2 计算题（15 分）

2.1 给定两个类别的样本分别为:

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$$\omega_1 : \{(3,1),(2,2),(4,3),(3,2)\}$$

$$\omega_2 : \{(1,3),(1,2),(-1,1),(-1,2)\}$$

$N_1=4, \mu_1=(3,2)$

$N_2=4, \mu_2=(0,2)$

$\mu=(1.5,2)$

试利用 LDA，将样本特征维数压缩为一维。

$$S_1=\frac{1}{4}\left(\begin{bmatrix}0\\-1\end{bmatrix}\begin{bmatrix}0 & -1\end{bmatrix}+\begin{bmatrix}-1\\0\end{bmatrix}\begin{bmatrix}-1, & 0\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix}\right)$$

$$=\frac{1}{4}\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}+\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}+0\right)=\frac{1}{4}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$$

$$S_2=\frac{1}{4}\left(\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}+\begin{bmatrix}-1\\-1\end{bmatrix}\begin{bmatrix}-1 & -1\end{bmatrix}+\begin{bmatrix}-1\\0\end{bmatrix}\begin{bmatrix}-1 & 0\end{bmatrix}\right)$$

$$=\frac{1}{4}\left(\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}+\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}+\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}+\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right)$$

$$=\frac{1}{4}\begin{bmatrix}4 & 2\\2 & 4\end{bmatrix}.$$

类内散度矩阵: $S_w = \frac{1}{8}\left(\begin{bmatrix}2&1\\1&2\end{bmatrix}+\begin{bmatrix}4&2\\2&4\end{bmatrix}\right)$

$= \frac{1}{8}\begin{bmatrix}6&3\\3&6\end{bmatrix} = \frac{3}{8}\begin{bmatrix}2&1\\1&2\end{bmatrix}$

类间散度矩阵 $S_b = \frac{4}{8}\left(\begin{bmatrix}3/2\\0\end{bmatrix}\begin{bmatrix}3/2&0\end{bmatrix} + \begin{bmatrix}-3/2\\0\end{bmatrix}\begin{bmatrix}-3/2&0\end{bmatrix}\right)$

$= \frac{1}{2}\begin{bmatrix}9/2&0\\0&0\end{bmatrix}$

$= \frac{9}{4}\begin{bmatrix}1&0\\0&0\end{bmatrix} = \frac{3}{8}\begin{bmatrix}6&0\\0&0\end{bmatrix}$

$S_T = S_w + S_b = \frac{3}{8}\begin{bmatrix}8&1\\1&2\end{bmatrix}$

对特征值分解.

$S_w^{-1}S_b v = \lambda v$, $\Rightarrow S_w^{-1}S_b - \lambda I = 0$.

$\left[\begin{array}{cc|cc}2&1&1&0\\1&2&0&1\end{array}\right] \rightarrow \left[\begin{array}{cc|cc}4&2&2&0\\1&2&0&1\end{array}\right] \rightarrow \left[\begin{array}{cc|cc}3&0&2&-1\\1&2&0&1\end{array}\right]$

$\rightarrow \left[\begin{array}{cc|cc}1&0&2/3&-1/3\\1&2&0&1\end{array}\right] \rightarrow \left[\begin{array}{cc|cc}1&0&2/3&-1/3\\0&2&-2/3&4/3\end{array}\right] \rightarrow \begin{matrix}2/3&-1/3\\-1/3&2/3\end{matrix}$

$S_w^{-1} = \frac{8}{3}\cdot\frac{1}{3}\begin{bmatrix}2&-1\\-1&2\end{bmatrix} = \frac{8}{9}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}$

$$S_w^{-1} \cdot S_b = \frac{8}{9} \times \frac{3}{8} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 12 & 0 \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -2 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & 0 \\ -2 & -\lambda \end{vmatrix} = 0. \Rightarrow (4-\lambda)\lambda = 0.$$

$$\lambda = \underset{(舍)}{0} \text{ 或 } 4.$$

$$\begin{bmatrix} 4 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

$$\parallel$$

$$\begin{bmatrix} 4v_1 \\ -2v_1 \end{bmatrix} = 4 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \qquad -2v_1 = 4v_2. \quad v_1 = -2v_2$$

