Quantum state exclusion for group-generated ensembles of pure states

A. Diebra[1], S. Llorens[1], E. Bagan[1], G. Sentís[1], and R. Munoz-Tapia[1]
[1]*Física Teòrica: Informació i Fenòmens Quàntics, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain*

We completely solve the quantum state exclusion task for group-generated sets of pure quantum states. We derive necessary and sufficient conditions alternative to existing ones. We obtain the optimal success probability for state exclusion when perfect exclusion is impossible. We derive the analytical expressions for both minimum error and unambiguous identification.

PACS numbers: 03.67.-a, 03.65.Ta,42.50.-p

Introduction.– In many scientific and practical scenarios, the ability to exclude certain possibilities is often more valuable than identifying the exact state of a system. This concept is well-illustrated in classical contexts. For instance, in medical diagnostics, ruling out potential diseases can be crucial for narrowing down treatment options, even if the exact disease remains unidentified. Similarly, in testing engineering systems, excluding certain failure modes is often more advantageous than trying to identify the exact faulty process. Also, in network security, excluding potential vulnerabilities is usually more efficient than trying to track down the exact nature of a threat.

This idea extends into the domain of quantum mechanics through the task of quantum state exclusion (QSE) [1]. Given a quantum system prepared in an unknown state from a predetermined set, the goal is to devise a protocol to exclude one possible state. This task is also termed quantum antidistinguishability [2] and it is particularly relevant in the context of quantum state assignments, where the compatibility of different state assignments must be assessed [3, 4]. It also plays a crucial role in the debate about the physical reality or not of quantum systems [5]. Contrasting the traditional approach of quantum state discrimination [6–8], where the objective is to identify the exact state, QSE focuses on ruling out certain states, providing a different perspective on the information that can be extracted from quantum systems. In the context of resource theory, it has been proven that the convex weight measure, which evaluates the advantage provided by the resource over any free device, corresponds to an exclusion task [9, 10]. QSE has been shown to give an operational interpretation of the Choi rank of a quantum channel [11]. This rank, among other properties, sets bounds on the minimum number of Kraus operators needed to decompose a channel. Non-contextual inequalities have also been obtained from quantum state exclusion [12] as well as implications in communication complexity [13, 14]. Very recently, Chernoff error exponents have been calculated in [15].

Discrimination problems are notoriously involved beyond the binary case (a.k.a. hypothesis testing). In a multiple hypotheses scenario, only cases with some symmetries that reduce the complexity of the problem have been solved [16–25]. This principle of symmetry has been considered in [26] to derive tight bounds on QSE for circulant sets of pure quantum states.

As far as we are aware, all the existing literature on QSE has focused on perfect exclusion, that is, on the study of protocols that allow the exclusion without error of a state (or states) in which a system could have been prepared. Furthermore, these results are often not constructive, i.e., the measurement attaining the optimal bound is not explicitly given, only its existence is stated [26]. In this letter, we provide for the first time a complete analytical solution for group-generated ensembles of pure states. Namely, we obtain the success probability for the minimum error approach for any group and any range of the parameter space, be it in the region of perfect exclusion or outside. We also prove that the optimal measurement can always be taken to contain rank-one operators [~~in the whole parameter region~~] and provide a constructive proof. Our approach is [~~based on an efficient use of~~ exploits] the duality properties of semidefinite programming [27, 28] and uses group representation theory [results] [29] applied to the Gram matrix [30]. We also investigate the zero-error (unambiguous) approach [31], which permits inconclusive answers but forbids erroneous ones. In this context, we also derive the complete analytical expression of the success probability for any group-generated ensemble. Our findings are summarized in Theorems I and II. The optimal probabilities are expressed in terms of the eigenvalues of the Gram matrix, which can then be extended to non-symmetric sources. Remarkably, [our expressions also provide a very accurate approximation in these general scenarios].

This letter is organized as follows. First, we present the formalism of the problem. We then derive the results for the minimum error approach and provide a proof of a crucial lemma that states that the optimal measurement can be constructed from rank-one operators. Next, we address the zero-error approach and, finally, we conclude with a summary of our findings.

Formalism of the problem.– The task of excluding a certain hypothesis from an ensemble of arbitrary quantum states, $\mathcal{E} = \{\eta_i, \rho_i\}_i$, where ρ_i represents the pure states composing the ensemble and η_i their respective prior probabilities, is determined by a quantum measurement with elements $\{\Pi_i\}$. Upon obtaining a given outcome j, the hypothesis ρ_j should be excluded. Minimizing the probability of incorrectly excluding the correct hypothesis, P_e, requires optimization over the set of all possible POVMs. This task can be formulated as a semidefinite program (SDP)

$$P_e = \min_{\{\Pi_i\}} \sum_i \eta_i \operatorname{Tr}(\Pi_i \rho_i)\,, \qquad \text{s.t.} \sum_i \Pi_i = \mathbb{1}, \quad \Pi_i \geq 0 \ \ \forall i\,. \tag{1}$$

~~This formulation corresponds to what is known as minimum error protocol. Alternatively, there is another protocol, the zero-error protocol, in which no error is allowed when excluding a hypothesis. However, this restriction comes with the cost of introducing an additional POVM element, the inconclusive one, $\Pi_?$. If this outcome occurs, no decision can be made. The goal of the optimization in this case is to minimize the probability of obtaining an inconclusive outcome~~

