CHAPTER 1

Graph Theoretic Foundations

1. Basic Definitions and Notations

Functions and Relations

Let X and Y be sets. A *function* (or *mapping*) f from X to Y, denoted

$$f: X \to Y,$$

is a rule which associates to each element x of X a corresponding element y of Y. It is usual to call y the *image* of x under f and denote it by $y = f(x)$. We call f an *injective* or *one-to-one* function if no pair of distinct members of X has the same image under f, that is,

$$x \neq x' \Rightarrow f(x) \neq f(x') \qquad (x, x' \in X),$$

or equivalently,

$$f(x) = f(x') \Rightarrow x = x' \qquad (x, x' \in X).$$

The function f is called *surjective* or *onto* if each y in Y is the image of some x in X, that is,

$$(\forall y \in Y)(\exists x \in X) \qquad \text{such that} \quad y = f(x).$$

A function which is both injective and surjective is called a *bijection.* A *permutation* is simply a bijection from a set to itself.

Following the usual notation of mathematics, $x \in X$ indicates that x is a member of the set X and $A \subseteq X$ means that A is a (not necessarily proper) subset of X. The *cardinality* or *size* of X is denoted by $|X|$. For subsets A and B of X, the notation $A \cap B$ and $A \cup B$ are the usual set intersection and set union operations. When A and B are disjoint subsets, we often write their union with a plus sign. That is,

$$C = A + B \qquad \text{indicates} \quad A \cap B = \varnothing \quad \text{and} \quad C = A \cup B,$$

where $\varnothing$ is the empty set. Throughout this book we will deal exclusively with finite sets. A collection $\{X_i\}_{i \in I}$ of subsets of a set X is said to *cover* X if their union equals X. The collection is called a *partition* of X if the subsets are pairwise disjoint and the collection covers X.

Let $\mathscr{P}(X)$ denote the *power set* of a set X, i.e., the collection of all subsets of X. It is well known that $|\mathscr{P}(X)| = 2^{|X|}$. A *binary relation* on X is defined to be a function

$$R: X \to \mathscr{P}(X)$$

from X to the power set of X. For each $x \in X$, the image of x under R is a subset $R(x) \subseteq X$ called the set of *relatives* of x. It is customary to represent the relation R as a collection of *ordered* pairs $\mathscr{R} \subseteq X \times X$, where

$$(x, x') \in \mathscr{R} \qquad \text{if and only if} \quad x' \in R(x).$$

In this case we say that x' is *related* to x. Notice that this does *not* necessarily imply that x is related to x'. (Perhaps one should read "will inherit from" instead of "is related to," as in the case of a poor nephew with ten children and his rich widowed childless aunt.)

A binary relation R on X may satisfy one or more of the following properties:

symmetric property

$$x' \in R(x) \Rightarrow x \in R(x') \qquad (x, x' \in X),$$

antisymmetric property

$$x' \in R(x) \Rightarrow x \notin R(x') \qquad (x, x' \in X),$$

reflexive property

$$x \in R(x) \qquad (x \in X),$$

irreflexive property

$$x \notin R(x) \qquad (x \in X),$$

transitive property

$$z \in R(y),\ y \in R(x) \Rightarrow z \in R(x) \qquad (x, y, z \in X).$$

Such a relation is said to be an *equivalence* if it is reflexive, symmetric, and transitive. A binary relation is called a *strict partial order* if it is irreflexive and transitive. It is a simple exercise to show that a strict partial order will also be antisymmetric.

Graphs

Let us formally define the notion of a graph. A *graph** G consists of a finite set V and an irreflexive binary relation on V. We call V the set of *vertices*. The binary relation may be represented either as a collection E of *ordered* pairs or as a function from V to its power set,

$$\text{Adj}: V \to \mathscr{P}(V).$$

Both of these representations will be used interchangeably. We call $\text{Adj}(v)$ the *adjacency set* of vertex v, and we call the ordered pair $(v, w) \in E$ an *edge*. Clearly

$$(v, w) \in E \qquad \text{if and only if} \qquad w \in \text{Adj}(v).$$

In this case we say that w is *adjacent* to v and v and w are *endpoints* of the edge (v, w). The assumption of irreflexivity implies that

$$(v, v) \notin E \qquad (v \in V),$$

or equivalently,

$$v \notin \text{Adj}(v) \qquad (v \in V).$$

We further denote

$$N(v) = \{v\} + \text{Adj}(v),$$

which is called the *neighborhood* of v.

In this book we will usually drop the parentheses and the comma when denoting an edge. Thus

$$xy \in E \qquad \text{and} \qquad (x, y) \in E$$

will have the same meaning. This convention, we believe, improves the clarity of exposition.

We have defined a graph as a set and a certain relation on that set. It is often convenient to draw a "picture" of the graph. This may be done in many ways. Usually one draws a circle for each vertex and connects vertex x and vertex y with a directed arrow whenever xy is an edge. If both xy and yx are edges, then sometimes a single line joins x and y without arrows. Figure 1.1 shows three of the many possible drawings that one could use to represent the same graph. In each case the adjacency structure remains unchanged. Occasionally, very intelligent persons will become extremely angry because one does not like the other's pictures. When this happens it is best to remember that our figures are meant simply as a tool to help understand the underlying mathematical structure or as an aid in constructing a mathematical model for some application.

* Some authors use the term *directed graph* or *digraph*.

