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Lower bounds of the success probability in quantum state exclusion for general ensembles

Author	:	Sergio Castañeiras Morales
Supervisor	:	Ramón Muñoz Tapia
Co-Supervisor	:	Santiago Llorens Fernández

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Ab ovo usque ad mala.

Horace

Abstract

Given a quantum state known to be prepared from an ensemble of two or more states, quantum state exclusion aims to rule out the possibility that it was prepared in a particular state from the ensemble. Using the known solution for group generated ensembles **?**, we study this result as a lower bound for randomly generated ensembles via semidefinite programming.

Keywords: *SDP, Quantum state exclusion , [Add keywords](#).*

I INTRODUCTION

In many real-world scenarios, excluding a certain hypothesis can be more practical than solving the problem entirely. For instance, in disease diagnosis, ruling out potential diseases often serves as the first step in identifying the actual condition. Similarly, when repairing a machine, it is sometimes more efficient to identify the components that are functioning correctly, which narrows down the search for the faulty part.

In this project, we project this idea into the quantum realm by focusing on Quantum State Exclusion (QSE). Rather than determining the exact state of a quantum system, we aim to eliminate one or more possible candidates from a known ensemble of states. Notice, this approach can be more suitable or efficient in certain quantum information tasks.

Given a quantum state known to be prepared from a finite ensemble, Quantum State Discrimination (QSD) seeks to identify which specific state from the ensemble corresponds to the given system. In contrast, QSE **?** adopts the opposite perspective: it aims to determine which states from the ensemble do not correspond to the prepared state. While QSD has been deeply studied in recent years**?**, with significant advances since its inception **?**, QSE offers a complementary framework with distinct advantages.

Although the tasks of exclusion and discrimi-

nation coincide for ensembles containing only two states [1], when dealing with ensembles of three or more states, the two problems diverge in both approach and complexity. One of the most significant features of QSE is the possibility of achieving *perfect exclusion*, where certain states can be ruled out with zero probability of error in cases where perfect discrimination is impossible**?**.

This capability opens new frontiers in quantum information theory, particularly in the context of partial information retrieval from quantum systems. By excluding certain states, it is possible to gain insight into the encoded information without needing to fully determine the original state.

As with QSD, obtaining a general analytical solution for QSE remains an open problem. However, analytical results have been found in specific cases when the ensemble of quantum states exhibits a certain degree of symmetry. In particular, when the ensemble is generated by the action of a finite group, the problem becomes more tractable and exact solutions have been derived.

The exclusion task can be carried out under two main protocols: *Minimum Error* and *Zero Error*[2]. In the Minimum Error scenario, the goal is to minimize the probability of mistakenly excluding the actual prepared state. In contrast, the Zero Error approach seeks to exclude a state with absolute certainty, even if

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[1]Since for the two states case excluding one necessarily implies identifying the other.

[2]Also known as *unambiguous exclusion.*

that means sometimes the procedure yields an inconclusive result.

Building on recent results that provide exact solutions for exclusion tasks in group generated ensembles **?**, this project undertakes a numerical study of such results as lower bounds for more general, randomly generated ensembles. To this end, we employ Semidefinite Program (SDP) to explore QSE performance in arbitrary settings. Furthermore, we investigate improved bounds for the general case based on how closely a given ensemble resembles a group-generated one[3].

II   FORMULATION OF THE PROBLEM

Let $\{(\rho_i, \eta_i)\}_{i=1}^n$ be a set of n states and n probabilities, where ρ_i dennotes the density matrix $\rho_i = |\psi_i\rangle\langle\psi_i|$, and η_i dennotes the prior probability in which ρ_i accours. Additionally, let ρ_j be a state belonging to the ensemble be our target state, we aim to develope a proceadure to find $\rho_k \in \{\rho_i\}_{i=1}^n$ such that $\rho_j \neq \rho_k$.

The quantum mesurement is represented by a Positive Operator-Valued Measure (POVM) set dennoted as $\{\Pi_j\}_j$ acting on quantum states' Hilbert space $\mathcal{H}$.

Hence, the formulation in a SDP problem for the Minimum Error (ME) protocol for the miminization of the probability of excluding our target state from our hypothesis in the minimum error protocol reads,

$$P_{\mathrm{ME}}^e = \min_{\Pi_i} \sum_{i=1}^n \mathrm{Tr}\,(\Pi\rho_i),$$

$$\text{subject to } \sum_{i=1}^n \Pi_i = \mathbb{1}, \Pi_i \geq 0 \quad \forall i \in \{1, ..., n\}$$

notice that both conditions $\sum_{i=1}^n \Pi_i = \mathbb{1}$ and $\Pi_i \geq 0$ stands for the POVM definition i.e. by demanding possible definitness and a sum equal to the identity we are demanding $\{\Pi_j\}_j$ to be a set of POVMs. Notice the e upperindex stands for the *error probability.*

Similarly the problem can trivially be formulated in an equivalent way in terms of the maximization of the *success probability,* defined as the probabilty of a successfull exclusion form the point of view

$$P_{\mathrm{ME}}^s = \max_{\Pi_i} 1 - \sum_{i=1}^n \mathrm{Tr}\,(\Pi\rho_i),$$

$$\text{subject to } \sum_{i=1}^n \Pi_i = \mathbb{1}, \Pi_i \geq 0 \quad \forall i \in \{1, ..., n\}$$

Since naturally, $P_{\mathrm{ME}}^s + P_{\mathrm{ME}}^e = 1$.

Moreover we refere an ensemble as a *group generated ensemble* when it is the outcome of one or more unitary transformations U_i applied onto a seed state such that U_i generates a finite group. For instance if $U_i^n = \mathbb{1}$ will lead up a $\mathbb{Z}/n\mathbb{Z}$ group generated ensemble.

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[3]The notion of "how close" will be formally defined in Section add section.

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LIST OF ABBREVIATIONS

ME	Minimum Error.
POVM	Positive Operator-Valued Measure.
QSD	Quantum State Discrimination.
QSE	Quantum State Exclusion.
SDP	Semidefinite Program.
ZE	Zero Error.