CHAPTER 3

Perfect Graphs

1. The Star of the Show

In this section we introduce the main character of the book—*the perfect graph.* He was "discovered" by Claude Berge, who has been his agent since the early 1960s. P.G. has appeared in such memorable works as "Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind" and "Caractérisation des graphes non orientés dont on peut orienter les arrêtes de manière à obtenir le graphe d'une relation d'ordre." Despite his seemingly assuming name, P.G. has mixed the highbrow glamorous life with an intense dedication to improving the plight of mankind. His feature role in "Perfect graphs and an application to optimizing municipal services" has won him admiration and respect around the globe. Traveling incognito, a further sign of his modesty, he has been spotted by fans disguised as a graph parfait or as a (banana) split graph in a local ice cream parlor. So, ladies and gentlemen, without further ado, the management proudly presents

THE PERFECT GRAPH

Let us recall the following parameters of an undirected graph, which were defined in Section 1.1.

$\omega(G)$, the *clique number* of G: the size of the largest complete subgraph of G.

$\chi(G)$, the *chromatic number* of G: the fewest number of colors needed to properly color the vertices of G, or equivalently, the fewest number of stable sets needed to cover the vertices of G.

$\alpha(G)$, the *stability number* of G: the size of the largest stable set of G.

$k(G)$, the *clique cover number* of G: the fewest number of complete subgraphs needed to cover the vertices of G.

The intersection of a clique and a stable set of a graph G can be at most one vertex. Thus, for any graph G,

$$\omega(G) \leq \chi(G)$$

and

$$\alpha(G) \leq k(G).$$

These equalities are dual to one another since $\alpha(G) = \omega(\bar{G})$ and $k(G) = \chi(\bar{G})$.

Let $G = (V, E)$ be an undirected graph. The main purpose of this book is to study those graphs satisfying the properties

$$(\mathrm{P}_1) \qquad \omega(G_A) = \chi(G_A) \qquad (\text{for all } A \subseteq V)$$

and

$$(\mathrm{P}_2) \qquad \alpha(G_A) = k(G_A) \qquad (\text{for all } A \subseteq V).$$

Such a graph is called *perfect*. It is clear by duality that a graph G satisfies (P_1) if and only if its complement $\bar{G}$ satisfies (P_2). A much stronger result was conjectured by Berge [1961], cultivated by Fulkerson [1969, 1971, 1972], and finally proven by Lovász [1972a], namely, that (P_1) and (P_2) are equivalent. This has become known as the Perfect Graph theorem, which will be proved in the next section along with a third equivalent condition, due to Lovász [1972b],

$$(\mathrm{P}_3) \qquad \omega(G_A)\alpha(G_A) \geq |A| \qquad (\text{for all } A \subseteq V).$$

In subsequent chapters it will be sufficient to show that a graph satisfies any (P_i) in order to conclude that it is perfect, and a perfect graph will satisfy all properties (P_i).

A fourth characterization of perfect graphs, due to Chvátal [1975], will be discussed in Section 3.3, and we shall encounter still another formulation in the chapter on superperfect graphs.

It is traditional to call a graph χ-*perfect* if it satisfies (P_1) and α-*perfect* if it satisfies (P_2). The Perfect Graph theorem then states that a graph is χ-perfect if and only if it is α-perfect. However, *the equivalence of* (P_1) *and* (P_2) *fails for uncountable graphs.* The interested reader may consult the following references on infinite perfect graphs: Hajnal and Surányi [1958], Perles [1963], and Nash-Williams [1967], Baumgartner, Malitz, and Reinhardt [1970], Trotter [1971], and Wagon [1978].

2. The Perfect Graph Theorem

In this section we shall show the equivalence of properties (P_1)–(P_3). A key to the proof is that multiplication of the vertices of a graph, as defined below, preserves each of the properties (P_i).

Let G be an undirected graph with vertex x. The graph $G \circ x$ is obtained from G by adding a new vertex x' which is connected to all the neighbors of x. We leave it to the reader to prove the elementary property

$$(G \circ x) - y = (G - y) \circ x \qquad \text{for distinct vertices } x \text{ and } y.$$

More generally, if $x_1, x_2, \ldots, x_n$ are the vertices of G and $\mathbf{h} = (h_1, h_2, \ldots, h_n)$ is a vector of non-negative integers, then $H = G \circ \mathbf{h}$ is constructed by substituting for each x_i a stable set of h_i vertices $x_i^1, \ldots, x_i^{h_i}$ and joining x_i^s with x_j^t iff x_i and x_j are adjacent in G. We say that H is obtained from G by *multiplication of vertices.*

Remark. The definition allows $h_i = 0$, in which case H includes no copy of x_i. Thus, every induced subgraph of G can be obtained by multiplication of the appropriate (0, 1)-valued vector.

Lemma 3.1 (Berge [1961]). Let H be obtained from G by multiplication of vertices.

(i) If G satisfies (P_1), then H satisfies (P_1).
(ii) If G satisfies (P_2), then H satisfies (P_2).

Proof. The lemma is true if G has only one vertex. We shall assume that (i) and (ii) are true for all graphs with fewer vertices than G. Let $H = G \circ \mathbf{h}$. If one of the coordinates of $\mathbf{h}$ equals zero, say $h_i = 0$, then H can be obtained from $G - x_i$ by multiplication of vertices. But, if G satisfies (P_1) [resp. (P_2)], then $G - x_i$ also satisfies (P_1) [resp. (P_2)]. In this case the induction hypothesis implies (i) and (ii).

Thus, we may assume that each coordinate $h_i \geq 1$, and since H can be built up from a sequence of smaller multiplications (Exercise 2), it is sufficient to prove the result for $H = G \circ x$. Let x' denote the added "copy" of x.

Assume that G satisfies (P_1). Since x and x' are nonadjacent, $\omega(G \circ x) = \omega(G)$. Let G be colored using $\omega(G)$ colors. Color x' the same color as x. This will be a coloring of $G \circ x$ in $\omega(G \circ x)$ colors. Hence, $G \circ x$ satisfies (i).

Next assume that G satisfies (P_2). We must show that $\alpha(G \circ x) = k(G \circ x)$. Let $\mathscr{K}$ be a clique cover of G with $|\mathscr{K}| = k(G) = \alpha(G)$, and let K_x be the clique of $\mathscr{K}$ containing x. There are two cases.

Case 1: *x is contained in a maximum stable set S of G, i.e.*, $|S| = \alpha(G)$. In this case $S \cup \{x'\}$ is a stable set of $G \circ x$, so

$$\alpha(G \circ x) = \alpha(G) + 1.$$

Since $\mathscr{K} \cup \{\{x'\}\}$ covers $G \circ x$, we have that

$$k(G \circ x) \leq k(G) + 1 = \alpha(G) + 1 = \alpha(G \circ x) \leq k(G \circ x).$$

Thus, $\alpha(G \circ x) = k(G \circ x)$.

Case 2: *No maximum stable set of G contains x*. In this case,

$$\alpha(G \circ x) = \alpha(G).$$

Since each clique of $\mathscr{K}$ intersects a maximum stable set exactly once, this is true in particular for K_x. But x is not a member of any maximum stable set. Therefore, $D = K_x - \{x\}$ intersects each maximum stable set of G exactly once, so

$$\alpha(G_{V-D}) = \alpha(G) - 1.$$

This implies that

$$k(G_{V-D}) = \alpha(G_{V-D}) = \alpha(G) - 1 = \alpha(G \circ x) - 1.$$

Taking a clique cover of G_{V-D} of cardinality $\alpha(G \circ x) - 1$ along with the extra clique $D \cup \{x'\}$, we obtain a cover of $G \circ x$. Therefore,

$$k(G \circ x) = \alpha(G \circ x). \quad \blacksquare$$

Lemma 3.2 (Fulkerson [1971], Lovász [1972b]). Let G be an undirected graph each of whose proper induced subgraphs satisfies (P_2), and let H be obtained from G by multiplication of vertices. If G satisfies (P_3), then H satisfies (P_3).

Proof. Let G satisfy (P_3) and choose H to be a graph having the smallest possible number of vertices which can be obtained from G by multiplication of vertices but which fails to satisfy (P_3) itself. Thus,

$$\omega(H)\alpha(H) < |X|, \tag{1}$$

where X denotes the vertex set of H, yet (P_3) does hold for each proper induced subgraph of H.

As in the proof of the preceding lemma, we may assume that each vertex of G was multiplied by at least 1 and that some vertex u was multiplied by $h \geq 2$. Let $U = \{u^1, u^2, \ldots, u^h\}$ be the vertices of H corresponding to u. The vertex u^1 plays a distinguished role in the proof. By the minimality of H, (P_3) is satisfied by H_{X-u^1}, which gives

$$\begin{aligned}|X| - 1 = |X - u^1| &\le \omega(H_{X-u^1})\alpha(H_{X-u^1}) \qquad &[\text{by } (P_3)]\\ &\le \omega(H)\alpha(H)\\ &\le |X| - 1 &[\text{by } (1)].\end{aligned}$$

Thus, equality holds throughout, and we can define

$$p = \omega(H_{X-u^1}) = \omega(H),$$
$$q = \alpha(H_{X-u^1}) = \alpha(H),$$

and

$$pq = |X| - 1. \tag{2}$$

Since H_{X-U} is obtained from $G - u$ by multiplication of vertices, Lemma 3.1 implies that H_{X-U} satisfies (P_2). Thus, H_{X-U} can be covered by a set of q complete subgraphs of H, say $K_1, K_2, \ldots, K_q$. We may assume that the K_i are pairwise disjoint and that $|K_1| \ge |K_2| \ge \cdots \ge |K_q|$. Obviously,

$$\sum_{i=1}^{q} |K_i| = |X - U| = |X| - h = pq - (h - 1) \qquad [\text{by } (2)].$$

Since $|K_i| \le p$, at most $h - 1$ of the K_i fail to contribute p to the sum. Hence,

$$|K_1| = |K_2| = \cdots = |K_{q-h+1}| = p.$$

Let H' be the subgraph of H induced by $X' = K_1 \cup \cdots \cup K_{q-h+1} \cup \{u^1\}$. Thus

$$|X'| = p(q - h + 1) + 1 < pq + 1 = |X| \qquad [\text{by } (2)], \tag{3}$$

so by the minimality of H,

$$\omega(H')\alpha(H') \ge |X'| \qquad [\text{by } (P_3)]. \tag{4}$$

But $p = \omega(H) \ge \omega(H')$, so

$$\begin{aligned}\alpha(H') &\ge |X'|/p \qquad &[\text{by } (4)]\\ &> q - h + 1 &[\text{by } (3)].\end{aligned}$$

Let S' be a stable set of H' of cardinality $q - h + 2$. Certainly $u^1 \in S'$, for otherwise S' would contain two vertices of a clique (by the definition of H'). Therefore, $S = S' \cup U$ is a stable set of H with $q + 1$ vertices, contradicting the definition of q. ∎

Theorem 3.3 The Perfect Graph Theorem (Lovász [1972b]). For an undirected graph $G = (V, E)$, the following statements are equivalent:

$$(\mathrm{P}_1) \quad \omega(G_A) = \chi(G_A) \quad (\text{for all } A \subseteq V),$$

$$(\mathrm{P}_2) \quad \alpha(G_A) = k(G_A) \quad (\text{for all } A \subseteq V),$$

$$(\mathrm{P}_3) \quad \omega(G_A)\alpha(G_A) \geq |A| \quad (\text{for all } A \subseteq V).$$

Proof. We may assume that the theorem is true for all graphs with fewer vertices than G.

$(\mathrm{P}_1) \Rightarrow (\mathrm{P}_3)$. Suppose we can color G_A in $\omega(G_A)$ colors. Since there are at most $\alpha(G_A)$ vertices of a given color it follows that $\omega(G_A)\alpha(G_A) \geq |A|$.

$(\mathrm{P}_3) \Rightarrow (\mathrm{P}_1)$. Let $G = (V, E)$ satisfy (P_3); then by induction each proper induced subgraph of G satisfies (P_1)–(P_3). It is sufficient to show that $\omega(G) = \chi(G)$.

If we had a stable set S of G such that $\omega(G_{V-S}) < \omega(G)$, we could then paint S orange and paint G_{V-S} in $\omega(G) - 1$ other colors, and we would have $\omega(G) = \chi(G)$.

Suppose G_{V-S} has an $\omega(G)$-clique $K(S)$ for every stable set S of G. Let $\mathscr{S}$ be the collection of all stable sets of G, and keep in mind that $S \cap K(S) = \varnothing$. For each $x_i \in V$, let h_i denote the number of cliques $K(S)$ which contain x_i. Let $H = (X, F)$ be obtained from G by multiplying each x_i by h_i. On the one hand, by Lemma 3.2,

$$\omega(H)\alpha(H) \geq |X|.$$

On the other hand, using some simple counting arguments we can easily show that

$$\begin{aligned} |X| &= \sum_{x_i \in V} h_i \\ &= \sum_{S \in \mathscr{S}} |K(S)| = \omega(G)|\mathscr{S}|, && (5) \\ \omega(H) &\leq \omega(G), && (6) \\ \alpha(H) &= \max_{T \in \mathscr{S}} \sum_{X_i \in T} h_i && (7) \\ &= \max_{T \in \mathscr{S}} \left[\sum_{S \in \mathscr{S}} |T \cap K(S)| \right] && (8) \\ &\leq |\mathscr{S}| - 1, && (9) \end{aligned}$$

which together imply that

$$\omega(H)\alpha(H) \le \omega(G)(|\mathcal{S}| - 1) < |X|,$$

a contradiction.*

$(P_2) \Leftrightarrow (P_3)$. By what we have already proved, we have the following implications:

$$\begin{aligned} G \text{ satisfies } (P_2) &\Leftrightarrow \bar{G} \text{ satisfies } (P_1) \\ &\Leftrightarrow \bar{G} \text{ satisfies } (P_3) \Leftrightarrow G \text{ satisfies } (P_3). \end{aligned}$$ ∎

Corollary 3.4. A graph G is perfect if and only if its complement $\bar{G}$ is perfect.

Corollary 3.5. A graph G is perfect if and only if every graph H obtained from G by multiplication of vertices is perfect.

Historical note. The equivalence of (P_1) and (P_2) was almost proved by Fulkerson. He heard the news of the success of Lovász, who was not aware of Fulkerson's work at that time, from a postcard sent by Berge. Fulkerson immediately returned to his previous results on pluperfection and, within a few hours, obtained his own proof. Such are the joys and sorrows of research. His consolation, to our benefit, was that in the process of his investigations, Fulkerson invented and developed the notion of antiblocking pairs of polyhedra, an idea which has become an important topic in the rapidly growing field of polyhedral combinatorics.†

Briefly, and in our terminology, Fulkerson had proved the following:

Let $\mathcal{M}(G)$ be the collection of all graphs H which can be constructed from a graph G by multiplication of vertices. Then, H satisfies (P_1) for all $H \in \mathcal{M}(G)$ if and only if H satisfies (P_2) for all $H \in \mathcal{M}(G)$.

* Equations (5)–(9) are justified as follows:

(5) Consider the incidence matrix whose rows are indexed by the vertices $x_1, x_2, \ldots, x_n$ and whose columns correspond to the cliques $K(S)$ for $S \in \mathcal{S}$. Then, h_i equals the number of nonzeros in row i, and $|K(S)|$ equals the number of nonzeros in its corresponding column, which is by definition equal to $\omega(G)$.

(6) At most one "copy" of any vertex of G could be in a clique of H.

(7) If a maximum stable set of H contains some of the "copies" of x_i, then it will contain all of the "copies."

(8) Restrict attention to those rows of the matrix pertinent to (5) which belong to elements of T.

(9) $|T \cap K(S)| \le 1$ and $|T \cap K(T)| = 0$.

† Polyhedral combinatorics deals with the interplay between concepts from combinatorics and mathematical programming.

Clearly, this result together with Lemma 3.1 would give a proof of the equivalence of (P_1) and (P_2) for G.

3. p-Critical and Partitionable Graphs*

An undirected graph G is called *p-critical* if it is minimally imperfect, that is, G is *not* perfect but every proper induced subgraph of G *is* a perfect graph. Such a graph, in particular, satisfies the inequalities

$$\alpha(G - x) = k(G - x) \qquad \text{and} \qquad \omega(G - x) = \chi(G - x)$$

for all vertices x, where $G - x$ denotes the resulting graph after deleting x. The following properties of p-critical graphs are easy consequences of the Perfect Graph theorem.

