Controlled Quadrotor ODEs

Aircraft Dynamics
UNIVERSITY OF COLORADO **BOULDER**

Last time: Linearization

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \frac{1}{m}\begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L \\ \frac{1}{I_y} M \\ \frac{1}{I_z} N \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

Nonlinear Quadrotor Equations of Motion

Hover Trim Condition
+
Taylor Series Expansion

$$\begin{pmatrix} \Delta\dot{x}_E \\ \Delta\dot{y}_E \\ \Delta\dot{z}_E \end{pmatrix} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \end{pmatrix}$$

$$\begin{pmatrix} \Delta\dot{\phi} \\ \Delta\dot{\theta} \\ \Delta\dot{\psi} \end{pmatrix} = \begin{pmatrix} \Delta p \\ \Delta q \\ \Delta r \end{pmatrix}$$

$$\begin{pmatrix} \Delta\dot{u} \\ \Delta\dot{v} \\ \Delta\dot{w} \end{pmatrix} = g \begin{pmatrix} -\Delta\theta \\ \Delta\phi \\ 0 \end{pmatrix} + \frac{1}{m}\begin{pmatrix} 0 \\ 0 \\ \Delta Z_c \end{pmatrix}$$

$$\begin{pmatrix} \Delta\dot{p} \\ \Delta\dot{q} \\ \Delta\dot{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{I_x}\Delta L_c \\ \frac{1}{I_y}\Delta M_c \\ \frac{1}{I_z}\Delta N_c \end{pmatrix}$$

Longitudinal

$$\begin{pmatrix} \Delta\dot{x}_E \\ \Delta\dot{u} \\ \Delta\dot{\theta} \\ \Delta\dot{q} \end{pmatrix} = \begin{pmatrix} \Delta u \\ -g\Delta\theta \\ \Delta q \\ \frac{1}{I_y}\Delta M_c \end{pmatrix}$$

Lateral

$$\begin{pmatrix} \Delta\dot{y}_E \\ \Delta\dot{v} \\ \Delta\dot{\phi} \\ \Delta\dot{p} \end{pmatrix} = \begin{pmatrix} \Delta v \\ g\Delta\phi \\ \Delta p \\ \frac{1}{I_x}\Delta L_c \end{pmatrix}$$

Vertical

$$\begin{pmatrix} \Delta\dot{z}_E \\ \Delta\dot{w} \end{pmatrix} = \begin{pmatrix} \Delta w \\ \frac{1}{m}\Delta Z_c \end{pmatrix}$$

Spin

$$\begin{pmatrix} \Delta\dot{\psi} \\ \Delta\dot{r} \end{pmatrix} = \begin{pmatrix} \Delta r \\ \frac{1}{I_z}\Delta N_c \end{pmatrix}$$

Linearized Quadrotor EOM

Longitudinal

$$\begin{pmatrix} \Delta\dot{x}_E \\ \Delta\dot{u} \\ \Delta\dot{\theta} \\ \Delta\dot{q} \end{pmatrix} = \begin{pmatrix} \Delta u \\ -g\Delta\theta \\ \Delta q \\ \frac{1}{I_y}\Delta M_c \end{pmatrix}$$

Lateral

$$\begin{pmatrix} \Delta\dot{y}_E \\ \Delta\dot{v} \\ \Delta\dot{\phi} \\ \Delta\dot{p} \end{pmatrix} = \begin{pmatrix} \Delta v \\ g\Delta\phi \\ \Delta p \\ \frac{1}{I_x}\Delta L_c \end{pmatrix}$$

Today: Solution of 2nd order linear ODEs

Vertical

$$\begin{pmatrix} \Delta\dot{z}_E \\ \Delta\dot{w} \end{pmatrix} = \begin{pmatrix} \Delta w \\ \frac{1}{m}\Delta Z_c \end{pmatrix}$$

Spin

$$\begin{pmatrix} \Delta\dot{\psi} \\ \Delta\dot{r} \end{pmatrix} = \begin{pmatrix} \Delta r \\ \frac{1}{I_z}\Delta N_c \end{pmatrix}$$

Linear ODEs

$$\dot{x} = A\,x + B\,u$$
$$y = C\,x + D\,u$$

Aircraft dynamics (mostly) uses this one
It is called a “state space” model.

These contain (roughly) the same information!

