CHAPTER 9

Superperfect Graphs

1. Coloring Weighted Graphs

In this chapter we turn our attention to a notion of perfection in weighted graphs. In the process, a more general type of coloring the vertices of a graph will be introduced, suggesting many interesting applications. The concept of superperfection, introduced in Section 2, is due to Alan Hoffman and Ellis Johnson. They were motivated by the shipbuilding problem (Application 9.1), and most of the early results are theirs.

To each vertex x of a graph $G = (V, E)$ we associate a non-negative number $w(x)$, and we define the *weight* of a subset $S \subseteq V$ to be the quantity

$$w(S) = \sum_{x \in S} w(x).$$

(Without loss of generality we may assume throughout that all weights are integral.) The pair $(G; w)$ is called a *weighted graph.* The subset S will often be the vertices of a simple cycle or a clique or a stable set.

An *interval coloring* of a weighted graph $(G; w)$ maps each vertex x onto an (open) interval I_x of the real line of width (or measure) $w(x)$ such that adjacent vertices are mapped to disjoint intervals, that is, $xy \in E$ implies $I_x \cap I_y = \varnothing$. Figure 9.1 shows two colorings of a weighted graph. The *number of hues* of a coloring (i.e., its total *width*) is defined to be $|\bigcup_x I_x|$. The *interval chromatic number* $\chi(G; w)$ is the least number of hues needed to color the vertices with intervals. For the graph in Figure 9.1, $\chi(G; w) = 10$.

Example 1. If $w(x) = 1$ for every vertex $x \in V$, then $\chi(G; w) = \chi(G)$. That is, the notion of interval coloring reduces to the usual definition of coloring when all weights are equal.

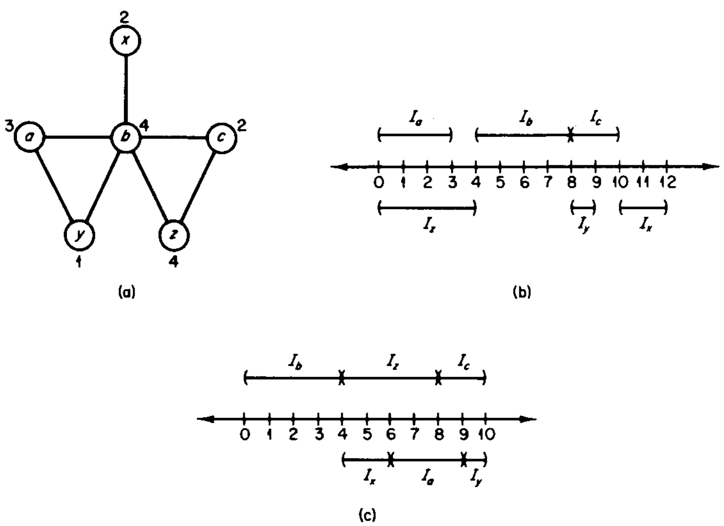

Figure 9.1. Two interval colorings of a weighted graph: (a) $(G; w)$; (b) a coloring of $(G; w)$ using 12 hues; and (c) a coloring of $(G; w)$ using 10 hues.

Application 9.1 (The Shipbuilding Problem). In certain shipyards the sections of a ship are constructed on a dry dock, called the welding plane, according to a rigid time schedule. Each section s requires a certain width $w(s)$ on the dock during construction. Can the sections be assigned space on a welding plane of total width k so that no spot is reserved for two sections at the same time?

Let the sections be represented by the vertices of a graph G and connect two vertices if their corresponding sections have intersecting time intervals. Thus $(G; w)$ is a weighted interval graph. An interval coloring of $(G; w)$ will provide the assignment of the sections to spaces, of appropriate size, on the welding plane. This assignment will be consistent with the intersecting time restrictions. (The reader must be careful to distinguish between the *time intervals* which produced the edges of G and the *coloring intervals* which provide a solution to the assignment of space on the dock.)

Remark 1. Larry Stockmeyer has shown that determining whether $\chi(G; w) \leq k$ is an NP-complete problem, even if w is restricted to the values 1 and 2 and G is an interval graph. It follows that the shipbuilding problem is NP-complete.

Application 9.2 (The Banquet Problem). The menu for a banquet includes a number of cooked dishes which must be prepared in advance. A dish

d must be baked for $m(d)$ minutes at a temperature (not necessarily constant) between $t_1(d)$ and $t_2(d)$. Unfortunately, there is only one oven. How can we schedule the dishes so that the total cooking time is minimized?

