Lateral Mode Approximations and Control Surfaces

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix} \qquad \mathbf{c}_{lat} = \begin{pmatrix} \dfrac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix} = Bu$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

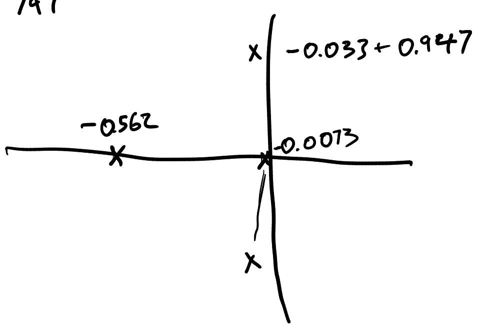
$$\begin{array}{c} \\ \end{array}\left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & A_{lat} & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 0 & 0 & \sec\theta_0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u_0\cos\theta_0 & 0 \end{array}\right] \begin{array}{c} v \\ p \\ r \\ \phi \\ \Delta\Psi \\ \Delta Y_E \end{array}$$

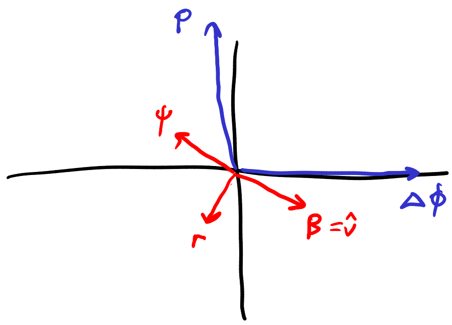
λ_i	ζ	ω_n
$-7.30e-03$	$1.00e+00$	$7.30e-03$
$-5.62e-01$	$1.00e+00$	$5.62e-01$
$-3.30e-02+9.47e-01i$	$3.49e-02$	$9.47e-01$
$-3.30e-02-9.47e-01i$	$3.49e-02$	$9.47e-01$

$$\mathbf{v}_1 = \begin{pmatrix} 0.9821 \\ -0.0014 \\ 0.0078 \\ 0.1880 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -0.9972 \\ -0.0367 \\ 0.0021 \\ 0.0652 \end{pmatrix} \qquad \mathbf{v}_{3/4} = \begin{pmatrix} -1.0000 \\ 0.0019 \mp 0.0032i \\ -0.0001 \pm 0.0011i \\ -0.0035 \mp 0.0019i \end{pmatrix}$$

Spiral — Roll — Dutch Roll

747





$$A_{lat} = \begin{bmatrix} \mathcal{Y}_v & \mathcal{Y}_p & \mathcal{Y}_r & g\cos\theta_0 \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

2x2 Spiral Approx.

$$\tilde{v}_1 = \begin{bmatrix} -0.0012 \\ 0.0013 \\ -0.0073 \\ -0.1768 \\ 1.0 \end{bmatrix}$$

$p = 0$

$\dot{p} = 0$

Ignore side force

$$\begin{bmatrix} \dot{v} \\ \dot{p} = 0 \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_v & \mathcal{Y}_p & \mathcal{Y}_r & g\cos\theta_0 \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_v & \mathcal{L}_r \\ \mathcal{N}_v & \mathcal{N}_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$0 = \mathcal{L}_v v + \mathcal{L}_r r \quad \therefore \quad v = -\frac{\mathcal{L}_r}{\mathcal{L}_v} r$$

$$\dot{r} = -\mathcal{N}_v \frac{\mathcal{L}_r}{\mathcal{L}_v} r + \mathcal{N}_r r = \left(\frac{\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r}{\mathcal{L}_v} \right) r$$

$$\dot{x} = A x$$

$|A - \lambda I| = 0 \qquad \lambda = A$ if A is a scalar

$$\boxed{\lambda_{s,\,approx} = \left(\frac{\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r}{\mathcal{L}_v} \right)}$$

$= -0.0296$ for B747

$\tau = \frac{1}{\lambda} = 33.8\,s$

-0.0073

$\tau = 137\,s$

Not a great approximation

Characteristic Eqn-Based Spiral Approx

$$|A_{lat} - \lambda I| = A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

since $\lambda_s << 1$

$$D\lambda + E = 0$$

$$\boxed{\lambda_{s,\,approx} = -\frac{E}{D}}$$

$$\rightarrow E = g\left[(\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r)\cos\theta_0 + (\mathcal{N}_v \mathcal{L}_p - \mathcal{L}_v \mathcal{N}_p)\sin\theta_0 \right]$$

$$D = -g(\mathcal{L}_v \cos\theta_0 + \mathcal{N}_v \sin\theta_0) + u_0(\mathcal{L}_v \mathcal{N}_p - \mathcal{L}_p \mathcal{N}_v)$$

for B747 $\quad \lambda_{s,\,approx} = -0.00725 \qquad$ < 1% error

One necessary condition for stability is $E > 0$

$$(C_{l_\beta} C_{n_r} - C_{l_r} C_{n_\beta}) \cos\theta_0 + (C_{l_p} C_{n_\beta} - C_{l_\beta} C_{n_p}) \sin\theta_0 > 0$$