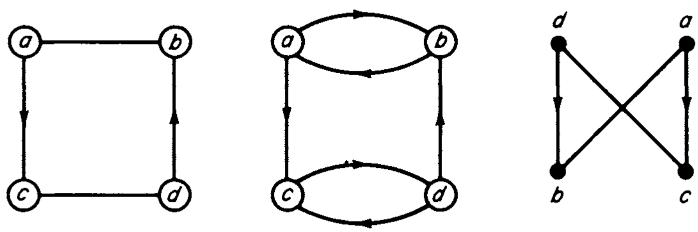


Figure 1.1. Three pictures of the same graph.

Two graphs $G = (V, E)$ and $G' = (V', E')$ are called *isomorphic*, denoted $G \cong G'$, if there is a bijection $f\colon V \to V'$ satisfying, for all $x, y \in V$,

$$(x, y) \in E \Leftrightarrow (f(x), f(y)) \in E'.$$

Two edges are *adjacent* if they share a common endpoint; otherwise they are *nonadjacent*.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The graph $G^{-1} = (V, E^{-1})$ is said to be the *reversal* of G, where

$$E^{-1} = \{(x, y) \mid (y, x) \in E\},$$

that is,

$$xy \in E^{-1} \Leftrightarrow yx \in E \qquad (x, y \in V).$$

We define *symmetric closure* of G to be the graph $\hat{G} = (V, \hat{E})$, where

$$\hat{E} = E \cup E^{-1}.$$

A graph $G = (V, E)$ is called *undirected* if its adjacency relation is symmetric, i.e., if

$$E = E^{-1},$$

or equivalently,

$$E = \hat{E}.$$

We occasionally denote an undirected edge by $\widehat{ab} = \{ab\} \cup \{ba\}$. A graph $H = (V, F)$ is called an *oriented* graph if its adjacency relation is antisymmetric, i.e., if

$$F \cap F^{-1} = \varnothing.$$

If, in addition, $F + F^{-1} = E$, then H (or F) is called an *orientation* of G (or E). The four nonisomorphic orientations of the pentagon are given in Figure 1.2.

Let $G = (V, E)$ be an undirected graph. We define the *complement* of G to be the graph $\bar{G} = (V, \bar{E})$, where

$$\bar{E} = \{(x, y) \in V \times V \mid x \neq y \text{ and } (x, y) \notin E\}.$$

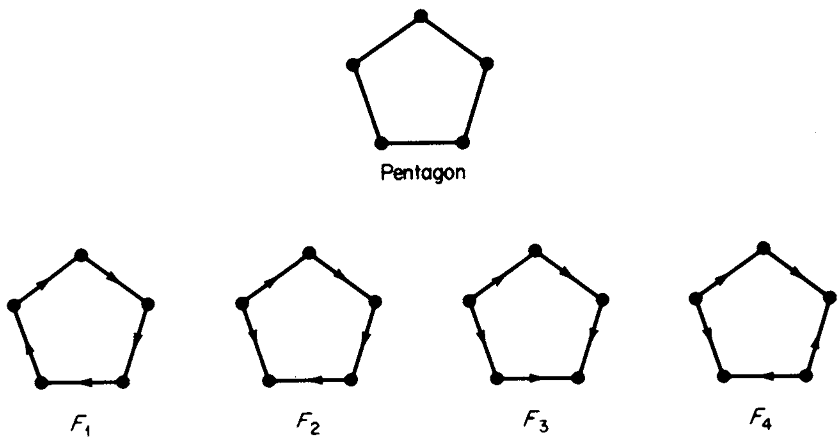


Figure 1.2. The four nonisomorphic orientations of the pentagon.

Intuitively, the edges of G become the nonedges of $\bar{G}$ and vice versa. A graph is *complete* if every pair of distinct vertices is adjacent. Thus, the complement $\bar{G} = (V, \bar{E})$ of G could equivalently be defined as that set $\bar{E}$ satisfying $E \cap \bar{E} = \varnothing$ and $E + \bar{E}$ complete. The complete graph on n vertices is usually denoted by K_n (see Figure 1.3).

A *(partial) subgraph* of a graph $G = (V, E)$ is defined to be any graph $H = (V', E')$ satisfying $V' \subseteq V$ and $E' \subseteq E$. Two types of subgraphs are of particular importance, namely, the subgraph spanned by a given subset of edges and the subgraph induced by a given subset of vertices. They will now be described.

A subset $S \subseteq E$ of the edges *spans* the subgraph $H = (V_S, S)$, where $V_S = \{v \in V \mid v$ is an endpoint of some edge of $S\}$. We call H the (partial) subgraph spanned by S.

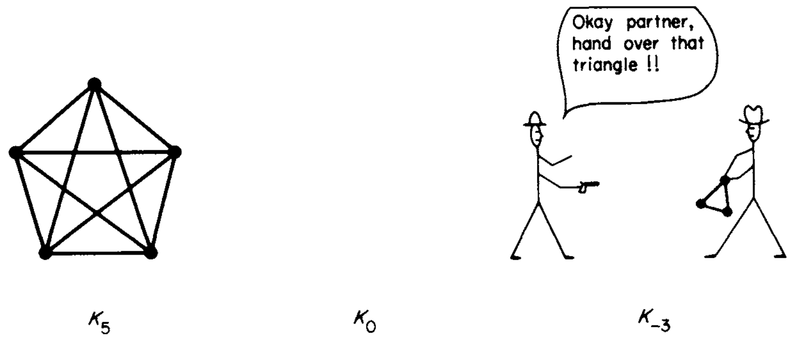


Figure 1.3. Some complete graphs.