Theorem 3.6. If G is a p-critical graph on n vertices, then

$$n = \alpha(G)\omega(G) + 1,$$

and for all vertices x of G,

$$\alpha(G) = k(G - x) \qquad \text{and} \qquad \omega(G) = \chi(G - x).$$

Proof. By Theorem 3.3, since G is p-critical we have $n > \alpha(G)\omega(G)$ and $n - 1 \leq \alpha(G - x)\omega(G - x)$ for all vertices x. Thus,

$$n - 1 \leq \alpha(G - x)\omega(G - x) \leq \alpha(G)\omega(G) < n.$$

Hence, $n - 1 = \alpha(G)\omega(G)$, $\alpha(G) = \alpha(G - x) = k(G - x)$, and

$$\omega(G) = \omega(G - x) = \chi(G - x). \qquad \blacksquare$$

Let $\alpha, \omega \geq 2$ be arbitrary integers. An undirected graph G on n vertices is called *(α, ω)-partitionable* if $n = \alpha\omega + 1$ and for all vertices x of G

$$\alpha = k(G - x), \qquad \omega = \chi(G - x).$$

We have shown in Theorem 3.6 that every p-critical graph is (α, ω)-partitionable with $\alpha = \alpha(G)$ and $\omega = \omega(G)$. A more general result holds.

Remark 3.7. After removing any vertex x of an (α, ω)-partitionable graph, the remaining graph has $\alpha\omega$ vertices, chromatic number ω, and clique cover number α. So an ω-coloring of $G - x$ will partition the vertices into ω stable sets, one of which must be at least of size α. Similarly, a minimum clique

* Sections 3.3–3.5 were written jointly with Mark Buckingham.

cover of $G - x$ will partition the vertices into α cliques, one of which must be at least of size ω.

Theorem 3.8. If G is an (α, ω)-partitionable graph, then $\alpha = \alpha(G)$ and $\omega = \omega(G)$.

Proof. Let $G = (V, E)$ be (α, ω)-partitionable. By Remark 3.7, $\alpha \leq \alpha(G)$ and $\omega \leq \omega(G)$. Conversely, take a maximum stable set S of G and let $y \in V - S$. Then S is also a maximum stable set of $G - y$, so

$$\alpha(G) = |S| = \alpha(G - y) \leq k(G - y) = \alpha.$$

Thus, $\alpha(G) \leq \alpha$. Similarly, $\omega(G) \leq \omega$. Therefore, $\alpha = \alpha(G)$ and $\omega = \omega(G)$. ▮

Theorem 3.8 shows that the integers α and ω for a partitionable graph are unique. Therefore, we shall simply use the term partitionable graph and assume that $\alpha = \alpha(G)$ and $\omega = \omega(G)$. The class of p-critical graphs is properly contained in the class of partitionable graphs which, in turn, is properly contained in the class of imperfect graphs (Exercise 10).

Lemma 3.9. If G is a partitionable graph on n vertices, then the following conditions hold:

(i) G contains a set of n maximum cliques $K_1, K_2, \ldots, K_n$ that cover each vertex of G exactly $\omega(G)$ times;
(ii) G contains a set of n maximum stable sets $S_1, S_2, \ldots, S_n$ that cover each vertex of G exactly $\alpha(G)$ times; and
(iii) $K_i \cap S_j = \varnothing$ if and only if $i = j$.

Proof. Choose a maximum clique K of G and, for each $x \in K$, choose a minimum clique cover $\mathscr{K}_x$ of $G - x$. By Remark 3.7, all of the members of $\mathscr{K}_x$ must be cliques of size ω. Finally, let $\mathbf{A}$ be the $n \times n$ matrix whose first row is the characteristic vector of K and whose subsequent rows are the characteristic vectors of each of the cliques in $\mathscr{K}_x$ for all $x \in K$. (Note that the number of rows is $1 + \alpha\omega = n$.)

Each vertex $y \notin K$ is covered once by $\mathscr{K}_x$ for all $x \in K$. Each vertex $z \in K$ is covered once by K and once by $\mathscr{K}_x$ for all $z \neq x \in K$. Therefore, every vector is covered ω times. For each row $\mathbf{a}_i$ of $\mathbf{A}$ we let K_i be the clique whose characteristic vector is $\mathbf{a}_i$. We may express (i) by the matrix equation $\mathbf{1A} = \omega\mathbf{1}$, where $\mathbf{1}$ is the row vector containing all ones. Condition (i) will be satisfied once we show that the K_i are distinct.

For each i, pick a vertex $v \in K_i$ and let $\mathscr{S}$ denote a minimum stable set covering (coloring) of $G - v$. By Remark 3.7 and an easy counting exercise, there must be some stable set $S_i \in \mathscr{S}$ such that $K_i \cap S_i = \varnothing$. Let $\mathbf{b}_i$ be the characteristic vector of S_i, and let $\mathbf{B}$ denote the $n \times n$ matrix having rows $\mathbf{b}_i$ for $i = 1, \dots, n$. Since $\mathbf{1} \cdot \mathbf{b}_i^{\mathrm{T}} = \alpha$, we have

$$\mathbf{1AB}^{\mathrm{T}} = \omega \mathbf{1B}^{\mathrm{T}} = \omega\alpha\mathbf{1} = (n-1)\mathbf{1}.$$

But $\mathbf{a}_i \cdot \mathbf{b}_i^{\mathrm{T}} = 0$, so $\mathbf{AB}^{\mathrm{T}} = \mathbf{J} - \mathbf{I}$, where $\mathbf{J}$ is the matrix containing all ones and $\mathbf{I}$ is the identity matrix. This proves (iii).

Finally, both $\mathbf{A}$ and $\mathbf{B}$ are nonsingular matrices since $\mathbf{J} - \mathbf{I}$ is nonsingular. Thus, the K_i are distinct and the S_i are distinct. Furthermore,

$$\begin{aligned}\mathbf{1B} &= \mathbf{1BA}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}})^{-1} = \mathbf{1}(\mathbf{J} - \mathbf{I})(\mathbf{A}^{\mathrm{T}})^{-1} = (n-1)\mathbf{1}(\mathbf{A}^{\mathrm{T}})^{-1} \\ &= [(n-1)/\omega]\mathbf{1} = \alpha\mathbf{1},\end{aligned}$$

which proves (ii). ∎

The next result shows that *all* the maximum cliques and stable sets of G are among those in Lemma 3.9.

Lemma 3.10. A partitionable graph G contains exactly n maximum cliques and n maximum stable sets.

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be the matrices whose rows are the characteristic vectors of the cliques and stable sets, respectively, satisfying $\mathbf{AB}^{\mathrm{T}} = \mathbf{J} - \mathbf{I}$ as specified in Lemma 3.9. Suppose that $\mathbf{c}$ is the characteristic vector of some maximum clique of G. We will show that $\mathbf{c}$ is a row of $\mathbf{A}$.

We first observe that $\mathbf{A}^{-1} = \omega^{-1}\mathbf{J} - \mathbf{B}^{\mathrm{T}}$ since

$$\mathbf{A}(\omega^{-1}\mathbf{J} - \mathbf{B}^{\mathrm{T}}) = \omega^{-1}\mathbf{AJ} - \mathbf{AB}^{\mathrm{T}} = \mathbf{J} - \mathbf{AB}^{\mathrm{T}} = \mathbf{I}.$$

A solution $\mathbf{t}$ to the equation $\mathbf{tA} = \mathbf{c}$ will satisfy

$$\mathbf{t} = \mathbf{cA}^{-1} = \omega^{-1}\mathbf{cJ} - \mathbf{cB}^{\mathrm{T}} = \omega^{-1}(\omega\mathbf{1}) - \mathbf{cB}^{\mathrm{T}} = \mathbf{1} - \mathbf{cB}^{\mathrm{T}}.$$

Therefore, $\mathbf{t}$ is a (0, 1)-valued vector. Also,

$$\mathbf{t} \cdot \mathbf{1}^{\mathrm{T}} = (\mathbf{1} - \mathbf{cB}^{\mathrm{T}}) \cdot \mathbf{1}^{\mathrm{T}} = n - \alpha\mathbf{c} \cdot \mathbf{1}^{\mathrm{T}} = n - \alpha\omega = 1.$$

Therefore, $\mathbf{t}$ is a unit vector. This implies that $\mathbf{c}$ is a row of $\mathbf{A}$.

Similarly, every characteristic vector of a maximum stable set is a row of $\mathbf{B}$. ∎

Theorem 3.11. Let G be an undirected graph on n vertices, and let $\alpha = \alpha(G)$ and $\omega = \omega(G)$. Then G is partitionable if and only if the following conditions hold:

(i) $n = \alpha\omega + 1$;
(ii) G has exactly n maximum cliques and n maximum stable sets;

(iii) every vertex of G is contained in exactly ω maximum cliques and in exactly α maximum stable sets;

(iv) each maximum clique intersects all but one maximum stable set and vice versa.

Proof. ($\Rightarrow$) This implication follows from Lemmas 3.9 and 3.10.

($\Leftarrow$) Following our previous notation, conditions (ii)–(iv) imply that

$$\mathbf{AJ} = \mathbf{JA} = \omega\mathbf{J}, \qquad \mathbf{BJ} = \mathbf{JB} = \alpha\mathbf{J}, \qquad \mathbf{AB}^{\mathrm{T}} = \mathbf{J} - \mathbf{I},$$

where $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices whose rows are the characteristic vectors of the maximum cliques and maximum stable sets, respectively. Let x_i be a vertex of G and let $\mathbf{h}_i^{\mathrm{T}}$ be its corresponding column in $\mathbf{A}$. Since

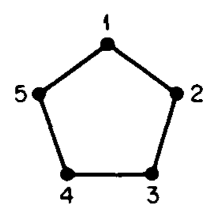
$$\begin{aligned}\mathbf{A}^{\mathrm{T}}\mathbf{B} &= \mathbf{B}^{-1}\mathbf{B}\mathbf{A}^{\mathrm{T}}\mathbf{B} = \mathbf{B}^{-1}(\mathbf{J} - \mathbf{I})\mathbf{B} = \mathbf{B}^{-1}(\alpha\mathbf{J} - \mathbf{B})\\ &= \alpha\alpha^{-1}\mathbf{J} - \mathbf{I} = \mathbf{J} - \mathbf{I},\end{aligned}$$

we obtain $\mathbf{h}_i\mathbf{B} = \mathbf{1} - \mathbf{e}_i$, where $\mathbf{e}_i$ is the ith unit vector. Thus, $\mathbf{h}_i$ designates ω rows of $\mathbf{B}$ (i.e., stable sets of G) which cover $G - x_i$. Thus, $\chi(G - x_i) \le \omega$. By a similar argument, $k(G - x_i) \le \alpha$ for all x_i. But since $n - 1 = \alpha\omega$, we must have $\chi(G - x_i) = \omega$ and $k(G - x_i) = \alpha$. Therefore, G is partitionable. ∎

Corollary 3.12 (Padberg [1974]). If G is a p-critical graph, then conditions (i)–(iv) of Theorem 3.11 hold.

Padberg's investigation of the facial structure of polyhedra associated with (0, 1)-valued matrices first led him to a proof of Corollary 3.12. (We shall discuss some of Padberg's work in Section 3.5.) The proof presented here, using only elementary linear algebra, is due to Bland, Huang, and Trotter [1979]. Additional results on p-critical graphs can be found in Section 3.6.

The only p-critical graphs known are the chordless cycles of odd length and their complements. Figures 3.1 and 3.2 illustrate the conditions of Theorem 3.11 for the graphs C_5 and $\bar{C}_7$.



$K_1 = \{1, 2\}, K_2 = \{2, 3\}, K_3 = \{3, 4\}, K_4 = \{4, 5\}, K_5 = \{5, 1\}$

$S_1 = \{3, 5\}, S_2 = \{1, 4\}, S_3 = \{2, 5\}, S_4 = \{1, 3\}, S_5 = \{2, 4\}$

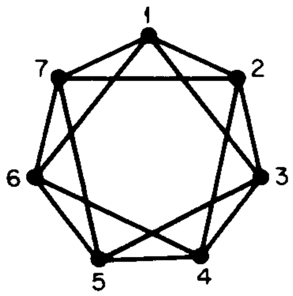
$\{1\} \in K_1, K_5, S_2, S_4;\quad \{2\} \in K_1, K_2, S_3, S_5;$

$\{3\} \in K_2, K_3, S_1, S_4;\quad \{4\} \in K_3, K_4, S_2, S_5;$

$\{5\} \in K_4, K_5, S_1, S_3$

$n = 5,\ \alpha(G) = 2,\ \omega(G) = 2$

Figure 3.1. The graph C_5 and its maximum clique and stable set structure as specified in Theorem 3.11.



$n = 7 \qquad \alpha(G) = 2 \qquad \omega(G) = 3$

$K_1 = \{1, 2, 3\}$, $K_2 = \{2, 3, 4\}$, $K_3 = \{3, 4, 5\}$, $K_4 = \{4, 5, 6\}$
$K_5 = \{5, 6, 7\}$, $K_6 = \{6, 7, 1\}$, $K_7 = \{7, 1, 2\}$
$S_1 = \{4, 7\}$, $S_2 = \{1, 5\}$, $S_3 = \{2, 6\}$, $S_4 = \{3, 7\}$
$S_5 = \{1, 4\}$, $S_6 = \{2, 5\}$, $S_7 = \{3, 6\}$
$\{1\} \in K_1, K_6, K_7, S_2, S_5$; $\{2\} \in K_1, K_2, K_7, S_3, S_6$;
$\{3\} \in K_1, K_2, K_3, S_4, S_7$; $\{4\} \in K_2, K_3, K_4, S_1, S_5$;
$\{5\} \in K_3, K_4, K_5, S_2, S_6$; $\{6\} \in K_4, K_5, K_6, S_3, S_7$;
$\{7\} \in K_5, K_6, K_7, S_1, S_4$

Figure 3.2. The graph $\bar{C}_7$ and its maximum clique and stable set structure as specified in Theorem 3.11.

4. A Polyhedral Characterization of Perfect Graphs

Let **A** be an $m \times n$ matrix. We consider the two polyhedra

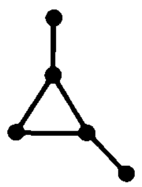
$$P(\mathbf{A}) = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{1}, \mathbf{x} \geq \mathbf{0}\}$$

and

$$P_I(\mathbf{A}) = \text{convex hull}(\{\mathbf{x} \mid \mathbf{x} \in P(\mathbf{A}), \mathbf{x} \text{ integral}\}),$$

where $\mathbf{x}$ is an n-vector and $\mathbf{1}$ is the m-vector of all ones. Clearly $P_I(\mathbf{A}) \subseteq P(\mathbf{A})$, and for (0, 1)-valued matrices **A** having no zero column, $P(\mathbf{A})$ and $P_I(\mathbf{A})$ are bounded and are within the unit hypercube in $\mathbb{R}^n$. An important example of such a matrix is the maximal cliques-versus-vertices incidence matrix of an undirected graph G. This is called the *clique matrix* if all the maximal cliques are included. The clique matrix of G is unique up to permutations of the rows and columns (see Figure 3.3).

Let **A** be any $m \times n$ (0, 1)-valued matrix having no zero columns. The *derived graph* of **A** has n vertices $v_1, v_2, \ldots, v_n$ corresponding to the columns of **A**, and an edge connecting v_i and v_j whenever the ith and jth columns of **A** have a 1 in some row $\mathbf{a}_k$. Clearly every row of **A** forms a (not necessarily



$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Figure 3.3. A graph and its clique matrix.

maximal) clique in its derived graph. Many matrices have the same derived graph. For example, if $\mathbf{A}$ is either the clique matrix or the edge incidence matrix of G, then the derived graph of $\mathbf{A}$ will be G.

Lemma 3.13. Let G be an undirected graph, and let $\mathbf{A}$ be any (0, 1)-valued matrix having no zero column whose derived graph equals G. Then $\mathbf{x}$ is an extremum of $P_I(\mathbf{A})$ if and only if $\mathbf{x}$ is the characteristic vector of some stable set of G.

Proof. If $\mathbf{x}$ is an extremum of $P_I(\mathbf{A})$, then $\mathbf{x}$ must be integral, and since $\mathbf{A}$ is (0, 1)-valued without a zero column, $\mathbf{x} \leq \mathbf{1}$. Thus, $\mathbf{x}$ is the characteristic vector of some set of vertices S. Suppose there exist vertices u and v of S that are connected in G; hence some row $\mathbf{a}_k$ of $\mathbf{A}$ has a 1 in columns u and v. This yields $\mathbf{a}_k \cdot \mathbf{x} \geq 2$, yet $\mathbf{Ax} \leq \mathbf{1}$. Therefore, S must be a stable set.

Conversely, given that $\mathbf{x}$ is a characteristic vector of a stable set of G, certainly $\mathbf{x} \in P_I(\mathbf{A})$. Let $\mathbf{x}$ be expressed as a convex combination of some set of extrema $\{\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \ldots, \mathbf{b}^{(s)}\}$ of $P_I(\mathbf{A})$; that is,

$$x_k = \sum_i c^{(i)} b_k^{(i)}, \qquad 1 = \sum_i c^{(i)}, \qquad 0 \leq c^{(i)} \leq 1.$$

Thus, if $x_k = 1$, then $b_k^{(i)} = 1$ for all i, and if $x_k = 0$, then $b_k^{(i)} = 0$ for all i. Therefore, $\mathbf{x} = \mathbf{b}^{(i)}$ and $\mathbf{x}$ is an extremum of $P_I(\mathbf{A})$. ∎

Theorem 3.14 (Chvátal [1975]). Let $\mathbf{A}$ be the clique matrix of an undirected graph G. Then G is perfect if and only if $P_I(\mathbf{A}) = P(\mathbf{A})$.

