CHAPTER 5

Comparability Graphs

1. Γ-Chains and Implication Classes

This chapter is devoted to the class of perfect graphs known as comparability graphs or transitively orientable graphs. These graphs were encountered in Section 1.3 in connection with interval graphs (Proposition 1.3), but our treatment here will be independent of that brief introduction.

An undirected graph $G = (V, E)$ is a *comparability graph* if there exists an orientation (V, F) of G satisfying

$$F \cap F^{-1} = \varnothing, \qquad F + F^{-1} = E, \qquad F^2 \subseteq F,$$

where $F^2 = \{ac \mid ab, bc \in F \text{ for some vertex } b\}$. The relation F is a strict partial ordering of V whose comparability relation is exactly E, and F is called a *transitive orientation* of G (or of E). Comparability graphs are also known as *transitively orientable* graphs and *partially orderable* graphs. Examples of some comparability graphs can be found in Section 1.3.

Let us see what happens when we try to assign a transitive orientation to the 4-cycle (Figure 5.1a). Arbitrarily choosing $ab \in F$ *forces* us to orient the bottom edge toward b and the top edge toward d (for otherwise transitivity would be violated). These in turn, force the remaining edge to be oriented toward d. Applying the same idea to the graph in Figure 5.1b, we find that a contradiction arises, namely, choosing $ab \in F$ forces successively the orientations cb, cd, cf, ef, bf, and ba. This graph is not a comparability graph. We now make the notion of forcing more precise.

Define the binary relation Γ on the edges of an undirected graph $G = (V, E)$ as follows:

$$ab\,\Gamma\,a'b' \quad \text{iff} \quad \begin{cases} \text{either} & a = a' \text{ and } bb' \notin E \\ \text{or} & b = b' \text{ and } aa' \notin E \end{cases}$$

(a) (b)

Figure 5.1. Examples of forcing. The arbitrary choice of $ab \in F$ forces the other indicated orientations.

We say that *ab directly forces a'b'* whenever $ab\,\Gamma\,a'b'$. Since E is irreflexive, $ab\,\Gamma\,ab$; however, $ab\,\not\Gamma\,ba$. The reader should not continue until he is convinced of this fact.

The reflexive, transitive closure Γ^* of Γ is easily shown to be an equivalence relation on E and hence partitions E into what we shall call the *implication classes* of G. Thus edges ab and cd are in the same implication class if and only if there exists a sequence of edges

$$ab = a_0 b_0 \,\Gamma\, a_1 b_1 \,\Gamma \cdots \Gamma\, a_k b_k = cd, \qquad \text{with} \quad k \geq 0.$$

Such a sequence is called a *Γ-chain* from ab to cd, and we say that ab (eventually) *forces cd* whenever $ab\,\Gamma^*\,cd$.

The reader can easily verify the properties

$$ab\,\Gamma\,a'b' \Leftrightarrow ba\,\Gamma\,b'a',$$

$$ab\,\Gamma^*\,a'b' \Leftrightarrow ba\,\Gamma^*\,b'a',$$

which follow directly from the definitions.

Let $\mathscr{I}(G)$ denote the collection of implication classes of G. We define

$$\hat{\mathscr{I}}(G) = \{\hat{A} \mid A \in \mathscr{I}(G)\},$$

where $\hat{A} = A \cup A^{-1}$ is the symmetric closure of A. The members of $\hat{\mathscr{I}}(G)$ are called the *color classes* of G for reasons that will become evident later.

Examples. The graph G in Figure 5.2 has eight implication classes:

$$\begin{array}{llll} A_1 = \{ab\}, & A_2 = \{cd\}, & A_3 = \{ac, ad, ae\}, & A_4 = \{bc, bd, be\}, \\ A_1^{-1} = \{ba\}, & A_2^{-1} = \{dc\}, & A_3^{-1} = \{ca, da, ea\}, & A_4^{-1} = \{cb, db, eb\}. \end{array}$$

So we have $\hat{\mathscr{I}}(G) = \{\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4\}$. On the other hand, the graph in Figure 5.1b has only one implication class:

$$A = \{ab, cb, cd, cf, ef, bf, ba, bc, dc, fc, fe, fb\}$$

and $A = \hat{A}$.

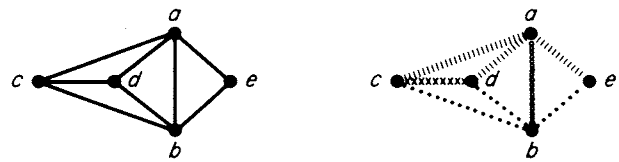

Figure 5.2. An undirected graph G and a coloring of its edges according to the classes of $\mathscr{I}(G)$.

Theorem 5.1. Let A be an implication class of an undirected graph G. If G has a transitive orientation F, then either $F \cap \hat{A} = A$ or $F \cap \hat{A} = A^{-1}$ and, in either case, $A \cap \hat{A} = \varnothing$.

Proof. We defined Γ in order to capture the fact that, for any transitive orientation F of G,

$$\text{if} \quad ab\,\Gamma\, a'b' \quad \text{and} \quad ab \in F, \qquad \text{then} \quad a'b' \in F.$$

Applying this property repeatedly, we obtain $F \cap A = \varnothing$ or $A \subseteq F$. Since (i) $A \subseteq F + F^{-1}$ and (ii) $F \cap F^{-1} = \varnothing$, we have the implications

$$\begin{aligned} F \cap A = \varnothing &\Rightarrow A \subseteq F^{-1} \qquad \text{[by (i)]} \\ &\Rightarrow A^{-1} \subseteq F \Rightarrow F \cap \hat{A} = A^{-1}, \end{aligned}$$

and

$$\begin{aligned} A \subseteq F &\Rightarrow A^{-1} \subseteq F^{-1} \Rightarrow F \cap A^{-1} = \varnothing \qquad \text{[by (ii)]} \\ &\Rightarrow F \cap \hat{A} = A. \end{aligned}$$

In either case $A \cap A^{-1} = \varnothing$. ∎

The converse of Theorem 5.1 is also valid, namely, *if $A \cap A^{-1} = \varnothing$ for every implication class A, then G has a transitive orientation.* This result will be proved as part of Theorem 5.27. Theorem 5.27 also provides the justification for an algorithm which assigns a transitive orientation to a comparability graph.

Remark. Many readers may wonder whether an arbitrary union of implication classes $F = \bigcup_i A_i$ satisfying $F \cap F^{-1} = \varnothing$ and $F + F^{-1} = E$ is necessarily a transitive orientation of G. The answer is no. As a counterexample, consider a triangle which has $8 = 2^3$ such orientations, two of which fail to be transitive.

Next we present two lemmas which will be useful throughout this chapter.

Let $ab = a_0b_0\,\Gamma\, a_1b_1\,\Gamma \cdots \Gamma\, a_kb_k = cd$ be given. For each $i = 1, \ldots, k$ we have

$$a_{i-1}b_{i-1}\,\Gamma\, a_ib_{i-1}\,\Gamma\, a_ib_i,$$

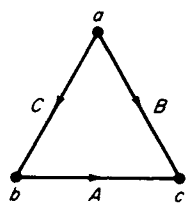

Figure 5.3.

since the added middle edge equals one of the other two. Hence we may state the following:

Lemma 5.2. If $ab\ \Gamma^*\ cd$, then there exists a Γ-chain from ab to cd of the form

$$ab = a_0b_0\ \Gamma\ a_1b_0\ \Gamma\ a_1b_1\ \Gamma\ a_2b_1\ \Gamma \cdots \Gamma\ a_kb_k = cd.$$

Such a chain will be called a *canonical* Γ*-chain.*

Lemma 5.3 (The Triangle Lemma). Let A, B, and C be implication classes of an undirected graph $G = (V, E)$ with $A \neq B$ and $A \neq C^{-1}$ and having edges $ab \in C$, $ac \in B$, and $bc \in A$ (see Figure 5.3).

(i) If $b'c' \in A$, then $ab' \in C$ and $ac' \in B$.
(ii) If $b'c' \in A$ and $a'b' \in C$, then $a'c' \in B$.
(iii) No edge in A touches the vertex a.

Proof. By Lemma 5.2 there exists a canonical Γ-chain

$$bc = b_0c_0\ \Gamma\ b_1c_0\ \Gamma\ b_1c_1\ \Gamma \cdots \Gamma\ b_kc_k = b'c'.$$

By induction on i, we have the following implications:

$$[B \ni ac_i \not\Gamma\ b_{i+1}c_i \in A] \Rightarrow ab_{i+1} \in E,$$

$$b_{i+1}b_i \notin E \Rightarrow ab_{i+1}\ \Gamma\ ab_i \in C,$$

$$[C^{-1} \ni b_{i+1}a \not\Gamma\ b_{i+1}c_{i+1} \in A] \Rightarrow ac_{i+1} \in E,$$

$$c_{i+1}c_i \notin E \Rightarrow ac_{i+1}\ \Gamma\ ac_i \in B.$$

Therefore, in particular, $ab' = ab_k \in C$ and $ac' = ac_k \in B$. This proves (i).

Next, let us assume that $b'c' \in A$ and $a'b' \in C$. By part (i), $ac' \in B$. Consider a new canonical Γ-chain,

$$ab = a_0b_0\ \Gamma\ a_1b_0\ \Gamma\ a_1b_1\ \Gamma \cdots \Gamma\ a_lb_l = a'b'.$$

This chain gives rise to the chain

$$ac' = a_0 c' \,\Gamma\, a_1 c' \,\Gamma \cdots \Gamma\, a_l c' = a'c'.$$

Thus, $ac' \,\Gamma^*\, a'c'$ and $a'c' \in B$, which proves (ii).

Finally, part (i) immediately implies (iii). ∎

Theorem 5.4. Let A be an implication class of an undirected graph $G = (V, E)$. Exactly one of the following alternatives holds:

(i) $A = \hat{A} = A^{-1}$;

(ii) $A \cap A^{-1} = \varnothing$, A and A^{-1} are transitive, and they are the only transitive orientations of $\hat{A}$.

Proof. (i) Assume $A \cap A^{-1} \neq \varnothing$. Let $ab \in A \cap A^{-1}$, so $ab \,\Gamma^*\, ba$. For any $cd \in A$, $cd \,\Gamma^*\, ab$ and $dc \,\Gamma^*\, ba$. Since Γ^* is an equivalence relation, $cd \,\Gamma^*\, dc$ and $dc \in A$. Thus $A = \hat{A}$.

(ii) Assume $A \cap A^{-1} = \varnothing$ and let $ab, bc \in A$. Now $ac \notin E \Rightarrow ab \,\Gamma\, cb \Rightarrow cb \in A \Rightarrow bc \in A^{-1}$, a contradiction. Thus $ac \in E$.

Let B be the implication class of G containing ac, and suppose $A \neq B$. Since $A \neq A^{-1}$ and $ab \in A$, the Triangle Lemma 5.3(i) implies that $ab \in B$, a contradiction. Thus $ac \in A$, and A is transitive. Moreover, A being transitive implies that A^{-1} is transitive.

Finally, A is an implication class of $\hat{A}$, so by Theorem 5.1 A and A^{-1} are the only transitive orientations of $\hat{A}$. ∎

Corollary 5.5. Each color class of an undirected graph G either has exactly two transitive orientations, one being the reversal of the other, or has no transitive orientation. If in G there is a color class having no transitive orientation, then G fails to be a comparability graph.

2. Uniquely Partially Orderable Graphs

Let H_0 be a graph with n vertices $v_1, v_2, \ldots, v_n$ and let $H_1, H_2, \ldots, H_n$ be n disjoint graphs.* The *composition* graph $H = H_0[H_1, H_2, \ldots, H_n]$ is formed as follows: For all $1 \leq i, j \leq n$, replace vertex v_i in H_0 with the graph H_i and make each vertex of H_i adjacent to each vertex of H_j whenever v_i is

* The graphs may be directed or undirected.

