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# Modellierung & Simulation I

## Serie 09

### Chapter 1 - Fundamentals of Simulation Programs

### Problem 9.1.1

It should be used for precision, because float and real have a different precision than long int. For example it holds that for real and float $10^{20} + 0.01 \hat{=} 10^{20}$ which leads to the simulation time being unprecise for long simulations. Therefore using a long int is more practical.

### Problem 9.1.2

A product-oriented simulation is for example the simulation of the processing-time in a plant of a given product. In comparison towards that a machine-oriented approach is for example the simulation of different scheduling algorithms in one machine.

### Chapter 2 - Fundamentals of Statistics I

### Problem 9.1.1

For a continuous random variable $X$ with the probability density function $f(x)$ the mean is defined as:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

### Problem 9.1.2

For a discrete random variable $X$ with $x_i$ being the value and $p(x_i)$ being the probability of the value, the mean is defined as:

$$E(X) = \sum_{i=1}^{n} x_i p\left(x_i\right)$$

**Problem 9.1.3**

The definition of the sample mean differs based one the fact if the process is time-discrete or time-continuous:

$$\overline{X^k} = \begin{cases} \overline{X^k}(n) = \frac{1}{n} \cdot \sum_{0 \le i \le n} X_i^k & \text{for time-discrete process} \\ \overline{X^k}(T) = \frac{1}{T} \cdot \int_0^T X(t)^k dt & \text{for time-continuous process} \end{cases}$$
$$\overline{X} = \overline{X^1}$$

**Problem 9.1.4**

For discrete distributions the convolution (dt. "Faltung") is used for the sum of two random variables. For $X_1, X_2$ being random variables the distribution $X = X_1 + X_2$ is given by:

$$P(X = i) = x(i) = \sum_{j=0}^{N} x_1(j) \cdot x_2(i - j) = (x_1 * x_2)(i)$$

**Problem 9.1.5**

The distribution function of the minimum of two random variables $A_0, A_1$ is calculated like:

$$A = min(A_0, A_1)$$
$$A^c(t) = A_1^c(t) \cdot A_2^c(t)$$
$$a(t) = a_1(t) \cdot A_2^c(t) + a_2(t) \cdot A_1^c(t)$$

**Problem 9.1.6**

A histogram is a graphical representation of the distribution of a process. For this cause, a histogram needs to contain multiple segments (bins) with a constant or variable width. For each bin the number of values, which lie in the range of the respective bin, get counted. An example would be to see how the waiting times in a simulation of a queuing system are distributed.

**Problem 9.1.7**

Time-weighting counters are counters, which take the time into account by weighting the sum $\sum^k$ of the observed values with the time at $t_n \le t < t_{n+1}$

like the following:

$$\Sigma^k = \sum_{0 \le i < n} (X_i)^k \cdot (t_{i+1} - t_i)$$

An example for time-weighting counters would be the mean waiting time over a given interval.

**Chapter 3 – Distributions I**

**Problem 9.1.1**

For the exponential distribution the distribution function $F(x)$, the probability density function $f_\lambda(x)$, the mean $E(X)$, the standard deviation $\sigma(X)$, and the coefficient of variation $c_{var}(X)$ are the following:

$$\begin{aligned}
\text{pdf} : f_\lambda(x) &= \begin{cases} \lambda \mathrm{e}^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases} \\
\text{cdf} : \mathrm{F(x)} &= 1 - \mathrm{e}^{-\lambda \cdot \mathrm{x}} \\
E(X) &= \int_0^\infty t \cdot f_\lambda dt = \int_0^\infty t \cdot \lambda \cdot e^{-\lambda \cdot t} dt \\
&= \left[ t \cdot \left( -e^{-\lambda \cdot t} \right) \right]_0^\infty - \int_0^\infty \left( -e^{-\lambda \cdot t} \right) dt = 0 - 0 - \left( \frac{1}{\lambda} \left[ e^{-\lambda \cdot t} \right]_0^\infty \right) = \boxed{\boldsymbol{\frac{1}{\lambda}}} \\
E\left(X^2\right) &= \int_0^\infty t^2 \cdot f_\lambda dt = \int_0^\infty t^2 \cdot \lambda \cdot e^{-\lambda \cdot t} dt = \cdots = \boxed{\boldsymbol{\frac{2}{\lambda^2}}} \\
VAR(X) &= E\left(X^2\right) - E(X)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\boldsymbol{\frac{1}{\lambda^2}}} \\
\sigma(X) &= \sqrt{\frac{1}{\lambda^2}} = \boxed{\boldsymbol{\frac{1}{\lambda}}} \\
c_{var}(X) &= \frac{\sigma(X)}{E(X)} = \frac{1}{\lambda} / \frac{1}{\lambda} = \boxed{\mathbf{1}}
\end{aligned}$$

The mean of the Erlang-K distribution is $\frac{k}{\lambda}$, because it is the sum of $k$ exponential distributed random variables.

