CHAPTER 10

Threshold Graphs

In this chapter we discuss a particularly simple technique for distinguishing between stable and nonstable subsets of vertices in a special class of graphs. The graphs that admit this technique, which involves assigning certain weights to the vertices, are called *threshold graphs*. Threshold graphs were introduced by Chvátal and Hammer [1973]. Their results form the basis for much of the next two sections. We begin by introducing the more general notion of threshold dimension.

1. The Threshold Dimension

Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of an undirected graph G. Any subset $X \subseteq V$ can be represented by its *characteristic vector* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where for all i

$$x_i = \begin{cases} 1 & \text{if } v_i \in X, \\ 0 & \text{if } v_i \notin X. \end{cases}$$

Thus, the subsets of vertices are in one-to-one correspondence with the corners of the unit hypercube in $\mathbb{R}^n$ according to the coordinates of their characteristic vectors.

Let us consider the collection of all stable sets of G. We ask the following: Is there a hyperplane that cuts n-space in half in such a way that on one side all corners of the hypercube (characteristic vectors) correspond to stable sets of G and on the other side all corners correspond to nonstable sets? Equivalently, can we distinguish which subsets of V are stable sets using a single

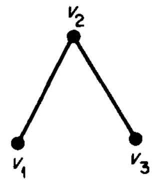

Figure 10.1.

linear inequality? If the answer is affirmative, then the graph under consideration is a *threshold graph.* If not, we shall want to know how many inequalities are needed to distinguish between stable and nonstable sets.

Example. Consider the graph in Figure 10.1. Its stable sets correspond to the solid corners of the unit 3-cube in Figure 10.2. The inequality $x_1 + 2x_2 + x_3 \leq 2$ is satisfied only by the characteristic vectors of the stable sets. Thus, a separating plane does exist, namely, $x + 2y + z = 2$.

The *threshold dimension* $\theta(G)$ of the graph $G = (V, E)$ is defined to be the minimum number k of linear inequalities

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq t_1, \\ &\vdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n &\leq t_k, \end{aligned} \tag{1}$$

such that X is a stable set if and only if its characteristic vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ satisfies (1). Regarding each inequality of (1) as a hyperplane in n-space, X is stable iff $\mathbf{x}$ lies on or within the "good" side of each of those k hyperplanes. Since G is finite, $\theta(G)$ is finite and well defined.

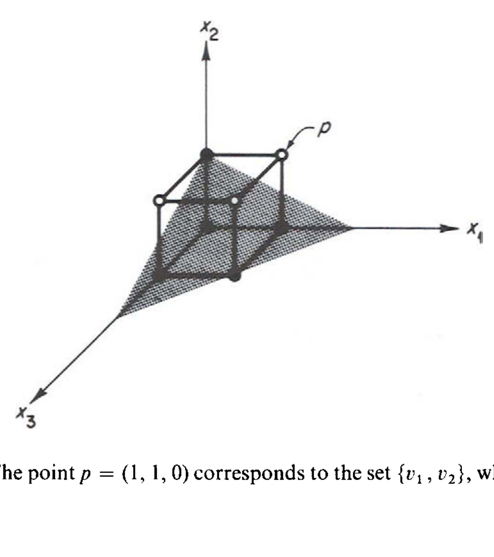

Figure 10.2. The point $p = (1, 1, 0)$ corresponds to the set $\{v_1, v_2\}$, which is not stable.

Remark. The only graphs G for which $\theta(G) = 0$ are those having no edges. In this case, the empty set of constraints suffices.

Let us first notice that *without loss of generality, we may assume that all the numbers a_{ij} and t_i in* (1) *are non-negative integers.* Suppose we are given a set of linear inequalities (1). Since the zero vector represents a stable set, we must have each $t_i \geq 0$. Furthermore, any negative coefficient a_{ij} can be changed to zero because for sets X not containing v_j the sum $\mathrm{COUNT}_i(X)$, defined by

$$\mathrm{COUNT}_i(X) = \sum_{v_p \in X} a_{ip} = \sum_{p=1}^{n} a_{ip} x_p,$$

would remain unchanged, whereas for sets X containing v_j this sum would be increased to $\mathrm{COUNT}_i(X - \{v_j\})$ which is $\leq t_i$ if and only if X is stable. Finally, since the graph is finite and the x_i are integral, we can perturb the system by a small ε here and there to make all the numbers non-negative rationals. Then we multiply by the least common divisor in order to obtain integers.

