Universitat Autònoma de Barcelona

Notes: Quantum State Exclusion - Temporal Name -

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1 Definitions

Definition 1. ***Completely positive map***: *"Let A and B be C*-algebras. A linear map $\phi : A \to B$ is called a positive map if ϕ maps positive elements to positive elements:*

$$a \geq 0 \implies \phi(a) \geq 0. \tag{1}$$

Any linear map $\phi : A \to B$ induces another map

$$\mathrm{id} \otimes \phi : \mathbb{C}^{k\times k} \otimes A \to \mathbb{C}^{k\times k} \otimes B \tag{2}$$

in a natural way. If $\mathbb{C}^{k\times k} \otimes A$ is identified with the C-algebra $A^{k\times k}$ of $k \times k$-matrices with entries in A, then $\mathrm{id} \otimes \phi$ acts as*

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} \mapsto \begin{pmatrix} \phi(a_{11}) & \cdots & \phi(a_{1k}) \\ \vdots & \ddots & \vdots \\ \phi(a_{k1}) & \cdots & \phi(a_{kk}) \end{pmatrix}. \tag{3}$$

The map ϕ is called k-positive if $\mathrm{id}_{\mathbb{C}^{k\times k}} \otimes \phi$ is a positive map, and completely positive if ϕ is k-positive for all k."[1]

Definition 2. ***Kraus operator:***

Definition 3. ***Inner product***: *Given a vector space A and a field F we define the an inner product $\langle\cdot,\cdot\rangle$ as $\langle\cdot,\cdot\rangle : A \times A \to F$ that satisfies the following properties. Given $x, y, z \in A$ and $\lambda, \mu \in F$:*

1. *Conjugate symmetry: $\langle x, y\rangle = (\langle y, x\rangle)^*$.*
2. *Linearity: $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$*
3. *Positive-definiteness: $\langle x, x\rangle > 0$ if $x \neq 0$*

Definition 4. ***Inner product space***: *Given a vector space A and a field F we define the inner product space as the duple $(A, \langle\cdot,\cdot\rangle)$ such that $\langle\cdot,\cdot\rangle : A \times A \to F$ is an inner product.*

Definition 5. ***Gram matrix***: *Given a vector space A, a field F , a set of vectors $\{v_i\}_{i=0}^n \subset A$ and an inner product space such that $\langle\cdot,\cdot\rangle : A \times A \to F$ the Gram matrix G is defined as the matrix $G \in F^{n\times n}$ whose entries are $g_{i,j} = \langle v_i, v_j\rangle$ i.e,*

$$G = \begin{pmatrix} \langle v_1, v_1\rangle & \dots & \langle v_1, v_n\rangle \\ \vdots & \ddots & \vdots \\ \langle v_n, v_1\rangle & \dots & \langle v_n, v_n\rangle \end{pmatrix}. \tag{4}$$

Due to the first property of the inner product definition 3 it is immediate that $G^\dagger = G$ in other word G is hemitian (trivial proof).

References

[1] Wikipedia contributors. Completely positive map, 2023. Accessed: 2024-10-27.