1	2	Σ
40	50	90/100

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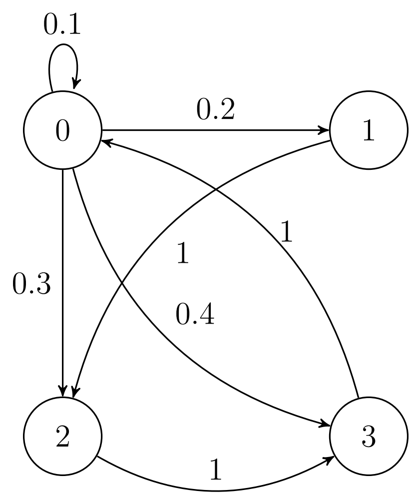
Modellierung & Simulation I

Serie 07

Problem 7.1.1

The state transition matrix and state transition diagram look pretty straight forward and how we expected them to be:



✓

10/10

State transition matrix: $P = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ ✓

Problem 7.1.2

Using matrix multiplication in R with the source code provided gives the following:

$$\begin{aligned}
x_0 &= (1, 0, 0, 0)\\
x_1 &= (0.1, 0.2, 0.3, 0.4)\\
x_2 &= (0.41, 0.02, 0.23, 0.34)\\
x_3 &= (0.381, 0.082, 0.143, 0.394)\\
x_4 &= (0.4321, 0.0762, 0.1963, 0.2954)\\
x_5 &= (0.33861, 0.08642, 0.20583, 0.36914)\\
x_6 &= (0.403001, 0.067722, 0.188003, 0.341274)\\
x_7 &= (0.3815741, 0.0806002, 0.1886223, 0.3492034)\\
x_8 &= (0.38736081, 0.07631482, 0.19507243, 0.34125194)\\
x_9 &= (0.379988021, 0.077472162, 0.192523063, 0.350016754)\\
x_{10} &= (0.3880155561, 0.0759976042, 0.1914685683, 0.3445182714)
\end{aligned}$$

√ 10/10

Here we can see that they slowly converge to their respective values and do not oscillate, like we see in a later exercise.

Problem 7.1.3

Using matrix powering in R with the source code provided gives the following:

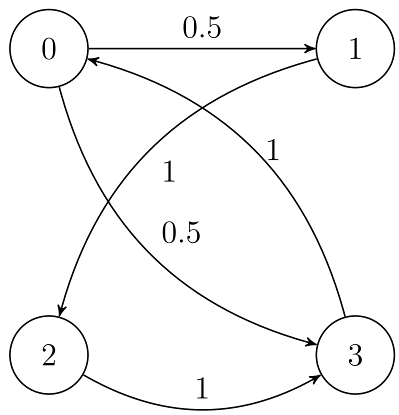
$$P^{32} = \begin{bmatrix} 0.3846154 & 0.07692306 & 0.1923077 & 0.3461538 \\ 0.3846154 & 0.07692306 & 0.1923077 & 0.3461538 \\ 0.3846154 & 0.07692306 & 0.1923077 & 0.3461538 \\ 0.3846154 & 0.07692306 & 0.1923077 & 0.3461538 \end{bmatrix}$$

√ 5/5

Which doesn't really change if we increase the number of powers any further. So the probability for being in state 0 in the long run is $p(0) \approx 0.3846154$, for state 1 it is $p(1) \approx 0.07692306$, for state 2 it is $p(2) \approx 0.1923077$ and for state 3 it is $p(3) \approx 0.3461538$. Also the probability of the states do not oscillate, like we see in following exercises.

Problem 7.1.5

The state transition matrix and state transition diagram still look pretty straight forward and how we expected them to be. However now we only have 2 possibilities to leave state 0, each with a 50% probability. This causes the oscillating behaviour we see in the following exercises:



$$\text{State transition matrix: } P = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

5/5

Problem 7.1.6

Using matrix multiplication in R with the source code provided gives the following:

$$
\begin{aligned}
x_0 &= (1, 0, 0, 0) \\
x_1 &= (0, 0.5, 0, 0.5) \\
x_2 &= (0.5, 0, 0.5, 0) \\
x_3 &= (0, 0.25, 0, 0.75) \\
x_4 &= (0.75, 0, 0.25, 0) \\
x_5 &= (0, 0.375, 0, 0.625) \\
x_6 &= (0.625, 0, 0.375, 0) \\
x_7 &= (0, 0.3125, 0, 0.6875) \\
x_8 &= (0.6875, 0, 0.3125, 0) \\
x_9 &= (0, 0.34375, 0, 0.65625) \\
x_{10} &= (0.65625, 0, 0.34375, 0)
\end{aligned}
$$

5/5

The consecutive state distributions do not converge. For explanation of the observed oscillation effect, please see the following exercise.