An undirected graph $G = (V, E)$ whose threshold dimension $\theta(G)$ is ≤ 1 is a *threshold graph.* Equivalently, $G = (V, E)$ is *threshold* if there exists a threshold assignment $[a; t]$ consisting of a labeling a of the vertices by non-negative integers and an integer threshold t such that

$$X \text{ is stable} \Leftrightarrow \sum_{x \in X} a(x) \leq t \qquad (X \subseteq V). \tag{2}$$

Examples. The star graph $K_{1,n}$ is easily seen to be a threshold graph by assigning $a(v)$ to be the degree of v and $t = n$, (Figure 10.3). Labeling by degree, however, does not always work. The labeling in Figure 10.4a fails to satisfy (2) for any value of t since there is a stable set of weight 7 and a non-stable set of weight 6. It is not a threshold assignment. On the other hand, the

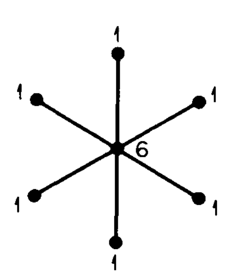

Figure 10.3. The graph $K_{1,6}$ and a threshold assignment with $t = 6$.

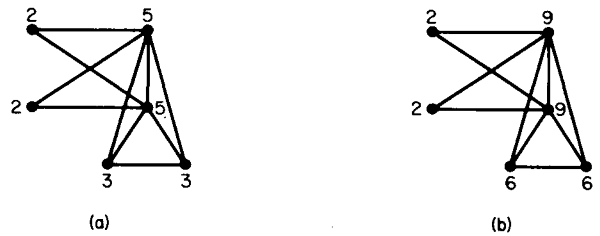

Figure 10.4. (a) Degree labeling. (b) A threshold assignment for $t = 10$.

labeling in Figure 10.4b of the same graph is a threshold assignment for $t = 10$. There are graphs which are not threshold graphs. For example, the chordless cycle C_n with $n \geq 4$ is not threshold and neither is the path P_n for $n \geq 4$ (Figure 10.5).

It should also be noted that an induced subgraph of a threshold graph is a threshold graph. Therefore, any graph which contains an induced subgraph isomorphic to one of those in Figure 10.5 is not threshold.

The threshold dimension $\theta(G)$ of an arbitrary graph G can be defined in an alternate but equivalent manner using threshold graphs. Take $\theta(G)$ to be the *minimum number of threshold graphs needed to cover the edges of* G, i.e., partial subgraphs of G, which are themselves threshold, and include every edge at least once. For example, $\theta(C_4) = 2$ since C_4 can be covered by two copies of $K_{1,2}$. The formalities of proving the definitions equivalent are left to the reader (Exercise 13). However, one can easily see that each inequality of (1) corresponds to one threshold graph and vice versa, and taken together they determine the adjacencies of the graph. This idea of covering by threshold graphs can be used to prove the following theorem. Let $\alpha(G)$ denote the size of the largest stable set of G.

Theorem 10.1 (Chvátal and Hammer [1973]). If G is an undirected graph with n vertices, then $\theta(G) \leq n - \alpha(G)$. Moreover, equality holds if (but not only if) G contains no triangle.

Proof. Let X be a stable set of cardinality $\alpha(G)$. For each vertex $v \notin X$ let S_v be the star graph with v at the center and having edges vv' for each v'

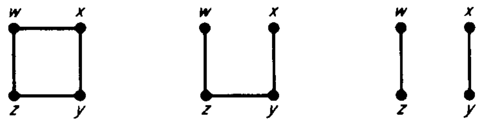

Figure 10.5. The graphs C_4, P_4, and $2K_2$ are not threshold since any assignment would require the inequalities $w + y \leq t$, $w + z > t$, $x + z \leq t$, and $x + y > t$, which are inconsistent.

adjacent to v in G. Thus, $\{S_v | v \notin X\}$ forms an edge covering of cardinality $n - \alpha(G)$, proving the first assertion.

Since C_m and P_m are not threshold for $m \geq 4$ and since being threshold is a property inherited by induced subgraphs, it follows that any minimum cover $\{G_i | i = 1, \ldots, \theta(G)\}$ of a *triangle-free* graph G consists only of stars G_i, whose center vertices we denote by u_i. Moreover, any edge of G has at least one endpoint in $U = \{u_i | i = 1, \ldots, \theta(G)\}$, implying that $V - U$ is stable and $\alpha(G) \geq |V - U| = n - \theta(G)$. Combining this with the first assertion, we obtain $\alpha(G) = n - \theta(G)$ for triangle-free graphs. ∎

Corollary 10.2. For the following graphs we have

(i) $\theta(C_n) = \lceil n/2 \rceil \quad (n > 3)$,
(ii) $\theta(K_{m,n}) = \mathrm{MIN}\{m, n\}$,
(iii) $\theta(P_n) = \lfloor n/2 \rfloor$.