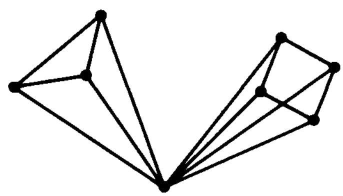

Figure 5.4. The composition of some undirected graphs.

adjacent to v_j in H_0. Formally, for $H_i = (V_i, E_i)$ we define $H = (V, E)$ as follows:

$$V = \bigcup_{i \geq 1} V_i;$$

$$E = \bigcup_{i \geq 1} E_i \cup \{xy \mid x \in V_i, y \in V_j \text{ and } v_i v_j \in E_0\}.$$

We may also denote $E = E_0[E_1, E_2, \ldots, E_n]$. We call H_0 the *outer factor* and $H_1, \ldots, H_n$ the *inner factors* (see Figures 5.4 and 5.5).

Theorem 5.6. Let $G = G_0[G_1, G_2, \ldots, G_n]$, where the G_i are disjoint undirected graphs. Then G is a comparability graph if and only if each G_i $(0 \leq i \leq n)$ is a comparability graph.

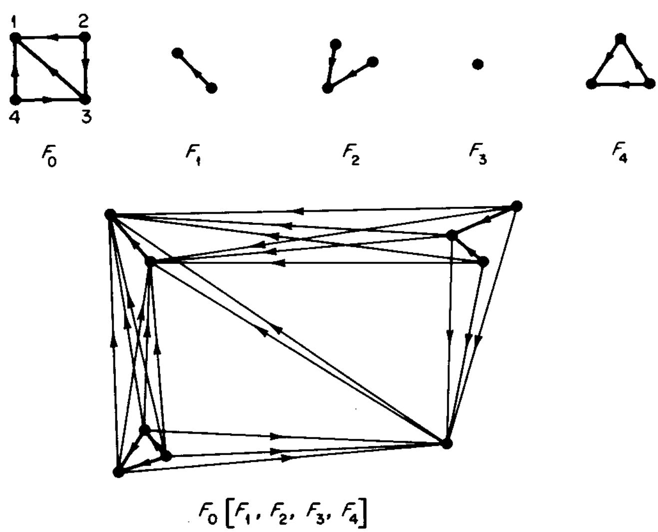

Figure 5.5. The composition of some transitively oriented graphs.

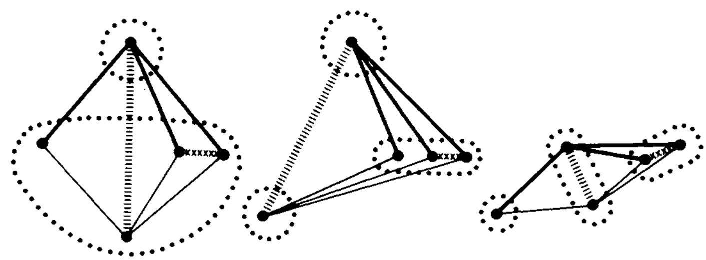

Figure 5.6. Three decompositions of the same graph. The edges are marked according to their color classes.

Proof. Let $F_0, F_1, \ldots, F_n$ be transitive orientations of $G_0, G_1, \ldots, G_n$, respectively. It is easy to show that $F_0[F_1, \ldots, F_n]$ is a transitive orientation of G. The converse follows from the hereditary property of comparability graphs. ∎

A graph is called *decomposable* if it can be expressed as a nontrivial composition of some of its induced subgraphs; otherwise, it is called *indecomposable*. Three decompositions of the same graph are illustrated in Figure 5.6. Of course, any graph G has the trivial decompositions $G = K_1[G]$ and $G = G[K_1, K_1, \ldots, K_1]$. Formally, $G = (V, E)$ is *decomposable* if there exists a partition $V = V_1 + V_2 + \cdots + V_r$ of the vertices into nonempty pairwise disjoint subsets with $1 < r < |V|$ such that

$$G = G_R[G_{V_1}, G_{V_2}, \ldots, G_{V_r}]$$

for any set of representatives $R = \{x_1, x_2, \ldots, x_r\}$, $x_i \in V_i$. Such a partition is said to *induce* a *proper decomposition* of G. Theorem 5.6 may be reinterpreted as follows.

Corollary 5.7. Let F be a transitive orientation of a comparability graph G. If $G = G_R[G_{V_1}, \ldots, G_{V_r}]$ is a proper decomposition of G, then $F = F_R[F_{V_1}, \ldots, F_{V_r}]$.

Let us examine the effect of this decomposition on the color classes. Notice in Figure 5.6 that each color class occurs either entirely within one internal factor or entirely within the external edges. This phenomenon is true in general.

Theorem 5.8. Let $G = G_0[G_1, \ldots, G_n]$ be the composition of disjoint undirected graphs $G_i = (V_i, E_i)$ $(i = 0, 1, \ldots, n)$. If $\hat{A}$ is a color class of G, then one of the following alternatives holds:

(i) $\hat{A} \subseteq E_j$ for exactly one index $j \geq 1$, or
(ii) $\hat{A} \cap E_j = \varnothing$ for all indices $j \geq 1$.

Proof. By our original definition of forcing, every color class $\hat{A}$ is a connected (partial) subgraph of G. Suppose that $\hat{A} \cap E_j \neq \varnothing$ for some $j \geq 1$. Let $ab \in \hat{A} \cap E_j$ and consider an edge $a'b' \,\Gamma\, ab$. Clearly $a'b' \notin E_k$ for any $k \neq j$, $k \geq 1$, since edges in different internal components never share a vertex. Moreover, $a'b'$ cannot be an external edge because if it were then by the definition of composition the vertices a, a', b, b' would induce a triangle in G, implying that $a'b' \not\Gamma\, ab$. Hence, $a'b'$ must also be in E_j. Thus, by connectivity, $\hat{A} \subseteq E_j$. ∎

Let $G = (V, E)$ be an undirected graph. A subset $Y \subseteq V$ is called *partitive* if for each $x \in V - Y$ either $Y \cap \operatorname{Adj}(x) = \varnothing$ or $Y \subseteq \operatorname{Adj}(x)$. A partitive set Y is *nontrivial* if $1 < |Y| < |V|$. On the one hand, any internal factor of a decomposition of G is partitive. On the other hand, a partitioning of the vertices $V = \{v_1\} + \cdots + \{v_k\} + Y$ where Y is partitive induces a proper decomposition of G. Therefore, we may conclude the following remark.

Remark 5.9. *G has a nontrivial partitive set if and only if G is decomposable.*

Before continuing, we present two simple consequences of the Triangle lemma.

Proposition 5.10. If Y is the set of vertices spanned by a color class $\hat{A}$ of an undirected graph $G = (V, E)$, then Y is partitive.

Proof. If $Y = V$, then the result is trivial. Otherwise, let $a \in V - Y$, and suppose that $b \in Y \cap \operatorname{Adj}(a)$. Then, $ab \in E - \hat{A}$ and $bc \in \hat{A}$ for some $c \in Y$, which implies that $ac \in E - \hat{A}$. Applying Lemma 5.3(i), we obtain that $Y \subseteq \operatorname{Adj}(a)$. ∎

Proposition 5.11. An undirected graph $G = (V, E)$ may have at most one color class which spans all of V.

Proof. Suppose that two distinct color classes $\hat{A}$ and $\hat{B}$ both span V. Then for every vertex b there exist edges $ab \in \hat{B}$ and $bc \in \hat{A}$. Since $\hat{A} \neq \hat{B}$, the edge ac is in E. What color is it? Let $\hat{C}$ denote the color class containing ac. If $\hat{C} \neq \hat{A}$, then Lemma 5.3(iii) implies that no edge from $\hat{A}$ may touch vertex a, a contradiction. Hence $\hat{C} = \hat{A} \neq \hat{B}$, and Lemma 5.3(iii) now implies that no edge from $\hat{B}$ may touch vertex c, another contradiction. Therefore, $\hat{A}$ and $\hat{B}$ cannot both span all of V. ∎

A comparability graph G is called *uniquely partially orderable* (UPO) if it has exactly two transitive orientations, one being the reversal of the other. Clearly, a comparability graph is UPO if and only if it has exactly one color class (see Corollary 5.5).

Theorem 5.12 (Shevrin and Filippov [1970]; Trotter, Moore, and Sumner [1976]). Let G be a connected comparability graph. The following conditions are equivalent.

(i) G is UPO.
(ii) Every nontrivial partitive set of G is a stable set.
(iii) For every proper decomposition of G, each internal factor is a stable set (i.e., all edges are external).

Proof. The following proof is due to Arditti [1976a]. By the comments preceding Remark 5.9, (ii) and (iii) are equivalent. If G is UPO, then G has exactly one color class, and this class spans V. Therefore, by Theorem 5.8 any proper decomposition of G must make all edges external. Thus (i) implies (iii). Next, suppose G is not UPO; then by Proposition 5.11 G has a color class which only spans a proper subset Y of V. By Proposition 5.10, Y is a nontrivial partitive set which is not a stable set. Thus (ii) implies (i). ∎

Corollary 5.13. Let G be a comparability graph. If G is indecomposable, then G is UPO.

Proof. If G is indecomposable, then G is connected and it satisfies condition (iii) of Theorem 5.12 vacuously. Hence G is UPO. ∎

3. The Number of Transitive Orientations

In this section we shall examine the interaction between implication classes. In the process we will obtain a formula for the number $t(G)$ of transitive orientations of a comparability graph G and a procedure for constructing them. Our treatment follows Golumbic [1977a], in which most of this theory was developed. An alternate method for calculating $t(G)$ appears in Shevrin and Filippov [1970].

Example. A transitive orientation of any graph partially orders its vertices. Consider a transitive orientation F of the complete graph K_{r+1} on $r + 1$ vertices. Since in F each pair of distinct vertices is comparable, the partial ordering is actually a linear ordering (total ordering). Conversely, any linear ordering of the vertices of K_{r+1} yields a transitive orientation by directing each edge from smaller to larger. Therefore,

$$\begin{aligned} t(K_{r+1}) &= \text{the number of linear orderings of } r+1 \text{ elements} \\ &= (r+1)! \end{aligned}$$

Let $G = (V, E)$ be an undirected graph. A complete subgraph (V_S, S) on $r + 1$ vertices is called a *simplex* of *rank r* if each undirected edge $\widehat{ab}$ of S is contained in a different color class of G. For example, each undirected edge $\widehat{ab}$ of E is itself a simplex of rank 1. A simplex is *maximal* if it is not properly contained in any larger simplex.

The *multiplex* generated by a simplex S of rank r is defined to be the following undirected (partial) subgraph: (V_M, M), where

$$M = \{ab \in E \mid ab\, \Gamma^*\, xy \text{ for some } xy \in S\},$$

or alternatively,

$$M = \bigcup \hat{A},$$

where the union is over all color classes $\hat{A} \in \hat{\mathscr{I}}(G)$ satisfying $\hat{A} \cap S \neq \varnothing$. Thus, M is the union of the $\frac{1}{2}r(r + 1)$ color classes represented by the edges of the simplex S. (This number is due to S being a complete graph on $r + 1$ vertices.) Anticipating Corollary 5.15 we say that the multiplex M also has rank r. A multiplex is *maximal* if it is not properly contained in any larger multiplex. We will soon see that M is a maximal multiplex if and only if S is a maximal simplex.

Remark. If we actually assign a different color to each class of $\hat{\mathscr{I}}(G)$ and paint the edges of G accordingly, then a complete subgraph S whose edges are each painted a different color is a *simplex*. The collection of edges of E painted the same color as some edge of S is a *multiplex*. For example, if there is a red, white, and blue triangle in the graph, then the set of all red, white, and blue edges is a multiplex of rank 2. The graph in Figure 5.2 has two disjoint maximal multiplexes, one of rank 2 and one of rank 1. The expressions *tricolored triangle* and simplex of rank 2 are synonymous. Finally, notice that *the edges and implication classes of a tricolored triangle satisfy the hypotheses of the Triangle Lemma 5.3.*

An *isomorphism* between two simplices (V_1, S_1) and (V_2, S_2) of an undirected graph is a bijection $f\colon V_1 \to V_2$ such that $ab\, \Gamma^*\, f(a)f(b)$ for each distinct pair $a, b \in V_1$. It is thus possible to lay S_1 on top of S_2 so that the colors of their edges match.

