CHAPTER 12

Perfect Gaussian Elimination

1. Perfect Elimination Matrices

Let $\mathbf{M}$ be a nonsingular $n \times n$ matrix with entries m_{ij} in some field (like the real numbers). We *reduce* $\mathbf{M}$ to the identity matrix $\mathbf{I}$ by repeatedly

(a) choosing a nonzero entry m_{ij} to act as *pivot* and
(b) updating the matrix by using elementary row and column operations to change m_{ij} to 1 and to make all other entries in the ith row and jth column equal to 0.

This familiar technique is called Gaussian elimination* and can be found in most books on linear algebra.

When performing Gaussian elimination on a sparse matrix, an arbitrary choice of pivots may result in the filling in of some zero positions with nonzeros. One may ask, when is there a sequence of pivots which induces no fill-in? A *perfect elimination scheme* for $\mathbf{M}$ is a sequence of pivots which reduces $\mathbf{M}$ to $\mathbf{I}$ without ever changing a zero entry (even temporarily) to a nonzero. Such a sequence does not exist for every matrix. When $\mathbf{M}$ has a perfect elimination scheme and $\mathbf{M}$ is also sparse, then the sparseness can be preserved throughout the reduction. This is important for the storage requirements of $\mathbf{M}$ since a sparse matrix is most efficiently represented in a computer by listing its nonzero entries.

* In practice one usually "zeros out" just the jth column postponing calculations on the ith row until the end, at which time back substitution is used. For our purposes the methods are the same.

$$
\begin{array}{lccccccc}
\text{Bad Choice:} & \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} & \to & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{pmatrix} & \to & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 8 & -4 \\ 0 & 0 & -4 & 8 \end{pmatrix} & \to & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 12 \end{pmatrix} \\
 & & & & & & & \downarrow \\
\text{Good Choice:} & \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} & \to & \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \to & \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \to & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

Figure 12.1. Two reductions of a matrix.

Examples. Two reductions of the same matrix are given in Figure 12.1; the first introduces new nonzeros but the second does not. What causes the fill-in? It is easy to see that pivoting on a nonzero entry m_{ij} will introduce some fill-in if $m_{sj} \neq 0$ and $m_{it} \neq 0$ but $m_{st} = 0$ for some s and t (see Figure 12.2). The matrix in Figure 12.3 is nonsingular but has no perfect elimination scheme. Any entry chosen as the first pivot will cause some fill-in. A tridiagonal matrix, as indicated in Figure 12.4, has a number of perfect elimination schemes, one being the positions on the main diagonal ordered from top to bottom.

$$
\begin{pmatrix} \boxed{*}_{i,j} & & \boxed{*}_{i,t} \\ & & \\ \boxed{*}_{s,j} & & \boxed{0}_{s,t} \end{pmatrix}
$$

Figure 12.2. Choosing position (i, j) as pivot results in filling in position (s, t). Thus the choice is unacceptable. The asterisks indicate nonzeros.

$$
\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}
$$

Figure 12.3. A matrix with no perfect elimination scheme.

$$
\begin{pmatrix} * & * & & & & \\ * & * & * & & & \\ & * & * & * & & \\ & & * & * & \ddots & \\ & & & \ddots & \ddots & * \\ & & & & \ddots & * \\ & & & & & * \end{pmatrix}
$$

Figure 12.4. A tridiagonal matrix. The asterisks indicate the nonzero entities.

Throughout this chapter we shall assume that $\mathbf{M}$ is nonsingular and that arithmetic coincidence does not cause new zeros to occur during the reductions.*

Let us look at the problem graph theoretically.

The *graph* $G(\mathbf{M})$ *of* $\mathbf{M}$ has vertices $v_1, \ldots, v_n$, with $v_i v_j$ being an edge if and only if $i \neq j$ and entry $m_{ij} \neq 0$. The *bipartite graph* $B(\mathbf{M})$ of $\mathbf{M}$ has vertices $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, corresponding to the rows and columns, respectively, where x_i is adjacent to y_j if and only if $m_{ij} \neq 0$. We call x_i and y_i partners; their correspondence with vertex v_i of $G(\mathbf{M})$ is obvious.

