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Modellierung & Simulation I

Serie 06

Problem 6.1.1

We use the assumption, that the arrivals at the three interfaces are independent and the underlying poisson processes have the same rate λ. Then the superposition of these three poisson processes with rate λ is a poisson process with rate 3λ. Therefore the interarrival times follow an exponential distribution and the cumulative distribution function is $A(t) = 1 - e^{-3\lambda t}$.

Problem 6.1.2

Using the same assumptions as above, the interarrival times at the switch are exponentially distributed. The parameter is 3λ.

Problem 6.1.3

For an interval Δ of finite duration, the number of arrivals $\#$ within this timeframe is poisson distributed. With the above assumption the parameter is 3λ and the probability formula becomes:

$$P(\# = n) = \frac{(3\lambda)^n}{n!} e^{-3\lambda t}$$

Problem 6.2.1

If the traffic arrives with exponential inter-arrival times and only every n-th packet is extracted, the time between 2 extracted packets is Erlang-n distributed. This is, because the sum of n (independent) samples from the exponential distribution with rate λ determine the time t, after which the next extraction occurs. Therefor

$$P(D_A = t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$$

Problem 6.2.2

The convolution of 2 functions f and g is defined by

$$(f \otimes g) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

We use the proposition, that, if X and Y are independent random variables with PDFs f and g, then $X+Y$ has the probability density function $f \otimes g$. Per definition the sum of 2 exponential distributions with the same parameter λ gives the Erlang 2 distribution (with parameter λ). Then because these functions are only nonzero for nonnegative arguments the PDF of the Erlang 2 distribution is calculated as:

$$\begin{aligned} g(x,2,\lambda) &= (f \otimes f)(x,\lambda) \\ &= \int_0^x \lambda e^{-\lambda(x-y)} \lambda e^{-\lambda y} dy \\ &= \lambda^2 \int_0^x e^{-\lambda x} dy \\ &= \lambda^2 x e^{-\lambda x} \end{aligned}$$

Using the formula from the script for the Erlang-k distribution (3.2.3) we get:

$$\begin{aligned} g(x) &= \frac{(\lambda x)^{k-1}}{(k-1)!} \cdot \lambda \cdot e^{-\lambda x} \quad \text{for } k=2 \\ &= \frac{(\lambda x)^{2-1}}{(2-1)!} \cdot \lambda \cdot e^{-\lambda x} \\ &= \frac{\lambda x}{1!} \cdot \lambda \cdot e^{-\lambda x} \\ &= \lambda x \cdot \lambda \cdot e^{-\lambda x} \\ &= \lambda^2 x e^{-\lambda x} \end{aligned}$$

Which shows that the two formulas are identical.

Problem 6.2.3

With $p = \frac{1}{n}$ and the rate $p \cdot \lambda$ the pdf of the exponential distribution is:

$$f(x;\lambda) = \frac{\lambda}{n} e^{-\frac{\lambda}{n}x}$$

We get this by using the conditional probability that D_B takes k exponential inter-arrival times:

$$\begin{aligned} p(D_B) &= \sum_{i=1}^{k} p(D_B|e_i) \cdot p(e_i) \\ &= \sum_{i=1}^{k} \frac{1}{n} e^{-\frac{1}{n}x} \cdot (\lambda_i e^{-\lambda_i x}) \\ &= \sum_{i=1}^{k} \frac{\lambda_i}{n} e^{-\frac{\lambda_i}{n}x} \end{aligned}$$

Problem 6.2.4

Approach A: Erlang distribution with parameters n and λ

$$\begin{aligned} \mathbb{E}[D_A] &= \frac{n}{\lambda} \\ \mathbb{V}ar[D_A] &= \frac{n}{\lambda^2} \\ c_{var}[D_A] &= \frac{\sqrt{\mathbb{V}ar(D_A)}}{\mathbb{E}(D_A)} \\ &= \frac{\sqrt{\frac{n}{\lambda^2}}}{\frac{n}{\lambda}} \\ &= \frac{1}{\sqrt{n}} \end{aligned}$$

Approach B: Exponential distribution with parameters $p \cdot \lambda$

$$\begin{aligned}
\mathbb{E}[D_B] &= \frac{1}{p \cdot \lambda} \\
\mathbb{V}ar[D_B] &= \frac{1}{p^2 \cdot \lambda^2} \\
c_{var}[D_B] &= \frac{\sqrt{\mathbb{V}ar(D_B)}}{\mathbb{E}(D_B)} \\
&= \frac{\sqrt{\frac{1}{p^2 \cdot \lambda^2}}}{\frac{1}{p \cdot \lambda}} \\
&= 1
\end{aligned}$$

Problem 6.2.5

The mean time between two extracted packets of both approaches is the same:

$$\frac{1}{p \cdot \lambda} = \frac{1}{\frac{1}{n}\lambda} = \frac{n}{\lambda}$$

But because the coefficient of variation for approach A is usually lower than for approach B, the latter needs a faster packet processing engine.