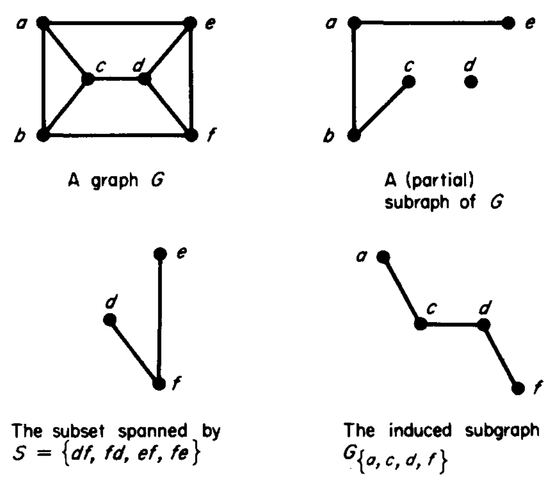


Figure 1.4. Examples of subgraphs.

Given a subset $A \subseteq V$ of the vertices, we define the *subgraph induced* by A to be $G_A = (A, E_A)$, where

$$E_A = \{xy \in E \mid x \in A \text{ and } y \in A\}.$$

For $v \in A$ we denote $\mathrm{Adj}_A(v) = \mathrm{Adj}(v) \cap A$. Obviously not every subgraph of G is an induced subgraph of G (Figure 1.4).

Let $G = (V, E)$ be an undirected graph. Consider the following definitions.

Clique: A subset $A \subseteq V$ of r vertices is an *r-clique* if it induces a complete subgraph, i.e., if $G_A \cong K_r$. A single vertex is a 1-clique. A clique A is *maximal* if there is no clique of G which properly contains A as a subset. A clique is *maximum* if there is no clique of G of larger cardinality. Some authors use the term *complete set* to indicate a clique.

$\omega(G)$ is the number of vertices in a maximum clique of G; it is called the *clique number* of G.

A *clique cover* of size k is a partition of the vertices $V = A_1 + A_2 + \cdots + A_k$ such that each A_i is a clique.

$k(G)$ is the size of a smallest possible clique cover of G; it is called the *clique cover number* of G.

A *stable set* is a subset X of vertices no two of which are adjacent. Some authors use the term *independent set* to indicate a stable set.

$\alpha(G)$ is the number of vertices in a stable set of maximum cardinality; it is called the *stability number* of G.

A *proper c-coloring* is a partition of the vertices $V = X_1 + X_2 + \cdots + X_c$ such that each X_i is a stable set. In such a case, the members of X_i are "painted" with the color i and adjacent vertices will receive different colors. We say that G is *c-colorable*. It is common to omit the word proper; a *coloring* will always be assumed to be a proper coloring.

$\chi(G)$ is the smallest possible c for which there exists a proper c-coloring of G; it is called the *chromatic number* of G.

It is easy to see that

$$\omega(G) \leq \chi(G) \qquad \text{and} \qquad \alpha(G) \leq k(G),$$

since every vertex of a maximum clique (maximum stable set) must be contained in a different partition segment in any minimum proper coloring (minimum clique cover). There is an obvious duality to these notions, namely,

$$\omega(G) = \alpha(\bar{G}) \qquad \text{and} \qquad \chi(G) = k(\bar{G}).$$

Let $G = (V, E)$ be an arbitrary graph. The *out-degree* of a vertex x, denoted by $d^+(x)$, is defined as $d^+(x) = |\mathrm{Adj}(x)|$. The *in-degree* $d^-(x)$ of x is defined similarly:

$$d^-(x) = |\{y \in V \mid x \in \mathrm{Adj}(y)\}|.$$

Although in general $d^+(x)$ and $d^-(x)$ will not be equal, we do have

$$\sum_{x \in V} d^+(x) = \sum_{x \in V} d^-(x) = |E|,$$

each ordered pair in E contributing 1 to both summands. A vertex whose out-degree (in-degree) equals zero is called a *sink* (*source*). If both $d^+(x) = 0$ and $d^-(x) = 0$, then x is an *isolated vertex*.

When G is an undirected graph the situation is somewhat special. In such a case $d^+(x) = d^-(x)$ for each $x \in V$, and we call this number simply the *degree* of x, denoted $d(x)$. That is, the degree of x in an undirected graph is the size of its adjacency set. Finally, defining $\|E\| = \frac{1}{2}|E|$ we obtain the familiar formula

$$\tfrac{1}{2} \sum_{x \in V} d(x) = \|E\|.$$

Let $G = (V, E)$ be an arbitrary graph. We present some fairly standard definitions.

Chain: A sequence of vertices $[v_0, v_1, v_2, \ldots, v_l]$ is a *chain of length l* in G if $v_{i-1}v_i \in E$ or $v_iv_{i-1} \in E$ for $i = 1, 2, \ldots, l$.

Path: A sequence of vertices $[v_0, v_1, v_2, \ldots, v_l]$ is a *path* from v_0 to v_l *of length l* in G provided that $v_{i-1}v_i \in E$ for $i = 1, 2, \ldots, l$.