2. Symmetric Matrices

If $\mathbf{M}$ is a symmetric matrix, then $G(\mathbf{M})$ is an undirected graph. In this case a nonzero diagonal entry m_{ii} is acceptable as a pivot if and only if v_i is a simplicial vertex of $G(\mathbf{M})$. (Why?) In fact, pivoting on m_{ii} is equivalent to making $\mathrm{Adj}(v_i)$ into a complete subgraph by adding any missing edges and deleting v_i. Therefore, the perfect elimination schemes for $\mathbf{M}$ under the restriction that

(R) all pivots are chosen along the main diagonal whose entries are each nonzero

correspond precisely to the perfect vertex elimination schemes of $G(\mathbf{M})$. By Theorem 4.1 we obtain the equivalence of statements (ii) and (iii), first obtained by Rose [1970], in the following theorem. We present a generalization due to Golumbic [1978].

Theorem 12.1. Let $\mathbf{M}$ be a symmetric matrix with nonzero diagonal entries. The following conditions are equivalent:

(i) $\mathbf{M}$ has a perfect elimination scheme;
(ii) $\mathbf{M}$ has a perfect elimination scheme under restriction (R);
(iii) $G(\mathbf{M})$ is a triangulated graph.

Before proving the theorem, we must introduce a bipartite graph model of the elimination process. This model will be used here and throughout the chapter.

An edge $e = xy$ of a bipartite graph $H = (U, E)$ is *bisimplicial* if $\mathrm{Adj}(x) + \mathrm{Adj}(y)$ induces a complete bipartite subgraph of H. Take note that *the*

* Actually it is sufficient to assume that no coincidental new zero will be found in a position which we want to choose as pivot (expecting it to be nonzero) at the time when we want to choose it.

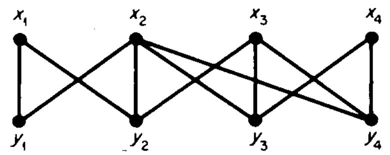

Figure 12.5.

bisimpliciality of an edge is retained as a hereditary property in an induced subgraph. Let $\sigma = [e_1, e_2, \ldots, e_k]$ be a sequence of pairwise nonadjacent edges of H. Denote by S_i the set of endpoints of the edges $e_1, \ldots, e_i$, and let $S_0 = \emptyset$. We say that σ is a *perfect edge elimination scheme* for H if each edge e_i is bisimplicial in the remaining induced subgraph $H_{U-S_{i-1}}$ and H_{U-S_n} has no edge. Thus, we regard the elimination of an edge as the removal of all edges adjacent to e. For example, the graph in Figure 12.5 has the perfect edge elimination scheme $[x_1y_1, x_2y_2, x_3y_3, x_4y_4]$. Notice that initially x_2y_2 is not bisimplicial. A subsequence $\sigma' = [e_1, e_2, \ldots, e_k]$ of σ ($k \leq n$) is called a *partial scheme*. The notations $H - \sigma'$ and H_{U-S_k} will be used to indicate the same subgraph.

Consider the bipartite matrix $B(\mathbf{M})$ of $\mathbf{M}$. It is evident that bisimplicial edges of $B(\mathbf{M})$ correspond precisely to acceptable pivots of $\mathbf{M}$ and that perfect edge elimination schemes for $B(\mathbf{M})$ correspond to perfect elimination schemes for $\mathbf{M}$. (See Figure 12.2 again in conjunction with Figure 12.6.)