又 $v_1^2 + v_2^2 = 1 \Rightarrow v_1 = 2/\sqrt{5}$, $v_2 = -1/\sqrt{5}$.

$$\therefore W_{LDA} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}.$$

投影后 $w_1 = \{\sqrt{5}, 2/\sqrt{5}, \sqrt{5}, 4/\sqrt{5}\}$

$w_2 = \{-1/\sqrt{5}, 0, -3/\sqrt{5}, -4/\sqrt{5}\}.$

2.2 模型训练通常需要大量的数据，假设某采集的数据集包含 80% 的有效数据和 20% 的无效数据。采用一种算法判断数据是否有效，其中无效数据被成功判别为无效数据的概率为 90%，而有效数据被误判为无效数据的概率为 5%。如果某条数据经过该算法被判别为无效数据，则根据贝叶斯定理，这条数据是无效数据的概率是多少？

(提示：全概率公式 $P(Y) = \sum_{i=1}^{N} P(Y|X_i)P(X_i)$)

ω_1　ω_2　x_1, x_2

$P(\omega_1) = 0.8$. $P(\omega_2) = 0.2$.

$P(x_2|\omega_2) = 0.9$　$P(x_2|\omega_1) = 0.05$.　$P(\omega_2|x_2)$?

$$P(x_2) = P(x_2|\omega_1)P(\omega_1) + P(x_2|\omega_2)P(\omega_2)$$
$$= 0.05 \times 0.8 + 0.9 \times 0.2 = 0.04 + 0.18 = 0.22.$$

$$P(\omega_2|x_2) = \frac{P(\omega_2 \cap x_2)}{P(x_2)} = \frac{P(x_2|\omega_2)P(\omega_2)}{P(x_2)}$$
$$= \frac{0.9 \times 0.2}{0.22} = \frac{0.9}{1.1} = \boxed{81.82\%}$$

2.3 设有两类正态分布的样本集，第一类均值为 $\mu_1 = [2, -1]^T$，第二类均值为 $\mu_2 = [1, 1]^T$。两类样本集的协方差矩阵和出现的先验概率都相等：$\Sigma_1 = \Sigma_2 = \Sigma = \begin{bmatrix} 4 & 2 \\ 2 & \frac{4}{3} \end{bmatrix}$，$p(\omega_1) = p(\omega_2)$。试计算分类界面，并对特征向量 $x = [6, 2]^T$ 分类。

$\Rightarrow$ 判决函数：$g_i(x) = -\frac{1}{2}(x - \mu_i)^T \Sigma_i^{-1} (x - \mu_i) + \ln P(\omega_i)$

$$\Rightarrow g_{i\,LDA}(x) = (\Sigma^{-1}\mu_i)^T \vec{x} - \frac{1}{2}\mu_i^T \Sigma^{-1}\mu_i$$

Σ^{-1}: $\Sigma = \frac{2}{3}\begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 6 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 4 & 2 & 2/3 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 2/3 & -1 \\ 3 & 2 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 2/3 & -1 \\ 0 & 2 & -2 & 4 \end{array}\right] \rightarrow \Sigma^{-1} = \frac{3}{2}\begin{bmatrix} 2/3 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}$$

$$\Rightarrow g_1(x) = \frac{1}{2}\left(\begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)^T \vec{x}$$

$$- \frac{1}{2}\cdot\frac{1}{2}\begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 7 & -12 \end{bmatrix}\vec{x} - \frac{1}{4}\begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ -12 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 7 & -12 \end{bmatrix}\vec{x} - \frac{1}{4}(14+12)$$

$$= \frac{1}{2}\begin{bmatrix} 7 & -12 \end{bmatrix}\vec{x} - \frac{13}{2}$$

$$g_2(x) = \frac{1}{2}\left(\begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^T \vec{x}$$

$$- \frac{1}{2}\times\frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & 3 \end{bmatrix}\vec{x} - \frac{1}{4}\times 2$$

$$= \frac{1}{2}\begin{bmatrix} -1 & 3 \end{bmatrix}\vec{x} - \frac{1}{2}.$$

分类界面：$\boxed{g(x) = g_1(x) - g_2(x) = \frac{1}{2}[8 \; -15]\vec{x} - 6.}$

当 $\vec{x} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$g_1\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix}\right) = [7 \; -12]\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{13}{2} = 21 - 12 - \frac{13}{2} = \frac{5}{2}$