[RMT: This formulation is known as the minimum error protocol. Alternatively, the zero-error protocol allows no errors when excluding a hypothesis. However, this requires an additional POVM element, the inconclusive outcome, $\Pi_?$. If this occurs, no decision can be made. The goal of optimization here is to minimize the probability of an inconclusive outcome.]

$$P_? = \min_{\{\Pi_i\}} \sum_i \eta_i \operatorname{Tr}(\Pi_? \rho_i)\,, \qquad \text{s.t.} \sum_i \Pi_i = \mathbb{1} - \Pi_?, \qquad \Pi_? \geq 0\,,\ \Pi_i \geq 0\,,\ \operatorname{Tr}(\Pi_i\rho_i) = 0\ \forall i\,. \tag{2}$$

In this work, we consider quantum ensembles that are group-generated, [~~meaning that~~ i.e.,] all states are obtained by applying the transformations of all group elements to a seed state $|\psi\rangle$: $|\psi_g\rangle = U_g|\psi\rangle$, where $g \in \mathcal{G}$ and U_g is its unitary representation in the Hilbert space. We assume that all hypotheses are equally probable, $\eta_g = 1/|\mathcal{G}|$. Analogous to quantum state discrimination tasks, all relevant information is encoded in the Gram matrix of the ensemble, whose entries are the overlaps between the hypotheses, $G_{gh} = \langle\psi_g|\psi_h\rangle$, with $g, h \in \mathcal{G}$. ~~The underlying symmetries of our problem determine the structure of the Gram matrix, namely, the Gram matrix belongs to the algebra of the so-called regular representation of the group [32]. In the supplemental material [?] we construct an explicit example for an abelian group.~~ [RMT: The symmetries of our problem determine the structure of the Gram matrix, which belongs to the algebra of the regular representation of the group [32]. An explicit example for an abelian group is provided in the supplemental material [?].]

[RMT: ~~To better understand~~ Here we describe? the structure of the problem for a general group. Consider] a finite-dimensional Hilbert space $\mathcal{H}$ over which the unitary representations $\{U_g\}$ act. This space can be decomposed into orthogonal subspaces as [32]

$$\mathcal{H} = \bigoplus_\mu \mathcal{H}_\mu \otimes \mathbb{C}^{m_\mu}\,, \tag{3}$$

where the sum runs over all irreducible representations, irreps. for short, into which the $\mathcal{U}_\mathcal{G}$ can be decomposed, and m_μ denotes their respective multiplicities. Here, the subspace $\mathcal{H}_\mu$ carries the irrep. μ, while $\mathbb{C}^{m_\mu}$ denotes the multiplicity space, over which $\mathcal{U}_\mathcal{G}$ acts trivially, namely $U_g = \bigoplus_\mu U_g^\mu \otimes \mathbb{1}_{m_\mu}$, where each representation $\{U_g^\mu\}$ is irreducible and $\mathbb{1}_d$ denotes the projector onto a d-dimensional subspace.

A key aspect of this decomposition is that it allows us to write the seed state as $|\psi\rangle = 1/|\mathcal{G}|^{[1/2]} \sum_\mu \sqrt{d_\mu}|\psi_\mu\rangle$, ~~with respect to~~ [RMT: for] the decomposition of Eq. (3). Here, d_μ denotes the dimension of the irrep μ, and $|\psi_\mu\rangle \in \mathcal{H}_\mu \otimes \mathbb{C}^{m_\mu}$ are non-normalized bipartite vectors ensuring that $\||\psi\rangle\|^2 = 1$. It is convenient to express each bipartite vector $|\psi_\mu\rangle$ in a Schmidt-like form

$$|\psi_\mu\rangle = \sum_{k=1}^{r_\mu} \sqrt{\alpha_k^\mu}\, |v_k^\mu\rangle|u_k^\mu\rangle\,, \tag{4}$$

where $r_\mu = \min\{d_\mu, m_\mu\}$ and $\alpha_k^\mu \geq 0$, $\forall \mu, k$. Both $\{|v_k^\mu\rangle\}_{k=1}^{d_\mu}$ and $\{|u_k^\mu\rangle\}_{k=1}^{m_\mu}$ are orthonormal basis of $\mathcal{H}_\mu$ and $\mathbb{C}^{m_\mu}$ respectively. Expressing $|\psi\rangle$ in this way, the operator $\Omega = \sum_g U_g|\psi\rangle\langle\psi|U_g^\dagger$ is diagonal, namely

$$\Omega = \bigoplus_\mu \mathbb{1}_{d_\mu} \otimes \sum_{k=1}^{r_\mu} \alpha_k^\mu |u_k^\mu\rangle\langle u_k^\mu|\,, \tag{5}$$

where α_k^μ are the different eigenvalues, each one with multiplicity d_μ.

Results.– Most of the efforts on QSE [RMT: have focused] on quantum state ensembles that can be perfectly excluded[[1, 11, 14, 26, 31]], that is, when the SDP in Eq. (1) outputs a null value. In this work, we do not restrict ourselves to the region where the exclusion is conclusive. [~~Moreover,~~] [We] consider two different protocols for the region of non-perfect exclusion: minimum and zero error protocols, analogous to their homonymous in quantum state discrimination.

Our first result gives the optimal error probability for excluding one of the hypotheses from our ensemble with the minimum error protocol.

