Nondimensional Longitudinal Stability Derivatives

Table 5.1
Summary—Longitudinal Derivatives

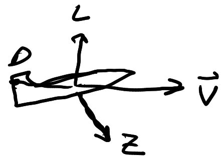
	C_x	C_z	C_m
$\hat{u}$†	$\mathbf{M}_0\left(\frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}}\right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u}\left(1 - \frac{\partial C_D}{\partial C_T}\right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u} \frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0 \frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u} \frac{\partial C_m}{\partial C_T}$
α	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot{\alpha}$	Neg.	$*-2a_t V_H \frac{\partial \epsilon}{\partial \alpha}$	$*-2a_t V_H \frac{l_t}{\bar{c}} \frac{\partial \epsilon}{\partial \alpha}$
$\hat{q}$	Neg.	$*-2a_t V_H$	$*-2a_t V_H \frac{l_t}{\bar{c}}$

Neg. means usually negligible.
*means contribution of the *tail only,* formula for wing-body not available.
$\dagger C_{T_u} = \frac{(\partial T/\partial u)_0}{\frac{1}{2}\rho u_0 S} - 2C_{T_0};\ C_{T_0} = C_{D_0} + C_{w_0}\sin\theta_0$

α derivatives

$$C_{m_\alpha} = C_{L_\alpha}(h - h_N)$$

C_{z_α}



$$Z = -L\cos\alpha - D\sin\alpha$$
$$C_Z = -(C_L\cos\alpha + C_D\sin\alpha)$$
$$= -(C_L + C_D\alpha)$$
$$C_{z_\alpha} \equiv \left.\frac{\partial C_z}{\partial \alpha}\right|_0 = -\left(C_{L_\alpha} + C_{D_0} + \cancel{\alpha_0 \frac{\partial C_D}{\partial \alpha}}^{\,0}\right)$$
$$\boxed{C_{z_\alpha} = -(C_{L_\alpha} + C_{D_0})}$$

u derivatives

3 important factors:
- Compressibility : Mach Number — $M \equiv \frac{V}{a}$ ← speed of sound
- Dynamic Pressure : $p_d = \frac{1}{2}\rho V^2$
- Thrust

- Different from the dynamic pressure in nondimensionalization
Changes in C_L, C_D, etc. due to changes in dynamic pressure

$$C_{x_u} \equiv \left.\frac{\partial C_x}{\partial \hat{u}}\right|_0$$

$* \in \{x, z, m\}$

$$C_{*_u} = \left.\frac{\partial C_*}{\partial M}\right|_0 \left.\frac{\partial M}{\partial \hat{u}}\right|_0 + \left.\frac{\partial C_*}{\partial p_d}\right|_0 \left.\frac{\partial p_d}{\partial \hat{u}}\right|_0 + \left.\frac{\partial C_*}{\partial C_T}\right|_0 \left.\frac{\partial C_T}{\partial \hat{u}}\right|_0$$

$$\rightarrow \left.\frac{\partial M}{\partial \hat{u}}\right|_0 = u_0 \left.\frac{\partial M}{\partial u}\right|_0 = \frac{u_0}{a} \left.\frac{\partial V}{\partial u}\right|_0 = M_0$$

$$\frac{\partial p_d}{\partial \hat{u}}\Big|_0 = u_0 \frac{\partial p_d}{\partial u}\Big|_0 = u_0 \frac{1}{2}\rho \frac{\partial V^2}{\partial u}\Big|_0 = u_0 \rho u_0 = \rho u_0^2$$

$C_T = \frac{T}{\frac{1}{2}\rho V^2 S}$

$$\frac{\partial C_T}{\partial \hat{u}}\Big|_0 = u_0 \frac{\partial C_T}{\partial u}\Big|_0 = u_0 \left(\frac{\partial T/\partial u}{\frac{1}{2}\rho V^2 S} - \frac{2T}{\frac{1}{2}\rho V^3 S} \right)\Big|_0 = \frac{\partial T/\partial u}{\frac{1}{2}\rho u_0 S}\Big|_0 - 2C_{T_0}$$

$$\frac{\partial}{\partial x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$C_{T_0} = C_{D_0} + C_{W_0}\sin\theta_0$