$g_2\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix}\right) = [-1 \; 3]\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{1}{2} = -3 + 3 - \frac{1}{2} = -\frac{1}{2}.$

$g_1\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix}\right) > g_2\left(\begin{bmatrix} 6 \\ 2 \end{bmatrix}\right)$

$\therefore$ $\boxed{\vec{x} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \in \text{第1类}.}$

2.4 给定异或的样本集

$D = \{((0,0)^T, -1), ((0,1)^T, 1), ((1,0)^T, 1), ((1,1)^T, -1)\}$

该样本集是线性不可分的，可采用如下所示的多项式函数 $\phi(\mathbf{x})$ 将样本 $D = \{(\mathbf{x}_n, y_n)\}$ 映射为 $D_\phi = \{(\phi(\mathbf{x}_n), y_n)\}$，其中 $\phi(\mathbf{x})$ 满足

$$\phi_1(\mathbf{x}) = 2(x_1 - 0.5) = 2x_1 - 1$$
$$\phi_2(\mathbf{x}) = 4(x_1 - 0.5)(x_2 - 0.5) = 4x_1x_2 - 2(x_1 + x_2) + 1.$$

(1) 给出映射后的样本集；

(2) 在映射后的样本集中，设计一个线性 SVM 分类器，给出支持向量及分类界面。

1) 对于 $\begin{bmatrix}0\\0\end{bmatrix}$, 。 $\phi_1(\vec{x}_1) = -1$. $\phi_2(\vec{x}_2) = 1$.

$$\phi(\vec{x}_1) = \begin{bmatrix}-1\\1\end{bmatrix}.$$

$\vec{x}_2 = \begin{bmatrix}0\\1\end{bmatrix}$, $\phi(\vec{x}_2) = \begin{bmatrix}-1\\-1\end{bmatrix}$.

$\vec{x}_3 = \begin{bmatrix}1\\0\end{bmatrix}$, $\phi(\vec{x}_3) = \begin{bmatrix}1\\-1\end{bmatrix}$.

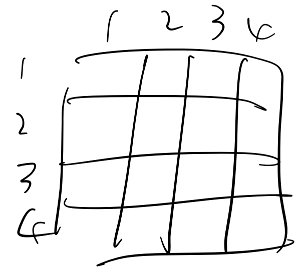
$\vec{x}_4 = \begin{bmatrix}1\\1\end{bmatrix}$, $\phi(\vec{x}_4) = \begin{bmatrix}1\\1\end{bmatrix}$

$$\boxed{D_\phi = \left\{\left(\begin{bmatrix}-1\\1\end{bmatrix}, -1\right), \left(\begin{bmatrix}-1\\-1\end{bmatrix}, 1\right), \left(\begin{bmatrix}1\\-1\end{bmatrix}, 1\right), \left(\begin{bmatrix}1\\1\end{bmatrix}, -1\right)\right\}.}$$

2) 有目标fit

$$L(\vec{w}, b, \vec{\alpha}) = \sum_{i=1}^{4}\alpha_i - \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\alpha_i\alpha_j y_i y_j \vec{x}_i^T\vec{x}_j.$$

求解 $\underset{\alpha}{\text{maximize}}(L(\vec{w}, b, \vec{\alpha}))$



$$\frac{\partial L}{\partial \alpha_1} = 1 - \frac{1}{2}\left(\sum_{j\neq 1, j=1}^{4}\alpha_j y_1 y_j \vec{x}_1^T\vec{x}_j\right)$$
$$- \frac{1}{2}\left(\sum_{i\neq 1, i=1}^{4}\alpha_i y_i y_1 \vec{x}_i^T\vec{x}_1\right)$$ 一样的

$$-2\cdot\frac{1}{2}\,2\alpha_1 \underbrace{y_1^2\vec{x}_1^T\vec{x}_1}_{=2}$$ ($\vec{x}_i^T\vec{x}_i = 2$. $y_i^2 = 1$.

