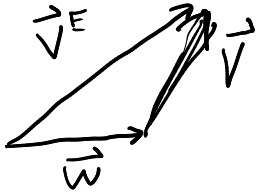


Vector Clarity

\vec{V}_B^E ← "frame of reference", "with respect to", "relative to"
 "coordinate frame", "expressed in", "written in".

→ $\vec{V}^E = \vec{V}^{(w)} + \vec{W}^{(E)}$ ← by convention



True in Any coordinate system
 as long as all in same coordinate system

$$\vec{V}_B^E = \vec{V}_B + \vec{W}_B$$

$$\begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} \text{scribble} \\ \text{scribble} \\ \text{scribble} \end{bmatrix}$$

$$\vec{V}_E^E = \vec{V}_E + \vec{W}_E$$

Kinematics

Kinematics: "Geometry of Motion" (no forces)

Translational
Rotational

Dynamics: Effects of forces and moments on objects

Vector Derivatives

→ $\frac{d}{dt} \vec{p}$ \equiv time rate of change of \vec{p}

$$\frac{d}{dt} \vec{p} = \vec{v}^E$$

$$\vec{p}_B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} \rightarrow \underline{\dot{\vec{p}}_B} = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$$

$$\vec{v}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} \stackrel{?}{=} \dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$$

Not Always True

$$\begin{aligned} \hat{e}_B^B \\ \hat{e}_B^B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \hat{e}_B^B \equiv \hat{e}_B^B \end{aligned}$$

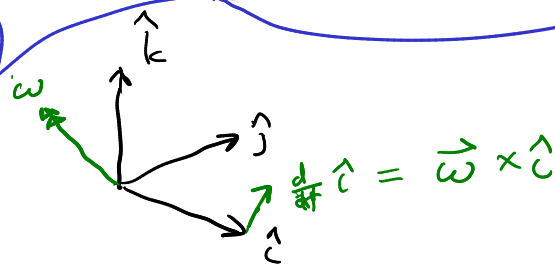
$$\vec{p}_B = p_x \hat{e}_B^B + p_y \hat{j}_B^B + p_z \hat{k}_B^B$$

$$\dot{\vec{p}}_B = \dot{p}_x \hat{e}_B^B + \dot{p}_y \hat{j}_B^B + \dot{p}_z \hat{k}_B^B$$

$$\frac{d}{dt}(uv) = u \frac{d}{dt} v + v \frac{d}{dt} u$$

$$\left(\frac{d}{dt} \vec{p} \right)_B = \dot{p}_x \hat{e}_B^B + p_x \frac{d}{dt} \hat{e}_B^B + \dot{p}_y \hat{j}_B^B + p_y \frac{d}{dt} \hat{j}_B^B + \dot{p}_z \hat{k}_B^B + p_z \frac{d}{dt} \hat{k}_B^B$$

What is $\frac{d}{dt} \hat{e}_B^B$



$$\frac{d}{dt} \hat{e}_B^B = \vec{\omega} \times \hat{e}_B^B$$

$$\vec{\omega}_B = \vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\begin{aligned} &= p_x (\vec{\omega}_B \times \hat{e}_B^B) + p_y (\vec{\omega}_B \times \hat{j}_B^B) + p_z (\vec{\omega}_B \times \hat{k}_B^B) \\ &= \vec{\omega}_B \times \vec{p}_B \end{aligned}$$

$$\boxed{\frac{d}{dt} \vec{p}_B = \dot{\vec{p}}_B + \vec{\omega}_B \times \vec{p}_B} = \dot{\vec{p}}_B + \tilde{\omega}_B \vec{p}_B$$

$$\tilde{\omega}_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

Kinematic Transport Theorem

Equations of motion for a 3D body

$$\ddot{\vec{x}} = f(t, \vec{x})$$

Aircraft State

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \psi \\ \theta \\ u_E \\ v_E \\ w_E \\ p \\ q \\ r \end{bmatrix} \begin{matrix} \vec{p}_E \equiv \vec{p}_E \\ \vec{\omega} \text{ pseudo vector} \\ \vec{v}_B^E \\ \vec{\omega}_B^E \end{matrix}$$

