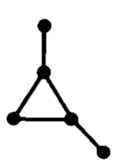
$$P(\mathbf{A}) = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{1}, \mathbf{x} \geq \mathbf{0}\}$$

and

$$P_I(\mathbf{A}) = \text{convex hull}(\{\mathbf{x} \mid \mathbf{x} \in P(\mathbf{A}), \mathbf{x} \text{ integral}\}),$$

where $\mathbf{x}$ is an n-vector and $\mathbf{1}$ is the m-vector of all ones. Clearly $P_I(\mathbf{A}) \subseteq P(\mathbf{A})$, and for (0, 1)-valued matrices **A** having no zero column, $P(\mathbf{A})$ and $P_I(\mathbf{A})$ are bounded and are within the unit hypercube in $\mathbb{R}^n$. An important example of such a matrix is the maximal cliques-versus-vertices incidence matrix of an undirected graph G. This is called the *clique matrix* if all the maximal cliques are included. The clique matrix of G is unique up to permutations of the rows and columns (see Figure 3.3).

Let **A** be any $m \times n$ (0, 1)-valued matrix having no zero columns. The *derived graph* of **A** has n vertices $v_1, v_2, \ldots, v_n$ corresponding to the columns of **A**, and an edge connecting v_i and v_j whenever the ith and jth columns of **A** have a 1 in some row $\mathbf{a}_k$. Clearly every row of **A** forms a (not necessarily



$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Figure 3.3. A graph and its clique matrix.

maximal) clique in its derived graph. Many matrices have the same derived graph. For example, if $\mathbf{A}$ is either the clique matrix or the edge incidence matrix of G, then the derived graph of $\mathbf{A}$ will be G.

Lemma 3.13. Let G be an undirected graph, and let $\mathbf{A}$ be any (0, 1)-valued matrix having no zero column whose derived graph equals G. Then $\mathbf{x}$ is an extremum of $P_I(\mathbf{A})$ if and only if $\mathbf{x}$ is the characteristic vector of some stable set of G.

Proof. If $\mathbf{x}$ is an extremum of $P_I(\mathbf{A})$, then $\mathbf{x}$ must be integral, and since $\mathbf{A}$ is (0, 1)-valued without a zero column, $\mathbf{x} \le \mathbf{1}$. Thus, $\mathbf{x}$ is the characteristic vector of some set of vertices S. Suppose there exist vertices u and v of S that are connected in G; hence some row $\mathbf{a}_k$ of $\mathbf{A}$ has a 1 in columns u and v. This yields $\mathbf{a}_k \cdot \mathbf{x} \ge 2$, yet $\mathbf{A}\mathbf{x} \le \mathbf{1}$. Therefore, S must be a stable set.

Conversely, given that $\mathbf{x}$ is a characteristic vector of a stable set of G, certainly $\mathbf{x} \in P_I(\mathbf{A})$. Let $\mathbf{x}$ be expressed as a convex combination of some set of extrema $\{\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \ldots, \mathbf{b}^{(s)}\}$ of $P_I(\mathbf{A})$; that is,

$$x_k = \sum_i c^{(i)} b_k{}^{(i)}, \qquad 1 = \sum_i c^{(i)}, \qquad 0 \le c^{(i)} \le 1.$$

Thus, if $x_k = 1$, then $b_k{}^{(i)} = 1$ for all i, and if $x_k = 0$, then $b_k{}^{(i)} = 0$ for all i. Therefore, $\mathbf{x} = \mathbf{b}^{(i)}$ and $\mathbf{x}$ is an extremum of $P_I(\mathbf{A})$. ∎

Theorem 3.14 (Chvátal [1975]). Let $\mathbf{A}$ be the clique matrix of an undirected graph G. Then G is perfect if and only if $P_I(\mathbf{A}) = P(\mathbf{A})$.

To prove the theorem we shall use a result from linear programming used by Edmonds [1965] and others:

Lemma 3.15. Given bounded polyhedra S and T, where S has a finite number of extrema,

$$S = T \qquad \text{iff} \qquad \max_{\text{subj } \mathbf{x} \in S} \mathbf{c} \cdot \mathbf{x} = \max_{\text{subj } \mathbf{x} \in T} \mathbf{c} \cdot \mathbf{x} \quad (\forall \mathbf{c}, \text{ integral}).$$

Proof of Theorem 3.14. Assume that $P_I(\mathbf{A}) = P(\mathbf{A})$. Let G_U be an induced subgraph of G, and let $\mathbf{u}$ denote the characteristic vector of U. We have,

$$\alpha(G_U) = \max_{\text{subj } \mathbf{x} \in P_I(\mathbf{A})} \mathbf{u}\mathbf{x} = \max_{\text{subj } \mathbf{A}\mathbf{x} \le \mathbf{1},\, \mathbf{x} \ge \mathbf{0}} \mathbf{u}\mathbf{x} = \min_{\text{subj } \mathbf{y}\mathbf{A} \ge \mathbf{u},\, \mathbf{y} \ge \mathbf{0}} \mathbf{y} \cdot \mathbf{1}.$$

The first equality follows from the fact that maximums are always achievable at some extremum and the extrema of $P_I(\mathbf{A})$ correspond to stable sets (Lemma 3.13). The second equality follows from Lemma 3.15 setting $\mathbf{c} = \mathbf{u}$, and the third equality comes from the duality theorem of linear programming.

Therefore, choose $\mathbf{y} \geq \mathbf{0}$ such that $\sum y_i = \alpha(G_U)$ and $\mathbf{u} \leq \mathbf{yA}$. Denoting the jth column of $\mathbf{A}$ by $\mathbf{a}^j$, we obtain

$$|U| = \sum_{j \in U} u_j \leq \sum_{j \in U} \mathbf{y} \cdot \mathbf{a}^j = \mathbf{y} \cdot \sum_{j \in U} \mathbf{a}^j \leq \mathbf{y} \cdot (\omega(G_U)\mathbf{1}) = \alpha(G_U)\omega(G_U).$$

Thus, by Theorem 3.3, G is perfect.

Conversely, assume that G is perfect. For any integer vector $\mathbf{c}$, form the graph H by multiplying the ith vertex of G by $\max(0, c_i)$ for each i. By Lemma 3.1, H is perfect. We have the following:

$\alpha(H) = \alpha_{\mathbf{c}}(G)$	The maximum weighted stable set of G given by $\mathbf{c}$.
$= \max\limits_{\text{subj } \mathbf{x} \in P_I(\mathbf{A})} \mathbf{c} \cdot \mathbf{x}$	The maximum can always be found at an extremum, which corresponds to a stable set (Lemma 3.13).
$\leq \max\limits_{\text{subj } \mathbf{x} \in P(\mathbf{A})} \mathbf{c} \cdot \mathbf{x}$	$P_I(\mathbf{A}) \subseteq P(\mathbf{A})$.
$= \min\limits_{\text{subj } \mathbf{yA} \geq \mathbf{c},\, \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1}$	Duality theorem.
$\leq \min\limits_{\text{subj } \mathbf{yA} \geq \mathbf{c},\text{ non-negative integral } \mathbf{y}} \mathbf{y} \cdot \mathbf{1}$	The constraint set is smaller.
$= k_{\mathbf{c}}(G)$	The minimum clique covering of G such that vertex i is covered c_i times. The constraint $\mathbf{yA} \geq \mathbf{c}$, non-negative integral $\mathbf{y}$, specifies such a covering.
$= k(H)$.	Any clique of H corresponds to a clique of G, thus $k(H) \geq k_{\mathbf{c}}(G)$; if vertex i of G is covered by c_i cliques, then there are c_i cliques in H, each covering a different copy of i, so $k_{\mathbf{c}}(G) \geq k(H)$.