Proof of Theorem 12.1. We have already remarked that (ii) and (iii) are equivalent, and since (ii) trivially implies (i), it suffices to prove that (i) implies (iii). Let us assume that $\mathbf{M}$ is symmetric with nonzero diagonal entries, and let σ be a perfect edge elimination scheme for $B(\mathbf{M})$.

Suppose $G(\mathbf{M})$ has a chordless cycle $[v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_m}, v_{\alpha_1}]$. This corresponds in $B(\mathbf{M})$ to the configuration B_C (Figure 12.7) induced by $C = \{x_{\alpha_1}, y_{\alpha_1}, \ldots, x_{\alpha_m}, y_{\alpha_m}\}$. Consider the first edge e of σ involving a vertex of C. Clearly, e involves only *one* vertex of C since none of the edges of B_C is bisimplicial. Assume without loss of generality that $e = x_{\alpha_1}y_s$ for some vertex $y_s \notin C$, and let x_s be the partner of y_s.

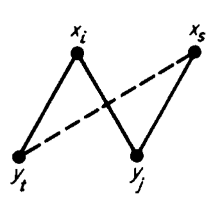

Figure 12.6. The edge x_iy_j is not simplicial. A broken line indicates a nonedge.

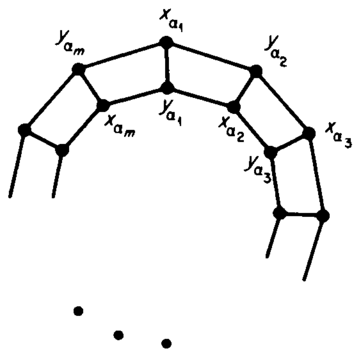

Figure 12.7. The indicated subgraph B_C of $B(\mathbf{M})$.

Since in B_C, $\mathrm{Adj}(x_{\alpha_1}) = \{y_{\alpha_1}, y_{\alpha_2}, y_{\alpha_m}\}$ and $\mathrm{Adj}(y_{\alpha_1}) \cap \mathrm{Adj}(y_{\alpha_2}) \cap \mathrm{Adj}(y_{\alpha_m}) = \{x_{\alpha_1}\}$, the simpliciality of $x_{\alpha_1}y_s$ implies that $C \cap \mathrm{Adj}(y_s) = \{x_{\alpha_1}\}$, and by symmetry $C \cap \mathrm{Adj}(x_s) = \{y_{\alpha_1}\}$. Thus, we have that $x_{\alpha_1}y_{\alpha_2}$ and $x_s y_s$ are edges but $x_s y_{\alpha_2}$ is not an edge, which contradicts the bisimpliciality of $x_{\alpha_1}y_s$. Therefore, $G(\mathbf{M})$ is triangulated. ∎

Corollary 12.2. A symmetric matrix with nonzero diagonal entries can be tested for possession of a perfect elimination scheme in time proportional to the number of nonzero entries.

Proof. Let m denote the number of nonzero entries of matrix $\mathbf{M}$. If $\mathbf{M}$ is stored in $O(m)$ space, then the data structures needed for applying Algorithms 4.1 and 4.2 to $G(\mathbf{M})$ can be initialized in $O(m)$ time. The result follows from Corollary 4.6. ∎

Theorem 12.1 characterized perfect elimination for symmetric matrices. Moreover, it says that it suffices to consider only the diagonal entries. Haskins and Rose [1973] treat the nonsymmetric case under (R), and Kleitman [1974] settles some questions left open by Haskins and Rose. The unrestricted case was finally solved by Golumbic and Goss [1978], who introduced perfect elimination bipartite graphs. These graphs will be discussed in the next section. Additional background on these and other matrix elimination problems can be found in the following survey articles and their references: Tarjan [1976], George [1977], and Reid [1977]. A discussion of the complexity of algorithms which calculate minimal and minimum fill-in under (R) can be found in Ohtsuki [1976], Ohtsuki, Cheung, and Fujisawa [1976], Rose, Tarjan, and Lueker [1976], and Rose and Tarjan [1978].