Always keep this in mind!

$$\ddot{y} + 2\zeta\,\omega_n\,\dot{y} + \omega_n^2\,y = \omega_n^2 u$$

$$u \longrightarrow \boxed{\dfrac{\omega_n^2}{s^2 + 2\zeta\,\omega_n\,s + \omega_n^2}} \longrightarrow y$$

Roll Angle State Space Model

$$\begin{pmatrix} \Delta\dot{\phi} \\ \Delta\dot{p} \end{pmatrix} = \begin{pmatrix} \Delta p \\ \frac{1}{I_x}\Delta L_c \end{pmatrix}$$

$$\begin{pmatrix} \Delta\dot{\phi} \\ \Delta\dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta\phi \\ \Delta p \end{pmatrix} + \begin{pmatrix} 0 \\ 1/I_x \end{pmatrix} \Delta L_c$$

$$\dot{x} \quad = \quad A \quad x \quad + \quad B \quad u$$

$$\phi = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta\phi \\ \Delta p \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} \Delta L_c$$

$$y \quad = \quad C \quad x \quad + \quad D \quad u$$

Open Loop Solution

$$\begin{pmatrix} \Delta\dot{\phi} \\ \Delta\dot{p} \end{pmatrix} = \begin{pmatrix} \Delta p \\ \frac{1}{I_x}\Delta L_c \end{pmatrix}$$

$$\Delta\ddot{\phi} = \Delta\dot{p}$$

$$\Delta\ddot{\phi} = \frac{1}{I_x}\Delta L_c$$

What are the solutions?

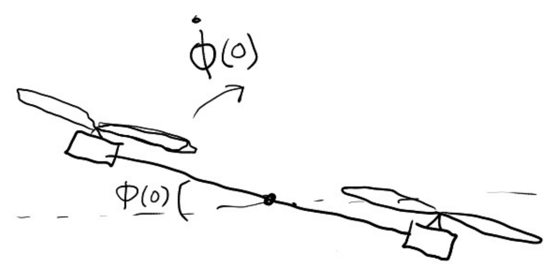
Depends on ΔL_c

$\Delta L_c = 0$
$\Rightarrow \Delta\ddot{\phi} = 0$

(hover control)

$$\Delta\dot{\phi}(t) = \int \Delta\ddot{\phi} = 0 + \Delta\dot{\phi}(0)$$

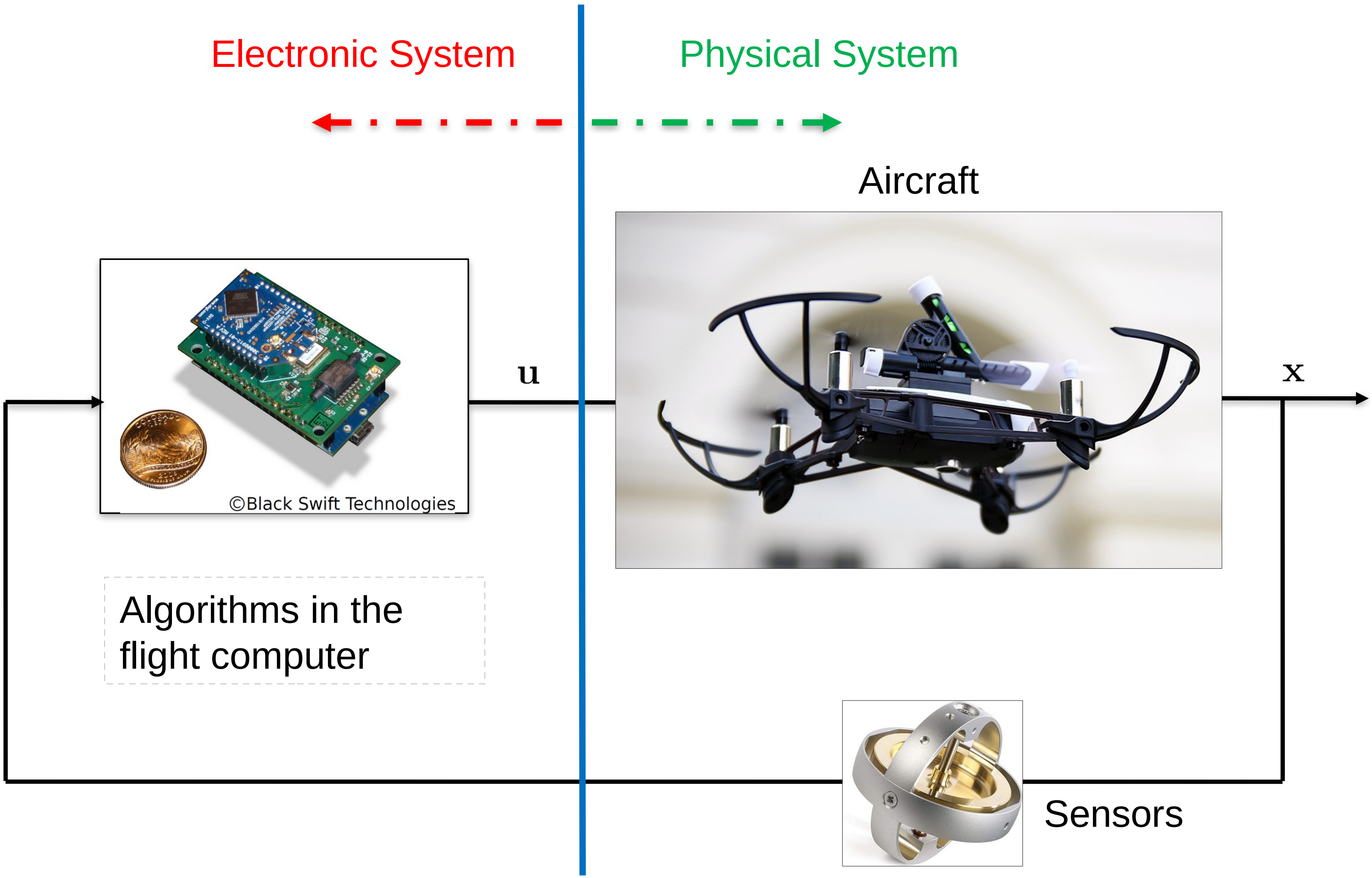
$$\Delta\phi(t) = \int \Delta\dot{\phi} = \Delta\dot{\phi}(0)t + \Delta\phi(0)$$