To prove the theorem we shall use a result from linear programming used by Edmonds [1965] and others:

Lemma 3.15. Given bounded polyhedra S and T, where S has a finite number of extrema,

$$S = T \qquad \text{iff} \qquad \max_{\text{subj } \mathbf{x} \in S} \mathbf{c} \cdot \mathbf{x} = \max_{\text{subj } \mathbf{x} \in T} \mathbf{c} \cdot \mathbf{x} \quad (\forall \mathbf{c}, \text{integral}).$$

Proof of Theorem 3.14. Assume that $P_I(\mathbf{A}) = P(\mathbf{A})$. Let G_U be an induced subgraph of G, and let $\mathbf{u}$ denote the characteristic vector of U. We have,

$$\alpha(G_U) = \max_{\text{subj } \mathbf{x} \in P_I(\mathbf{A})} \mathbf{ux} = \max_{\text{subj } \mathbf{Ax} \leq \mathbf{1},\, \mathbf{x} \geq \mathbf{0}} \mathbf{ux} = \min_{\text{subj } \mathbf{yA} \geq \mathbf{u},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1}.$$

The first equality follows from the fact that maximums are always achievable at some extremum and the extrema of $P_I(\mathbf{A})$ correspond to stable sets (Lemma 3.13). The second equality follows from Lemma 3.15 setting $\mathbf{c} = \mathbf{u}$, and the third equality comes from the duality theorem of linear programming.

Therefore, choose $\mathbf{y} \geq \mathbf{0}$ such that $\sum y_i = \alpha(G_U)$ and $\mathbf{u} \leq \mathbf{yA}$. Denoting the jth column of $\mathbf{A}$ by $\mathbf{a}^j$, we obtain

$$|U| = \sum_{j \in U} u_j \leq \sum_{j \in U} \mathbf{y} \cdot \mathbf{a}^j = \mathbf{y} \cdot \sum_{j \in U} \mathbf{a}^j \leq \mathbf{y} \cdot (\omega(G_U)\mathbf{1}) = \alpha(G_U)\omega(G_U).$$

Thus, by Theorem 3.3, G is perfect.

Conversely, assume that G is perfect. For any integer vector $\mathbf{c}$, form the graph H by multiplying the ith vertex of G by $\max(0, c_i)$ for each i. By Lemma 3.1, H is perfect. We have the following:

$\alpha(H) = \alpha_{\mathbf{c}}(G)$	The maximum weighted stable set of G given by $\mathbf{c}$.
$= \max\limits_{\text{subj } \mathbf{x} \in P_I(\mathbf{A})} \mathbf{c} \cdot \mathbf{x}$	The maximum can always be found at an extremum, which corresponds to a stable set (Lemma 3.13).
$\leq \max\limits_{\text{subj } \mathbf{x} \in P(\mathbf{A})} \mathbf{c} \cdot \mathbf{x}$	$P_I(\mathbf{A}) \subseteq P(\mathbf{A})$.
$= \min\limits_{\text{subj } \mathbf{yA} \geq \mathbf{c},\, \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1}$	Duality theorem.
$\leq \min\limits_{\text{subj } \mathbf{yA} \geq \mathbf{c},\ \text{non-negative integral } \mathbf{y}} \mathbf{y} \cdot \mathbf{1}$	The constraint set is smaller.
$= k_{\mathbf{c}}(G)$	The minimum clique covering of G such that vertex i is covered c_i times. The constraint $\mathbf{yA} \geq \mathbf{c}$, non-negative integral $\mathbf{y}$, specifies such a covering.
$= k(H)$.	Any clique of H corresponds to a clique of G, thus $k(H) \geq k_{\mathbf{c}}(G)$; if vertex i of G is covered by c_i cliques, then there are c_i cliques in H, each covering a different copy of i, so $k_{\mathbf{c}}(G) \geq k(H)$.

But $\alpha(H) = k(H)$. Thus,

$$\max_{\text{subj } \mathbf{x} \in P_I(\mathbf{A})} \mathbf{c} \cdot \mathbf{x} = \max_{\text{subj } \mathbf{x} \in P(\mathbf{A})} \mathbf{c} \cdot \mathbf{x}$$

and, by Lemma 3.15, $P_I(\mathbf{A}) = P(\mathbf{A})$. ∎

Remark. The first half of the proof of Theorem 3.14 still holds under a weakened hypothesis on $\mathbf{A}$:

If $\mathbf{A}$ is a (0, 1)-valued matrix having no zero column whose derived graph equals G, then $P_I(\mathbf{A}) = P(\mathbf{A})$ implies that G is perfect.

5. A Polyhedral Characterization of p-Critical Graphs

Manfred Padberg first discovered the properties shown in Section 3.3 of p-critical graphs while investigating the facial structure of the polyhedra $P(\mathbf{A})$ for general (0, 1)-valued matrices $\mathbf{A}$. In doing so, he also discovered a polyhedral characterization of p-critical graphs. In Padberg [1973, 1974], he used the results of Lovász and Chvátal to produce these results. In a later work, Padberg [1976b], he developed a more general approach, which enabled him to prove the same results directly and to prove the theorems of Lovász and Chvátal in a different manner.

The matrix $\mathbf{A}$ is said to be *perfect* if $P(\mathbf{A})$ is *integral*, that is, $P(\mathbf{A})$ has only integer extrema: $P_I(\mathbf{A}) = P(\mathbf{A})$. $\mathbf{A}$ is said to be *almost perfect* if $P(\mathbf{A})$ is *almost integral*, that is, (i) $P_I(\mathbf{A}) \neq P(\mathbf{A})$ ($P(\mathbf{A})$ has at least one nonintegral extremum), and (ii) the polyhedra $P_j(\mathbf{A}) = P(\mathbf{A}) \cap \{\mathbf{x} \in \mathbb{R}^n | x_j = 0\}$ are all integral, $j = 1, 2, \ldots, n$.

For the remainder of this section, $\mathbf{A}$ will always denote an $m \times n$ (0, 1)-valued matrix having no zero column, and P, P_I, and P_j will denote $P(\mathbf{A})$, $P_I(\mathbf{A})$, and $P_j(\mathbf{A})$, respectively.

Padberg's results, although not stated in the following manner, include the following six theorems.

Theorem 3.16. If $\mathbf{A}$ is perfect, then $\mathbf{A}$ is an augmented clique matrix of its derived graph, that is, $\mathbf{A}$ is the clique matrix possibly augmented with some redundant rows corresponding to nonmaximal cliques.

Let $\mathbf{J}$ denote the matrix of all ones and $\mathbf{I}$ the identity matrix. We say that $\mathbf{A}$ *contains* the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$ if some permutation of $\mathbf{J} - \mathbf{I}$ occurs as an $n \times n$ submatrix of $\mathbf{A}$.

Theorem 3.17. If $\mathbf{A}$ is almost perfect, then either (i) $\mathbf{A}$ is an augmented clique matrix of its derived graph or (ii) $\mathbf{A}$ contains the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$.

Theorem 3.18. Let G be the derived graph of $\mathbf{A}$. If $\mathbf{A}$ is almost perfect and does not contain the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$, then

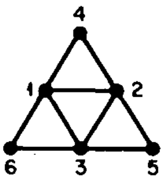
(i) $n = \alpha(G)\omega(G) + 1$;

(ii) every vertex of G is in exactly ω cliques of size ω and in exactly α stable sets of size α;

(iii) G has exactly n maximum cliques and n maximum stable sets;

(iv) there is a numbering of the maximum cliques $K_1, K_2, \ldots, K_n$ and maximum stable sets $S_1, S_2, \ldots, S_n$ of G such that $K_i \cap S_j = \varnothing$ if and only if $i = j$.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$G =$ 

$$\mathbf{A}' = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 3.4. The derived graph G of the matrix $\mathbf{A}$ is a perfect graph, yet $P(\mathbf{A})$ has $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0)$ as an extremum; thus $\mathbf{A}$ is an imperfect matrix. $\mathbf{A}'$ is the clique matrix of G and is perfect.

Theorem 3.19. $\mathbf{A}$ is perfect if and only if $\mathbf{A}$ is an augmented clique matrix of its derived graph and the derived graph is perfect.

Corollary 3.20. $\mathbf{A}$ is almost perfect if and only if either (i) $\mathbf{A}$ is an augmented clique matrix of its derived graph and the derived graph is almost perfect (p-critical) or (ii) $\mathbf{A}$ has no row of all ones and contains the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$ for $n \geq 3$. Furthermore, in (ii) the derived graph is complete.

Corollary 3.21. Every p-critical graph has the four properties of Theorem 3.18.

Note carefully the wording of Theorem 3.19. It is very possible that $\mathbf{A}$ is not a perfect matrix and yet its derived graph G is perfect and every row of $\mathbf{A}$ corresponds to a maximal clique of G. Of course, in this case, by Theorem 3.19, the matrix is missing a row corresponding to some other maximal clique (see Figure 3.4).

Theorems 3.16, 3.17, 3.19, and Corollary 3.20 are very useful when considering graphs as incidence matrices. Corollary 3.21 is a restatement of Corollary 3.12.

To show Theorems 3.16 and 3.17 we will turn to the concept of *antiblocking polyhedra* (Fulkerson [1971, 1972]). Two polyhedra P_1 and P_2 are an *antiblocking pair* if $P_1 = \{\mathbf{x} \mid \mathbf{x}P_2 \leq \mathbf{1}, \mathbf{x} \geq \mathbf{0}\}$ or $P_2 = \{\mathbf{y} \mid \mathbf{y}P_1 \leq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\}$, the conditions being equivalent. If P_2 is generated from a (0, 1)-valued matrix $\mathbf{A}_2$ having no zero column, then we have the property, among many others, that every extremum of P_1 is a projection of some row of $\mathbf{A}_2$ and every nonredundant row of $\mathbf{A}_2$ is an extremem of P_1 (Fulkerson [1972]). The same result holds if we interchange the indices.

Let $\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \ldots, \mathbf{b}^{(r)}$ be the extrema of P_I and denote the matrix having rows $\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \ldots, \mathbf{b}^{(r)}$ by $\mathbf{B}$. Define $Q = P(\mathbf{B})$, $Q_I = P_I(\mathbf{B})$ and $Q_j = P_j(\mathbf{B})$ for $j = 1, 2, \ldots, n$. The polyhedra P_I and Q are an antiblocking pair (Fulkerson [1972]). By Lemma 3.13, the rows of $\mathbf{B}$ correspond to all of the stable sets of G, the derived graph of $\mathbf{A}$. Thus, $\mathbf{B}$ is an augmented clique matrix of the complement $\bar{G}$. See also Monma and Trotter [1979].

Proof of Theorem 3.16. Let **A** be perfect ($P_I = P$) and G be its derived graph. Since P_I and Q are an antiblocking pair, P and Q are also an antiblocking pair. By the properties of antiblocking pairs, the extrema of Q must all be projections of the rows of **A**, so Q is integral. By Lemma 3.13, all the stable sets of $\bar{G}$, in other words all the cliques of G, are extrema of Q, since $Q = Q_I$. Thus, every clique of G must be a projection of some row of **A**. Therefore, **A** is an augmented clique matrix of its derived graph. ∎

Proof of Theorem 3.17. Assuming that **A** is almost perfect, P_j is integral for $j = 1, 2, \ldots, n$. By a similar argument to that for Theorem 3.16, each Q_j is also integral. This follows since $P(\mathbf{A}) \cap \{\mathbf{x} \in \mathbb{R}^n | x_j = 0\}$ is the same as removing the jth column from **A** and forming its polyhedron.

Case 1: Q is not integral and thus is almost integral. In this case Padberg was able to show by a direct analysis of the facets of P that P and Q_I are an antiblocking pair. As in the proof of Theorem 3.16, we have that **A** is an augmented clique matrix of its derived graph.

Case 2: Q is integral. In this case Padberg was able to show by the nonintegrality of P that **1** is an extremum of Q. This means that **B** must be the identity matrix (or a permutation of it). This in turn implies that the derived graph of **A** is complete. Therefore, for P_j to be integral, some row k of the matrix formed by deleting the jth column of **A** must be all ones (Theorem 3.16). Yet no row in **A** can have all ones since **A** is only almost perfect. Thus, row k in **A** must be all ones except for a zero in column j. Since this is true for all $j = 1, 2, \ldots, n$, **A** contains the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$. ∎

Although Theorem 3.18 is essentially contained among the results of Section 3.3, Padberg's proof does not use the Perfect Graph theorem and his technique is valuable in its own right. Before proving Theorem 3.18 we state Padberg's cornerstone lemma.

Lemma 3.22. If **x** is a nonintegral extremum of an almost integral polyhedron P, then for every $n \times n$ nonsingular submatrix $\mathbf{A}_1$ of **A** such that $\mathbf{A}_1\mathbf{x} = \mathbf{1}$, there exists an $n \times n$ nonsingular submatrix $\mathbf{B}_1$ of **B** satisfying the matrix equation

$$\mathbf{B}_1\mathbf{A}_1^{\mathrm{T}} = \mathbf{J} - \mathbf{I}.$$

Furthermore,

$$\mathbf{x} = (1/(n-1))\mathbf{B}_1^{\mathrm{T}}\mathbf{1}.$$

As a quick observation, we note that for any noninteger extremum **x** of P, $\mathbf{x} > \mathbf{0}$. If for some k, $x_k = 0$, then $\mathbf{x} \in P_k$ and thus is an extremum of P_k. But then **x** would have to be integral. The only way **x** could be an extremum of P

is to satisfy n linearly independent constraints of $\mathbf{Ax} \leq \mathbf{1}$. Let $\mathbf{A}_1$ be the $n \times n$ nonsingular submatrix of $\mathbf{A}$. Thus, for each $\mathbf{x}$ there does exist such an $\mathbf{A}_1$ as specified in Lemma 3.22.

Padberg was also able to show that $\mathbf{x}$, a noninteger extremum of P, is the unique noninteger extremum; that $\mathbf{y} = (1/(n-1))\mathbf{A}_1^{\mathrm{T}}\mathbf{1}$, for any $\mathbf{A}_1$ of Lemma 3.22, is an extremum of Q; and that for any $\mathbf{A}_1$ and corresponding $\mathbf{B}_1$ of Lemma 3.22, $\mathbf{x} = |\det \mathbf{A}_1^{-1}|\mathbf{1}$ and $\mathbf{y} = |\det \mathbf{B}_1^{-1}|\mathbf{1}$. Armed with these matrix equations, the proof of Theorem 3.18 is a straightforward exercise in linear algebra.

Proof of Theorem 3.18. Let G be the derived graph of $\mathbf{A}$, where $\mathbf{A}$ is almost perfect and does not contain the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$. By the definition of almost perfect we have the existence of a noninteger extremum $\mathbf{x}$ of P. By Lemma 3.22 and the previous discussion, $\mathbf{x}$ is unique and there exist $n \times n$ nonsingular submatrices $\mathbf{A}_1$ of $\mathbf{A}$ and $\mathbf{B}_1$ of $\mathbf{B}$ such that $\mathbf{A}_1\mathbf{x} = \mathbf{1}$ and $\mathbf{B}_1\mathbf{A}_1^{\mathrm{T}} = \mathbf{J} - \mathbf{I}$. Moreover, for all such $\mathbf{A}_1$ and $\mathbf{B}_1$, $\mathbf{x} = (1/(n-1))\mathbf{B}_1^{\mathrm{T}}\mathbf{1} = |\det \mathbf{A}_1^{-1}|\mathbf{1}$, and $\mathbf{y}$, defined by $\mathbf{y} = (1/(n-1))\mathbf{A}_1^{\mathrm{T}}\mathbf{1} = |\det \mathbf{B}_1^{-1}|\mathbf{1}$, is an extremum of Q.