## Problem 9.1.2

For the Bernoulli distribution the distribution $x(i)$ and the mean $E(X)$ are:

$$x(i) = \begin{cases} 1-p & i=0 \quad \text{(failure)} \\ p & i=1 \quad \text{(success)} \end{cases}$$
$$E(X) = \sum_i i \cdot x(i) = p$$

For the binomial distribution the distribution $x(i)$ and the mean $E(X)$ are:

$$x(i) = \begin{pmatrix} n \\ i \end{pmatrix} \cdot p^i \cdot (1-p)^{n-i}$$
$$E(X) = np$$

For the geometric distribution the distribution $x(i)$ and the mean $E(X)$ are:

$$x(i) = (1-p)^i \cdot p$$
$$E(X) = \frac{(1-p)}{p}$$

All three distributions are related to each other, due to the fact that the binomial distribution are $n$ consecutive Bernoulli experiments and the geometric distribution is the number of failures in consecutive Bernoulli experiments until the first success.

## Problem 9.1.3

Recalling the shape of the probability density function of exponential distributed random variables explains, that it is more likely to have arrivals of for example 1 or 2 people than it is to have arrivals of 4 or 5 people. This is due to the fact that for a small $x$ the probability density function is much higher than it is for larger $x$. If we try to generalize, following the probability density function of exponential distributed random variables, it is more likely for single or smaller group arrivals than it is for larger group arrivals. This is all under the assumption that the arrivals are random.

**Chapter 4 – Sampling Theory**

**Problem 9.1.1**

Confidence intervals show how precise a sample estimates the true mean. With a statistical security (probability) of $1 - \alpha$ (confidence level) a sample $Z_n$ lies in the confidence interval $[-z_{1-\alpha/2}; +z_{1-\alpha/2}]$. Here $\alpha$ is the significance level.

**Problem 9.1.2**

The Student-t confidence intervals are only exact for a small number of samples $n$.

**Problem 9.1.3**

Pivot tables are used to organize results for flexible presentation. For example if we have system factors $x_1, x_2, x_3$ for a system response $R(x_1, x_2, x_3)$ we can show it in a two dimensional diagram with the response on the y-axis and one of the system factors on the x-axis or a three dimensional diagram including two system factors on the two x-axes. For the third parameter $x_3$ we can either set it to fixed value or compute the system response for all values of $x_3$. The problem here is that the main system factor, which influences the system response heavily isn't known a priori and therefore needs to be found. Many visualization tools have the requirement that the $x$ and $y$ values are given in consecutive column vectors, which can be achieved through organizing with a pivot table.

**Chapter 5 – Fundamentals of Statistics II**

**Problem 9.1.1**

For the discrete case the sample covariance is given as:

$$\overline{COV}[X,Y] = \frac{1}{n-1} \cdot \sum_{0 \le i < n} \big(X_i - \overline{X}(n)\big) \cdot \big(Y_i - \overline{Y}(n)\big)$$

For the continous case the sample covariance is defined as:

$$\overline{\mathrm{COV}}[X,Y] = \frac{1}{T} \cdot \int_0^T (X(t) - \overline{X}) \cdot (Y(t) - \overline{Y}) dt$$

The sample autocorrelation $\rho_j(n)$ is defined as the following equation, containing the sample autocovariance $\hat{\mathrm{C}}_{\mathrm{j}}(\mathrm{n})$ for the discrete case or $\hat{C}_u(T)$ for the continous case:

$$\rho_j(n) = \frac{\hat{C}_j(n)}{S^2(n)}$$
$$\hat{\mathrm{C}}_{\mathrm{j}}(\mathrm{n}) = \frac{1}{(\mathrm{n}-\mathrm{j})} \sum_{0\leq \mathrm{i}<\mathrm{n}-\mathrm{j}} \left[\mathrm{X}_{\mathrm{i}} - \overline{\mathrm{X}}\right] \left[\mathrm{X}_{\mathrm{i+j}} - \overline{\mathrm{X}}\right]$$
$$\hat{C}_u(T) = \frac{1}{T} \cdot \int_0^{T-u} (X(t+u) - \overline{X}) \cdot (X(t) - \overline{X}) dt$$