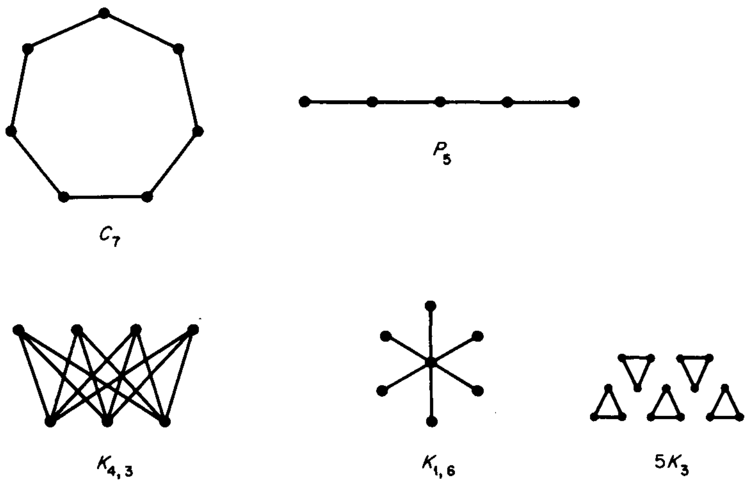


Figure 1.5.

A path or chain in G is called *simple* if no vertex occurs more than once. It is called *trivial* if $l = 0$.

Connected graph: A graph G is *connected* if between any two vertices there exists a chain in G joining them.

Strongly connected graph: A graph G is *strongly connected* if for any two vertices x and y there exists a path in G from x to y.

Remark. The notions of chain and path coincide when G is an undirected graph.

Cycle: A sequence of vertices $[v_0, v_1, v_2, \ldots, v_l, v_0]$ is called a *cycle* of length $l + 1$ (or closed path) if $v_{i-1}v_i \in E$ for $i = 1, 2, \ldots, l$ and $v_l v_0 \in E$.

Simple cycle: A cycle $[v_0, v_1, v_2, \ldots, v_l, v_0]$ is a *simple cycle* if $v_i \neq v_j$ for $i \neq j$.

Chordless cycle: A simple cycle $[v_0, v_1, v_2, \ldots, v_l, v_0]$ is *chordless* if $v_i v_j \notin E$ for i and j differing by more than 1 mod $l + 1$.

Bipartite graph: An undirected graph $G = (V, E)$ is *bipartite* if its vertices can be partitioned into two disjoint stable sets $V = S_1 + S_2$, i.e., every edge has one endpoint in S_1 and the other in S_2. Equivalently, G is bipartite if and only if it is 2-colorable. It is customary to use the notation $G = (S_1, S_2, E)$, which emphasizes the partition. Vertices $x \in S_i$ and $y \in S_j$ are of the *same parity* if $i = j$ and are of *opposite parity* if $i \neq j$.

Complete bipartite graph: A bipartite graph $G = (S_1, S_2, E)$ is *complete* if for every $x \in S_1$ and $y \in S_2$ we have $xy \in E$, i.e., every possible edge that could exist does exist.

Throughout the text certain graphs will occur many times. We give names to some of them (see Figure 1.5).

K_n: the *complete graph* on n vertices or *n-clique*.
C_n: the *chordless cycle* on n vertices or *n-cycle*.
P_n: the *chordless path graph* on n vertices or *n-path*.
$K_{m,n}$: the *complete bipartite graph* on $m + n$ vertices partitioned into an *m-stable set* and an *n-stable set*.
$K_{1,n}$: the *star graph* on $n + 1$ vertices.
mK_n: m disjoint copies of K_n.

There is obviously some overlap with these names. For example, $K_3 = C_3$ is called a *triangle*. Notice also that $\bar{C}_4 = 2K_2$ and $K_{n,n} = \overline{2K_n}$.

2. Intersection Graphs

Let $\mathscr{F}$ be a family of nonempty sets. The *intersection graph* of $\mathscr{F}$ is obtained by representing each set in $\mathscr{F}$ by a vertex and connecting two vertices by an edge if and only if their corresponding sets intersect. When $\mathscr{F}$ is allowed to be an arbitrary family of sets, the class of graphs obtained as intersection graphs is simply all undirected graphs (Marczewski [1945]). The problem of characterizing the intersection graphs of families of sets having some specific topological or other pattern is often very interesting and frequently has applications to the real world.

The intersection graph of a family of intervals on a linearly ordered set (like the real line) is called an *interval graph*. If these intervals are required to have unit length, then we have a *unit interval graph*; a *proper interval graph* is constructed from a family of intervals on a line such that no interval properly contains another. Roberts [1969a] showed that the classes of unit interval graphs and proper interval graphs coincide. Interval graphs are discussed in Section 1.3 and in Chapter 8.

Consider the following relaxation of the notion of intervals on a line. If we join the two ends of our line, thus forming a circle, the intervals will become arcs on the circle. Allowing arcs to slip over and include the point of connection, we obtain a class of intersection graphs called the *circular-arc graphs*, which properly contains the interval graphs. Circular-arc graphs have been extensively studied by A. C. Tucker and others. We will survey these results in Section 8.6. There are a number of interesting applications of circular-arc graphs, including computer storage allocation and the phasing of traffic lights. Let us look at an example of the latter application.