Theorem 5.14 (Golumbic [1977a]). Let (V_T, T) be a simplex generating the multiplex M, and let (V_S, S) be a simplex contained in M. Then (V_S, S) is isomorphic to a subsimplex of (V_T, T).

Proof. Choose an edge $bc \in S$. Since T generates M, there exists an edge $b'c' \in T$ such that $bc\, \Gamma^*\, b'c'$. Define $f(b) = b'$ and $f(c) = c'$. If rank $S = 1$, then the theorem is proved.

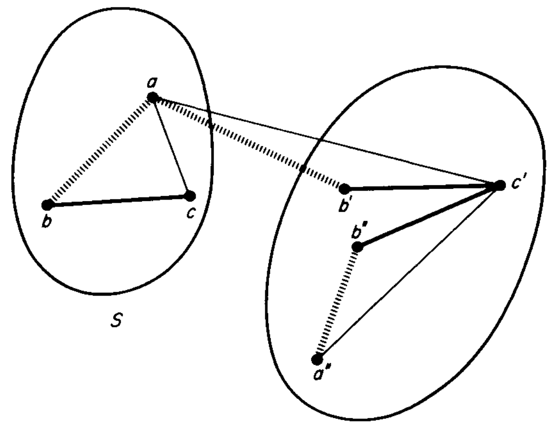

Figure 5.7. From a tricolored triangle in S we find an isomorphic tricolored triangle in T. The vertices b' and b'' must be equal since T is a simplex.

Otherwise, consider any other vertex $a \in V_S$, and let A, B, and C denote the implication classes such that $bc \in A$, $ac \in B$, and $ab \in C$. Since T generates M, there exists an edge $a''b'' \in T \cap C$. Applying the Triangle Lemma 5.3(i) twice we obtain (1) $ab' \in C$ and $ac' \in B$ and (2) $a''c' \in B$ and $b''c \in A$ (Figure 5.7). But the simplex T cannot contain two different edges $b'c'$ and $b''c'$ which are the same color; hence $b' = b''$. Define $f(a) = a''$. In this manner f is defined for all vertices of V_S. Choose distinct vertices a and d of V_S, different from b and c. Since $ab\,\Gamma^*\, f(a)f(b)$ and $db\,\Gamma^*\, f(d)f(b)$, then $f(a) = f(d)$ would imply $a = d$, since S is a simplex. Thus f is injective. Moreover, the Triangle Lemma 5.3(ii) implies that $ad\,\Gamma^*\, f(a)f(d)$. Therefore, f is an isomorphism from S to a subsimplex of T. ∎

The following is an immediate result of the preceding theorem.

Corollary 5.15. Simplices generating the same multiplex are isomorphic.

The next lemma shows us how to construct simplices.

Lemma 5.16. Let (V_S, S) be a simplex of an undirected graph $G = (V, E)$ generating a multiplex M. If G contains a tricolored triangle on vertices a, b, c such that $ab \notin M$ but $bc \in M$, then we may adjoin the vertex a to (V_S, S) to obtain the larger simplex (V_T, T) containing (V_S, S), where

$$V_T = V_S \cup \{a\},$$
$$T = S \cup \{\widehat{ad} \mid d \in V_S\}.$$

Proof. Let us assume that G contains a tricolored triangle on a, b, c satisfying $ab \notin M$ and $bc \in M$. Since S generates M, there is some edge $b'c' \in S$ for which $b'c' \, \Gamma^* \, bc$. The Triangle Lemma 5.3(i) implies that ab' and ac' are in the same two distinct color classes of G, respectively, as are ab and ac. Thus, $ab' \notin M$. Next we shall show that $ac' \notin M$ as well.

Suppose that $ac' \in M$; then $ac' \, \Gamma^* \, xy$ for some $xy \in S$ (because ac' must be the same color as some edge in S). Again by the Triangle lemma, $b'a \, \Gamma^* \, b'x$, however, $b'a \notin M$ while $b'x \in S$, a contradiction. Thus, $ac' \notin M$. This argument actually proves the stronger claim:

Fact 1. If a tricolored triangle has one side in M and another side not in M, then the third side is also not in M.

Next let $d \in V_S$, $d \neq b', c'$. Certainly $ad \in E$ since ab' and $b'd$ are in different color classes. Whereas the edges $b'c$ and $c'd$ are in different color classes, the edge ad is in a different class than at least one of them. Therefore, at least one of the triangles $G_{\{a,b',d\}}$ or $G_{\{a,c',d\}}$ is tricolored and satisfies the hypothesis of Fact 1, implying that $ad \notin M$. Thus, the set $\{\hat{ad} \mid d \in V_S\}$ shares no color classes with S.

Since ab' and ac' are in different color classes, to conclude the proof that (V_T, T) is a simplex it suffices to show the following claim:

Fact 2. Either the undirected edges $\hat{ad}$ (for $d \in V_S$) are all in different color classes, or all of the edges ad (for $d \in V_S$) are Γ^*-related.

Suppose that $ad, ad' \in \hat{A} \in \mathscr{I}(G)$. If $\hat{A}$ has no transitive orientation, then Theorem 5.4(i) implies that $ad \, \Gamma^* ad'$. If A has a transitive orientation, then Theorem 5.4(ii) implies that $ad \, \Gamma^* \, ad'$ since $dd' \notin \hat{A}$. Now let d'' be any vertex of V_S other than d or d'. If $ad'' \notin \hat{A}$, then $G_{\{a,d,d''\}}$ and $G_{\{a,d',d''\}}$ are both tricolored triangles sharing two common colors. So by the Triangle Lemma 5.3(i), $dd'' \, \Gamma^* \, d'd''$, which contradicts the definition of a simplex. Thus, $ad'' \in \hat{A}$ and, as before, $ad \, \Gamma^* \, ad''$. This proves Fact 2 and concludes the proof of the theorem. Obviously, rank $T = 1 +$ rank S. ∎

Lemma 5.3(ii) tells us that if an undirected graph contains a red, white, and blue triangle, then anywhere in the graph where we find a red edge ab and a white edge bc, the edge ac will be blue. Suppose there is a multiplex M containing a red, white, and blue triangle. The next theorem shows, in particular, that every red, white, and blue triangle is part of a simplex generating M.

Theorem 5.17 (Golumbic [1977a]). Let S be a simplex contained in a multiplex M. There exists a simplex S_M generating M such that $S \subseteq S_M$.

Proof. If rank S = rank M, then S itself generates M. We proceed by reverse induction, assuming the theorem to be true for any simplex of rank greater than rank S.

Let U be any simplex generating M. Since rank U = rank M, only some of the edges of U have "cousins" in S of the same color. These are the ones contained in M_1, defined here as the multiplex generated by S. Thus $M_1 \subset M$. Since U is connected it has a tricolored triangle on a, b, c with $bc \in M_1$, $ab \notin M_1$. By Lemma 5.16, we can adjoin the vertex a to S creating a simplex T containing S with rank $T = 1 +$ rank S. Thus, by induction, there is a simplex S_M generating M such that $S \subset T \subseteq S_M$. ∎

Theorems 5.14 and 5.17 can be summarized as follows:

Corollary 5.18. Let M_1, M_2 be multiplexes with $M_1 \subseteq M_2$.

(i) Every simplex generating M_1 is contained in a simplex generating M_2.

(ii) Every simplex generating M_2 contains a subsimplex which generates M_1.

Theorem 5.19. Let M be the multiplex generated by a simplex S. Then, M is a maximal multiplex if and only if S is a maximal simplex.

Proof. ($\Rightarrow$) This implication follows directly from the definition of multiplex.

($\Leftarrow$) Suppose S is maximal and $M \subseteq M'$, where M' is another multiplex. Since $S \subseteq M \subseteq M'$, Theorem 5.17 implies the existence of a simplex S' containing S with S' generating M'. But the maximality of S yields $S = S'$, so $M = M'$. ∎

By virtue of the preceding theorem and corollary we can now locate a maximal multiplex by a *local search* of the edges. We pick an edge at random and build up successively large simplices each containing its predecessor until the simplex we have is maximal. It then generates a maximal multiplex.

The next theorem implies that *the maximal multiplexes partition the edges of* G.

Theorem 5.20. If M_1 and M_2 are maximal multiplexes of an undirected graph G, then either $M_1 \cap M_2 = \varnothing$ or $M_1 = M_2$.

Proof. Let S_1 and S_2 be simplices generating M_1 and M_2, respectively. By Theorem 5.19, S_1 and S_2 are maximal. Suppose $M_1 \cap M_2 \neq \varnothing$ and $M_1 \neq M_2$, then some edges of S_2 are in M_1 and some are not. Because S_2 is connected, it must contain a tricolored triangle $G_{\{a,b,c\}}$ with $bc \in M_1$ and

$ab \notin M_1$. By Lemma 5.16, we can construct a large simplex T containing S_1, contradicting the maximality of S_1. Thus, one of the alternatives of the theorem must hold. ∎

Theorem 5.21. If A is an implication class of an undirected graph $G = (V, E)$ such that $A = \hat{A}$, then A itself is a maximal multiplex of rank 1.

The proof of Theorem 5.21 follows directly from the Triangle lemma and the definition of multiplex. It is left as an exercise for the reader. ∎

A simplex of rank r has $(r + 1)!$ transitive orientations, as we have seen in the example at the beginning of this section. Moreover, in the proof of the next theorem we will show that a transitive orientation of the simplex extends uniquely to a transitive orientation of the multiplex generated by it, *except* when the multiplex is itself an implication class and hence not transitively orientable (by Theorem 5.4). Conversely, a transitive orientation of a multiplex restricts uniquely to a transitive orientation of any simplex contained in it.

Theorem 5.22. Let M be a multiplex of rank r. If M is transitively orientable, then $t(M) = (r + 1)!$.

Remark. Theorem 5.21 shows that the *only* case in which M might fail to be transitively orientable is when $r = 1$.

Proof. Let S be a simplex of rank r generating M, and let F_S be a transitive orientation of S. Finally, let $A_1, \ldots, A_k$ $[k = \frac{1}{2}r(r + 1)]$ be the *implication classes* containing the edges of F_S. The corresponding color classes $\hat{A}_i$ are distinct, and $\hat{A}_1 + \cdots + \hat{A}_k = M$. If $r = 1$, then A_1 is a transitive orientation of $M = \hat{A}_1$ if and only if $A_1 \neq \hat{A}_1$ if and only if $t(M) = 2$. If $r > 1$, then $F = A_1 + \cdots + A_k$ is certainly an orientation of M by Theorems 5.4 and 5.21. We must show that F is transitive. Let $ab \in A_i$, $bc \in A_j$. If $i = j$, then $ac \in A_i$ by the transitivity of A_i [Theorem 5.4(ii)]. If $i \neq j$, then $ac \in E$ since $\hat{A}_i \cap \hat{A}_j = \varnothing$. Suppose $ca \in F$, then the individual transitivity of A_i and A_j implies that $ca \in A_t$ for some $i \neq t \neq j$. Theorem 5.14, however, implies that there exist edges $a'b'$, $b'c'$, $c'a' \in S$ such that $a'b' \in A_i$, $b'c' \in A_j$, and $c'a' \in A_t$, contradicting the transitivity of F_S. Therefore, $ac \in F$ and F is transitive. Thus, for each transitive orientation of S we obtain a unique transitive orientation of M, so $t(M) \geq t(S) = (r + 1)!$.

Conversely, given a transitive orientation F_2 of M, consider its restriction $F_2 \cap S$ to S. The three facts, $ab, bc \in F_2 \cap S$, F_2 being transitive and S being complete, collectively imply that $ac \in F_2 \cap S$. So $F_2 \cap S$ is a transitive orientation of S. Therefore, $t(S) \geq t(M)$ and Theorem 5.22 is proved. ∎

The partition of an undirected graph $G = (V, E)$ into its maximal multiplexes $E = M_1 + \cdots + M_k$ will be referred to as its *M-decomposition.* It is unique up to the order of the M_i. Having just examined the transitive orientability of a multiplex, let us now investigate the transitive orientability of all of E. The next major theorem shows a one-to-one correspondence between the transitive orientations of the M_i and those of E.

