# Longitudinal Dynamics

## Last Time

$$C_L = C_{L_\alpha}\alpha + C_{L_{\delta e}}\delta e \qquad = C_{L_{trim}}$$

$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_{\delta e}}\delta e \qquad = 0$$

$$C_{m_\alpha} = C_{L_\alpha}(h - h_n)$$

Today: Linear Longitudinal EOM

Stability Derivatives

$$\dot{\vec{p}}_E = R_B^E \vec{V}_B^E$$

$$\dot{\vec{O}} = T\vec{\omega}_B$$

$$\dot{\vec{V}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{V}_B^E$$

$$\dot{\vec{\omega}}_B = I^{-1}\left[\vec{G}_B - \vec{\omega}_x^{\times} I \vec{\omega}_B\right]$$

Symmetry about x-z axis $\qquad I_{xy} = I_{yz} = 0$



$$I_B^{-1} = \begin{bmatrix} \frac{I_z}{\Gamma} & 0 & \frac{I_{xz}}{\Gamma} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{\Gamma} & 0 & \frac{I_x}{\Gamma} \end{bmatrix} \qquad \Gamma = I_x I_z - I_{xz}^2$$

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma} \qquad \Gamma_4 = \frac{I_{xz}}{\Gamma} \qquad \Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma} \qquad \Gamma_5 = \frac{I_z - I_x}{I_y} \qquad \Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \qquad \Gamma_6 = \frac{I_{xz}}{I_y} \qquad \Gamma = I_x I_z - I_{xz}^2$$

$$\begin{pmatrix}\dot x_E\\ \dot y_E\\ \dot z_E\end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta\end{pmatrix}\begin{pmatrix}u^E\\ v^E\\ w^E\end{pmatrix}$$

$$\begin{pmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{pmatrix} = \begin{pmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}$$

$$\begin{pmatrix}\dot u^E\\ \dot v^E\\ \dot w^E\end{pmatrix} = \begin{pmatrix}rv^E - qw^E\\ pw^E - ru^E\\ qu^E - pv^E\end{pmatrix} + g\begin{pmatrix}-\sin\theta\\ \cos\theta\sin\phi\\ \cos\theta\cos\phi\end{pmatrix} + \frac{1}{m}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}$$

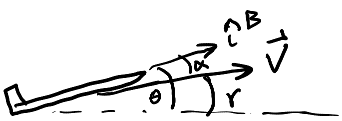
$$\begin{pmatrix}\dot p\\ \dot q\\ \dot r\end{pmatrix} = \begin{pmatrix}\Gamma_1 pq - \Gamma_2 qr\\ \Gamma_5 pr - \Gamma_6(p^2 - r^2)\\ \Gamma_7 pq - \Gamma_1 qr\end{pmatrix} + \begin{pmatrix}\Gamma_3 L + \Gamma_4 N\\ \frac{1}{I_y}M\\ \Gamma_4 L + \Gamma_8 N\end{pmatrix}$$

$V = u_0$

Trim State

Inputs $u_0$, $h_0$, $\gamma_{a_0}$ $\xrightarrow{\text{determine}}$ $\alpha_0$, $\delta e_0$, $\delta t_0$

speed  altitude  air relative flight path angle




$$\vec{x} = \begin{bmatrix}x_E\\ y_E\\ z_E\\ \phi\\ \theta\\ \psi\\ u^E\\ v^E\\ w^E\\ p\\ q\\ r\end{bmatrix} = \vec{x}_0 + \begin{bmatrix}\Delta x_E\\ \Delta y_E\\ \Delta z_E\\ \Delta\phi\\ \Delta\theta\\ \Delta\psi\\ \Delta u^E\\ \Delta v^E\\ \Delta w^E\\ \Delta p\\ \Delta q\\ \Delta r\end{bmatrix} \qquad \vec{x}_0 = \begin{bmatrix}\cdot\\ \cdot\\ -h_0\\ 0\\ \theta_0\\ \cdot\\ u_0^E\\ 0\\ w_0^E\\ 0\\ 0\\ 0\end{bmatrix} \qquad \vec{u}_0 = \begin{bmatrix}\delta e_0\\ 0\\ 0\\ \delta t_0\end{bmatrix}$$