Proof. Each of these graphs is triangle free, so the theorem provides the equalities. ∎

As pointed out in Chvátal and Hammer [1977], *the problem of computing* $\theta(G)$ *is NP-complete* in view of Poljak's proof (Theorem 2.1) that computing $\alpha(G)$ for triangle-free graphs is NP-complete. We shall see, however, that deciding whether or not $\theta(G) = 1$ can be done in linear time.

Unfortunately, since $\theta(K_n) = 1$, the bound of the theorem is sometimes useless. However, the next result shows that it is the best possible.

Corollary 10.3 (Chvátal and Hammer [1973]). For every $\varepsilon > 0$ there exists a graph G with n vertices such that $(1 - \varepsilon)n < \theta(G)$.

Proof. Erdös [1961] has proved that for any positive integer N there is a triangle-free graph G_N with $\alpha(G_N) < N$ and $n > c(N/\log N)^2$ vertices. (Here c is a positive constant independent of N.) Given $\varepsilon > 0$, choose N large enough so that $\varepsilon c N \geq (\log N)^2$ and consider the Erdös graph G_N. Since $\varepsilon \geq N(\log N)^2/cN^2 > N/n$, it follows that $(1 - \varepsilon)n < n - N < n - \alpha(G_N) = \theta(G_N)$. ∎

2. Degree Partition of Threshold Graphs

In this section we present a number of characterizations of threshold graphs. Let $G = (V, E)$ be a threshold graph with threshold assignment $[a; t]$. The following properties are immediate:

$$a(x) \leq t \qquad (x \in V), \tag{3}$$

$$xy \in E \Leftrightarrow a(x) + a(y) > t \qquad (x, y \in V, x \neq y). \tag{4}$$

A labeling satisfying (3) and (4) is not in general a threshold assignment since the sets being tested for stability are restricted to those of cardinality ≤ 2. However, condition (4) does imply the existence of a different labeling and threshold satisfying (2), as we shall prove in Theorem 10.4. For example, the labeling given in Figure 10.4a does satisfy (3) and (4) with $t = 5$, but it is not a threshold assignment. On the other hand, the labeling in Figure 10.4b is a threshold assignment for $t = 10$.

We begin by defining the degree partition of an undirected graph $G = (V, E)$ in which we associate vertices having the same degree. Let $0 < \delta_1 < \delta_2 < \cdots < \delta_m < |V|$ be the degrees of the nonisolated vertices; the δ_i are distinct and there may be many vertices of degree δ_i. Define $\delta_0 = 0$ and $\delta_{m+1} = |V| - 1$. The *degree partition* of V is given by

$$V = D_0 + D_1 + \cdots + D_m,$$

where D_i is the set of all vertices of degree δ_i. Only D_0 is possibly empty.

The following theorem is due to Chvátal and Hammer [1973]. The equivalence of (i) and (ii) was discovered independently by Henderson and Zalcstein [1977].

Theorem 10.4. Let $G = (V, E)$ be an undirected graph with degree partition $V = D_0 + D_1 + \cdots + D_m$. The following statements are equivalent:

(i) G is a threshold graph;

(ii) there exists an integer labeling c of V and an integer (threshold) t such that for distinct vertices x and y,

$$xy \in E \Leftrightarrow c(x) + c(y) > t;$$

(iii) for all distinct vertices $x \in D_i$ and $y \in D_j$,

$$xy \in E \Leftrightarrow i + j > m;$$

(iv) the recursions below are satisfied:

$$\begin{aligned} \delta_{i+1} &= \delta_i + |D_{m-i}| \qquad (i = 0, 1, \ldots, \lfloor m/2 \rfloor - 1), \\ \delta_i &= \delta_{i+1} - |D_{m-i}| \qquad (i = m, m, - 1, \ldots, \lfloor m/2 \rfloor + 1). \end{aligned}$$

Before proving the theorem let us understand its significance. Statement (iii) says that *the structure of the graph is entirely determined by the indices of the degree partition.* The vertices contained in the first half of the partition cells form a stable set, while those contained in the later half of the partition cells from a complete set. Furthermore, the adjacencies possess a natural containment, as illustrated in Figure 10.6. Statement (iv) is most important computationally for it allows us to verify that we have a threshold graph by using purely arithmetic operations without making reference to edges or