Example. The traffic flow at the corner of Holly, Vood, and Wine is pictured in Figure 1.6. Certain lanes are compatible with one another, such as c and j, or d and k, while others are incompatible, such as b and f. In order to avoid collisions, we wish to install a traffic light system to control the flow of vehicles. Each lane will be assigned an arc on a circle representing the time interval during which it has a green light. Incompatible lanes must be assigned disjoint arcs. The circle may be regarded as a clock representing an entire cycle which will be continually repeated. An arc assignment for our example is given in Figure 1.7. In general, if G is the intersection graph of the arcs of such an assignment (see Figure 1.8), and if H is the compatibility relation defined on the pairs of lanes, then clearly G is a (partial) subgraph of H. In our example, the compatible pairs (d, k), (h, j), and (i, j) are in H but are not in G. Additional aspects of this problem, such as how to choose an arc assignment which minimizes waiting time, can also be incorporated into the model. The reader is referred to Stoffers [1968] and Roberts [1976, pp. 129–134; 1978, Section 3.6] for more details.

A *proper circular-arc* graph is the intersection graph of a family of arcs none of which properly contains another. It can be shown (Theorem 8.18)

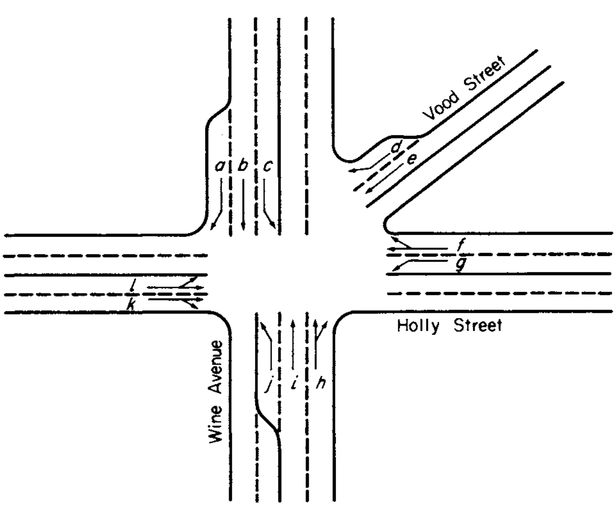


Figure 1.6.

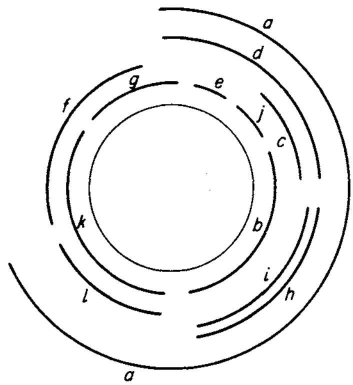


Figure 1.7. The clock cycle.

that every proper circular-arc graph has a representation as intersecting arcs of a circle in which not only is no arc properly contained in another but also no pair of arcs together cover the entire circle.

In a different generalization of interval graphs, Renz [1970] characterized the intersection graphs of paths in a tree, and Gavril [1978] gives a recognition algorithm for them. Walter [1972], Buneman [1974], and Gavril [1974] carried this idea further and showed that the intersection graphs of subtrees of a tree are exactly the triangulated graphs of Chapter 4. All of this is summarized in Figure 1.9.

A permutation diagram consists of n points on each of two parallel lines and n straight line segments matching the points. The intersection graph of the line segments is called a *permutation graph*. These graphs will be discussed

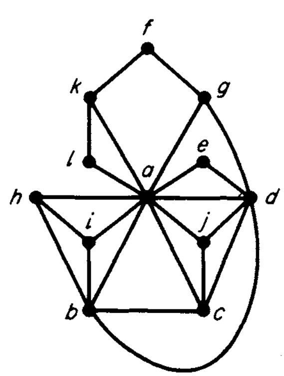


Figure 1.8. G, the circular-arc graph.

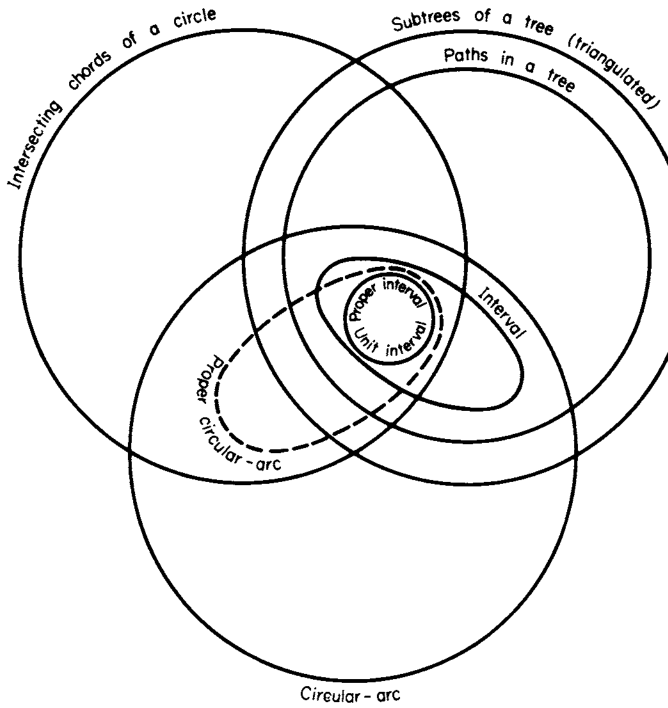


Figure 1.9.

in Chapter 7. If the $2n$ points are located randomly around a circle, then the matching segments will be *chords of the circle* and the resulting class of intersection graphs, studied in Chapter 11, properly contains the permutation graphs. A simple argument shows that *every proper circular-arc graph is also the graph of intersecting chords of a circle*: We may assume that no pair of arcs together covers the entire circle (Theorem 8.18). For each arc on the circle, draw the chord connecting its two endpoints. Clearly, two arcs overlap if and only if their corresponding chords intersect.