3. Perfect Elimination Bipartite Graphs

The general problem of deciding if a given (nonsymmetric) matrix **M** has a perfect elimination scheme can be answered using the bipartite graph model introduced in the preceding section. A perfect elimination bipartite graph is one for which there exists *some* perfect edge elimination scheme. But how do we construct such a scheme? Is it possible to choose *any* bisimplicial edge, eliminate it, and then continue from there to construct the remainder of a scheme? The next theorem answers this question in the affirmative. (Throughout this section the term *scheme* will mean perfect edge elimination scheme.)

Theorem 12.3 (Golumbic and Goss [1978]). If $e = xy$ is a bisimplicial edge of a perfect elimination bipartite graph $H = (X, Y, E)$, then $H_{X-\{x\}+Y-\{y\}}$ is also a perfect elimination bipartite graph.

Proof. We wish to show that if H has a scheme $[e_1, e_2, \ldots, e_n]$ then it also has a scheme beginning with e. Let $e_i = x_i y_i$ with $x_i \in X$ and $y_i \in Y$ $(i = 1, 2, \ldots, n)$, and define H_i to be the subgraph of H induced by $X - \{x_1, \ldots, x_{i-1}\} + Y - \{y_1, \ldots, y_{i-1}\}$.

Case 1. $x = x_i$ and $y = y_i$ for some i. Since bisimpliciality of an edge is preserved in induced subgraphs, it follows that $[e, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n]$ is a scheme.

Case 2. $x = x_i$ and $y = y_j$ for some $i \neq j$. We may assume that $i < j$ by interchanging X and Y if necessary; hence $[e, e_1, \ldots, e_{i-1}]$ is a partial scheme.

Consider an edge $x_h y_h$ for some $i < h < j$. Suppose there exists an $m > h$ such that $x_m y_h$ and $x_h y_i$ are edges in H. We would then have the following implications:

$$
\begin{aligned}
&x_i y_i \text{ bisimplicial in } H_i \text{ implies } x_h y_j \in E,\\
&x_h y_h \text{ bisimplicial in } H_h \text{ implies } x_m y_j \in E,\\
&x_i y_j \text{ bisimplicial in } H \text{ implies } x_m y_i \in E.
\end{aligned}
$$

This shows that $\sigma = [e, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{j-1}]$ is a partial scheme.

Similarly, the following argument shows that $e' = x_j y_i$ is in H and is simplicial in $H - \sigma$. If $x_j y_t$ and $x_s y_i$ are in E for $s, t > j$, then

$$
\begin{aligned}
&x_i y_j \text{ bisimplicial in } H \text{ implies } x_j y_i \in E,\\
&x_i y_i \text{ bisimplicial in } H_i \text{ implies } x_s y_j \in E,\\
&x_j y_j \text{ bisimplicial in } H_j \text{ implies } x_s y_t \in E.
\end{aligned}
$$

Since $(H - \sigma) - [e'] = H_{j+1}$, we conclude that

$$[e, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{j-1}, e', e_{j+1}, \ldots, e_n]$$

is a scheme.

Case 3. One of x and y is not among the x_i and y_j. Assume that $x = x_i$ and $y \neq y_j$ for some i and for all j. By an argument similar to, but shorter than, Case 2, $[e, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n]$ is a scheme. ∎

Corollary 12.4. The greedy algorithm of repeatedly eliminating a bisimplicial edge of the remaining graph (thus also removing all edges adjacent to it) until no edge remains will succeed if and only if H is a perfect elimination bipartite graph.