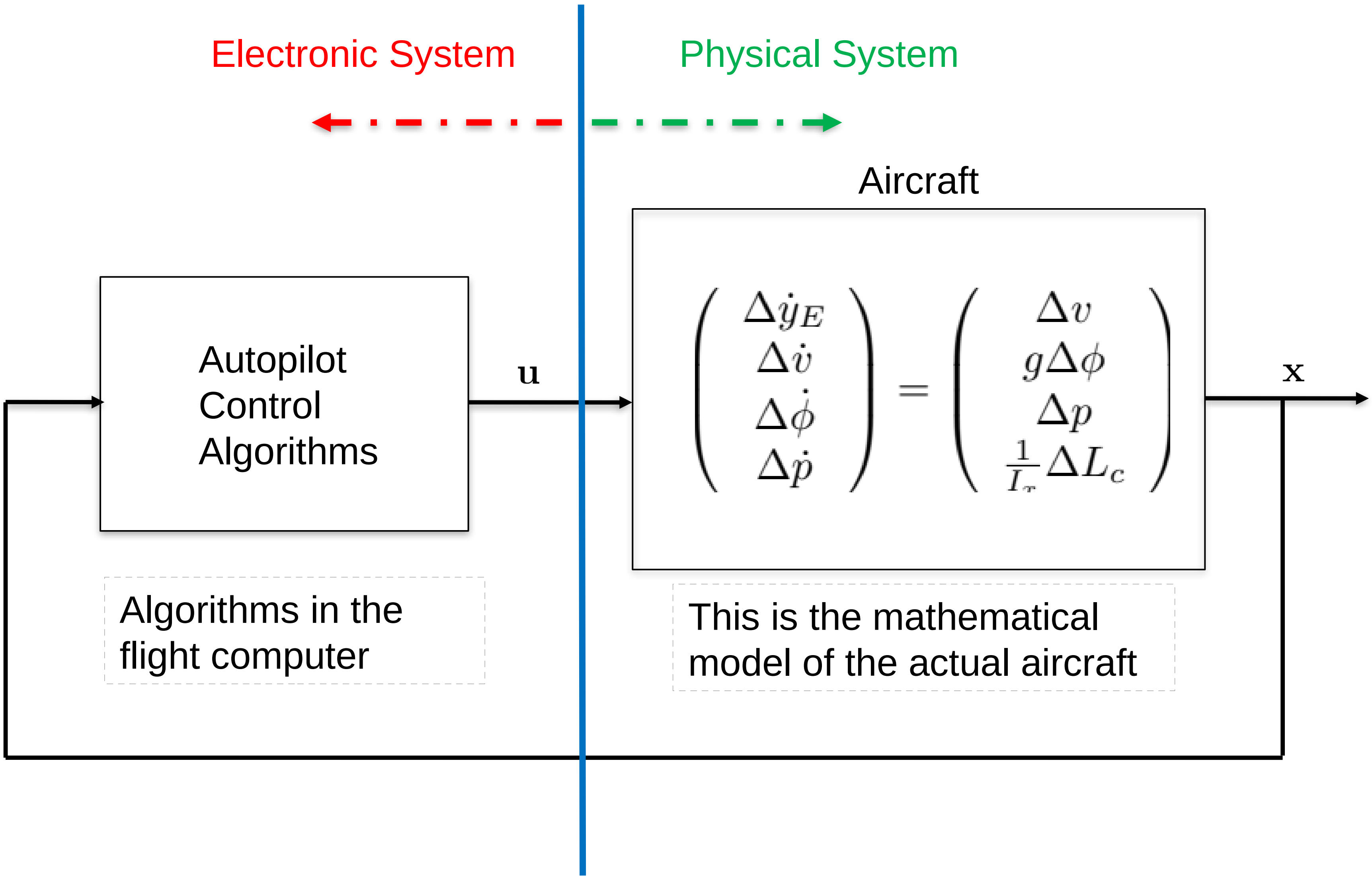
If initial conditions are nonzero, quad will crash!