Theorem 5.23 (Golumbic [1977a]). Let $G = (V, E)$ be an undirected graph, and let $E = M_1 + \cdots + M_k$, where each M_i is a maximal multiplex of E.

(i) If F is a transitive orientation of G, then $F \cap M_i$ is a transitive orientation of M_i.

(ii) If $F_1, \ldots, F_k$ are transitive orientations of $M_1, \ldots, M_k$, respectively, then $F_1 + \cdots + F_k$ is a transitive orientation of G.

(iii) $t(G) = t(M_1)t(M_2)\cdots t(M_k)$.

(iv) If G is a comparability graph and $r_i = \text{rank } M_i$, then $t(G) = \prod_{i=1}^{k} (r_i + 1)!$.

Proof. Statement (iii) follows from (i) and (ii), while (iv) is implied by (iii) and Theorem 5.22.

(i) Assume F is a transitive orientation of G and let $ab, bc \in F \cap M_i$. Suppose that $ac \notin M_i$; then $G_{\{a,b,c\}}$ must not be a tricolored triangle. Therefore, $ab, bc \in \hat{A}$ for some $\hat{A} \in \mathscr{I}(G)$. Thus $ab, bc \in F \cap \hat{A}$, and $F \cap \hat{A}$ *equals* either A or A^{-1}, both of which are transitive by Theorems 5.1 and 5.4. Hence $ac \in \hat{A}$, which is a contradiction.

(ii) Assume that $F_1, \ldots, F_k$ are transitive orientations of $M_1, \ldots, M_k$, respectively. We shall show that $F_1 + \cdots + F_k$ is transitive. Let $ab \in F_i$, $bc \in F_j$. If $i = j$, then $ac \in F_i$ by transitivity of F_i. If $i \neq j$, then ab and bc are in different color classes, so $ac \in E$. Since $G_{\{a,b,c\}}$ cannot be a tricolored triangle and hence cannot be contained in a single multiplex, it follows that $ac \in M_i + M_j$. But if $ca \in F_i + F_j$, then transitivity gives a contradiction. Thus, $ac \in F_i + F_j$. ∎

Summarizing the results of this section, we have shown that the maximal multiplexes partition the edges and act independently with respect to transitive orientation. They are generated by maximal simplices which can be built up from a single edge by a local search. Simplices generating the same multiplex are isomorphic. Finally, the number of transitive orientations of an undirected graph is a product of factorials depending on the ranks of its maximal multiplexes. Thus, every comparability graph behaves as if it were a disjoint collection of complete graphs.

4. Schemes and G-Decompositions—An Algorithm for Assigning Transitive Orientations

In this section we describe an algorithm for calculating transitive orientations and for determining whether or not a graph is a comparability graph. This technique is a modification of one first presented by Pnueli, Lempel, and Even [1971]. Our version uses the notions introduced in Section 5.1; the proof of its correctness relies on some of the results of Section 5.3. A discussion of its computational complexity will follow in Section 5.6.

Let $G = (V, E)$ be an undirected graph. A partition of the edge set $E = \hat{B}_1 + \hat{B}_2 + \cdots + \hat{B}_k$ is called a *G-decomposition* of E if B_i is an implication class of $\hat{B}_i + \cdots + \hat{B}_k$ for all $i = 1, 2, \ldots, k$. A sequence of edges $[x_1y_1, x_2y_2, \ldots, x_ky_k]$ is called a *decomposition scheme* for G if there exists a G-decomposition $E = \hat{B}_1 + \hat{B}_2 + \cdots + \hat{B}_k$ satisfying $x_iy_i \in B_i$ for all $i = 1, 2, \ldots, k$. In this chapter the term *scheme* will always mean a decomposition scheme.

For a given G-decomposition there will be many corresponding schemes (any set of representatives from the B_i). However, for a given scheme there exists exactly one corresponding G-decomposition. A scheme and G-decomposition can be constructed by the following procedure:

Algorithm 5.1 (Decomposition Algorithm).
Let $G = (V, E)$ be an undirected graph.
Initially, let $i = 1$ and $E_1 = E$.

Step (1): Arbitrarily pick an edge $e_i = x_iy_i \in E_i$.
Step (2): Enumerate the implication class B_i of E_i containing x_iy_i.
Step (3): Define $E_{i+1} = E_i - \hat{B}_i$.
Step (4): If $E_{i+1} = \varnothing$, then let $k = i$ and Stop; otherwise, increase i by 1 and go back to Step (1).

Clearly, the decomposition algorithm yields a scheme $[x_1y_1, \ldots, x_ky_k]$ and corresponding G-decomposition $\hat{B}_1 + \cdots + \hat{B}_k$ for any undirected graph G. Moreover, if y_ix_i had been chosen instead of x_iy_i for some i, then B_i^{-1} would replace B_i in the G-decomposition. Applying the algorithm to the graph in Figure 5.2, the scheme $[ac, bc, dc]$ gives the G-decomposition for which $B_1 = A_3$, $B_2 = A_4 + A_1^{-1}$ and $B_3 = A_2^{-1}$ (see p. 106 and Figure 5.8*). In this example notice that although ba and bc were not Γ-related in the original graph, once $\hat{B}_1$ is removed they become Γ-related in the remaining subgraph and their implication classes merge. In general, each implication

* Another example is given in Exercise 8.

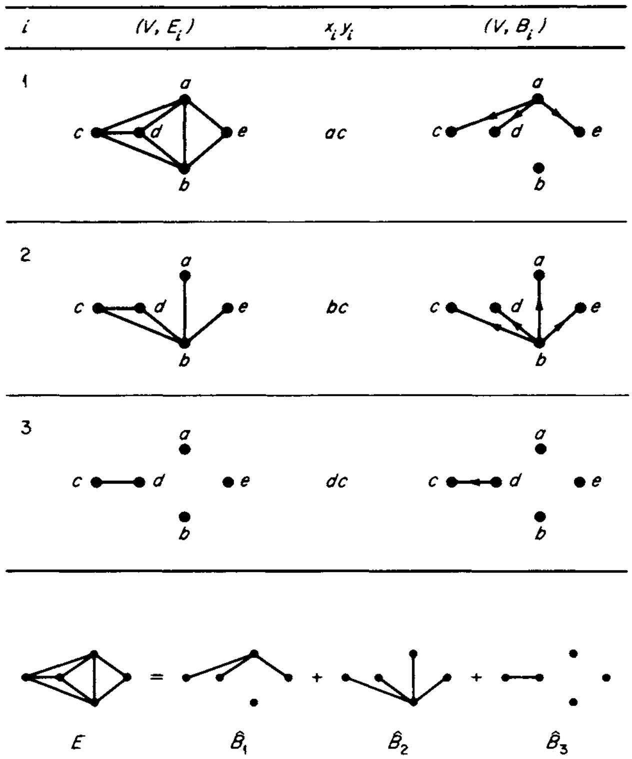

Figure 5.8. An illustration of the decomposition algorithm.

class of E_{i+1} will be the union of *some* number of implication classes of E_i. We now examine exactly how the old classes merge.

Theorem 5.24 (Golumbic [1977a]). Let A be an implication class of an undirected graph $G = (V, E)$, and let D be an implication class of $E - \hat{A}$. Either

(i) D is an implication class of E, and A is an implication class of $E - \hat{D}$,

or

(ii) $D = B + C$ where B and C are implication classes of E, and $\hat{A} + \hat{B} + \hat{C}$ is a multiplex of E of rank 2.

Proof. Removing $\hat{A}$ from E may cause some implication classes of E to merge. Let D be the union of k implication classes of E.

Assume $k \geq 2$; then there exists a triangle on vertices a, b, c with $bc \in \hat{A}$ and either $ac \in B$ and $ab \in C$ or $ca \in B$ and $ba \in C$, where B and C are distinct implication classes of E contained in D. Without loss of generality we may assume $ac \in B$ and $ab \in C$ since the other case is identical for D^{-1}. Suppose $B = C^{-1}$, then ba, $ac \in B$. But $bc \notin B$, so by Theorem 5.4 $B = \hat{B} = B^{-1}$, implying $B = C$, a contradiction. Therefore $\hat{B} \cap \hat{C} = \varnothing$ and $G_{\{a,b,c\}}$ is a tricolored triangle, making $\hat{A} + \hat{B} + \hat{C}$ a multiplex of rank 2.

Furthermore, any Γ-chain in $E - \hat{A}$ containing edges from $\hat{B}$ and $\hat{C}$ could not contain edges from other implication classes since all triangles in E with one edge in $\hat{A}$ and a second edge in $\hat{B}$ (resp. $\hat{C}$) must have its third side in $\hat{C}$ (resp. $\hat{B}$) and would be isomorphic as a simplex to $G_{\{a,b,c\}}$. Thus $k = 2$ and $D = B + C$.

Finally, we shall show that if $k = 1$, then A is an implication class of $E - \hat{D}$. By what we have already proved, if A is not an implication class of $E - \hat{D}$, then $\hat{D} + \hat{A} + \hat{A}_1$ is a multiplex of rank 2 in E for some third implication class A_1 of E. However, this implies that D alone is not an implication class of $E - \hat{A}$, contradicting $k = 1$. So indeed A is an implication class of $E - \hat{D}$. ∎

Corollary 5.25. Let A be an implication class of an undirected $G = (V, E)$. If $A = \hat{A}$, then all other implication classes of E are again implication classes of $E - \hat{A}$.

Corollary 5.26. Let A be an implication class of an undirected graph $G = (V, E)$. Then $|\hat{\mathscr{I}}(E)| = r + |\hat{\mathscr{I}}(E - \hat{A})|$, where r is the rank of the maximal multiplex of E containing A.

The proof of the first corollary follows directly from Theorem 5.21, while the second corollary is a result of $\hat{A}$ being a part of exactly $r - 1$ different multiplexes of rank 2. ∎

The next theorem is of major importance since it legitimizes the use of G-decompositions as a constructive tool for deciding whether an undirected graph is a comparability graph, and if so, producing a transitive orientation. Condition (iv) is the traditional characterization due to Gilmore and Hoffman [1964] and Ghouila-Houri [1962].

Theorem 5.27 (TRO Theorem). Let $G = (V, E)$ be an undirected graph with G-decomposition $E = \hat{B}_1 + \cdots + \hat{B}_k$. The following statements are equivalent:

(i) $G = (V, E)$ is a comparability graph;
(ii) $A \cap A^{-1} = \varnothing$ for all implication classes A of E;
(iii) $B_i \cap B_i^{-1} = \varnothing$ for $i = 1, \ldots, k$;
(iv) every "circuit" of edges $v_1v_2, v_2v_3, \ldots, v_qv_1 \in E$ such that $v_{q-1}v_1, v_qv_2, v_{i-1}v_{i+1} \notin E$ (for $i = 2, \ldots, q - 1$) has even length.

Furthermore, when these conditions hold, $B_1 + \cdots + B_k$ is a transitive orientation of E.

Proof. (i) $\Rightarrow$ (ii) This is precisely Theorem 5.1.

(ii) $\Rightarrow$ (iii) We shall proceed by induction. Since B_1 is an implication class of E, we have $B_1 \cap B_1^{-1} = \varnothing$. If $k = 1$, then we are done. Assume the implication is true for all G-decompositions of graphs of length less than k. Then, in particular, it is true for $E - \hat{B}_1$.

Let D be an implication class of $E - \hat{B}_1$. By Theorem 5.24, either D is an implication class of E, in which case $D \cap D^{-1} = \varnothing$, or $D = B + C$, where B and C are implication classes of E such that $\hat{B} \cap \hat{C} = \varnothing$, implying that

$$\begin{aligned} D \cap D^{-1} &= (B + C) \cap (B^{-1} + C^{-1}) \\ &= (B \cap B^{-1}) + (C \cap C^{-1}) \\ &= \varnothing. \end{aligned}$$

Therefore, by induction, $B_i \cap B_i^{-1} = \varnothing$, for $i = 2, \ldots, k$.

(iii) $\Rightarrow$ (i) Let $E = \hat{B}_1 + \cdots + \hat{B}_k$ be a G-decomposition of E with $B_i \cap B_i^{-1} = \varnothing$. By Theorem 5.4, B_1 is transitive. If $k = 1$, then the implication holds. Assume the implication is true for all G-decompositions of graphs of length less than k. By this assumption, $F = B_2 + \cdots + B_k$ is a transitive orientation of $E - \hat{B}_1$. We must show that $B_1 + F$ is transitive.