**Chapter 6 – Statistical Analysis of Simulation Data**

**Problem 9.1.1**

Since we are observing long term system parameters during non-terminating simulations, like "throughput", "queue size", etc. the beginning phase (transient phase) of such simulations isn't typical for the system and therefore should be excluded. A good example for this problem is the simulation of a manufacturing plant. Since the plant is empty at the start of the simulation, waiting times for products are very small. Due to that, only the "typical" state of the plant (steady state) should be included, which is most likely after few iterations (the end of the transient phase needs to be calculated).

**Problem 9.1.2**

The two methods for the calculation of confidence intervals in case of correlated simulation output presented in the lecture were the Batch-means method and the Replicate-Delete method.

**Chapter 7 – Stochastic Processes**

**Problem 9.1.1**

For $c_{var}(A) < 1$ the mean recurrence time $E(R)$ is smaller than the mean inter-arrival time $E(A)$ (which is essentially: $E(R) < E(A)$). For a $c_{var}(A) > 1$ the mean recurrence time $E(R)$ is larger than the mean inter-arrival time $E(A)$ (which is essentially: $E(R) > E(A)$).

**Chapter 8 – Discrete-Time Markov Chains**

**Problem 9.1.1**

**input:** factor distribution $y$
  initialize $P$ with zeros
  **for all** $i \in \mathcal{X}$ **do**
    **for all** $j \in \mathcal{Y}$ **do**
      $p_{i,f(i,j)} = p_{i,f(i,j)} + y[j]$
    **end for**
  **end for**
**output:** $P$

**Problem 9.1.2**

Probability that system changes after $k$ transitioin steps from state $i$ to state $j$: $p_{ij}^k$

**Problem 9.1.3**

One possibility is the Cesàro limit with $x_S = \lim_{n\to\infty}\left(\frac{1}{n+1}\cdot\sum_{i=0}^{n} x_i\right)$. If $x_i$ converges, the resulting limit is equal to the avergage state distribution. Another possibility is matrix powering, which is efficient for small matrices.

**Problem 9.1.4**

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 1 \\ 0.9 & 0 & 0.1 & 0 \end{pmatrix}$$

**Problem 9.1.5**

The sojourn time distribution is how long the markov chain stays in the state $j$. Basically we can see it as a shifted geometric distribution with $p_{jj}$ being a failure and $1 - p_{jj}$ being the success. This yields the desired distribution of $P(k \text{ steps }) = p_{jj}^{k-1} \cdot (1 - p_{jj})$.

## Chapter 9 – Continuous-Time Markov Chains

### Problem 9.1.1

Poisson Arrivals See Time Averages (PASTA): Blocking probability $p_B = x(n)$

### Problem 9.1.2

The Kaufman & Roberts formula describes the blocking probabilites in a multi dimensional loss system ($M/M/C-0$). It is used for the computation of service-specific blocking probabilities or for link dimensioning.

### Problem 9.1.3

Generally the transition probability from a process in state $i$ to a state $j$ is given by:

$$\frac{q_{ij}}{\sum_{k \neq i} q_{ik}}$$

Using this for our example we get:

$$\frac{q_{01}}{q_{01} + q_{02} + q_{03} + q_{04}}$$

### Problem 9.1.4

Upward transition rate $\lambda_i$ and downward transition rate $\mu_i$ for $M/M/n-0$:

$$\lambda_i = \lambda$$
$$\mu_i = i \cdot \mu$$

Upward transition rate $\lambda_i$ and downward transition rate $\mu_i$ for $M/M/1-K$:

$$\lambda_i = \lambda$$
$$\mu_i = \mu$$

Upward transition rate $\lambda_i$ and downward transition rate $\mu_i$ for $M/M/s-K$:

$$\lambda_i = \lambda$$
$$\mu_i = \begin{cases} i \cdot \mu & \text{for } i \leq s \\ s \cdot \mu & \text{for } i > s \end{cases}$$

**Chapter 10 – Distributions II**

**Problem 9.1.1**

Gamma distribution, Weibull distribution and Lognormal distribution