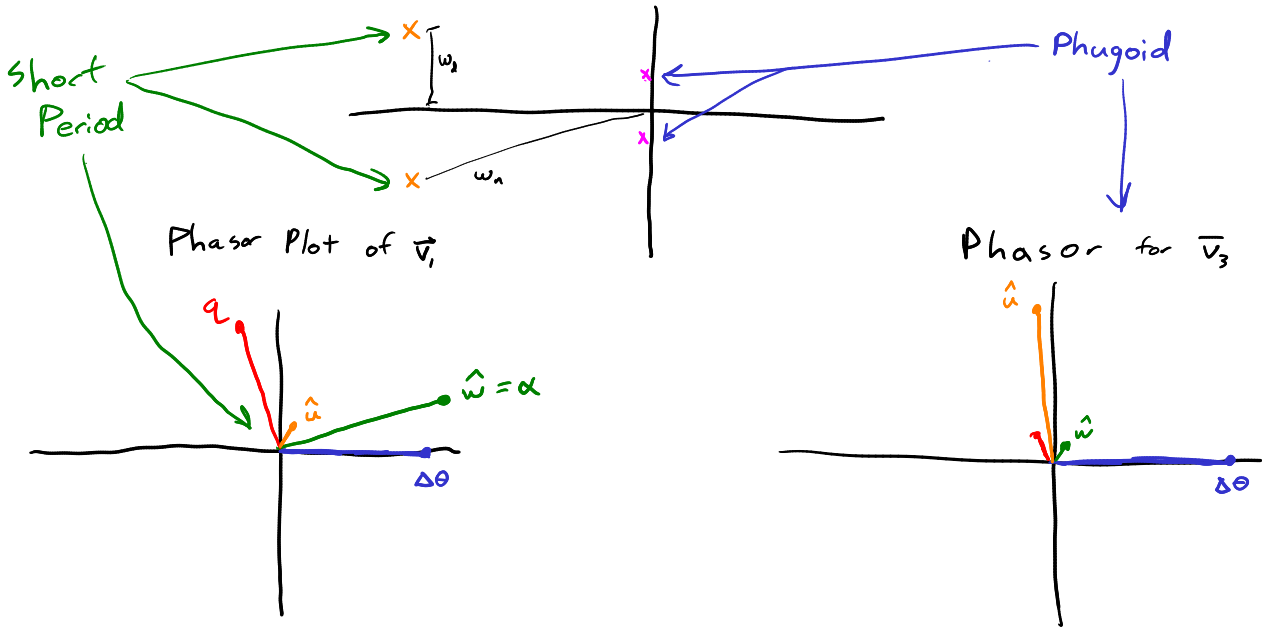


# Longitudinal Modal Approximations

$$\dot{\vec{x}}_{lon} = \underbrace{A_{lon}}_{\text{Eigen}} \vec{x}_{lon} + \vec{c}_{lon}$$

$$\vec{x}_m = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$



## Short Period Approx

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\vec{x}}_{lon} = A_{lon} \vec{x}_{lon} + \vec{c}_{lon}$$

$$\vec{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \vec{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$A_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + m u_0}{m - Z_{\dot{w}}} & \frac{-m g \sin \theta_0}{m - Z_{\dot{w}}} \\ \frac{1}{I_y} \left[ M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_{\dot{w}} (Z_q + m u_0)}{m - Z_{\dot{w}}} \right] & \frac{-M_{\dot{w}} m g \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

Assume:  $\Delta u = 0$

$\theta_0 = 0$

$Z_{\dot{w}} \ll m$

$Z_q \ll m u_0$

If we also assume no vertical motion,  
 $\theta_0 = 0$  implies  $\Delta \theta = \alpha \approx \frac{\Delta w}{u_0}$

$$\begin{bmatrix} \Delta \dot{w} \\ \Delta \dot{q} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{Z_w}{m} & u_0 \\ \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m} \right] & \frac{1}{I_y} [M_q + M_{\dot{w}} u_0] \end{bmatrix}}_{A_{sp}} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}$$

$$|A_{sp} - \lambda I| = \lambda^2 - \underbrace{\left[ \frac{Z_w}{m} + \frac{1}{I_y} [M_q + M_{\dot{w}} u_0] \right]}_{-2\zeta \omega_n} \lambda - \underbrace{\frac{1}{I_y} (u_0 M_w - \frac{M_{\dot{w}} Z_w}{m})}_{-\omega_n^2} = 0$$

How does this relate to size and shape?

