CHAPTER 6

Split Graphs

1. An Introduction to Chapters 6–8: Interval, Permutation, and Split Graphs

An undirected graph G may possess one or more of these familiar properties:

Property C: G is a comparability graph.
Property $\bar{C}$: $\bar{G}$ is a comparability graph (i.e., G is a *cocomparability* graph).
Property T: G is a triangulated graph.
Property $\bar{T}$: $\bar{G}$ is a triangulated graph (i.e., G is a *cotriangulated* graph).

These four properties are independent of one another. Examples of all 16 possible combinations are given in Appendix F.

Chapters 6–8 deal with the classes of graphs which have been characterized in terms of these four properties. In particular, we shall show the following:

$$\text{interval graphs} \equiv T + \bar{C};$$
$$\text{permutation graphs} \equiv C + \bar{C};$$
$$\text{split graphs} \equiv T + \bar{T}.$$

We begin our study with split graphs, which are defined in the next section. Chapters 6–8 are independent of one another; they may be read in any order without loss of continuity.

2. Characterizing Split Graphs

An undirected graph $G = (V, E)$ is defined to be *split* if there is a partition $V = S + K$ of its vertex set into a stable set S and a complete set K. There is

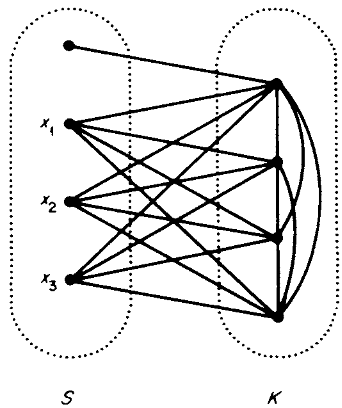

Figure 6.1. A split graph with one of its four partitions indicated. The other partitions are $(S - \{x_i\}) + (K \cup \{x_i\})$ for $i = 1,2,3$.

no restriction on edges between vertices of S and vertices of K. In general, the partition $V = S + K$ of a split graph will not be unique; neither will S (resp. K) necessarily be a maximal stable set (resp. clique). For example, the graph G in Figure 6.1 has four partitions, one of which is indicated. Notice also that $|S| = \alpha(G) = 4$ whereas $4 = |K| < \omega(G) = 5$; S is the only maximum stable set of G, and $K \cup \{x_i\}$ (for $i = 1, 2, 3$) are the only maximum cliques.

Since a stable set of G is a complete set of the complement $\bar{G}$ and vice versa, we have an immediate result.

Theorem 6.1. An undirected graph G is a split graph if and only if its complement $\bar{G}$ is a split graph.

The next theorem follows from the work of Hammer and Simeone [1977].

Theorem 6.2. Let G be a split graph whose vertices have been partitioned into a stable set S and a complete set K. Exactly one of the following conditions holds:

(i) $|S| = \alpha(G)$ and $|K| = \omega(G)$
(in this case the partition $S + K$ is unique),
(ii) $|S| = \alpha(G)$ and $|K| = \omega(G) - 1$
(in this case there exists an $x \in S$ such that $K + \{x\}$ is complete),
(iii) $|S| = \alpha(G) - 1$ and $|K| = \omega(G)$
(in this case there exists a $y \in K$ such that $S + \{y\}$ is stable).

Proof. Since a stable set and a complete set can have at most one common vertex, it follows that a split graph has the sum $\alpha(G) + \omega(G)$ equal to either $|V|$ or $|V| + 1$.

If $\alpha(G) + \omega(G) = |V|$, then we are in case (i). Suppose, in this case, there is another partition $V = S' + K'$. Let $\{x\} = S \cap K'$ and $\{y\} = S' \cap K$. If x and y are adjacent in G, then $\{x\} + K$ is a clique of size $\omega(G) + 1$, which is impossible. If x and y are not adjacent in G, then $\{y\} + S$ is a stable set of size $\alpha(G) + 1$, which is impossible. Hence, the partition $V = S + K$ must be unique.