But $\alpha(H) = k(H)$. Thus,

$$\max_{\text{subj } \mathbf{x} \in P_I(\mathbf{A})} \mathbf{c} \cdot \mathbf{x} = \max_{\text{subj } \mathbf{x} \in P(\mathbf{A})} \mathbf{c} \cdot \mathbf{x}$$

and, by Lemma 3.15, $P_I(\mathbf{A}) = P(\mathbf{A})$. ∎

Remark. The first half of the proof of Theorem 3.14 still holds under a weakened hypothesis on $\mathbf{A}$:

If $\mathbf{A}$ is a (0, 1)-valued matrix having no zero column whose derived graph equals G, then $P_I(\mathbf{A}) = P(\mathbf{A})$ implies that G is perfect.

5. A Polyhedral Characterization of p-Critical Graphs

Manfred Padberg first discovered the properties shown in Section 3.3 of p-critical graphs while investigating the facial structure of the polyhedra $P(\mathbf{A})$ for general (0, 1)-valued matrices **A**. In doing so, he also discovered a polyhedral characterization of p-critical graphs. In Padberg [1973, 1974], he used the results of Lovász and Chvátal to produce these results. In a later work, Padberg [1976b], he developed a more general approach, which enabled him to prove the same results directly and to prove the theorems of Lovász and Chvátal in a different manner.

The matrix **A** is said to be *perfect* if $P(\mathbf{A})$ is *integral*, that is, $P(\mathbf{A})$ has only integer extrema: $P_I(\mathbf{A}) = P(\mathbf{A})$. **A** is said to be *almost perfect* if $P(\mathbf{A})$ is *almost integral*, that is, (i) $P_I(\mathbf{A}) \neq P(\mathbf{A})$ ($P(\mathbf{A})$ has at least one nonintegral extremum), and (ii) the polyhedra $P_j(\mathbf{A}) = P(\mathbf{A}) \cap \{\mathbf{x} \in \mathbb{R}^n \mid x_j = 0\}$ are all integral, $j = 1, 2, \ldots, n$.

For the remainder of this section, **A** will always denote an $m \times n$ (0, 1)-valued matrix having no zero column, and P, P_I, and P_j will denote $P(\mathbf{A})$, $P_I(\mathbf{A})$, and $P_j(\mathbf{A})$, respectively.

Padberg's results, although not stated in the following manner, include the following six theorems.

Theorem 3.16. If **A** is perfect, then **A** is an augmented clique matrix of its derived graph, that is, **A** is the clique matrix possibly augmented with some redundant rows corresponding to nonmaximal cliques.

Let **J** denote the matrix of all ones and **I** the identity matrix. We say that **A** *contains* the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$ if some permutation of $\mathbf{J} - \mathbf{I}$ occurs as an $n \times n$ submatrix of **A**.

Theorem 3.17. If **A** is almost perfect, then either (i) **A** is an augmented clique matrix of its derived graph or (ii) **A** contains the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$.

Theorem 3.18. Let G be the derived graph of **A**. If **A** is almost perfect and does not contain the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$, then

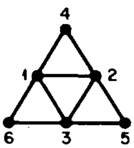
(i) $n = \alpha(G)\omega(G) + 1$;

(ii) every vertex of G is in exactly ω cliques of size ω and in exactly α stable sets of size α;

(iii) G has exactly n maximum cliques and n maximum stable sets;

(iv) there is a numbering of the maximum cliques $K_1, K_2, \ldots, K_n$ and maximum stable sets $S_1, S_2, \ldots, S_n$ of G such that $K_i \cap S_j = \varnothing$ if and only if $i = j$.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$G =$ 

$$\mathbf{A}' = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 3.4. The derived graph G of the matrix $\mathbf{A}$ is a perfect graph, yet $P(\mathbf{A})$ has $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)$ as an extremum; thus $\mathbf{A}$ is an imperfect matrix. $\mathbf{A}'$ is the clique matrix of G and is perfect.

Theorem 3.19. $\mathbf{A}$ is perfect if and only if $\mathbf{A}$ is an augmented clique matrix of its derived graph and the derived graph is perfect.

Corollary 3.20. $\mathbf{A}$ is almost perfect if and only if either (i) $\mathbf{A}$ is an augmented clique matrix of its derived graph and the derived graph is almost perfect (p-critical) or (ii) $\mathbf{A}$ has no row of all ones and contains the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$ for $n \geq 3$. Furthermore, in (ii) the derived graph is complete.

Corollary 3.21. Every p-critical graph has the four properties of Theorem 3.18.

Note carefully the wording of Theorem 3.19. It is very possible that $\mathbf{A}$ is not a perfect matrix and yet its derived graph G is perfect and every row of $\mathbf{A}$ corresponds to a maximal clique of G. Of course, in this case, by Theorem 3.19, the matrix is missing a row corresponding to some other maximal clique (see Figure 3.4).

Theorems 3.16, 3.17, 3.19, and Corollary 3.20 are very useful when considering graphs as incidence matrices. Corollary 3.21 is a restatement of Corollary 3.12.

To show Theorems 3.16 and 3.17 we will turn to the concept of *antiblocking polyhedra* (Fulkerson [1971, 1972]). Two polyhedra P_1 and P_2 are an *antiblocking pair* if $P_1 = \{\mathbf{x} \mid \mathbf{x}P_2 \leq \mathbf{1}, \mathbf{x} \geq \mathbf{0}\}$ or $P_2 = \{\mathbf{y} \mid \mathbf{y}P_1 \leq 1, \mathbf{y} \geq \mathbf{0}\}$, the conditions being equivalent. If P_2 is generated from a (0, 1)-valued matrix $\mathbf{A}_2$ having no zero column, then we have the property, among many others, that every extremum of P_1 is a projection of some row of $\mathbf{A}_2$ and every nonredundant row of $\mathbf{A}_2$ is an extremem of P_1 (Fulkerson [1972]). The same result holds if we interchange the indices.

Let $\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \ldots, \mathbf{b}^{(r)}$ be the extrema of P_I and denote the matrix having rows $\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \ldots, \mathbf{b}^{(r)}$ by $\mathbf{B}$. Define $Q = P(\mathbf{B})$, $Q_I = P_I(\mathbf{B})$ and $Q_j = P_j(\mathbf{B})$ for $j = 1, 2, \ldots, n$. The polyhedra P_I and Q are an antiblocking pair (Fulkerson [1972]). By Lemma 3.13, the rows of $\mathbf{B}$ correspond to all of the stable sets of G, the derived graph of $\mathbf{A}$. Thus, $\mathbf{B}$ is an augmented clique matrix of the complement $\bar{G}$. See also Monma and Trotter [1979].

Proof of Theorem 3.16. Let **A** be perfect ($P_I = P$) and G be its derived graph. Since P_I and Q are an antiblocking pair, P and Q are also an antiblocking pair. By the properties of antiblocking pairs, the extrema of Q must all be projections of the rows of **A**, so Q is integral. By Lemma 3.13, all the stable sets of $\bar{G}$, in other words all the cliques of G, are extrema of Q, since $Q = Q_I$. Thus, every clique of G must be a projection of some row of **A**. Therefore, **A** is an augmented clique matrix of its derived graph. ∎

Proof of Theorem 3.17. Assuming that **A** is almost perfect, P_j is integral for $j = 1, 2, \ldots, n$. By a similar argument to that for Theorem 3.16, each Q_j is also integral. This follows since $P(\mathbf{A}) \cap \{\mathbf{x} \in \mathbb{R}^n | x_j = 0\}$ is the same as removing the jth column from **A** and forming its polyhedron.