There are many other interesting classes of intersection graphs. We have introduced you to only some of them, specifically those classes which will be developed further in the text. To the reader who wishes to investigate other intersection graphs we offer the following references:

Cubes and boxes in n-space:	Danzer and Grunbaum [1967], Roberts [1969b].
Convex sets in n-space:	Wegner [1967], Ogden and Roberts [1970].

3. Interval Graphs—A Sneak Preview of the Notions Coming Up

Our intention in this section is to arouse the reader's curiosity by presenting some basic ideas that will be pursued in greater detail in later chapters. We also hope to imbue the reader with a sense of how the subject matter is relevant to applied mathematics and computer science.

An undirected graph G is called an *interval graph* if its vertices can be put into one-to-one correspondence with a set of intervals $\mathscr{I}$ of a linearly ordered set (like the real line) such that two vertices are connected by an edge of G if and only if their corresponding intervals have nonempty intersection. We call $\mathscr{I}$ an *interval representation* for G. (It is unimportant whether we use open intervals or closed intervals; the resulting class of graphs will be the same.) An interval representation of the windmill graph is given in Figure 1.10.

Let us discuss one application of interval graphs. Many other such applications will be presented in Section 8.4.

Application to Scheduling

Consider a collection $C = \{c_i\}$ of courses being offered by a major university. Let T_i be the time interval during which course c_i is to take place. We would like to assign courses to classrooms so that no two courses meet in the same room at the same time.

This problem can be solved by properly coloring the vertices of the graph $G = (C, E)$ where

$$c_i c_j \in E \Leftrightarrow T_i \cap T_j \neq \varnothing.$$

Each color corresponds to a different classroom. The graph G is obviously an interval graph, since it is represented by time intervals.

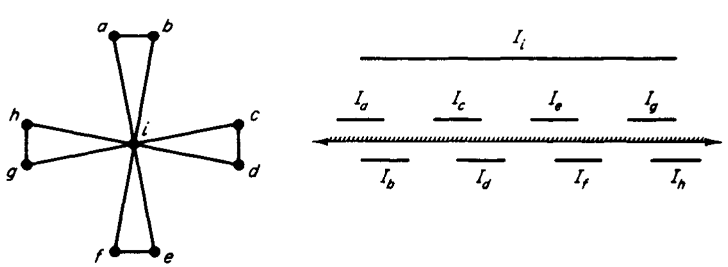


Figure 1.10. An interval graph—the windmill graph (at left)—and an interval representation for it.

This example is especially interesting because efficient, linear-time algorithms are known for coloring interval graphs with a minimum number of colors. (The minimum coloring problem is NP-complete for general graphs, Section 2.1.) We will discuss these algorithms in subsequent chapters.

Remark. The determination of whether a given graph is an interval graph can also be carried out in linear time (Section 8.3).

We have chosen interval graphs as an introduction to our studies because they satisfy so many interesting properties. The first fact that we notice is that being an interval graph is a *hereditary property.*

Proposition 1.1. An induced subgraph of an interval graph is an interval graph.

Proof. If $\{I_v\}_{v \in V}$ is an interval representation for a graph $G = (V, E)$, then $\{I_v\}_{v \in X}$ is an interval representation for the induced subgraph $G_X = (X, E_X)$. ∎

Hereditary properties abound in graph theory. Some of our favorites include planarity, bipartiteness, and any "forbidden subgraph" characterization. The next property of interval graphs is also a hereditary property.

Triangulated graph property. Every simple cycle of length strictly greater than 3 possesses a chord.

Graphs which satisfy this property are called *triangulated graphs.* The graph in Figure 1.10 is triangulated, but the house graph in Figure 1.11 is not triangulated because it contains a chordless 4-cycle.

Proposition 1.2 (Hajös [1958]). An interval graph satisfies the triangulated graph property.

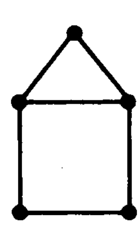


Figure 1.11. A graph which is not triangulated: The house graph.

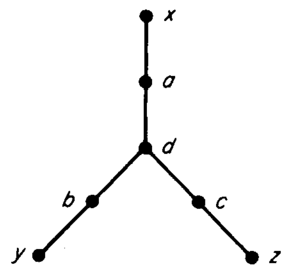


Figure 1.12. A triangulated graph which is not an interval graph.

Proof. Suppose the interval graph G contains a chordless cycle $[v_0, v_1, v_2, \ldots, v_{l-1}, v_0]$ with $l > 3$. Let I_k denote the interval corresponding to v_k. For $i = 1, 2, \ldots, l-1$, choose a point $p_i \in I_{i-1} \cap I_i$. Since I_{i-1} and I_{i+1} do not overlap, the p_i constitute a strictly increasing or strictly decreasing sequence. Therefore, it is impossible for I_0 and I_{l-1} to intersect, contradicting the criterion that $v_0 v_{l-1}$ is an edge of G. ∎

Not every triangulated graph is an interval graph. Consider the tree T given in Figure 1.12, which certainly has no chordless cycles. The intervals I_a, I_b, and I_c of a representation for T would have to be disjoint, and I_d would properly include the middle interval, say I_b. Where, then, could we put I_y so that it intersects I_b but not I_d? Clearly we would be stuck. So there must be more to the story of interval graphs than we have told so far.

