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Lower bounds of the success probability in quantum state exclusion for general ensembles

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Ab ovo usque ad mala.

Horace

Abstract

Given a quantum state known to be prepared from an ensemble of two or more states, quantum state exclusion aims to rule out the possibility that it was prepared in a particular state from the ensemble. Using the known solution for group generated ensembles [4], we study this result as a lower bound for randomly generated ensembles via semidefinite programming.

Keywords: *SDP, Quantum state exclusion , Add keywords.*

I INTRODUCTION

In many real-world scenarios, excluding a certain hypothesis can be more practical than solving the problem entirely. For instance, in disease diagnosis, ruling out potential diseases often serves as the first step in identifying the actual condition. Similarly, when repairing a machine, it is sometimes more efficient to identify the components that are functioning correctly, which narrows down the search for the faulty part.

In this project, we project this idea into the quantum realm by focusing on Quantum State Exclusion (QSE). Rather than determining the exact state of a quantum system, we aim to eliminate one or more possible candidates from a known ensemble of states. Notice, this approach can be more suitable or efficient in certain quantum information tasks.

Given a quantum state known to be prepared from a finite ensemble, Quantum State Discrimination (QSD) seeks to identify which specific state from the ensemble corresponds to the given system. In contrast, QSE [2] adopts the opposite perspective: it aims to determine which states from the ensemble do not correspond to the prepared state. While QSD has been deeply studied in recent years[1], with significant advances since its inception [5], QSE offers a complementary framework with distinct advantages.

Although the tasks of exclusion and discrimination coincide for ensembles containing only two states [1], when dealing with ensembles of three or more states, the two problems diverge in both approach and complexity. One of the most significant features of QSE is the possibility of achieving *perfect exclusion*, where certain states can be ruled out with zero probability of error in cases where perfect discrimination is impossible[3].

This capability opens new frontiers in quantum information theory, particularly in the context of partial information retrieval from quantum systems. By excluding certain states, it is possible to gain insight into the encoded information without needing to fully determine the original state.

As with QSD, obtaining a general analytical solution for QSE remains an open problem. However, analytical results have been found in specific cases when the ensemble of quantum states exhibits a certain degree of symmetry. In particular, when the ensemble is generated by the action of a finite group, the problem becomes more tractable and exact solutions have been derived.

The exclusion task can be carried out under two main protocols: *Minimum Error* and *Zero Error* [2]. In the Minimum Error scenario, the goal is to minimize the probability of mistakenly excluding the actual prepared state. In contrast, the Zero Error approach seeks to exclude a state with absolute certainty, even if that means sometimes the procedure yields an inconclusive result.

Building on recent results that provide ex-

[1] Since for the two states case excluding one necessarily implies identifying the other.

[2] Also known as *unambiguous exclusion*.

act solutions for exclusion tasks in group generated ensembles [4], this project undertakes a numerical study of such results as lower bounds for more general, randomly generated ensembles. To this end, we employ Semidefinite Program (SDP) to explore QSE performance in arbitrary settings. Furthermore, we investigate improved bounds for the general case based on how closely a given ensemble resembles a group-generated one[3].

II FORMULATION OF THE PROBLEM

Let $\{(\rho_i, \eta_i)\}_{i=1}^n$ be an ensemble of n quantum states, where each ρ_i denotes a pure-state density matrix, $\rho_i = |\psi_i\rangle\langle\psi_i|$, and η_i represents the prior probability of occurrence of state ρ_i. Let ρ_j be the target state from this ensemble. Our goal is to develop a procedure to find another state $\rho_k \in \{\rho_i\}_{i=1}^n$, such that $\rho_k \neq \rho_j$.

Quantum measurements are represented by a set of Positive Operator-Valued Measures (POVMs), denoted $\{\Pi_i\}_{i=1}^n$, acting on the Hilbert space $\mathcal{H}$ of the quantum system.