Theorem I. *Let $\mathcal{E}$ be an ensemble of $N = |\mathcal{G}|$ group-generated pure states, and let all the hypotheses be equally probable with Gram matrix G, whose eigenvalues are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{|\mathcal{G}|}$. Then, the probability of excluding the wrong hypothesis is given by*

$$P_e = \begin{cases} 0 & \text{if} \quad \sqrt{\lambda_1} \leq \sum_{j\neq 1} \sqrt{\lambda_j} \\ \frac{\left(2\sqrt{\lambda_1} - \operatorname{Tr}\sqrt{G}\right)^2}{|\mathcal{G}|^2} & \text{otherwise}\,. \end{cases} \tag{6}$$

To prove Theorem I we use the following Lemma, which establishes the projective form of the optimal POVM for the minimum error protocol.

Lemma 1. *The optimal POVM for minimum error state exclusion can be constructed from only rank-1 operators, with symmetry $\Pi_g = U_g \Pi U_g^\dagger$, and $\Pi = |\omega\rangle\langle\omega|$. Moreover, if the condition $\sqrt{\lambda_1} \leq \sum_{j\neq 1} \sqrt{\lambda_j}$ does not hold the seed projector has the simple form*

$$|\omega\rangle = \frac{1}{\sqrt{|\mathcal{G}|}} \left(|v^1\rangle|u^1\rangle - \sum_{\mu\neq 1} \sqrt{d_\mu} \sum_{k=1}^{r_\mu} |v_k^\mu\rangle|u_k^\mu\rangle \right), \tag{7}$$

with $|v^1\rangle$ and $|u^1\rangle$ the decomposition for the maximum eigenvalue of Ω in Eq. (5)

Proof. Consider the POVM elements [$\Pi_g = U_g|\omega\rangle\langle\omega|U_g^\dagger$] for some $|\omega\rangle \in \mathcal{H}$, with $\sum_{g\in\mathcal{G}} U_g|\omega\rangle\langle\omega|U_g^\dagger = \mathbb{1}_\psi$, where $\mathbb{1}_\psi$ is the projector onto the subspace spanned by the orbit of the seed state. Therefore, by Eqs. (4) and (5), the state $|\omega\rangle$ is of the form

$$|\omega\rangle = \frac{1}{\sqrt{|\mathcal{G}|}} \sum_\mu \sqrt{d_\mu} \sum_{j=1}^{r_\mu} |e_j^\mu\rangle|f_j^\mu\rangle\,, \tag{8}$$

where $\{|e_j^\mu\rangle\}_{j=1}^{d_\mu}$ and $\{|f_j^\mu\rangle\}_{j=1}^{m_\mu}$ form orthonormal bases of $\mathcal{H}_\mu$ and $\mathbb{C}^{m_\mu}$, respectively, with the extra condition that $\operatorname{span}\{|f_j^\mu\rangle\}_j = \operatorname{span}\{|u_k^\mu\rangle\}_k\,, \forall\mu$.

The average error probability $P_e = |\langle\omega|\psi\rangle|^2$ takes then the following form

$$P_e = \frac{1}{|\mathcal{G}|^2} \left| \sum_\mu d_\mu \sum_{j,k=1}^{r_\mu} \sqrt{\alpha_j^\mu}\, |\langle v_j^\mu|e_k^\mu\rangle\langle u_j^\mu|f_k^\mu\rangle|\, e^{i\theta_{jk}^\mu} \right|^2, \tag{9}$$

where $e^{i\theta_{jk}^\mu}$ is taken out from the overlap for simplicity.

We minimize the error probability by choosing the proper decomposition of $|\omega\rangle$. Let us consider directly the case for the right regular representation, as working in this representation the ensemble presents the same (anti-) distinguishability properties as in any other given representation (see Appendix A for a detailed explanation). [RMT: AQUI S'HAURIA DE DIR ALGUNA COSA DE LA GRAM, NO? HO TROBO A FALTAR] In the right regular representation, the dimension of the irreps. is equal to their multiplicities, $d_\mu = m_\mu$. Therefore, the decomposition in Eq. (4), $\{|u_k^\mu\rangle\}_{k=1}^{r_\mu}$ spans the space $\mathbb{C}^{d_\mu}$, since $r_\mu = d_\mu$. Then, we are free to choose any basis $\{|f_j^\mu\rangle\}_{j=1}^{d_\mu}$ of $\mathbb{C}^{d_\mu}$. We choose the bases $|e_j^\mu\rangle$ and $|f_j^\lambda\rangle$ to be mutually unbiased with $|v_k^\mu\rangle$ and $|u_k^\mu\rangle$ respectively, i.e.,

$$|\langle e_j^\mu|v_k^\mu\rangle| = |\langle f_j^\mu|u_k^\mu\rangle| = \frac{1}{\sqrt{d_\mu}} \quad \forall j,k\,, \tag{10}$$

which we can always do [33]. Then, the average error probability reads

$$P_e = \frac{1}{|\mathcal{G}|^2} \left| \sum_\mu \sum_{j,k=1}^{d_\mu} \sqrt{\alpha_j^\mu}\, e^{i\theta_{kj}^\mu} \right|^2, \tag{11}$$

and since each eigenvalue α_k^μ appears in Ω with multiplicity d_μ, the sum inside the absolute value is the same as the sum over all the eigenvalues of Ω, each multiplied by an arbitrary phase, namely

$$P_e = \frac{1}{|\mathcal{G}|^2} \left| \sum_{k=1}^{|\mathcal{G}|} \sqrt{\lambda_k}\, e^{i\theta_k} \right|^2 \tag{12}$$

where λ_k are all the non-zero eigenvalues of Ω (where repeated eigenvalues count as different ones), which are the same as those of G.