Transitive orientation property. Each edge can be assigned a one-way direction in such a way that the resulting oriented graph (V, F) satisfies the following condition:

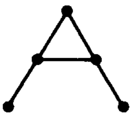
$$ab \in F \text{ and } bc \in F \quad \text{imply} \quad ac \in F \qquad (\forall a, b, c \in V). \tag{1}$$

An undirected graph which is transitively orientable is sometimes called a *comparability graph.* Figure 1.13 shows a transitive orientation of the A graph and of the suspension bridge graph. The odd length chordless cycles $C_5, C_7, C_9, \ldots$ and the bull's head graph (see Figure 1.14) cannot be transitively oriented.

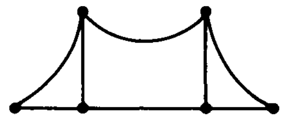
Proposition 1.3 (Ghouila-Houri [1962]). The complement of an interval graph satisfies the transitive orientation property.

Proof. Let $\{I_v\}_{v \in V}$ be an interval representation for $G = (V, E)$. Define an orientation F of the complement $\bar{G} = (V, \bar{E})$ as follows:

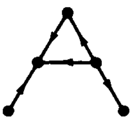
$$xy \in F \Leftrightarrow I_x < I_y \qquad (\forall xy \in \bar{E}).$$



The A graph



The suspension bridge graph



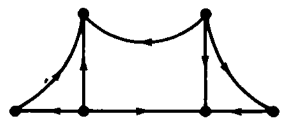


Figure 1.13. Transitive orientations of two comparability graphs.

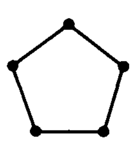
Here $I_x < I_y$ means that the interval I_x lies entirely to the left of the interval I_y. (Remember, they are disjoint.) Clearly (1) is satisfied, since $I_x < I_y < I_z$ implies $I_x < I_z$. Thus, F is a transitive orientation of $\bar{G}$. ∎

As in the case of triangulated graphs, there are graphs whose complements are comparability graphs but which fail to be interval graphs. So it seems that Propositions 1.2 and 1.3 simply provide necessary (but not sufficient) conditions for interval graphs. Rather than wait any longer, we state an important result that says, if we put these two properties together, we get (drum roll, please) exactly all interval graphs.

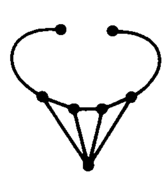
Theorem 1.4 (Gilmore and Hoffman [1964]). An undirected graph G is an interval graph if and only if G is a triangulated graph and its complement $\bar{G}$ is a comparability graph.

The proof of sufficiency is postponed until Chapter 8, primarily because this is a "getting acquainted with" section.

Looking back, each of the graphs in Figures 1.10, 1.11, and 1.13 can be properly colored using three colors and each contains a triangle. Therefore,



C_5



The bull's head graph

Figure 1.14. Two graphs which are not transitively orientable. Why?

for these graphs, their chromatic number equals their clique number. This is not an accident. In Chapters 4 and 5 we will show that any triangulated graph and any comparability graph also satisfies the following properties.

χ-Perfect property. For each induced subgraph G_A of G,

$$\chi(G_A) = \omega(G_A).$$

The chordless cycles C_5, C_7, C_9 are not χ-perfect. A dual notion of χ-perfection is the following:

α-Perfect property. For each induced subgraph G_A of G,

$$\alpha(G_A) = k(G_A).$$

A very important theorem in Chapter 3 states that a graph is χ-perfect if and only if it is α-perfect. This equivalence was originally conjectured by Claude Berge, and it was proved some ten years later by László Lovász.

4. Summary

The reader has been introduced to the graph theoretic foundations needed for the remainder of the book. In addition, he has had a taste of some of the particular notions that we intend to investigate further. Returning to the table of contents at this point, he will recognize many of the topics listed. The chapter dependencies are given in Figure 1.15.

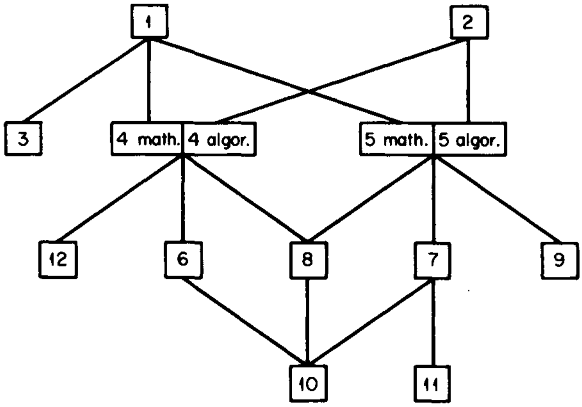


Figure 1.15. The chapter dependencies. The reader may wish to read Chapters 1 and 2 quickly and refer back to them as needed.

In the next chapter we will present the foundations of algorithmic design and analysis. As was the case in this chapter, many examples will be given which will introduce the reader to the ideas and techniques that he will find helpful in subsequent chapters.

EXERCISES

1. Show that the graphs in Figures 1.16 and 1.17 are both intersection graphs of a family of chords of a circle but that neither is a circular-arc graph.

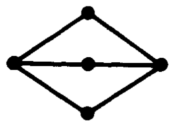


Figure 1.16.

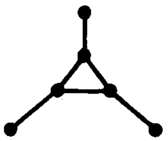


Figure 1.17.

