Longitudinal Stability

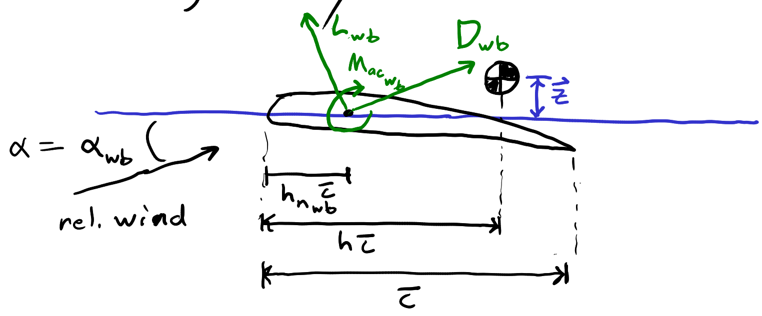
So far : C_L, C_D, C_m — Steady state forces

Now : C_{L_α}, C_{m_α}, $C_{L_{\delta_e}}$, $C_{m_{\delta_e}}$ — trim, static stability

static margin : $h_n - h$

Three contributors
1. Wing/body
2. Propulsion (often small)
3. Tail

Wing/body



Aerodynamic Center
$\frac{\partial \text{Moment}}{\partial \alpha} = 0$
Location stays constant
Moment usually < 0

Center of Pressure
Moment $= 0$
Changes w/ α

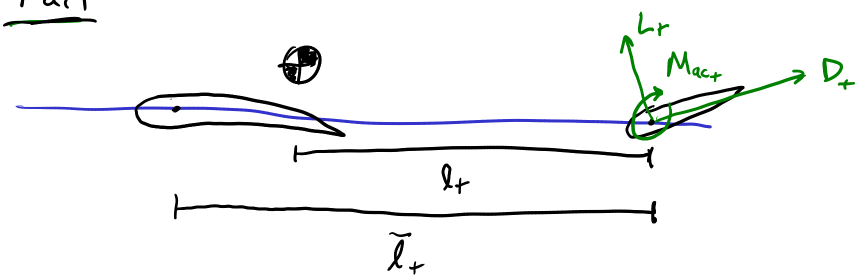
Neutral Point
C.G. location that yields $C_{m_\alpha} = 0$
A.C. of entire A/c

$$M_{wb} = M_{ac_{wb}} + (L\cos\alpha + D\sin\alpha)(h - h_{n_{wb}})\bar{c} + (L\sin\alpha - D\cos\alpha)\bar{z}$$

only keep most important terms + nondimensionalize

$$C_{m_{wb}} = C_{m_{ac_{wb}}} + C_{L_{wb}}(h - h_{n_{wb}})$$

Tail



$$L = L_{wb} + L_t$$
$$= C_{L_{wb}}\left(\tfrac{1}{2}\rho V^2 S\right) + C_{L_t}\left(\tfrac{1}{2}\rho V^2 S_t\right)$$
$$= C_L\left(\tfrac{1}{2}\rho V^2 S\right)$$

$$\Longrightarrow \quad C_L = C_{L_{wb}} + \frac{S_t}{S} C_{L_t}$$

$$M_t = -l_t L_t = -l_t C_{L_t}\left(\tfrac{1}{2}\rho V^2 S_t\right) \Longrightarrow C_{m_t} = \underbrace{\frac{-l_t}{\bar{c}}\frac{S_t}{S}}\, C_{L_t}$$

$$= -V_H C_{L_t}$$

"volume ratio"

more convenient b/c C.G. can change

$$\bar{V}_H = \frac{\bar{l}_t}{\bar{c}}\frac{S_t}{S} \Longrightarrow V_H = \bar{V}_H - \frac{S_t}{S}(h - h_{n_{wb}})$$

$$C_{m_t} = -\bar{V}_H C_{L_t} + C_{L_t}\frac{S_t}{S}(h - h_{n_{wb}})$$

$$\rightarrow\; C_m = \overbrace{C_{m_{ac_{wb}}} + C_L(h - h_{n_{wb}})}^{\text{Wing/body}} \underbrace{- \bar{V}_H C_{L_t}}_{\text{Tail}} + \underbrace{C_{m_p}}_{\text{propulsion}}$$

$$C_{m_\alpha} = \underbrace{\cancelto{0}{\frac{\partial C_{m_{ac_{wb}}}}{\partial \alpha}}} + C_{L_\alpha}(h - h_{n_{wb}}) - \bar{V}_H \frac{\partial C_{L_t}}{\partial \alpha} + \frac{\partial C_{m_p}}{\partial \alpha}$$

