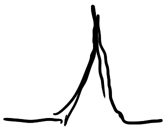
# State Space $\leftrightarrow$ Laplace

$\dot{x} = Ax + Bu$
$y = Cx + Du$

$G_{yu}(s)$

## Review Properties of Laplace Transforms



$$\mathcal{L}[x(t)](s) = \int_0^\infty e^{-st} x(t)\, dt$$

Appendix A1

$$x(t) \Longleftrightarrow x(s)$$

$$\dot{x}(t) \Longleftrightarrow \underline{sx(s)} - x(0)$$

$$\int_0^t x(\tau)\, d\tau \Longleftrightarrow \frac{1}{s} x(s)$$

$$\alpha x(t) + \beta y(t) \Longleftrightarrow \alpha x(s) + \beta y(s)$$

## Transfer Function

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \omega_n^2 u$$

$$\ddot{x} = -2\zeta\omega_n \dot{x} - \omega_n^2 x + \omega_n^2 u$$

$$s^2 x(s) = -2\zeta\omega_n s x(s) - \omega_n^2 x(s) + \omega_n^2 u(s)$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)\, x(s) = \omega_n^2 u(s)$$

$$G_{xu}(s) \equiv \frac{X(s)}{u(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

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$\times$ $\ddot{x} = -\dot{x}^2 + u$ No transfer function for nonlinear systems

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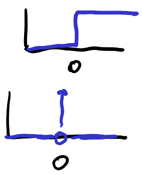
step

impulse

$\delta(0)$



$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

$u(t) \longrightarrow u(s)$

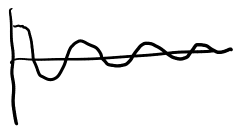
step $u(s) = \frac{1}{s}$

impulse $u(s) = 1$

$$x(s) = G_{xu}(s)\, u(s)$$

$x(s) \longrightarrow x(t)$ table, method of partial fractions

$$\frac{s+b}{s^3 + a_1 s^2 + a_2 s + a_3} \longrightarrow \frac{c_1}{s^2 + d_1} + \frac{c_2}{s^2 + d_2}$$



## Final Value Theorem

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s\, x(s)$$

Only if $s\,x(s)$ is stable

# Two representations of a Linear System

$\rightarrow \dot{x} = Ax + Bu$

$\rightarrow y = Cx + Du$

$G_{yu}(s)$

$|A - \lambda I|$

State space to TF — ss2tf

given SS want $G_{yu}$

$$s\,x(s) = A\,x(s) + B\,u(s)$$

$$(sI - A)\,x(s) = B\,u(s)$$

$$x(s) = (sI - A)^{-1} B\,u(s)$$

$$y(s) = \left(C(sI - A)^{-1}B + D\right) u(s)$$

(from here on assume $D = 0$)

$$M^{-1} = \frac{\mathrm{adj}(M)}{|M|} \qquad (sI - A)^{-1} = \frac{\mathrm{adj}(sI - A)}{|sI - A|}$$

Adjugate: transpose of cofactor matrix $F$

$$F_{ij} = (-1)^{i+j} \left| M_{-i-j} \right|$$

$-i$ = all rows except $i$

If $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

then $F = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \\ & & \\ & & \end{bmatrix}$

$\mathrm{adj}(M) = F^T$

$$G_{yu}(s) = \frac{y(s)}{u(s)} = C(sI - A)^{-1}B = \boxed{\frac{C\,\mathrm{adj}(sI - A)\,B}{|sI - A|}} = \frac{N(s)}{D(s)}$$

$|sI - A| = 0$

Roots of $D(s)$ are the eigenvalues of $A$

stability $\iff$ all roots of $D(s)$ in LHP

all eigenvalues of $A$ in LHP

# TF to SS

tf2ss

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^n + b_1 s^{n-1} \dots + b_{n-1} s + b_n}{a_0 s^n + \dots a_{n-1} s + a_n}$$

e.g. $$\frac{Y(s)}{U(s)} = G(s) = \frac{b_0 s + b_1}{s^2 + a_1 + a_2}$$

multiply by $\frac{x(s)}{x(s)}$

$$\frac{y}{u} = \frac{b_0 s x + b_1 x}{s^2 x + a_1 s x + a_2 x}$$

$\rightarrow$ $y(t) = b_0 \dot{x}(t) + b_1 x(t)$

$u(t) = \ddot{x}(t) + a_1 \dot{x}(t) + a_2 x(t)$

$\rightarrow$ $\ddot{x}(t) = -a_1 \dot{x}(t) - a_2 x(t) + u$

$$\dot{\vec{x}} = A \vec{x} + B u$$

$\rightarrow$ $$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = C \vec{x} + D u$$

