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Modellierung & Simulation I

Serie 09

Chapter 1 - Fundamentals of Simulation Programs

Problem 9.1.1

It should be used for precision, because float and real have a different precision than long int. For example it holds that for real and float $10^{20} + 0.01 \triangleq 10^{20}$ which leads to the simulation time being unprecise for long simulations. Therefore using a long int is more practical. ✓

Problem 9.1.2

A product-oriented simulation is for example the simulation of the processing-time in a plant of a given product. In comparison towards that a machine-oriented approach is for example the simulation of different scheduling algorithms in one machine. ✓

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Chapter 2 - Fundamentals of Statistics I

Problem 9.1.1

For a continuous random variable X with the probability density function $f(x)$ the mean is defined as:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx \quad \checkmark$$

Problem 9.1.2

For a discrete random variable X with x_i being the value and $p(x_i)$ being the probability of the value, the mean is defined as:

$$E(X) = \sum_{i=1}^n x_i p(x_i) \quad \checkmark$$

Problem 9.1.3

The definition of the sample mean differs based on the fact if the process is time-discrete or time-continuous:

$$\overline{X^k} = \begin{cases} \overline{X^k}(n) = \frac{1}{n} \cdot \sum_{0 \leq i \leq n} X_i^k & \text{for time-discrete process} \\ \overline{X^k}(T) = \frac{1}{T} \cdot \int_0^T X(t)^k dt & \text{for time-continuous process} \end{cases}$$

$$\overline{X} = \overline{X^1} \quad \int$$

Problem 9.1.4

For discrete distributions the convolution (dt. "Faltung") is used for the sum of two random variables. For X_1, X_2 being random variables the distribution $X = X_1 + X_2$ is given by:

$$P(X = i) = x(i) = \sum_{j=0}^N x_1(j) \cdot x_2(i - j) = (x_1 * x_2)(i) \quad \checkmark$$

Problem 9.1.5

The distribution function of the minimum of two random variables A_0, A_1 is calculated like:

$$A = \min(A_0, A_1)$$

$$A^c(t) = A_1^c(t) \cdot A_2^c(t)$$

$$\left(a(t) = a_1(t) \cdot A_2^c(t) + a_2(t) \cdot A_1^c(t) \right) \quad \int \quad - - \checkmark$$

Problem 9.1.6

A histogram is a graphical representation of the distribution of a process. For this cause, a histogram needs to contain multiple segments (bins) with a constant or variable width. For each bin the number of values, which lie in the range of the respective bin, get counted. An example would be to see how the waiting times in a simulation of a queuing system are distributed. ✓

Problem 9.1.7

Time-weighting counters are counters, which take the time into account by weighting the sum \sum^k of the observed values with the time at $t_n \leq t < t_{n+1}$

like the following:

$$\Sigma^k = \sum_{0 \leq i < n} (X_i)^k \cdot (t_{i+1} - t_i)$$

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An example for time-weighting counters would be the mean waiting time over a given interval. ✓

Chapter 3 – Distributions I

Problem 9.1.1

For the exponential distribution the distribution function $F(x)$, the probability density function $f_\lambda(x)$, the mean $E(X)$, the standard deviation $\sigma(X)$, and the coefficient of variation $c_{var}(X)$ are the following:

$$\text{pdf : } f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad \checkmark$$

$$\text{cdf : } F(x) = 1 - e^{-\lambda x} \quad \checkmark$$

$$E(X) = \int_0^\infty t \cdot f_\lambda dt = \int_0^\infty t \cdot \lambda \cdot e^{-\lambda t} dt$$

$$= [t \cdot (-e^{-\lambda t})]_0^\infty - \int_0^\infty (-e^{-\lambda t}) dt = 0 - 0 - \left(\frac{1}{\lambda} [e^{-\lambda t}]_0^\infty \right) = \boxed{\frac{1}{\lambda}} \quad \checkmark$$

$$E(X^2) = \int_0^\infty t^2 \cdot f_\lambda dt = \int_0^\infty t^2 \cdot \lambda \cdot e^{-\lambda t} dt = \dots = \boxed{\frac{2}{\lambda^2}}$$

$$VAR(X) = E(X^2) - E(X)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

$$\sigma(X) = \sqrt{\frac{1}{\lambda^2}} = \boxed{\frac{1}{\lambda}} \quad \checkmark$$

$$c_{var}(X) = \frac{\sigma(X)}{E(X)} = \frac{1}{\lambda} / \frac{1}{\lambda} = \boxed{1} \quad \checkmark$$

