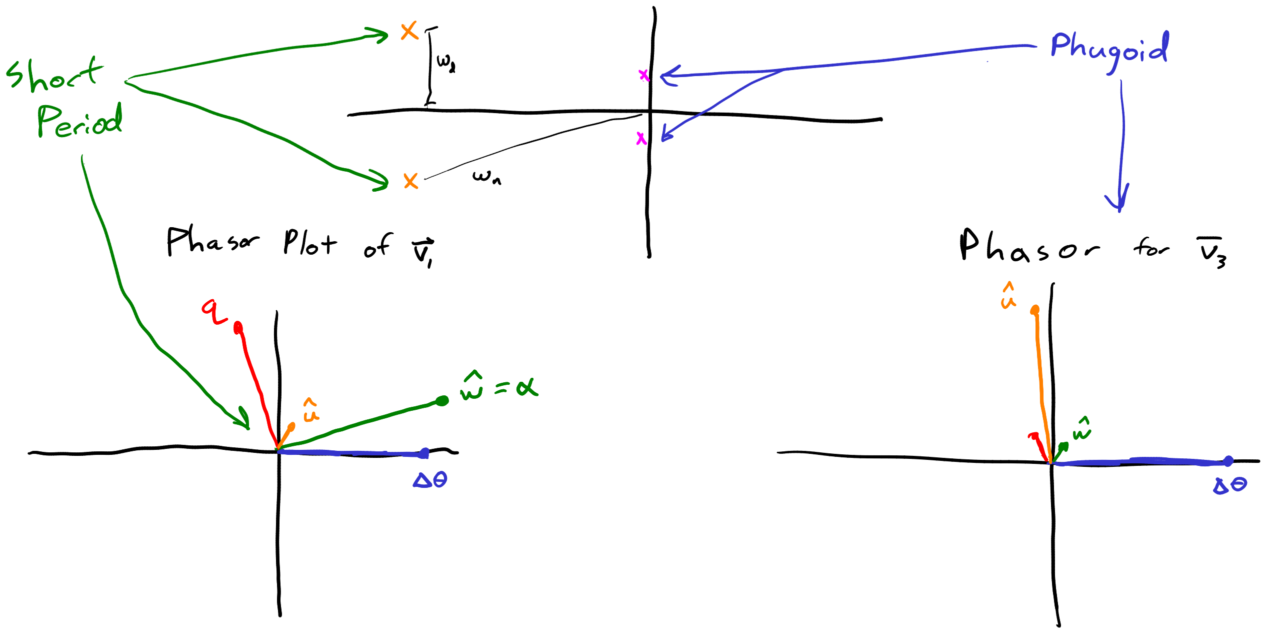
Longitudinal Modal Approximations

$$\dot{\vec{x}}_{lon} = \underbrace{A_{lon}}_{\text{Eigen}}\vec{x}_{lon} + \vec{c}_{lon} \qquad \vec{x}_{lon} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$



Short Period Approx

Dynamics of Flight, Eq. (4.9,18) $\qquad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - \cancelto{0}{Z_{\dot{w}}}} & \dfrac{\cancelto{0}{Z_q} + mu_0}{m - \cancelto{0}{Z_{\dot{w}}}} & \dfrac{-mg\cancelto{0}{\sin\theta_0}}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(\cancelto{0}{Z_q} + mu_0)}{m - \cancelto{0}{Z_{\dot{w}}}}\right] & \dfrac{-M_{\dot{w}}mg\cancelto{0}{\sin\theta_0}}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

Assume: $\Delta u = 0$
$\theta_0 = 0$
$Z_{\dot{w}} \ll m$
$Z_q \ll m u_0$

If we also assume no vertical motion, $\theta_0 = 0$ implies $\Delta\theta = \alpha \approx \frac{\Delta w}{u_0}$

$$\boxed{\begin{bmatrix} \Delta\dot{w} \\ \Delta\dot{q} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{Z_w}{m} & u_0 \\ \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m}\right] & \dfrac{1}{I_y}\left[M_q + M_{\dot{w}}u_0\right] \end{bmatrix}}_{A_{sp}} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}}$$

$$|A_{sp} - \lambda I| = \lambda^2 - \underbrace{\left[\frac{Z_w}{m} + \frac{1}{I_y}\left[M_q + M_{\dot{w}}u_0\right]\right]}_{-2\zeta\omega_n}\lambda - \underbrace{\frac{1}{I_y}\left(u_0 M_w - \frac{M_q Z_w}{m}\right)}_{-\omega_n^2} = 0$$

How does this relate to size and shape?