Dimensional Stab. Deriv.

Nondim. Stab. Deriv.

A/C Params

$$\bar{Z}_w \equiv \frac{\partial \bar{Z}}{\partial w} \bigg|_0 = \frac{1}{2} \rho u_0 S \bar{C}_{Z\alpha}$$

$$C_{Z\alpha}$$

$$C_{Z\alpha} = -C_{D\alpha} - C_{L\alpha}$$

$$M_w$$

$$C_{m\alpha}$$

$$= C_{L\alpha} (h - h_n)$$

$$M_{\dot{w}}$$

$$C_{m\dot{\alpha}}$$

$$M_q$$

$$C_{mq}$$

$$-Z_{\alpha+} V_H \frac{l+}{\bar{c}}$$

How accurate is this approximation?

For 747  
@ cruise

Full A<sub>bn</sub>

$$\lambda_{1,2} = -0.372 \pm 0.889i$$

$$\xi = 0.387$$

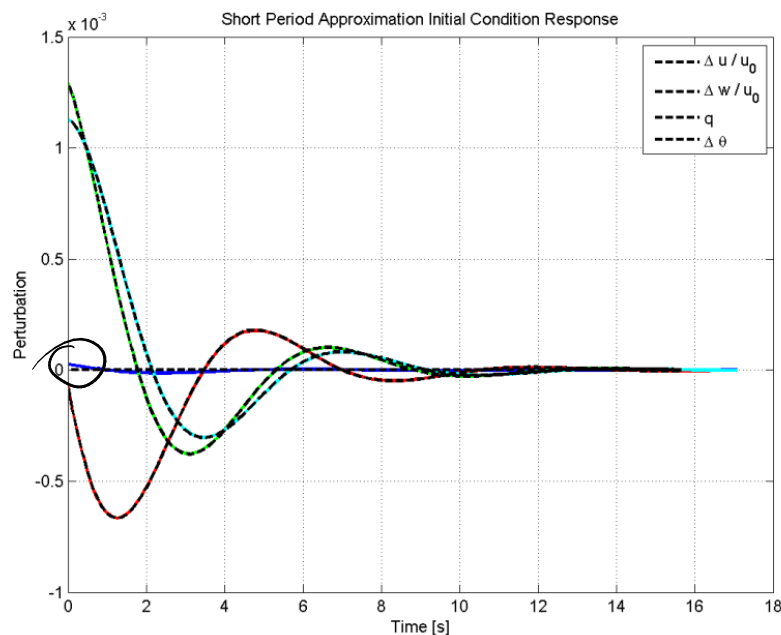
$$\omega_n = 0.962$$

S.P. approx

$$\lambda_{sp} = -0.371 \pm 0.889i$$

$$\xi = 0.385$$

$$\omega_n = 0.963$$



(note: here since  $\Delta \theta \neq \frac{\Delta w}{u_0}$ , there is some vertical motion)

Phugoid Mode

Lanchester (1908)

Assume conservation of energy

$$E = \frac{1}{2} m V^2 - m g \Delta z_E = \frac{1}{2} m u_0^2$$

$$V^2 = 2g \Delta z_E + u_0^2$$

$$C_L = C_{L0} = C_{W0}$$

$$L = \frac{1}{2} \rho V^2 S C_L = \frac{1}{2} \rho u_0^2 S C_{W0} + \rho g S C_{W0} \Delta z_E = W + \rho g S C_{W0} \Delta z_E$$