Proof. Assume the corollary is true for all subgraphs of H. If H is not perfect elimination, then the algorithm will surely fail (i.e., at some time before all edges are removed, there will be no bisimplicial edge). If H is perfect elimination, then eliminate some bisimplicial edge e. Since $H - [e]$ is also perfect elimination, we have, by induction, that the algorithm will succeed. ∎

A pair of edges ab and cd of $H = (U, E)$ is *separable* if the subgraph induced by them is isomorphic to $2K_2$. The graph H is said to be *separable* if it contains a pair of separable edges; otherwise H is *nonseparable.* Clearly a nonseparable graph has at most one nontrivial connected component. Furthermore, any induced subgraph of a nonseparable graph is nonseparable.

Theorem 12.5 (Golumbic and Goss [1978]). If $H = (X, Y, E)$ is a nonseparable bipartite graph, then each *nonisolated vertex z is the endpoint of some bisimplicial edge of H.*

Suppose that z is a nonisolated vertex which is not the endpoint of any bisimplicial edge. We may assume that $z \in Y$; let $x_0 z$ be any edge. We shall construct an infinite chain of subsets of X

$$X_0 \subset X_1 \subset \cdots X_k \subset \cdots$$

which will contradict the finiteness of X. Assume we are given subsets

$$X_k = \{x_0, x_1, \ldots, x_k\} \subseteq X$$

and

$$Y_k = \{z, y_1, \ldots, y_k\} \subseteq Y$$

such that

$$x_i y_j \in E \Leftrightarrow i < j \qquad \text{for all } 0 \leq i, j \leq k$$

and

$$x_i z \in E \qquad \text{for all } 0 \le i \le k.$$

(The arbitrary edge $x_0 z$ will start the induction when $k = 0$.)

Since $x_k z$ is not bisimplicial, there exist vertices x and y ($\neq z$) such that $x_k y$, $xz \in E$ but $xy \notin E$. Hence, $y \notin Y_k$. Moreover, for all $0 \le i < k$ the edges $x_i y_{i+1}$ and $x_k y$ are not separable, implying that $x_i y \in E$. But $xy \notin E$, so $x \notin X_k$. Therefore, by renaming $x = x_{k+1}$ and $y = y_{k+1}$ and setting $X_{k+1} = X_k \cup \{x_{k+1}\}$ and $Y_{k+1} = Y_k \cup \{y_{k+1}\}$, we are ready for the next iteration of our construction. This algorithm goes on indefinitely, but X and Y are finite, a contradiction. Thus H must have a bisimplicial edge with z as one of its endpoints. ∎

Corollary 12.6. Every nonseparable bipartite graph H is a perfect elimination bipartite graph.

Proof. By Theorem 12.3, it suffices to show that H has a bisimplicial edge. The corollary follows from Theorem 12.5. ∎

We have accomplished two things in regard to perfect elimination bipartite graphs: We have provided an algorithm for recognizing them, and we have proven a sufficient (but not necessary) condition for them. Being perfect elimination, however, cannot tell us much about the structure of a graph. Indeed, let H be any bipartite graph with vertices $u_1, u_2, \ldots, u_n$; add new vertices $w_1, w_2, \ldots, w_n$ and connect u_i with w_i for each $i = 1, \ldots, n$. This augmented graph is a perfect elimination bipartite graph and completely masks the structure of H. It follows from this negative result that there cannot exist a characterization of perfect elimination bipartite graphs in terms of some forbidden configurations or subgraphs.

4. Chordal Bipartite Graphs

In the preceding section we have successfully generalized the perfect elimination aspect of triangulated graphs. This raises the following question: Is there an appropriate notion of chordality for bipartite graphs? A triangulated graph may have 3-cycles, but any longer cycle must have a chord. In bipartite graphs the smallest allowable cycle has length 4, so we make the following definition. A bipartite graph is *chordal* if every cycle of length strictly greater than 4 has a chord.

Remark. Every nonseparable bipartite graph is chordal bipartite.

Separable edges can be equivalently defined as follows: A pair of edges ab and cd of $G = (V, E)$ is *separable* if there exists a set S of vertices whose removal from G causes ab and cd to lie in distinct connected components of the remaining subgraph G_{V-S}. The set S is called an *edge separator* for ab and cd; S is *minimal* if no proper subset of S is an edge separator for ab and cd. The next result is analogous to Theorem 4.1(iii).