The goal of the minimum-error (Minimum Error (ME)) protocol is miminizing the probability of excluding the target state from our hypothesis. In an SDP formulation the problem reads,

$$P^e_{\mathrm{ME}} = \min_{\{\Pi_i\}} \sum_{i=1}^n \mathrm{Tr}(\Pi_i \rho_i),$$
$$\text{subject to} \sum_{i=1}^n \Pi_i = \mathbb{1}, \Pi_i \geq 0 \quad \forall i \in \{1, \dots, n\}.$$

Note that the conditions $\sum_{i=1}^n \Pi_i = \mathbb{1}$ and $\Pi_i \geq 0$ define the POVM elements, ensuring positivity and completeness. The superscript e in P^e_{ME} stands for the *error probability*.

Alternatively, the problem can be formulated in terms of the *success probability* P^s_{ME}, which represents the probability of successful exclusion. This is other perspective is nothing but,

$$P^s_{\mathrm{ME}} = \max_{\{\Pi_i\}} \left(1 - \sum_{i=1}^n \mathrm{Tr}(\Pi_i \rho_i)\right),$$
$$\text{subject to} \sum_{i=1}^n \Pi_i = \mathbb{1}, \Pi_i \geq 0 \quad \forall i \in \{1, \dots, n\}.$$

Naturally, $P^s_{\mathrm{ME}} + P^e_{\mathrm{ME}} = 1$.

In the case of the zero-error (Zero Error (ZE)) protocol, the POVMs must additionally satisfy the unambiguity condition i.e. each measurement operator Π_i must be orthogonal to the corresponding state ρ_i, in other words,

$$\mathrm{Tr}(\Pi_i \rho_i) = 0 \quad \forall i \in \{1, \dots, n\}.$$

An additional inconclusive outcome, denoted by a POVM element $\Pi_?$ (representing failure to exclude any state), must be introduced to ensure the completeness of the measurement,

$$\Pi_? = \mathbb{1} - \sum_{i=1}^n \Pi_i.$$

If the measurement outcome corresponds to $\Pi_?$ (i.e., "?" is observed), the result is inconclusive. The corresponding SDP for minimizing the error probability (i.e., the probability of an inconclusive outcome) in the ZE protocol is:

$$P^e_{\mathrm{ZE}} = \min_{\{\Pi_i\}} \sum_{i=1}^n \mathrm{Tr}(\Pi_? \rho_i),$$
$$\text{subject to} \quad \sum_{i=1}^n \Pi_i + \Pi_? = \mathbb{1}, \quad \Pi_? \geq 0,$$
$$\mathrm{Tr}(\Pi_i \rho_i) = 0, \quad \Pi_i \geq 0 \quad \forall i \in \{1, \dots, n\}.$$

Similarly, the corresponding success probability is:

[3] The notion of "how close" will be formally defined in Section add section.

$$P^s_{\mathrm{ZE}} = \max_{\{\Pi_i\}} \left(1 - \sum_{i=1}^{n} \mathrm{Tr}(\Pi_? \rho_i) \right),$$

$$\text{subject to} \quad \sum_{i=1}^{n} \Pi_i + \Pi_? = \mathbb{1}, \quad \Pi_? \geq 0,$$
$$\mathrm{Tr}(\Pi_i \rho_i) = 0, \quad \Pi_i \geq 0 \quad \forall i \in \{1, \ldots, n\}.$$

Note that this formulation is analogous to the ME protocol, except for the constraint $\mathrm{Tr}(\Pi_i \rho_i) = 0$, which enforces unambiguous discrimination.

Finally, we define a **group-generated ensemble** as one constructed by applying a finite group of unitary transformations to a fixed seed state. For example, let U be a unitary operator such that $U^n = \mathbb{1}$. Then the states of the ensemble are given by,

$$\rho_i = U^i \rho (U^\dagger)^i, \quad i = 1, \ldots, n,$$

where ρ is the seed state. This forms a $\mathbb{Z}/n\mathbb{Z}$ generated ensemble.

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LIST OF ABBREVIATIONS

ME	Minimum Error.
POVM	Positive Operator-Valued Measure.
QSD	Quantum State Discrimination.
QSE	Quantum State Exclusion.
SDP	Semidefinite Program.
ZE	Zero Error.