When the condition $\sqrt{\lambda_1} \leq \sum_{k>1} \sqrt{\lambda_k}$ holds, the phases in Eq. (12) can be set in such a way that the error probability is null (see Supplemental Material REF), proving Lemma 1 in this region. Note that if the maximum eigenvalue λ_1 has a multiplicity greater than one, the condition is immediately fulfilled, and therefore the set can be perfectly excluded. Thus, ~~always that~~ [when] the maximum α_k^μ corresponds to an irrep μ whose dimension is greater than one, [it is always possible to exclude a state conclusively].

On the contrary, when the condition is not fulfilled, i.e., $\sqrt{\lambda_1} > \sum_{k>1} \sqrt{\lambda_k}$, the error probability in Eq. (12) cannot be zero. ~~Therefore~~[Then, the absolute minimum takes place for phases] $\theta_1 = 0$ and $\theta_k = \pi, \forall k \neq 1$, leading to a POVM of the form ~~of the one~~ in Eq. (7), and a success probability ~~of~~

$$P_e = \frac{1}{|\mathcal{G}|^2}\left(\sqrt{\lambda_1} - \sum_{k\neq 1}^{|\mathcal{G}|}\sqrt{\lambda_k}\right)^2 . \tag{13}$$

~~We check the optimality of this protocol by considering the dual problem of the task~~ [The optimality of this POVM is proven by considering the dual program of Eq. (1)]

$$\begin{aligned} &P_e = \max_Y \ \mathrm{Tr}(Y)\,, \\ &\text{s.t. } Y \geq 0\,,\quad \eta_g\rho_g - Y \geq 0 \ \ \forall g \in \mathcal{G}\,. \end{aligned} \tag{14}$$

~~In our particular case~~. [In our symmetric case, we construct the dual operator Y from the seed state as]

$$Y = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} U_g|\omega\rangle\langle\omega|\psi\rangle\langle\psi|U_g^\dagger\,, \tag{15}$$

[The operator Y so constructed fulfills] the (Holevo) condition: $\eta_g\rho_g - Y \geq 0 \ \forall g$ as the matrix $M := |\mathcal{G}|^2\left(|\psi\rangle\langle\psi| - Y\right)$ can be decomposed as a convex combination of positive semidefinite operators (see Supplemental Material REF).

~~With this we conclude the proof for Lemma 1~~. [This concludes the proof of Lemma 1]

■

This Lemma, together with Eq. (12), directly proves Theorem I, establishing the ultimate bound on the error probability P_e for any Gram matrix G of a group-generated ensemble.

From Theorem I, we derive the following corollary:

Corollary 1. *Let G be the Gram matrix of an ensemble $\mathcal{E}$ of $|\mathcal{G}|$ equally probable group-generated pure states, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|\mathcal{G}|}$. Then, the condition*

$$\sqrt{\lambda_1} \leq \sum_{k=2}^{|\mathcal{G}|}\sqrt{\lambda_k},\,, \tag{16}$$

is both necessary and sufficient for $\mathcal{E}$ to be perfectly excluded.

[This corollary has Theorem 5.2 in [26] as a particular case for circulant matrices, which here is extended to any group of symmetry].

~~Outside the region defined by the eigenvalue condition, we introduce another protocol that allows no error when excluding a hypothesis. This zero-error protocol is relevant only in cases of non-perfect exclusion, as the solution for perfect exclusion in the minimum error protocol coincides with the zero-error protocol optimal measurement. The main result for this protocol is presented in the following theorem.~~

[We next address the zero-error (or unambiguous) protocol. Here our answers can either contain no error, i.e. a state is excluded with certainty, or be inconclusive. Obviously, the region of perfect exclusion is the same as in the minimum error approach. Outside this region, the goal is to minimize the probability of having an inconclusive outcome, or, alternatively, to maximize the probability of zero-error outcomes. Our main result for this protocol is given in the following theorem:]

Theorem II. *Let $\mathcal{E}$ be an ensemble of $N = |\mathcal{G}|$ group-generated pure states, and let all the hypotheses be equally probable with Gram matrix G, whose eigenvalues are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|\mathcal{G}|}$. Then, the (optimal) probability of obtaining an inconclusive outcome is given by*

$$P_? = \begin{cases} 0 & \text{if } \sqrt{\lambda_1} \leq \sum_{j\neq 1}\sqrt{\lambda_j} \\ \frac{\mathrm{Tr}\sqrt{G}}{|\mathcal{G}|}\left(2\sqrt{\lambda_1} - \mathrm{Tr}\sqrt{G}\right) & \text{otherwise}\,. \end{cases} \tag{17}$$

Proof. The minimum probability of obtaining an inconclusive outcome is the result of the SDP in Eq. (2). Notice that the solution [indeed] coincides with the minimum error in the region of perfect exclusion. Hence, [we only need to consider] the scenario where $\sqrt{\lambda_1} > \sum_{j\neq 1}\sqrt{\lambda_j}$.