Let G be the graph whose vertices represent dishes, with two vertices being connected if their corresponding dishes have disjoint temperature intervals (and therefore can never be in the oven at the same time). A $\chi(G; m)$-coloring* of $(G; m)$ will provide a solution to the scheduling problem by assigning an appropriate time interval to each dish during which it is to be in the oven. The Helly property for (the temperature) intervals insures that there is always a common acceptable temperature for all dishes being simultaneously baked. (Our solution does not take into account the size limitation of the oven.)

Remark 2. The banquet problem *can* be solved in polynomial time. This is particularly interesting in the context of Remark 1. The reason for the tractability here is that the graph obtained in the banquet problem is the *complement* of an interval graph. We will show in the next section that $\chi(G; w)$ can be calculated in polynomial time whenever G is a comparability graph.

Application 9.3 (Computer Storage Optimization). Most compilers maintain a one-to-one mapping between the variables in a program and their locations in storage. Therefore, in a tight storage situation, the programmer may have to overlay storage by deliberately using the same variable for more than one purpose, much to the detriment of clarity and reliability of the program. Using the notion of interval coloring, Fabri [1979] has investigated freeing the programmer from the task of overlaying by having the processor perform all storage allocation decisions. Thus, we want an automatic construction of a many-to-one correspondence between the variables and storage which guarantees the integrity of the variables. It is assumed that the variables have differing size requirements (as with arrays).

Let G be an undirected graph whose vertices correspond to the variables of a program. We connect two vertices v and u by an edge if and only if there is some node in the program flow graph at which v and u are simultaneously live† and thus enjoined from sharing storage. Associated with each vertex of G is a weight corresponding to the size of the variable. Since nonconflicting variables may overlay one another in storage, an interval coloring of G corresponds to a linear storage layout, and the interval chromatic number corresponds to the size of the optimum (i.e., smallest) such storage layout.

* From this point on, we will use the term *coloring* to mean *interval coloring* whenever the context allows.

†This can be determined by global data flow analysis.

We may regard an interval coloring in another manner. Associated with any such coloring of a weighted undirected graph $G = (V, E)$ is an implicit acyclic orientation F of G. This orientation is obtained by directing an edge toward the vertex whose coloring interval is to the right of the other, on the real line, that is,

$$xy \in F \Leftrightarrow I_x < I_y \qquad (\text{for all} \quad xy \in E).$$

This suggests the following alternative definition of $\chi(G; w)$.

Proposition 9.1. Let $(G; w)$ be a weighted undirected graph. Then

$$\chi(G; w) = \min_F \left(\max_\mu w(\mu) \right)$$

where F is an acyclic orientation of G and μ is a path in F.

Proof. Given F we define a coloring h of $(G; w)$ in the same way that one usually constructs a height function in a partial order. For a sink x, let $h(x) = (0, w(x))$. Proceeding inductively, for a vertex y let t be the largest endpoint of the intervals corresponding to the sons of y, and define $h(y) = (t, t + w(y))$. Thus, h is a coloring and its number of hues equals $\max_\mu w(\mu)$. This proves that $\chi(G; w) \le \min_F(\max_\mu w(\mu))$.

Conversely, a minimum coloring gives us an acyclic orientation F' as mentioned above, and clearly $\chi(G; w) \ge w(\mu)$ for any path μ in F'. This proves the reverse inequality, and hence equality holds. ∎

2. Superperfection

The *clique number* of a weighted graph $(G; w)$ is defined as

$$\omega(T; w) = \max\{w(K) \mid K \text{ is a clique of } G\}.$$

As one might expect, $\omega(G; w) \le \chi(G; w)$, which follows from Proposition 9.1.

An undirected graph G is *superperfect* if for every non-negative weighting

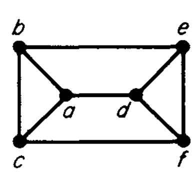

Figure 9.2. A superperfect graph.

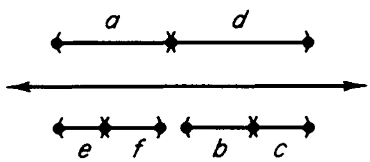

Figure 9.3.

w of the vertices $\omega(G; w) = \chi(G; w)$. Equivalently, G is superperfect if for every weighting w there exists an acyclic orientation F of G such that $w(\mu) \leq w(K)$ for every path μ in F and some clique K of G. In particular, the weight of the "heaviest" clique in G will equal the weight of the "heaviest" path in F. Thus, we have two basic methods for demonstrating superperfection: providing a suitable coloring or giving a suitable acyclic orientation. We shall illustrate these techniques on a few examples.