We shall first show that $\mathbf{A}_1$ is unique, in that any row $\mathbf{a}_k$ of $\mathbf{A}$ satisfying $\mathbf{a}_k \cdot \mathbf{x} = 1$ is in $\mathbf{A}_1$. We have the following implications:

$$\mathbf{B}_1\mathbf{A}_1^{\mathrm{T}} = \mathbf{J} - \mathbf{I} \Rightarrow \mathbf{A}_1\mathbf{B}_1^{\mathrm{T}} = \mathbf{J} - \mathbf{I} \Rightarrow \mathbf{B}_1^{\mathrm{T}} = \mathbf{X} - \mathbf{A}_1^{-1} \Rightarrow \mathbf{A}_1^{-1} = \mathbf{X} - \mathbf{B}_1^{\mathrm{T}},$$

where $\mathbf{X}$ is the $n \times n$ matrix having n columns of $\mathbf{x}$. Thus, if $\mathbf{a}_k \cdot \mathbf{x} = \mathbf{1}$, then $\mathbf{a}_k\mathbf{A}_1^{-1} = \mathbf{a}_k\mathbf{X} - \mathbf{a}_k\mathbf{B}_1^{\mathrm{T}}$ is 0 or 1, yet $\mathbf{a}_k\mathbf{A}_1^{-1} \cdot \mathbf{1} = \mathbf{a}_k \cdot \mathbf{x} = \mathbf{1}$. Therefore, $\mathbf{a}_k\mathbf{A}_1^{-1} = \mathbf{e}_j$, the jth unit vector, for some $j \in \{1, 2, \dots, n\}$. This implies that $\mathbf{a}_k$ is equal to the jth row of $\mathbf{A}_1$, that is, $\mathbf{a}_k$ is in $\mathbf{A}_1$. Finally, since $\mathbf{x} = |\det \mathbf{A}_1^{-1}|\mathbf{1}$, we have that $\mathbf{A}_1$ contains exactly all the rows of $\mathbf{A}$ having the maximum number of ones. By Theorem 3.17, $\mathbf{A}$ is an augmented clique matrix of G. Therefore, $\mathbf{A}_1$ must contain exactly all the maximum cliques of G.

A similar argument holds for $\mathbf{y}$, $\mathbf{B}_1$, $\mathbf{B}$ and $\bar{G}$. Since

$$\mathbf{B}_1\mathbf{y} = \mathbf{B}_1((1/(n-1))\mathbf{A}_1^{\mathrm{T}}\mathbf{1}) = \mathbf{1},$$

we have $\mathbf{B}_1^{-1} = \mathbf{Y} - \mathbf{A}_1^{\mathrm{T}}$. Thus for any row $\mathbf{b}_k$ of $\mathbf{B}$ satisfying $\mathbf{b}_k \cdot \mathbf{y} = 1$, we have $\mathbf{b}_k\mathbf{B}_1^{-1} = \mathbf{b}_k\mathbf{Y} - \mathbf{b}_k\mathbf{A}_1^{\mathrm{T}}$, and yet $\mathbf{b}_k\mathbf{B}_1^{-1} \cdot \mathbf{1} = \mathbf{b}_k \cdot \mathbf{y} = 1$. So $\mathbf{b}_k$ is in $\mathbf{B}_1$. Since $\mathbf{y} = |\det \mathbf{B}_1^{-1}|\mathbf{1}$, and since by construction $\mathbf{B}$ is an augmented clique matrix of $\bar{G}$, we have that $\mathbf{B}_1$ must contain exactly all the maximum cliques of $\bar{G}$.

(i) The row sum of $\mathbf{A}_1$ is $\omega(G)$, yet $\mathbf{A}_1\mathbf{1} = \mathbf{A}_1|\det \mathbf{A}_1|\mathbf{x} = |\det \mathbf{A}_1|\mathbf{1}$; thus $|\det \mathbf{A}_1| = \omega(G)$. Similarly for $\mathbf{B}_1$, the row sum is $\alpha(G)$, yet

$$\mathbf{B}_1\mathbf{1} = \mathbf{B}_1|\det \mathbf{B}_1|\mathbf{y} = |\det \mathbf{B}_1|\mathbf{1};$$

so $|\det \mathbf{B}_1| = \alpha(G)$. Therefore, $\alpha(G)\omega(G) = |\det \mathbf{B}_1\mathbf{A}_1^{\mathrm{T}}| = |\det(\mathbf{E} - \mathbf{I})| = |(-1)^{n-1}(n-1)| = n - 1$. Thus, $n = \alpha(G)\omega(G) + 1$.

(ii) Since $\mathbf{y} = (1/(n-1))\mathbf{A}_1^T\mathbf{1}$, we have $(1/\alpha)\mathbf{1} = (1/\alpha\omega)$, $\mathbf{A}_1^T\mathbf{1}$, and thus $\omega\mathbf{1} = \mathbf{A}_1^T\mathbf{1}$. That is, all the column sums of $\mathbf{A}_1$ are ω. Therefore, every vertex is in exactly ω cliques of size ω.

Similarly for $\mathbf{x}$, $\mathbf{x} = (1/(n-1))\mathbf{B}_1^T\mathbf{1}$ implies $(1/\omega)\mathbf{1} = (1/\alpha\omega)\mathbf{B}_1^T\mathbf{1}$, and hence $\alpha\mathbf{1} = \mathbf{B}_1^T\mathbf{1}$. Therefore, every vertex is in exactly α stable sets of size α.

(iii) $\mathbf{A}_1$ is an $n \times n$ nonsingular matrix containing exactly all the maximum cliques of G; therefore, G has exactly n maximum cliques. By a similar argument on $\mathbf{B}_1$, G has exactly n maximum stable sets.

(iv) Let K_i correspond to the ith row of $\mathbf{A}_1$ for $i = 1, 2, \ldots, n$, and S_j correspond to the jth row of $\mathbf{B}_1$ for $j = 1, 2, \ldots, n$. Since $\mathbf{B}_1\mathbf{A}_1^T = \mathbf{J} - \mathbf{I}$, the maximum cliques $K_1, K_2, \ldots, K_n$ and maximum stable sets $S_1, S_2, \ldots, S_n$ of G are numbered such that $K_i \cap S_j = \varnothing$ if and only if $i = j$. ▌

The "only if" condition of Theorem 3.19 is a stronger statement than Theorem 3.16; it states that the derived graph itself is perfect, which also turns out to be a sufficient condition for $\mathbf{A}$ to be perfect. In fact, Theorem 3.19 is precisely Chvátal's Theorems 3.14 and 3.16 put together. A more direct proof here will be instructive. Again we need an intermediate result of Padberg's.

Lemma 3.23. *P is integral if and only if $\max_{\text{subj } \mathbf{x}\in P} \mathbf{q}\cdot\mathbf{x} \equiv 0 \bmod 1$ for all $(0, 1)$-valued $\mathbf{q}$.*

It is well known that for a general matrix $\mathbf{A}$ with non-negative entries and no zero column, satisfying $\max_{\text{subj } \mathbf{x}\in P} \mathbf{c}\cdot\mathbf{x} \equiv 0 \bmod 1$ for all non-negative $\mathbf{c}$ is equivalent to P being integral. But for our matrix $\mathbf{A}$, considering only $(0, 1)$-valued $\mathbf{q}$ is sufficient.

Proof of Theorem 3.19. ($\Leftarrow$) Let $\mathbf{A}$ be an augmented clique matrix of its derived graph G, where G is perfect. Let $\mathbf{q}$ be a $(0, 1)$-valued vector and G' its corresponding induced subgraph of G. Then

$$\alpha(G') = \max_{\text{subj } \mathbf{x}\in P_I} \mathbf{q}\cdot\mathbf{x} \le \max_{\text{subj } \mathbf{x}\in P} \mathbf{q}\cdot\mathbf{x} = \min_{\text{subj } \mathbf{y}\mathbf{A}\ge\mathbf{q},\ \mathbf{y}\ge\mathbf{0}} \mathbf{y}\cdot\mathbf{1}$$
$$\le \min_{\text{subj } \mathbf{y}\mathbf{A}\ge\mathbf{q},\ \mathbf{y}\ge\mathbf{0},\ \text{integral}} \mathbf{y}\cdot\mathbf{1} = k(G').$$

The first equality is clear because of Lemma 3.13 and the fact that an optimal solution can always be found at an extremum. The last equality is true since $\mathbf{A}$ is an augmented clique matrix and any optimal $\mathbf{y}$ is $(0, 1)$-valued. The inequalities have been seen before in Section 3.4.

Now since G is perfect, we must have equality everywhere. Thus,

$$\max_{\text{subj } \mathbf{x}\in P} \mathbf{q}\cdot\mathbf{x} \equiv 0 \bmod 1.$$

Finally, since $\mathbf{q}$ was arbitrary, Lemma 3.23 implies that P is integral, and thus $\mathbf{A}$ is perfect.

($\Rightarrow$) Let $\mathbf{A}$ be perfect. By Theorem 3.16, $\mathbf{A}$ is an augmented clique matrix of its derived graph. To show that G is perfect we shall use induction on the size of the induced subgraphs.

For $|G'| = 0$ it is clear that $\alpha(G') = k(G')$. Assume that every k-vertex induced subgraph is perfect. Given $|G'| = k + 1$, let $\mathbf{q}$ be the characteristic vector of G'. Since P is integral and $\mathbf{A}$ is an augmented clique matrix of G,

$$\alpha(G') = \max_{\text{subj } \mathbf{x} \in P} \mathbf{q} \cdot \mathbf{x} = \min_{\text{subj } \mathbf{yA} \geq \mathbf{q},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1} \leq \min_{\text{subj } \mathbf{yA} \geq \mathbf{q},\ \mathbf{y} \geq \mathbf{0},\ \text{integral}} \mathbf{y} \cdot \mathbf{1}$$
$$= k(G'). \tag{10}$$

We claim that there is an integer optimal solution for $\min_{\text{subj } \mathbf{yA} \geq \mathbf{q},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1}$. We know that an optimal solution $\bar{\mathbf{y}}$ exists. If $\bar{\mathbf{y}}$ is integral we are done; otherwise there is a k such that $0 < \bar{y}_k < 1$. Clearly the kth row $\mathbf{a}_k$ of $\mathbf{A}$ has the property $\mathbf{a}_k \cdot \mathbf{q} > 0$, for otherwise $\bar{\mathbf{y}}$ would not be optimal. Define $\bar{\bar{q}}_i = q_i$ for $a_{ki} = 0$ and $\bar{\bar{q}}_i = 0$ for $a_{ki} = 1$. Since $\bar{\bar{\mathbf{q}}}$ is the characteristic vector of a smaller induced subgraph, and since (10) still holds, there is an integer optimal solution $\bar{\bar{\mathbf{y}}}$ for $\bar{\bar{\mathbf{q}}}$. Clearly any optimal solution for $\bar{\bar{\mathbf{q}}}$ has its kth component zero; thus $\bar{\mathbf{y}}$ is feasible but not optimal for $\bar{\bar{\mathbf{q}}}$. Yet $\mathbf{y}^*$, where $\mathbf{y}^* = \bar{\bar{\mathbf{y}}}$ except for $y_k^* = 1$, is feasible for $\mathbf{q}$. That is,

$$\min_{\text{subj } \mathbf{yA} \geq \bar{\bar{\mathbf{q}}},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1} < \min_{\text{subj } \mathbf{yA} \geq \mathbf{q},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1} \leq \mathbf{y}^* \cdot \mathbf{1} = \min_{\text{subj } \mathbf{yA} \geq \bar{\bar{\mathbf{q}}},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1} + 1.$$

Therefore, $\mathbf{y}^*$ is an integer optimal solution for $\min_{\text{subj } \mathbf{yA} \geq \mathbf{q},\ \mathbf{y} \geq \mathbf{0}} \mathbf{y} \cdot \mathbf{1}$ and thus $\alpha(G') = k(G')$. ∎

The observant reader will notice that the same "only if" proof could have been used in Theorem 3.14.

The proofs of Corollaries 3.20 and 3.21 are now easy.

Proof of Corollary 3.20. ($\Leftarrow$) *Case 1*: Let $\mathbf{A}$ be an augmented clique matrix of its derived graph G, where G is p-critical. Since deleting any vertex j of G results in a perfect graph, all the P_j are integral. Yet by Theorem 3.19, $\mathbf{A}$ is imperfect because G is imperfect; therefore $\mathbf{A}$ is almost perfect.

Case 2: Let $\mathbf{A}$ have no row of all ones and contain the $n \times n$ submatrix $\mathbf{J} - \mathbf{I}$ for $n \geq 3$. Since each P_j is obtained from the matrix $\mathbf{A}$ with its jth column deleted, and since this submatrix has a row containing all ones, all P_j are integral. Yet $(1/(n-1))\mathbf{1}$ is an extremum of P, since every row has at most $n - 1$ ones and $\mathbf{J} - \mathbf{I}$ is an $n \times n$ submatrix. Therefore $\mathbf{A}$ is almost perfect.

($\Rightarrow$) Given that $\mathbf{A}$ is almost perfect, we apply Theorem 3.17 to obtain two cases.

Case 1: **A** *is an augmented clique matrix of its derived graph G.* By Theorem 3.19 each P_j is integral, the submatrix of **A** obtained by deleting the jth column is perfect, and thus the deletion of any vertex j of G results in a perfect graph. Yet by Theorem 3.19 again, G itself is not perfect since **A** is not perfect. Therefore, G is p-critical.

Case 2: **A** *contains the* $n \times n$ *submatrix* $\mathbf{J} - \mathbf{I}$. Clearly **A** does not contain a row of all ones, for otherwise **A** would be perfect. Finally, we must certainly have $n \geq 3$, thus G is complete. ∎

Proof of Corollary 3.21. Given a p-critical graph G, form **A**, its clique matrix. By Corollary 3.20, case 1, **A** is almost perfect. Certainly G is the derived graph of **A**, and thus the hypothesis of Theorem 3.18 is satisfied. ∎

6. The Strong Perfect Graph Conjecture

The odd cycle C_{2k+1} (for $k \geq 2$) is not a perfect graph since $\alpha(C_{2k+1}) = k$ and $k(C_{2k+1}) = k + 1$ (or, alternatively, since $\omega(C_{2k+1}) = 2$ and $\chi(C_{2k+1}) = 3$). However, every proper subgraph of C_{2k+1} is perfect. Thus, C_{2k+1} is a p-critical graph (i.e., minimally imperfect) and by the Perfect Graph theorem its complement $\bar{C}_{2k+1}$ is also p-critical. To date, these are the only known p-critical graphs.

During the second international meeting on graph theory, held at Halle-on-Saal in March 1960, Claude Berge raised the question of whether or not other p-critical graphs besides the odd cycles and their complements exist. He conjectured that there are none, and this has come to be known as the *strong perfect graph conjecture* (*SPGC*). (Actually, the word "conjecture" first appeared in Berge [1962].)

The strong perfect graph conjecture may be stated in several equivalent forms:

SPGC$_1$. An undirected graph is perfect if and only if it contains no induced subgraph isomorphic to C_{2k+1} or $\bar{C}_{2k+1}$ (for $k \geq 2$).

SPGC$_2$. An undirected graph G is perfect if and only if in G and in $\bar{G}$ every odd cycle of length ≥ 5 has a chord.

SPGC$_3$. The only p-critical graphs that exist are C_{2k+1} and $\bar{C}_{2k+1}$ (for $k \geq 2$).

The graphs C_{2k+1} and $\bar{C}_{2k+1}$ are commonly referred to as the *odd hole* and the *odd antihole*, respectively.

We have seen in Sections 3.3 and 3.5 that p-critical graphs reflect an extraordinary amount of symmetry (as indeed they should if the SPGC turns out to be true). Let G be a p-critical graph on n vertices, and let $\alpha = \alpha(G)$ and $\omega = \omega(G)$. Then the following conditions hold for G.

Lovász condition

$$n = \alpha\omega + 1$$

Padberg conditions

Every vertex is in exactly ω maximum cliques (of size ω).
Every vertex is in exactly α maximum stable sets (of size α).
G has exactly n maximum cliques (of size ω).
G has exactly n maximum stable sets (of size α).
The maximum cliques and maximum stable sets can be indexed K_1, $K_2, \ldots, K_n$ and $S_1, S_2, \ldots, S_n$, respectively, so that $|K_i \cap S_j| = 1 - \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Clearly, any p-critical graph must be connected. But C_n is the only connected graph on n vertices for which $\omega = 2$ and having exactly n undirected edges such that each vertex is an endpoint of exactly two of these edges. So, by Padberg's conditions we obtain another equivalent form of the strong perfect graph conjecture:

SPGC$_4$. There is no p-critical graph with $\alpha > 2$ and $\omega > 2$.

Recall from Section 3.3 that a partitionable graph on n vertices satisfies the Lovász and Padberg conditions.

Figures 3.5 and 3.9 give two examples of (3, 3)-partitionable graphs which fail to be p-critical. For this reason, the Lovász and Padberg conditions alone are not sufficient to prove the SPGC. Nevertheless, partitionable graphs do give us further reductions of the SPGC.

One special type of partitionable graph is easy to describe. The undirected graph C_n^d has vertices $v_1, v_2, v_3, \ldots, v_n$ with v_i and v_j joined by an edge if and only if i and j differ by at most d. (Here and in the next theorem all subscript arithmetic is taken modulo n.) It is easy to see that the graph $C_{\alpha\omega+1}^{\omega-1}$ is an (α, ω)-partitionable graph. When $\omega = 2$, then $C_{\alpha\omega+1}^{\omega-1}$ is simply the odd hole $C_{2\alpha+1}$; when $\alpha = 2$, then $C_{\alpha\omega+1}^{\omega-1}$ is the odd antihole $\bar{C}_{2\omega+1}$.

Theorem 3.24 (Chvátal [1976]). For any integers $\alpha \geq 3$ and $\omega \geq 3$, the partitionable graph $C_{\alpha\omega+1}^{\omega-1}$ is not p-critical.