Want $h_n \equiv$ cg location where $C_{m_\alpha} = 0$

$$0 = C_{L_\alpha}(h_n - h_{n_{wb}}) - \bar{V}_H \frac{\partial C_{L_t}}{\partial \alpha} + \frac{\partial C_{m_p}}{\partial \alpha}$$

$$h_n = h_{n_{wb}} + \frac{1}{C_{L_\alpha}}\left(\underbrace{\bar{V}_H \frac{\partial C_{L_t}}{\partial \alpha}}_{\text{tail correction}} - \frac{\partial C_{m_p}}{\partial \alpha}\right)$$

$$\boxed{C_{m_\alpha} = C_{L_\alpha}(h - h_n)}$$

$C_{m_\alpha} < 0$ for stability

static margin

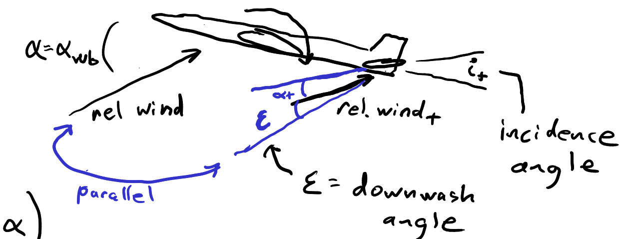
$K_n \equiv h_n - h$ must be > 0 for stability

back to h_n eq.

Lift curve slope $a \equiv C_{L_\alpha}$

$$C_{L_{wb}} = a_{wb}\alpha_{wb} = a_{wb}\alpha$$

$$C_{L_t} = a_t \alpha_t$$



$$\alpha_t = \alpha - i_t - \left(\varepsilon_{zero} + \frac{\partial \varepsilon}{\partial \alpha}\alpha\right)$$

$$\frac{\partial C_{L_t}}{\partial \alpha} = a_t\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right)$$

$$\boxed{h_n = h_{n_{wb}} + \frac{a_t}{a}\bar{V}_H\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right) - \frac{1}{a}\frac{\partial C_{m_p}}{\partial \alpha}}$$

$$\boxed{C_{L_\alpha} = a = a_{wb}\left[1 + \frac{a_t S_t}{a_{wb} S}\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right)\right]}$$

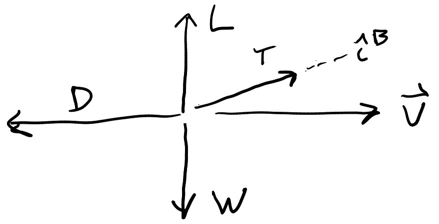
Longitudinal Control

δ_e changes C_{L_t}

$$C_{L_{\delta_e}} = \frac{\partial C_{L_t}}{\partial \delta_e}\frac{S_t}{S} = a_e \frac{S_t}{S}$$

$$C_{m_{\delta_e}} = -a_e \bar{V}_H + C_{L_{\delta_e}}(h - h_{n_{wb}})$$

Linear Trim Estimation



For Linear trim, $T = D$, $L = W$

0 for Linear Trim Estimation

$$C_{L_{trim}} = \frac{W}{\frac{1}{2}\rho V^2 S} = \cancel{C_{L_{zero}}} + C_{L_\alpha}\alpha_{trim} + C_{L_{\delta e}}\delta_{e_{trim}}$$

$$C_{m_{trim}} = C_{m_{zero}} + C_{m_\alpha}\alpha_{trim} + C_{m_{\delta e}}\delta_{e_{trim}} = 0$$

$$\begin{bmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{bmatrix}\begin{bmatrix} \alpha_{trim} \\ \delta_{e_{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_{trim}} \\ -C_{m_{zero}} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Cramer's rule)

$$\begin{bmatrix} \alpha_{trim} \\ \delta_{e_{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{bmatrix}^{-1}\begin{bmatrix} C_{L_{trim}} \\ -C_{m_{zero}} \end{bmatrix}$$

$$\alpha_{trim} = \frac{C_{m_{zero}} C_{L_{\delta e}} + C_{m_{\delta e}} C_{L_{trim}}}{\Delta}$$

$$\Delta = C_{L_\alpha} C_{m_{\delta e}} - C_{L_{\delta e}} C_{m_\alpha}$$

$$\delta_{e_{trim}} = -\frac{C_{m_{zero}} C_{L_\alpha} + C_{m_\alpha} C_{L_{trim}}}{\Delta}$$