Example 2. The graph in Figure 9.2 is superperfect. The heaviest clique is either (i) one of the two triangles or (ii) one of the three edges not contained in a triangle. Suppose (ii) is the case for some weighting w, and assume, without loss of generality, that $\{a, d\}$ is the heaviest. Then $w(b) + w(c) \leq w(d)$ and $w(e) + w(f) \leq w(a)$, so that the coloring in Figure 9.3 will do. Otherwise, suppose (ii) is *not* the case, and assume that $\{d, e, f\}$ is the heaviest clique. Therefore, $w(a) + w(b) + w(c) \leq w(d) + w(e) + w(f)$ and, since (ii) has been ruled out,

$$w(a) < w(e) + w(f), \qquad w(b) < w(d) + w(f), \qquad w(c) < w(d) + w(e).$$

By cyclically permuting the vertices of each triangle if necessary, we may also assume that $w(b) \leq w(d)$. If $w(a) \geq w(f)$, then the coloring in Figure 9.4a gives a solution; otherwise Figure 9.4b works. Therefore, in every case, we have exhibited a coloring whose number of hues equals the weight of the heaviest clique. We conclude that the graph is superperfect.

Example 3. An undirected graph is perfect if and only if for every (0, 1)-valued weighting w of the vertices $\omega(G; w) = \chi(G; w)$. Thus *every superperfect graph is a perfect graph.*

Figure 9.4.

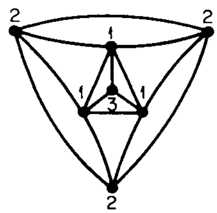

Figure 9.5. G: not superperfect.

Example 4. The graph G in Figure 9.5 is not superperfect since any acyclic orientation with the weighting shown would have a path of weight > 6. Its complement $\bar{G}$ (Figure 9.6), however is superperfect.

Example 5. By extending a weighting w of $X \subseteq V$ to all of V, defining $w(v) = 0$ for all $v \in V - X$, it follows that *each induced subgraph of a superperfect graph is itself superperfect.*

Let F be a transitive orientation of a comparability graph G. By transitivity, every path in F is contained in a clique of G. So, in particular, for any weighting of the vertices of G, the weight of heaviest path in F equals the weight of the heaviest clique in G. This argument proves the following theorem.

Theorem 9.2. A comparability graph is superperfect.

Theorem 9.2 was first noted by Alan Hoffman, and he raised the question of the existence of superperfect graphs which are not comparability graphs. Such a graph was found by the author in 1974; it is the graph in Figure 9.2. We shall explore this question further in Sections 3 and 4.

Theorem 9.2 has an algorithmic aspect as well. The interval chromatic number $\chi(G; w)$ of a weighted comparability graph can be calculated in polynomial time. One must simply obtain a transitive orientation F, for which Algorithm 5.2 may be used, and then apply Algorithm 5.4 to find a

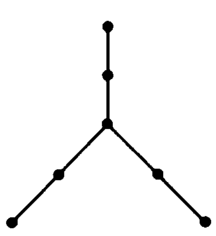

Figure 9.6. $\bar{G}$: superperfect.

maximum weighted clique. In fact, the optimal coloring may be calculated efficiently by a depth-first search procedure utilizing the method described in the proof of Proposition 9.1.

3. An Infinite Class of Superperfect Noncomparability Graphs

Before describing our class of graphs, we will prove the following useful lemma.

Lemma 9.3. Let $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{n-1}$ be sequences of real numbers such that

$$\sum_{i=0}^{n-1} a_i \leq \sum_{i=0}^{n-1} b_i.$$

There exists a cyclic permutation π of $\{0, 1, \ldots, n-1\}$ such that

$$\sum_{i=0}^{m} a_{\pi_i} \leq \sum_{i=0}^{m} b_{\pi_i} \qquad (m = 0, 1, \ldots, n-1).$$

Proof. Let $c_i = b_i - a_i$. If each of the partial sums $\sum_{i=0}^{m} c_i \geq 0$ ($m = 0, 1, \ldots, n-1$), then the result holds. Otherwise, let $\sum_{i=0}^{j} c_i$ be the smallest of these partial sums (i.e, the most negative).