Again, the POVM can be chosen to be covariant, such that the inconclusive POVM element reads

$$E_? = \mathbb{1} - \sum_{g\in\mathcal{G}} U_g\tilde{E}U_g^\dagger\,, \tag{18}$$

for a given $\tilde{E} \geq 0$. Let us consider the ansatz $\tilde{E} = |\tilde{\omega}\rangle\langle\tilde{\omega}|$, where

$$|\tilde{\omega}\rangle = \frac{1}{\sqrt{|\mathcal{G}|}}\left(\gamma|v^1\rangle|u^1\rangle - \sum_{\mu\neq 1}\sqrt{d_\mu}\sum_{k=1}^{r_\mu}|v_k^\mu\rangle|u_k^\mu\rangle\right)\,, \tag{19}$$

with $\gamma = \sum_{\mu\neq 1}\sum_{k=1}^{r_\mu} d_\mu\sqrt{\alpha_k^\mu}/\sqrt{\alpha^1} < 1$. With this choice we have that $\langle\tilde{\omega}|\psi\rangle = 0$, and $E_? = (1-\gamma^2)|v^1\rangle\langle v^1| \otimes |u^1\rangle\langle u^1| \geq 0$, making it a feasible solution with the probability of obtaining an inconclusive outcome of

$$P_?^{\mathrm{prim}} = \frac{\alpha^1(1-\gamma^2)}{|\mathcal{G}|} = \frac{\mathrm{Tr}\sqrt{G}\left(2\sqrt{\lambda_1} - \mathrm{Tr}\sqrt{G}\right)}{|\mathcal{G}|}\,. \tag{20}$$

To now proof that this POVM choice is optimal we solve the dual SDP of the zero error exclusion task, which for the particular case of group-generated ensembles reads

$$\begin{aligned} \max_{X} \quad & 1 - \mathrm{Tr}(X)\,, \\ \text{s.t.} \quad & \sum_{g\in\mathcal{G}} U_g X U_g^\dagger + \nu\rho - \Omega \geq 0\,, \\ & X \geq 0\,,\, \nu \in \mathbb{R}\,. \end{aligned} \tag{21}$$

Recall that ρ is the seed state and $\Omega = \sum_g \rho_g$, and notice that only one hypothesis appears in the constraints, due to the underlying symmetries of the ensemble. Also, due to the group symmetry of the constraints of the dual SDP (21), we construct an operator X that remains invariant under the action of the group $\mathcal{G}$. Additionally, we restrict X to be orthogonal to the inconclusive POVM element $E_?$, ensuring that [COMPROBAR: Slater's condition is satisfied] [YLF: REF:GENERALSDP] [27, 28]. With this construction, the dual SDP outputs an inconclusive probability of $P_?^{\mathrm{dual}} = P_?^{\mathrm{prim}} - \varepsilon$, where $\varepsilon > 0$ is an arbitrary positive real number. Furthermore, due to the freedom when choosing ν, for each value of ε there exists a corresponding ν that makes the solution feasible. Therefore, in the limit of ε approaching zero, we recover $P_?^{\mathrm{dual}} = P_?^{\mathrm{prim}}$, proving its optimality (see Supplemental Material for the detailed constructive proof).

■

Conclusions. [We have obtained the complete solution of the minimum and zero error protocols for any group-generated source of states. We have given a constructive proof that also furnishes an explicit form of the optimal POVM in both cases. Some previous results follow from our findings as particular cases. We have extensively exploited the fact that the discrimination properties of any source of states (symmetric or not) are all contained in the Gram matrix and have found a convenient basis for the source states for any group. Although we have focused on symmetric sources, our expressions of the optimal error probability are written solely in terms of the eigenvalues of the Gram matrix, and therefore allow a direct extension to the non-symmetric sources. In these general cases, our results also a provide a very good approximation of the true value. We have numerical evidence to conjecture that Eqs. 6 and 17 are lower bounds of the error probability for the minimum and zero error cases, respectively, for arbitrary sources. Heuristically it reflects that the discrimination of states of symmetric sources is expected to be the hardest setting. We are currently investigating these matters to devise what could be considered the pretty good measurement [34] for quantum state exclusion.]

Acknowledgments.– This work has been financially supported by MCIN with funding from European Union NextGenerationEU (PRTR-C17.I1) and by Generalitat de Catalunya. We also acknowledge support from the Ministry of Economic Affairs and Digital Transformation of the Spanish Government through the QUANTUM ENIA project: Quantum Spain, by the European Union through the Recovery, Transformation and Resilience Plan - NextGenerationEU within the framework of the "Digital Spain 2026 Agenda", and by grant PID2022-141283NB-I00 funded by MICIU/AEI/10.13039/501100011033.