Let $ab, bc \in B_1 + F$. If both these edges are in B_1 or both in F, then by the individual transitivity of B_1 and F, $ac \in B_1 + F$. Assume, therefore, that $ab \in B_1$ and $bc \in F$, which implies that $ab\ \Gamma^*\ cb$, so $ac \in E$. What would happen if $ac \notin B_1 + F$? Then $ca \in B_1 + F$. However,

$$ca \in B_1,\ ab \in B_1 \Rightarrow cb \in B_1, \qquad \text{a contradiction,}$$

and

$$ca \in F,\ bc \in F \Rightarrow ba \in F, \qquad \text{a contradiction.}$$

Thus $ac \in B_1 + F$. Similarly, $ab \in F$ and $bc \in B_1$ imply $ac \in B_1 + F$. So indeed $B_1 + \cdots + B_k$ is a transitive orientation of E.

(iv) $\Leftrightarrow$ (i) Suppose $v_1v_2 \in A \cap A^{-1} \neq \varnothing$. By Lemma 5.2, there exists a Γ-chain

$$v_1v_2\ \Gamma\ v_3v_2\ \Gamma\ v_3v_4\ \Gamma \cdots \Gamma\ v_qv_{q-1}\ \Gamma\ v_qv_{q+1} = v_2v_1.$$

By construction, q is odd, since all first coordinates have odd index. Furthermore, $v_1v_2, v_2v_3, \ldots, v_qv_1$ is such a circuit, a contradiction.

Conversely, if E has such a circuit of odd length q, then

$$v_1v_2\ \Gamma\ v_3v_2\ \Gamma\ v_3v_4\ \Gamma \cdots \Gamma\ v_qv_{q-1}\ \Gamma\ v_qv_1\ \Gamma\ v_2v_1$$

is a Γ-chain in E, implying that $A \cap A^{-1} \neq \varnothing$ for the implication class A containing $v_1 v_2$, a contradiction. ∎

By combining the TRO theorem with the decomposition algorithm, we obtain an algorithm for recognizing comparability graphs and assigning a transitive orientation.

Algorithm 5.2 (TRO Algorithm).

Input: An undirected graph $G = (V, E)$.
Output: A transitive orientation F of edges of G, or a message that G is not a comparability graph.
Method: The entire algorithm is as follows:

```
    begin
      Initialize: i ← 1; E_i ← E; F ← ∅;
1.    Arbitrarily pick an edge x_i y_i ∈ E_i;
2.    Enumerate the implication class B_i of E_i containing x_i y_i;
        if B_i ∩ B_i^{-1} = ∅ then
          add B_i to F;
        else
          print "G is not a comparability graph";
          STOP;
3.    Define: E_{i+1} ← E_i − B̂_i;
4.      if E_{i+1} = ∅ then
          k ← i; output F;
          STOP;
        else
          i ← i + 1;
          go to 1;
    end
```

The sequence of free choices made in line 1 of the algorithm determines which of the many transitive orientations of G is produced by the algorithm. A different scheme may give a different transitive orientation. But when you try out a few different schemes you will notice a remarkable phenomenon: No matter how the free choices for G are made, the number of iterations k will always be the same. A proof that this is actually true for any graph G and, more importantly, a characterization of the underlying mathematical structure which causes it are the subject of the next section.

5. The Γ^*-Matroid of a Graph

The Decomposition Algorithm 5.1 emphasizes that the order in which the edges appear in a scheme is extremely important. The free choices made in earlier iterations affect which edges remain to be chosen in latter iterations. If the algorithm once gave us a scheme $[e_1, e_2, e_3, \ldots, e_k]$, what will happen if we rerun the algorithm by choosing e_2 first and e_1 second? Is there any reason for believing that e_3 will not have been removed and will therefore be available as the third free choice? The answer to the latter question is yes.

All the results in this section are due to Golumbic [1977a].

Theorem 5.28. Let $[e_1, e_2, \ldots, e_k]$ be a scheme for an undirected graph G, and let π be a permutation of the numbers $\{1, \ldots, k\}$. Then $[e_{\pi(1)}, e_{\pi(2)}, \ldots, e_{\pi(k)}]$ is also a scheme for G.

Proof. If $k = 1$, then there is nothing to prove. Assume therefore that $k \geq 2$. Let $\hat{B}_1 + \hat{B}_2 + \cdots + \hat{B}_k$ be the G-decomposition corresponding to the given scheme. Theorem 5.24 allows us to commute edges occurring next to each other in a scheme in the following manner. Fix $i < k$. Let

$$E_i = \hat{B}_i + \cdots + \hat{B}_k,$$

$$C_i = \text{implication class of } E_i \text{ containing } e_{i+1},$$

$$C_{i+1} = \text{implication class of } E_i - \hat{C}_i \text{ containing } e_i.$$

By Theorem 5.24, either (i) $B_{i+1} = C_i$ and $B_i = C_{i+1}$, so that $\hat{B}_i + \hat{B}_{i+1} = \hat{C}_i + \hat{C}_{i+1}$, or (ii) there exists an implication class A of E_i such that $\hat{B}_{i+1} = \hat{A} + \hat{C}_i$ and $\hat{C}_{i+1} = \hat{A} + \hat{B}_i$, also implying that $\hat{B}_i + \hat{B}_{i+1} = \hat{C}_i + \hat{C}_{i+1}$. Consequently, in either case, $\hat{B}_i + \cdots + \hat{C}_i + \hat{C}_{i+1} + \cdots + \hat{B}_k$ is a G-decomposition of E with scheme $[e_1, \ldots, e_{i+1}, e_i, \ldots, e_k]$.

However, every permutation can be expressed as a composition of such local commutations (often called transpositions), from which the theorem follows. ∎

Theorem 5.29 (Golumbic [1977a]). Let $G = (V, E)$ be an undirected graph.

(i) Each scheme for G has the same length.
(ii) Each G-decomposition of G has the same length.
(iii) If $[e_1, e_2, \ldots, e_k]$ and $[f_1, f_2, \ldots, f_k]$ are schemes for G, then for any e_i there exists f_j such that $[e_1, \ldots, e_{i-1}, f_j, e_{i+1}, \ldots, e_k]$ is also a scheme for G.

Proof. If G has an implication class A such that $E = \hat{A}$, then any scheme has length 1 and any edge can be chosen as a scheme. Therefore, assume that the theorem is true for all graphs having fewer implication classes than G, and let $[e_1, e_2, \ldots, e_k]$ and $[f_1, f_2, \ldots, f_m]$ be schemes for G with $k, m \geq 2$. Choose e_i and (using Theorem 5.28 if necessary) make sure that it is *not* in the first position. If $E = \hat{C}_1 + \hat{C}_2 + \cdots + \hat{C}_m$ is the G-decomposition corresponding to $[f_1, f_2, \ldots, f_m]$, then $e_1 \in \hat{C}_p$ for some p. Thus $[f_1, \ldots, f_{p-1}, e_1, f_{p+1}, \ldots, f_m]$ is also a scheme. Theorem 5.28 then implies that $[e_1, f_1, \ldots, f_{p-1}, f_{p+1}, \ldots, f_m]$ is a scheme for G.

Finally, both $[e_2, \ldots, e_i, \ldots, e_k]$ and $[f_1, \ldots, f_{p-1}, f_{p+1}, \ldots, f_m]$ are schemes for $E - \hat{B}$, where B is the implication class of E containing e_1. Since $E - \hat{B}$ has fewer implication classes than E, by induction the lengths $k - 1$ and $m - 1$ are equal and there exists some f_j which can replace e_i in its scheme. In conclusion, since corresponding G-decompositions and schemes have the same length, all G-decompositions must have the same length. ∎

Thus we have found a number associated with an undirected graph G which is invariant over all schemes and G-decompositions of the graph, namely the length of any scheme or G-decomposition of G. We shall denote this number by $r(G)$.

Theorem 5.30. Let $G = (V, E)$ be an undirected graph, and let $E = M_1 + \cdots + M_k$, where M_i is a maximal multiplex of E of rank r_i. Then $r(G) = r_1 + \cdots + r_k$.

Proof. Let $\hat{A} \in \hat{\mathscr{I}}(G)$ satisfy $\hat{A} \subseteq M_1$. Now $M_1 - \hat{A}$ is a multiplex of rank $r_1 - 1$, and $E - \hat{A} = (M_1 - \hat{A}_1) + M_2 + \cdots + M_k$ is an M-decomposition of $G' = (V, E - \hat{A})$. Since $|\mathscr{I}(G)| > |\mathscr{I}(G')|$, we may assume by induction that $r(G') = (r_1 - 1) + r_2 + \cdots + r_k$. Therefore, $r(G) = r_1 + r_2 + \cdots + r_k$. ∎

Let $G = (V, E)$ be a comparability graph with G-decomposition $E = \hat{B}_1 + \cdots + \hat{B}_k$ and corresponding scheme $[e_1, \ldots, e_k]$. By Theorem 5.27, $B_1 + \cdots + B_k$ will be a transitive orientation of G. Replacing e_i by e_i^{-1} in the scheme will have the effect of replacing B_i by B_i^{-1}, thus giving a new transitive orientation of G. In this manner we obtain $2^{r(G)}$ transitive orientations of G, since $k = r(G)$. There may, however, be others; the scheme $[e_{\pi(1)}, \ldots, e_{\pi(k)}]$ may even give a transitive orientation of G different from the $2^{r(G)}$ above. In fact, the only time when these $2^{r(G)}$ represent all the transitive orientations of G is when each maximal multiplex is of rank one. (This follows from Theorems 5.22 and 5.30 and the inequality $2^r < (r + 1)!$ for $r > 1$.) For example, $r(K_{r+1}) = r$ and $t(K_{r+1}) = (r + 1)!$ for the complete graph on $r + 1$ vertices. On the other hand, the graph G in Figure 5.9 has $t(G) = 2^{r(G)}$.

Story

The owner of a large railroad decided to introduce his sons into the business. He asked his eldest to choose any two cities between which they provide train service, and the father would give him control of that run. The

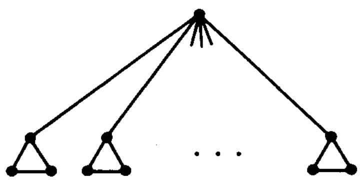

Figure 5.9. The number of triangles is $r(G) - 1$.

lad chose New York and Philadelphia. But the boy was clever and reasoned with his father saying, "Since you operate service between Harrisburg and Philadelphia and I operate the New York–Philadelphia trains, and since we don't offer any direct service between Harrisburg and New York, why not give me also the Harrisburg–Philadelphia run for the convenience of our passengers who would otherwise be burdened with their heavy luggage in changing trains!"

The father was convinced by the son's argument and gave him the extra rail link. The son, encouraged by his success, continued this type of reasoning for triples of cities that fit the above pattern and accumulated more rail lines until finally no more triples of that form were left. His father handed him the corresponding deeds; they embraced and the son left to go out on his own.

The father continued the same process with his other sons, giving one rail line and then also giving any other link A–B when the son already controlled B–C provided they did not operate A–C between the two of them. Finally, the father had given away his entire rail system.

Theorem 5.29 shows that no matter how each son chooses his initial free choice, *exactly* $r(G)$ sons get portions of the railroad, where G is the graph whose vertices are the cities and edges the rail links. ∎

We will now describe the underlying mathematical structure that causes the invariant $r(G)$ to arise.

A matroid $\langle E, \mathscr{B} \rangle$ consists of a nonempty (finite) set E of *elements* together with a nonempty collection $\mathscr{B}$ of subsets of E, called *bases*, satisfying the following axioms.

(i) No base properly contains another base.

(ii) If $\beta_1, \beta_2 \in \mathscr{B}$ and $x \in \beta_1$, then there exists an element $y \in \beta_2$ such that $(\beta_1 - \{x\} + \{y\}) \in \mathscr{B}$.

Theorem 5.31. Let $G = (V, E)$ be an undirected graph.