Need

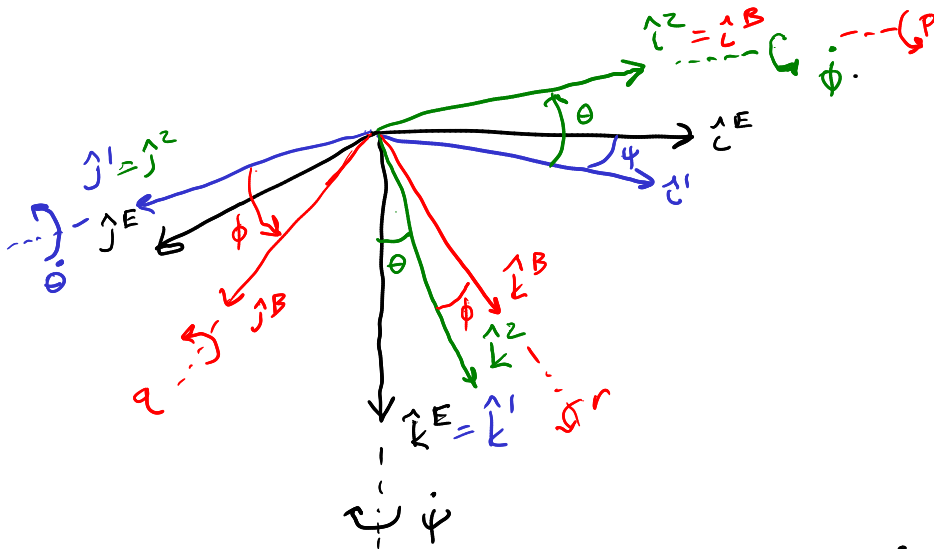
$$\begin{matrix} \dot{\vec{p}}_E & \swarrow & \text{Kinematic} \\ \dot{\vec{\omega}} & & \\ \dot{\vec{v}}_B^E & \swarrow & \text{Dynamics} \\ \dot{\vec{\omega}}_B \end{matrix}$$

Translational Kinematics

$$\dot{\vec{p}}_E = \frac{d}{dt} \vec{p}_E - \vec{\omega}_E^E \times \vec{p}_E = \vec{v}_E^E = R_B^E \vec{v}_B^E$$

$$\dot{\vec{p}}_E^E = R_B^E \vec{v}_B^E$$

Rotational Kinematics



$$\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\vec{\omega} = \dot{\psi} \hat{k}^E + \dot{\theta} \hat{j}^I + \dot{\phi} \hat{i}^B$$

$$\vec{\omega}_B = \dot{\psi} \hat{k}_B^E + \dot{\theta} \hat{j}_B^I + \dot{\phi} \hat{i}_B^B$$

$$= \dot{\psi} R_E^B \hat{k}_E^E + \dot{\theta} R_I^B \hat{j}_I^I + \dot{\phi} \hat{i}_B^B$$

$$= R_1(\phi) R_2(\theta) R_3(\psi) \begin{bmatrix} 0 \\ 0 \\ \psi \end{bmatrix} + R_1(\phi) R_2(\theta) \begin{bmatrix} \theta \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$R_E^B = R_1(\phi) R_2(\theta) R_3(\psi)$$

$$R_I^B = R_1(\phi) R_2(\theta)$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \overset{\text{invert}}{\begin{bmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{bmatrix}} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\sec x = \frac{1}{\cos x}$$

$$\dot{\vec{\theta}} = \underset{\substack{\uparrow \\ \text{attitude,} \\ \text{influence matrix}}}{T} \vec{\omega}_B$$