Consider the permutation $\pi_i = i + j + 1 \pmod n$. For $m = j+1, \ldots, n-1$ we have

$$0 \leq \sum_{i=0}^{m} c_i - \sum_{i=0}^{j} c_i = \sum_{i=j+1}^{m} c_i,$$

where, for $m = 1, \ldots, j$,

$$0 \leq \sum_{i=0}^{n-1} c_i = \sum_{i=0}^{j} c_i + \sum_{i=j+1}^{n-1} c_i \leq \sum_{i=0}^{m} c_i + \sum_{i=j+1}^{n-1} c_i,$$

thus proving the lemma. ∎

Let n and k be arbitrary positive integers, $n \geq k$. Consider the undirected graph $G_{n,k} = (A + B, E)$, where

(i) $A = \{a_0, a_1, \ldots, a_{n-1}\}$ is a clique,
(ii) $B = \{b_0, b_1, \ldots, b_{n-1}\}$ is a clique, and
(iii) a_i is adjacent to $b_{i+j (\bmod n)}$ for $j = 1, 2, \ldots, k$.

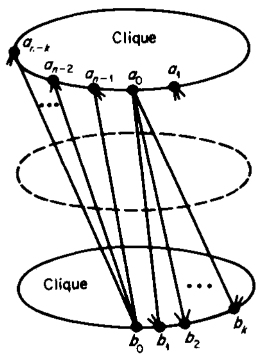

Figure 9.7. $G_{n,k}$.

Figure 9.7 illustrates these adjacencies. We remark here that $G_{n,n-2}$ is the same graph as $\bar{C}_{2n}$, the complement of a chordless cycle of length $2n$. The vertices of the cycle going around in order are $[a_0, b_0, a_1, b_1, \ldots, a_{n-1}, b_{n-1}]$.

Theorem 9.4 (Golumbic [1974]). For arbitrary integers $n \geq k \geq 0$, the graph $G_{n,k}$ is superperfect.

Proof. Assume that $n > k > 0$, for the other cases are trivial. Our method of showing superperfection has three steps.

Step I. Assign an arbitrary weight to each vertex.
Step II. Describe a particular acyclic orientation F.
Step III. Show that every maximal path in F is either (i) contained in some clique, or (ii) has weight less than or equal to a path (already shown to be) in class (i).

We call F a *superperfect orientation* with respect to this weighting.

Step I. We assign an arbitrary weight to each vertex. For simplicity, denote the weight of a_i and b_i by $\hat{a}_i$ and $\hat{b}_i$, respectively. We may assume that $\hat{a}_0 + \hat{a}_1 + \cdots + \hat{a}_{n-1} \leq \hat{b}_0 + \hat{b}_1 + \cdots + \hat{b}_{n-1}$ by interchanging the sets A and B, if necessary. Furthermore, applying Lemma 9.3, we may assume that the vertices have been indexed so that the partial sums satisfy

$$\hat{a}_0 + \cdots + \hat{a}_m \leq \hat{b}_0 + \cdots + \hat{b}_m \qquad (m = 0, 1, \ldots, n-1).$$

Step II. Let us assign the acyclic orientation F of $G_{n,k}$ as follows:

$$\begin{aligned} a_i a_j,\ b_i b_j \in F \qquad & (0 \leq i < j \leq n-1), \\ a_i b_j \in F \qquad & (0 \leq i < j \leq \min(n-1, i+k)), \\ b_i a_j \in F \qquad & (0 \leq i \leq k-1, n-k+i \leq j \leq n-1) \end{aligned}$$

(see Figure 9.8).

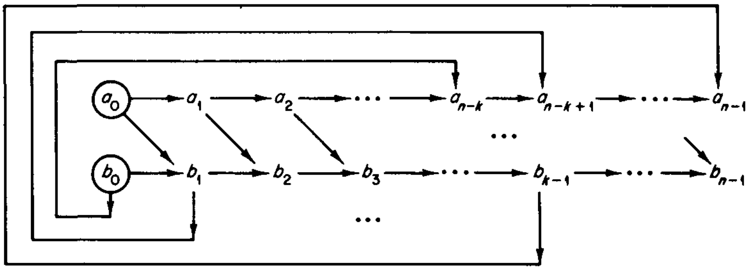

Figure 9.8. Maximal paths in F.