Theorem 12.7 (Golumbic and Goss [1978]). A bipartite graph $H = (X, Y, E)$ is chordal bipartite if and only if every minimal edge separator induces a complete bipartite subgraph.

Proof. Let $C = [v_1, v_2, \ldots, v_k, v_1]$ be a cycle of H having even length $k \geq 6$. Consider the set $S = \mathrm{Adj}(v_2) + \mathrm{Adj}(v_3) - \{v_2, v_3\}$. Clearly S separates $v_2 v_3$ from $v_5 v_6$, and $S \cap C = \{v_1, v_4\}$. Let $S' \subseteq S$ be a minimal edge separator for $v_2 v_3$ and $v_5 v_6$. Thus $v_1 \in S'$ and $v_4 \in S'$. If S' is complete bipartite, then $v_1 v_4$ is a chord of C due to the opposite parity of the subscripts.

Conversely, let T be a minimal edge separator and let H_A and H_B be connected components of the graph remaining after removing T. Let x and y be any pair of vertices of T of opposite parity. Since H_A and H_B are connected, there exist minimum length paths $[x, a_1, a_2, \ldots, y]$ and $[y, b_1, b_2, \ldots, x]$ with $a_i \in A$ and $b_i \in B$. Because these paths are of odd length ≥ 3, they join to give a cycle of length ≥ 6. If this cycle has a chord, it must be the edge xy, since by construction no other pair may be adjacent. Hence, T will be a complete bipartite set. ∎

The next theorem generalizes Lemma 4.2 with separability in bipartite graphs corresponding to nonadjacency in undirected graphs.

Theorem 12.8 (Golumbic [1979]). Let H be a chordal bipartite graph. If H is separable, then it has at least two separable bisimplicial edges.

Proof. Assume that $H = (X, Y, E)$ has separable edges α and β and that the theorem is true for all graphs with fewer vertices than H. Let S be a minimal edge separator for α and β with H_A and H_B being the connected components of H_{X+Y-S} containing α and β, respectively. We claim H_{A+S} has a bisimplicial edge whose endpoints are both in A.

Case 1. H_{A+S} is separable. By induction H_{A+S} has two separable bisimplicial edges $x_1 y_1$ and $x_2 y_2$. Since S is complete, at most two of the four endpoints are in S, either those with the same parity or those with the same subscript. Suppose $x_1, x_2 \in S$ and $y_1, y_2 \in A$. Take a minimum length path $[y_1, a_1, \ldots, a_j, y_2]$ in H_A and a minimum length path $[x_2, b_1, \ldots, b_k, x_1]$ with the b_i in H_B. Gluing these together we obtain a cycle of length at least 6 which must have a chord. But minimality permits only the chords $x_1 y_2$ or

x_2y_1, contradicting the separability of x_1y_1 and x_2y_2. Similarly y_1 and y_2 cannot both be in S. Therefore, H_{A+S} has a bisimplicial edge whose endpoints are both in A.

Case 2. H_{A+S} is nonseparable. Let x_1y_2 be any edge of H_A. By Theorem 12.5 there exist vertices y_1 and x_2 such that x_1y_1 and x_2y_2 are bisimplicial in H_{A+S}. Suppose both y_1 and x_2 are in S, for otherwise the claim is true. Then x_2 and y_1 are adjacent since S is complete. The bisimpliciality of x_1y_1 implies that $\text{Adj}(x_1) \subseteq \text{Adj}_{A+S}(x_2)$, and the bisimpliciality of x_2y_2 implies that $\text{Adj}_{A+S}(x_2) \subseteq \text{Adj}(x_1)$. Hence, $\text{Adj}(x_1) + \text{Adj}(y_2) = \text{Adj}_{A+S}(x_2) + \text{Adj}(y_2)$, which we know induces a complete bipartite subgraph. Thus, x_1y_2 is also bisimplicial in H_{A+S}, so the claim for this case is proven.