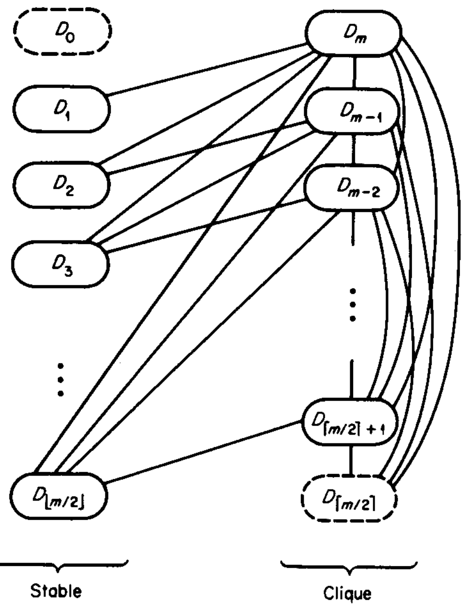

Figure 10.6. The typical structure of a threshold graph. A line between cells D_i and D_j indicates that each vertex in D_i is adjacent to each vertex of D_j. Cell D_0 contains all isolated vertices and may be empty. Cell $D_{\lceil m/2 \rceil}$ only exists if m is odd.

adjacency sets—a very unusual situation in graph theory.* Since these recursive relations can be verified within $O(n)$ computational steps for a graph with n vertices, we obtain the following.

Corollary 10.5. Given only the degrees of the vertices of an undirected graph G, there is an algorithm which decides whether or not G is a threshold graph and which runs in time proportional to the number of vertices of G.

Proof of this corollary is given as Exercise 7.

Proof of Theorem 10.4. (i) $\Rightarrow$ (ii) This is just Property (4).

(ii) $\Rightarrow$ (iii) The proof is by induction on the length of the degree partition. We may assume that $0 < i \leq j \leq m$.

Let y be a vertex of largest label $c(y)$. For any other vertex x, if x is adjacent to some vertex w (i.e., $x \notin D_0$), then $t < c(x) + c(w) \leq c(x) + c(y)$, implying that x is adjacent to y. Hence, $y \in D_m$, $\delta_m = |V| - |D_0| - 1$ and each vertex

* The reader will notice that the two sets of recursions actually use the same equation. They are stated separately to emphasize the method of calculation (δ_0 and δ_{m+1} are known), and to indicate how they may be proved inductively.

in D_m is adjacent to all nonisolated vertices. This proves (iii) in the case $j = m$.

Furthermore, *the vertices of D_1 are adjacent only to those in D_m*. For suppose $\delta_1 > |D_m|$, then *each* vertex $x \in V - D_0$ would be adjacent to some vertex in $V - D_m$; hence x would also be adjacent to z where z has the largest label in $V - D_m$. This forces z to be in D_m, a contradiction. Thus, $\delta_1 = |D_m|$.

Finally, let $V' = V - D_0 - D_m$ and consider the induced subgraph $G_{V'} = (V', E_{V'})$ which also satisfies (ii). Since its degree partition $V' = D_1 + \cdots + D_{m-1}$ is shorter by 2 than our original, the induction hypothesis proves the claim for $j < m$.

(iii) $\Rightarrow$ (iv) After some reflection (iv) is seen simply as a restatement of (iii).

(iv) $\Rightarrow$ (i) We shall assign an integer label a_i to each $x \in D_i$ such that the sum of the labels of the vertices in $X \subseteq V$ is less than or equal to a designated integer t if and only if X is a stable set. Now $D_0 + \cdots + D_{\lfloor m/2 \rfloor}$ is stable, and if X is a stable set containing a vertex $y \in D_j$ with $j > \lfloor m/2 \rfloor$, then $X - y \subseteq D_0 + \cdots + D_{m-j}$.

The reader may verify that the following labeling is a threshold assignment. (He should do the arithmetic base $|V|$.)

$$\begin{aligned} a_i &= |V|^i && (i = 0, 1, \ldots, \lfloor m/2 \rfloor), \\ t &= 2|V|^{\lfloor m/2 \rfloor + 1}, \\ a_j &= t + 1 - |V|^{m-j+1} && (j = \lfloor m/2 \rfloor + 1, \ldots, m). \end{aligned}$$ ∎

Remark. Orlin [1977] has given a construction of the unique *integral* threshold assignment which minimizes the threshold t.

Notice that almost the mirror image of Figure 10.6 will appear if we replace edges by nonedges in that illustration. This is not surprising in light of the following corollary.

Corollary 10.6. The complement of a threshold graph is a threshold graph.