Step III. Any maximal path in F will start in a source node a_0 or b_0. (i) Consider a maximal path μ starting in b_0. Either $\mu = [b_0, b_1, \ldots, b_{n-1}]$, in which case it is contained in a clique, or, for some indices i and p, $0 \leq i \leq k-1$, $1 \leq p < k - i$,

$$\mu = [b_0, \ldots, b_i, a_{n-k+i}, \ldots, a_{n-p}, b_{n-p+1}, \ldots, b_{n-1}].$$

(Obviously if $p = 1$, then there are no b's at the end.) Now $b_t a_j \in E$ for $t = 0, \ldots, i$ and $j = n - k + i, \ldots, n - 1$ and $a_{n-q} b_{n-q+j} \in E$ for $q \leq k - i$ and $j < q + 1$. Thus, μ is contained in a clique.

(ii) Consider a maximal path v starting in a_0. Now v is of the form $v = [a_0, \ldots, a_r, b_{r+1}, \text{remainder}]$ where b_{r+1} is the first b_j in v. Since

$$\hat{a}_0 + \cdots + \hat{a}_r \leq \hat{b}_0 + \cdots + \hat{b}_r,$$

the weight of v is no more than the weight of the path

$$[b_0, \ldots, b_r, b_{r+1}, \text{remainder}],$$

which is contained in a clique, concluding the proof of the theorem. ∎

Corollary 9.5. The complement of an even-length cycle with no chords is superperfect.

The next result shows that the graphs $G_{n,k}$ constitute a class of superperfect graphs distinct from the comparability graphs.

Theorem 9.6. $G_{n,k}$ is not a comparability graph, for $1 \leq k \leq n - 2$.

Proof. Recall that an undirected graph is a comparability graph if and only if every closed path with no triangular chords has even length (see Theorem 5.27). However,

$$[a_0, a_1, b_{k+1}, b_0, a_{k+1}, a_0, b_k, a_0, a_k, a_0, \ldots, a_0, b_1, a_0]$$

is such a closed path in $G_{n,k}$ and has odd length. ∎

4. When Does Superperfect Equal Comparability?

Figure 9.9 illustrates part of the world of superperfect graphs. We have shown in Section 3 that the superperfect graphs properly contain the comparability graphs. This leads us to ask *under what conditions these two classes coincide*. In this section we shall give one answer to this question and we shall discuss some open problems.

Földes and Hammer [1977] have proved the following:

Theorem 9.7. If G is a split graph, then G is a comparability graph if and only if G contains no induced subgraph isomorphic to H_1, H_2, or H_3 of Figure 9.10.

Proof. The forward implication is immediate since none of the graphs in Figure 9.10 is a comparability graph. We shall show the reverse implication. Let G be a split graph whose vertices are partitioned into a stable set X and a complete set Y. An edge of G is called *pure* if both its endpoints are in Y and called *mixed* otherwise. A vertex from X (resp. Y) is denoted by a subscripted lower-case x (resp. y). The key to the proof is the observation that a minimal Γ-chain (i.e., one that does not properly contain another Γ-chain) will alternate between mixed and pure edges and will involve only two vertices of X. Assume that G contains no induced copy of H_1, H_2, or H_3.

Let γ be a minimal Γ-chain. Since no two pure edges are Γ-related, how many mixed edges may separate consecutive pure edges e_1 and e_2 in γ? All such mixed edges will share a common vertex from Y, and hence they are

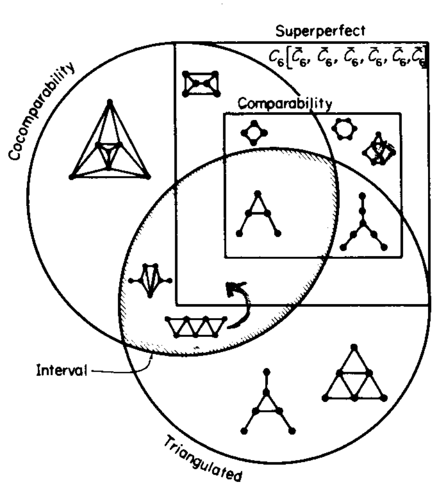

Figure 9.9.