Newton's 2nd Law in  $z$

$$W - L = m \Delta \ddot{z}_E$$

$$W - (W + \rho g S C_{W0} \Delta z_E) = m \Delta \ddot{z}_E$$

$$\Delta \ddot{z}_E + \underbrace{\frac{\rho g S C_{W0}}{m}}_{\omega_n^2} \Delta z_E = 0$$

$$T = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m}{\rho g S C_{W_0}}} = \boxed{\pi \sqrt{2} \frac{u_0}{g}} = \begin{matrix} 0.138 u_0 \leftarrow \text{if } u_0 \text{ in ft/s} \\ 0.453 u_0 \leftarrow \text{if } u_0 \text{ in m/s} \end{matrix}$$

for 747

$$\frac{\text{Full } A_{lon}}{T = 93s}$$

$$\frac{\text{Lanchester}}{T = 107s}$$

"Zxz"

Phugoid Approx

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} -\frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mu_0}{m - Z_{\dot{w}}} & \frac{-mg \sin \theta_0}{m - Z_{\dot{w}}} \\ \frac{1}{I_y} \left[ M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_{\dot{w}}} \right] & \frac{M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

Assume  $Z_{\dot{w}} \ll m$

$Z_q \ll mu_0$

$\Delta \dot{q}$  small

$\Delta \alpha = 0$

$\theta_0 = 0$

$$\begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{w} \\ \Delta \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & 0 & -g \\ \frac{Z_u}{m} & u_0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta \theta \end{bmatrix}$$

$$\Rightarrow 0 = \frac{Z_u}{m} \Delta u + u_0 \Delta q \rightarrow \Delta q = -\frac{Z_u}{mu_0} \Delta u$$

$$\begin{bmatrix} \Delta \dot{u} \\ \Delta \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{mu_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix}$$

$$\lambda^2 - \frac{X_u}{m} \lambda - \frac{Z_u g}{mu_0} = 0$$

$\underbrace{\quad}_{-2\zeta\omega_n} \quad \underbrace{\quad}_{-\omega_n^2}$

$$Z_u = -\rho u_0 S C_{W_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{Z_u}$$

$$C_{Z_u} = \underbrace{-M_0 \frac{\partial C_L}{\partial \alpha}}_{\text{small}} \Big|_0 - \underbrace{\rho u_0^2 \frac{\partial C_L}{\partial p_1}}_{\text{small}} \Big|_0 - \underbrace{C_{T_u} \frac{\partial C_L}{\partial C_T}}_{\text{small}} \Big|_0$$

$$-C_{T_u} \frac{\partial C_L}{\partial C_T} \Big|_0$$

$$C_{Z_u} = 0$$

$$Z_u \approx -\rho u_0 S C_{W_0}$$

$$X_u = \rho u_0 S C_{w_0} \sin \theta_0^0 + \frac{1}{2} \rho u_0 S C_{x_u}$$

$$C_{x_u} = -2 C_{T_0} \quad (\text{constant thrust})$$

$$C_{T_0} = C_{D_0} + C_{w_0} \sin \theta_0^0$$

$$X_u \approx -\rho u_0 S C_{D_0}$$

$$\omega_n = \sqrt{-\frac{Z_u g}{m u_0}} = \sqrt{\frac{\rho S C_{w_0} g}{m}} \quad \text{Same as Lanchester}$$

$$\zeta = -\frac{X_u}{2} \sqrt{\frac{u_0}{m Z_u g}} = \frac{\rho u_0 S C_{D_0}}{2} \sqrt{\frac{u_0}{m g \rho u_0 S C_{L_0}}}$$

$$= \frac{C_{D_0}}{\sqrt{2}} \sqrt{\frac{\frac{1}{2} \rho u_0^2 S}{m g}} \frac{1}{C_{L_0}}$$

$$\zeta = \frac{1}{\sqrt{2}} \frac{C_{D_0}}{C_{L_0}} \quad \text{High } L/D = \text{less energy loss} = \text{less damping}$$

747

Full  $\Delta_{lon}$

$$\lambda_{3,4} = -3.29 \times 10^{-3} \pm 6.72 \times 10^{-2}i$$

$$\zeta = 0.0489$$

$$\omega_n = 0.0673$$

Ph Approx

$$\lambda_{ph} = -3.43 \times 10^{-3} \pm 6.11 \times 10^{-2}i$$

$$\zeta = 0.0561$$

$$\omega_n = 0.0612$$