3 cases

Gliding case: $\boxed{C_{T_u} = 0}$

Constant Thrust (Jet): $\boxed{C_{T_u} = -2C_{T_0}}$

Constant Power (Prop): $\boxed{C_{T_u} = -3C_{T_0}}$

TV = constant

$$\frac{\partial T}{\partial u}\Big|_0 = -\frac{T_0}{u_0}$$

C_{x_u} $\qquad C_x \approx C_T - C_D$

$$\frac{\partial C_x}{\partial M}\Big|_0 = \frac{\partial C_T}{\partial M}\Big|_0 - \frac{\partial C_D}{\partial M}\Big|_0$$

$$\frac{\partial C_x}{\partial p_d}\Big|_0 = \cancel{\frac{\partial C_T}{\partial p_d}}^{\text{Neg}}\Big|_0 - \frac{\partial C_D}{\partial p_d}\Big|_0$$

$$\frac{\partial C_x}{\partial C_T}\Big|_0 = 1 - \frac{\partial C_D}{\partial C_T}\Big|_0$$

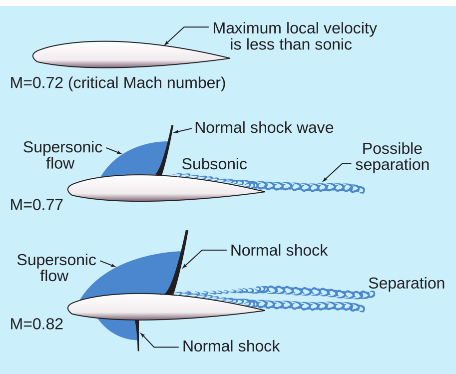
$$\boxed{C_{x_u} = M_0\left(\frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M}\right)\Big|_0 - \rho u_0^2 \frac{\partial C_D}{\partial p_d}\Big|_0 + C_{T_u}\left(1 - \frac{\partial C_D}{\partial C_T}\Big|_0\right)}$$

C_{z_u} $\qquad$ Assume $C_z = -C_L$

$$\boxed{C_{z_u} = -\underbrace{M_0 \frac{\partial C_L}{\partial M}\Big|_0}_{\text{small except transonic}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d}\Big|_0 - C_{T_u}\frac{\partial C_L}{\partial C_T}\Big|_0}$$

C_{m_u}

$$\boxed{C_{m_u} = \underbrace{M_0 \frac{\partial C_m}{\partial M}\Big|_0}_{\text{Mach Tuck}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d}\Big|_0 + C_{T_u}\frac{\partial C_m}{\partial C_T}\Big|_0}$$



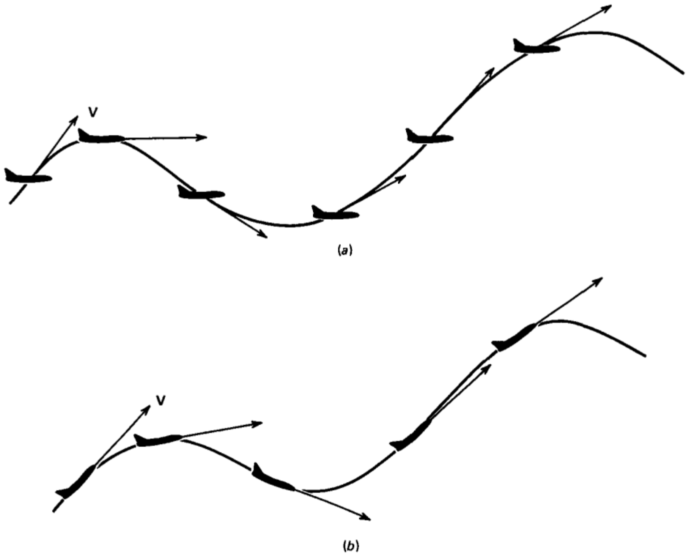
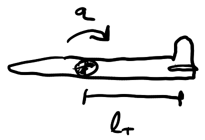


Figure 5.2 (a) Motion with zero q, but varying α_x. (b) Motion with zero α_x but varying q.