(i) $\langle E, \mathscr{B} \rangle$ is a matroid, where $\{e_1, \ldots, e_k\} \in \mathscr{B}$ if and only if $[e_1, \ldots, e_k]$ is a scheme for G.

(ii) $\langle \hat{\mathscr{I}}(G), \mathscr{B}(G)\rangle$ is a matroid, where $\hat{\mathscr{I}}(G)$ is the set of color classes of G and $\{\hat{A}_1, \ldots, \hat{A}_k\} \in \mathscr{B}(G)$ if and only if $\{e_1, \ldots, e_k\} \in \mathscr{B}$ for $e_i \in \hat{A}_i$.

Proof. The order in which the edges appear in a scheme is important for the G-decomposition it will produce. Theorem 5.28, however, allows us to treat schemes as sets of chosen representative edges in which order is not relevant. By Theorem 5.29, these subsets satisfy the axioms of a matroid. This proves (i). Condition (ii) follows easily from (i). ∎

The matroid $\langle \hat{\mathscr{I}}(G), \mathscr{B}(G)\rangle$ may be regarded as the quotient of the matroid $\langle E, \mathscr{B}\rangle$. For those readers familiar with matroids, the invariant $r(G)$ equals the rank (in the usual matroid sense) of $\langle E, \mathscr{B}\rangle$ and of $\langle \hat{\mathscr{I}}(G), \mathscr{B}(G)\rangle$. These matroids are of a very special type. Let us see exactly what class of matroids is produced in this manner.

By Theorem 5.24, the free choices taken from one maximal multiplex in no way influence choices taken from any other maximal multiplex. Therefore, it suffices to restrict our attention to applying the decomposition algorithm to a maximal simplex (V_S, S). Let $r = \text{rank } S$. Its free choices (r of them) constitute the edges of a spanning tree of (V_S, S). Why is that? It is certainly true if $r = 1$ or $r = 2$. If it were false, then there would be a scheme β containing a simple cycle of edges $v_1v_2, v_2v_3, \ldots, v_lv_1$ of minimal length l over all schemes. By Theorem 5.24, $l \neq 3$. Again by Theorem 5.24, v_2v_3 could be replaced by v_1v_3 in β, forming another scheme with a cycle of length less than l, contradicting minimality. Therefore, the r edges contain no simple cycles and must be a spanning tree of (V_S, S), since there are r edges and $r + 1$ vertices. Furthermore, any spanning tree of (V_S, S) is a scheme since it contains r edges, and for every other edge ab the tree provides a path $e_1, e_2, \ldots, e_q$ from a to b which, when used successively in the construction of a G-decomposition, will also eliminate the edge ab.

Two matroids $\langle E_1, \mathscr{B}_1\rangle$ and $\langle E_2, \mathscr{B}_2\rangle$ are *isomorphic* if there exists a bijection $f\colon E_1 \to E_2$ such that

$$f(\beta_1) \in \mathscr{B}_2 \qquad \text{for all} \quad \beta_1 \in \mathscr{B}_1$$

and

$$f^{-1}(\beta_2) \in \mathscr{B}_1 \qquad \text{for all} \quad \beta_2 \in \mathscr{B}_2.$$

Let $\mathscr{M}$ denote the family of matroids

$$\mathscr{M} = \{\langle \hat{\mathscr{I}}(G), \mathscr{B}(G)\rangle \mid G \text{ is an undirected graph}\}.$$

From the above discussion we may state the following characterization of the matroids in $\mathscr{M}$.

Theorem 5.32. A matroid is in the family $\mathscr{M}$ if and only if it is isomorphic to the matroid of spanning trees of a set of disjoint complete graphs.

6. The Complexity of Comparability Graph Recognition

A version of the decomposition algorithms of Section 5.4 is presented here in a pseudo-computer-language. It will suggest to us how we may actually enumerate the implication classes of a graph. We shall show that one can find a G-decomposition and test for transitive orientability of an undirected graph $G = (V, E)$ in $O(\delta \cdot |E|)$ time and $O(|V| + |E|)$ space, where δ is the maximum degree of a vertex.

Let $G = (V, E)$ be an undirected graph with vertices $v_1, v_2, \ldots, v_n$. In the algorithm below we use the function

$$\text{CLASS}(i, j) = \begin{cases} 0 & \text{if } v_i v_j \notin E, \\ k & \text{if } v_i v_j \text{ has been assigned to } B_k, \\ -k & \text{if } v_i v_j \text{ has been assigned to } B_k^{-1}, \\ \text{undefined} & \text{if } v_i v_j \in E \text{ has not yet been assigned,} \end{cases}$$

and $|\text{CLASS}(i, j)|$ denotes the absolute value of CLASS(i, j). As usual, the set E is always assumed to be a collection of ordered pairs and the *degree* d_i of vertex v_i is taken here to mean the number of edges with v_i as first coordinate (i.e., the out-degree). We freely use the identity

$$|E| = \sum_{i=1}^{n} d_i$$

in our analysis.

Algorithm 5.3 (Decomposition Algorithm—Alternate Version).

Input: An undirected graph $G = (V, E)$ with vertices $v_1, v_2, \ldots, v_n$ whose adjacency sets obey $j \in \text{Adj}(i)$ if and only if $v_i v_j \in E$.

Output: A G decomposition of the graph given by the final values of CLASS and a variable FLAG which is 0 if the graph is a comparability graph and 1 otherwise. If the algorithm terminates with FLAG equal to zero, then a transitive orientation of G is obtained by combining all edges having positive CLASS.

Method: The algorithm proceeds until all edges have been explored. In the kth iteration an unexplored edge is placed in B_k. (Its CLASS is changed to k.) Whenever an edge is placed into B_k it is explored using the recursive procedure of Figure 5.10 by adding to B_k those edges Γ-related to it in the

```
procedure EXPLORE(i, j):
for each m ∈ Adj(i) such that [m ∉ Adj(j) or |CLASS(j, m)| < k] do
  begin
    if CLASS (i, m) is undefined then
      begin
        CLASS(i, m) ← k; CLASS(m, i) ← −k;
        EXPLORE(i, m);
      end
    else
      if CLASS(i, m) = −k then
        begin
          CLASS(i, m) ← k; FLAG ← 1;
          EXPLORE(i, m);
        end
  end
for each m ∈ Adj(j) such that [m ∉ Adj(i) or |CLASS(i, m)| < k] do
  begin
    if CLASS(m, j) is undefined then
      begin
        CLASS(m, j) ← k; CLASS(j, m) ← −k;
        EXPLORE(m, j);
      end
    else
      if CLASS(m, j) = −k then
        begin
          CLASS(m, j) ← k; FLAG ← 1;
          EXPLORE(m, j);
        end
  end
return
```

Figure 5.10.

graph E_k. (Notice that $v_i v_j \in E_k$ if and only if either $|\text{CLASS}(i, j)|$ equals k or is undefined throughout the kth iteration.)

The variable FLAG is changed from 0 to 1 the first time a B_k is found such that $B_k \cap B_k^{-1} \neq \varnothing$. At that point it is known that G is not a comparability graph (by Theorem 5.27).

The algorithm is as follows:

```
begin
  initialize: k ← 0; FLAG ← 0;
  for each edge v_i v_j in E do
    if CLASS (i, j) is undefined then
      begin
        k ← k + 1;
        CLASS (i, j) ← k; CLASS (j, i) ← −k;
        EXPLORE (i, j);
      end
end
```

Complexity Analysis

We begin by specifying an appropriate data structure. The adjacency sets are stored as linked lists sorted into increasing order. The element of the list Adj(i) which represents edge $v_i v_j$ will have four fields containing j, CLASS(i, j), pointer to CLASS(j, i), and pointer to next element on Adj(i) (see Figure 5.11). The storage requirement for this data structure is $O(|V| + |E|)$, and if sorting the lists is done using Algorithm 2.1, then the entire initialization of the data structure can be accomplished in linear time.

The crucial factor in the analysis of our algorithm is the time required to access or assign the CLASS function. Ordinarily finding CLASS(i, m) could take $O(d_i)$ steps by scanning Adj(i), but if a temporary pointer happened to be in the neighborhood, then a reference to CLASS(i, m) or CLASS(m, i) would take a fixed number of steps. Consider the first loop of EXPLORE(i, j). Two temporary pointers simultaneously scan Adj(i) and Adj(j) looking for values of m which satisfy the condition in the **for** statement. Since the lists are sorted and thanks to these neighborly pointers, this loop can be executed in $O(d_i + d_j)$ steps. The second loop is done similarly; hence the time complexity of EXPLORE(i, j) is $O(d_i + d_j)$.

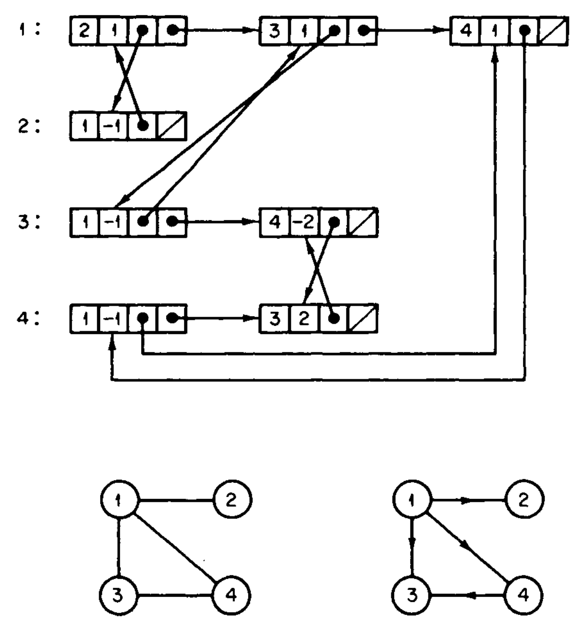

Figure 5.11. An undirected graph, the transitive orientation generated by the scheme [(1, 2), (4, 3)] and its data structure after running the algorithm.

In the main program, a temporary pointer scans each adjacency list successively in the **for** loop, implying a time complexity of $O(|E|)$. Finally, the algorithm calls EXPLORE once for each edge or its reversal (both if their implication classes are not disjoint). Therefore, since

$$\sum_{v_i v_j \in E} (d_i + d_j) = 2\sum_{i=1}^{n} d_i^2 \leq 2\delta \sum_{i=1}^{n} d_i = 2\delta |E|,$$

it follows that the time complexity for the entire algorithm (including preprocessing the input) is at most $O(\delta \cdot |E|)$. Thus we have proved the following:

Theorem 5.33. Comparability graph recognition and finding a transitive orientation can be done in $O(\delta \cdot |E|)$ time and $O(|V| + |E|)$ space, where δ is the maximum degree of a vertex.

The algorithm as presented in this section explores the edges in a depth-first search. Replacing each recursive call EXPLORE(x, y) by placing xy in a queue of edges to be explored would change the algorithm to breadth-first search. Some future application may lead us to prefer one over the other.

7. Coloring and Other Problems on Comparability Graphs

To any *acyclic* orientation F (not necessarily transitive) of an undirected graph $G = (V, E)$ we may associate a strict partial ordering of the vertices, namely, $x > y$ iff there exists a nontrivial path in F from x to y. A *height* function h can then be placed on V as follows: $h(v) = 0$ if v is a sink; otherwise, $h(v) = 1 + \max\{h(w) \mid vw \in F\}$. We have already seen, in Chapter 2, Exercise 8, that the height function can be assigned in linear time using a recursive depth-first search. The function h is always a proper vertex coloring of G, but it is not necessarily a minimum coloring. The number of colors used will be equal to the number of vertices in the longest path of F. This is also equal to $1 + \max\{h(v) \mid v \in V\}$ since we started at height (color) zero. A poor choice of F may result in an overly colorful coloring. However, the situation is guaranteed to be better if F happens also to be transitive.

Suppose that G is a comparability graph, and let F be a transitive orientation of G. In such a case, every path in F corresponds to a clique of G because of transitivity. Thus, the height function will yield a coloring which uses exactly $\omega(G)$ colors, which is the best possible. Moreover, since being a comparability graph is a hereditary property, we find that $\omega(G_A) = \chi(G_A)$ for all induced subgraphs G_A of G. This proves the following result.

