Taste of Optimal Control

Techniques for choosing gains

1. Poll Assignment
   $A^{cl} = A - BK$ solve for entries in K to achieve desired A^{cl} eigenvalues
2. Hand Tuning (often PID)
3. Root Locus
4. Optimal Control →

Optimization Problem

$$\underset{x}{\text{minimize}} \quad f(x) \leftarrow \text{objective}$$

optimization variables

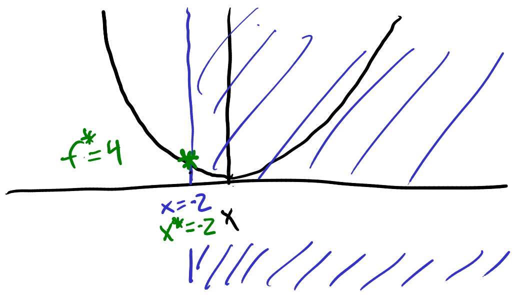
$$\text{subject to} \quad g(x) \le 0 \leftarrow \text{constraints}$$

Engineer specifies f and g; computer finds x^* that minimizes $f(x)$

Example

$$\underset{x}{\text{minimize}} \quad x^2$$

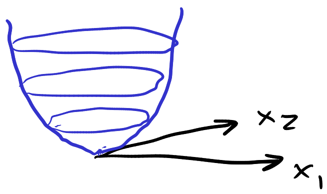
$$\text{subject to} \quad x + 2 \le 0 \qquad x \le -2$$



Example

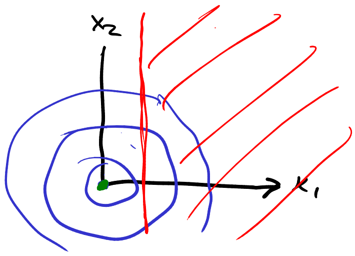
$$\underset{\vec{x}}{\text{minimize}} \quad x_1^2 + x_2^2$$

$$\text{subject to} \quad x_1 - 1 \le 0$$



$$\nabla f = 0$$

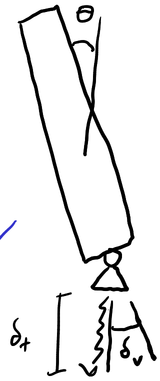
$$\vec{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Optimal Control

$$\underset{\vec{x}(t),\vec{u}(t)}{\text{minimize}} \quad J(\vec{x},\vec{t})$$

$$\text{subject to} \quad \dot{\vec{x}} = f(\vec{x},\vec{u})$$
$$\vec{x}_{min} \le \vec{x} \le \vec{x}_{max}$$
$$\vec{u}_{min} \le \vec{u} \le \vec{u}_{max}$$



$$\underset{x(t)\ u(t)}{\text{minimize}} \quad \| \vec{x}(T) - \vec{x}_{target} \|$$

$$\text{subject to} \quad fuel(t) \ge 0$$
$$\theta_{min} \le \theta(t) \le \theta_{max}$$

Special Problem : LQR

"Linear Quadratic Regulator"

Linear Analytic Solution !

$u(t) = -K\,x(t)$

$$\underset{\vec{u}(t),\vec{x}(t)}{\text{minimize}} \quad J(\vec{x},\vec{u}) = \int_0^\infty \vec{x}(t)^T Q \vec{x}(t) + \vec{u}(t)^T R \vec{u}(t)\, dt$$

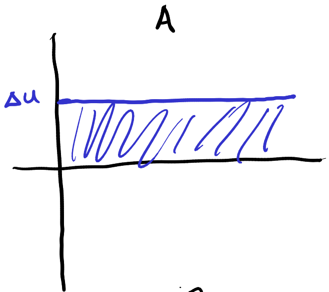
$$\text{subject to} \quad \dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$$
$$\vec{x}(0) = x_0$$

