Domácí cvičení 6

(l'Hospitalovo pravidlo, Taylorův polynom,
Lagrangeova věta o střední hodnotě)

6/ 1) Pomocí l'Hospitalova pravidla najděte $\lim_{x\to x_0} f(x)$:

a) $f(x) = \dfrac{1-\cos 2x}{1-\cos 5x}\,, \quad x_0 = 0,$

b) $f(x) = \dfrac{x^4-16}{x^3+5x^2-6x-16}\,, \quad x_0 = 2,$

c) $f(x) = x\cdot \operatorname{cotg} 2x\,, \quad x_0 = 0,$

d) $f(x) = \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right), \quad x_0 = 0,$

e) $f(x) = x^{3/(x^2-1)}\,, \quad x_0 = 1,$

f) $f(x) = \dfrac{3\cdot 4^x - 4\cdot 3^x}{5^x - 5\cdot 2^{x-1}}\,, \quad x_0 = 1.$

6/ 2) Pomocí l'Hospitalova pravidla najděte $\lim_{x\to x_0} f(x)$:

a) $f(x) = \dfrac{\sqrt[3]{x}}{\ln^3 x}\,, \quad x_0 = \infty,$

b) $f(x) = \dfrac{\sqrt[3]{\mathrm{e}^x}}{x^3}\,, \quad x_0 = \infty,$

c) $f(x) = \sqrt[3]{x}\cdot \ln^3 x\,, \quad x_0 = 0^+,$

d) $f(x) = \sqrt[3]{\mathrm{e}^x}\cdot x^3\,, \quad x_0 = -\infty.$

6/ 3) Kde je to (rozumně) možné, spočítejte limity funkcí z 3. a 4. domácího cvičení pomocí l'Hospitalova pravidla.

6/ 4) Najděte Taylorův polynom řádu n funkce f v bodě x_0:

a) $n = 3, \quad f(x) = \arcsin x, \quad x_0 = \dfrac{\sqrt{3}}{2},$

b) $n = 4, \quad f(x) = x^2\cdot \ln(-2x) - 3, \quad x_0 = -\dfrac{1}{2}.$

6/ 5) Ukažte, že pro všechna $x\in\mathbb{R}$ platí: $\quad \operatorname{arccotg} x + \arcsin \dfrac{x}{\sqrt{1+x^2}} = \dfrac{\pi}{2}.$

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Výsledky:

6/ 1) a) $\dfrac{4}{25}$ (2 × l'H) $\left(\lim\limits_{x\to 0} f(x) = \left\langle\!\!\left\langle \dfrac{0}{0} \right\rangle\!\!\right\rangle \overset{\text{l'H}}{=} \lim\limits_{x\to 0} \dfrac{2\sin 2x}{5\sin 5x} = \left\langle\!\!\left\langle \dfrac{0}{0} \right\rangle\!\!\right\rangle \overset{\text{l'H}}{=} \lim\limits_{x\to 0} \dfrac{4\cos 2x}{25\cos 5x}\right)$

b) $\dfrac{16}{13}$ (1 × l'H) $\left(\lim\limits_{x\to 2} f(x) = \left\langle\!\!\left\langle \dfrac{0}{0} \right\rangle\!\!\right\rangle \overset{\text{l'H}}{=} \lim\limits_{x\to 2} \dfrac{4x^3}{3x^2+10x-6}\right)$

c) $\dfrac{1}{2}$ (1 × l'H) $\left(\lim\limits_{x\to 0} f(x) = \left\langle\!\!\left\langle 0\cdot(\pm\infty) \right\rangle\!\!\right\rangle = \lim\limits_{x\to 0} \dfrac{x}{\operatorname{tg} 2x} = \left\langle\!\!\left\langle \dfrac{0}{0} \right\rangle\!\!\right\rangle \overset{\text{l'H}}{=} \lim\limits_{x\to 0} \dfrac{1}{2\frac{1}{\cos^2 2x}}\right)$