$$= 1 - \left(\alpha_2(-1)\cdot 1\cdot[-1\ 1]\begin{bmatrix}-1\\-1\end{bmatrix}\right.$$
$$+\alpha_3(-1)\cdot 1\cdot[-1\ 1]\begin{bmatrix}1\\-1\end{bmatrix}$$
$$\left.+\alpha_4(-1)(-1)[-1\ 1]\begin{bmatrix}1\\1\end{bmatrix}\right) - \alpha_1\cdot 1\cdot \cancel{2}4$$

$$= 1 - (2\alpha_3) - 4\alpha_1 = 0 \Rightarrow 2\alpha_1 + \alpha_3 = 1/2.$$

$$\frac{\partial L}{\partial \alpha_2} = 1 - \Big(\sum_{1,3,4} \alpha_j y_2 y_j \vec{x}_2^{T} \vec{x}_j\Big) - 2\alpha_2$$

$$= 1 - \Big(\alpha_1 (-1) \cancel{[-1\ -1]\begin{bmatrix}-1\\1\end{bmatrix}} + \alpha_3 \cdot (+1)\, [-1\ -1]\cancel{\begin{bmatrix}-1\\-1\end{bmatrix}} + \alpha_4 (-1)\, [-1\ -1]\begin{bmatrix}1\\1\end{bmatrix}\Big) - 2\alpha_2$$

$$= 1 - 2\alpha_4 - 2\alpha_2 = 0. \Rightarrow 2\alpha_2 + \alpha_4 = 1/2.$$

$$\frac{\partial L}{\partial \alpha_3} = 1 - y_3\Big(\sum_{1,2,4} \alpha_j y_j \vec{x}_3^{T} \vec{x}_j\Big) - 2\alpha_3$$

$$= 1 - \Big(\alpha_1 (-1)\, [1\ -1]\begin{bmatrix}-1\\1\end{bmatrix} + \alpha_2 \cdot (+1) \cdot \cancel{[1\ -1]\begin{bmatrix}-1\\-1\end{bmatrix}} + \alpha_4 (-1)\, \cancel{[1\ -1]\begin{bmatrix}1\\1\end{bmatrix}}\Big) - 2\alpha_3$$

$$= 1 - 2\alpha_1 - 2\alpha_3 = 0 \Rightarrow 2\alpha_3 + \alpha_1 = 1/2.$$

$$\frac{\partial L}{\partial \alpha_4} = 1 - y_4\Big(\sum_{1,2,3} \alpha_j y_j \vec{x}_4^{T} \vec{x}_j\Big) - 2\alpha_4$$

$$= 1 + \Big(\alpha_1 \cdot (-1)\, \cancel{[1\ 1]\begin{bmatrix}-1\\1\end{bmatrix}} + \alpha_2 \cdot 1 \cdot [1\ 1]\begin{bmatrix}-1\\-1\end{bmatrix} + \alpha_3 \cdot 1 \cdot \cancel{[1\ 1]\begin{bmatrix}1\\-1\end{bmatrix}}\Big) - 2\alpha_4$$

$$= 1 - 2\alpha_2 - 2\alpha_4 = 0 \Rightarrow \alpha_2 + 2\alpha_4 = 1/2.$$

$$\Rightarrow 2\alpha_1 + \alpha_3 = 2\alpha_2 + \alpha_4 = 2\alpha_3 + \alpha_1 = 2\alpha_4 + \alpha_2 = 1/2$$

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/6.$$ $\boxed{\text{均为支持向量}}$

$\therefore \vec{w} = \sum_{i=1}^{4} \alpha_i y_i \vec{x_i}$

$= \frac{1}{6}\left(-\begin{bmatrix}-1\\1\end{bmatrix} + \begin{bmatrix}-1\\-1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}\right)$

$= \frac{1}{6}\begin{bmatrix}0\\-4\end{bmatrix} = \begin{bmatrix}0\\-2/3\end{bmatrix}$.

取 $\vec{x_{\phi 1}}, y_1 = \left(\begin{bmatrix}-1\\1\end{bmatrix}, -1\right)$.

$-1 = \begin{bmatrix}0 & -2/3\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} + b$.

$= -\frac{2}{3} + b$. $b = -\frac{1}{3}$.

$\therefore g(\vec{x_\phi}) = \vec{w}^T \vec{x_\phi} + b$

$= \begin{bmatrix}0 & -2/3\end{bmatrix} \vec{x_\phi} - \frac{1}{3}$

分类界面 $g(\vec{x_\phi}) = 0$.