Proof. Assume that a labeling satisfying condition (ii) is given. The labeling $\bar{c}(x) = t - c(x)$ (for all $x \in V$) with the threshold $\bar{t} = t - 1$ satisfies (ii) for the complement since

$$\begin{aligned} xy \notin E &\Leftrightarrow 0 \le t - c(x) - c(y) \\ &\Leftrightarrow t \le \bar{c}(x) + \bar{c}(y) \\ &\Leftrightarrow \bar{t} < \bar{c}(x) + \bar{c}(y). \end{aligned}$$ ∎

From this corollary we conclude that a graph with n vertices is threshold if and only if there exists a hyperplane in $\mathbb{R}^n$ separating the characteristic vectors of the complete sets of vertices from the characteristic vectors of the noncomplete sets. An alternative proof of Corollary 10.6 follows from the next result.

Theorem 10.7 (Chvátal and Hammer [1973]). A graph is threshold if and only if it has no induced subgraph isomorphic to $2K_2$, P_4, or C_4.

Proof. The graphs $2K_2$, P_4, and C_4 are not threshold graphs (Figure 10.4), hence no threshold graph can contain one of them. Conversely, suppose $G = (V, E)$ is not threshold, then there exists a subset $X \subseteq V$ with $|X| \geq 4$ such that $\varnothing \neq \text{Adj}(x) \cap X \neq X - \{x\}$ for each $x \in X$. (This is a straightforward consequence of Theorem 10.4; see Exercise 3.) Choose $x_1 \in X$ to have the smallest degree in G_X and pick $x_2, x_3 \in X$ so that $x_1x_2 \in E$ but $x_2x_3 \notin E$. Since $|\text{Adj}(x_1) \cap X| \leq |\text{Adj}(x_3) \cap X|$ there exists an $x_4 \in X$ such that $x_3x_4 \in E$ but $x_1x_4 \notin E$. Thus, the set $\{x_1, x_2, x_3, x_4\}$ induces one of the three forbidden graphs $2K_2$, P_4, or C_4, which proves the theorem. ▌

Benzaken and Hammer [1978] have studied an analogous threshold problem for absorbent (or dominating) sets. A subset X of vertices is *absorbent* if every vertex not in X is adjacent to some member of X. The class of graphs obtained properly contains the threshold graphs. They give a number of characterizations of this class.

3. A Characterization Using Permutations

Where does a threshold graph G fit into the world of perfect graphs? First of all, G is a split graph since its vertices can be partitioned into a stable set and a complete set; the edges between these sets are structured in a manner that has already been described. Secondly, the edges of G can be transitively oriented; let the vertices of G be numbered according to ascending degree and orient each edge toward its larger numbered endpoint. By Corollary 10.6, the complement $\bar{G}$ can also be transitively oriented, so G is a special kind of permutation graph. In the nomenclature of Section 6.1, every threshold graph is a triangulated–cotriangulated–comparability–cocomparability graph, or symbolically,

$$\text{THRESHOLD} \subset T \cap \bar{T} \cap C \cap \bar{C}.$$

This inclusion is proper as demonstrated by the graph P_4.

Let us characterize threshold graphs in the context of permutation graphs.

Let π be a permutation of the numbers $\{1, 2, \ldots, n\}$. In Chapter 7 we defined the graph of π, denoted by $G[\pi]$, to have vertices numbered v_1,

$v_2, \ldots, v_n$, with v_i and v_j adjacent if and only if $(i - j)(\pi_i^{-1} - \pi_j^{-1}) < 0$. For example, writing π as the sequence $\pi_1\pi_2 \cdots \pi_n$ we see that $G[1, 2, \ldots, n]$ has no edges whereas $G[n, n - 1, \ldots, 1]$ is the complete graph. Recall that the graphs $G[\pi]$ and $G[\pi^\rho]$ are complementary, where π^ρ denotes π written in reversed sequential order. Finally, an undirected graph G is a *permutation graph* if it is isomorphic to $G[\pi]$ for some π.

Let σ and τ be two sequences over some alphabet. The *shuffle product* is defined as follows:

$$\sigma \sqcup\!\sqcup\, \tau = \{\sigma_1\tau_1 \cdots \sigma_k\tau_k \mid \sigma = \sigma_1 \cdots \sigma_k \quad \text{and} \quad \tau = \tau_1 \cdots \tau_k\}.$$

Here the σ_i and τ_i are subsequences, k ranges over all integers, and juxtaposition means concatenation. The notion of shuffle product appears in automata theory (see Eilenberg [1974]).

Theorem 10.8 (Golumbic [1978a]). The threshold graphs are precisely those permutation graphs corresponding to sequences contained in

$$[1, 2, \ldots, p] \sqcup\!\sqcup\, [n, n - 1, \ldots, p + 1], \tag{5}$$

where p and n are positive integers and $\sqcup\!\sqcup$ denotes shuffle product.