Case 1: *Q is not integral* and thus is almost integral. In this case Padberg was able to show by a direct analysis of the facets of P that P and Q_I are an antiblocking pair. As in the proof of Theorem 3.16, we have that **A** is an augmented clique matrix of its derived graph.

Case 2: *Q is integral.* In this case Padberg was able to show by the nonintegrality of P that **1** is an extremum of Q. This means that **B** must be the identity matrix (or a permutation of it). This in turn implies that the derived graph of **A** is complete. Therefore, for P_j to be integral, some row k of the matrix formed by deleting the jth column of **A** must be all ones (Theorem 3.16). Yet no row in **A** can have all ones since **A** is only almost perfect. Thus, row k in **A** must be all ones except for a zero in column j. Since this is true for all $j = 1, 2, \ldots, n$, **A** contains the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$. ∎

Although Theorem 3.18 is essentially contained among the results of Section 3.3, Padberg's proof does not use the Perfect Graph theorem and his technique is valuable in its own right. Before proving Theorem 3.18 we state Padberg's cornerstone lemma.

Lemma 3.22. If **x** is a nonintegral extremum of an almost integral polyhedron P, then for every $n \times n$ nonsingular submatrix $\mathbf{A}_1$ of **A** such that $\mathbf{A}_1\mathbf{x} = \mathbf{1}$, there exists an $n \times n$ nonsingular submatrix $\mathbf{B}_1$ of **B** satisfying the matrix equation

$$\mathbf{B}_1\mathbf{A}_1^{\mathrm{T}} = \mathbf{J} - \mathbf{I}.$$

Furthermore,

$$\mathbf{x} = (1/(n - 1))\mathbf{B}_1^{\mathrm{T}}\mathbf{1}.$$

As a quick observation, we note that for any noninteger extremum **x** of P, $\mathbf{x} > \mathbf{0}$. If for some k, $x_k = 0$, then $\mathbf{x} \in P_k$ and thus is an extremum of P_k. But then **x** would have to be integral. The only way **x** could be an extremum of P

is to satisfy n linearly independent constraints of $\mathbf{Ax} \le \mathbf{1}$. Let $\mathbf{A}_1$ be the $n \times n$ nonsingular submatrix of $\mathbf{A}$. Thus, for each $\mathbf{x}$ there does exist such an $\mathbf{A}_1$ as specified in Lemma 3.22.

Padberg was also able to show that $\mathbf{x}$, a noninteger extremum of P, is the unique noninteger extremum; that $\mathbf{y} = (1/(n-1))\mathbf{A}_1^{\mathrm{T}}\mathbf{1}$, for any $\mathbf{A}_1$ of Lemma 3.22, is an extremum of Q; and that for any $\mathbf{A}_1$ and corresponding $\mathbf{B}_1$ of Lemma 3.22, $\mathbf{x} = |\det \mathbf{A}_1^{-1}|\mathbf{1}$ and $\mathbf{y} = |\det \mathbf{B}_1^{-1}|\mathbf{1}$. Armed with these matrix equations, the proof of Theorem 3.18 is a straightforward exercise in linear algebra.

Proof of Theorem 3.18. Let G be the derived graph of $\mathbf{A}$, where $\mathbf{A}$ is almost perfect and does not contain the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$. By the definition of almost perfect we have the existence of a noninteger extremum $\mathbf{x}$ of P. By Lemma 3.22 and the previous discussion, $\mathbf{x}$ is unique and there exist $n \times n$ nonsingular submatrices $\mathbf{A}_1$ of $\mathbf{A}$ and $\mathbf{B}_1$ of $\mathbf{B}$ such that $\mathbf{A}_1\mathbf{x} = \mathbf{1}$ and $\mathbf{B}_1\mathbf{A}_1^{\mathrm{T}} = \mathbf{J} - \mathbf{I}$. Moreover, for all such $\mathbf{A}_1$ and $\mathbf{B}_1$, $\mathbf{x} = (1/(n-1))\mathbf{B}_1^{\mathrm{T}}\mathbf{1} = |\det \mathbf{A}_1^{-1}|\mathbf{1}$, and $\mathbf{y}$, defined by $\mathbf{y} = (1/(n-1))\mathbf{A}_1^{\mathrm{T}}\mathbf{1} = |\det \mathbf{B}_1^{-1}|\mathbf{1}$, is an extremum of Q.

We shall first show that $\mathbf{A}_1$ is unique, in that any row $\mathbf{a}_k$ of $\mathbf{A}$ satisfying $\mathbf{a}_k \cdot \mathbf{x} = 1$ is in $\mathbf{A}_1$. We have the following implications:

$$\mathbf{B}_1\mathbf{A}_1^{\mathrm{T}} = \mathbf{J} - \mathbf{I} \Rightarrow \mathbf{A}_1\mathbf{B}_1^{\mathrm{T}} = \mathbf{J} - \mathbf{I} \Rightarrow \mathbf{B}_1^{\mathrm{T}} = \mathbf{X} - \mathbf{A}_1^{-1} \Rightarrow \mathbf{A}_1^{-1} = \mathbf{X} - \mathbf{B}_1^{\mathrm{T}},$$

where $\mathbf{X}$ is the $n \times n$ matrix having n columns of $\mathbf{x}$. Thus, if $\mathbf{a}_k \cdot \mathbf{x} = \mathbf{1}$, then $\mathbf{a}_k\mathbf{A}_1^{-1} = \mathbf{a}_k\mathbf{X} - \mathbf{a}_k\mathbf{B}_1^{\mathrm{T}}$ is 0 or 1, yet $\mathbf{a}_k\mathbf{A}_1^{-1} \cdot \mathbf{1} = \mathbf{a}_k \cdot \mathbf{x} = \mathbf{1}$. Therefore, $\mathbf{a}_k\mathbf{A}_1^{-1} = \mathbf{e}_j$, the jth unit vector, for some $j \in \{1, 2, \dots, n\}$. This implies that $\mathbf{a}_k$ is equal to the jth row of $\mathbf{A}_1$, that is, $\mathbf{a}_k$ is in $\mathbf{A}_1$. Finally, since $\mathbf{x} = |\det \mathbf{A}_1^{-1}|\mathbf{1}$, we have that $\mathbf{A}_1$ contains exactly all the rows of $\mathbf{A}$ having the maximum number of ones. By Theorem 3.17, $\mathbf{A}$ is an augmented clique matrix of G. Therefore, $\mathbf{A}_1$ must contain exactly all the maximum cliques of G.

A similar argument holds for $\mathbf{y}$, $\mathbf{B}_1$, $\mathbf{B}$ and $\bar{G}$. Since

$$\mathbf{B}_1\mathbf{y} = \mathbf{B}_1((1/(n-1))\mathbf{A}_1^{\mathrm{T}}\mathbf{1}) = \mathbf{1},$$

we have $\mathbf{B}_1^{-1} = \mathbf{Y} - \mathbf{A}_1^{\mathrm{T}}$. Thus for any row $\mathbf{b}_k$ of $\mathbf{B}$ satisfying $\mathbf{b}_k \cdot \mathbf{y} = 1$, we have $\mathbf{b}_k\mathbf{B}_1^{-1} = \mathbf{b}_k\mathbf{Y} - \mathbf{b}_k\mathbf{A}_1^{\mathrm{T}}$, and yet $\mathbf{b}_k\mathbf{B}_1^{-1} \cdot \mathbf{1} = \mathbf{b}_k \cdot \mathbf{y} = 1$. So $\mathbf{b}_k$ is in $\mathbf{B}_1$. Since $\mathbf{y} = |\det \mathbf{B}_1^{-1}|\mathbf{1}$, and since by construction $\mathbf{B}$ is an augmented clique matrix of $\bar{G}$, we have that $\mathbf{B}_1$ must contain exactly all the maximum cliques of $\bar{G}$.