[1] S. Bandyopadhyay, R. Jain, J. Oppenheim, and C. Perry, Conclusive exclusion of quantum states, Physical Review A **89**, 022336 (2014).
[2] T. Heinosaari and O. Kerppo, Antidistinguishability of pure quantum states, Journal of Physics A: Mathematical and Theoretical **51**, 365303 (2018).
[3] T. A. Brun, J. Finkelstein, and N. D. Mermin, How much state assignments can differ, Physical Review A **65**, 032315 (2002).
[4] C. M. Caves, C. A. Fuchs, and R. Schack, Conditions for compatibility of quantum-state assignments, Physical Review A **66**, 062111 (2002).
[5] M. F. Pusey, J. Barrett, and T. Rudolph, On the reality of the quantum state, Nature Physics **8**, 475 (2012).
[6] A. Chefles, Quantum state discrimination, Contemporary Physics **41**, 401 (2000).
[7] J. A. Bergou, Discrimination of quantum states, Journal of Modern Optics **57**, 160 (2010).
[8] J. Bae and L.-C. Kwek, Quantum state discrimination and its applications, Journal of Physics A: Mathematical and Theoretical **48**, 083001 (2015).
[9] A. F. Ducuara and P. Skrzypczyk, Operational interpretation of weight-based resource quantifiers in convex quantum resource theories, Physical Review Letters **125**, 110401 (2020).
[10] R. Uola, T. Bullock, T. Kraft, J.-P. Pellonpää, and N. Brunner, All quantum resources provide an advantage in exclusion tasks, Physical Review Letters **125**, 110402 (2020).
[11] B. Stratton, C.-Y. Hsieh, and P. Skrzypczyk, Operational interpretation of the choi rank through k-state exclusion, arXiv preprint arXiv:2406.08360 (2024).
[12] M. Leifer and C. Duarte, Noncontextuality inequalities from antidistinguishability, Physical Review A **101**, 062113 (2020).
[13] V. Havlíček and J. Barrett, Simple communication complexity separation from quantum state antidistinguishability, Physical Review Research **2**, 013326 (2020).
[14] T. Heinosaari, O. Kerppo, L. Leppäjärvi, and M. Plávala, Simple information-processing tasks with unbounded quantum advantage, Physical Review A **109**, 032627 (2024).
[15] H. K. Mishra, M. Nussbaum, and M. M. Wilde, On the optimal error exponents for classical and quantum antidistinguishability, Letters in Mathematical Physics **114**, 76 (2024).
[16] M. Sasaki, S. M. Barnett, R. Jozsa, M. Osaki, and O. Hi-

rota, Accessible information and optimal strategies for real symmetrical quantum sources, Physical Review A **59**, 3325 (1999).
[17] S. M. Barnett, Minimum-error discrimination between multiply symmetric states, Physical Review A **64**, 030303 (2001).
[18] Y. C. Eldar and G. D. Forney, On quantum detection and the square-root measurement, IEEE Transactions on Information Theory **47**, 858 (2001).
[19] Y. C. Eldar, A. Megretski, and G. C. Verghese, Optimal detection of symmetric mixed quantum states, IEEE Transactions on Information Theory **50**, 1198 (2004).
[20] K. Nakahira and T. S. Usuda, Minimum-bayes-cost discrimination for symmetric quantum states, Physical Review A—Atomic, Molecular, and Optical Physics **86**, 062305 (2012).
[21] K. Nakahira and T. S. Usuda, Quantum measurement for a group-covariant state set, Physical Review A—Atomic, Molecular, and Optical Physics **87**, 012308 (2013).
[22] G. Sentís, E. Bagan, J. Calsamiglia, G. Chiribella, and R. Munoz-Tapia, Quantum change point, Physical Review Letters **117**, 150502 (2016).
[23] G. Sentís, J. Calsamiglia, and R. Munoz-Tapia, Exact identification of a quantum change point, Physical review letters **119**, 140506 (2017).
[24] S. Llorens, G. Sentís, and R. Muñoz-Tapia, Quantum multi-anomaly detection, Quantum **8**, 1452 (2024).
[25] M. Skotiniotis, S. Llorens, R. Hotz, J. Calsamiglia, and R. Muñoz Tapia, Identification of malfunctioning quantum devices, Phys. Rev. Res. **6**, 033329 (2024).
[26] N. Johnston, V. Russo, and J. Sikora, Tight bounds for antidistinguishability and circulant sets of pure quantum states, arXiv preprint arXiv:2311.17047 (2023).
[27] S. Boyd and L. Vandenberghe, *Convex optimization* (Cambridge university press, 2004).
[28] J. Watrous, *The theory of quantum information* (Cambridge university press, 2018).
[29] G. Sentís, A. Monràs, R. Munoz-Tapia, J. Calsamiglia, and E. Bagan, Unsupervised classification of quantum data, Phys. Rev. X **9**, 041029 (2019).
[30] R. A. Horn and C. R. Johnson, *Matrix analysis* (Cambridge university press, 2012).
[31] J. Crickmore, I. V. Puthoor, B. Ricketti, S. Croke, M. Hillery, and E. Andersson, Unambiguous quantum state elimination for qubit sequences, Physical Review Research **2**, 013256 (2020).
[32] S. Llorens, W. González, G. Sentís, J. Calsamiglia, E. Bagan, and R. Muñoz-Tapia, Quantum edge detection, arXiv preprint arXiv:2405.11373 (2024).
[33] T. Durt, B.-G. Englert, I. Bengtsson, and K. Życzkowski, On mutually unbiased bases, International Journal of Quantum information **8**, 535 (2010).
[34] P. Hausladen and W. K. Wootters, A 'pretty good'measurement for distinguishing quantum states, Journal of Modern Optics **41**, 2385 (1994).

Appendix A: Representation theory of finite groups

[ADH: Tot això és provisional]

In group theory, a representation of a group refers to a homomorphism from a group $\mathcal{G}$ to the group of invertible linear transformations (or matrices) acting on a vector space V. This provides a way to understand abstract group elements by associating them with matrices, thus allowing the study of the properties of abstract groups through linear algebra methods. Formally, if $GL(V)$ is the general linear group of V, a representation is a map $\mathcal{U} : G \to GL(V)$ such that for all $g, h \in \mathcal{G}$, $\mathcal{U}(gh) = \mathcal{U}(g)\mathcal{U}(h)$ and $\mathcal{U}(e) = \mathbb{1}$, where e is the identity element in $\mathcal{G}$ and $\mathbb{1}$ is the identity matrix in $GL(V)$. Here we only consider unitary representations, which we denote as $U_g := \mathcal{U}(g) \in U(V)$, where $U(V)$ is the unitary group on V.