Dimensional Stab. Deriv.	Nondim. Stab. Deriv.	A/C Params
$Z_w \equiv \left.\frac{\partial Z}{\partial w}\right\vert_0 = \frac{1}{2}\rho u_0 S C_{Z_\alpha}$	C_{Z_α}	$C_{Z_\alpha} = -C_{D_0} - C_{L_\alpha}$
M_w	C_{m_α}	$= C_{L_\alpha}(h - h_n)$
$M_{\dot{w}}$	$C_{m_{\dot{\alpha}}}$	...
M_q	C_{m_q}	$-2a_t V_H \frac{l_t}{\bar{c}}$

How accurate is this approximation?

For 747 @ cruise

Full A_{lon}

$\lambda_{1,2} = -0.372 \pm 0.888i$

$\zeta = 0.387$

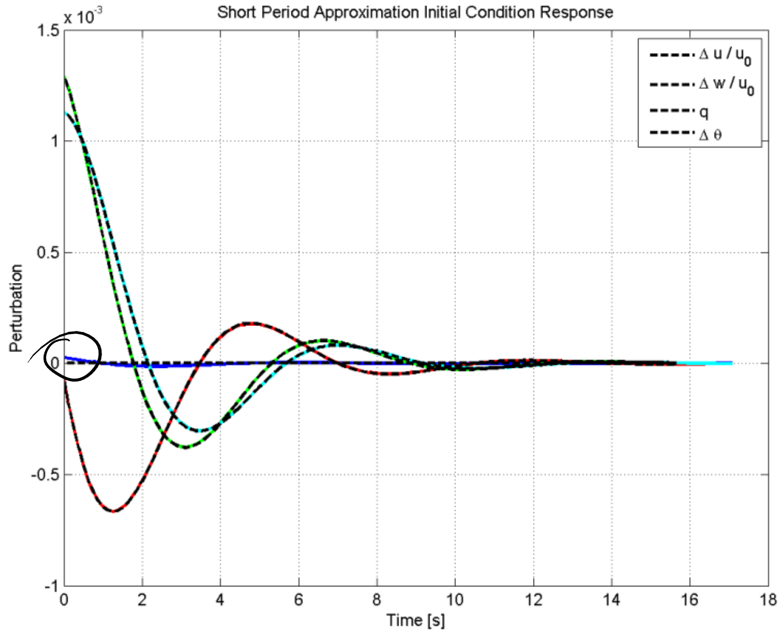
$\omega_n = 0.962$

S.P. approx

$\lambda_{sp} = -0.371 \pm .889i$

$\zeta = 0.385$

$\omega_n = 0.963$



(note: here since $\Delta\theta \neq \frac{\Delta w}{u_0}$, there is some vertical motion)

Phugoid Mode

Lanchester (1908) Assume conservation of energy

$$E = \frac{1}{2}mV^2 - mg\Delta z_E = \frac{1}{2}m u_0^2$$

$$V^2 = 2g\Delta z_E + u_0^2$$

$$C_L = C_{L_0} = C_{W_0}$$

$$L = \frac{1}{2}\rho V^2 S C_L = \frac{1}{2}\rho u_0^2 S C_{W_0} + \rho g S C_{W_0}\Delta z_E = W + \rho g S C_{W_0}\Delta z_E$$

Newton's 2nd Law in z

$$W - L = m\Delta\ddot{z}_E$$

$$W - (W + \rho g S C_{W_0}\Delta z_E) = m\Delta\ddot{z}_E$$

$$\Delta\ddot{z}_E + \underbrace{\frac{\rho g S C_{W_0}}{m}}_{\omega_n^2}\Delta z_E = 0$$