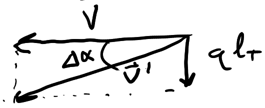
q- derivatives

Wing-body || Tail

Tail



velocity observed by tail



$$\Delta C_{L_t} = a_t \Delta\alpha = a_t \tan^{-1}\frac{q l_t}{u_0} \approx a_t \frac{q l_t}{u_0}$$

$$\Delta C_L = \frac{S_t}{S}\Delta C_{L_t} = \frac{S_t}{S} a_t \frac{q l_t}{u_0}$$

$$V_H = \frac{S_t l_t}{S\bar{c}}$$

$(C_{Z_q})_{tail}$

$$C_{Z_q} = \frac{\partial C_Z}{\partial \hat{q}}\bigg|_0 = \frac{2u_0}{\bar{c}}\frac{\partial C_Z}{\partial q}\bigg|_0 = -\frac{2u_0}{\bar{c}}\frac{\partial C_L}{\partial q}\bigg|_0$$

$$(C_{Z_q})_{tail} = \frac{-2u_0}{\bar{c}} a_t \frac{S_t l_t}{S u_0} = \boxed{-2 a_t V_H}$$

$(C_{m_q})_{tail}$

$$\Delta C_m = -V_H \Delta C_{L_t} = a_t V_H \frac{q l_t}{u_0}$$

$$C_{m_q} = \frac{\partial C_m}{\partial \hat{q}}\bigg|_0 = \frac{2u_0}{\bar{c}}\frac{\partial C_m}{\partial q}\bigg|_0$$

$$\boxed{(C_{m_q})_{tail} = -2 a_t V_H \frac{l_t}{\bar{c}}}$$

Wing-Body

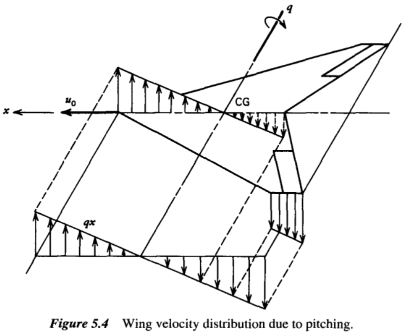


Figure 5.4 Wing velocity distribution due to pitching.

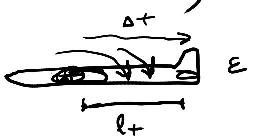
$\dot{\alpha}$ derivatives

Unsteady effects

Wing-Body $\rightarrow$ Determined by initial response or oscillation of wing in wind tunnel or flight test

Tail

Downwash Lag



$$\Delta\varepsilon = -\frac{\partial\varepsilon}{\partial\alpha}\dot{\alpha}\,\Delta t = -\frac{\partial\varepsilon}{\partial\alpha}\dot{\alpha}\frac{l_t}{u_0}$$

$$= -\Delta\alpha_t$$

$(C_{z_{\dot{\alpha}}})_{tail}$

$$\Delta C_{L_t} = a_t\Delta\alpha_t = a_t\dot{\alpha}\frac{l_t}{u_0}\frac{\partial\varepsilon}{\partial\alpha}$$

$$\rightarrow \Delta C_L = a_t\dot{\alpha}\frac{l_t S_t}{u_0 S}\frac{\partial\varepsilon}{\partial\alpha}$$

$$C_{z_{\dot{\alpha}}} = \left.\frac{\partial C_z}{\partial \frac{\dot{\alpha}\bar{c}}{2u_0}}\right|_0 = \frac{2u_0}{\bar{c}}\frac{\partial C_z}{\partial\dot{\alpha}} = -2a_t\frac{l_t S_t}{\bar{c} S}\frac{\partial\varepsilon}{\partial\alpha}$$

$$\boxed{(C_{z_{\dot{\alpha}}})_{tail} = -2a_t V_H\frac{\partial\varepsilon}{\partial\alpha}}$$

$$\boxed{(C_{m_{\dot{\alpha}}})_{tail} = -2a_t V_H\frac{l_t}{\bar{c}}\frac{\partial\varepsilon}{\partial\alpha}}$$