Longitudinal Linear Model

$$\dot{\vec{p}}_E = R_B^E \vec{V}_B^E$$
$$\dot{\vec{o}} = T \vec{\omega}_B$$
$$\dot{\vec{V}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{V}_B^E$$
$$\dot{\vec{\omega}} = I^{-1}\left[\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B\right]$$

2 differences
1. Aerodynamic Forces
2. I more complex

symmetry about x-z axis $\Rightarrow I_{xy} = I_{yz} = 0$

$I_{xz} \neq 0$

$$I_B^{-1} = \begin{bmatrix} \frac{I_z}{\Gamma} & 0 & \frac{I_{xz}}{\Gamma} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{\Gamma} & 0 & \frac{I_x}{\Gamma} \end{bmatrix} \qquad \Gamma = I_x I_z - I_{xz}^2$$

$$\Gamma_1 = \frac{I_{xz}\left(I_x - I_y + I_z\right)}{\Gamma} \qquad \Gamma_4 = \frac{I_{xz}}{\Gamma} \qquad \Gamma_7 = \frac{I_x\left(I_x - I_y\right) + I_{xz}^2}{\Gamma}$$
$$\Gamma_2 = \frac{I_z\left(I_z - I_y\right) + I_{xz}^2}{\Gamma} \qquad \Gamma_5 = \frac{I_z - I_x}{I_y} \qquad \Gamma_8 = \frac{I_x}{\Gamma}$$
$$\Gamma_3 = \frac{I_z}{\Gamma} \qquad \Gamma_6 = \frac{I_{xz}}{I_y} \qquad \Gamma = I_x I_z - I_{xz}^2$$

$I_{yz} \neq 0$
$I_{xy} \neq 0$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

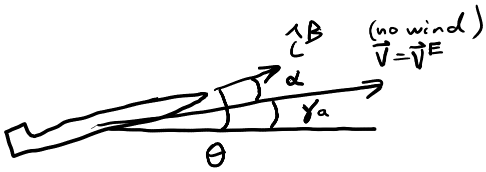
$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

Linearize about trim state

Inputs: $u_0, h_0, \gamma_{a_0} \longrightarrow \alpha_0, \delta_{e_0}, \delta_{t_0}$

airspeed $= V = V_a$; altitude ; air-relative flight-path angle



$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} = \vec{x}_0 + \Delta \vec{x} \qquad \vec{x}_0 = \begin{bmatrix} \cdot \\ \cdot \\ -h_0 \\ 0 \\ \theta_0 \\ \cdot \\ u_0^E \\ 0 \\ w_0^E \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For linearization: $\vec{V} = \vec{V}^E$ (no wind)

assume $w_0^E = 0$

$$\vec{u} = \begin{bmatrix} \delta_e \\ \delta_a \\ \delta_r \\ \delta_t \end{bmatrix} \qquad \vec{u}_0 = \begin{bmatrix} \delta_{e_0} \\ 0 \\ 0 \\ \delta_{t_0} \end{bmatrix}$$

Examples to Linearize

$$\dot{\theta} = \cos\phi\, q - \sin\phi\, r$$

$$\cancelto{0}{\dot{\theta}_0} + \Delta\dot{\theta} = \cancelto{0}{\cos\phi_0 q_0} - \cancelto{0}{\sin\phi_0 r_0} + \frac{\partial}{\partial\phi}(\cos\phi q - \sin\phi r)\Big|_0 \Delta\phi$$
$$+ \frac{\partial}{\partial q}(\cos\phi q - \sin\phi r)\Big|_0 \Delta q$$
$$+ \frac{\partial}{\partial r}(\cos\phi q - \sin\phi r)\Big|_0 \Delta r$$

$$= -\cancelto{0}{\sin\phi_0 q_0} - \cancelto{0}{\cos\phi_0 r_0} + \cancelto{1}{\cos\phi_0}\Delta q - \cancelto{0}{\sin\phi_0}\Delta r$$

$$\boxed{\Delta\dot{\theta} = \Delta q}$$

$$\dot{u} = rv - qw - g\sin\theta + \frac{X}{m}$$

$$\cancelto{0}{\dot{u}_0} + \Delta\dot{u} = \cancelto{0}{r_0 v_0} - \cancelto{0}{q_0 w_0} \cancelto{0}{\underbrace{- g\sin\theta_0 + \frac{X_0}{m}}_{\text{cancel}}} + \frac{\partial}{\partial r} rv\Big|_0 \Delta r + \frac{\partial}{\partial v} rv\Big|_0 \Delta v + \frac{\partial}{\partial q}(-qw)\Big|_0 \Delta q$$
$$+ \frac{\partial}{\partial w}(-qw)\Big|_0 \Delta w + \frac{\partial}{\partial\theta}(g\sin\theta)\Big|_0 \Delta\theta + \frac{\partial}{\partial X}\left(\frac{X}{m}\right)\Big|_0 \Delta X$$