(i) The row sum of $\mathbf{A}_1$ is $\omega(G)$, yet $\mathbf{A}_1\mathbf{1} = \mathbf{A}_1|\det \mathbf{A}_1|\mathbf{x} = |\det \mathbf{A}_1|\mathbf{1}$; thus $|\det \mathbf{A}_1| = \omega(G)$. Similarly for $\mathbf{B}_1$, the row sum is $\alpha(G)$, yet

$$\mathbf{B}_1\mathbf{1} = \mathbf{B}_1|\det \mathbf{B}_1|\mathbf{y} = |\det \mathbf{B}_1|\mathbf{1};$$

so $|\det \mathbf{B}_1| = \alpha(G)$. Therefore, $\alpha(G)\omega(G) = |\det \mathbf{B}_1\mathbf{A}_1^{\mathrm{T}}| = |\det(\mathbf{E} - \mathbf{I})| = |(-1)^{n-1}(n-1)| = n - 1$. Thus, $n = \alpha(G)\omega(G) + 1$.

(ii) Since $\mathbf{y} = (1/(n-1))\mathbf{A}_1^{\mathrm{T}}\mathbf{1}$, we have $(1/\alpha)\mathbf{1} = (1/\alpha\omega)$, $\mathbf{A}_1^{\mathrm{T}}\mathbf{1}$, and thus $\omega\mathbf{1} = \mathbf{A}_1^{\mathrm{T}}\mathbf{1}$. That is, all the column sums of $\mathbf{A}_1$ are ω. Therefore, every vertex is in exactly ω cliques of size ω.

Similarly for $\mathbf{x}$, $\mathbf{x} = (1/(n-1))\mathbf{B}_1^{\mathrm{T}}\mathbf{1}$ implies $(1/\omega)\mathbf{1} = (1/\alpha\omega)\mathbf{B}_1^{\mathrm{T}}\mathbf{1}$, and hence $\alpha\mathbf{1} = \mathbf{B}_1^{\mathrm{T}}\mathbf{1}$. Therefore, every vertex is in exactly α stable sets of size α.

(iii) $\mathbf{A}_1$ is an $n \times n$ nonsingular matrix containing exactly all the maximum cliques of G; therefore, G has exactly n maximum cliques. By a similar argument on $\mathbf{B}_1$, G has exactly n maximum stable sets.

(iv) Let K_i correspond to the ith row of $\mathbf{A}_1$ for $i = 1, 2, \ldots, n$, and S_j correspond to the jth row of $\mathbf{B}_1$ for $j = 1, 2, \ldots, n$. Since $\mathbf{B}_1\mathbf{A}_1^{\mathrm{T}} = \mathbf{J} - \mathbf{I}$, the maximum cliques $K_1, K_2, \ldots, K_n$ and maximum stable sets $S_1, S_2, \ldots, S_n$ of G are numbered such that $K_i \cap S_j = \varnothing$ if and only if $i = j$. ▮

The "only if" condition of Theorem 3.19 is a stronger statement than Theorem 3.16; it states that the derived graph itself is perfect, which also turns out to be a sufficient condition for $\mathbf{A}$ to be perfect. In fact, Theorem 3.19 is precisely Chvátal's Theorems 3.14 and 3.16 put together. A more direct proof here will be instructive. Again we need an intermediate result of Padberg's.

Lemma 3.23. P is integral if and only if $\max_{\text{subj } \mathbf{x} \in P} \mathbf{q} \cdot \mathbf{x} \equiv 0 \bmod 1$ for all (0, 1)-valued $\mathbf{q}$.

It is well known that for a general matrix $\mathbf{A}$ with non-negative entries and no zero column, satisfying $\max_{\text{subj } \mathbf{x} \in P} \mathbf{c} \cdot \mathbf{x} \equiv 0 \bmod 1$ for all non-negative $\mathbf{c}$ is equivalent to P being integral. But for our matrix $\mathbf{A}$, considering only (0, 1)-valued $\mathbf{q}$ is sufficient.

Proof of Theorem 3.19. ($\Leftarrow$) Let $\mathbf{A}$ be an augmented clique matrix of its derived graph G, where G is perfect. Let $\mathbf{q}$ be a (0, 1)-valued vector and G' its corresponding induced subgraph of G. Then

$$\alpha(G') = \max_{\text{subj } \mathbf{x} \in P_I} \mathbf{q} \cdot \mathbf{x} \le \max_{\text{subj } \mathbf{x} \in P} \mathbf{q} \cdot \mathbf{x} = \min_{\text{subj } \mathbf{y}\mathbf{A} \ge \mathbf{q},\ \mathbf{y} \ge \mathbf{0}} \mathbf{y} \cdot \mathbf{1}$$
$$\le \min_{\text{subj } \mathbf{y}\mathbf{A} \ge \mathbf{q},\ \mathbf{y} \ge \mathbf{0},\ \text{integral}} \mathbf{y} \cdot \mathbf{1} = k(G').$$

The first equality is clear because of Lemma 3.13 and the fact that an optimal solution can always be found at an extremum. The last equality is true since $\mathbf{A}$ is an augmented clique matrix and any optimal $\mathbf{y}$ is (0, 1)-valued. The inequalities have been seen before in Section 3.4.

Now since G is perfect, we must have equality everywhere. Thus,

$$\max_{\text{subj } \mathbf{x} \in P} \mathbf{q} \cdot \mathbf{x} \equiv 0 \bmod 1.$$

Finally, since $\mathbf{q}$ was arbitrary, Lemma 3.23 implies that P is integral, and thus $\mathbf{A}$ is perfect.

($\Rightarrow$) Let $\mathbf{A}$ be perfect. By Theorem 3.16, $\mathbf{A}$ is an augmented clique matrix of its derived graph. To show that G is perfect we shall use induction on the size of the induced subgraphs.

For $|G'| = 0$ it is clear that $\alpha(G') = k(G')$. Assume that every k-vertex induced subgraph is perfect. Given $|G'| = k + 1$, let $\mathbf{q}$ be the characteristic vector of G'. Since P is integral and $\mathbf{A}$ is an augmented clique matrix of G,

$$\begin{aligned}\alpha(G') = \max_{\text{subj } \mathbf{x} \in P} \mathbf{q} \cdot \mathbf{x} = \min_{\text{subj } \mathbf{yA} \geq \mathbf{q},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1} \leq \min_{\text{subj } \mathbf{yA} \geq \mathbf{q},\ \mathbf{y} \geq \mathbf{0},\ \text{integral}} \mathbf{y} \cdot \mathbf{1} \\ = k(G'). \qquad (10)\end{aligned}$$