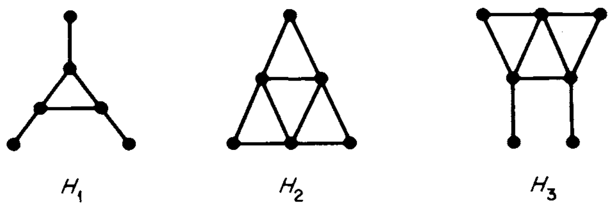

Figure 9.10.

Γ-related to one another. Thus, if there were more than two, the chain γ could be shortened, contradicting minimality. Suppose there are exactly two; then we have the following segment of γ,

$$\cdots f_0 \,\Gamma\, e_1 \,\Gamma\, f_1 \,\Gamma\, f_2 \,\Gamma\, e_2 \,\Gamma\, f_3 \cdots,$$

which corresponds to the diagram in Figure 9.11. By minimality of γ, y_1 and y_3 are adjacent, respectively, to x_2 and x_1. If $y_1 \in \mathrm{Adj}(x_3)$ or $y_3 \in \mathrm{Adj}(x_0)$, which includes the possibility of x_0 and x_3 coinciding, then G contains an induced copy of H_2; otherwise, G contains a copy of H_3. Therefore, γ *alternates between pure and mixed edges*, as claimed.

If G is not a comparability graph, then there exists a minimal Γ-chain γ from some mixed edge $x_0 y_1$ to its reversal, namely,

$$x_0 y_1 \,\Gamma\, y_2 y_1 \,\Gamma\, y_2 x_1 \,\Gamma\, y_2 y_3 \,\Gamma\, x_2 y_3 \,\Gamma\, y_4 y_3 \,\Gamma\, y_4 x_3 \,\Gamma \cdots \Gamma\, y_n x_{n+1} = y_1 x_0.$$

Now $x_0 \neq x_1$ and γ involves only these two vertices from X, since G has no induced copy of H_1. Thus, $x_0 = x_2 = x_4 = \cdots$ and $x_1 = x_3 = x_5 = \cdots$, and by the parity of the indices x_{n+1} equals x_1 and not x_0, a contradiction. This proves the theorem. ∎

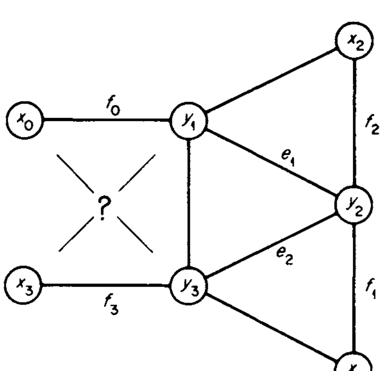

Figure 9.11.

One can easily verify that H_1, H_2, and H_3 are not superperfect (Exercise 1). From this observation we obtain a new result.

Corollary 9.8. For split graphs, G is a comparability graph if and only if G is superperfect.

Proof. Suppose G is not a comparability graph; then G contains one of the forbidden induced subgraphs of Figure 9.10. Since superperfection is a hereditary property, we deduce that G is not superperfect. The opposite implication is true for all graphs. ∎

The class of split graphs is very restrictive. We wonder how much it is possible to weaken the hypothesis of Corollary 9.8 and yet obtain the same conclusion. For example, all the superperfect noncomparability graphs of Section 3 were neither triangulated nor cotriangulated. Is it true or false that for triangulated (or cotriangulated) graphs G is a comparability graph if and only if G is superperfect?

5. Composition of Superperfect Graphs

Recall from Section 5.2 the definition of the *composition* of graphs. In this section we investigate how this operation affects superperfection. Let $G_0, G_1, \ldots, G_n$ be undirected graphs, where G_0 has n vertices $v_1, v_2, \ldots, v_n$.

Theorem 9.9. If $G_0, G_1, \ldots, G_n$ are superperfect, then their composition $G = G_0[G_1, \ldots, G_n]$ is superperfect; i.e., superperfection is preserved under composition.

Proof. Let $G_i = (V_i, E_i)$ for $i = 0, 1, \ldots, n$ be disjoint superperfect graphs, and let w be a weighting of $V_1 + V_2 + \cdots + V_n$. (The vertices in V_0 are not weighted since they will be replaced.) Suppose further that F_i is a superperfect orientation of G_i with respect to w (restricted to G_i) for each $i = 0, 1, \ldots, n$. We claim that $F = F_0[F_1, \ldots, F_n]$ is a superperfect orientation of G with respect to w.