$\Rightarrow -\frac{2}{3} x_{\phi 2} - \frac{1}{3} = 0$.

$2x_{\phi 2} = 1$. $\boxed{x_{\phi 2} = \frac{1}{2}}$.

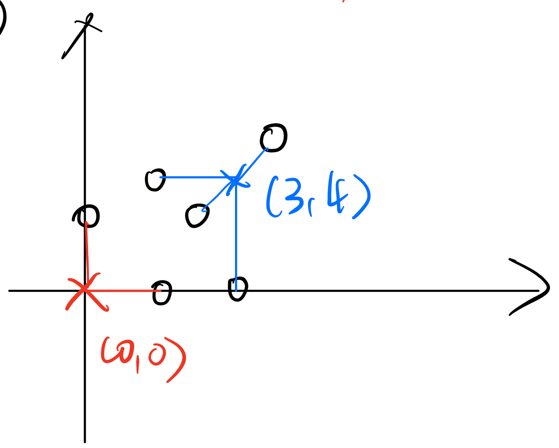
在原空间中. 即 $4x_1x_2 - 2(x_1 + x_2) + 1 = \frac{1}{2}$.

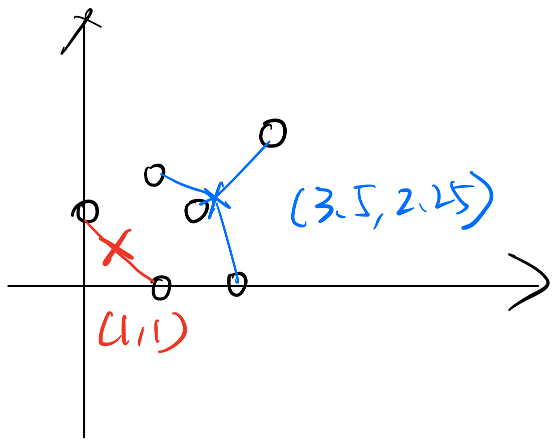
2.5 使用 KMeans 算法对 2 维空间中的 6 个点 $(0,2),(2,0),(2,3),(3,2),(4,0),(5,4)$ 进行聚类，距离函数选择欧氏距离 $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$。

(1) 起始聚类中心选择 (0,0) 和 (4,3)，计算聚类中心；

(2) 起始聚类中心选择 (1,4) 和 (3,1)，计算聚类中心。

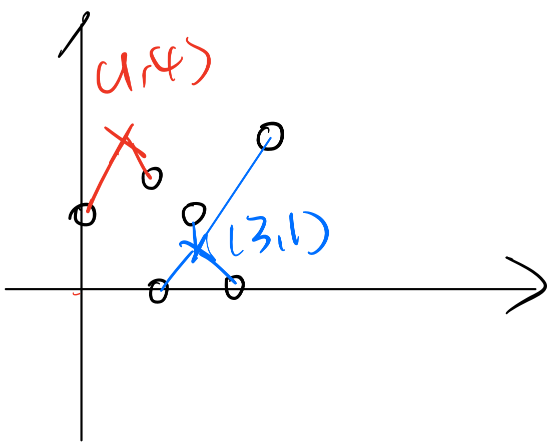
1)

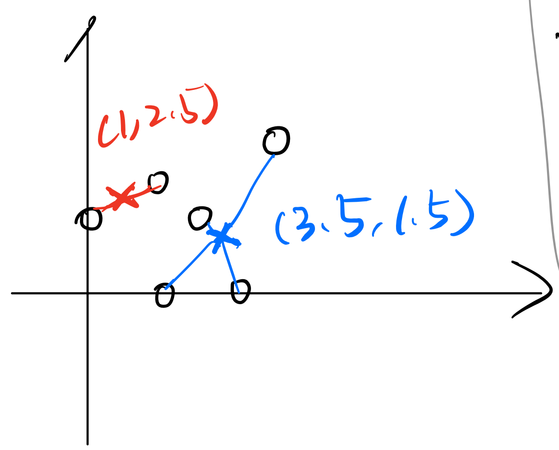




中心：
(1,1)
(3.5,2.25)

2)





中心：
(1,2.5)
(3.5,1.5)