(This is not usually what we want)

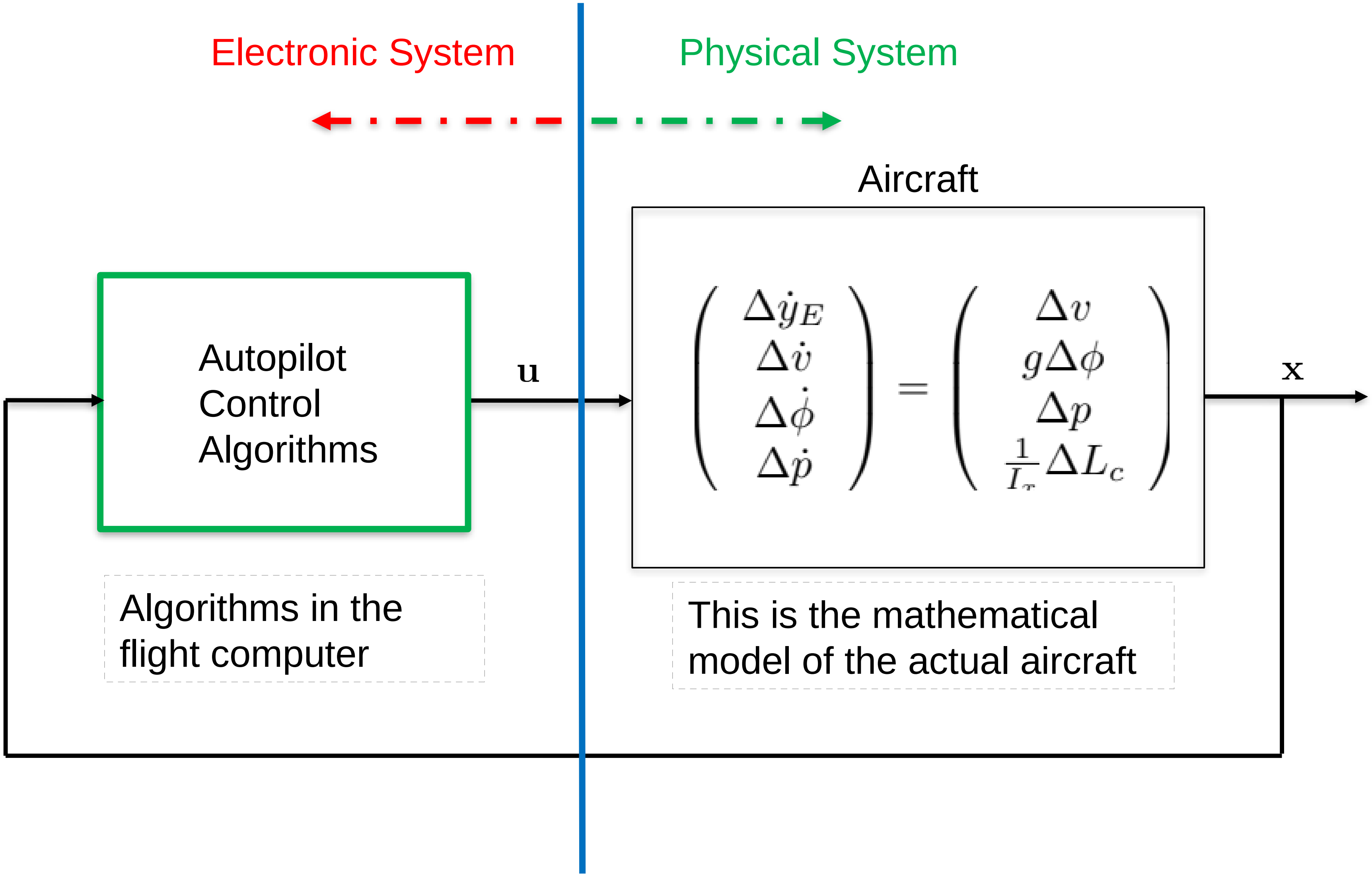
Feedback Control



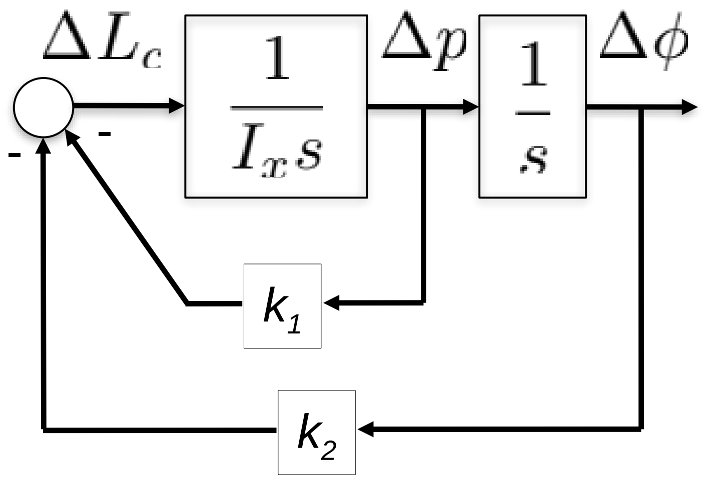
Feedback Control



Feedback Control



Control Strategy for Roll Angle



$$\Delta L_c = -k_1 \Delta p - k_2 \Delta\phi$$

Derivative   Proportional

$$\dot{x} = A x + B u$$

$$u = -K x = -\begin{pmatrix} k_1 & k_2 \end{pmatrix} \begin{pmatrix} \Delta p \\ \Delta\phi \end{pmatrix}$$

$$\dot{x} = A x - B K x$$
$$= (A - BK)\, x$$

$$= A^{cl}\, x$$

$A^{cl} = A - BK$ is the **closed loop dynamics matrix**

$$\dot{x} = A^{cl} x$$

is the **closed loop state space model** with controller integrated

Closed Loop Behavior

There are two ways mathematically to describe the resulting "closed loop behavior"

$$\begin{pmatrix} \Delta\dot{\phi} \\ \Delta\dot{p} \end{pmatrix} = \begin{pmatrix} \Delta p \\ \frac{1}{I_x}\Delta L_c \end{pmatrix}$$

$$\Delta\ddot{\phi} = \Delta\dot{p} = \frac{1}{I_x}\Delta L_c$$

$$\Delta\ddot{\phi} = \frac{1}{I_x}\left(-k_1\Delta p - k_2\Delta\phi\right)$$

$$\Delta\ddot{\phi} + \frac{k_1}{I_x}\Delta\dot{\phi} + \frac{k_2}{I_x}\Delta\phi = 0$$

$$\begin{pmatrix} \Delta\dot{\phi} \\ \Delta\dot{p} \end{pmatrix} = \begin{pmatrix} \Delta p \\ \frac{1}{I_x}\Delta L_c \end{pmatrix}$$

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$$\begin{pmatrix} \Delta\dot{\phi} \\ \Delta\dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{pmatrix}\begin{pmatrix} \Delta\phi \\ \Delta p \end{pmatrix}$$

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Mathematically equivalent!