The mean of the Erlang-K distribution is $\frac{k}{\lambda}$, because it is the sum of k exponential distributed random variables. ✓

Problem 9.1.2

For the Bernoulli distribution the distribution $x(i)$ and the mean $E(X)$ are:

$$x(i) = \begin{cases} 1-p & i=0 \quad (\text{failure}) \\ p & i=1 \quad (\text{success}) \end{cases}$$
$$E(X) = \sum_i i \cdot x(i) = p$$

For the binomial distribution the distribution $x(i)$ and the mean $E(X)$ are:

$$x(i) = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$$
$$E(X) = np$$

For the geometric distribution the distribution $x(i)$ and the mean $E(X)$ are:

$$x(i) = (1-p)^{i-1} \cdot p$$
$$E(X) = \frac{1-p}{p}$$

All three distributions are related to each other, due to the fact that the binomial distribution are n consecutive Bernoulli experiments and the geometric distribution is the number of failures in consecutive Bernoulli experiments until the first success.

Problem 9.1.3

Recalling the shape of the probability density function of exponential distributed random variables explains, that it is more likely to have arrivals of for example 1 or 2 people than it is to have arrivals of 4 or 5 people. This is due to the fact that for a small x the probability density function is much higher than it is for larger x . If we try to generalize, following the probability density function of exponential distributed random variables, it is more likely for single or smaller group arrivals than it is for larger group arrivals. This is all under the assumption that the arrivals are random.

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Zusammenhang
zu 1.1.1?
- ✓

Chapter 4 – Sampling Theory

Problem 9.1.1

Confidence intervals show how precise a sample estimates the true mean. With a statistical security (probability) of $1 - \alpha$ (confidence level) a sample Z_n lies in the confidence interval $[-z_{1-\alpha/2}; +z_{1-\alpha/2}]$. Here α is the significance level. (v)

Problem 9.1.2

The Student-t confidence intervals are only exact for a small number of samples n . f - 1

Problem 9.1.3

Pivot tables are used to organize results for flexible presentation. For example if we have system factors x_1, x_2, x_3 for a system response $R(x_1, x_2, x_3)$ we can show it in a two dimensional diagram with the response on the y-axis and one of the system factors on the x-axis or a three dimensional diagram including two system factors on the two x-axes. For the third parameter x_3 we can either set it to fixed value or compute the system response for all values of x_3 . The problem here is that the main system factor, which influences the system response heavily isn't known a priori and therefore needs to be found. Many visualization tools have the requirement that the x and y values are given in consecutive column vectors, which can be achieved through organizing with a pivot table. 7/8 ✓

Chapter 5 – Fundamentals of Statistics II

Problem 9.1.1

For the discrete case the sample covariance is given as:

$$\overline{COV}[X, Y] = \frac{1}{n-1} \cdot \sum_{0 \leq i < n} (X_i - \bar{X}(n)) \cdot (Y_i - \bar{Y}(n)) \quad \checkmark$$

For the continuous case the sample covariance is defined as:

$$\overline{COV}[X, Y] = \frac{1}{T} \cdot \int_0^T (X(t) - \bar{X}) \cdot (Y(t) - \bar{Y}) dt$$

The sample autocorrelation $\rho_j(n)$ is defined as the following equation, containing the sample autocovariance $\hat{C}_j(n)$ for the discrete case or $\hat{C}_u(T)$ for the continuous case:

$$\rho_j(n) = \frac{\hat{C}_j(n)}{S^2(n)} \quad \checkmark$$

$$\hat{C}_j(n) = \frac{1}{(n-j)} \sum_{0 \leq i < n-j} [X_i - \bar{X}] [X_{i+j} - \bar{X}]$$

$$\hat{C}_u(T) = \frac{1}{T} \cdot \int_0^{T-u} (X(t+u) - \bar{X}) \cdot (X(t) - \bar{X}) dt$$