Proof. Let $\alpha \geq 3$ and $\omega \geq 3$ be given. We will show that $C_{\alpha\omega+1}^{\omega-1}$ contains a proper induced subgraph H which is not perfect.

If we index the $n = \alpha\omega - 1$ maximal cliques $\{K_i\}$ of $C_{\alpha\omega+1}^{\omega-1}$ so that $K_i = \{v_i, v_{i+1}, \ldots, v_{i+\omega-1}\}$ for each $1 \leq i \leq n$, then the clique matrix of the graph has the familiar cyclical pattern, as shown in Figure 3.6. Let H denote the

	0	1	2	3	4	5	6	7	8	9
K_1	1	1								1
K_2	1	1	1							
K_3		1	1	1						
K_4				1	1	1				
K_5					1	1	1			
K_6						1	1	1		
K_7			1				1	1		
K_8								1	1	1
K_9				1	1				1	
K_{10}	1								1	1

Clique matrix

	0	1	2	3	4	5	6	7	8	9
S_1			1			1			1	
S_2				1			1			1
S_3	1				1			1		
S_4		1					1		1	
S_5	1			1				1		
S_6			1		1					1
S_7		1				1			1	
S_8	1			1			1			
S_9			1			1				1
S_{10}		1			1			1		

Stable set matrix

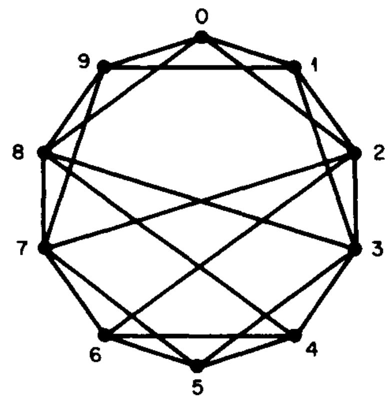


Figure 3.5. A graph satisfying the Lovász and Padberg conditions which fails to be p-critical. The clique matrix and stable set matrix indicate the required indexing of the maximum cliques and maximum stable sets. This example was discovered independently by Huang [1976] and by Chvátal, Graham, Perold, and Whitesides [1979].

subgraph remaining after deleting the $\alpha + 2$ vertices $v_n, v_2, v_{\omega+1}, v_{\omega+3}$, and all $v_{t\omega+2}$ for $t = 2, 3, \ldots, \alpha - 1$. In the deleting process every maximum clique has lost at least one of its members, so $\omega(H) \leq \omega - 1$. Therefore, it suffices to show that H cannot be colored using $\omega - 1$ colors.

Suppose that H is $\omega - 1$ colorable. Let v_1 be colored *black* and let the $\omega - 2$ additional colors be called the *rainbow*. We have the following series of implications:

$$\{v_1, v_3, \ldots, v_\omega\} \subset K_1 \Rightarrow \{v_3, \ldots, v_\omega\} \text{ requires the entire rainbow;}$$

$$\{v_3, \ldots, v_\omega, v_{\omega+2}\} \subset K_3 \Rightarrow v_{\omega+2} \text{ is black;}$$

$$\{v_{\omega+2}, v_{\omega+4}, \ldots, v_{2\omega+1}\} \subset K_{\omega+2}$$
$$\Rightarrow \{v_{\omega+4}, \ldots, v_{2\omega+1}\} \text{ requires the entire rainbow;}$$

$$\{v_{\omega+4}, \ldots, v_{2\omega+1}, v_{2\omega+3}\} \subset K_{\omega+4} \Rightarrow v_{2\omega+3} \text{ is black;}$$

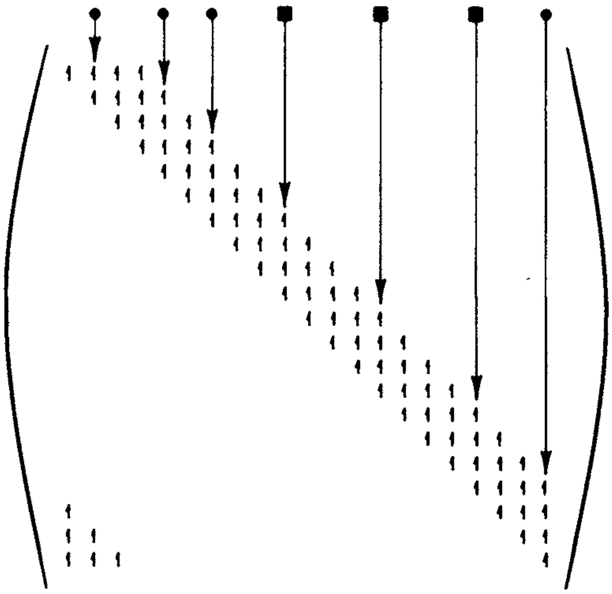


Figure 3.6. The clique matrix of $C^{\omega-1}_{\alpha\omega+1}$, where $\omega = 4$ and $\alpha = 5$. The markers designate which vertices are to be deleted to obtain an imperfect subgraph.

and finally, by induction on t,

$$\begin{aligned} &v_{t\omega+3} \text{ is black} \\ &\qquad \Rightarrow \{v_{t\omega+4}, \ldots, v_{(t+1)\omega+1}\} \text{ requires the entire rainbow} \\ &\qquad \Rightarrow \{v_{(t+1)\omega+3}\} \text{ is black,} \end{aligned}$$

for $t = 2, \ldots, \alpha - 2$.

Therefore, both v_1 and $v_{(\alpha-1)\omega+3}$ are black, but they are both contained in the clique $K_{(\alpha-1)\omega+3}$, a contradiction. Hence, $\chi(H) > \omega - 1 \geq \omega(H)$ and H is imperfect as required. ∎

As a corollary of Theorem 3.24 we obtain another equivalent version of the strong perfect graph conjecture:

SPGC$_5$. If G is p-critical with $\alpha(G) = \alpha$ and $\omega(G) = \omega$, then G contains an induced subgraph isomorphic to $C^{\omega-1}_{\alpha\omega+1}$.

Chvátal, Graham, Perold, and Whitesides [1979] have presented two procedures for constructing (α, ω)-partitionable graphs other than $C^{\omega-1}_{\alpha\omega+1}$.

If we we restrict the universe of graphs being considered by making an extra assumption about their structure, then, in certain cases, the SPGC can be shown to hold. Table 3.1 lists some successful restrictions. For the most part the original proofs cited do not make use of the Padberg conditions. Tucker [1979] has incorporated the Padberg conditions into new proofs of the SPGC for $K_{1,3}$-free graphs and 3-chromatic graphs.

Table 3.1

Classes of graphs for which the strong perfect graph conjecture is known to hold

Planar graphs	Tucker [1973a]
$K_{1,3}$-free graphs	Parthasarathy and Ravindra [1976]
Circular-arc graphs	Tucker [1975]
$\boxtimes$-free graphs	Parthasarathy and Ravindra [1979]
3-chromatic graphs (actually, any graph with $\omega \leq 3$)	Tucker [1977]
Toroidal graphs; graphs having maximum vertex degree ≤ 6	Grinstead [1978]

The strong perfect graph conjecture remains a formidable challenge to us. Its solution has eluded researchers for two decades. Perhaps in the third decade a reader of this book will settle the problem.

EXERCISES

1. Let x and y be distinct vertices of a graph G. Prove that $(G \circ x) - y = (G - y) \circ x$.

2. Let $x_1, x_2, \ldots, x_n$ be the vertices of a graph G and let $H = G \circ \mathbf{h}$ where $\mathbf{h} = (h_1, h_2, \ldots, h_n)$ is a vector of non-negative integers.

Verify that H can be constructed by the following procedure:

```
begin
  H ← G;
  for i ← 1 to n do
    if h_i = 0 then H ← H − x_i;
    else while h_i > 0 do
      begin
        H ← H ∘ x_i;
        h_i ← h_i − 1;
      end
end
```

3. Give an example of a graph G for which $\alpha(G) = k(G)$ and $\omega(G) < \chi(G)$. Why does this not contradict the Perfect Graph theorem?

4. Suppose G satisfies $\alpha(G) = k(G)$. Let $\mathscr{K}$ be a clique cover of G where $|\mathscr{K}| = k(G)$, and let $\mathscr{S}$ be the collection of all stable sets of cardinality $\alpha(G)$. Show that

$$|S \cap K| = 1 \qquad \text{for all } S \in \mathscr{S} \text{ and } K \in \mathscr{K}.$$

Give a dual statement for a graph satisfying $\omega(G) = \chi(G)$.

5. Prove the following: For any integer k, there exists a graph G such that $\omega(G) = 2$ and $\chi(G) = k$. Thus, the gap between the clique number and the chromatic number can be arbitrarily large (Tutte [1954], Kelly and Kelly [1954], Zykov [1952]; see also Sachs [1969]).
6. Prove that an n-vertex graph G is an odd chordless cycle if and only if $n = 2k + 1$, $\alpha(G) = k$, and $\alpha(G - v - w) = k$ for all vertices v and w of G (Melnikov and Vising [1971], Greenwell [1978]).
7. An undirected graph G is *unimodular* if its clique matrix **A** has the property that every square submatrix of **A** has determinant equal to 0, +1, or −1. Prove the following:

(i) The graph in Figure 3.7 is unimodular;
(ii) unimodularity is a hereditary property;
(iii) a bipartite graph is unimodular;
(iv) a unimodular graph is perfect (if necessary, for (iv) see Berge [1975]).

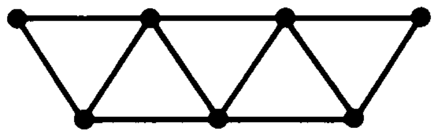


Figure 3.7

8. Show that the five versions of the strong perfect graph conjecture given in this chapter are equivalent.
9. Prove that G is p-critical if and only if G is partitionable but no proper induced subgraph of G is partitionable.
10. Show that the graph in Figure 3.8 is partitionable but not p-critical. Show that the graph in Figure 3.9 is imperfect but not partitionable.

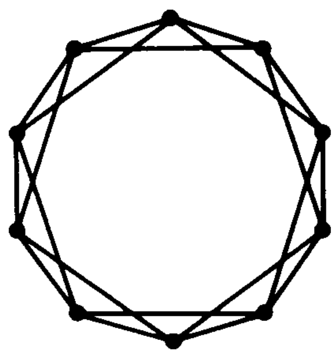


Figure 3.8

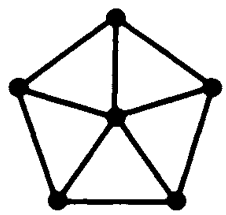


Figure 3.9

11. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices and let α and ω be integers. Using matrix operations give a short proof of the following: If $\mathbf{AJ} = \mathbf{JA} = \omega\mathbf{J}$, $\mathbf{BJ} = \mathbf{JB} = \alpha\mathbf{J}$, and $\mathbf{AB}^{\mathrm{T}} = \mathbf{J} - \mathbf{I}$, then $\alpha\omega = n - 1$.

12. Let $G = (X, E)$ and $H = (Y, F)$ be undirected graphs. Their normal product is defined to be the graph $G \cdot H$ whose vertex set is the Cartesian product $X \times Y$ with vertices (x, y) and (x', y') adjacent if and only if

$$x = x' \quad \text{and} \quad yy' \in F \qquad \text{or} \qquad xx' \in E \quad \text{and} \quad y = y'$$

or

$$xx' \in E \quad \text{and} \quad yy' \in F.$$

Prove the following:

(i) $\chi(G \cdot h) \geq \max\{\chi(G), \chi(H)\}$;
(ii) $\omega(G \cdot H) = \omega(G)\omega(H)$;
(iii) $\alpha(G \cdot H) \geq \alpha(G)\alpha(H)$;
(iv) $k(G \cdot H) \leq k(G)k(H)$.

13. Let G^r denote the normal product of G with itself $r - 1$ times, i.e., $G^1 = G$ and $G^r = G \cdot G^{r-1}$. Let

$$c(G) = \sup \sqrt[r]{\alpha(G^r)}.$$

Prove that $\alpha(G) = k(G)$ implies $c(G) = \alpha(G)$. For an application of this to zero-capacity codes, see Berge [1973, p. 382; 1975, p. 13].

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CHAPTER 4

Triangulated Graphs

1. Introduction

One of the first classes of graphs to be recognized as being perfect was the class of triangulated graphs. Hajnal and Surányi [1958] showed that triangulated graphs satisfy the *perfect property* P_2 (α-perfection), and Berge [1960] proved that they satisfy P_1 (χ-perfection). These two results, in large measure, inspired the conjecture that P_1 and P_2 were equivalent, a statement that we now know to be true (Theorem 3.3). Thus, the study of triangulated graphs can well be thought of as the beginning of the theory of perfect graphs.

We briefly looked at the triangulated graph property in the sneak preview Section 1.3. For completeness' sake, we shall repeat the definition here and mention a few basic properties.

An undirected graph G is called *triangulated* if every cycle of length strictly greater than 3 possesses a chord, that is, an edge joining two nonconsecutive vertices of the cycle. Equivalently, G does not contain an induced subgraph isomorphic to C_n for $n > 3$. Being triangulated is a hereditary property inherited by all the induced subgraphs of G. You may recall from Section 1.3 that the interval graphs constitute a special type of triangulated graph. Thus we have our first example of triangulated graphs.

In the literature, triangulated graphs have also been called *chordal*, *rigid-circuit*, *monotone transitive*, and *perfect elimination* graphs.

2. Characterizing Triangulated Graphs

A vertex x of G is called *simplicial* if its adjacency set Adj(x) induces a complete subgraph of G, i.e., Adj(x) is a clique (not necessarily maximal).

Dirac [1961], and later Lekkerkerker and Boland [1962], proved that a triangulated graph always has a simplicial vertex (in fact at least two of them), and using this fact Fulkerson and Gross [1965] suggested an iterative procedure to recognize triangulated graphs based on this and the hereditary property. Namely, *repeatedly locate a simplicial vertex and eliminate it from the graph, until either no vertices remain and the graph is triangulated or at some stage no simplicial vertex exists and the graph is not triangulated.* The correctness of this procedure is proved in Theorem 4.1. Let us state things more algebraically.

Let $G = (V, E)$ be an undirected graph and let $\sigma = [v_1, v_2, \ldots, v_n]$ be an ordering of the vertices. We say that σ is a *perfect vertex elimination scheme* (or *perfect scheme*) if each v_i is a simplicial vertex of the induced subgraph $G_{\{v_i, \ldots, v_n\}}$. In other words, each set

$$X_i = \{v_j \in \mathrm{Adj}(v_i) \mid j > i\}$$

is complete. For example, the graph G_1 in Figure 4.1 has a perfect vertex elimination scheme $\sigma = [a, g, b, f, c, e, d]$. It is not unique; in fact G_1 has 96 different perfect elimination schemes. In contrast to this, the graph G_2 has no simplicial vertex, so we cannot even start constructing a perfect scheme—it has none.

A subset $S \subset V$ is a *vertex separator* for nonadjacent vertices a and b (or an *a–b separator*) if the removal of S from the graph separates a and b into distinct connected components. If no proper subset of S is an a–b separator, then S is a *minimal vertex separator* for a and b. Consider again the graphs of Figure 4.1. In G_2, the set $\{y, z\}$ is a minimal vertex separator for p and q, whereas $\{x, y, z\}$ is a minimal vertex separator for p and r. (How is it possible that both are *minimal* vertex separators, yet one is contained in the other?) In G_1, every minimal vertex separator has cardinality 2. This is an unusual phenomenon. However, notice also that the two vertices of such a separator of G_1 are adjacent, in every case. This latter phenomenon actually occurs for all triangulated graphs, as you will see in Theorem 4.1.

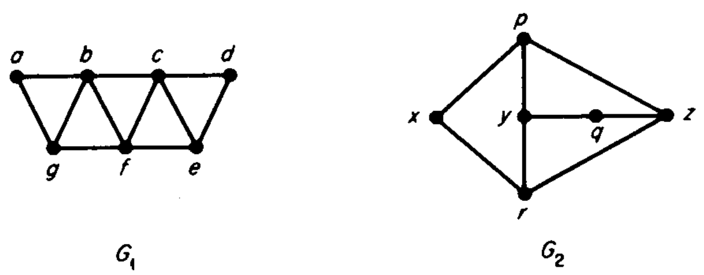


Figure 4.1. Two graphs, one triangulated and one not triangulated.

We now give two characterizations of triangulated graphs, one algorithmic (Fulkerson and Gross [1965]) and the other graph theoretic (Dirac [1961]).

Theorem 4.1. Let G be an undirected graph. The following statements are equivalent:

(i) G is triangulated.

(ii) G has a perfect vertex elimination scheme. Moreover, *any* simplicial vertex can start a perfect scheme.

(iii) Every minimal vertex separator induces a complete subgraph of G.