Ex: Long Dynamics

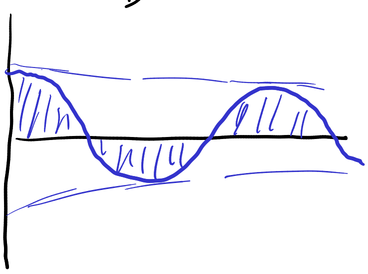
$$Q = \begin{bmatrix} c_u & 0 & 0 & 0 \\ 0 & c_w & 0 & 0 \\ 0 & 0 & c_q & 0 \\ 0 & 0 & 0 & c_\theta \end{bmatrix} \Rightarrow \vec{x}^T Q \vec{x} = c_u \Delta u^2 + c_w \Delta w^2 + c_q q^2 + c_\theta \Delta\theta^2$$

$$R = \begin{bmatrix} c_{\delta_e} & 0 \\ 0 & c_{\delta_t} \end{bmatrix} \Rightarrow \vec{u}^T R \vec{u} = c_{\delta_e} \Delta\delta_e^2 + c_{\delta_t} \Delta\delta_t^2$$

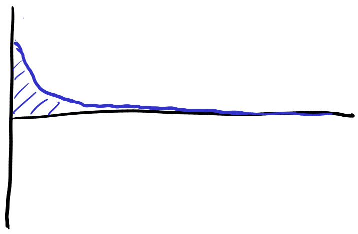
A



B



C



$$J(\vec{x},\vec{u}) = \int_0^\infty \underbrace{x^T Q x}_{c_u \Delta u^2} + u^T R u \, dt$$

Relatively Large $Q \Rightarrow$ smaller state deviations

Relatively Large $R \Rightarrow$ smaller control deviations

$$\underset{x\ u}{\text{minimize}}\ J(x,u) = \int_0^\infty x^TQx + u^TRu\,dt$$

$$J_0(x,u) = x(0)^TQx(0) + u(0)^TRu(0)$$

can't change

if this were the cost function

$$V^*(x(\tau)) = \min \int_\tau^\infty x^TQx + u^TRu\,dt$$

$$J^* = J^*(x^*, u^*) = V^*(x(0))$$

$$V^*(x(\tau)) = \min\left\{ \underbrace{\int_\tau^{\tau+d\tau} x^TQx + u^TRu\,dt}_{\text{immediate}} + \underbrace{\int_{\tau+d\tau}^\infty x^TQx + u^TRu\,dt}_{\text{future}} \right\}$$

$$= \min\left\{ \left(x(\tau)^TQx(\tau) + u(\tau)^TRu(\tau)\right)d\tau + V^*(x(\tau+d\tau)) \right\}$$

$$\cancel{V^*(x(\tau))} = \min\left\{ \text{''} \quad \text{''} \quad \text{''} + \cancel{V^*(x(\tau))} + \frac{dx}{dt}\cdot \nabla V^*\Big|_{x(\tau)} d\tau \right\}$$

$$\frac{dx}{dt} = Ax + Bu$$

$$0 = \min_u \left\{ x^TQx + u^TRu + (Ax + Bu)\cdot \nabla V^* \right\}$$

assumption $\quad V^*(x(\tau)) = x(\tau)^T S\, x(\tau)$

$$\nabla V^* = 2Sx$$

$$0 = \min_{u(\tau)} \underbrace{\left(x(\tau)^TQx(\tau) + u(\tau)^TRu(\tau) + (Ax(\tau) + Bu(\tau))\cdot 2Sx(\tau) \right)}_{f(u)}$$

minimized when $\nabla f = 0$

$$\nabla f = 2Ru + B^T 2Sx = 0$$

$$u = -\underbrace{R^{-1}B^TS}_{K}\, x$$

$$\boxed{u = -Kx}$$

S is a solution to

$$0 = Q + A^TS + SA + SBR^{-1}B^TS$$

In matlab

$$\boxed{K = \text{lqr}(A, B, Q, R)}$$

Traditional Methods:

Use RL, hand tuning, Pole assignment $\longrightarrow$ choose K directly

LQR

Choose Q, R $\longrightarrow$ computer calculates K