$$\dot{\vec{x}} = f(\vec{x}, \vec{u})$$

- differential
- 1st order
- ordinary
- nonlinear
- coupled

general form

$$\begin{cases} \dot{\vec{p}}^E_E = R^E_B \vec{V}^E_B \\ \dot{\vec{\Theta}} = T\,\vec{\omega}_B \\ \dot{\vec{V}}^E_B = \dfrac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{V}^E_B \\ \dot{\vec{\omega}}_B = I_B^{-1}\left[\vec{G}_B - \vec{\omega}_B \times I_B \vec{\omega}_B\right] \end{cases}$$

$$T = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix}$$

$\phi = 90°$
$\theta = 0$

$$\begin{bmatrix} 0 \\ 0 \\ 10°/s \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q = 10°/s \\ r \end{bmatrix}$$

Quadrotors

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \frac{1}{m}\begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L \\ \frac{1}{I_y} M \\ \frac{1}{I_z} N \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

$0 = \frac{1}{m}X + \frac{0}{m}$ $\therefore$ $X = 0$
$Y = 0$

$0 = g + \frac{1}{m}Z + \frac{1}{m}Z_c$

$Z_{c,0} = -mg$

"Hover" trim condition

trim condition

$\dot{x}_{E,0} = 0$	$u^E_0 = 0$	$\dot{u}^E_0 = 0$
$\dot{y}_{E,0} = 0$	$v^E_0 = 0$	$\dot{v}^E_0 = 0$
$\dot{z}_{E,0} = 0$	$w^E_0 = 0$	$\dot{w}^E_0 = 0$
$p_0 = 0$	$\dot{p}_0 = 0$	$\phi_0 = 0$
$q_0 = 0$	$\dot{q}_0 = 0$	$\theta_0 = 0$
$r_0 = 0$	$\dot{r}_0 = 0$	$\psi_0 = \cdot$

any constant value

$f_{1,0} = \frac{mg}{4}$
$f_{2,0} =$ "
$f_{3,0} =$ "
$f_{4,0} =$ "

$\boxed{Z_{c,0} = mg}$
$L_{c,0} = 0$
$M_{c,0} = 0$
$N_{c,0} = 0$

Linearazation

$$\vec{x} = \vec{x}_0 + \Delta\vec{x} \qquad \vec{u} = \vec{u}_0 + \Delta\vec{u}$$

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} = \begin{bmatrix} x_{E,0} + \Delta x_E \\ y_{E,0} + \Delta y_E \\ z_{E,0} + \Delta z_E \\ \Delta\phi \\ \Delta\theta \\ \Delta\psi \\ \Delta u^E \\ \Delta v^E \\ \Delta w^E \\ \Delta p \\ \Delta q \\ \Delta r \end{bmatrix}$$

$$\Delta\dot{x} = A\,\Delta x + B\,\Delta u$$

Taylor Series

$y = f(x,u)$ $\qquad x_0 + \Delta x \qquad y_0 = f(x_0, u_0)$

$$y_0 + \Delta y = f(x_0 + \Delta x,\ u_0 + \Delta u)$$

$$= \underbrace{f(x_0, u_0)}_{y_0} + \left.\frac{\partial f}{\partial x}\right|_0 \Delta x + \left.\frac{\partial f}{\partial u}\right|_0 \Delta u + \text{H.O.T.}$$

$$\Delta y = \left.\frac{\partial f}{\partial x}\right|_0 \Delta x + \left.\frac{\partial f}{\partial u}\right|_0 \Delta u + \text{H.O.T.}$$