d) 0 (2 × l'H) $\left(\lim\limits_{x\to 0} f(x) = \left\langle\!\!\left\langle (\pm\infty)-(\pm\infty) \right\rangle\!\!\right\rangle = \lim\limits_{x\to 0} \dfrac{x-\sin x}{x\cdot\sin x} = \left\langle\!\!\left\langle \dfrac{0}{0} \right\rangle\!\!\right\rangle \overset{\text{l'H}}{=} \lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin x + x\cos x} =\right.$
$\left. = \left\langle\!\!\left\langle \dfrac{0}{0} \right\rangle\!\!\right\rangle \overset{\text{l'H}}{=} \lim\limits_{x\to 0} \dfrac{\sin x}{\cos x + \cos x - x\sin x}\right)$

e) $\mathrm{e}^{\frac{3}{2}}$ (1 × l'H) $\left(x^{3/(x^2-1)} = \exp\Big((3/(x^2-1))\ln x\Big) \overset{\text{ozn.}}{=} \exp(h(x));\;\; \lim\limits_{x\to 1} h(x) = \lim\limits_{x\to 1} \dfrac{3\ln x}{x^2-1} =\right.$
$\left. = \left\langle\!\!\left\langle \dfrac{0}{0} \right\rangle\!\!\right\rangle \overset{\text{l'H}}{=} \lim\limits_{x\to 1} \dfrac{\frac{3}{x}}{2x} = \dfrac{3}{2}\right)$

f) $\dfrac{12}{5}\cdot\dfrac{\ln 4 - \ln 3}{\ln 5 - \ln 2}$ (1 × l'H) $\left(\lim\limits_{x\to 1} f(x) = \left\langle\!\!\left\langle \dfrac{0}{0} \right\rangle\!\!\right\rangle \overset{\text{l'H}}{=} \lim\limits_{x\to 1} \dfrac{3\cdot 4^x\cdot\ln 4 - 4\cdot 3^x\cdot \ln 3}{5^x\cdot\ln 5 - 5\cdot 2^{x-1}\cdot\ln 2}\right)$

6/2) a) ∞ $(3\times\ \mathrm{l'H})$

b) ∞ $(3\times\ \mathrm{l'H})$

c) 0 $(3\times\ \mathrm{l'H})$ $\left(\sqrt[3]{x}\cdot\ln^3 x = \dfrac{\ln^3 x}{x^{-\frac{1}{3}}}\right)$

d) 0 $(3\times\ \mathrm{l'H})$ $\left(\sqrt[3]{\mathrm{e}^x}\cdot x^3 = \dfrac{x^3}{\mathrm{e}^{-\frac{x}{3}}}\right)$

6/3) l'H lze (rozumně) použít a pomůže v těchto případech:

3/1) a) (2× l'H)

b) (2×, příp. 3× l'H)

c) (2× l'H)

d) (5× l'H) (možné, ale zbytečně pracné)

3/3) a) $x_0 = 5$ $(1\times\ \mathrm{l'H})$;
$x_0 = -\infty$ $(2\times\ \mathrm{l'H})$

b) $x_0 = 1$ $(2\times\ \mathrm{l'H})$

c) $x_0 = 1$ $(1\times\ \mathrm{l'H})$;
$x_0 = -\infty$ $(2\times\ \mathrm{l'H})$ (oboje po převodu na jeden zlomek)

3/4) a) $x_0 = 1$ $(1\times\ \mathrm{l'H})$;
$x_0 = +\infty$ $(1\times\ \mathrm{l'H})$

4/2) b) (1×, příp. 2× l'H)

4/3) a) $x_0 = \dfrac{\pi}{4}$ $(1\times\ \mathrm{l'H})$

b) $x_0 = \dfrac{\pi}{2}$ $(1\times\ \mathrm{l'H})$ $\left(\lim\limits_{x\to\frac{\pi}{2}} f(x) = \left\langle\!\!\left\langle (\mp\infty)-(\mp\infty)\right\rangle\!\!\right\rangle = \lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sin x - 1}{\cos x} = \left\langle\!\!\left\langle \dfrac{0}{0}\right\rangle\!\!\right\rangle \overset{\mathrm{l'H}}{=}\right.$
$\left.\overset{\mathrm{l'H}}{=} \lim\limits_{x\to\frac{\pi}{2}} \dfrac{\cos x}{-\sin x} = \dfrac{0}{-1} = 0\right)$