Since each of the F_i $(i = 0, 1, \ldots, n)$ are acyclic, so is F. Let K_i $(i = 1, \ldots, n)$ be a clique of G_i whose weight $w(K_i)$ is greater than or equal to that of any path in F_i. Define $w'(v_i) = w(K_i)$ for all $v_i \in V_0$, and let K_0 be a clique of G_0 whose weight $w'(K_0)$ is greater than or equal to that of any path in F_0. Now any path μ in $F_0[F_1, \ldots, F_n]$ is of the form $\mu = [\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_t}]$, where the

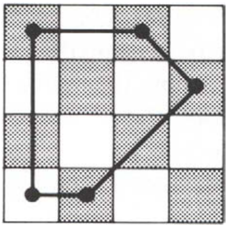

Figure 9.12. A chordless 5-cycle in (V_4, Q).

μ_{i_j} are paths in distinct F_{i_j}, and the sequence of vertices $[v_{i_1}, v_{i_2}, \ldots, v_{i_t}]$ is a path in F_0. Hence, we have the following inequalities:

$$\begin{aligned} w(\mu) = w(\mu_{i_1}) + \cdots + w(\mu_{i_t}) &\leq w(K_{i_1}) + \cdots + w(K_{i_t}) \\ &= w'(v_{i_1}) + \cdots + w'(v_{i_t}) \\ &\leq w'(K_0). \end{aligned}$$

But the vertices of $\bigcup\{K_i \mid v_i \in K_0\}$ induce a clique K of $G_0[G_1, \ldots, G_n]$ whose weight $w(K)$ equals $w'(K_0)$. Thus, we have shown that G is superperfect. ∎

Example 6. Let X_k be the set of positions of a $k \times k$ chessboard, and let Q be the binary relation defined on X_k as follows: $xy \in Q$ iff a queen can be moved from position x to position y in a single chess move.

Consider, for the moment, the graph (X_3, Q). Let x be the middle position and let $X = X_3 - x$. Notice that (X_3, Q) is the composition of the single vertex x and the induced subgraph (X, Q_X) with external factor K_2. However, (X, Q_X) is the complement of a chordless 8-cycle and is therefore a superperfect graph. Hence, (X_3, Q) *is also a superperfect noncomparability graph.*

Clearly (X_k, Q) is an induced subgraph of (X_{k+1}, Q), so (X_k, Q) is a noncomparability graph for all $k \geq 3$. Moreover, Figure 9.12 shows that (X_4, Q) is not perfect since it contains a chordless 5-cycle. Thus, (X_k, Q) is not perfect and hence not superperfect for $k \geq 4$.

6. A Representation Using the Consecutive 1's Property

We now relate the concept of superperfection to some ideas of linear programming. The material presented here is due to Alan J. Hoffman and Ellis L. Johnson.

Recall that a (0, 1)-valued matrix is said to have the *consecutive 1's property* (for columns) if the rows can be permuted so that all the 1's in each column occur consecutively. Let $\mathbf{M}$ denote the stable sets-versus-vertices incidence matrix of an undirected graph G.

Theorem 9.10. G is superperfect if and only if for every row vector $\mathbf{w} \geq \mathbf{0}$ the linear programming problem

$$\mathbf{yM} \geq \mathbf{w}, \quad \mathbf{y} \geq \mathbf{0}, \tag{1a}$$

$$\text{minimize} \sum_i y_i, \tag{1b}$$

has an optimum (row vector) solution $\mathbf{y}$ such that

the submatrix of $\mathbf{M}$ consisting of those rows S_i with $y_i \neq 0$ has the consecutive 1's property. (2)

Assume that the vertex set V is indexed $v_1, v_2, \ldots, v_n$, and let us interpret what the theorem says.

Interpretation. Each stable set S_i is assigned a plot on the real line of width y_i for use only by its members. (Recall that no two members of S_i will need this communal plot simultaneously.)

Feasibility: By (1a) the sum of the widths of the plots available to a given vertex v_j must be at least w_j.
Minimality: By (1b) the combined width of the plots is smallest possible.
Consecutive 1's: By (2) the nonempty plots can be arranged so that those for each vertex are contiguous.