If $\alpha(G) + \omega(G) = |V| + 1$, then we are in either case (ii) or case (iii). We will prove the claim in case (ii) only, case (iii) being analogous. Let $|S| = \alpha(G)$, $|K| = \omega(G) - 1$ and let K' be a clique of size $\omega(G)$. Since $S + K$ is a partition and K' is larger than K, $S \cap K'$ must be nonempty and therefore of cardinality 1. Let $\{x\} = S \cap K'$, it follows that $K' = K + \{x\}$, which is complete. ∎

Theorem 6.3 (Földes and Hammer [1977b]). Let G be an undirected graph. The following conditions are equivalent:

(i) G is a split graph,
(ii) G and $\bar{G}$ are triangulated graphs,
(iii) G contains no induced subgraph isomorphic to $2K_2$, C_4, or C_5.

Proof. (i) $\Rightarrow$ (ii) Let $G = (V, E)$ have vertex partition $V = S + K$ with S stable and K complete. Suppose G contained a chordless cycle C of length ≥ 4. At least one and at most two (adjacent) vertices of C would be in K. Both cases would imply that S contains a pair of adjacent vertices, a contradiction. Therefore, G must be triangulated. By Theorem 6.1, $\bar{G}$ is split, so $\bar{G}$ is triangulated.

(ii) $\Rightarrow$ (iii) Immediate.

(iii) $\Rightarrow$ (i) Let K be a maximum clique of G chosen (among all maximum cliques) so that G_{V-K} has fewest possible edges. We must show that $S = V - K$ is stable.

Suppose, on the contrary, that G_S has an edge xy. By the maximality of K, no vertex of S could be adjacent to every member of K. Moreover, if both x and y were adjacent to every vertex of K with the exception of the same single vertex z, then $K - \{z\} + \{x\} + \{y\}$ would be a complete set larger than K. Thus, there must exist distinct vertices $u, v \in K$ such that $xu \notin E$ and $yv \notin E$.

Since G contains neither an induced copy of $2K_2$ nor C_4, it follows that exactly one of the edges xv or yu is in G. Assume, without loss of generality, that $xv \notin E$ and $yu \in E$. For any $w \in K - \{u, v\}$, if $yw \notin E$ and $xw \notin E$, then $G_{\{x,y,v,w\}} \cong 2K_2$, whereas if $yw \notin E$ and $xw \in E$, then $G_{\{x,y,u,w\}} \cong C_4$. Thus,

y is adjacent to every vertex of $K - \{v\}$, and $K' = K - \{v\} + \{y\}$ is a maximal clique.

Since $G_{V-K'}$ can have no fewer edges than G_{V-K} has, it follows from the fact that x is adjacent to y but not to v that there exists a vertex $t \neq y$ in $V - K$ which is adjacent to v but not to y. Now tx must be an edge of G, for otherwise $\{t, x, y, v\}$ would induce a copy of $2K_2$. Similarly, $tu \notin E$, for otherwise $\{t, x, y, u\}$ would induce a copy of C_4. However, this implies that $\{t, x, y, u, v\}$ induces a copy of C_5, a contradiction. Therefore, $S = V - K$ is stable, and G is a split graph. ∎

A characterization of when a split graph is also a comparability graph appears in Chapter 9 (Theorem 9.7).

3. Degree Sequences and Split Graphs

A sequence $\Delta = [d_1, d_2, \ldots, d_n]$ of integers, $n - 1 \geq d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$, is called *graphic* if there exists an undirected graph having Δ as its degree sequence. For example, the sequence [2, 2, 2, 2] corresponds to the chordless 4-cycle C_4, while the sequence [2, 2, 2, 2, 2, 2] corresponds to both $2K_3$ and C_6. It is easy to construct sequences which are not graphic, such as [1, 1, 1] and [4, 4, 2, 1, 1].

A simple necessary condition for a sequence to be graphic comes from Euler's theorem: The sum $\sum d_i$ must be even. However, as the preceding example shows, an even sum is not sufficient to insure graphicness. Two classical theorems characterizing graphic sequences will now be stated.