Finally, a bisimplicial edge of H_{A+S} whose endpoints lie in A is also bisimplicial in H since $\text{Adj}(A) \subseteq A + S$. Therefore, by the claim, H has a bisimplicial edge α' whose endpoints lie in A and, similarly, a bisimplicial edge β' whose endpoints lie in B, and α' and β' are separable. ▮

The proof of Theorem 12.8 actually gives a slightly stronger result.

Corollary 12.9. Let $H = (X, Y, E)$ be a chordal bipartite graph. If S is a minimal edge separator for some pair of edges, then H has a simplicial edge in each nontrivial connected component of H_{X+Y-S}.

Theorem 12.10 (Golumbic and Goss [1978]). Every chordal bipartite graph is a perfect elimination bipartite graph.

Proof. Since chordal bipartiteness is a hereditary property, it is sufficient to show that a chordal bipartite graph H has a bisimplicial edge. Applying Theorem 12.5 for H nonseparable or Theorem 12.8 for H separable, we obtain the desired result. ▮

Unlike the case of triangulated graphs (Theorem 4.1), the converse of Theorem 12.10 is false. Each of the edges u_iw_i in Figure 12.8 is bisimplicial, and the elimination of any one of them breaks the 6-cycle. Nevertheless, we do have a necessary and sufficient condition for chordality in terms of perfect elimination by adding a hereditary condition.

Corollary 12.11. *A graph is chordal bipartite if and only if every induced subgraph is perfect elimination bipartite.*

Proof. If H possesses a chordless cycle C of length strictly greater than 4, then C would be an induced subgraph which is not perfect elimination. Conversely, if H is chordal bipartite, then so is every induced subgraph H', and by Theorem 12.10, H' is perfect elimination. ▮

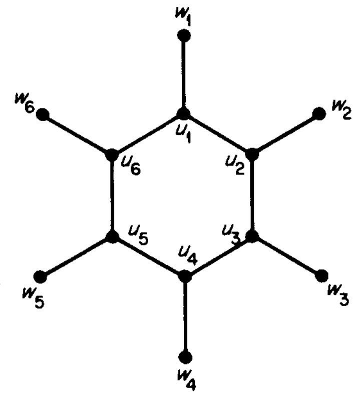

Figure 12.8. A perfect elimination bipartite graph which is not chordal. The cycle is broken when one of the edges $u_i w_i$ is eliminated.

Summary

We have presented bipartite generalizations of triangulated graphs according to two important properties: perfect elimination and chordality. Although these notions do not coincide in the general setting, both do extend certain aspects of triangulated graphs. Perfect elimination bipartite graphs correctly model the application to Gaussian elimination with no fill-in. Chordal bipartite graphs satisfy the separation theorems analogous to those of Dirac. Alan Hoffman and Michel Sakarovich have recently discovered that the chordal bipartite graphs give a characterization of the matrices in an important class of linear programming problems for which the greedy heuristic approach gives an optimum solution.

EXERCISES

1. Verify that the matrix below has a perfect elimination scheme but does *not* have one under restriction (R).

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 & 1 \\ 5 & 1 & 1 & 0 & 1 \end{pmatrix}$$

2. Let $G = (V, E)$ be an undirected graph with vertices $v_1, v_2, \ldots, v_n$. The *bipartite graph* $B(G)$ of G has vertices $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, with x_i adjacent to y_j if and only if $i = j$ or $v_i v_j \in E$.

(i) Show that if $B(G)$ is chordal bipartite, then G is a triangulated graph.

(ii) For which of the triangulated graphs in Figure 12.9 is $B(G)$ *not* chordal bipartite? Find a minimal edge separator which does not induce a bipartite clique.