Proof. From the above interpretation, it is clear that any $\mathbf{y}$ satisfying both (1a) and (2) gives a coloring of $(G; \mathbf{w})$ of width $\sum y_i$. The following converse also holds:

For every coloring c of $(G; \mathbf{w})$ there exists a vector $\mathbf{y}$ (to be constructed below) satisfying (1a) and (2) such that $\sum y_i$ equals the width of c. (3)

Let c map V onto the interval from 0 to t. We may assume that c is *left justified*, that is, that no interval can be shifted to the left without disturbing the validity of c as an interval coloring.

Divide each interval $c(v_i)$ into subintervals labeled with exactly those vertices assigned to that subinterval (see Figure 9.12). Each of these labels is some stable set. Suppose there are two subintervals, I_1 and I_2, with the same label S_i. If they are adjacent, then combine them into one larger subinterval.

If they are nonadjacent, there is a vertex v such that $c(v)$ is wholly contained between I_1 and I_2 and whose left endpoint coincides with the right endpoint of I_1 (assume I_1 is to the left of I_2). However, shifting $c(v)$ to the left by the width of I_1 yields another coloring, contradicting left justification. Thus, we may assume that for each stable set S_i there is at most one subinterval with label S_i, and we define y_i to equal the width of that subinterval if it exists and zero otherwise. Clearly, $\mathbf{y}$ satisfies (1a) and (2) and $\sum y_i$ equals the width of the coloring c. This proves claim (3).

Consider the linear programming problem

$$\begin{aligned} &\mathbf{Mx} \leq 1, \qquad \mathbf{x} \geq \mathbf{0}, \\ &\text{maximize} \quad \sum_j w_j x_j. \end{aligned} \tag{4}$$

By the Duality theorem, the optimum solutions of (1) and (4) are equal. Furthermore, if $\mathbf{x}$ is the characteristic vector of a clique of G, then $\mathbf{x}$ is a feasible solution to (4). Conversely, any integral feasible solution to (4) is the characteristic vector of a clique. Thus, an optimum solution $\bar{\mathbf{y}}$ to (1) satisfies

$$\sum \bar{y}_i \geq \omega(G; \mathbf{w}). \tag{5}$$

We do not necessarily have equality in (5) since (4) may not have an optimum solution which is integral. (For example, consider the graph C_5.)

We are now ready to prove the theorem in one direction. Suppose that G is superperfect, and let $\mathbf{w} \geq \mathbf{0}$ be given. Choose a coloring c of $(G; \mathbf{w})$ of width $\omega(G; \mathbf{w})$. By (3) we obtain a vector $\mathbf{y}$ satisfying (1a) and (2) with $\sum y_i = \omega(G; \mathbf{w})$; and by (5), $\mathbf{y}$ is optimum.

To prove the converse of the theorem, we need the following lemmas.

> If $\mathbf{A}$ is a (0, 1)-valued matrix whose columns have the consecutive 1's property, then $\mathbf{A}$ is totally unimodular (i.e., every subdeterminant is 0, 1 or -1). (6)

Hence, if $\mathbf{w}$ is integral, then every optimum solution to (1) which satisfies (2) is integral. (See Hoffman and Kruskal [1956].)

> If for every integral $\mathbf{w}^* \geq \mathbf{0}$ (1) has an optimum solution which is integral, then for every $\mathbf{w} \geq \mathbf{0}$ (4) has an optimum solution which is integral.[1] (See Hoffman [1974] for analogous theorem.) (7)

Suppose that for all $\mathbf{w} \geq \mathbf{0}$ (1) has an optimum solution $\bar{\mathbf{y}}$ satisfying (2). Then $\chi(G; \mathbf{w}) = \sum_i \bar{y}_i$. By (6) and (7), there is an optimum solution $\bar{\mathbf{x}}$ to (4) which is integral. But this optimum solution $\bar{\mathbf{x}}$ is the characteristic vector of a clique of G, so $\omega(G, \mathbf{w}) = \sum_i \bar{x}_i$. Finally, by the duality of (1) and (4) we obtain $\chi(G; \mathbf{w}) = \omega(G; \mathbf{w})$. ∎

EXERCISES

1. Using the technique of Example 5, prove that the graphs H_1, H_2, and H_3 of Figure 9.10 are not superperfect.
2. Prove the following: If H is obtained from G by multiplication of vertices, then H is superperfect if and only if G is superperfect.
3. Prove that the shipbuilding problem is NP-complete.
4. Write a polynomial-time algorithm to solve the banquet problem. Analyze the complexity of your algorithm.
5. Show that the bull's head graph (Figure 1.14) is an interval graph which is not superperfect.

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