$$\cancel{\dot\theta_0}^{\,0} + \Delta\dot\theta = \cancel{\cos\theta_0 q_0}^{\,0} + \cancel{\frac{\partial\cos\phi q}{\partial\phi}\Big|_0}^{\,0}\Delta\phi + \cancel{\frac{\partial\cos\phi q}{q}\Big|_0}^{\,1}\Delta q - \cancel{\sin\theta_0 r_0}^{\,0} - \cancel{\frac{\partial\sin\phi r}{\partial\phi}\Big|_0}^{\,0}\Delta\phi - \cancel{\frac{\partial\sin\phi r}{\partial r}\Big|_0}^{\,0}\Delta r$$

$$\Delta\dot\theta = \Delta q$$

$$\cancel{\dot u_0}^{\,0} + \Delta\dot u = \cancel{r_0 v_0}^{\,0} + \cancel{r_0\Delta v}^{\,0} + \cancel{v_0\Delta r}^{\,0} + \cancel{\Delta r\Delta v}^{\,0} - \cancel{q_0 w_0}^{\,0} - \cancel{q_0\Delta w}^{\,0} - \cancel{w_0\Delta q}^{\,0} - \cancel{\Delta q\Delta w}^{\,0}$$
$$\underbrace{-g\sin\theta_0}_{\text{cancel}} - g\cos\theta_0\Delta\theta + \underbrace{\tfrac{1}{m}(X_0}_{\text{cancel}} + \Delta X)$$

$$\boxed{\Delta\dot u = -g\cos\theta_0\Delta\theta + \frac{1}{m}\Delta X}$$

Longitudinal

Lat-d

$X_u \equiv \frac{\partial X}{\partial u}$

$$\begin{aligned}
\Delta\dot\phi &= \Delta p + \Delta r\tan\theta_0\\
\Delta\dot\theta &= \Delta q\\
\Delta\dot u &= -g\cos\theta_0\Delta\theta + \frac{\Delta X}{m}\\
\Delta\dot v &= -u_0\Delta r + g\cos\theta_0\Delta\phi + \frac{\Delta Y}{m}\\
\Delta\dot w &= u_0\Delta q - g\sin\theta_0\Delta\theta + \frac{\Delta Z}{m}\\
\Delta\dot p &= \Gamma_3\Delta L + \Gamma_4\Delta N\\
\Delta\dot q &= \frac{\Delta M}{I_y}\\
\Delta\dot r &= \Gamma_4\Delta L + \Gamma_8\Delta N
\end{aligned}$$