$\rightarrow$ $$y = \begin{bmatrix} b_1 & b_0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + [0] u$$

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$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

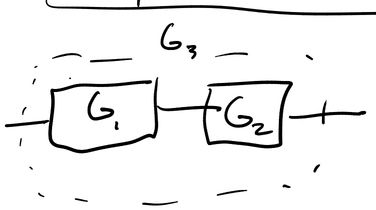
$$\begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ & & \ddots & & \vdots \\ & & & \ddots & 0 \\ & & & & 1 \\ a_n & \dots & & & a_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

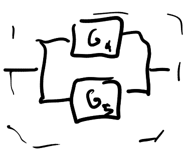
$$C = \begin{bmatrix} b_n - a_n b_0 & \dots & b_1 - a_1 b_0 \end{bmatrix} \qquad D = [0]$$

$\vec{x}$ may not correspond to interpretable physical variables

## Why transfer function

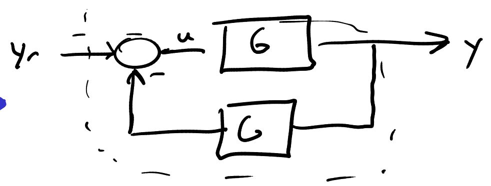



$$G_3 = G_1 G_2$$




$$G_6 = G_4 + G_5$$

rlocus $\rightarrow$



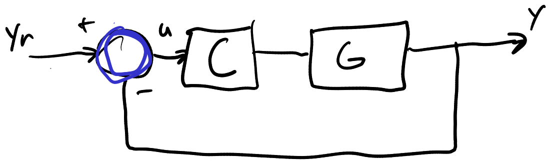

$y = Gu$

$u = y_r - Cy$

$y = G(y_r - Cy)$

$(1 + GC) y = G y_r$

$$\boxed{\frac{y}{y_r} = \frac{G}{1 + GC}}$$




$$y = GCu$$

$$u = y_r - y$$

$$y = GC(y_r - y)$$

$$y(1+GC) = GCy_r$$

$$\boxed{\frac{Y}{y_r} = \frac{GC}{1+GC}}$$

## Example

$$A_{ph} = \begin{bmatrix} -0.0025 & -30 \\ 0.0001 & 0 \end{bmatrix} \qquad x = \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix}$$

$$B_{ph,\delta_T} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \qquad C_{ph,\theta} = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad D = [0]$$

1) find $G_{\theta\delta_T}(s)$

$$G_{\theta\delta_T}(s) = \frac{C\,\mathrm{adj}(sI-A)\,B}{|sI-A|}$$

$F^T$

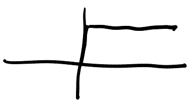
$$sI - A = \begin{bmatrix} s+0.0025 & 30 \\ -0.0001 & s \end{bmatrix}$$

$$F = \begin{bmatrix} & 0.0001 \\ & \end{bmatrix}$$

$$G_{\theta\delta_T}(s) = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} & \\ 0.0001 & \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix}}{(s+0.0025)s - 0.0001\cdot(-30)}$$

$$\boxed{= \frac{0.001}{s^2 + 0.0025s + 0.003}}$$

2) $\lim_{t\to\infty} \Delta\theta(t)$ in response to a $\delta_T$ step function with magnitude $\varepsilon$



$$\delta_T(t) = \mathbb{1}(t)\,\varepsilon$$

$$\delta_T(s) = \frac{1}{s}\varepsilon$$

$$\Delta\theta(s) = G_{\theta\delta_T}(s)\,\delta_T(s) = \frac{0.001\varepsilon}{s(s^2 + 0.0025s + 0.003)}$$

$$\lim_{s\to 0} s\Delta\theta(s) = \lim_{s\to 0} \frac{\cancel{s}\,0.001\varepsilon}{\cancel{s}(s^2 + 0.0025s + 0.003)} = \frac{0.001\varepsilon}{0.003} = \frac{\varepsilon}{3}$$

$$\begin{bmatrix} G_{y_1u_1}(s) & G_{y_1u_2}(s) \\ G_{y_2u_1}(s) & G_{y_2u_2}(s) \end{bmatrix} = \frac{C\,\mathrm{adj}(sI-A)B}{|sI-A|}$$

$$= \frac{\begin{bmatrix} N_{y_1u_1}(s) & N_{y_1u_2}(s) \\ N_{y_2u_1}(s) & N_{y_2u_2}(s) \end{bmatrix}}{|sI-A| = D(s)}$$

$$G_{y_1u_2}(s) = \frac{N_{y_1u_2}(s)}{D(s)}$$