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{\rho g S C_{W_0}}} = \boxed{\pi\sqrt{2}\,\frac{u_0}{g}} = \begin{cases} 0.138\, u_0 & \leftarrow \text{if } u_0 \text{ in ft/s} \\ 0.453\, u_0 & \leftarrow \text{if } u_0 \text{ in m/s} \end{cases}$$

for 747

Full A_{lon}	Lanchester
$T = 93\,s$	$T = 107\,s$

"2×2" **Phugoid Approx**

Dynamics of Flight, Eq. (4.9,18) $\quad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \cancel{\dfrac{X_w}{m}} & 0 & -g\cancel{\cos\theta_0}^{\,1} \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \cancel{\dfrac{Z_w}{m - Z_{\dot{w}}}} & \dfrac{Z_q + m u_0}{m - Z_{\dot{w}}} & \dfrac{-mg\cancel{\sin\theta_0}^{\,0}}{m - Z_{\dot{w}}} \\ \cancel{\dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}}\right]} & \cancel{\dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}}\right]} & \cancel{\dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + m u_0)}{m - Z_{\dot{w}}}\right]} & \cancel{\dfrac{-M_{\dot{w}} m g \sin\theta_0}{I_y(m - Z_{\dot{w}})}} \\ 0 & \cancel{0} & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

Assume $Z_{\dot{w}} \ll m$

$Z_q \ll m u_0$

$\Delta\dot{q}$ small

$\Delta\alpha = 0$

$\theta_0 = 0$

$$\begin{bmatrix} \Delta\dot{u} \\ \cancel{\Delta\dot{\alpha}} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & 0 & -g \\ \frac{Z_u}{m} & u_0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

$$0 = \frac{Z_u}{m}\Delta u + u_0 \Delta q \longrightarrow \Delta q = -\frac{Z_u}{m u_0}\Delta u$$

$$\boxed{\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{m u_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix}}$$

$$\lambda^2 - \underbrace{\frac{X_u}{m}}_{-2\zeta\omega_n}\lambda - \underbrace{\frac{Z_u g}{m u_0}}_{-\omega_n^2} = 0$$

$$Z_u = -\rho u_0 S C_{W_0} \cancel{\cos\theta_0}^{\,1} + \tfrac{1}{2}\rho u_0 S C_{Z_u}$$

$$C_{Z_u} = \underbrace{-M_0 \frac{\partial C_L}{\partial M}\Big|_0}_{\text{small}} \underbrace{- \rho u_0^2 \frac{\partial C_L}{\partial p_d}\Big|_0}_{\text{small}} \underbrace{- C_{T_u}\frac{\partial C_L}{\partial C_T}\Big|_0}_{\text{small}} \qquad -C_{T_u}\frac{\partial C_x}{\partial C_T}\Big|_0$$

$$C_{Z_u} = 0 \qquad Z_u \approx -\rho u_0 S C_{W_0}$$

$$X_u = \rho u_0 S C_{w_0} \cancel{\sin\theta_0}^{\,0} + \frac{1}{2}\rho u_0 S \underline{C_{x_u}}$$

$$C_{x_u} = -2C_{T_0} \qquad \text{(constant thrust)}$$

$$C_{T_0} = C_{D_0} + C_{w_0}\cancel{\sin\theta_0}^{\,0}$$

$$X_u \approx -\rho u_0 S C_{D_0}$$

$$\omega_n = \sqrt{\frac{-Z_u g}{m u_0}} = \sqrt{\frac{\rho S C_{w_0} g}{m}}$$ Same as Lanchester

$$\zeta = -\frac{X_u}{2}\sqrt{\frac{-u_0}{m Z_u g}} = \rho\,\frac{u_0 S C_{D_0}}{2}\sqrt{\frac{u_0}{\frac{1}{2} m g \rho u_0 S C_{L_0}}}$$

$$= C_{D_0}\sqrt{\frac{\frac{1}{4}\rho u_0^2 S}{\frac{1}{2} m g}\,\frac{1}{C_{L_0}}}$$

$$\zeta = \frac{C_{D_0}}{C_{L_0}}$$ High L/D = less energy loss = less damping

747

Full A_{lon}

$\lambda_{3,4} = -3.29\times10^{-3} \pm 6.72\times10^{-2} i$

$\zeta = 0.0489$

$\omega_n = 0.0673$

Ph Approx

$\lambda_{ph} = -3.43\times10^{-3} \pm 6.11\times10^{-2} i$

$\zeta = 0.0561$

$\omega_n = 0.0612$