We claim that there is an integer optimal solution for $\min_{\text{subj } \mathbf{yA} \geq \mathbf{q},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1}$. We know that an optimal solution $\bar{\mathbf{y}}$ exists. If $\bar{\mathbf{y}}$ is integral we are done; otherwise there is a k such that $0 < \bar{y}_k < 1$. Clearly the kth row $\mathbf{a}_k$ of $\mathbf{A}$ has the property $\mathbf{a}_k \cdot \mathbf{q} > 0$, for otherwise $\bar{\mathbf{y}}$ would not be optimal. Define $\bar{\bar{q}}_i = q_i$ for $a_{ki} = 0$ and $\bar{\bar{q}}_i = 0$ for $a_{ki} = 1$. Since $\bar{\bar{\mathbf{q}}}$ is the characteristic vector of a smaller induced subgraph, and since (10) still holds, there is an integer optimal solution $\bar{\bar{\mathbf{y}}}$ for $\bar{\bar{\mathbf{q}}}$. Clearly any optimal solution for $\bar{\bar{\mathbf{q}}}$ has its kth component zero; thus $\bar{\mathbf{y}}$ is feasible but not optimal for $\bar{\bar{\mathbf{q}}}$. Yet $\mathbf{y}^*$, where $\mathbf{y}^* = \bar{\bar{\mathbf{y}}}$ except for $y_k^* = 1$, is feasible for $\mathbf{q}$. That is,

$$\min_{\text{subj } \mathbf{yA} \geq \bar{\bar{\mathbf{q}}},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1} < \min_{\text{subj } \mathbf{yA} \geq \mathbf{q},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1} \leq \mathbf{y}^* \cdot \mathbf{1} = \min_{\text{subj } \mathbf{yA} \geq \bar{\bar{\mathbf{q}}},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1} + 1.$$

Therefore, $\mathbf{y}^*$ is an integer optimal solution for $\min_{\text{subj } \mathbf{yA} \geq \mathbf{q},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1}$ and thus $\alpha(G') = k(G')$. ∎

The observant reader will notice that the same "only if" proof could have been used in Theorem 3.14.

The proofs of Corollaries 3.20 and 3.21 are now easy.

Proof of Corollary 3.20. ($\Leftarrow$) *Case 1*: Let $\mathbf{A}$ be an augmented clique matrix of its derived graph G, where G is p-critical. Since deleting any vertex j of G results in a perfect graph, all the P_j are integral. Yet by Theorem 3.19, $\mathbf{A}$ is imperfect because G is imperfect; therefore $\mathbf{A}$ is almost perfect.

Case 2: Let $\mathbf{A}$ have no row of all ones and contain the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$ for $n \geq 3$. Since each P_j is obtained from the matrix $\mathbf{A}$ with its jth column deleted, and since this submatrix has a row containing all ones, all P_j are integral. Yet $(1/(n - 1))\mathbf{1}$ is an extremum of P, since every row has at most $n - 1$ ones and $\mathbf{J} - \mathbf{I}$ is an $n \times n$ submatrix. Therefore $\mathbf{A}$ is almost perfect.

($\Rightarrow$) Given that $\mathbf{A}$ is almost perfect, we apply Theorem 3.17 to obtain two cases.

Case 1: **A** *is an augmented clique matrix of its derived graph* G. By Theorem 3.19 each P_j is integral, the submatrix of **A** obtained by deleting the jth column is perfect, and thus the deletion of any vertex j of G results in a perfect graph. Yet by Theorem 3.19 again, G itself is not perfect since **A** is not perfect. Therefore, G is p-critical.

Case 2: **A** *contains the* $n \times n$ *submatrix* $\mathbf{J} - \mathbf{I}$. Clearly **A** does not contain a row of all ones, for otherwise **A** would be perfect. Finally, we must certainly have $n \geq 3$, thus G is complete. ∎

Proof of Corollary 3.21. Given a p-critical graph G, form **A**, its clique matrix. By Corollary 3.20, case 1, **A** is almost perfect. Certainly G is the derived graph of **A**, and thus the hypothesis of Theorem 3.18 is satisfied. ∎

6. The Strong Perfect Graph Conjecture

The odd cycle C_{2k+1} (for $k \geq 2$) is not a perfect graph since $\alpha(C_{2k+1}) = k$ and $k(C_{2k+1}) = k + 1$ (or, alternatively, since $\omega(C_{2k+1}) = 2$ and $\chi(C_{2k+1}) = 3$). However, every proper subgraph of C_{2k+1} is perfect. Thus, C_{2k+1} is a p-critical graph (i.e., minimally imperfect) and by the Perfect Graph theorem its complement $\bar{C}_{2k+1}$ is also p-critical. To date, these are the only known p-critical graphs.

During the second international meeting on graph theory, held at Halle-on-Saal in March 1960, Claude Berge raised the question of whether or not other p-critical graphs besides the odd cycles and their complements exist. He conjectured that there are none, and this has come to be known as the *strong perfect graph conjecture* (*SPGC*). (Actually, the word "conjecture" first appeared in Berge [1962].)

The strong perfect graph conjecture may be stated in several equivalent forms:

SPGC$_1$. An undirected graph is perfect if and only if it contains no induced subgraph isomorphic to C_{2k+1} or $\bar{C}_{2k+1}$ (for $k \geq 2$).

SPGC$_2$. An undirected graph G is perfect if and only if in G and in $\bar{G}$ every odd cycle of length ≥ 5 has a chord.

SPGC$_3$. The only p-critical graphs that exist are C_{2k+1} and $\bar{C}_{2k+1}$ (for $k \geq 2$).

The graphs C_{2k+1} and $\bar{C}_{2k+1}$ are commonly referred to as the *odd hole* and the *odd antihole*, respectively.

We have seen in Sections 3.3 and 3.5 that p-critical graphs reflect an extraordinary amount of symmetry (as indeed they should if the SPGC turns out to be true). Let G be a p-critical graph on n vertices, and let $\alpha = \alpha(G)$ and $\omega = \omega(G)$. Then the following conditions hold for G.

Lovász condition

$$n = \alpha\omega + 1$$

Padberg conditions

Every vertex is in exactly ω maximum cliques (of size ω).
Every vertex is in exactly α maximum stable sets (of size α).
G has exactly n maximum cliques (of size ω).
G has exactly n maximum stable sets (of size α).
The maximum cliques and maximum stable sets can be indexed $K_1, K_2, \ldots, K_n$ and $S_1, S_2, \ldots, S_n$, respectively, so that $|K_i \cap S_j| = 1 - \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Clearly, any p-critical graph must be connected. But C_n is the only connected graph on n vertices for which $\omega = 2$ and having exactly n undirected edges such that each vertex is an endpoint of exactly two of these edges. So, by Padberg's conditions we obtain another equivalent form of the strong perfect graph conjecture:

SPGC$_4$. There is no p-critical graph with $\alpha > 2$ and $\omega > 2$.

Recall from Section 3.3 that a partitionable graph on n vertices satisfies the Lovász and Padberg conditions.

Figures 3.5 and 3.9 give two examples of (3, 3)-partitionable graphs which fail to be p-critical. For this reason, the Lovász and Padberg conditions alone are not sufficient to prove the SPGC. Nevertheless, partitionable graphs do give us further reductions of the SPGC.

One special type of partitionable graph is easy to describe. The undirected graph C_n^d has vertices $v_1, v_2, v_3, \ldots, v_n$ with v_i and v_j joined by an edge if and only if i and j differ by at most d. (Here and in the next theorem all subscript arithmetic is taken modulo n.) It is easy to see that the graph $C_{\alpha\omega+1}^{\omega-1}$ is an (α, ω)-partitionable graph. When $\omega = 2$, then $C_{\alpha\omega+1}^{\omega-1}$ is simply the odd hole $C_{2\alpha+1}$; when $\alpha = 2$, then $C_{\alpha\omega+1}^{\omega-1}$ is the odd antihole $\overline{C}_{2\omega+1}$.