Proof. Let $G = (V, E)$ be a given threshold graph with degree partition $V = D_0 + D_1 + \cdots + D_m$. Let $s_i = \sum_{q=0}^{i} |D_q|$ and rename the vertices $v_1, v_2, \ldots, v_n$ such that $\deg v_i < \deg v_j$ implies $i < j$. We define a permutation π as follows:

$$\gamma_0 = \begin{cases} [1, \ldots, D_0] & \text{if} \quad |D_0| > 0, \\ \text{empty sequence} & \text{otherwise}; \end{cases}$$

$$\gamma_i = \begin{cases} [1 + s_{i-1}, \ldots, s_i] & \text{for} \quad 1 \le i \le \lfloor m/2 \rfloor, \\ [s_i, \ldots, 1 + s_{i+1}] & \text{for} \quad \lfloor m/2 \rfloor < i \le m; \end{cases}$$

$$\pi = \gamma_0\gamma_m\gamma_1\gamma_{m-1}\gamma_2\gamma_{m-2} \cdots \gamma_{\lceil m/2 \rceil}.$$

Note that π is of the form (5), and that

$$v_z \in D_k \Leftrightarrow s_{k-1} < z \le s_k \Leftrightarrow z \in \gamma_k. \tag{6}$$

We will show that $G = G[\pi]$.

Choose vertices $v_x \in D_i$ and $v_y \in D_j$; we may assume that $x < y$ and hence, by construction, $i \le j$. By (6), v_x and v_y are adjacent in $G[\pi]$ if and only if y appears to the left of x in π. This will occur if and only if either (i) $i \le \lfloor m/2 \rfloor$ and γ_j is strictly to the left of γ_i or (ii) $\lfloor m/2 \rfloor < i \le j$. But conditions (i) and (ii) together are equivalent with $i + j > m$. Hence, by Theorem 10.4(iii), v_x and v_y are adjacent in $G[\pi]$ if and only if $v_xv_y \in E$, proving that $G = G[\pi]$.

Conversely, any permutation of the form (5) yields a threshold graph. ∎

Teng and Liu [1978] use the shuffle product for the integration of several logically independent and concurrently operating transmission grammars. Transition grammars describe the rules, or *protocols*, which regulate the interactions between the attached entities in a computer network to ensure that they proceed in an orderly fashion.

4. An Application to Synchronizing Parallel Processes

Threshold graphs were rediscovered and studied by others, including Henderson and Zalcstein [1977]; they are responsible for the application presented here. See also Vantilborgh and van Lamsweede [1972].

A hypergraph $H = (S, \mathscr{E})$ consisting of a vertex set S and a hyperedge collection $\mathscr{E}$ of subsets of S, is called a *threshold hypergraph* if there exists a non-negative integer labeling c of S and an integer threshold t such that for all $X \subseteq S$,

$$X \text{ contains no hyperedge} \Leftrightarrow \sum_{x \in X} c(x) \leq t.$$

As before, we call the pair $[c; t]$ a *threshold assignment* for H.

Unlike the special case when H is a graph for which many results are known, the problem of characterizing threshold hypergraphs is unsolved and appears to be quite difficult. Nevertheless, threshold graphs and hypergraphs can be useful in an application to computing which we will now present.

Consider a set of computer programs $\mathscr{P} = \{P_i\}$ to be run in parallel. (Some of the P_i might actually be subroutines of larger programs.) Because of overall memory constraints or common memory location requirements some conflict may arise when a certain subset $\mathscr{P}'$ of $\mathscr{P}$ is not able to run simultaneously. Let $\mathscr{E}$ denote the collection of all such forbidden $\mathscr{P}'$. Hence, the programs in $X \subseteq \mathscr{P}$ can be run together without conflict if and only if X contains no member of $\mathscr{E}$.

When $(\mathscr{P}, \mathscr{E})$ is a threshold hypergraph a particularly simple programming technique can be applied to let the computer prevent conflicts automatically and control the traffic of programs running and waiting. Let $[c; t]$ be a threshold assignment for $(\mathscr{P}, \mathscr{E})$ and denote $c_i = c(P_i)$. The technique is as follows.

(1) Precede each program P_i with a call to procedure $P(s, c_i)$.
(2) Follow each program P_i with a call to procedure $V(s, c_i)$.
(3) Initialize a new global variable s with the value t.

```
procedure P(s, c):              procedure V(s, c):
  if s ≥ c then                   s ← s + c
    s ← s − c                   return
    enter P_i
  else
    call again
return
```

Figure 10.7. Subroutine P requests permission to begin and subroutine V informs the traffic controller that the program is finished. Variable s records how much "room" is currently available.