Closed Loop Behavior

2nd Order System perspective

$$\Delta\ddot{\phi} + \frac{k_1}{I_x}\Delta\dot{\phi} + \frac{k_2}{I_x}\Delta\phi = 0$$

$$\Downarrow$$

$$\Delta\ddot{\phi} + 2\zeta\omega_n\Delta\dot{\phi} + \omega_n^2\Delta\phi = 0$$

$$\omega_n = \sqrt{\frac{k_2}{I_x}} \qquad \zeta = \frac{k_1}{2\sqrt{k_2 I_x}}$$

which has roots $\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$

The response to an initial condition is determined by the natural frequency and damping ratio.

Review: Poles of a system

The roots of the characteristic equation are called the “poles” of the system and determine the dynamic stability properties

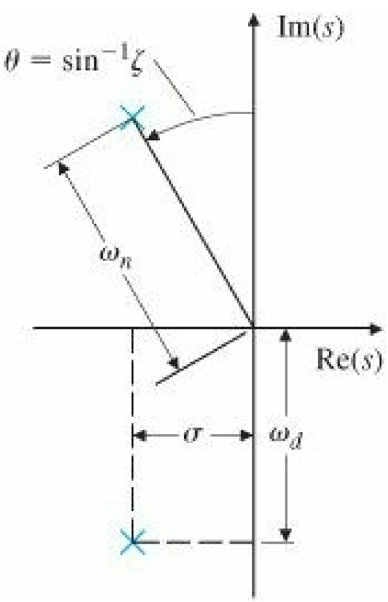
$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