Theorem 6.4 (Havel [1955], Hakimi [1962]). A sequence Δ of integers $n - 1 \geq d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$ is graphic if and only if the modified sequence

$$\Delta' = [d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n]$$

(sorted into decreasing order) is graphic.

Theorem 6.5 (Erdös and Gallai [1960]). A sequence of integers $n - 1 \geq d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$ is graphic if and only if

(i) $\sum_{i=1}^{n} d_i$ is even, and

(ii) $\sum_{i=1}^{r} d_i \leq r(r - 1) + \sum_{i=r+1}^{n} \min\{r, d_i\}$,

for $r = 1, 2, \ldots, n - 1$.

The inequality (ii) will be called the *rth Erdös–Gallai inequality* (EGI). Shortly, we shall give a characterization of split graphs in terms of these inequalities. We shall not prove Theorem 6.4 or Theorem 6.5 here since a very readable treatment can be found in Harary [1969, Chapter 6]. Both of these theorems suggest algorithms for testing whether or not a given sequence is graphic (Exercise 6).

A third classical theorem on graphic sequences depends partly on the following observation. Let x, y, z, and w be distinct vertices of G with xy and zw edges of G and xz and yw nonedges of G. If we replace the two edges by the two nonedges, the resulting graph G' will have the same degree sequence as G (see Figure 6.2). Such a replacement will be called an *interchange*. A stronger result holds, which we now state.

Remark 6.6. Provided that we allow graphs to have multiple edges, if two graphs have the same degree sequence, then each can be obtained from the other by a finite sequence of interchanges.

A proof of Remark 6.6 can be found in Ryser [1963, Chapter 6, Theorem 3.1] by applying his technique to the edges-versus-vertices incidence matrix of G.

A general question arises: What graph theoretic properties can be determined solely from the degree sequence? In Section 2.5 we remarked that transitive tournaments could be recognized by the in-degrees of the vertices. Also, a characterization of trees in terms of degree sequences is known. We will now discuss this problem as applied to split graphs.

Let $\Delta = [d_1, d_2, \ldots, d_n]$ be an integer sequence with $n - 1 \geq d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$, and let $\zeta = [0, 1, 2, \ldots, n - 1]$. Comparing the decreasing sequence Δ with the increasing sequence ζ, let us draw attention to the position just prior to ζ overtaking Δ. Let m be the largest index i such that $d_i \geq i - 1$. Thus, either $m = n$ and Δ is the degree sequence of K_n, or $d_m \geq m - 1$ and $d_{m+1} < m$.

The next result characterizes split graphs as those for which equality holds in the mth Erdös–Gallai inequality, where m is defined as above.

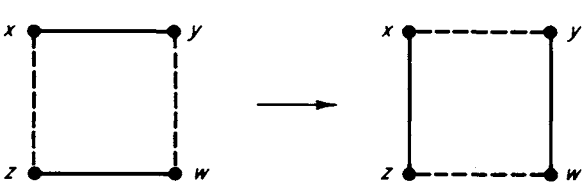

Figure 6.2. A solid line denotes an edge of G; a broken line denotes a nonedge of G. An *interchange* replaces the two edges with the two nonedges.

Theorem 6.7 (Hammer and Simeone [1977]). Let $G = (V, E)$ be an undirected graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, and let $m = \max\{i \mid d_i \geq i - 1\}$. Then, G is a split graph if and only if

$$\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i.$$

Furthermore, if this is the case, then $\omega(G) = m$.

Proof. The theorem is true if G is a complete graph, so we may assume that $d_m \geq m - 1$ and $d_{m+1} < m$. Since Δ is nonincreasing, $\min\{m, d_i\} = d_i$ for $i \geq m + 1$. Therefore, the mth EGI simplifies to

$$s = \sum_{i=1}^{m} d_i \leq m(m-1) + \sum_{i=m+1}^{n} d_i. \tag{1}$$

Let K denote the first m vertices of largest degree. The left summand of (1) splits into two contributions $s = s_1 + s_2$, where

$$s_1 = \sum_{x \in K} |\{z \in K \mid xz \in E\}| \leq m(m-1), \tag{2}$$

$$s_2 = \sum_{x \in K} |y \notin K \mid xy \in E\}|$$

$$= \sum_{y \notin K} |\{x \in K \mid xy \in E| \leq \sum_{i=m+1}^{n} d_i. \tag{3}$$

Equality holds in (2) if and only if K is complete. Equality holds in (3) if and only if $V - K$ is stable. Therefore, if equality holds in (1), then G is a split graph.