$$= \cancelto{0}{v_0}\Delta r + \cancelto{0}{r_0}\Delta v - \cancelto{0}{w_0 \Delta q} - \cancelto{0}{q_0}\Delta w - g\cos\theta_0 \Delta\theta + \frac{1}{m}\Delta X$$

$$\boxed{\Delta\dot{u} = -g\cos\theta_0 \Delta\theta + \frac{1}{m}\Delta X}$$

Lateral

$$\Delta\dot{\phi} = \Delta p + \Delta r \tan\theta_0$$

$$\Delta\dot{\theta} = \Delta q$$

Long.

$$\Delta\dot{u} = -g\cos\theta_0\Delta\theta + \frac{\Delta X}{m}$$

$$\Delta\dot{v} = -u_0\Delta r + g\cos\theta_0\Delta\phi + \frac{\Delta Y}{m}$$

$$\Delta\dot{w} = u_0\Delta q - g\sin\theta_0\Delta\theta + \frac{\Delta Z}{m}$$

$$\Delta\dot{p} = \Gamma_3\Delta L + \Gamma_4\Delta N$$

$$\Delta\dot{q} = \frac{\Delta M}{I_y}$$

$$\Delta\dot{r} = \Gamma_4\Delta L + \Gamma_8\Delta N$$

Dimensional stab derivs.

$$\Delta X = X_u\Delta u + X_w\Delta w + \Delta X_c$$

$$\Delta Y = Y_v\Delta v + Y_p\Delta p + Y_r\Delta r + \Delta Y_c$$

$$\Delta Z = Z_u\Delta u + Z_w\Delta w + Z_{\dot{w}}\Delta\dot{w} + Z_q\Delta q + \Delta Z_c$$

$$\Delta L = L_v\Delta v + L_p\Delta p + L_r\Delta r + \Delta L_c$$

$$\Delta M = M_u\Delta u + M_w\Delta w + M_{\dot{w}}\Delta\dot{w} + M_q\Delta q + \Delta M_c$$

$$\Delta N = N_v\Delta v + N_p\Delta p + N_r\Delta r + \Delta N_c$$

Dimensional Stability Derivative

$$X_u \equiv \left.\frac{\partial X}{\partial u}\right|_0$$

Nondim Stab Deriv.

$$C_{x_u} \equiv \left.\frac{\partial C_x}{\partial \hat{u}}\right|_0$$

Table 4.4
Longitudinal Dimensional Derivatives

	X X_u	Z	M
u	$\rho u_0 S C_{w_0}\sin\theta_0 + \frac{1}{2}\rho u_0 S C_{x_u}$	$-\rho u_0 S C_{w_0}\cos\theta_0 + \frac{1}{2}\rho u_0 S C_{z_u}$	$\frac{1}{2}\rho u_0 \bar{c} S C_{m_u}$
w	$\frac{1}{2}\rho u_0 S C_{x_\alpha}$	$\frac{1}{2}\rho u_0 S C_{z_\alpha}$	$\frac{1}{2}\rho u_0 \bar{c} S C_{m_\alpha}$
q	$\frac{1}{4}\rho u_0 \bar{c} S C_{x_q}$	$\frac{1}{4}\rho u_0 \bar{c} S C_{z_q}$	$\frac{1}{4}\rho u_0 \bar{c}^2 S C_{m_q}$
$\dot{w}$	$\frac{1}{4}\rho \bar{c} S C_{x_{\dot{\alpha}}}$	$\frac{1}{4}\rho \bar{c} S C_{z_{\dot{\alpha}}}$	$\frac{1}{4}\rho \bar{c}^2 S C_{m_{\dot{\alpha}}}$

$$Z_u \equiv \left.\frac{\partial Z}{\partial u}\right|_0 \qquad Z = \frac{1}{2}\rho \underline{V^2} S \underline{C_Z}$$

$$\left.\frac{\partial Z}{\partial u}\right|_0 = \frac{1}{2}\rho S\left(\left.\frac{\partial V^2}{\partial u}\right|_0 C_Z + V^2 \left.\frac{\partial C_Z}{\partial u}\right|_0\right)$$

$$= \frac{1}{2}\rho S\left(2u_0 C_{Z_0} + u_0^2 \left.\frac{\partial C_Z}{\partial u}\right|_0\right)$$

$$C_{Z_u} \equiv \left.\frac{\partial C_Z}{\partial \hat{u}}\right|_0 = \underbrace{u_0\left.\frac{\partial C_Z}{\partial u}\right|_0}$$