Theorem 5.34. Every comparability graph is a perfect graph.

Theorem 5.34 coupled with the Perfect Graph Theorem 3.3 implies that the stability number of a comparability graph is equal to the clique cover number of the graph. This proves the following classical result.

Theorem 5.35 (Dilworth [1950]). Let $(X, \leq)$ be a partially ordered set. The minimum number of linearly ordered subsets (usually called *chains*) needed to partition X is equal to the maximum cardinality of a subset of X having no two members comparable (usually called an *antichain*).

Many proofs of Dilworth's theorem can be found in the literature. Among them, those of Fulkerson [1956] and Perles [1963] seem most elegant. The reader is referred also to Dilworth [1950], Pretzel [1979], Trotter [1975], and Tverberg [1967]. Greene and Kleitman [1976] have recently extended Dilworth's theorem to more general partitions of a poset into chains. Some related references include Greene [1974, 1976], Griggs [1979], and Hoffman and Schwartz [1977].

We direct our attention next to some algorithmic aspects of problems on comparability graphs. In Section 5.6 we showed that a transitive orientation F could be constructed for a comparability graph G in $O(\delta e + n)$ steps, where δ is the maximum degree of a vertex, e is the number of edges, and n is the number of vertices. From the transitive orientation F we can assign a minimum coloring of G using the height function in $O(n + e)$ additional steps. At the same time a maximum clique could also be calculated. We shall illustrate this by solving a slightly more general problem.

MAXIMUM WEIGHTED CLIQUE.

Instance: An undirected graph G and an assignment of a weight $w(v)$ to each vertex v.

Question: Find a clique of G for which the sum of the weights of its vertices is largest possible.

If all vertices have the same weight, then the problem is reduced to the usual problem of finding a clique of maximum cardinality. In general the MAXIMUM WEIGHTED CLIQUE problem is NP-complete, but when restricted to comparability graphs it becomes tractable.

Algorithm 5.4. Maximum weighted clique of a comparability graph.

Input: A transitive orientation F of a comparability graph $G = (V, E)$ and a weight function w defined on V.

Output: A clique K of G whose weight is maximum.

```
procedure EXPLORE(v):
  if Adj(v) = ∅ then
    W(v) = w(v);
    POINTER(v) ← Λ;
    return;;
  for all x ∈ Adj(v) do
    if x is unexplored then
      EXPLORE(x);
  end for all;
  select y ∈ Adj(v) such that W(y) = max{W(x) | x ∈ Adj(v)};
  W(v) ← w(v) + W(y);
  POINTER(v) ← y;
  return
end
```

Figure 5.12.

Method: We use a modification of the height calculation technique employing the recursive depth-first search procedure EXPLORE in Figure 5.12. To each vertex v we associate its cumulative weight $W(v)$, which equals the weight of the heaviest path from v to some sink. A pointer is assigned to v designating its successor on that heaviest path. Lines 4–10 calculate K once the cumulative weights are assigned. The algorithm is given as a procedure.

```
    procedure MAXWEIGHT CLIQUE(V, F):
 1.  for all v ∈ V do
 2.    if v is unexplored then
 3.      EXPLORE (v);
     end for all;
 4.  select y ∈ V such that W(y) = max{W(v) | v ∈ V};
 5.  K ← {y};
 6.  y ← POINTER (y);
 7.  while y ≠ Λ do
 8.    K ← K ∪ {y};
 9.    y ← POINTER (y);;
10.  return K;
    end
```

Proving the correctness of Algorithm 5.4 and displaying an implementation whose complexity is linear in the size of the graph (assuming that F is provided to the algorithm in the proper data structure) are left as exercises for the reader.

We conclude with an interesting polynomial-time method for finding $\alpha(G)$, the size of the largest stable set of a comparability graph G. We transform a transitive orientation (V, F) of G into a transportation network by adding two new vertices s and t and edges sx and yt for each source x and sink y of F. Assigning a lower capacity of 1 to each vertex, we initialize a compatible integer-valued flow and then call a minimum-flow algorithm. The value of the minimum flow will equal the size of the smallest covering of the vertices by cliques, which in turn will equal the size of the largest independent set since every comparability graph is perfect. Such a minimum-flow algorithm can run in polynomial time. (See Figure 2.1 for the complexities of various maximum-flow algorithms.)

8. The Dimension of Partial Orders

Szpilrajn [1930] first noted that any partial order (X, P) could always be extended to a linear ordering L of X. In Section 2.4 we called such a linear extension a topological sorting. Let $\mathscr{L}(P)$ denote the collection of all linear extensions of P. Any subset $\mathscr{L} \subseteq \mathscr{L}(P)$ satisfying $\bigcap_{L \in \mathscr{L}} L = P$ is called a *realizer* of P, and its *size* is $|\mathscr{L}|$. The intersection is that of sets of ordered pairs, that is,

$$ab \in \bigcap_{L \in \mathscr{L}} L \Leftrightarrow ab \in L \qquad \text{for every} \quad L \in \mathscr{L}.$$

Clearly, $\mathscr{L}(P)$ itself is a realizer of P. We define the *dimension* of P, dim P, to be the size of the smallest possible realizer for P. Such a realizer is called a *minimum realizer* for P. The notion of dimension of a partial order first appeared in Dushnik and Miller [1941].

Examples. The partial order P whose Hasse diagram is illustrated in Figure 5.13 has dimension 2. A minimum realizer for P is also shown. Notice

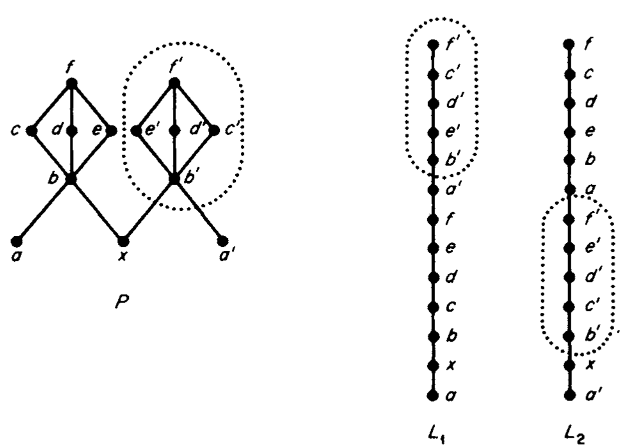

Figure 5.13. A partial order P of dimension 2. We have $P = L_1 \cap L_2$.

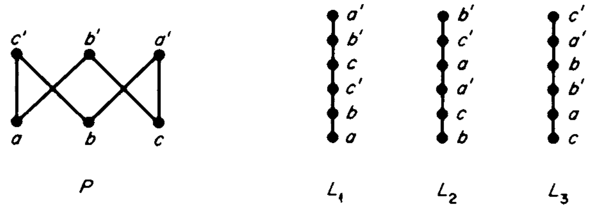

Figure 5.14. A partial order P of dimension 3. We have $P = L_1 \cap L_2 \cap L_3$. Why would two linear orders be insufficient to realize P?

that the subposet P' which is circled also has dimension 2 and that it must appear above element a in one of the linear orders and below element a in the other. Figure 5.14 shows the Hasse diagram of a partial order whose dimension is 3 (see Exercise 16).

Lemma 5.36. Let (X, P) be a poset. For each $Y \subseteq X$, we have

$$\dim P_Y \leq \dim P.$$

Proof. Clearly, restricting the linear extensions in a realizer $\mathscr{L}$ of P to the elements of Y yields a realizer (not necessarily minimum) of P_Y. Choosing $\mathscr{L}$ to be minimum for P we obtain the result. ∎

Theorem 5.37 (Hiraguchi [1951]). Let $P = P_0[P_1, P_2, \ldots, P_k]$ be the composition of disjoint partial orders (X_i, P_i) $(0 \leq i \leq k)$. Then

$$\dim P = \max\{\dim P_i \mid 0 \leq i \leq k\}.$$

Proof. For each i, let $L_{i,1}, L_{i,2}, \ldots, L_{i,m}$ be a realizer for P_i, where $m = \max\{\dim P_i \mid i = 0, 1, \ldots, k\}$. Define

$$\Lambda_j = L_{0,j}[L_{1,j}, L_{2,j}, \ldots, L_{k,j}].$$

Then $\{\Lambda_j \mid j = 1, 2, \ldots, m\}$ is a realizer of P, so $\dim P \leq m$.

Next, observe that P contains each of the P_i as a subposet. (To obtain P_0 take a set of representatives from $X_1, \ldots, X_k$.) Hence, by Lemma 5.36, $m \leq \dim P$. ∎

As noted earlier, the dimension of a partial order was introduced by Dushnik and Miller [1941]. They showed that there exist partial orders of dimension d for all positive integers d, and they gave the first characterization of the posets of dimension 2. We shall briefly mention some other known results on dimension theory. A special bibliography on the subject appears at the end of this chapter. In addition, W. T. Trotter is currently completing a book on the subject.

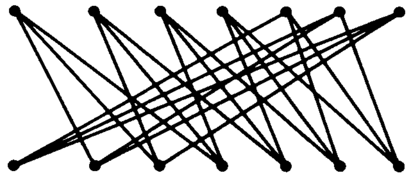

Figure 5.15. The Hasse diagram of the crown B_7^3.

Let S be a nonempty set and let $\mathscr{P}(S)$ denote its power set ordered by inclusion. Komm [1948] proved that dim $\mathscr{P}(S) = |S|$. Hiraguchi [1951] showed that dim $P \leq \frac{1}{2}|X|$ for any partial order (X, P) and gave examples of posets for which equality holds. Another proof of this result can be found in Bogart [1973].

Sedmak [1952–1954] investigated the poset $P(\pi)$ consisting of the empty set and the points, lines, faces, etc., of a polyhedron π in $\mathbb{R}^k$. He proved the following implications.

(1) If π is a polygon in $\mathbb{R}^2$, then dim $P(\pi) = 3$.

(2) If π is a polyhedron in $\mathbb{R}^3$, then dim $P(\pi) \geq 4$, with equality holding for regular polyhedra, pyramids, prisms, and their duals in $\mathbb{R}^3$.

(3) There exist polyhedra in $\mathbb{R}^3$ with arbitrarily high dimension.

This problem was originally posed by Kurepa [1951].

Ducamp [1967] showed that finding a minimum realizer for a partial order is equivalent to a certain bipartite covering problem. However, for all but small posets the method is intractable.

Let G be a connected undirected graph, and let $P(G)$ denote the collection of connected induced subgraphs of G ordered by inclusion. Trotter and Moore [1976a] proved that dim $P(G)$ equals the number of nonarticulation vertices of G. (A nonarticulation vertex is one whose removal from G leaves it connected.) This result generalizes a result of Leclerc [1976], namely, the dimension of the collection of subtrees of a tree T ordered by inclusion equals the number of leaves of T. The special case of dim $P(G) = 2$ was done by Dushnik and Miller [1941].

Trotter [1974a] studied the class of partial orders called *crowns*, obtaining an exact formula for their dimension. Briefly, let B_m^l be a poset on $2m$ elements split into an incomparable set $\{x_0, x_1, \ldots, x_{m-1}\}$ and another incomparable set $\{y_0, y_1, \ldots, y_{m-1}\}$ with $x_i y_j \in B_m^l$ for $j = i + 1, i + 2, \ldots, i + l$ (addition modulo m) (see Figure 5.15). Trotter proved that for $0 < l < m$ and $m \geq 3$

$$\dim B_m^l = \lceil 2m/(m - l + 1) \rceil.$$

Baker, Fishburn, and Roberts [1972] used the family $\{B_m^2\}_{m\geq 3}$ to show that, for any $n \geq 1$, the collection of all posets of dimension $\leq n$ is not axiomatizable by a sentence in first-order logic and cannot be characterized by a finite collection of forbidden subconfigurations.

Ore [1962] observed that the dimension of a partial order could be viewed in another, equivalent manner. The points in the Euclidean space $\mathbb{R}^k$ of dimension k can be partially ordered in a natural way: $(x_1, x_2, \ldots, x_k) \leq (y_1, y_2, \ldots, y_k)$ iff $x_i \leq y_i$ for each i. Then the dimension of a poset P is the smallest nonnegative integer k for which P can be embedded in $\mathbb{R}^k$. In some sense this justifies the choice of the term *dimension* for partial orders.

