PhD Diary

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1 Exact Solution to diffusion equation

Found a decent explanation here: ((https://physics.stackexchange.com/users/17320/michael-brown)).

The diffusion equation is a partial differential equation. The unknown quantity is a function $C(x, t)$. To complete the problem statement you need to specify an initial condition (at $t = 0$) and boundary conditions. I'm guessing that your boundary conditions are at infinity, so we take

$$C(x, t) \to 0, \ x \to \pm\infty.$$

We take a delta function initial condition:

$$C(x, 0) = \delta(x).$$

The equation can be solved by using the Fourier transform:

$$C(x, t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{ikx}\, C_k(t).$$

The inverse transform is

$$C_k(t) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ikx}\, C(x, t).$$

So the transform of the initial condition is

$$C_k(0) = 1.$$

Substituting $C(x, t)$ in the diffusion equation gives

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{ikx} \left(\dot{C}_k(t) + Dk^2 C_k(t)\right) = 0.$$

This simplifies to

$$\dot{C}_k(t) + Dk^2 C_k(t) = 0,$$

with the solution

$$C_k(t) = C_k(0)\mathrm{e}^{-Dk^2 t} = \mathrm{e}^{-Dk^2 t}.$$

Putting it all together

$$C(x, t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{ikx}\, \mathrm{e}^{-Dk^2 t},$$

and all that's left is to do the k integral. Note that the k integral is a Gaussian so, with a little massaging, you can do it with the formula

$$\int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-y^2} = \sqrt{\pi}.$$

You should get

$$C(x, t) = \frac{1}{\sqrt{4\pi D t}} \mathrm{e}^{-\frac{x^2}{4Dt}}.$$

Figure 1: Diffusion Equation explanation

2 Links

https://medium.com/@codingfreak/top-50-dynamic-programming-practice-problems-4208fed71aa3

References

Michael Brown (https://physics.stackexchange.com/users/17320/michael-brown). Solving the diffusion equation.