A key concept in the theory of representations is that of irreducibility. A representation $\mathcal{U}$ is called irreducible if the only subspaces $W \subseteq V$ that remain invariant under the action of all U_g are the trivial subspaces, i.e., the null space $W = \emptyset$ and the whole space $W = V$. Irreducible representations, or irreps. in short, are fundamental as any representation $\mathcal{U}$ of a finite group $\mathcal{G}$ is either irreducible or can be decomposed into a direct sum of irreducible representations, i.e., $\mathcal{U} \cong \bigoplus_\mu \mathcal{U}^\mu$, where μ label the different irreps. $\mathcal{U}^\mu$ of $\mathcal{G}$.

In our particular problem we consider group-generated ensembles, that is, given a Hilbert space $\mathcal{H}$ and a unitary representation $\mathcal{U}$ of a finite group $\mathcal{G}$, we define the ensemble of states generated by $\mathcal{G}$ as the set of $|\mathcal{G}|$ states $\mathcal{E}_\mathcal{U} = \{|\psi_g\rangle = U_g|\psi\rangle\,,\, g \in \mathcal{G}\}$ for an arbitrary seed state $|\psi\rangle$. The decomposition into irreps. of $\mathcal{U}$ establishes a decomposition of the Hilbert space into orthogonal subspaces as

$$\mathcal{H} = \bigoplus_\mu \mathcal{H}_\mu \otimes \mathbb{C}^{m_\mu}\,, \tag{A1}$$

where recall that μ labels the different irreps. into which $\mathcal{U}$ can be decomposed, and m_μ denotes their respective multiplicities. Here, the subspace $\mathcal{H}_\mu$ carries the irrep. μ while $\mathbb{C}^{m_\mu}$ denotes the multiplicity space, over which $\mathcal{U}$ acts trivially, namely, each of the matrix representation can be written as

$$U_g = \bigoplus_\mu U_g^\mu \otimes \mathbb{1}_{m_\mu}\,, \tag{A2}$$

where each matrix representation U_g^μ is irreducible and $\mathbb{1}_d$ denotes the projector onto a d-dimensional subspace.

A key aspect of this decomposition is that it allows us to write the seed state as $|\psi\rangle = 1/|\mathcal{G}|\sum_\mu \sqrt{d_\mu}|\psi_\mu\rangle$, with respect to the decomposition of Eq. (3). Here, d_μ denotes the dimension of the irrep μ, and $|\psi_\mu\rangle \in \mathcal{H}_\mu \otimes \mathbb{C}^{m_\mu}$ are non-normalized bipartite vectors ensuring that $\||\psi\rangle\|^2 = 1$. It is convenient to express each bipartite vector $|\psi_\mu\rangle$ in a Schmidt-like form

$$|\psi_\mu\rangle = \sum_{k=1}^{r_\mu} \sqrt{\alpha_k^\mu}\,|v_k^\mu\rangle|u_k^\mu\rangle\,, \tag{A3}$$

where $r_\mu = \min\{d_\mu, m_\mu\}$ and $\alpha_k^\mu \geq 0$, $\forall \mu, k$. Both $\{|v_k^\mu\rangle\}_{k=1}^{d_\mu}$ and $\{|u_k^\mu\rangle\}_{k=1}^{m_\mu}$ are orthonormal basis of $\mathcal{H}_\mu$ and $\mathbb{C}^{m_\mu}$ respectively. Expressing $|\psi\rangle$ in this way, the projector onto the subspace spanned by the orbit of the seed state reads

$$\mathbb{1}_\psi = \bigoplus_\mu \mathbb{1}_{d_\mu} \otimes \sum_{k=1}^{r_\mu} |u_k^\mu\rangle\langle u_k^\mu|\,, \tag{A4}$$

which satisfies that $\mathbb{1}_\psi U_g|\psi\rangle = U_g|\psi\rangle$ for all $g \in \mathcal{G}$. One operator of great relevance when studying the (anti-) distinguishability of an ensemble of states is the ensemble operator, which in our case is written as $\Omega = \sum_{g\in\mathcal{G}} U_g|\psi\rangle\langle\psi|U_g^\dagger$. Notice that with this definition $[\Omega, U_g] = 0$, $\forall\, g \in \mathcal{G}$, thus, by Schur's Lemma [], Ω has to act trivially over the different irrep. subspaces, namely, $\Omega = \bigoplus_\mu \mathbb{1}_{d_\mu} \otimes \Omega^\mu$. Furthermore, since Ω is the sum of all states in our ensemble, it also satisfies that $[\Omega, \mathbb{1}_\psi] = 0$, meaning that the operators Ω^μ have to be diagonal in the basis $\{|u_k^\mu\rangle\}$, and it can be seen that the eigenvalues are just the coefficients α_k^μ that appear in the decomposition of $|\psi\rangle$, each of them with multiplicity d_μ, i.e.,

$$\Omega = \bigoplus_\mu \mathbb{1}_{d_\mu} \otimes \sum_{k=1}^{r_\mu} \alpha_k^\mu |u_k^\mu\rangle\langle u_k^\mu|\,. \tag{A5}$$