Theorem 3.24 (Chvátal [1976]). For any integers $\alpha \geq 3$ and $\omega \geq 3$, the partitionable graph $C_{\alpha\omega+1}^{\omega-1}$ is not p-critical.

Proof. Let $\alpha \geq 3$ and $\omega \geq 3$ be given. We will show that $C_{\alpha\omega+1}^{\omega-1}$ contains a proper induced subgraph H which is not perfect.

If we index the $n = \alpha\omega - 1$ maximal cliques $\{K_i\}$ of $C_{\alpha\omega+1}^{\omega-1}$ so that $K_i = \{v_i, v_{i+1}, \ldots, v_{i+\omega-1}\}$ for each $1 \leq i \leq n$, then the clique matrix of the graph has the familiar cyclical pattern, as shown in Figure 3.6. Let H denote the

	0	1	2	3	4	5	6	7	8	9
K_1	1	1								1
K_2	1	1	1							
K_3		1	1	1						
K_4				1	1	1				
K_5					1	1	1			
K_6						1	1	1		
K_7			1				1	1		
K_8								1	1	1
K_9				1	1				1	
K_{10}	1								1	1

Clique matrix

	0	1	2	3	4	5	6	7	8	9
S_1			1			1			1	
S_2				1			1			1
S_3	1				1			1		
S_4		1					1		1	
S_5	1			1				1		
S_6			1		1					1
S_7		1				1			1	
S_8	1			1			1			
S_9			1			1				1
S_{10}		1			1			1		

Stable set matrix

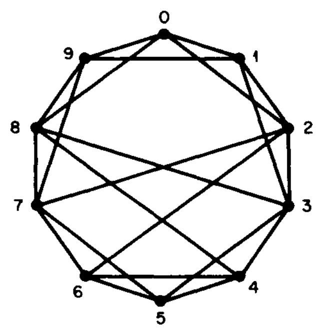


Figure 3.5. A graph satisfying the Lovász and Padberg conditions which fails to be p-critical. The clique matrix and stable set matrix indicate the required indexing of the maximum cliques and maximum stable sets. This example was discovered independently by Huang [1976] and by Chvátal, Graham, Perold, and Whitesides [1979].

subgraph remaining after deleting the $\alpha + 2$ vertices $v_n, v_2, v_{\omega+1}, v_{\omega+3}$, and all $v_{t\omega+2}$ for $t = 2, 3, \ldots, \alpha - 1$. In the deleting process every maximum clique has lost at least one of its members, so $\omega(H) \leq \omega - 1$. Therefore, it suffices to show that H cannot be colored using $\omega - 1$ colors.

Suppose that H is $\omega - 1$ colorable. Let v_1 be colored *black* and let the $\omega - 2$ additional colors be called the *rainbow*. We have the following series of implications:

$$\{v_1, v_3, \ldots, v_\omega\} \subset K_1 \Rightarrow \{v_3, \ldots, v_\omega\} \text{ requires the entire rainbow;}$$

$$\{v_3, \ldots, v_\omega, v_{\omega+2}\} \subset K_3 \Rightarrow v_{\omega+2} \text{ is black;}$$

$$\{v_{\omega+2}, v_{\omega+4}, \ldots, v_{2\omega+1}\} \subset K_{\omega+2}$$
$$\Rightarrow \{v_{\omega+4}, \ldots, v_{2\omega+1}\} \text{ requires the entire rainbow;}$$

$$\{v_{\omega+4}, \ldots, v_{2\omega+1}, v_{2\omega+3}\} \subset K_{\omega+4} \Rightarrow v_{2\omega+3} \text{ is black;}$$

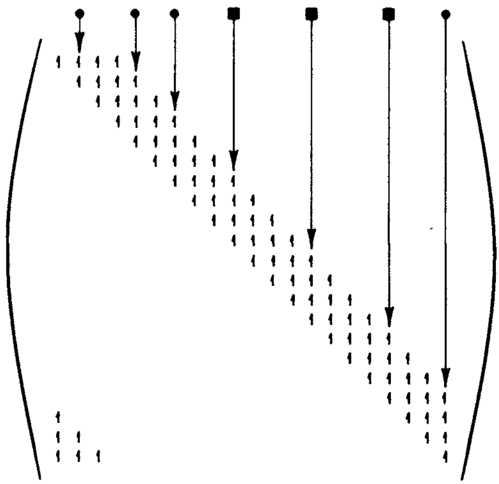


Figure 3.6. The clique matrix of $C_{\alpha\omega+1}^{\omega-1}$, where $\omega = 4$ and $\alpha = 5$. The markers designate which vertices are to be deleted to obtain an imperfect subgraph.

and finally, by induction on t,

$$
\begin{aligned}
v_{t\omega+3} &\text{ is black}\\
&\Rightarrow \{v_{t\omega+4}, \ldots, v_{(t+1)\omega+1}\} \text{ requires the entire rainbow}\\
&\Rightarrow \{v_{(t+1)\omega+3}\} \text{ is black,}
\end{aligned}
$$

for $t = 2, \ldots, \alpha - 2$.

Therefore, both v_1 and $v_{(\alpha-1)\omega+3}$ are black, but they are both contained in the clique $K_{(\alpha-1)\omega+3}$, a contradiction. Hence, $\chi(H) > \omega - 1 \geq \omega(H)$ and H is imperfect as required. ∎

As a corollary of Theorem 3.24 we obtain another equivalent version of the strong perfect graph conjecture:

SPGC$_5$. If G is p-critical with $\alpha(G) = \alpha$ and $\omega(G) = \omega$, then G contains an induced subgraph isomorphic to $C_{\alpha\omega+1}^{\omega-1}$.

Chvátal, Graham, Perold, and Whitesides [1979] have presented two procedures for constructing (α, ω)-partitionable graphs other than $C_{\alpha\omega+1}^{\omega-1}$.

If we we restrict the universe of graphs being considered by making an extra assumption about their structure, then, in certain cases, the SPGC can be shown to hold. Table 3.1 lists some successful restrictions. For the most part the original proofs cited do not make use of the Padberg conditions. Tucker [1979] has incorporated the Padberg conditions into new proofs of the SPGC for $K_{1,3}$-free graphs and 3-chromatic graphs.

Table 3.1

Classes of graphs for which the strong perfect graph conjecture is known to hold

Planar graphs	Tucker [1973a]
$K_{1,3}$-free graphs	Parthasarathy and Ravindra [1976]
Circular-arc graphs	Tucker [1975]
⊠-free graphs	Parthasarathy and Ravindra [1979]
3-chromatic graphs (actually, any graph with $\omega \leq 3$)	Tucker [1977]
Toroidal graphs; graphs having maximum vertex degree ≤ 6	Grinstead [1978]

The strong perfect graph conjecture remains a formidable challenge to us. Its solution has eluded researchers for two decades. Perhaps in the third decade a reader of this book will settle the problem.

EXERCISES

1. Let x and y be distinct vertices of a graph G. Prove that $(G \circ x) - y = (G - y) \circ x$.

2. Let $x_1, x_2, \ldots, x_n$ be the vertices of a graph G and let $H = G \circ \mathbf{h}$ where $\mathbf{h} = (h_1, h_2, \ldots, h_n)$ is a vector of non-negative integers.

Verify that H can be constructed by the following procedure:

```
begin
  H ← G;
  for i ← 1 to n do
    if h_i = 0 then H ← H − x_i;
    else while h_i > 0 do
      begin
        H ← H ∘ x_i;
        h_i ← h_i − 1;
      end
end
```

3. Give an example of a graph G for which $\alpha(G) = k(G)$ and $\omega(G) < \chi(G)$. Why does this not contradict the Perfect Graph theorem?