c) $x_0 = 0$ $(1\times\ \mathrm{l'H})$ $\left(\lim\limits_{x\to 0} f(x) = \left\langle\!\!\left\langle \dfrac{0}{0}\right\rangle\!\!\right\rangle \overset{\mathrm{l'H}}{=} \lim\limits_{x\to 0} \dfrac{\cos x}{\frac{1}{2}\,\frac{1}{\sqrt{1+\mathrm{tg}\,x}}\cdot\frac{1}{\cos^2 x} - \frac{1}{2}\,\frac{1}{\sqrt{1-\mathrm{tg}\,x}}\cdot\frac{-1}{\cos^2 x}} = \right.$
$\left. = \lim\limits_{x\to 0} \dfrac{2\cos^3 x}{\frac{1}{\sqrt{1+\mathrm{tg}\,x}} + \frac{1}{\sqrt{1-\mathrm{tg}\,x}}} = \dfrac{2}{2} = 1\right)$

4/4) a) $x_0 = -1$ $(1\times\ \mathrm{l'H})$

b) $x_0 = 0$ $(2\times\ \mathrm{l'H})$ (možné, ale zbytečně pracné) $\left(\lim\limits_{x\to 0} f(x) = \left\langle\!\!\left\langle \dfrac{0}{0}\right\rangle\!\!\right\rangle \overset{\mathrm{l'H}}{=} \lim\limits_{x\to 0} \dfrac{2x}{2\mathrm{tg}\,\frac{x}{2}\,\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}} = \right.$
$\left. = \lim\limits_{x\to 0} \dfrac{2x\cos^2\frac{x}{2}}{\mathrm{tg}\,\frac{x}{2}} = \left\langle\!\!\left\langle \dfrac{0}{0}\right\rangle\!\!\right\rangle \overset{\mathrm{l'H}}{=} \lim\limits_{x\to 0} \dfrac{2\cos^2\frac{x}{2} + 2x\cdot 2\cos\frac{x}{2}\left(-\sin\frac{x}{2}\right)\cdot\frac{1}{2}}{\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}} = \dfrac{2+0}{\frac{1}{2}} = 4\right)$

c) $x_0 = 0$ $(1\times\ \mathrm{l'H})$ $\left(\lim\limits_{x\to 0} f(x) = \left\langle\!\!\left\langle 0\cdot(\pm\infty)\right\rangle\!\!\right\rangle = \lim\limits_{x\to 0} \dfrac{\sin 3x}{\mathrm{tg}\,5x} = \left\langle\!\!\left\langle \dfrac{0}{0}\right\rangle\!\!\right\rangle \overset{\mathrm{l'H}}{=} \lim\limits_{x\to 0} \dfrac{3\cos 3x}{5\frac{1}{\cos^2 5x}} = \dfrac{3}{5}\right)$

4/5) a) $x_0 = \pi$ $(1\times\ \mathrm{l'H})$ $\left(f(x) = \mathrm{e}^{h(x)},\ \text{kde } h(x) = \mathrm{cotg}\,x\cdot\ln(1+3\mathrm{tg}\,x),\quad \lim\limits_{x\to\pi} h(x) = \left\langle\!\!\left\langle \pm\infty\cdot 0\right\rangle\!\!\right\rangle = \right.$
$\left. = \lim\limits_{x\to\pi} \dfrac{\ln(1+3\mathrm{tg}\,x)}{\mathrm{tg}\,x} = \left\langle\!\!\left\langle \dfrac{0}{0}\right\rangle\!\!\right\rangle \overset{\mathrm{l'H}}{=} \lim\limits_{x\to\pi} \dfrac{\frac{1}{1+3\mathrm{tg}\,x}\cdot\frac{3}{\cos^2 x}}{\frac{1}{\cos^2 x}} = 3,\ \text{ tedy } \lim\limits_{x\to\pi} f(x) = \mathrm{e}^3\right)$