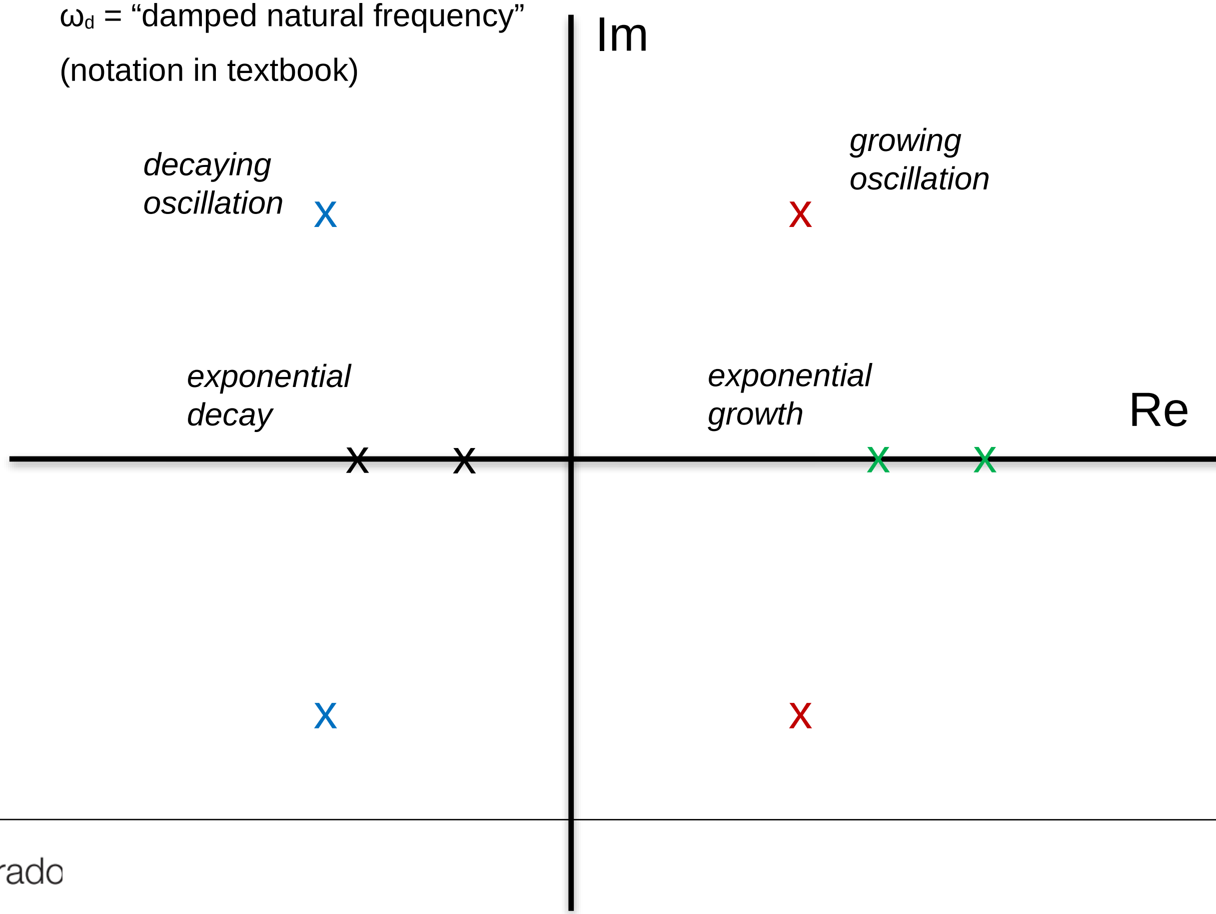
$$\lambda = -\sigma \pm \omega_d i$$