4. Suppose G satisfies $\alpha(G) = k(G)$. Let $\mathscr{K}$ be a clique cover of G where $|\mathscr{K}| = k(G)$, and let $\mathscr{S}$ be the collection of all stable sets of cardinality $\alpha(G)$. Show that

$$|S \cap K| = 1 \qquad \text{for all } S \in \mathscr{S} \text{ and } K \in \mathscr{K}.$$

Give a dual statement for a graph satisfying $\omega(G) = \chi(G)$.

5. Prove the following: For any integer k, there exists a graph G such that $\omega(G) = 2$ and $\chi(G) = k$. Thus, the gap between the clique number and the chromatic number can be arbitrarily large (Tutte [1954], Kelly and Kelly [1954], Zykov [1952]; see also Sachs [1969]).

6. Prove that an n-vertex graph G is an odd chordless cycle if and only if $n = 2k + 1$, $\alpha(G) = k$, and $\alpha(G - v - w) = k$ for all vertices v and w of G (Melnikov and Vising [1971], Greenwell [1978]).

7. An undirected graph G is *unimodular* if its clique matrix **A** has the property that every square submatrix of **A** has determinant equal to 0, +1, or −1. Prove the following:

(i) The graph in Figure 3.7 is unimodular;
(ii) unimodularity is a hereditary property;
(iii) a bipartite graph is unimodular;
(iv) a unimodular graph is perfect (if necessary, for (iv) see Berge [1975]).

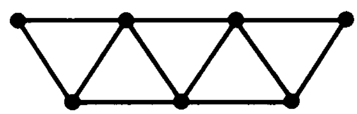


Figure 3.7

8. Show that the five versions of the strong perfect graph conjecture given in this chapter are equivalent.

9. Prove that G is p-critical if and only if G is partitionable but no proper induced subgraph of G is partitionable.

10. Show that the graph in Figure 3.8 is partitionable but not p-critical. Show that the graph in Figure 3.9 is imperfect but not partitionable.

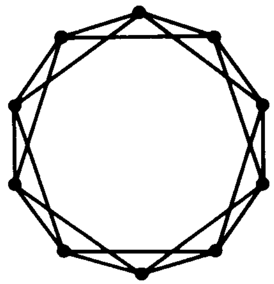


Figure 3.8

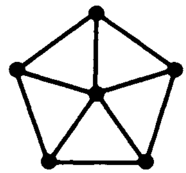


Figure 3.9

11. Let **A** and **B** be $n \times n$ matrices and let α and ω be integers. Using matrix operations give a short proof of the following: If $\mathbf{AJ} = \mathbf{JA} = \omega\mathbf{J}$, $\mathbf{BJ} = \mathbf{JB} = \alpha\mathbf{J}$, and $\mathbf{AB}^{\mathrm{T}} = \mathbf{J} - \mathbf{I}$, then $\alpha\omega = n - 1$.
12. Let $G = (X, E)$ and $H = (Y, F)$ be undirected graphs. Their normal product is defined to be the graph $G \cdot H$ whose vertex set is the Cartesian product $X \times Y$ with vertices (x, y) and (x', y') adjacent if and only if

$$x = x' \quad \text{and} \quad yy' \in F \qquad \text{or} \qquad xx' \in E \quad \text{and} \quad y = y'$$

or

$$xx' \in E \quad \text{and} \quad yy' \in F.$$

Prove the following:

(i) $\chi(G \cdot h) \geq \max\{\chi(G), \chi(H)\}$;
(ii) $\omega(G \cdot H) = \omega(G)\omega(H)$;
(iii) $\alpha(G \cdot H) \geq \alpha(G)\alpha(H)$;
(iv) $k(G \cdot H) \leq k(G)k(H)$.

13. Let G^r denote the normal product of G with itself $r - 1$ times, i.e., $G^1 = G$ and $G^r = G \cdot G^{r-1}$. Let

$$c(G) = \sup \sqrt[r]{\alpha(G^r)}.$$

Prove that $\alpha(G) = k(G)$ implies $c(G) = \alpha(G)$. For an application of this to zero-capacity codes, see Berge [1973, p. 382; 1975, p. 13].

Bibliography

Balinski, M. L., and Hoffman, A. J., eds.
[1978] "*Polyhedral Combinatorics*," Math. Programming Studies, Vol. 8. North-Holland, Amsterdam.

Baumgartner, J. E., Malitz, J., and Reinhardt, W.
[1970] Embedding trees in the rationals. *Proc. Nat. Acad. Sci. U.S.A.* **67**, 1748–1753.

Berge, Claude
[1961] Farbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind, *Wiss. Z. Martin-Luther-Univ., Halle-Wittenberg Math.-Natur, Reihe*, 114–115.

[1962] Sur une conjecture relative au problème des codes optimaux, *Comm. 13ieme Assemblee Gen. URSI*, Tokyo, 1962.

[1967] Some classes of perfect graphs, "Graph Theory and Theoretical Physics," pp. 155–165. Academic Press, London–New York. MR38 #1017.

[1969] Some classes of perfect graphs, "Combinatorial Mathematics and its Applications," pp. 539–552. Univ. North Carolina Press, Chapel Hill. MR42 #100.

[1973] "Graphs and Hypergraphs," Chapter 16. North-Holland, Amsterdam. MR50 #9640.

[1975] Perfect graphs, *in* "Studies in Graph Theory," Part I (D. R. Fulkerson, ed.), M.A.A. Studies in Mathematics Vol. 11, pp. 1–22. Math. Assoc. Amer., Washington, D.C. MR53 #10585.

[1976] Short note about the history of the perfect graph conjecture. (mimeographed).

Bland, Robert, G., Huang, H.-C., and Trotter, Leslie E., Jr.

[1979] Graphical properties related to minimal imperfection, *Discrete Math.* **27**, 11–22.

Bollobás, Béla

[1978] "Extremal Graph Theory," pp. 263–270. Academic Press, London.

Chvátal, Václav

[1973] Edmonds polytopes and a hierarchy of combinatorial problems, *Discrete Math.* **4**, 305–337. MR47 #1635.

[1975] On certain polytopes associated with graphs, *J. Combin. Theory B* **18**, 138–154. MR51 #7949.

[1976] On the strong perfect graph conjecture, *J. Combin. Theory B* **20**, 139–141. MR54 #129.

Chvátal, Václav, Graham, R. L., Perold, A. F., and Whitesides, S. H.

[1979] Combinatorial designs related to the strong perfect graph conjecture, *Discrete Math.* **26**, 83–92.

Commoner, F. G.

[1973] A sufficient condition for a matrix to be totally unimodular, *Networks* **3**, 351–365. MR49 #331.

de Werra, D.

[1978] On line perfect graphs, *Math. Programming* **15**, 236–238.
Gives alternate proof of result of L. E. Trotter [1977].

Edmonds, Jack

[1965] Maximum matching and a polyhedron with 0,1-vertices, *J. Res. Nat. Bur. Standards Sect. B* **69**, 125–130. MR32 #1012.

Fulkerson, Delbert Ray

[1969] The perfect graph conjecture and pluperfect graph theorem, *2nd Chapel Hill Conf. on Combin. Math. and its Appl.*, 171–175.

[1971] Blocking and anti-blocking pairs of polyhedra, *Math. Programming* **1**, 168–194. MR45 #3222.

[1972] Anti-blocking polyhedra, *J. Combin. Theory* **12**, 50–71. MR44 #2629.