Problem 6.2.6

We proof the claims by squaring the terms.
For approach A the discrete random variable N is a deterministic one, because only every n-th packet is extracted. Therefor we obtrain:

$$\begin{aligned}
c_{var}^n[A]^2 &= \frac{\mathbb{V}ar^n[A]}{\mathbb{E}^n[A]^2} \\
&= \frac{\mathbb{E}[N]\mathbb{V}ar[X] + \mathbb{V}ar[N]\mathbb{E}[X]^2}{\mathbb{E}[N]^2\mathbb{E}[X]^2} \\
&= \frac{n\mathbb{V}ar[X] + 0\mathbb{E}[X]^2}{n^2\mathbb{E}[X]^2} \\
&= \frac{\mathbb{V}ar[X]}{n\mathbb{E}[X]^2} \\
&= \frac{c_{var}[X]^2}{n}
\end{aligned}$$

For approach B the discrete random variable N is geometric distributed with $p = \frac{1}{n}$.

$$\begin{aligned}
c_{var}^{n}[B]^2 &= \frac{\mathbb{V}ar^n[B]}{\mathbb{E}^n[B]^2} \\
&= \frac{\mathbb{E}[N]\mathbb{V}ar[X] + \mathbb{V}ar[N]\mathbb{E}[X]^2}{\mathbb{E}[N]^2\mathbb{E}[X]^2} \\
&= \frac{\frac{1}{p}\mathbb{V}ar[X] + \frac{1-p}{p^2}\mathbb{E}[X]^2}{(\frac{1}{p})^2\mathbb{E}[X]^2} \\
&= \frac{n\mathbb{V}ar[X] + (1-\frac{1}{n})n^2\mathbb{E}[X]^2}{n^2\mathbb{E}[X]^2} \\
&= \frac{\frac{\mathbb{V}ar[X]}{\mathbb{E}[X]^2} + (1-\frac{1}{n})}{n} \\
&= \frac{c_{var}[X]^2 + (1-\frac{1}{n})}{n}
\end{aligned}$$

Problem 6.2.7

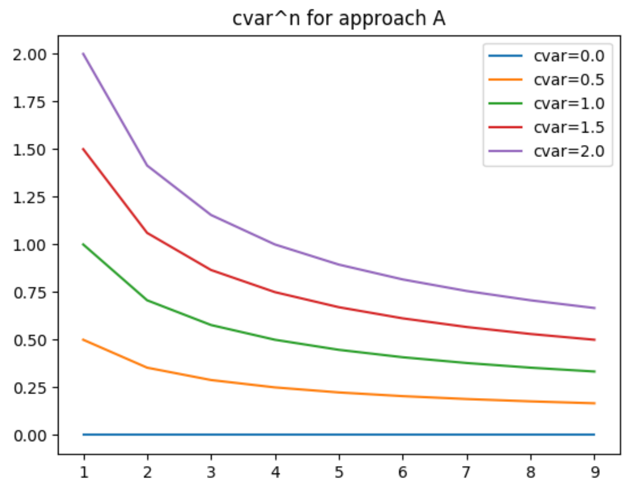

Figure 1: Plot for Approach A

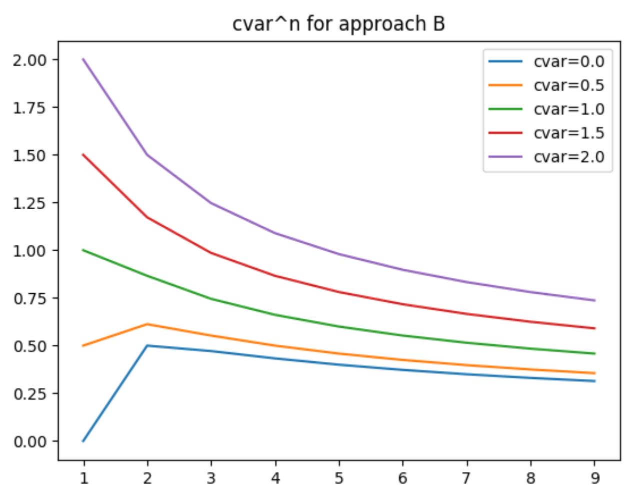

Figure 2: Plot for Approach B

Problem 6.2.8

This exercise was moved to part II of the lecture.