Roll Approximation

$$\dot{p} = \mathcal{L}_p p$$

$$\lambda_{r,approx} = \mathcal{L}_p = -0.434$$

$\lambda_r = -0.562$

23% difference

Roll + Spiral Approximation

Assume side force due to gravity produces same yaw rate that would exist with $\beta = 0$

$$0 = -u_0 r + g\phi \leftarrow$$

Also assume $Y_p = Y_r = 0$

$$\begin{bmatrix} 0 \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & -u_0 & g \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix}$$

$$|\tilde{A} - \lambda I| = C\lambda^2 + D\lambda + E = 0$$

$$C = u_0 \mathcal{N}_v$$

$$D = u_0(\mathcal{L}_v \mathcal{N}_p - \mathcal{L}_p \mathcal{N}_v) - g\mathcal{L}_v$$

$$E = g(\mathcal{L}_v \mathcal{N}_r - \mathcal{L}_r \mathcal{N}_v)$$

B747 $\lambda_{s,approx} = -0.00734$ $\lambda_{r,approx} = -0.597$

"true" -0.0073 -0.562

Dutch Roll Approx

Assume $\phi = p = 0$ $Y_r = 0$

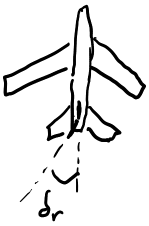
$$\begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_v & -u_0 \\ \mathcal{N}_v & \mathcal{N}_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

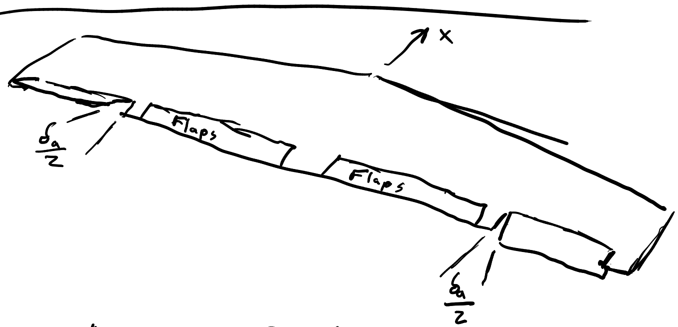
$$\lambda^2 - (\mathcal{Y}_v + \mathcal{N}_r)\lambda + (\mathcal{Y}_v \mathcal{N}_r + u_0 \mathcal{N}_v) = 0$$

$$\lambda_{dr,approx} = -0.1008 \pm 0.9157i$$

$$\lambda_{dr} = -0.033 \pm 0.947i$$

Lateral Control Surfaces





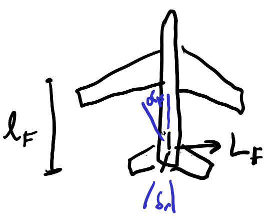
$$\dot{\vec{x}}_{lat} = A_{lat}\vec{x}_{lat} + \underbrace{B_{lat}\vec{u}_{lat}}_{\vec{\tau}_{lat}} \qquad \vec{u}_{lat} = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

$$B_{lat} = \begin{array}{c} \\ \dot{p} \\ \\ \\ \end{array} \begin{bmatrix} 0 & \frac{Y_{\delta_r}}{m} \\ \Gamma_3 L_{\delta_a} + \Gamma_4 N_{\delta_a} & \Gamma_3 L_{\delta_r} + \Gamma_4 N_{\delta_r} \\ \Gamma_4 L_{\delta_a} + \Gamma_8 N_{\delta_a} & \Gamma_4 L_{\delta_r} + \Gamma_8 N_{\delta_r} \\ 0 & 0 \end{bmatrix}$$

Table 7.1
Dimensional Control Derivatives

	X	Z	M
δ_e	$C_{x_{\delta_e}}\frac{1}{2}\rho u_0^2 S$	$C_{z_{\delta_e}}\frac{1}{2}\rho u_0^2 S$	$C_{m_{\delta_e}}\frac{1}{2}\rho u_0^2 S\bar{c}$
δ_p	$C_{x_{\delta_p}}\frac{1}{2}\rho u_0^2 S$	$C_{z_{\delta_p}}\frac{1}{2}\rho u_0^2 S$	$C_{m_{\delta_p}}\frac{1}{2}\rho u_0^2 S\bar{c}$

	Y	L	N
δ_a	$C_{y_{\delta_a}}\frac{1}{2}\rho u_0^2 S$	$C_{l_{\delta_a}}\frac{1}{2}\rho u_0^2 Sb$	$C_{n_{\delta_a}}\frac{1}{2}\rho u_0^2 Sb$
δ_r	$C_{y_{\delta_r}}\frac{1}{2}\rho u_0^2 S$	$C_{l_{\delta_r}}\frac{1}{2}\rho u_0^2 Sb$	$C_{n_{\delta_r}}\frac{1}{2}\rho u_0^2 Sb$



$$N_F = -l_F L_F = -l_F \tfrac{1}{2}\rho V_F^2 S_F C_{L_F}(\alpha_F, \delta_r)$$

$$C_{n_F} = \frac{N_F}{\frac{1}{2}\rho V^2 S b} = -\underbrace{\frac{l_F S_F}{S b}}_{V_V}\left(\frac{V_F^2}{V^2}\right) C_{L_F} = -V_V C_{L_F}\left(\frac{V_F^2}{V^2}\right)$$

$$C_{n_{\delta_r}} \equiv \left.\frac{\partial C_{n_F}}{\partial \delta_r}\right|_0 = -V_V\left(\frac{V_F^2}{V^2}\right)\underbrace{\left.\frac{\partial C_{L_F}}{\partial \delta_r}\right|_0}_{a_r} = \boxed{-a_r V_V\left(\frac{V_F^2}{V^2}\right)}$$

$Y_{\delta_r}, N_{\delta_r}, L_{\delta_r}$