(See Figure 10.7.) The variable s, called a *semaphore*, never allows the sum of the c_i for those programs currently running to exceed t. The number c_i resembles the space required to do P_i. Every time we wish to execute a routine P_i, the procedure P checks whether or not there is sufficient space (i.e., is $s \geq c_i$). If so, we reduce s by c_i and begin; if not, then we wait (in a queue) until there is enough space. When we finish P_i the procedure V releases c_i units of space.

Example 1. Given a set of programs $\{P_i\}$ such that at most 12 of them can be executed simultaneously, assign $t = 12$ and $c(P_i) = 1$ for each i.

Example 2. Let $\mathscr{P}$ consist of three types of processes: the readers $R_1, \ldots, R_r$; the writers $W_1, \ldots, W_w$; and the mathematicians $M_1, \ldots, M_m$. Assume that we may execute simultaneously either at most one mathematician plus an unlimited number of readers or at most one writer.* This problem has the threshold assignment

$$\begin{aligned} c(R_i) &= 1 && (i = 1, \ldots, r), \\ c(M_j) &= r + 1 && (j = 1, \ldots, m), \\ c(W_k) &= 2r + 1 && (k = 1, \ldots, w), \\ t &= 2r + 1. \end{aligned}$$

Although there is no accurate graph theoretic formulation for Example 1, Example 2 can be viewed as a graph G with edges connecting the readers with the writers, the mathematicians with each other and the writers, and the writers with everyone. In this case the stable sets of G correspond to the subsets which can be executed simultaneously.

Example 3. If we add some bureaucrats $B_1, \ldots, B_b$ to Example 2 who can work with writers but cannot work with mathematicians, then the system no longer has a threshold assignment.

* If someone is writing on the system, no one else may have access since changes are being made. Otherwise, as many readers can work as want, but only one mathematician can work because there is only one calculator and he needs a calculator.

Finally, suppose we have a system with threshold dimension k. We can proceed similarly using P's and V's except that k semaphores will be needed. One semaphore handles each inequality (or labeling), and a segment P_i can be entered if and only if there is sufficient resource according to each of the k semaphores.

EXERCISES

1. Prove that $\theta(G) \le n + 1 - \omega(G)$, where $\theta(G)$ and $\omega(G)$ denote the threshold dimension of G and the size of the largest clique of G, respectively.
2. Show that $\theta(H) \le \theta(G)$ for any induced subgraph H of G.
3. Prove the following: $G = (V, E)$ is a threshold graph if and only if for each subset $X \subseteq V$ there exists a vertex $x \in X$ such that $\mathrm{Adj}(x) \cap X = \varnothing$ or $\mathrm{Adj}(x) \cap X = X - \{x\}$ (i.e., x is adjacent to all the vertices of $X - \{x\}$ or to none of them; Chvátal and Hammer [1973].)
4. Show that the following procedure will recognize threshold graphs. What is its complexity?