The Gram matrix G of an ensemble $\mathcal{E} = \{|\psi_i\rangle\}_i$ of pure states contains all necessary information about its (anti-) distinguishability, and it is defined as the matrix of overlaps $G_{ij} = \langle\psi_i|\psi_j\rangle$, where the priors are all equal. The Gram matrix G is tightly related to the ensemble operator Ω. Ω can be seen as a non-normalized mixed state $\Omega = \sum_i |\psi_i\rangle\langle\psi_i|$, which can be purified as

$$|\Psi\rangle = \sum_i |\psi_i\rangle|i\rangle \in \mathcal{H} \otimes \mathcal{H}_{aux}\,, \tag{A6}$$

such that $\{|i\rangle\}$ forms an orthonormal basis and $\mathrm{Tr}_2(|\Psi\rangle\langle\Psi|) = \Omega$. But the Gram matrix also arises from this purification, in particular $\mathrm{Tr}_1(|\Psi\rangle\langle\Psi|) = G$. That means that Ω and G share the same non-zero eigenvalues.

In the case of group-generated ensembles, we write $G_{gh} = \langle\psi_g|\psi_h\rangle$, $g, h \in \mathcal{G}$. Notice that in that case, the choice of the seed state and how it decomposed into the different irreps. spaces, determines the spectral properties of the Gram matrices. Furthermore, group-generated Gram matrices have a well-defined structure. Namely, from the group structure we have that $G_{gh} = \langle\psi_g|\psi_h\rangle = \langle\psi|\psi_{g^{-1}h}\rangle$, where recall that $|\psi\rangle = |\psi_e\rangle$ is the seed state. Thus, G will contain at most $|\mathcal{G}|$ different matrix entries $a_g := \langle\psi|\psi_g\rangle$ in its first row, and each row is a permutation of the first. In particular, we can write

$$G = \sum_{g\in\mathcal{G}} a_g R_g\,, \tag{A7}$$

where R_g are matrices of zeroes and ones such that

$$(R_g)_{hf} = \begin{cases} 1 & \text{if} \quad g = f^{-1}h \\ 0 & \text{otherwise} \end{cases}\,. \tag{A8}$$

Notice that the structure of such Gram matrices depends only on the group which generates the ensemble, meaning that the decomposition of Eq.(A7) is independent of the election of the seed state or the representation which is used. That gives us an easy way of directly constructing Gram matrices with a given group symmetry without writing explicitly the states which generates them. One possible way of doing that is to write the multiplication table of the group but which column correspond to the inverse of the elements in the rows. In that way, from that multiplication table, the Gram matrix can be read by associating yo each group element g a complex number a_g, and writing the matrix keeping the structure of the resulting table. Finally, the allowed values of the coefficients a_g will be constrained by the positive semidefiniteness of the G.

Take as an example the symmetric group S_3, which has $|S_3| = 6$ elements.

[ADH: Here we put an example of the group table and the Gram matrix for a given group]

Furthermore, the set of matrices $\{R_g\}$ defines a unitary representation known as the right regular representation of the group $\mathcal{G}$. The regular representation is the linear representation obtained by the action of the group $\mathcal{G}$ on itself. That is, if we identify each element g of a group with the elements $\{|g\rangle\}$ of an orthonormal basis of a Hilbert space of dimension $|\mathcal{G}|$, the so called right regular representation is defined by its action on such basis by right translation, i.e., $R_g|h\rangle = |hg^{-1}\rangle$. Similarly, the left regular representation, that we denote by L_g, is defined by its action on the basis by left translation, i.e., $L_g|h\rangle = |gh\rangle$. Notice that from this definition we recover the matrices defined in Eq.(A8).[ADH: Comment properties of the regular representation as that its decomposition into irreps. is cool]

In addition, the set of matrices $\mathcal{R}_{\mathcal{G}} = \{R_g\, ,\, g \in \mathcal{G}\}$ form an associative algebra, the is, because of being a representation, the product of any two matrix is also within the set of matrices, $R_g R_h = R_{gh} \in \mathcal{R}_{\mathcal{G}}$, and the identity matrix, $\mathbb{1} = R_e$, is in the set. Notice that this is true for any representation, however the regular representation has a couple extra properties that make it interesting. In particular, all matrices in $\mathcal{R}_{\mathcal{G}}$ are $(0,1)$-matrices and satisfy that

$$\sum_{g\in\mathcal{G}} R_g = J\,, \tag{A9}$$

where J is the all-ones matrix. This makes the set of the matrices of the right (and left) regular representation to form what is called a Bose-Mesner algebra [REF]. That is, the different matrices R_g can be seen as adjacency matrices of an association scheme. Such algebras have been broadly studied, and several results about their spectral properties and structure are known. However, the fact that is of great relevance for us is that if we denote with $\mathcal{A}$ the vector space consisting of all matrices $\sum_{g\in\mathcal{G}} a_g R_g$, with $a_g \in \mathbb{C}$, the Gram matrix belongs to that space, as seen in Eq.(A7). That means that giving a group, all Gram matrices corresponding to the generated ensembles coincide with the set of positive semidefinite matrices in $\mathcal{A}$ (with a proper normalization). Moreover, since the set $\mathcal{R}_{\mathcal{G}}$ forms an algebra, we have that $\mathcal{A}$ is closed under matrix multiplication, i.e., $AB \in \mathcal{A}\ \forall A, B \in \mathcal{A}$. In particular, any function of G is also in $\mathcal{A}$.