Problem 7.1.7

We observe oscillation with period $p = 2$. This is because all states can be either reached again after 2 (for state 0) or 4 (for states $1 - 3$) transition steps. The greatest common divisor of all cycle lengths in state transition graph is therefore 2, which causes the period p to be exactly 2. This causes consecutive state distributions to not converge, which we will also need to recall when doing matrix powering to determine the probability of the states for the long in the following exercise.

5/5

Problem 7.1.8

Using matrix powering in R with the source code provided gives the following:

$$P^{32} = \begin{bmatrix} 0.6666718 & 0.0000000 & 0.3333282 & 0.0000000 \\ 0.0000000 & 0.3333435 & 0.0000000 & 0.6666565 \\ 0.6666565 & 0.0000000 & 0.3333435 & 0.0000000 \\ 0.0000000 & 0.3333282 & 0.0000000 & 0.6666718 \end{bmatrix}$$

$$P^{33} = \begin{bmatrix} 0.0000000 & 0.3333359 & 0.0000000 & 0.6666641 \\ 0.6666565 & 0.0000000 & 0.3333435 & 0.0000000 \\ 0.0000000 & 0.3333282 & 0.0000000 & 0.6666718 \\ 0.6666718 & 0.0000000 & 0.3333282 & 0.0000000 \end{bmatrix}$$

$$P^{34} = \begin{bmatrix} 0.6666641 & 0.0000000 & 0.3333359 & 0.0000000 \\ 0.0000000 & 0.3333282 & 0.0000000 & 0.6666718 \\ 0.6666718 & 0.0000000 & 0.3333282 & 0.0000000 \\ 0.0000000 & 0.3333359 & 0.0000000 & 0.6666641 \end{bmatrix}$$

Here we can also see the oscillating behavior which causes shifts in the powered matrices. We observe the following probabilities for the different states, depending on the probabilites of the other states. Overall we can see that either state 1 and 3 have a non-zero probability or state 2 and 4:

$$\begin{aligned} p(0) &= 0.6666641 \quad \text{or} \quad 0 \\ p(1) &= 0.3333282 \quad \text{or} \quad 0 \\ p(2) &= 0.3333359 \quad \text{or} \quad 0 \\ p(3) &= 0.6666718 \quad \text{or} \quad 0 \end{aligned}$$

0/5

Problem 7.1.9

Yes, we could've just removed the loop for state 0 to observe the effect, so all states have a period p which is > 1. This causes the greatest common divisor of all cycle lengths in the state transition graph to be greater than 1, which then corresponds to a overall period p larger than 1. However because of the loop on state 0 it wouldn't be enough to just remove the transition from state 0 to state 2 to observe the wanted effect.

0/5

Problem 7.2.1

The state space for this problem is $X = \{0, 1, 2, 3, 4, 5, 6\}$. The set of of terminating states is $X_t = \{5, 6\}$. This is caused by the fact that after state 4, which corresponds to 4 points in the game, it is possible to either flip heads or tails. This will either put us exactly at 5 points (state 5) which exactly fulfills the termination condition of 5 points, or we will get one point more than needed with 6 (state 6). This also fulfills our termination condition.

5/5

Problem 7.2.2

With X the state space from above and $Y = \{1, 2\}$ the factor space with underlying uniform distribution, the state transition function/equation is $f(X, Y) = X + Y$.