b) $x_0 = 0$ $(1\times\ \mathrm{l'H})$ $\left(f(x) = \mathrm{e}^{h(x)},\ \text{kde } h(x) = \dfrac{x+1}{x}\cdot\ln(1-2x),\quad \lim\limits_{x\to 0} h(x) = \left\langle\!\!\left\langle \pm\infty\cdot 0\right\rangle\!\!\right\rangle = \right.$
$= \lim\limits_{x\to 0} \dfrac{\ln(1-2x)}{\frac{x}{x+1}} = \left\langle\!\!\left\langle \dfrac{0}{0}\right\rangle\!\!\right\rangle \overset{\mathrm{l'H}}{=} \lim\limits_{x\to 0} \dfrac{\frac{1}{1-2x}\cdot(-2)}{\frac{x+1-x}{(x+1)^2}} = -2,\ \text{ tedy } \lim\limits_{x\to 0} f(x) = \mathrm{e}^{-2}$ (při výpočtu limity exponentu h by bylo vhodnější použít l'H jen na limitu podílu $\frac{\ln(1-2x)}{x}$ a získanou limitu vynásobit limitou výrazu $x+1$) $\Big)$

4/6) a) $x_0 = -5$ $(1\times\ \mathrm{l'H})$ $\left(\lim\limits_{x\to -5} f(x) = \left\langle\!\!\left\langle \dfrac{0}{0}\right\rangle\!\!\right\rangle \overset{\mathrm{l'H}}{=} \lim\limits_{x\to -5} \dfrac{\frac{1}{2}\,\frac{1}{\sqrt{x^2-9}}\cdot 2x}{1} = -\dfrac{5}{4}\right)$

6/ 4) a) $T_3(x) = \frac{\pi}{3} + 2\left(x - \frac{\sqrt{3}}{2}\right) + 2\sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)^2 + \frac{40}{3}\left(x - \frac{\sqrt{3}}{2}\right)^3$

$$\Big(\, f'(x) = \frac{1}{\sqrt{1-x^2}}\,,\ f''(x) = -\frac{1}{2}\,\frac{1}{\sqrt{(1-x^2)^3}}\,(-2x) = \frac{x}{\sqrt{(1-x^2)^3}}\,,$$

$$f'''(x) = \frac{\sqrt{(1-x^2)^3} - x\cdot\frac{3}{2}\sqrt{1-x^2}\,(-2x)}{(1-x^2)^3} = \frac{\sqrt{1-x^2}\,(1+2x^2)}{(1-x^2)^3}\ ;$$

$$f\Big(\frac{\sqrt{3}}{2}\Big) = \frac{\pi}{3}\,,\ f'\Big(\frac{\sqrt{3}}{2}\Big) = 2,\ f''\Big(\frac{\sqrt{3}}{2}\Big) = 4\sqrt{3},\ f'''\Big(\frac{\sqrt{3}}{2}\Big) = 80\Big)$$

b) $T_4(x) = -3 - \frac{1}{2}\left(x + \frac{1}{2}\right) + \frac{3}{2}\left(x + \frac{1}{2}\right)^2 - \frac{2}{3}\left(x + \frac{1}{2}\right)^3 - \frac{1}{3}\left(x + \frac{1}{2}\right)^4$

$$\Big(\, f'(x) = 2x\,\ln(-2x) + x,\ f''(x) = 2\ln(-2x) + 3,\ f'''(x) = \frac{2}{x}\,,\ f^{(4)}(x) = -\frac{2}{x^2}\,;$$

$$f\Big(-\frac{1}{2}\Big) = -3,\ f'\Big(-\frac{1}{2}\Big) = -\frac{1}{2}\,,\ f''\Big(-\frac{1}{2}\Big) = 3,\ f'''\Big(-\frac{1}{2}\Big) = -4\ \ f^{(4)}\Big(-\frac{1}{2}\Big) = -8\Big)$$

6/ 5) **Návod:** Ukažte, že funkce (f) na levé straně je definována na celém $\mathbb{R}$, její derivace je nulová, tedy funkce je konstantní, a spočítejte hodnotu funkce např. pro $x = 0$ nebo $x = \pm 1$ (lze též použít limity v $\pm\infty$ – pro procvičení doporučuji vyzkoušet včechny navrhovné možnosti).

$$\left(\ f'(x) = -\frac{1}{1+x^2} + \frac{1}{\sqrt{1-(\frac{x}{\sqrt{1+x^2}})^2}}\cdot\frac{\sqrt{1+x^2} - x\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{1+x^2}}\cdot 2x}{1+x^2} = 0\ \right)$$