2. Can you find graphs for each zone of the Venn diagram in Figure 1.9?
3. Let $\mathscr{F}$ be a family of intervals on a line such that no interval contains another. Show that none of the left endpoints coincide. Give a procedure which constructs a family $\mathscr{F}'$ of unit intervals such that the intersection graphs of $\mathscr{F}$ and $\mathscr{F}'$ are isomorphic.
4. Let $G = (V, E)$ be any undirected graph. Show that there is a family $\mathscr{F}$ of subsets of V such that G is the intersection graph of $\mathscr{F}$.
5. Let G be the intersection graph of a family of paths in a tree and let v be a vertex of G. Show that the induced subgraph $G_{\{v\}+\mathrm{Adj}(v)}$ is an interval graph.
6. Prove directly (using only the definition) that the graph in Figure 1.17 does not have an interval representation and is therefore not an interval graph.
7. Give an interval representation for the graph in Figure 1.18. Show that it is not a comparability graph. Why is this not in conflict with the Gilmore–Hoffman theorem?

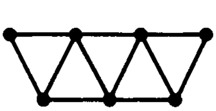


Figure 1.18.

8. Give a graph theoretic solution to the following problem: A group of calculus teaching assistants each gives two office hours weekly which are chosen in advance. Because of budgetary reasons, the TAs must share offices. Since each office has only one blackboard, how can office space be assigned so that at any particular time no more than one TA is meeting with students?

9. Give an example to show that the graph you obtain in Exercise 8 is not necessarily an interval graph. How could we alter the problem so that we would obtain only interval graphs?

10. Is the bull's head graph (Figure 1.14) an interval graph? Is the complement of the suspension bridge graph (Figure 1.13) an interval graph? What is a good name for this last graph?

11. An undirected graph is *self-complementary* if it is isomorphic to its complement. Show that there are exactly two self-complementary graphs having five vertices. How many are there for four vertices? Six vertices?

12. Let $G = (V, E)$ be an undirected graph. A subset $A \subseteq V$ is called an *edge cover* of G if for every edge $xy \in E$, either $x \in A$ or $y \in A$ or both. Prove that A is a minimum edge cover if and only if $V - A$ is a maximum stable set.

13. Let $\mathscr{F} = \{S_x\}_{x \in V}$ be a family of subsets of a set. Two members S_x and S_y of $\mathscr{F}$ *overlap*, denoted $S_x \between S_y$, if $S_x \cap S_y \neq \varnothing$, $S_x \nsubseteq S_y$, and $S_y \nsubseteq S_x$. The *overlap graph* of $\mathscr{F}$ is the undirected graph $G = (V, E)$ where

$$xy \in E \quad \text{if and only if} \quad S_x \between S_y \qquad (x, y \in V).$$

(i) Show that if x and y are in separate connected components G_A and G_B of G, then

$$S_x \subseteq S_y \Rightarrow S_a \subseteq S_y \qquad (a \in A).$$

(ii) Let $\mathscr{G}$ be the collection of (maximal) connected components of G. Show that the relation $<$, defined for all $G_A, G_B \in \mathscr{G}$ as

$$G_A < G_B \Leftrightarrow \exists x \in A, y \in B \text{ such that } S_X \subseteq S_Y,$$

is a strict partial order of $\mathscr{G}$.

14. A family $\mathscr{F}$ of distinct nonempty subsets of a set S is a *representation* of a graph G if the intersection graph of $\mathscr{F}$ is isomorphic to G. A representation is *minimum* if the set S is of smallest possible cardinality over all representations of G. A graph G is *uniquely intersectable* if for all minimum representations $\mathscr{F}_1$ and $\mathscr{F}_2$ of G, $\mathscr{F}_1$ and $\mathscr{F}_2$ are isomorphic.

(i) Prove that every triangle-free graph is uniquely intersectable.

A *star n-gon* is constructed from the cycle C_n by adjoining new vertices to the endpoints of each edge. Figure 1.19 illustrates a star 7-gon.

(ii) Verify that the family $\mathscr{F} = \{S_0, S_1, \ldots, S_{n-1}, D_0, D_1, \ldots, D_{n-1}\}$ is a minimum representation of the star n-gon, where $S_i = \{i\}$ and $D_i = \{i, i + 1 \bmod n\}$.

(iii) Prove that every star n-gon is uniquely intersectable (Alter and Wang [1977]).

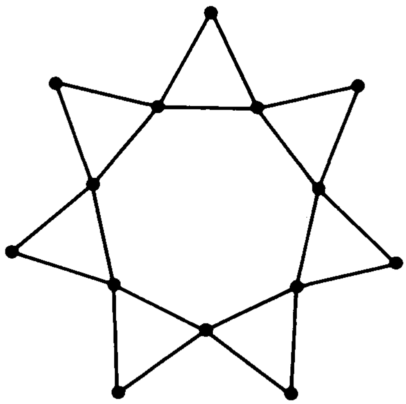


Figure 1.19. A star 7-gon.

15. *The Berge mystery story.* Six professors had been to the library on the day that the rare tractate was stolen. Each had entered once, stayed for some time, and then left. If two were in the library at the same time, then at least one of them saw the other. Detectives questioned the professors and gathered the following testimony: Abe said that he saw Burt and Eddie in the library; Burt said that he saw Abe and Ida; Charlotte claimed to see Desmond and Ida; Desmond said that he saw Abe and Ida; Eddie testified to seeing Burt and Charlotte; Ida said that she saw Charlotte and Eddie.

One of the professors lied!! Who was it?

Research Problem. Characterize uniquely intersectable graphs and/or give a recognition algorithm.

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