$$\lambda = n \pm \omega i$$

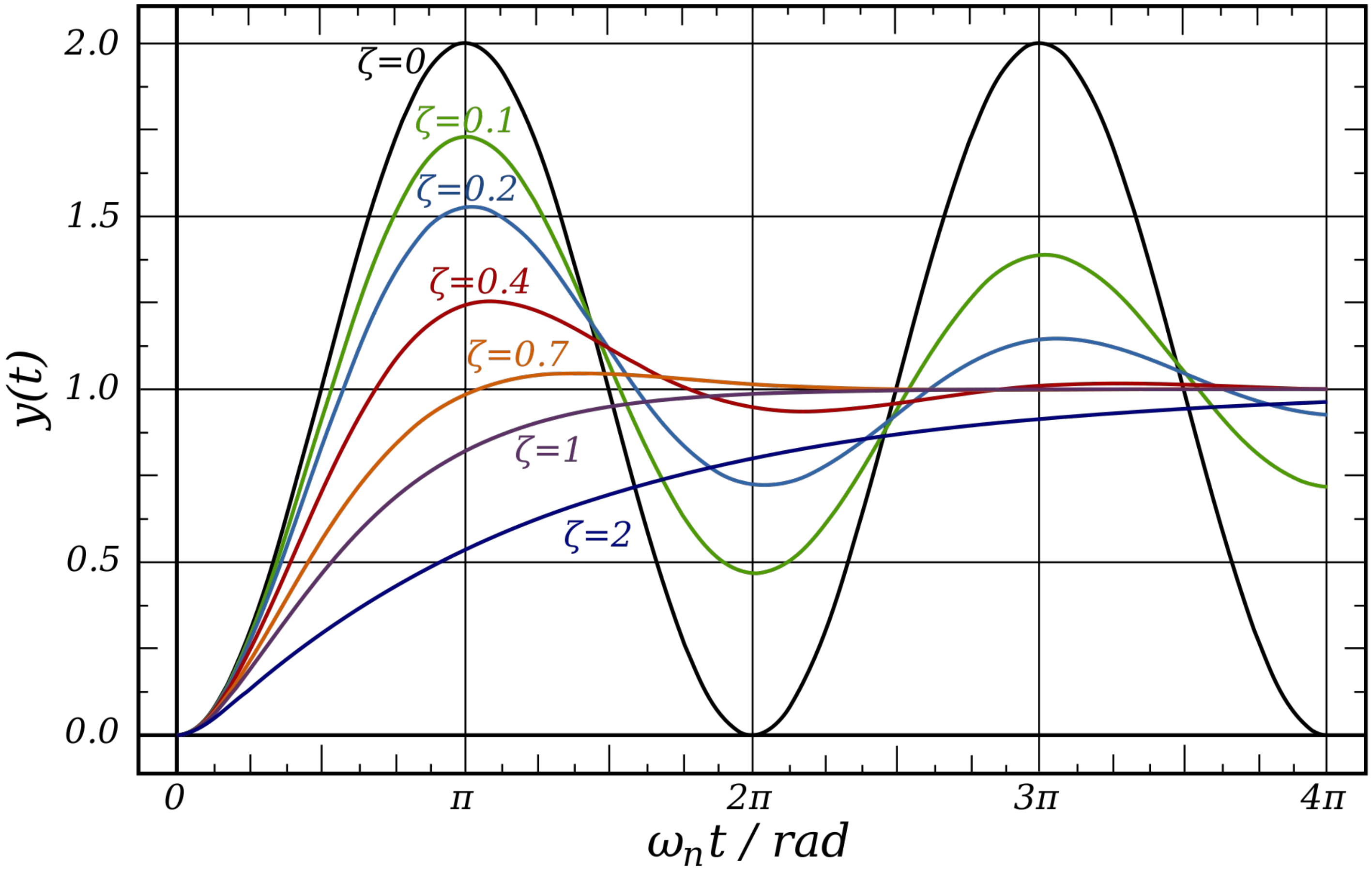
ω_d = “damped natural frequency”