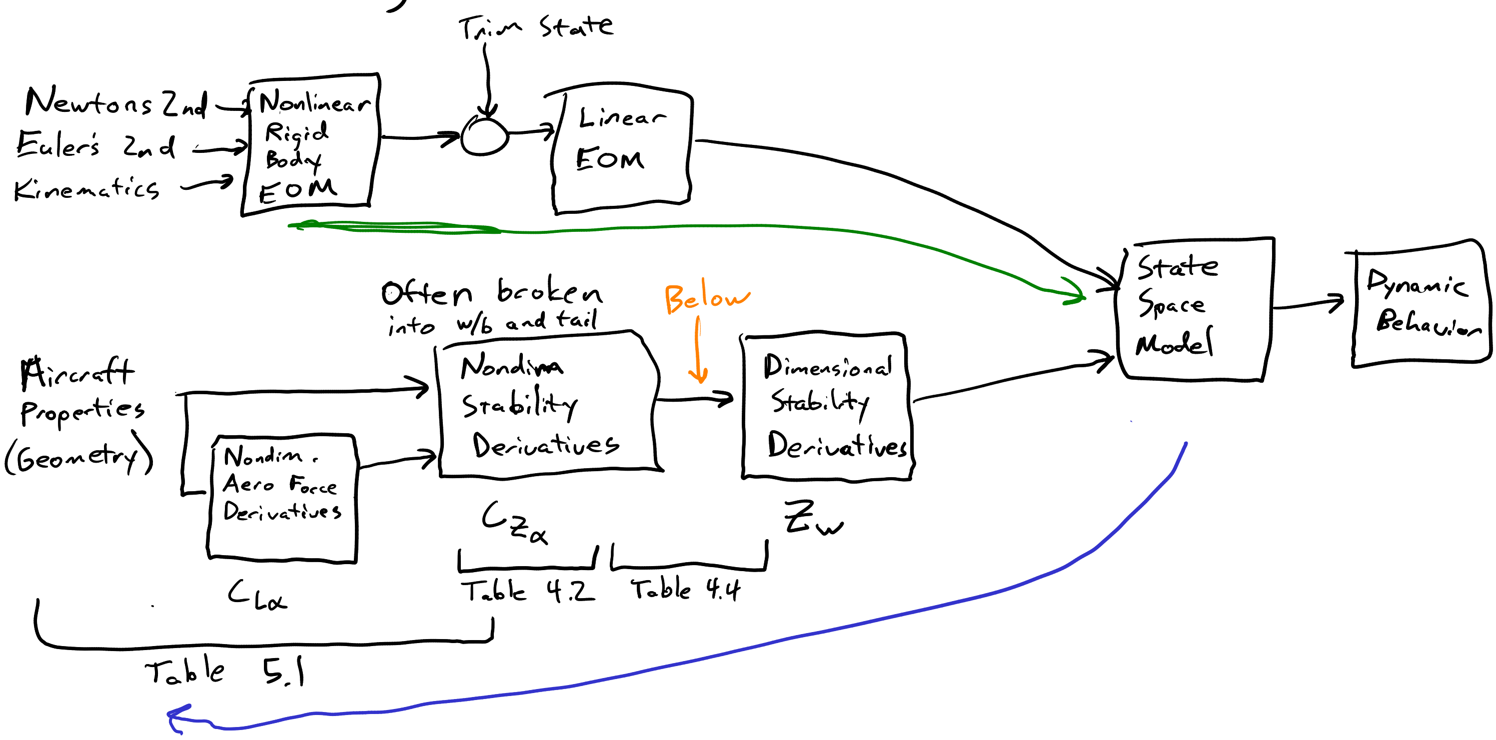
$$\begin{aligned}
\Delta X &= X_u\Delta u + X_w\Delta w + \Delta X_c\\
\Delta Y &= Y_v\Delta v + Y_p\Delta p + Y_r\Delta r + \Delta Y_c\\
\Delta Z &= Z_u\Delta u + Z_w\Delta w + Z_{\dot w}\Delta\dot w + Z_q\Delta q + \Delta Z_c\\
\Delta L &= L_v\Delta v + L_p\Delta p + L_r\Delta r + \Delta L_c\\
\Delta M &= M_u\Delta u + M_w\Delta w + M_{\dot w}\Delta\dot w + M_q\Delta q + \Delta M_c\\
\Delta N &= N_v\Delta v + N_p\Delta p + N_r\Delta r + \Delta N_c
\end{aligned}$$

*Dynamics of Flight*, Eq. (4.9,18) $\quad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + mu_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + mu_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}}mg\sin\theta_0}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

## Big Picture




| Variable | Divisor | Non-dim Variable |
|---|---|---|
| $X, Y, Z$ | $\frac{1}{2}\rho V^2 S$ | $C_x, C_y, C_z$ |
| $W$ | $\frac{1}{2}\rho V^2 S$ | $C_W$ |
| $M$ | $\frac{1}{2}\rho V^2 S\bar{c}$ | $C_m$ |
| $L, N$ | $\frac{1}{2}\rho V^2 S\bar{b}$ | $C_l, C_n$ |
| $u, v, w$ | $V$ | $\hat{u}, \hat{v}, \underline{\hat{w}}$ |
| $\dot{\alpha}, q$ | $2V/\bar{c}$ | $\hat{\dot{\alpha}}, \hat{q}$ |
| $\dot{\beta}, p, r$ | $2V/b$ | $\hat{\dot{\beta}}, \hat{p}, \hat{r}$ |
| $m$ | $\rho S\bar{c}/2$ | $\mu$ |
| $I_y$ | $\rho S(\bar{c}/2)^3$ | $\hat{I}_y$ |
| $I_x, I_z, I_{xz}$ | $\rho S(b/2)^3$ | $\hat{I}_x, \hat{I}_z, \hat{I}_{xz}$ |