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Chapter 6 – Statistical Analysis of Simulation Data

Problem 9.1.1

Since we are observing long term system parameters during non-terminating simulations, like "throughput", "queue size", etc. the beginning phase (transient phase) of such simulations isn't typical for the system and therefore should be excluded. A good example for this problem is the simulation of a manufacturing plant. Since the plant is empty at the start of the simulation, waiting times for products are very small. Due to that, only the "typical" state of the plant (steady state) should be included, which is most likely after few iterations (the end of the transient phase needs to be calculated). \checkmark

Problem 9.1.2

The two methods for the calculation of confidence intervals in case of correlated simulation output presented in the lecture were the Batch-means method and the Replicate-Delete method. \checkmark

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Chapter 7 – Stochastic Processes

Problem 9.1.1

For $c_{var}(A) < 1$ the mean recurrence time $E(R)$ is smaller than the mean inter-arrival time $E(A)$ (which is essentially: $E(R) < E(A)$). For a $c_{var}(A) > 1$ the mean recurrence time $E(R)$ is larger than the mean inter-arrival time $E(A)$ (which is essentially: $E(R) > E(A)$). \checkmark

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Chapter 8 – Discrete-Time Markov Chains

Problem 9.1.1

```
input: factor distribution  $y$   
        initialize  $P$  with zeros  
        for all  $i \in \mathcal{X}$  do  
            for all  $j \in \mathcal{Y}$  do  
                 $p_{i,f(i,j)} = p_{i,f(i,j)} + y[j]$   
            end for  
        end for  
output:  $P$ 
```

Problem 9.1.2

Probability that system changes after k transition steps from state i to state j : p_{ij}^k

Problem 9.1.3

One possibility is the Cesàro limit with $x_S = \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \cdot \sum_{i=0}^n x_i \right)$. If x_i converges, the resulting limit is equal to the average state distribution. Another possibility is matrix powering, which is efficient for small matrices.

Problem 9.1.4

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 1 \\ 0.9 & 0 & 0.1 & 0 \end{pmatrix}$$

Problem 9.1.5

The sojourn time distribution is how long the markov chain stays in the state j . Basically we can see it as a shifted geometric distribution with p_{jj} being

a failure and $1 - p_{jj}$ being the success. This yields the desired distribution of $P(k \text{ steps}) = p_{jj}^{k-1} \cdot (1 - p_{jj})$. ✓

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Chapter 9 – Continuous-Time Markov Chains

Problem 9.1.1

Poisson Arrivals See Time Averages (PASTA): Blocking probability $p_B = x(n)$ ✓

Problem 9.1.2

The Kaufman & Roberts formula describes the blocking probabilities in a multi dimensional loss system ($M/M/C - 0$). It is used for the computation of service-specific blocking probabilities or for link dimensioning. ✓

Problem 9.1.3

Generally the transition probability from a process in state i to a state j is given by:

$$\frac{q_{ij}}{\sum_{k \neq i} q_{ik}}$$

Using this for our example we get:

$$\frac{q_{01}}{q_{01} + q_{02} + q_{03} + q_{04}} \quad \checkmark$$

Problem 9.1.4

Upward transition rate λ_i and downward transition rate μ_i for $M/M/n - 0$:

$$\begin{aligned} \lambda_i &= \lambda \\ \mu_i &= i \cdot \mu \end{aligned} \quad \checkmark$$

Upward transition rate λ_i and downward transition rate μ_i for $M/M/1 - K$:

$$\begin{aligned} \lambda_i &= \lambda \\ \mu_i &= \mu \end{aligned} \quad \checkmark$$

Upward transition rate λ_i and downward transition rate μ_i for $M/M/s - K$:

$$\lambda_i = \lambda$$
$$\mu_i = \begin{cases} i \cdot \mu & \text{for } i \leq s \\ s \cdot \mu & \text{for } i > s \end{cases}$$

✓

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Chapter 10 – Distributions II

Problem 9.1.1

Gamma distribution, Weibull distribution and Lognormal distribution

✓

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