---

since $\Delta x, \Delta u$ are small

$\Delta x \Delta u \approx 0$

$\sin(\Delta x) \approx \Delta x$

$\cos(\Delta x) \approx 1$

$$x + y = x_0 + \Delta x + y_0 + \Delta y$$

$$xu \approx x_0 u_0 + \left.\frac{\partial xu}{\partial x}\right|_0 \Delta x + \left.\frac{\partial xu}{\partial u}\right|_0 \Delta u$$

$$= x_0 u_0 + u_0 \Delta x + x_0 \Delta u$$

alternative

$$xu = (x_0 + \Delta x)(u_0 + \Delta u) = x_0 u_0 + u_0 \Delta x + x_0 \Delta u + \cancel{\Delta x \Delta u} \to 0$$

$$\sin(\theta) \approx \sin\theta_0 + \left.\frac{\partial \sin\theta}{\partial \theta}\right|_0 \Delta\theta$$

$$= \sin\theta_0 + \cos\theta_0 \Delta\theta$$

alternatively

$$\sin(\theta) = \sin(\theta_0 + \Delta\theta) = \sin(\theta_0)\cancel{\cos(\Delta\theta)}^{1} + \cos(\theta_0)\cancel{\sin(\Delta\theta)}^{\Delta\theta} \approx \sin\theta_0 + \cos\theta_0 \Delta\theta$$

$$\cos(\theta) \approx \cos(\theta_0) - \sin(\theta_0)\Delta\theta$$

Linearize QR equations of motion

Examples

$$\dot\theta = \cos\phi\, q - \sin\phi\, r$$

$$\cancel{\dot\theta_0}^{0} + \Delta\dot\theta = \cos(\cancel{\phi_0}^{0} + \Delta\phi)(\cancel{q_0}^{0} + \Delta q) - \sin(\cancel{\phi_0}^{0} + \Delta\phi)(\cancel{r_0}^{0} + \Delta r)$$

$$\Delta\dot\theta \approx \cancel{\cos(\Delta\phi)}^{1}\Delta q - \cancel{\sin\Delta\phi}^{\Delta\phi}\Delta r$$

$$\approx \Delta q - \cancel{\Delta\phi\Delta r}^{0}$$

$$\boxed{\Delta\dot\theta = \Delta q}$$

$$\dot w^E = q u^E - p v^E + g\cos\theta\cos\phi + \frac{1}{m} Z + \frac{1}{m} Z_c$$

$$\Delta\dot w^E = \cancel{\Delta q \Delta u^E}^{0} - \cancel{\Delta p \Delta v^E}^{0} + g + \frac{1}{m}\Delta Z + \frac{1}{m}(Z_{c,0} + \Delta Z_c)$$

cancel

$$\Delta\dot w^E = \frac{1}{m}\Delta Z + \frac{1}{m}\Delta Z_c$$

Linearized Aerodynamics

~~In real life $\nu(p,q,r)$~~ $\qquad \nu$ constant

$$Z = -\nu w \sqrt{u^2 + v^2 + w^2}$$

$$\Delta Z = \quad = Z_0 + \left.\frac{\partial Z}{\partial u}\right|_0 \Delta u + \left.\frac{\partial Z}{\partial v}\right|_0 \Delta v + \left.\frac{\partial Z}{\partial w}\right|_0 \Delta w$$

Looks like $\frac{0}{0}$ take limit as $u, v, w \to 0$

$$\left.\frac{\partial Z}{\partial u}\right|_0 = -2\nu w u (u^2 + v^2 + w^2)^{-1/2}$$

$\to 0 \to 0 \qquad \to 0 \quad \to 0 \quad \to 0$

$$\boxed{\left.\frac{\partial Z}{\partial u}\right|_0 = 0}$$

$$\boxed{\left.\frac{\partial Z}{\partial v}\right|_0 = 0}$$

$$\frac{\partial Z}{\partial w} = -\nu \frac{u^2 + v^2 + 2w^2}{\sqrt{u^2 + v^2 + w^2}}$$

$$\boxed{\left.\frac{\partial Z}{\partial w}\right|_0 = 0}$$

$$\boxed{\begin{array}{l}\Delta L = 0 \\ \Delta M = 0 \\ \Delta N = 0\end{array}}$$

$$\boxed{\Delta Z = 0}$$

$Z = -\nu w^2$

$\frac{\partial Z}{\partial w} = -\nu 2 \cancel{w}^{0}$