Rabinovitch [1973, 1978a] has shown that the dimension of a semiorder is at most three. Semiorders arise naturally in psychology* and are discussed in Chapter 8. Kelly [1977] and Trotter and Moore [1976b] have characterized all posets of dimension 3.

Application. Let (X, P) be a partially ordered set, perhaps obtained as the transitive closure of an acyclic graph, and let $|X| = n$. The dim P may be regarded as the minimum number k of attributes needed to distinguish between the comparability and incomparability of pairs from X. The technique is the following: To each item $x \in X$ we associate a k-tuple $(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, where x_i is the relative position of x in L_i and $\mathscr{L} = \{L_i\}$ is a minimum realizer of P. In such a setup, (X, P) would be stored using $O(kn)$ storage locations, and a query of the form "Is $xy \in P$?" will require at most k comparisons. This technique is advantageous when n is large and k is very small provided that the preprocessing needed to obtain a minimum realizer is not too expensive. This is always the case when dim $P \leq 2$.†

Theorem 5.38 (Dushnik and Miller [1941]). Let G be the comparability graph of a poset P. Then dim $P \leq 2$ if and only if the complementary graph $\bar{G}$ is transitively orientable.

Proof. Let F be a transitive orientation of $\bar{G}$. It is easy to show that $\mathscr{L} = \{P + F, P + F^{-1}\}$ is a realizer of P. Conversely, if $\mathscr{L} = \{L_1, L_2\}$ is any realizer of P, then $F = L_1 - P = (L_2 - P)^{-1}$ is a transitive orientation of $\bar{G}$. For, suppose $ab, bc \in F$ but $ac \notin F$. The transitivity of L_1 implies that $ac \in P$; similarly, the transitivity of L_2 implies that $ca \in P$, a contradiction. ∎

* Some psychologists believe that preference is based on a single criterion with some degree of fuzziness; this viewpoint is modeled in Section 8.5. Other psychologists believe that the brain is actually comparing multiple criteria; this viewpoint is modeled by the realizers described in this section.

† To date the complexity of computing dim P for an arbitrary poset P is unknown. It may or may not be NP-complete.

From the preceding theorem it follows that two partial orders which have the same comparability graph either both have dimension ≤ 2 or both have dimension >2. A stronger result holds, which we shall now present.

Theorem 5.39 (Trotter, Moore, and Sumner [1976]). If two partial orders P and Q have the same comparability graph G, then $\dim P = \dim Q$.

Proof. The theorem is certainly true for posets of one element. We proceed by induction. Let (X, P) and (X, Q) be partial orders having the same comparability graph G, and let us assume that for all proper subsets Y of X, $\dim Q_Y = \dim P_Y$. There are two cases to consider.

Case 1: G is indecomposable. In this case, Corollary 5.13 implies that G is UPO. Therefore, either $P = Q$ or $P = Q^{-1}$, both implying that $\dim P = \dim Q$.

Case 2: G is decomposable. Let $G = G_R[G_{V_1}, \ldots, G_{V_r}]$ be a proper decomposition of G. By Corollary 5.7, $P = P_R[P_{V_1}, \ldots, P_{V_r}]$ and $Q = Q_R[Q_{V_1}, \ldots, Q_{V_r}]$. Applying Theorem 5.37 and the induction hypothesis, we can obtain

$$\begin{aligned}\dim P &= \max\{\dim P_R, \dim P_{V_1}, \ldots, \dim P_{V_r}\}\\ &= \max\{\dim Q_R, \dim Q_{V_1}, \ldots, \dim Q_{V_r}\}\\ &= \dim Q\end{aligned}$$ ∎

Theorem 5.39 also appears in Gysin [1977].

In a personal communication, Richard Stanley has reported that two partial orders P and Q having the same comparability graph also have the same number of linear extensions, i.e., $|\mathscr{L}(P)| = |\mathscr{L}(Q)|$. His proof is based on the results of Section 5.3.

EXERCISES

1. (i) Prove that the forcing relation Γ^* is an equivalence relation.

(ii) Prove that the following properties hold:

$$ab \;\Gamma\; a'b' \Leftrightarrow ba \;\Gamma\; b'a'$$

$$ab \;\Gamma^*\; a'b' \Leftrightarrow ba \;\Gamma^*\; b'a'$$

2. The complete graph K_2 has two implication classes. Give a formula for $|\mathscr{I}(K_n)|$ for $n \geq 2$.

3. Which of the graphs in Figure 5.16 are comparability graphs? How many implication classes and color classes do they have?

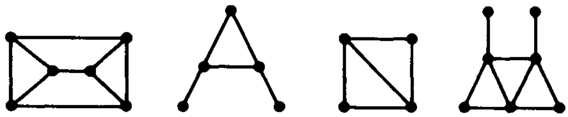

Figure 5.16.

4. Let G be a connected comparability graph whose complement $\bar{G}$ is connected and contains no induced subgraph isomorphic to $K_{1,3}$. Prove that G is UPO. [Hint: Use Theorem 5.12 (Aigner and Prins [1971]).]

5. Prove the following result for an undirected graph G. If F_1 and F_2 are transitive orientations of G and $\bar{G}$, respectively, then $F_1 + F_2$ is a transitive tournament.

6. Draw the graph $G = H_0[H_1, H_2, H_3, H_4]$ for the graphs in Figure 5.17. Verify that it has 16 color classes: 9 within the internal factors, 6 among the external edges connecting H_1 with H_2, and 1 consisting of the remaining external edges. Prove that G has 1440 transitive orientations.

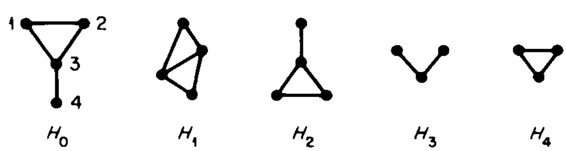

Figure 5.17.

7. Show that if an undirected graph G has no induced subgraph isomorphic to the path P_4, then both G and $\bar{G}$ are comparability graphs.

8. Verify that the graph in Figure 5.18 has four color classes partitioned into two maximal multiplexes of rank 1 and 2, respectively. Use the decomposition algorithm of Section 5.4 to obtain a G-decomposition of this graph. (One solution is given in Appendix D.) Is this graph a comparability graph?

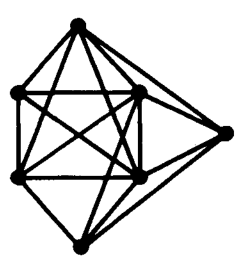

Figure 5.18.

9. Calculate $r(G)$ for the graphs in Exercises 3, 6, and 8.

10. Let $\alpha(G)$ be the stability number of an undirected graph $G = (V, E)$. Prove that $r(G) \leq |V| - \alpha(G)$ (Golumbic [1977a]).

11. A binary relation R is vacuously transitive if $R^2 = \varnothing$. (Vacuously transitive relations have been studied by Sharp [1973].) Prove that an undirected graph has a vacuously transitive orientation if and only if it is bipartite.

12. Prove that every transitive orientation of a comparability graph G is obtainable from some G-decomposition of G.

13. Let $G = (V, E)$ be an undirected graph, and consider the equivalence relation $\sim$, defined on V as follows:

$$a \sim a' \qquad \text{iff} \qquad \text{Adj}(a) = \text{Adj}(a').$$

By irreflexivity, equivalent vertices are not adjacent. We form the quotient graph $\tilde{G} = (\tilde{V}, \tilde{E})$ by merging equivalent vertices. Formally, let V be the set of all equivalence classes under $\sim$, and let $\tilde{a}$ denote the $\sim$-class containing the vertex a. For any subset of edges $A \subseteq E$ we define

$$\tilde{A} = \{\tilde{a}\tilde{b} \mid ab \in A\}.$$

(i) Prove that $\tilde{a}\tilde{b} \in \tilde{E} \Leftrightarrow ab \in E$. Give an example of a graph G and a subset of edges A such that $cd \in A$ but $\tilde{c}\tilde{d} \notin \tilde{A}$ for some edge cd.

(ii) Prove that the following conditions are equivalent:

(1) $\tilde{a}\tilde{b} = \tilde{c}\tilde{d}$,
(2) $a \sim c$ and $b \sim d$,
(3) $\tilde{a} = \tilde{c}$ and $\tilde{b} = \tilde{d}$.

(iii) Prove that (1)–(3) above imply that $ab\,\Gamma^*\,cd$ but not conversely (Golumbic [1977b]).

14. Let $G = (V, E)$ be an undirected graph and let $\tilde{G} = (\tilde{V}, \tilde{E})$ be its quotient graph as defined in Exercise 13. Prove the following.

(i) If $A \in \mathscr{I}(G)$, then $\tilde{A} \in \mathscr{I}(\tilde{G})$.

(ii) If $[e_1, e_2, \ldots, e_k]$ is a scheme for G with corresponding G decomposition $\hat{B}_1 + \hat{B}_2 + \cdots + \hat{B}_k$, then $[\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k]$ is a scheme for $\tilde{E}$ with corresponding G-decomposition $\hat{\tilde{B}}_1 + \hat{\tilde{B}}_2 + \cdots + \hat{\tilde{B}}_k$.

(iii) If (V, F) is a transitive orientation of G, then $(\tilde{V}, \tilde{F})$ is a transitive orientation of $\tilde{G}$.

(iv) Every implication class, scheme, G-decomposition, and transitive orientation of G is of the form indicated in (i)–(iii) (Golumbic [1977b]).

15. Prove that Algorithm 5.4 correctly computes a maximum weighted clique of a comparability graph. Show that the algorithm can be implemented to run in linear time in the size of the graph.

16. Prove that the partial order in Figure 5.14 has dimension 3.

17. Let Q be a subset of $\mathbb{R}^k$, and for $1 \leq i \leq k$ let Q_i consist of those numbers which appear as the ith coordinate in some k-tuple in Q. Consider the natural partial order on Q as defined in Section 8.

(i) Show that if Q is the Cartesian product $Q = Q_1 \times Q_2 \times \cdots \times Q_k$ and $|Q_i| \geq 2$ for each i, then $\dim Q = k$.

(ii) Prove Komm's theorem, namely, that $\dim \mathscr{P}(S) = k$ for a set S.

18. Complete the proof of Theorem 5.38.

19. A partial order (X, P) is an *interval inclusion order* if X can be put into one-to-one correspondence with a family $\{I_x\}_{x \in X}$ of intervals on a linearly ordered set such that

$$x < y \quad \text{iff } I_x \subset I_y \qquad (\forall x, y \in X).$$

Prove the following: $\dim P \leq 2$ if and only if P is an interval inclusion order (Dushnik and Miller [1941]).

20. Let $G = (V, E)$ be an undirected graph. Show the following statements are equivalent:

(i) G has a transitive orientation whose Hasse diagram is a rooted tree;

(ii) G is a comparability graph and the Hasse diagram of every transitive orientation of G is a rooted tree;

(iii) if $a, b, c, d \in V$ are distinct vertices satisfying $ab, bc, cd \in E$, then either $ac \in E$ or $bd \in E$;

(iv) G contains no induced subgraph isomorphic to C_4 or P_4.

Give an example of a comparability graph which is triangulated and whose complement is a comparability graph but which fails to satisfy the conditions above (Wolk [1962, 1965]). Arditti [1975b] has investigated comparability graphs whose Hasse diagram is a tree.

21. Show that the leaves of a rooted tree can be linearly ordered so that the set of decendent leaves of any vertex occur consecutively. Use this result to show that any graph G which is the comparability graph of a rooted tree is also an interval graph.

22. If $\bar{B}_1 + \bar{B}_2 + \cdots + \bar{B}_k$ is a G-decomposition of an undirected graph $G = (V, E)$, then $\bar{B}_1 + \cdots + \bar{B}_j$ is called a *partial* G-decomposition for each $j = 0, 1, \ldots, k$. Show that the subgraphs of G obtained as partial G-decompositions (including $\varnothing$ and E) form a lattice. Show that this lattice is modular but not necessarily distributive.

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