(notation in textbook)





Review: Damping Ratio



Closed Loop Behavior

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$$\Delta\ddot{\phi} = \frac{1}{I_x}\left(-k_1\Delta p - k_2\Delta\phi\right)$$

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This closed loop state space model can be analyzed with tools from linear algebra

Break

Closed Loop Behavior

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This "state space model" can be analyzed with tools from linear algebra

Linear System Natural Response

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

Consider natural response

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

Take the Laplace transform

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

$$(s\mathbf{I} - \mathbf{A})\,\mathbf{X}(s) = \mathbf{x}(0)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{x}(0)$$

Taylor series expansion

$$\mathbf{X}(s) = \left(\frac{\mathbf{I}}{s} + \frac{\mathbf{A}}{s^2} + \frac{\mathbf{A}^2}{s^3}\cdots\right)\mathbf{x}(0)$$

Take the inverse Laplace transform

$$\mathbf{x}(t) = \left(\mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2\cdots\right)\mathbf{x}(0)$$

This looks familiar…

$$e^{at} = 1 + ta + \frac{t^2}{2!}a^2 + \frac{t^3}{3!}a^3\cdots$$

We make the definition (need to be careful not to infer too many properties

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2\cdots$$

So we have

$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)}$$

Modal Analysis

Recall eigenvectors and eigenvalue

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

Consider the initial condition vector as one of the eigenvalues

$$\begin{aligned}
\mathbf{x}(t) &= \left(\mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 \cdots\right)\mathbf{x}(0) \\
&= \left(\mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 \cdots\right)\mathbf{v}_i \\
&= \left(\mathbf{v}_i + t\mathbf{A}\mathbf{v}_i + \frac{t^2}{2!}\mathbf{A}^2\mathbf{v}_i \cdots\right) \\
&= \left(\mathbf{v}_i + t\lambda_i\mathbf{v}_i + \frac{t^2}{2!}\lambda_i^2\mathbf{v}_i \cdots\right) \\
&= \left(1 + t\lambda_i + \frac{t^2}{2!}\lambda_i^2 \cdots\right)\mathbf{v}_i = \boxed{e^{\lambda_i t}\mathbf{v}_i}
\end{aligned}$$

When initial condition is

$$\mathbf{x}(0) = \sum_{i=1}^{n} k_i \mathbf{v}_i$$

The solution is

$$\boxed{\mathbf{x}(t) = \sum_{i=1}^{n} k_i e^{\lambda_i \cdot t}\mathbf{v}_i}$$

This is a critical fact! For eigenvector, matrix exponential solution becomes scalar exponential times the vector -> components stay in relative proportion

Review: Eigen Analysis

$$A v_i = \lambda_i v_i$$

$$\left(A - \lambda_i I \right) v_i = 0$$

Only has solutions when the determinant of A – λ I is zero.

$$\left| A - \lambda_i I \right| = 0$$

For a 2x2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

$$\left| A - \lambda I \right| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$$

$$= \lambda^2 - \left(a + d \right) \lambda + ad - bc$$

Solve via quadratic formula

Control via Pole Assignment

Given: State-Space System, Desired pole locations (or natural frequency and damping ratio)
Task: Find gains to place poles in desired locations

$$\begin{pmatrix} \Delta\dot{\phi} \\ \Delta\dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{pmatrix} \begin{pmatrix} \Delta\phi \\ \Delta p \end{pmatrix}$$

$$I_x = 7 \times 10^{-5}\, kg\, m^2$$

Desired: $\zeta = 0.7$ $\omega_n = 5\, rad/s$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ -k_2/I_x & -k_1/I_x - \lambda \end{vmatrix} = \lambda^2 + \frac{k_1}{I_x}\lambda + \frac{k_2}{I_x} = 0$$

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$\lambda = -3.5 \pm 3.57\, i$$

Quadratic Formula

$$\lambda = -\frac{k_1}{2 I_x} \pm \sqrt{\frac{k_1^2}{4 I_x^2} - \frac{k_2}{I_x}}$$

$$k_1 = 3.5 \times 2 I_x = 4.9 \times 10^{-4}$$

$$k_2 = -I_x\left((3.57\, i)^2 - \frac{k_1^2}{4 I_x^2}\right) = 0.00175$$

Next Time:
State space analysis of entire quadrotor linear system