$$C_{z_u} = \frac{\partial C_z}{\partial \hat{u}}$$

Wind-Angle Approximations

$$\Delta\alpha = \tan^{-1}\frac{\Delta w}{V} \approx \hat{w}$$

$$\Delta\beta = \sin^{-1}\frac{\Delta v}{V} \approx \hat{v}$$

***Table 4.4***
**Longitudinal Dimensional Derivatives**

$Z_u$

| | $X$ | $Z$ | $M$ |
|---|---|---|---|
| $u$ | $\rho u_0 S C_{w_0} \sin\theta_0 + \frac{1}{2}\rho u_0 S C_{x_u}$ | $\boxed{-\rho u_0 S C_{w_0} \cos\theta_0 + \frac{1}{2}\rho u_0 S C_{z_u}}$ | $\frac{1}{2}\rho u_0 \bar{c} S C_{m_u}$ |
| $w$ | $\frac{1}{2}\rho u_0 S C_{x_\alpha}$ | $\frac{1}{2}\rho u_0 S C_{z_\alpha}$ | $\frac{1}{2}\rho u_0 \bar{c} S C_{m_\alpha}$ |
| $q$ | $\frac{1}{4}\rho u_0 \bar{c} S C_{x_q}$ | $\frac{1}{4}\rho u_0 \bar{c} S C_{z_q}$ | $\frac{1}{4}\rho u_0 \bar{c}^2 S C_{m_q}$ |
| $\dot{w}$ | $\frac{1}{4}\rho \bar{c} S C_{x_{\dot\alpha}}$ | $\frac{1}{4}\rho \bar{c} S C_{z_{\dot\alpha}}$ | $\frac{1}{4}\rho \bar{c}^2 S C_{m_{\dot\alpha}}$ |

$$Z_u \equiv \left.\frac{\partial Z}{\partial u}\right|_0 \qquad Z = \frac{1}{2}\rho V^2 S C_Z$$

$$\left.\frac{\partial Z}{\partial u}\right|_0 = \frac{1}{2}\rho S\left(\left.\frac{\partial V^2}{\partial u}\right|_0 C_Z + \left.\frac{\partial C_Z}{\partial u}\right|_0 V^2\right)$$

$$= \frac{1}{2}\rho S\, 2u_0 C_{Z_0} + \frac{1}{2}\rho u_0^2 S \left.\frac{\partial C_Z}{\partial u}\right|_0$$

$$\boxed{Z_u = -\rho u_0 S C_{w_0}\cos\theta_0 + \frac{1}{2}\rho u_0 S C_{Z_u}}$$

$$\frac{\partial f(x)g(x)}{\partial x} = f(x)\frac{\partial g(x)}{\partial x} + \frac{\partial f(x)}{\partial x}g(x)$$

$$C_{Z_u} \equiv \frac{\partial C_Z}{\partial \hat{u}}$$

$C_{Z_u}$

$$\frac{\partial C_Z}{\partial u} = \frac{\partial C_Z}{\partial \hat{u} u_0} = \frac{1}{u_0}\frac{\partial C_Z}{\partial \hat{u}}$$

$$C_{Z_0} = -C_{w_0}\cos\theta_0$$

***Table 5.1***
**Summary—Longitudinal Derivatives**

| | $C_x$ | $C_z$ | $C_m$ |
|---|---|---|---|
| $\hat{u}$† | $\mathbf{M}_0\left(\frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}}\right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u}\left(1 - \frac{\partial C_D}{\partial C_T}\right)$ | $-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u}\frac{\partial C_L}{\partial C_T}$ | $\mathbf{M}_0 \frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u}\frac{\partial C_m}{\partial C_T}$ |
| $\alpha$ | $C_{l_0} - C_{D_\alpha}$ | $-(C_{L_\alpha} + C_{D_0})$ | $-a(h_n - h)$ |
| $\dot\alpha$ | Neg. | $*-2a_t V_H \frac{\partial \epsilon}{\partial \alpha}$ | $*-2a_t V_H \frac{l_t}{\bar{c}}\frac{\partial \epsilon}{\partial \alpha}$ |
| $\hat{q}$ | Neg. | $*-2a_t V_H$ | $*-2a_t V_H \frac{l_t}{\bar{c}}$ |

Neg. means usually negligible.

*means contribution of the *tail only,* formula for wing-body not available.

†$C_{T_u} = \dfrac{(\partial T/\partial u)_0}{\frac{1}{2}\rho u_0 S} - 2C_{T_0}$; $C_{T_0} = C_{D_0} + C_{w_0}\sin\theta_0$