[1973] On the perfect graph theorem, *in* "Mathematical Programming," (T. C. Hu and S. Robinson, eds.), pp. 68–76. Academic Press, New York. MR51 #10147.

Greenwell, Don

[1978] Odd cycles and perfect graphs, "Theory and Applications of Graphs," Lecture Notes in Math. Vol. 642, pp. 191–193. Springer-Verlag, Berlin.

Grinstead, Charles M.

[1978] Toroidal graphs and the strong perfect graph conjecture, Ph.D. thesis, UCLA.

Hajnal, Andras, and Surányi, Janos
[1958] Über die Auflösung von Graphen in vollständige Teilgraphen, *Ann. Univ. Sci. Budapest Eötvös. Sect. Math.* **1**, 113–121. MR21 #1944.
Huang, H.-C.
[1976] Investigations on combinatorial optimization, Ph.D. thesis, Yale Univ.
Jolivet, J. L.
[1975] Graphes parfaits pour une propriété, *P. Cahiers Centre Etudes Rech. Opér.* **17**, 253–256. MR53 #7841.
Karpetjan, I. A.
[1976] Critical and essential edges in perfect graphs (Russian; Armenian summary), *Akad. Nauk Armjan. SSR Dokl.* **63**, 65–70.
Kelly, John B., and Kelly, L. M.
[1954] Path and circuits in critical graphs, *Amer. J. Math.* **76**, 786–792. MR16, p. 387.
Lovász, László
[1972a] Normal hypergraphs and the perfect graph conjecture, *Discrete Math.* **2**, 253–267. MR46 #1624.
[1972b] A characterization of perfect graphs, *J. Combin. Theory B* **13**, 95–98. MR46 #8885.
Markosjan, S. E.
[1975] Perfect and critical graphs (Russian; Armenian summary), *Akad. Nauk Armjan. SSR Dokl.* **60**, 218–223. MR53 #10659.
Markosjan, S. E., and Karpetjan, I. A.
[1976] Perfect graphs (Russian; Armenian summary). *Akad. Nauk Armjan. SSR Dokl.* **63**, (1976), 292–296. MR56 #8427.
Melnikov, L. S., and Vising, V. G.
[1971] Solution to Toft's problem (Russian), *Diskret. Analiz.* **19**, 11–14. MR46 #3379. See Greenwell [1978].
Monma, C. L., and Trotter, Leslie E., Jr.
[1979] On perfect graphs and polyhedra with (0,1)-valued extreme points, *Math. Programming* **17**, 239–242.
Nash-Williams, C. St. J. A.
[1967] Infinite graphs—a survey, *J. Combin. Theory* **3**, 286–301. MR35 #5351.
Olaru, Elefterie
[1969] Über die Überdeckung von Graphen mit Cliquen, *Wiss. Z. Tech. Hochsch. Ilmenau* **15**, 115–121. MR43 #3162.
[1972] Beiträge zur Theorie der perfekten Graphen, *Elektron. Informationsverarb. Kybernet.* **8**, (1972), 147–172. MR47 #8338.
[1973a] Über perfekte und kritisch imperfekte Graphen, *Ann. St. Univ. Iasi* **19**, 477–486. MR54 #5053.
[1973b] Zur Charakterisierung perfekter Graphen, *Electron. Informationsverarb. Kybernet.* **9**, 543–548. MR51 #10167.
[1977] Zur Theorie der perfekten Graphen, *J. Combin. Theory B* **23**, 94–105.
Padberg, Manfred W.
[1973] On the facial structure of set packing polyhedra, *Math. Programming* **5**, 199–215. MR51 #4990.
[1974] Perfect zero–one matrices, *Math. Programming* **6**, 180–196. MR49 #4809.
[1975] Characterizations of totally unimodular, balanced and perfect matrices, *in* "Combinatorial Programming: Methods and Applications," Proc. NATO Advanced Study Inst., Versailles, France, 1974, pp. 275–284. Reidel, Dordrecht, MR53 #10291. An excellent survey article with good bibliography and clearly written.
[1976a] A note on the total unimodularity of matrices, *Discrete Math.* **14**, 273–278. MR54 #2685.

[1976b] Almost integral polyhedra related to certain combinatorial optimization problems, *Linear Algebra and Appl.* **15**, 69–88.

Parthasarathy, K. R., and Ravindra, G.

[1976] The strong perfect-graph conjecture is true for $K_{1,3}$-free graphs, *J. Combin. Theory B* **21**, 212–223. MR55 #10308.

[1979] The validity of the strong perfect-graph conjecture for $(K_4 - e)$-free graphs, *J. Combin. Theory B* **26**, 98–100.

Perles, M. A.

[1963] On Dilworth's theorem in the infinite case, *Israel J. Math.* **1**, 108–109. MR29 #5758.

Pretzel, Oliver

[1979] Another proof of Dilworth's decomposition theorem, *Discrete Math.* **25**, 91–92.

Ravindra, G.

[1975] On Berge's conjecture concerning perfect graphs, *Proc. Indian Nat. Sci. Acad.* **41A**, 294–296.

Sachs, Horst

[1969] Finite graphs (Investigations and generalizations concerning the construction of finite graphs having given chromatic number and no triangles), "Recent Progress in Combinatorics," pp. 175–184. Academic Press, New York. MR42 #2980.

[1970] On the Berge conjecture concerning perfect graphs, "Combinatorial Structures and their Applications," pp. 377–384. Gordon & Breach, New York. MR42 #7549.

Tamir, A.

[1976] On totally unimodular matrices, *Networks* **6**, 373–382.

Tomescu, Ioan

[1971] Sur le nombre des cliques maximales d'un graphe et quelques problèmes sur les graphes parfaits, *Rev. Roumaine Math. Pures Appl.* **16**, 1115–1126. MR45 #103.

Trotter, Leslie E., Jr.

[1977] Line perfect graphs, *Math. Programming* **12**, 255–259. MR56 #15501.
Characterizes those graphs G whose line graphs $L(G)$ are perfect.

Trotter, William Thomas, Jr.

[1971] A note on triangulated graphs, *Notices Amer. Math. Soc.* **18**, 1045 (A).

Tucker, Alan C.

[1972] The strong perfect graph conjecture and an application to a municipal routing problem, *in* "Graph Theory and Applications," Proc. Conf. Western Michigan Univ., Kalamazoo, Lecture Notes in Math., Vol. 303, pp. 297–303. Springer-Verlag, Berlin. MR49 #7181.

[1973a] The strong perfect graph conjecture for planar graphs, *Canad. J. Math.* **25**, 103–114. MR47 #4868.

[1973b] Perfect graphs and an application to optimizing municipal services, *SIAM Rev.* **15**, 585–590. MR48 #3817.

[1977] Critical perfect graphs and perfect 3-chromatic graphs, *J. Combin. Theory B* **23**, 143–149.

[1978] Circular arc graphs: new uses and a new algorithm, *in* "Theory and Application of Graphs," Lecture Notes in Math. 642; pp. 580–589. Springer-Verlag, Berlin.

[1979] On Berge's strong perfect graph conjecture, *Ann. N.Y. Acad. Sci.* **319**, 530–535.

Tutte, W. (alias B. Descartes)

[1954] Solution to advanced problem No. 4526, *Amer. Math. Monthly* **61**, 352.

Wagon, Stanley

[1978] Infinite triangulated graphs, *Discrete Math.* **22**, 183–189.

Zykov, A. A.

[1952] On some properties of linear complexes (Russian), *Math. Sbornik N.S.* **24** (1949), 163–188, English transl. *Amer. Math. Soc. Transl.* No. 79. MR11, p. 733.