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Boolean procedure THRESHOLD(G):
begin
  while the edge set is nonempty do
    begin
      delete all isolated vertices;
      if there is a vertex x adjacent to all remaining vertices then delete x;
      else
        return false;
    end
  return true;
end
```

5. Prove the following: A graph $G = (V, E)$ is threshold if and only if its vertices can be ordered and partitioned into a stable set $X = \{x_1, x_2, \ldots, x_s\}$ and a complete set $Y = \{y_1, \ldots, y_t\}$ such that

$$x_i y_j \in E \Rightarrow x_{i'} y_{j'} \in E \qquad (i' \ge i, \quad j' \ge j).$$

6. Prove that a threshold graph G with degree partition $V = D_0 + D_1 + \cdots + D_m$ has a Hamiltonian circuit if and only if the following relations are satisfied:

$$|D_0| = 0,$$

$$\sum_{i=1}^{k} |D_i| < \sum_{j=m+1-k}^{m} |D_j| \qquad (k = 1, 2, \ldots, \lfloor (m-1)/2 \rfloor),$$

$$\sum_{i=1}^{m/2} |D_i| \le \sum_{j=m/2+1}^{m} |D_j| \qquad (\text{if } m \text{ is even}).$$

Show how one may obtain the Hamiltonian circuit.

7. Let H have vertices $1, 2, \ldots, n$ and let DEG (i) equal the degree of vertex i. Write an algorithm which verifies the recurrence relations in Theorem 10.4(iv). Prove that your algorithm runs in $O(n)$ time.
8. Calculate the threshold dimension for the graphs in Figure 10.8 thus showing that $\theta(G)$ is not in general equal to $\theta(\bar{G})$ for nonthreshold graphs.

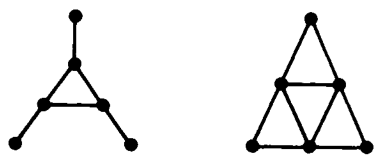

Figure 10.8.

9. Find necessary and sufficient conditions for a sequence $[a_1, a_2, \ldots, a_n; t]$ to be a threshold assignment for some threshold graph.
10. Prove that the number of mutually nonisomorphic n-vertex threshold graphs is 2^{n-1}.
11. Prove that G is a threshold graph if and only if equality holds in *each* of the Erdös–Gallai inequalities (see Section 6.3) (Hammer, Ibaraki, and Simeone [1978]).
12. Verify that the labeling given at the end of the proof of Theorem 10.4 is a threshold assignment.
13. Let $\theta'(G)$ denote the smallest integer k for which there exist partial subgraphs $(V_1, E_1), (V_2, E_2), \ldots, (V_k, E_k)$ of $G = (V, E)$ satisfying $E = E_1 \cup E_2 \cup \cdots \cup E_k$, where each (V_i, E_i) is a threshold graph. Prove that $\theta'(G)$ equals the threshold dimension $\theta(G)$ of G. (Note: You may assume $V_i = V$ for each i. Why?)
14. Let G be a threshold graph whose vertices are numbered according to increasing degree. Prove that the orientation obtained by directing each edge of G toward its larger numbered endpoint is transitive.
15. Let X be a set of propositions and let Y be a set of subjects in a psychological experiment. A subject either agrees or disagrees with a proposition. A *Guttman scale* is a linear ordering of $X \cup Y$ such that a subject agrees with all items following it and disagrees with all items preceding it. Let G be an undirected graph with vertex set $X \cup Y$ constructed as follows: X forms a stable set; Y forms a clique; subject y is adjacent to proposition x if and only if subject y agrees with proposition x. The following are from Leibowitz [1978]:

(i) Prove that there exists a Guttman scale for $X \cup Y$ if and only if G is a threshold graph.

(ii) Give an algorithm to construct a Guttman scale. For a discussion of Guttman scales, see Coombs [1964].

16. Prove that every threshold graph has an interval representation using intervals of at most two different lengths (Leibowitz [1978]).

17. Let $m(G)$ denote the number of maximal cliques of an undirected graph G and let $\alpha(G)$ be the *stability number*. Clearly,

$$\alpha(G) \leq m(G),$$

since there must be $\alpha(G)$ distinct cliques containing the members of a maximum stable set.

An undirected graph $G = (V, E)$ is said to be *trivially perfect* if for each $A \subseteq V$, the induced subgraph G_A of G satisfies $\alpha(G_A) = m(G_A)$. This name was chosen since it is trivial to show that such a graph is perfect. Prove the following (Golumbic [1978b]):

(i) A graph $G = (V, E)$ is trivially perfect if and only if it contains no induced subgraph isomorphic to C_4 or P_4.

(ii) A connected graph is trivially perfect if and only if it is a comparability graph whose Hasse diagram is a rooted tree.

(iii) G and $\bar{G}$ are both trivially perfect iff G is a threshold graph.

Research Problem. Characterize the graphs of threshold dimension 2.

Research Problem. Let S be a finite set and let $\mathscr{E}$ be a collection of subsets of S each of size r. The pair $H = (S, \mathscr{E})$ is usually called an r-regular hypergraph. If $r = 2$, then H is just an undirected graph. Consider the following properties:

(T_1) There exists a (positive integer) labeling c of S and an (integer) threshold t such that, for all subsets $X \subseteq S$,

$$X \text{ contains no member of } \mathscr{E} \Leftrightarrow \sum_{x \in X} c(x) \leq t.$$

(T_2) There exists a (positive integer) labeling c' of S and an (integer) threshold t' such that for all subsets $A \subseteq S$ of size r,

$$A \in \mathscr{E} \Leftrightarrow \sum_{x \in A} c'(x) > t'.$$

(T_3) For $x, y \in S$ define $x \succcurlyeq y$ if x can replace y in any hyperedge (member) of $\mathscr{E}$. That is, $x \succcurlyeq y$ if [$y \in A \in \mathscr{E}$ and $x \notin A$] imply $A - \{y\} + \{x\} \in \mathscr{E}$. Then, for all $x, y \in S$, either $x \succcurlyeq y$ or $y \succcurlyeq x$ or both.

Clearly (T_1) $\Rightarrow$ (T_2) $\Rightarrow$ (T_3). Either prove or disprove the reverse implications. [We know they are both true when $r = 2$. Perhaps a proof for $r = 3$ would generalize to arbitrary r.]

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