Proof. (iii) $\Rightarrow$ (i) Let $[a, x, b, y_1, y_2, \ldots, y_k, a]$ $(k \geq 1)$ be a simple cycle of $G = (V, E)$. Any minimal a–b separator must contain vertices x and y_i for some i, so $xy_i \in E$, which is a chord of the cycle.

(i) $\Rightarrow$ (iii) Suppose S is a minimal a–b separator with G_A and G_B being the connected components of G_{V-S} containing a and b, respectively. Since S is minimal, each $x \in S$ is adjacent to some vertex in A and some vertex in B. Therefore, for any pair $x, y \in S$ there exist paths $[x, a_1, \ldots, a_r, y]$ and $[y, b_1, \ldots, b_t, x]$, where each $a_i \in A$ and $b_i \in B$, such that these paths are chosen to be of smallest possible length. It follows that $[x, a_1, \ldots, a_r, y, b_1, \ldots, b_t, x]$ is a simple cycle whose length is at least 4, implying that it must have a chord. But $a_i b_j \notin E$ by the definition of vertex separator, and $a_i a_j \notin E$ and $b_i b_j \notin E$ by the minimality of r and t. Thus, the only possible chord is $xy \in E$. ∎

Remark. It also follows that $r = t = 1$, implying that for all $x, y \in S$ there exist vertices in A and B which are adjacent to both x and y. A stronger result is given in Exercise 12.

Before continuing with the remaining implications, we pause for a message from our lemma department.

Lemma 4.2 (Dirac [1961]). Every triangulated graph $G = (V, E)$ has a simplicial vertex. Moreover, if G is not a clique, then it has two nonadjacent simplicial vertices.

Proof. The lemma is trivial if G is complete. Assume that G has two nonadjacent vertices a and b and that the lemma is true for all graphs with fewer vertices than G. Let S be a minimal vertex separator for a and b with G_A and G_B being the connected components of G_{V-S} containing a and b, respectively.

By induction, either the subgraph G_{A+S} has two nonadjacent simplicial vertices one of which must be in A (since S induces a complete subgraph) or G_{A+S} is itself complete and any vertex of A is simplicial in G_{A+S}. Furthermore, since $\mathrm{Adj}(A) \subseteq A + S$, a simplicial vertex of G_{A+S} in A is simplicial in all of G. Similarly B contains a simplicial vertex of G. This proves the lemma.

We now rejoin the proof of the theorem which is still in progress.

(i) $\Rightarrow$ (ii) According to the lemma, if G is triangulated, then it has a simplicial vertex, say x. Since $G_{V-\{x\}}$ is triangulated and smaller than G, it has, by induction, a perfect scheme which, when adjoined as a suffix of x, forms a perfect scheme for G.

(ii) $\Rightarrow$ (i) Let C be a simple cycle of G and let x be the vertex of C with the smallest index in a perfect scheme. Since $|\mathrm{Adj}(x) \cap C| \geq 2$, the eventual simpliciality of x guarantees a chord in C. ∎

3. Recognizing Triangulated Graphs by Lexicographic Breadth-First Search

From Lemma 4.2 we learned that the Fulkerson–Gross recognition procedure affords us a choice of at least two vertices for each position in constructing a perfect scheme for a triangulated graph. Therefore, we can freely choose a vertex v_n to *avoid* during the whole process, saving it for the last position in a scheme. Similarly, we can pick any vertex v_{n-1} adjacent to v_n to save for the $(n-1)$st position. If we continued in this manner, we would be constructing a scheme *backwards*! This is exactly what Leuker [1974] and Rose and Tarjan [1975] have done in order to give a linear-time algorithm for recognizing triangulated graphs. The version presented in Rose, Tarjan, and Leuker [1976] uses a *lexicographic* breadth-first search in which the usual queue of vertices is replaced by a queue of (unordered) subsets of the vertices which is sometimes refined but never reordered. The method (Figure 4.2) is as follows:

```
   begin
1.   assign the label ∅ to each vertex;
2.   for i ← n to 1 step −1 do
3.      select: pick an unnumbered vertex v with largest label;
4.         σ(i) ← v; comment This assigns to v the number i.
5.      update: for each unnumbered vertex w ∈ Adj(v) do add i to label(w);
   end
```

Figure 4.2. Algorithm 4.1: Lex BFS.

Algorithm 4.1. Lexicographic breadth-first search.

Input: The adjacency sets of an undirected graph $G = (V, E)$.
Output: An ordering σ of the vertices.
Method: The vertices are numbered from n to 1 in the order that they are selected in line 3. This numbering fixes the positions of an elimination scheme

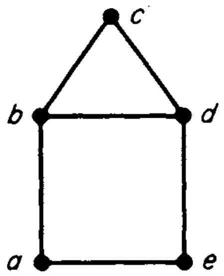


Figure 4.3.

σ. For each vertex x, the *label* of x will consist of a set of numbers listed in decreasing order. The vertices can then be lexicographically ordered according to their labels. (Lexicographic order is just dictionary order, so that $9761 < 985$ and $643 < 6432$.) Ties are broken arbitrarily.

Example. We shall apply Algorithm 4.1 to the graph in Figure 4.3. The vertex a is selected arbitrarily in line 3 during the first pass. The evolution of the labeling and the numbering are illustrated in Figure 4.4. Notice that the final numbering $\sigma = [c, d, e, b, a]$ is a perfect vertex elimination scheme. This is no accident.

For each value of i, let $L_i(x)$ denote the label of x when statement 4 is executed, i.e., when the ith vertex is numbered. Remember, the index is *decremented* at each successive iteration. For example, $L_n(x) = \emptyset$ for all x and $L_{n-1}(x) = \{n\}$ iff $x \in \operatorname{Adj}(\sigma(n))$. The following properties are of prime importance:

(L1) $L_i(x) \leq L_j(x) \quad (j \leq i)$;
(L2) $L_i(x) < L_i(y) \Rightarrow L_j(x) < L_j(y) \quad (j < i)$;
(L3) if $\sigma^{-1}(a) < \sigma^{-1}(b) < \sigma^{-1}(c)$ and $c \in \operatorname{Adj}(a) - \operatorname{Adj}(b)$, then there exists a vertex $d \in \operatorname{Adj}(b) - \operatorname{Adj}(a)$ with $\sigma^{-1}(c) < \sigma^{-1}(d)$.

	label	number
a	ϕ	–
b	ϕ	–
c	ϕ	–
d	ϕ	–
e	ϕ	–

	label	number
a	ϕ	5
b	{5}	–
c	ϕ	–
d	ϕ	–
e	{5}	–

	label	number
a	ϕ	5
b	{5}	4
c	{4}	–
d	{4}	–
e	{5,4}	–

	label	number
a	ϕ	5
b	{5}	4
c	{4}	–
d	{4,3}	–
e	{5,4}	3

	label	number
a	ϕ	5
b	{5}	4
c	{4,2}	–
d	{4,3}	2
e	{5,4}	3

	label	number
a	ϕ	5
b	{5}	4
c	{4,2}	1
d	{4,3}	2
e	{5,4}	3

Figure 4.4.

Property (L1) says that the label of a vertex may get larger but never smaller as the algorithm proceeds. Property (L2) states that once a vertex gets ahead of another vertex, they stay in that order. Finally, (L3) gives a condition under which there must be a suitable vertex d which was numbered before c (in time) and hence received a larger number.

Lexicographic breadth-first search can be used to recognize triangulated graphs as demonstrated by the next theorem.

Theorem 4.3. An undirected graph $G = (V, E)$ is triangulated if and only if the ordering σ produced by Algorithm 4.1 is a perfect vertex elimination scheme.

Proof. If $|V| = n = 1$, then the proof is trivial. Assume that the theorem is true for all graphs with fewer than n vertices and let σ be the ordering produced by Algorithm 4.1 when applied to a triangulated graph G. By induction, it is sufficient to show that $x = \sigma(1)$ is a simplicial vertex of G.

Suppose x is not simplicial. Choose vertices $x_1, x_2 \in \mathrm{Adj}(x)$ with $x_1x_2 \notin E$ so that x_2 is as large as possible (with respect to the ordering σ). (Remember, σ increases as you approach the root of the search tree.) Consider the following inductive procedure. Assume we are given vertices $x_1, x_2, \ldots, x_m$ with these properties: for all $i, j > 0$,

(1) $x, x_i \in E \Leftrightarrow i \leq 2$,
(2) $x_i x_j \in E \Leftrightarrow |i - j| = 2$,
(3) $\sigma^{-1}(x_1) < \sigma^{-1}(x_2) < \cdots < \sigma^{-1}(x_m)$,
(4) x_j is the largest vertex (with respect to σ) such that

$$x_{j-2}x_j \in E \qquad \text{but} \qquad x_{j-3}x_j \notin E.$$

(For notational reasons let $x_0 = x$ and $x_{-1} = x_1$.) The situation for $m = 2$ was constructed initially.

The vertices x_{m-2}, x_{m-1}, and x_m satisfy the hypothesis of property (L3) as a, b, and c, respectively. Hence, choose x_{m+1} to be the largest vertex (with respect to σ) larger than x_m which is adjacent to x_{m-1} but not adjacent to x_{m-2}. Now, if x_{m+1} were adjacent to x_{m-3}, then (L3) applied to the vertices x_{m-3}, x_{m-2}, x_{m+1} would imply the existence of a vertex larger than x_{m+1} (hence larger than x_m) which is adjacent to x_{m-2} but not to x_{m-3}, contradicting the maximality of x_m in (4). Therefore x_{m+1} is not adjacent to x_{m-3}. Finally, it follows from (1), (2), and chordality that $x_i x_{m+1} \notin E$ for $i = 0, 1, \ldots, m - 4, m$.

Clearly this inductive procedure continues indefinitely, but the graph is finite, a contradiction. Therefore, the vertex x must be simplicial, and the theorem is proved in one direction. The converse follows from Theorem 4.1. ∎

In an unpublished work, Tarjan [1976] has shown another method of searching a graph that can be used to recognize triangulated graphs. It is called *maximum cardinality search* (MCS), and it is defined as follows:

MCS: The vertices are to be numbered from n to 1.
The next vertex to be numbered is always one which is adjacent to the most numbered vertices, ties being broken arbitrarily.

Using an argument similar to the proof of Theorem 4.3, one can show that G is triangulated if and only if every MCS ordering of the vertices is a perfect ellimination scheme. It should be pointed out that there are MCS orderings which cannot be obtained by Lex BFS, there are Lex BFS orderings which are not MCS, and there exist perfect elimination schemes which are neither MCS nor Lex BFS. Exercises 27 and 28 develop some of the results on MCS. Both Lex BFS and MCS are special cases of a general method for finding perfect elimination schemes recently developed by Alan Hoffman and Michel Sakarovich.

4. The Complexity of Recognizing Triangulated Graphs

Having proved the correctness of Algorithm 4.1, let us now analyze its complexity. We first describe an implementation of Lex BFS, then show that it requires $O(|V| + |E|)$ time. We do not actually calculate the labels, but rather we keep the unnumbered vertices in lexicographic order.

Data Structure

We use a queue Q of sets

$$S_l = \{v \in V \mid \text{label}(v) = l \text{ and } \sigma^{-1}(v) \text{ undefined}\}$$

ordered lexicographically from smallest to largest; each set S_l is represented by a doubly linked list. Initially there is but one set, $S_\phi = V$. Each set S_l has a FLAG initially set at 0. For a vertex w, the array element SET(w) points to $S_{\text{label}(w)}$ and another array gives the address of w in SET(w) for deletion purposes. A list FIX LIST, initially empty, is also used, and simple arrays represent σ and σ^{-1}.

Implementation

Select as v in line 3 any vertex in the last set of Q and delete v from SET(v). Create a new set $S_{l \cdot i}$ for each old set S_l containing an unnumbered vertex

$w \in \text{Adj}(v)$. We delete from S_l all such vertices w and place them in the new set $S_{l \cdot i}$, which is inserted into the queue of sets immediately following S_l. Clearly this method maintains the proper lexicographic ordering without our actually having to calculate the labels. More specifically, *update* can be implemented as follows:

```
5. ┌ for all unnumbered w ∈ Adj(v) do
   │   begin
   │     if FLAG(SET(w)) = 0 then
   │       begin
   │         Create new set S′ and insert it in Q immediately in back of SET(w);
   │         FLAG(SET(w)) ← 1; FLAG(S′) ← 0; put a pointer to SET (w) on FIX LIST;
   │       end
   │     let S′ be the set immediately in back of SET(w) in Q; delete w from SET(w); add w to S′;
   │     SET(w) ← S′;
   │   end
   │ for each set S on FIX LIST do
   │   begin
   │     FLAG(S) ← 0;
   │     if S is empty then
   │       delete S from Q;
   └   end
```

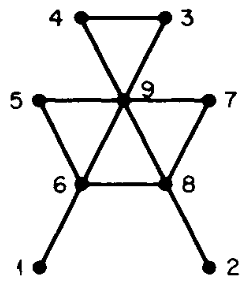
It is easy to verify that, as presented, statement 5 requires $O(|\text{Adj}(v)|)$ time. Consequently, the **for** loop between statements 2 and 5 uses $O(|V| + |E|)$ time. Initializing the data structure including statement 1 takes $O(|V|)$ time. This proves the following result.

Theorem 4.4. Algorithm 4.1 can be implemented to carry out lexicographic breadth-first search on an undirected graph $G = (V, E)$ in $O(|V| + |E|)$ time and space.

Example. Let Q_i denote the queue of sets of unnumbered vertices just before $\sigma(i)$ is defined in Algorithm 4.1. Figure 4.5b gives Q_9, Q_8, and Q_7 for the graph in Figure 4.5a. For convenience, the vertices are identified with their eventual position in σ. Figure 4.5c shows the data structure for Q_7 before the FIX LIST has been emptied and with the implicit labels in parentheses.

In order to use Lex BFS to recognize triangulated graphs, we need an efficient method to test whether or not a given ordering σ of the vertices is a perfect vertex elimination scheme. This is proved by the next algorithm.

Algorithm 4.2. Testing a perfect elimination scheme.
Input. The adjacency sets of an undirected graph $G = (V, E)$ and an ordering σ of V.



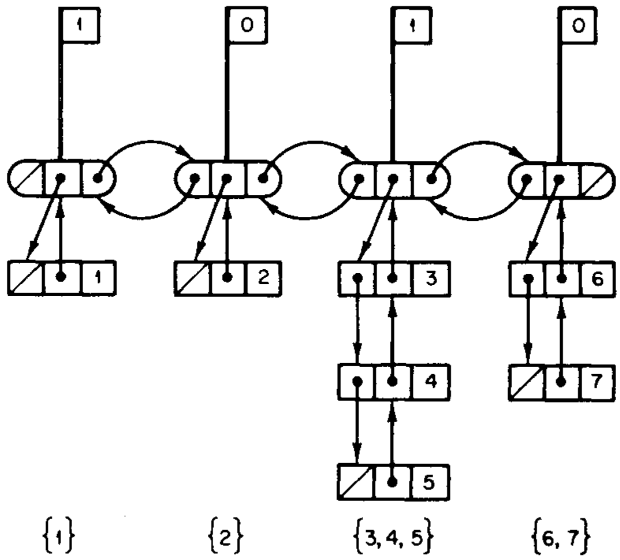
(a)

$$Q_9 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$Q_8 = \{1, 2\} < \{3, 4, 5, 6, 7, 8\}$$

$$Q_7 = \{1\} < \{2\} < \{3, 4, 5\} < \{6, 7\}$$

(b)



(c)

Figure 4.5.

Output. "True" if σ is a perfect vertex elimination scheme and "false" otherwise.

Method. A single call to the procedure PERFECT(σ), given in Figure 4.6. The list $A(u)$ collects all the vertices which will eventually have to be checked for adjacency with u. The actual checking is delayed until the iteration when $u = \sigma(i)$ in lines 8 and 9. This technique is used so that in the $\sigma^{-1}(v)$-th iteration there is no search of Adj(u).

Complexity. Arrays are used for σ and σ^{-1} and lists hold Adj(v) and $A(v)$. Lines 4–7 can be implemented simultaneously in one scan of Adj(v). The **go to** in line 5 will be executed exactly $j - 1$ times, where j is the number of connected components of G. The list $A(u)$ will represent a *set with repetitions.* The test in line 8 simply checks for a vertex w on the list $A(v)$ which is *not*

```
     boolean procedure PERFECT (σ):
     begin
1.     for all vertices v do A(v) ←∅;;
2.     for i ← 1 to n − 1 do
         begin
3.         v ← σ(i);
4.         X ← {x ∈ Adj(v) | σ⁻¹(v) < σ⁻¹(x)};
5.         if X = ∅ then go to 8;;
6.         u ← σ (min {σ⁻¹(x) | x ∈ X});
7.         concatenate X − {u} to A(u);
8.         if A(v) − Adj(v) ≠ ∅ then
9.           return "false";
         end
10.    return "true";
     end
```

Figure 4.6. Procedure to test a perfect vertex elimination scheme.

adjacent to v, can be done in $O(|\mathrm{Adj}(v)| + |\mathrm{A}(v)|)$ time by using an array TEST of size n initially set to all zeros as follows:

```
   ⎧ begin
   ⎪   for w ∈ Adj(v) do TEST(w) ← 1;;
   ⎪   for w ∈ A(v) do
8. ⎨     if TEST(w) = 0 then
   ⎪       return "nonempty";
   ⎪   for w ∈ Adj(v) do TEST(w) ← 0;;
   ⎪   return "empty";
   ⎩ end
```

Thus, the entire algorithm can be performed in time and space proportional to

$$|V| + \sum_{v \in V} |\mathrm{Adj}(v)| + \sum_{u \in V} |A(u)|,$$

where has $A(u)$ is its final value. Now, the middle summand is larger than the last since a given $\mathrm{Adj}(v)$ appears as part of at most one of the lists $A(u)$. Hence, both summands can be replaced by $O(|E|)$. This proves the complexity part of the next theorem.