Conversely, assume that $G = (V, E)$ is a split graph. By Theorem 6.2 we can partition V into a stable set S and a complete set K such that $|K| = \omega(G)$. Every vertex in K has degree at least $|K| - 1$, and, since K is maximum, every vertex in S has degree at most $|K| - 1$. Therefore, we may assume that the vertices are ordered so that $K = \{v_1, \ldots, v_{|K|}\}$ and $S = \{v_{|K|+1}, \ldots, v_n\}$, where $\deg v_i = d_i$. Moreover, $d_{|K|} \geq |K| - 1$ and $d_{|K|+1} \leq |K| - 1 < |K|$, so $\omega(G) = |K| = m$. Finally, since K is complete and $S = V - K$ is stable, we conclude that equality holds in (2) and (3) and therefore also in (1). ∎

Corollary 6.8. If G is a split graph, then every graph with the same degree sequence as G is also a split graph.

Remark. Hammer and Simeone [1977] investigated a more general problem on graphs. They define the *splittance* of an arbitrary undirected graph to be the minimum number of edges to be added or erased in order to produce a split graph. Of course, split graphs are just those graphs whose splittance is zero. Their main result shows that the splittance depends only on the degree sequence of the graph, and is given by the expression

$$\frac{1}{2}\left[m(m-1) - \sum_{i \le m} d_i + \sum_{i \ge m+1} d_i\right],$$

where m and the d_i are as in Theorem 6.7.

Those who have further interest in the topic of graphs and their degree sequences are encouraged to read the survey paper by Hakimi and Schmeichel [1978].

EXERCISES

1. Give necessary and sufficient conditions for a tree to be a split graph. Prove that your answer is correct.
2. Prove that the Hamiltonian circuit problem is NP-complete for split graphs. (Hint. Use the fact that the Hamiltonian circuit problem is NP-complete for bipartite graphs.)
3. How many nonisomorphic graphs are there with the following degree sequences: (i) [3, 3, 2, 2, 1, 1], (ii) [5, 5, 5, 4, 3, 2], (iii) [5, 5, 4, 3, 3, 3, 2, 1]?
4. Give an example of two nonisomorphic split graphs having the same degree sequence.
5. What is the splittance of graphs C_n, $K_{m,n}$, mK_n, and P_n?
6. Give an $O(n)$ time algorithm for determining whether or not a nonincreasing integer sequence $n - 1 \ge d_1 \ge d_2 \ge \cdots \ge d_n \ge 0$ is graphic. Prove that your algorithm is correct and that its complexity is linear.
7. Let $\Delta = [d_1, d_2, \ldots, d_n]$ be an integer sequence, and define $\bar{\Delta} = [\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_n]$ by the formula

$$\bar{d}_i = n - 1 - d_{n-i+1} \qquad (i = 1, \ldots, n).$$

Show that Δ is graphic if and only if $\bar{\Delta}$ is graphic. What can you say about the graphs corresponding to Δ and $\bar{\Delta}$?
8. Let $m = \max\{i \mid d_i \ge i - 1\}$ where $n - 1 \ge d_1 \ge \cdots \ge d_n \ge 0$. Show that if the mth EGI holds, then the rth EGI automatically holds for $r = m - 1, \ldots, n$ (Hammer, Ibaraki, and Simeone [1978]).
9. Prove Corollary 6.8 directly from Theorem 6.6.

Research problem. Characterize those graphs which are uniquely determined up to isomorphism by their degree sequence. R. H. Johnson has solved this problem for trees; the solution is the class obtained in Exercise 1.

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