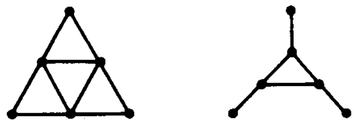

Figure 12.9.

3. Consider the graph H constructed as follows: $[a_1, \ldots, a_n, a_1]$ and $[b_1, \ldots, b_n, b_1]$ are vertex disjoint chordless cycles with n even, and $a_i b_j$ is an edge iff $i + j \equiv 1 \pmod 2$. Show that H is not a perfect elimination bipartite graph for $n \geq 6$.

4. Let $H = (X, Y, E)$ be a perfect elimination bipartite graph with perfect scheme σ. For $xy \in E$ define the *deficiency* of xy in G to be

$$D(xy) = \{ab \notin E \mid a, b \in \mathrm{Adj}(x) + \mathrm{Adj}(y)\}.$$

Show that σ is also a perfect scheme for the graph $H' = (X, Y, E + D(xy))$.

Suppose you made a stupid pivot choice and caused some fill-in on your perfect elimination bipartite graph; is all hope lost? No, you can still continue perfectly, as the next exercise shows.

5. If $H = (X, Y, E)$ is a perfect elimination bipartite graph and xy is any edge, then the *xy-elimination graph*

$$H_{xy} = (X - \{x\}, Y - \{y\}, E_{X - \{x\} + Y - \{y\}} + D(xy))$$

is also perfect elimination. (Hint: Use Theorem 12.3 or modify its proof.) If σ is the perfect scheme for H which was misplaced when xy was stupidly eliminated, how can σ be cleverly modified to give a perfect scheme for H_{xy}?

6. Prove the claim in case 3 of Theorem 12.3.

7. Let $H = (X, Y, E)$ be a bipartite graph and let $H' = (X, Y, E')$ denote its bipartite complement; that is, for all $x \in X$ and $y \in Y$, $xy \in E'$ iff $xy \notin E$. Prove the following: The graphs H and H' are both chordal bipartite if and only if H contains no induced subgraph isomorphic to C_6, $3K_2$, or C_8 (Golumbic and Goss [1978]).

8. Let $G = (V, E)$ be an undirected graph and let $B(G)$ be its bipartite graph (see Exercise 2). For $S \subset V$, let $B(S) = \{x_i \mid v_i \in S\} \cup \{y_i \mid v_i \in S\}$. Prove that S is a minimal vertex separator of G if and only if $B(S)$ is a minimal edge separator of $B(G)$ (Golumbic [1979]).

9. The Venn diagram in Figure 12.10 has nine regions representing all possibilities for a bipartite graph to satisfy or not satisfy the following properties.

P.E.B. The graph is a perfect elimination bipartite graph.

C.B. The graph is chordal bipartite.

(P.E.B.)′ The bipartite complement of the graph is perfect elimination bipartite.

(C.B.)′ The bipartite complement of the graph is chordal bipartite.

For each region give an example of a graph which lives in that region. (One solution is shown in Appendix E, but try to find your own examples without referring to it.)

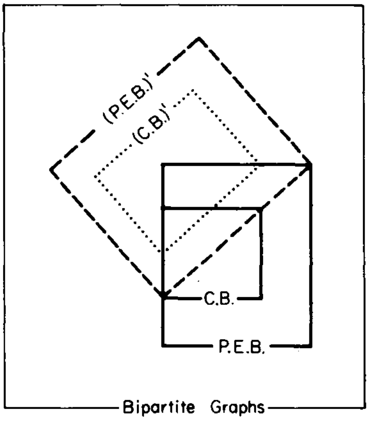

Figure 12.10. Bipartite graphs.

10. Let $H = (X, Y, E)$ be a bipartite graph, and let G be the split graph obtained from H by connecting every pair of vertices in Y. Prove that H is nonseparable if and only if G is a threshold graph. For an application, see Chapter 10, Exercise 15.

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