Theorem 4.5. Algorithm 4.2 correctly tests whether or not an ordering σ of the vertices is a perfect vertex elimination scheme. It can be implemented to run in time and space proportional to $|V| + |E|$.

Proof. The algorithm returns "false" during the $\sigma^{-1}(u)$-th iteration if and only if there exist vertices v, u, w ($\sigma^{-1}(v) < \sigma^{-1}(u) < \sigma^{-1}(w)$), where u is defined in line 4 during the $\sigma^{-1}(v)$-th iteration, and

$$u, w \in \mathrm{Adj}(v) \qquad \text{but } u \text{ is not adjacent to } w.$$

Clearly, if we get "false," then σ is not a perfect elimination scheme.

Conversely, suppose σ is not perfect elimination and the algorithm returns "true." Let v be the vertex with $\sigma^{-1}(v)$ largest possible such that $X = \{w \mid w \in \text{Adj}(v) \text{ and } \sigma^{-1}(v) < \sigma^{-1}(w)\}$ is *not* complete. Let u be the vertex of X defined in line 6 during the $\sigma^{-1}(v)$-th iteration, after which (in line 7) $X - \{u\}$ is added to $A(u)$. Since during the $\sigma^{-1}(u)$-th iteration line 9 is not executed,

$$\text{every } x \in X - \{u\} \text{ is adjacent to } u,$$

and

$$\text{every pair } x, y \in X - \{u\} \text{ is adjacent.}$$

The latter statement follows from the maximality of $\sigma^{-1}(v)$. Thus, X is complete, a contradiction. ∎

Corollary 4.6. Triangulated graphs can be recognized in linear time.

Proof. The proof follows from Theorems 4.3–4.5. ∎

5. Triangulated Graphs as Intersection Graphs

We have seen in Chapter 1 that the interval graphs are a proper subclass of the triangulated graphs. This leads naturally to the problem of characterizing triangulated graphs as the intersection graphs of some topological family slightly more general than intervals on a line. In this section we shall show that a graph is triangulated if and only if it is the intersection graph of a family of subtrees of a tree. (See Figure 4.7.)

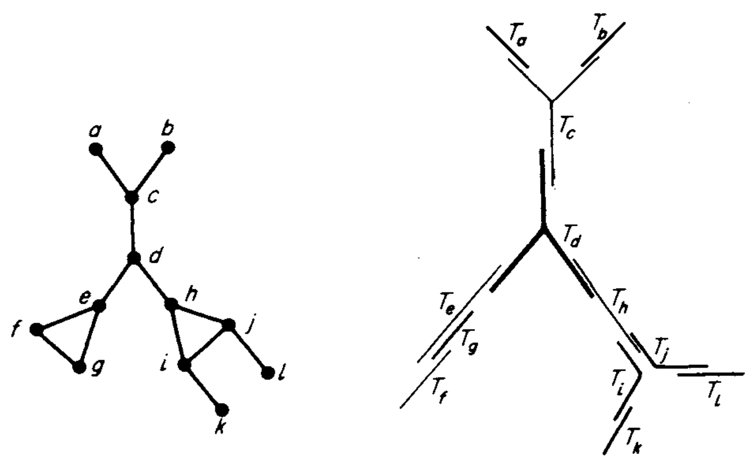


Figure 4.7. A triangulated graph and a subtree representation for it.

A family $\{T_i\}_{i \in I}$ of subsets of a set T is said to satisfy the *Helly property* if $J \subseteq I$ and $T_i \cap T_j \neq \varnothing$ for all $i, j \in J$ implies that $\bigcap_{j \in J} T_j \neq \varnothing$.

If we let T be a tree and let each T_i be a subtree of T, then we can prove the following result.

Proposition 4.7. A family of subtrees of a tree satisfies the Helly property.

Proof. Suppose $T_i \cap T_j \neq \varnothing$ for all $i, j \in J$. Consider three points a, b, c on T. Let S be the set of indices s such that T_s contains at least two of these three points, and let P_1, P_2, P_3 be the simple paths in T connecting a with b, b with c, and a with c, respectively. Since T is a tree, it follows that $P_1 \cap P_2 \cap P_3 \neq \varnothing$, but each $T_s (s \in S)$ contains one of these paths P_i. Therefore,

$$\bigcap_{s \in S} T_s \supseteq P_1 \cap P_2 \cap P_3 \neq \varnothing.$$

The lemma is proved by induction. Let us assume that

$$[T_i \cap T_j \neq \varnothing \qquad \text{for all} \quad i, j \in J] \Rightarrow \bigcap_{j \in J} T_j \neq \varnothing \tag{1}$$

for all index sets J of size $\leq k$. This is certainly true for $k = 2$. Consider a family of subtrees $\{T_{i_1}, \ldots, T_{i_{k+1}}\}$. By the induction hypothesis there exist points a, b, c on T such that

$$a \in \bigcap_{j=1}^{k} T_{i_j}, \qquad b \in \bigcap_{j=2}^{k+1} T_{i_j}, \qquad c \in T_{i_1} \cap T_{i_{k+1}}.$$

Moreover, every T_{i_j} contains at least two of the points a, b, c. Hence, by the preceeding paragraph, $\bigcap_{j=1}^{k+1} T_{i_j} \neq \varnothing$. ∎

Theorem 4.8 (Walter [1972], Gavril [1974a], and Buneman [1974]). Let $G = (V, E)$ be an undirected graph. The following statements are equivalent:

(i) G is a triangulated graph.
(ii) G is the intersection graph of a family of subtrees of a tree.
(iii) There exists a tree $T = (\mathscr{K}, \mathscr{E})$ whose vertex set $\mathscr{K}$ is the set of maximal cliques of G such that each of the induced subgraphs $T_{\mathscr{K}_v} (v \in V)$ is connected (and hence a subtree), where $\mathscr{K}_v$ consists of those maximal cliques which contain v.

Proof. (iii) $\Rightarrow$ (ii) Assume that there exists a tree $T = (\mathscr{K}, \mathscr{E})$ satisfying statement (iii). Let $v, w \in V$. Now

$$vw \in E, \quad v, w \in A \qquad \text{for some clique} \quad A \in \mathscr{K},$$

$$\mathscr{K}_v \cap \mathscr{K}_w \neq \varnothing, \qquad T_{\mathscr{K}_v} \cap T_{\mathscr{K}_w} \neq \varnothing.$$

Thus G is the intersection graph of the family of subtrees $\{T_{\mathscr{K}_v} \mid v \in V\}$ of T.

(ii) $\Rightarrow$ (i) Let $\{T_v\}_{v \in V}$ be a family of subtrees of a tree T such that $vw \in E$ iff $T_v \cap T_w \neq \emptyset$.

Suppose G contains a chordless cycle $[v_0, v_1, \ldots, v_{k-1}, v_0]$ with $k > 3$ corresponding to the sequence of subtrees $T_0, T_1, \ldots, T_{k-1}, T_0$ of the tree T; that is, $T_i \cap T_j \neq \emptyset$ if and only if i and j differ by at most one modulo k. All arithmetic will be done mod k.

Choose a point a_i from $T_i \cap T_{i+1}$ $(i = 0, \ldots, k-1)$. Let b_i be the last common point on the (unique) simple paths from a_i to a_{i-1} and a_i to a_{i+1}. These paths lie in T_i and T_{i+1}, respectively, so that b_i also lies in $T_i \cap T_{i+1}$. Let P_{i+1} be the simple path connecting b_i and b_{i+1}. Clearly $P_i \subseteq T_i$, so $P_i \cap P_j = \emptyset$ for i and j differing by more than 1 mod k. Moreover, $P_i \cap P_{i+1} = \{b_i\}$ for $i = 0, \ldots, k-1$. Thus, $\bigcup_i P_i$ is a simple cycle in T, contradicting the definition of a tree.

(i) $\Rightarrow$ (iii) We prove the implication by induction on the size of G. Assume that the theorem is true for all graphs having fewer vertices than G. If G is complete, then T is a single vertex and the result is trivial. If G is disconnected with components $G_1, \ldots, G_k$, then by induction there exists a corresponding tree T_i satisfying (iii) for each G_i. We connect a point of T_i with a point of T_{i+1} $(i = 1, \ldots, k-1)$ to obtain a tree satisfying (iii) for G.

Let us assume that G is connected but not complete. Choose a simplicial vertex a of G and let $A = \{a\} \cup \mathrm{Adj}(a)$. Clearly, A is a maximal clique of G. Let

$$U = \{u \in A \mid \mathrm{Adj}(u) \subset A\}$$

and

$$Y = A - U.$$

Note that the sets U, Y, and $V - A$ are nonempty since G is connected but not complete. Consider the induced subgraph $G' = G_{V-U}$, which is triangulated and has fewer vertices than G. By induction, let T' be a tree whose vertex set K' is the set of maximal cliques of G' such that for each vertex $v \in V - U$ the set $K'_v = \{X \in K' \mid v \in X\}$ induces a connected subgraph (subtree) of T'.

Remark. Either $K = K' + \{A\} - \{Y\}$ or $K = K' + \{A\}$ depending upon whether or not Y is a maximal clique of G'.

Let B be a maximal clique of G' containing Y.

Case 1. If $B = Y$, then we obtain T from T' by renaming B, A.

Case 2. If $B \neq Y$, then we obtain T from T' by connecting the new vertex A to B.

In either case, $K_u = \{A\}$ for all u in U and $K_v = K'_v$ for all v in $V - A$, each of which induces a subtree of T. We need only worry about the sets K_y $(y \in Y)$.

In case 1, $K_y = K'_y + \{A\} - \{B\}$, which induces the same subtree as K'_y since only names were changed. In case 2, $K_y = K'_y + \{A\}$, which clearly induces a subtree.

Thus, we have constructed the required tree T and the proof of the theorem is complete. ∎

Buneman [1972, 1974] discusses the application of the subtree intersection model in constructing evolutionary trees and in certain other classificatory problems.

An undirected graph $G = (V, E)$ is called a *path graph* if it is the intersection graph of a family of paths in a tree. Renz [1970] showed that G is a path graph if and only if G is triangulated and G is the intersection graph of a family $\mathscr{F}$ of paths in an undirected graph such that $\mathscr{F}$ satisfies the Helly property. Gavril [1978] presented an efficient algorithm for recognizing path graphs; he also proved a theorem for path graphs analogous to the equivalence of (ii) and (iii) in Theorem 4.8 (see Exercise 26).

6. Triangulated Graphs Are Perfect

Occasionally, the minimum graph coloring problem and the maximum clique problem can be simplified using the *principle of separation into pieces* (Berge [1973, p. 329]). This method is described in the following theorem and its proof. In particular, it is applicable to triangulated graphs.

Theorem 4.9. Let S be a vertex separator of a connected undirected graph $G = (V, E)$, and let $G_{A_1}, G_{A_2}, \ldots, G_{A_t}$ be the connected components of G_{V-S}. If S is a clique (not necessarily maximal), then

$$\chi(G) = \max_i \chi(G_{S+A_i})$$

and

$$\omega(G) = \max_i \omega(G_{S+A_i}).$$

Proof. Clearly $\chi(G) \geq \chi(G_{S+A_i})$ for each i, so $\chi(G) \geq k = \max_i \chi(G_{S+A_i})$. In fact, G can be colored using exactly k colors. First color G_S, then independently extend the coloring to each *piece* G_{S+A_i}. This composite will be a coloring of G. Thus, $\chi(G) = k$.

Next, certainly $\omega(G) \geq \omega(G_{S+A_i})$ for each i, so $\omega(G) \geq \max_i \omega(G_{S+A_i}) = m$. Let X be a maximum clique of G, i.e., $|X| = \omega(G)$. It is impossible that two vertices of X lie in G_{A_i} and G_{A_j} $(i \neq j)$ since the vertices are connected. Thus, X lies wholly in one of the pieces, say G_{S+A_r}. Hence, $m \geq \omega(G_{S+A_r}) \geq |X| = \omega(G)$.Therefore, $\omega(G) = m$. ∎

Corollary 4.10. Let S be a separating set of a connected undirected graph $G = (V, E)$, and let $G_{A_1}, G_{A_2}, \ldots, G_{A_t}$ be the connected components of G_{V-S}. If S is a clique, and if each subgraph G_{S+A_i} is perfect, then G is perfect.

Proof. Assume that the result is true for all graphs with fewer vertices than G. It suffices to show that $\chi(G) = \omega(G)$. Using Theorem 4.9 and the fact that each graph G_{S+A_i} is perfect, we have

$$\chi(G) = \max_i \chi(G_{S+A_i}) = \max_i \omega(G_{S+A_i}) = \omega(G).$$ ∎

We are now ready to state the main result.

Theorem 4.11 (Berge [1960], Hajnal and Surányi [1958]). Every triangulated graph is perfect.

Proof. Let G be a triangulated graph, and assume that the theorem is true for all graphs having fewer vertices than G. We may assume that G is connected, for otherwise we consider each component individually. If G is complete, then G is certainly perfect. If G is not complete, then let S be a minimal vertex separator for some pair of nonadjacent vertices. By Theorem 4.1, S is a clique. Moreover, by the induction hypothesis, each of the (triangulated) subgraphs G_{S+A_i}, as defined in Corollary 4.10, is perfect. Thus, by Corollary 4.10, G is perfect. ∎

Remark. The proofs in this section used only the perfect graph property (P_1) (Berge [1960]). Historically, however, until Theorem 3.3 was proved, the arguments had to be carried out for property (P_2) as well (Hajnal and Surányi [1958]).

Let $\mathscr{G}$ denote the class of all undirected graphs satisfying the property that every odd cycle of length greater than or equal to 5 has at least two chords. Clearly, every triangulated graph is in $\mathscr{G}$. Our ultimate goal in the remainder of this section is to prove that the graphs in $\mathscr{G}$ are perfect. The technique used to show this will be constructive in the following sense: Given a k-coloring of a graph $G \in \mathscr{G}$, we will show how to reduce it into an ω-coloring of G, where $k \geq \omega = \omega(G)$, by performing a sequence of color interchanges called *switchings.*

Let G be an undirected graph which has been properly colored. An (α, β)-*chain* in G is a chain whose vertices alternate between the colors α and β. Let

$G_{\alpha\beta}$ denote the subgraph induced by the vertices of G which are colored α or β. An $\langle\alpha, \beta\rangle$ *switch with respect to* G consists of the following operation:

Either interchange the colors in a nontrivial connected component of $G_{\alpha\beta}$ and leave all other colors unchanged, or recolor all isolated vertices of $G_{\alpha\beta}$ using β and leave all other colors unchanged.

Note that the result of an $\langle\alpha, \beta\rangle$ switch with respect to G is again a proper coloring of G.

Lemma 4.12. Let $G \in \mathscr{G}$ be properly colored, and let x be any vertex of G. Let vertices $y, z \in \mathrm{Adj}(x)$ be colored α and β, respectively, with $\alpha \neq \beta$. If y and z are linked by an (α, β)-chain in G, then they are linked by an (α, β)-chain in $G_{\mathrm{Adj}(x)}$.

Proof. Let $\mu = [y = x_0, x_1, x_2, \ldots, x_l = z]$ be an $[\alpha, \beta]$ chain in G of minimum length between y and z. Clearly, l must be odd. We claim that $\{x_0, x_1, x_2, \ldots, x_l\} \subseteq \mathrm{Adj}(x)$.

The claim is certainly true if $l = 1$. Let us assume that $l \geq 3$ and that the claim is true for all minimum (α, β)-chains of odd length strictly less than l. Now, the cycle $\bar{\mu} = [x, x_0, x_1, \ldots, x_l, x]$ has odd length $l + 2 \geq 5$, and all of its chords must have x as an endpoint since a chord between an α vertex and a β vertex of μ would give a shorter chain. Therefore, every subchain $\mu[x_s, x_t] = [x_s, \ldots, x_t]$ of μ is a minimum (α, β)-chain, and since $G \in \mathscr{G}$ the cycle $\bar{\mu}$ has at least two chords, xx_i and xx_j $(i < j)$.