This leads to the following state transition matrix:

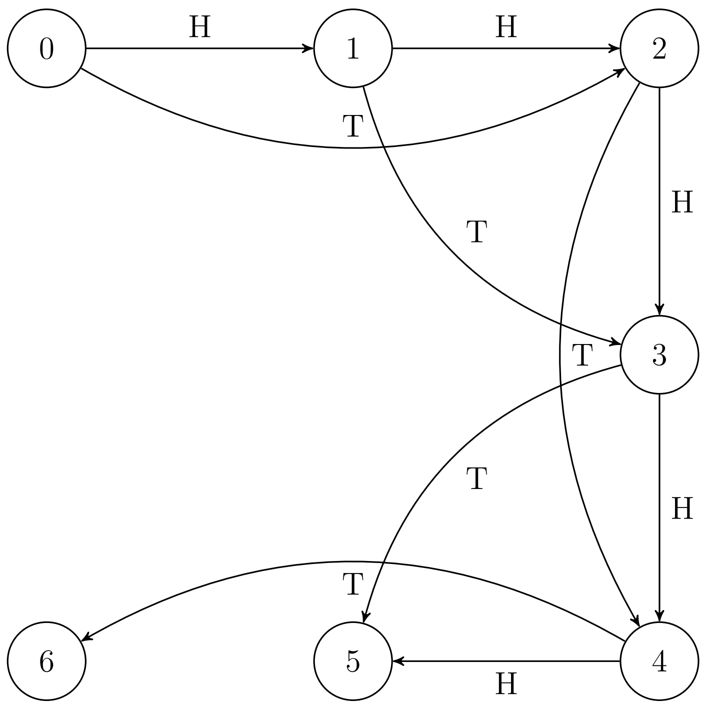
$$P = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

10/10

The ones at the positions $P_{5,5}$ and $P_{6,6}$ reflect the termination of the discrete markov chain in the states 5 and 6.

Problem 7.2.3

In our example of the state transition graph the node number represents the current points. The edge name displays the corresponding transition event (H: head, T: tail). We also have a start state $X_0 = 0$, which is due to the fact that the process always starts with 0 points. This gives the following state transition graph:



5/5

We omit the visualization of the transition from terminal states to themselves, because this contradicts the given view of markov chains with stopping times from the lectures.

Problem 7.2.4

We use the algorithm from lecture 8 page 148 and the transition matrix from above (compare Text in Figure 1 the first 8 lines).
With this we calculated the $x_n(i)$ (compare Text in Figure 1 lines 9 to 15).
The answer for task (a), the distribution that the game ends after n rounds with sum i is the submatrix of the $x_n(i)$ for $1 \le n \le 6$ and $i = \{6, 7\}$ printed in line 16 to 22.
Below that the mini table shows the answer to task (b). For every possible transition step it contains the probability p of termination.
In the last two lines the probabilities of terminating in one of the two terminal states is displayed.

```
     [,1] [,2] [,3] [,4] [,5] [,6] [,7]
[1,]    0  0.5  0.5  0.0  0.0  0.0  0.0
[2,]    0  0.0  0.5  0.5  0.0  0.0  0.0
[3,]    0  0.0  0.0  0.5  0.5  0.0  0.0
[4,]    0  0.0  0.0  0.0  0.5  0.5  0.0
[5,]    0  0.0  0.0  0.0  0.0  0.5  0.5
[6,]    0  0.0  0.0  0.0  0.0  1.0  0.0
[7,]    0  0.0  0.0  0.0  0.0  0.0  1.0
     [,1] [,2] [,3]  [,4]   [,5]    [,6]    [,7]
[1,]    1  0.0 0.00 0.000 0.0000 0.00000 0.00000
[2,]    0  0.5 0.50 0.000 0.0000 0.00000 0.00000
[3,]    0  0.0 0.25 0.500 0.2500 0.00000 0.00000
[4,]    0  0.0 0.00 0.125 0.3750 0.37500 0.12500
[5,]    0  0.0 0.00 0.000 0.0625 0.25000 0.18750
[6,]    0  0.0 0.00 0.000 0.0000 0.03125 0.03125
        [,1]    [,2]
[1,] 0.00000 0.00000
[2,] 0.00000 0.00000
[3,] 0.00000 0.00000
[4,] 0.37500 0.12500
[5,] 0.25000 0.18750
[6,] 0.03125 0.03125
[1] "terminating after n transitions | probability p"
[1] "--------------------------------|--------------"
[1] "                               1 |    0"
[1] "                               2 |    0"
[1] "                               3 |    0"
[1] "                               4 |    0.5"
[1] "                               5 |    0.4375"
[1] "                               6 |    0.0625"
[1] "probability of terminating in state 5: 0.65625"
[1] "probability of terminating in state 6: 0.34375"
```

Figure 1: Code output for a,b,c

30/30