If $\mu[x_0, x_i]$, $\mu[x_i, x_j]$, and $\mu[x_j, x_l]$ all have odd length, then applying the induction hypothesis to each of them we obtain $\{x_0, x_1, \ldots, x_l\} \subseteq \mathrm{Adj}(x)$. Otherwise, at least one of $\mu[x_0, x_i]$ or $\mu[x_j, x_l]$ has even length. Without loss of generality, assume that $\mu[x_0, x_i]$ has even length so that $\mu[x_i, x_l]$ has odd length. By induction, $\{x_i, x_{i+1}, \ldots, x_l\} \subseteq \mathrm{Adj}(x)$. In particular, $x_{i+1} \in \mathrm{Adj}(x)$, so $\mu[x_0, x_{i+1}]$ has odd length and by induction $\{x_0, x_1, \ldots, x_{i+1}\} \subseteq \mathrm{Adj}(x)$. This proves the claim. ∎

Let $G' = G_{\mathrm{Adj}(x)}$. Lemma 4.12 says that a nontrivial connected component of $G_{\alpha\beta}$ contains only one nontrivial connected component of $G'_{\alpha\beta}$ or only isolated α vertices of $G'_{\alpha\beta}$ or only isolated β vertices of $G'_{\alpha\beta}$.

Lemma 4.13. Let f be a proper coloring of a graph $G \in \mathscr{G}$, and let x be a vertex of G colored γ. Let $f_{G'}$ be the restriction of f to the subgraph G' induced by those vertices adjacent to x whose colors are from some arbitrary subset Q of colors with $\gamma \notin Q$. If $f_{G'}$ can be transformed into a coloring g' of G' by a sequence of switchings with respect to G' (using colors from Q), then f can be transformed into a coloring f' of G by a sequence of switchings with respect to G such that $f'_{G'} = g'$.

Proof. It is sufficient to consider the case of a single $\langle\alpha, \beta\rangle$ switch with respect to G', where $\alpha, \beta \neq \gamma$. Suppose that a connected component $H'_{\alpha\beta}$ of $G'_{\alpha\beta}$ was switched. If $H'_{\alpha\beta}$ is nontrivial, then by Lemma 4.12 the same result could be obtained by switching the component of $G_{\alpha\beta}$, containing $H'_{\alpha\beta}$. If $H'_{\alpha\beta}$ has only one vertex, then all isolated vertices of $G'_{\alpha\beta}$ were switched to β. In this case the same result could be obtained by switching all nontrivial components of $G_{\alpha\beta}$ which contain isolated α vertices of $G'_{\alpha\beta}$ plus switching all isolated vertices of $G_{\alpha\beta}$ to β. ∎

Theorem 4.14 (Meyniel [1976]). Let $G \in \mathscr{G}$ and let f be a k-coloring of G. Then there exists a q-coloring g of G with $q = \chi(G)$ which is obtainable from f by a sequence of switchings with respect to G.

Proof. The theorem is obviously true for graphs with one vertex. Assume that the theorem is true for all graphs with fewer vertices than G.

Consider a k-coloring f of G using the colors $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ with $k > q = \chi(G)$. Choose a vertex x with color $\alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_q$; if there is none, the proof is finished. Let G' be the subgraph induced by the vertices colored $\alpha_1, \alpha_2, \ldots, \alpha_q$ and adjacent to x. Clearly,

$$q' = \chi(G') \leq \chi(G_{\mathrm{Adj}(x)}) \leq q - 1.$$

Since $G' \in \mathscr{G}$, the induction hypothesis implies that there exists a q'-coloring g' of G' which is obtainable from $f_{G'}$ by a sequence of switchings with respect to G'. By Lemma 4.13, g' can also be obtained from f by a sequence of switchings with respect to G. After performing this sequence of switchings, we can recolor x with one of the colors $\alpha_1, \alpha_2, \ldots, \alpha_q$ which is unused by g' (since $q' \leq q - 1$). Thus, we have enlarged the set of vertices colored $\alpha_1, \alpha_2, \ldots, \alpha_q$. Repeating this process until all vertices of G are colored $\alpha_1, \alpha_2, \ldots, \alpha_q$ will yield a minimum coloring. ∎

We are now ready to show that the graphs in $\mathscr{G}$ are perfect. Gallai [1962] originally proved the case where each odd cycle has two noncrossing chords; a shorter proof appeared in Surányi [1968]. The case where each odd cycle has two crossing chords was proved by Olaru [1969] (see Sachs [1970]). The general case, as presented here, is due to Meyniel [1976].

Theorem 4.15. If G is an undirected graph such that every odd cycle has two chords, then G is perfect.

Proof. Let $G \in \mathscr{G}$ with $\chi(G) = q$, and let H be an induced subgraph of G satisfying

$$\chi(H) = q,$$
$$\chi(H - x) = q - 1 \qquad \text{for every vertex } x \text{ of } H.$$

Choose a vertex x of H and a $(q - 1)$-coloring f of $H - x$, and let H' be the subgraph induced by $\mathrm{Adj}_H(x)$. If H' were $(q - 2)$-colorable, then by Theorem 4.14 f restricted to H' could be transformed into a $(q - 2)$-coloring of H' by a sequence of switchings with respect to H'. Then by Lemma 4.13 there would exist a $(q - 1)$-coloring of $H - x$ using $q - 2$ colors for $\mathrm{Adj}_H(x)$. But this would imply that $\chi(H) = q - 1$, a contradiction.

Therefore, $\{x\} \cup \mathrm{Adj}_H(x)$ is not $(q - 1)$-colorable, and hence it must be the entire vertex set of H. Since this argument holds for all x, it follows that H is a q-clique. Thus, $\chi(G) = \omega(G) = q$. In like manner, $\chi(G') = \omega(G')$ for all induced subgraphs G' of G since being in $\mathscr{G}$ is a hereditary property. Thus G is perfect. ∎

7. Fast Algorithms for the COLORING, CLIQUE, STABLE SET, and CLIQUE-COVER Problems on Triangulated Graphs

Let $G = (V, E)$ be a triangulated graph, and let σ be a perfect elimination scheme for G. It was first pointed out by Fulkerson and Gross [1965] that every maximal clique was of the form $\{v\} \cup X_v$ where

$$X_v = \{x \in \mathrm{Adj}(v) \mid \sigma^{-1}(v) < \sigma^{-1}(x)\}.$$

This elementary fact is easily shown. By the definition of σ, each $\{v\} \cup X_v$ is complete. Let w be the first vertex in σ contained in an arbitrary maximal clique A; then $A = \{w\} \cup X_w$. Therefore, we have the following result.

Proposition 4.16 (Fulkerson and Gross [1965]). A triangulated graph on n vertices has at most n maximal cliques, with equality if and only if the graph has no edges.

It is easy enough to modify Algorithm 4.2 to print out each set $\{v\} \cup X_v$. However, some of these will not be maximal, and we would like to filter them out. The mechanism that we employ is the observation that $\{u\} \cup X_u$ is *not* maximal iff for some i, in line 7 of Algorithm 4.2, X_u is concatenated to $A(u)$ (Exercise 13). The modified algorithm is as follows:

Algorithm 4.3. Chromatic number and maximal cliques of a trangulated graph.

Input: The adjacency sets of a triangulated graph G and a perfect elimination scheme σ.
Output: All maximal cliques of G and the chromatic number $\chi(G)$.

Method: A single call to the procedure CLIQUES(σ) given in Figure 4.8. The number $S(v)$ indicates the size of the largest set that would have been concatenated to $A(v)$ in Algorithm 4.2. A careful comparison will reveal that Algorithm 4.3 is a modification of Algorithm 4.2.

Theorem 4.17. Algorithm 4.3 correctly calculates the chromatic number and all maximal cliques of a triangulated graph $G = (V, E)$ in $O(|V| + |E|)$ time.

The proof is similar to that of Theorem 4.5.

Next we tackle the problem of finding the stability number $\alpha(G)$ of a triangulated graph. Better yet, since G is perfect, let us demand that we produce both a stable set and clique cover of size $\alpha(G)$. A solution is given by Gavril.

Let σ be a perfect elimination scheme for $G = (V, E)$. We define inductively a sequence of vertices $y_1, y_2, \ldots, y_t$ in the following manner: $y_1 = \sigma(1)$; y_i is the first vertex in σ which follows y_{i-1} and which is not in $X_{y_1} \cup X_{y_2} \cup \cdots \cup X_{y_{i-1}}$; all vertices following y_t are in $X_{y_1} \cup \cdots \cup X_{y_t}$. Hence

$$V = \{y_1, y_2, \ldots, y_t\} \cup X_{y_1} \cup \cdots \cup X_{y_t}.$$

The following theorem applies.

Theorem 4.18 (Gavril [1972]). The set $\{y_1, y_2, \ldots, y_t\}$ is a maximum stable set of G, and the collection of sets $Y_i = \{y_i\} \cup X_{y_i}$ ($i = 1, 2, \ldots, t$) comprises a minimum clique cover of G.

```
     procedure CLIQUES (σ):
     begin
1.     χ ← 1;
2.     for all vertices v do S(v) ← 0;;
3.     for i ← 1 to n do
         begin
4.         v ← σ(i);
5.         X ← {x ∈ Adj(v) | σ⁻¹(v) < σ⁻¹(x)};
6.         if Adj(v) = ∅ then print {v};;
7.         if X = ∅ then go to 13;;
8.         u ← σ(min{σ⁻¹(x) | x ∈ X});
9.         S(u) ← max{S(u), |X| − 1};
10.        if S(v) < |X| then do
             begin
11.            print {v} ∪ X;
12.            χ = max{χ, 1+ |X|};
             end
13.      end
14.    print "The chromatic number is",χ;
     end
```

Figure 4.8. Procedure to list all maximal cliques of a triangulated graph, given a perfect elimination scheme.

Proof. The set $\{y_1, y_2, \ldots, y_t\}$ is stable since if $y_j y_i \in E$ for $j < i$, then $y_i \in X_{y_j}$, which cannot be. Thus $\alpha(G) \geq t$. On the other hand, each of the sets $Y_i = \{y_i\} \cup X_{y_i}$ is a clique, and so $\{Y_1, \ldots, Y_t\}$ is a clique cover of G. Thus, $\alpha(G) = k(G) = t$, and we have produced the desired maximum stable set and minimum clique cover. ∎

Implementing this procedure to run efficiently is a straightforward exercise and is left for the reader (Exercise 25). For a treatment of the maximum weighted stable set problem, see Frank [1976].

EXERCISES

1. Show that for $n \geq 5$ the graph $\bar{C}_n$ is not triangulated.

2. Using Theorem 4.1, condition (iii), prove that every interval graph is triangulated. What is the interpretation of a separator in an interval representation of a graph?

3. Prove properties (L1)–(L3) of lexicographic breadth-first search (Section 4.3).

4. Apply Algorithm 4.1 to the graph in Figure 3.3 by arbitrarily selecting the vertex of degree 2 in line 3 during the first pass of the algorithm. (i) What is the perfect scheme you get? (ii) Find a perfect scheme of G which cannot possibly arise from Algorithm 4.1.

The class of undirected graphs known as *k-trees* is defined recursively as follows: A k-tree on k vertices consists of a clique on k vertices (k-clique); given any k-tree T_n on n vertices, we construct a k-tree on $n + 1$ vertices by adjoining a new vertex x_{n+1} to T_n, which is made adjacent to each vertex of some k-clique of T_n and nonadjacent to the remaining $n - k$ vertices. Notice that a 1-tree is just a tree in the usual sense, and that a k-tree has at least k vertices. Exercises 5–7 below are due to Rose [1974]. Harary and Palmer [1968] discuss 2-trees.

5. Show that a k-tree has a perfect vertex elimination scheme and is therefore triangulated. Give an example of a triangulated graph which is not a k-tree for any k.

6. Prove the following result: An undirected graph $G = (V, E)$ is a k tree if and only if

(i) G is connected,
(ii) G has a k-clique but no $(k + 2)$-clique, and
(iii) every minimal vertex separator of G is a k-clique.

7. Let $G = (V, E)$ be a triangulated graph which has a k-clique but no $(k + 2)$-clique. Prove that $\|E\| \leq k|V| - \frac{1}{2}k(k + 1)$ with equality holding if and only if G is a k-tree.

8. Show that every 3-tree is planar.

9. Let G be an undirected graph and let H be constructed as follows. The vertices of H correspond to the edges of G, and two vertices of H are adjacent if their corresponding edges form two sides of a triangle in G. Prove that G is a 2-tree if and only if H is a cactus of triangles.

10. Show that every vertex of a minimal x–y separator is adjacent to some vertex in each of the connected components containing x and y, respectively.

11. Let S be a minimal x–y separator of a connected graph G. Show that every path in G from x to y contains a member of S and that every $s \in S$ is contained in some path μ from x to y which involves no other element of S, that is, $\mu \cap S = \{s\}$.

12. Prove the following: For any minimal vertex separator S of a triangulated graph $G = (V, E)$, there exists a vertex c in each connected component of G_{V-S} such that $S \subseteq \mathrm{Adj}(c)$. (Hint: Prove the inclusion for each subset $X \subseteq S$ using induction.)

13. Program Algorithms 4.1 and 4.2 using the data structures suggested and test some graphs for the triangulated graph property.

14. Give a representation of the graph in Figure 4.5a as intersecting subtrees of a tree.

15. Prove that G is triangulated if and only if G is the intersection graph of a family $\mathscr{T}$ of subtrees of a tree where no member of $\mathscr{T}$ contains another member of $\mathscr{T}$ (Gavril [1974a]).

16. Give an algorithm which constructs for any triangulated graph G a collection of subtrees of a tree whose intersection graph is isomorphic to G.

17. Prove the following: H is a tree if and only if every family of paths in H satisfies the Helly property.

18. Prove the following theorem of Renz [1970]: G is the intersection graph of a family of paths in a tree iff G is triangulated and is the intersection graph of a family of arcs of a graph satisfying the Helly property.

19. Using the Helly property for subtrees of a tree, show directly that (ii) implies (iii) in Theorem 4.8. (Hint: for each clique A of the intersection graph, paint the subtree corresponding to the intersection of all members of A red and paint the remainder of the tree green. What does it look like when you collapse each red piece to a point?)

20. Prove Corollary 4.10 using the perfect graph property (P_2) instead of (P_1).

21. The line graph $L(G)$ of G is defined to be the undirected graph whose vertices correspond to the edges of G, and two vertices of $L(G)$ are joined by an edge if and only if they correspond to adjacent edges in G. Prove that G is triangulated if and only if $L(G)$ is triangulated.

22. Prove that Algorithm 4.3 correctly calculates the chromatic number and all maximal cliques of a triangulated graph.

23. Let σ be a perfect vertex elimination scheme for a triangulated graph $G = (V, E)$. Let $H = (V, F)$ be an orientation of G, where $xy \in F$ iff $\sigma^{-1}(x) < \sigma^{-1}(y)$. Show that H is acyclic. Let τ be any topological sorting of H. Show that τ is also a perfect elimination scheme for G.

24. Prove that a height function h (see Chapter 2, Exercise 8) of the acyclic oriented graph H defined in the preceding exercise is a minimum coloring of the triangulated graph G. Thus, a triangulated graph can be colored with a minimum number of colors in time proportional to its size.

25. Modify Algorithm 4.3 so that, in addition, it prints out a maximum stable set and prints an asterisk next to those cliques which together comprise a minimum clique cover.

26. Prove the following: $G = (V, E)$ is a path graph if and only if there exists a tree T whose vertex set is $\mathscr{K}$ (the maximal cliques of G) such that for all $v \in V$, the induced subgraph $T_{\mathscr{K}_v}$ is a path in T. ($\mathscr{K}_v$ denotes the set of maximal cliques which contain v.) (Gavril [1978].)

27. Let $G = (V, E)$ be an undirected graph, and let $\sigma = [v_1, v_2, \ldots, v_n]$ be an ordering of V. Consider the following property:

(T): If $\sigma^{-1}(u) < \sigma^{-1}(v) < \sigma^{-1}(w)$ and $w \in \operatorname{Adj}(u) - \operatorname{Adj}(v)$, then there exists an x such that $\sigma^{-1}(v) < \sigma^{-1}(x)$ and $x \in \operatorname{Adj}(v) - \operatorname{Adj}(w)$.

Prove that if G is a triangulated graph and σ satisfies (T), then σ is a perfect elimination scheme for G (Tarjan [1976]).)

28. (i) Prove that any MCS order, as defined at the end of Section 4.3, satisfies property (T) from the preceding exercise.

(ii) Give an implementation of MCS to recognize triangulated graphs in $O(n + e)$ time. (Hint. To achieve linearity you may wish to link together all unnumbered vertices which are currently adjacent to the same number of numbered vertices (Tarjan [1976]).)

29. An undirected graph is called *i-triangulated* if every odd cycle with more than three vertices has a set of chords which form with the cycle a planar graph whose unbounded face is the exterior of the cycle and whose bounded faces are all triangles. Prove that a graph is *i*-triangulated if and only if every cycle of odd length k has $k - 3$ chords that do not cross one another (Gallai [1962]).

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