Domácí cvičení 4

(limity funkcí)

4/1) Najděte limity:

a) $\lim\limits_{x\to\infty} \dfrac{4^x+5^x}{4^{x+1}-5^{x+1}}$, b) $\lim\limits_{x\to\infty} \dfrac{4^{2x}+5^x}{4^{2x+1}-5^{x+1}}$.

4/2) Najděte limity:

a) $\lim\limits_{x\to\infty} \left(2+\dfrac{1}{x}\right)^x$, b) $\lim\limits_{x\to\infty} \left(1+\dfrac{4}{x}\right)^x$.

4/3) Najděte $\lim_{x\to x_0} f(x)$:

a) $f(x)=\dfrac{\cos 2x}{\sin x-\cos x}$, $x_0=0;\ \dfrac{\pi}{4}$, b) $f(x)=\operatorname{tg} x-\dfrac{1}{\cos x}$, $x_0=\dfrac{\pi}{2};\ \dfrac{\pi}{4}$,

c) $f(x)=\dfrac{\sin x}{\sqrt{1+\operatorname{tg} x}-\sqrt{1-\operatorname{tg} x}}$, $x_0=0;\ -\dfrac{\pi}{4}^+$.

4/4) Najděte $\lim_{x\to x_0} f(x)$:

a) $f(x)=\dfrac{x^2-1}{\sin(x+1)}$, $x_0=0;\ -1$, b) $f(x)=\dfrac{x^2}{\operatorname{tg}^2\left(\frac{x}{2}\right)}$, $x_0=0;\ \pi$,

c) $f(x)=\sin 3x\cdot\operatorname{cotg} 5x$, $x_0=0;\ \dfrac{\pi}{2}$.

4/5) Najděte $\lim_{x\to x_0} f(x)$:

a) $f(x)=(1+3\operatorname{tg} x)^{\operatorname{cotg} x}$, $x_0=\pi$, b) $f(x)=(1-2x)^{\frac{x+1}{x}}$, $x_0=0$.

4/6) Najděte limity funkce f v hraničních bodech množiny M:

a) $f(x)=\dfrac{\sqrt{x^2-9}-4}{x+5}$, $M=D(f)$, b) $f(x)=\dfrac{\sqrt{1+\operatorname{tg} x}-\sqrt{1+\sin x}}{x^3}$, $M=D(f)\cap\left\langle -\dfrac{\pi}{2},\dfrac{\pi}{2}\right\rangle$,

c) $f(x)=\left(\ln\dfrac{2x+1}{x}\right)^{\cos(\pi x)}$, $M=D(f)$.

4/7) Najděte limity:

a) $\lim\limits_{x\to 0} \operatorname{tg} x\cdot\operatorname{arctg}\dfrac{1}{x}$ b) $\lim\limits_{x\to 0}\left(x^2+\operatorname{arctg}\dfrac{1}{x}\right)$, c) $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}+\operatorname{arctg}\dfrac{1}{x}\right)$,

d) $\lim\limits_{x\to 0}\left(\operatorname{cotg} x+\operatorname{arctg}\dfrac{1}{x}\right)$, e) $\lim\limits_{x\to 0}\left(\operatorname{cotg} x\cdot\operatorname{arctg}\dfrac{1}{x}\right)$,

4/8) Najděte $\lim_{x\to x_0} f(x)$:

a) $f(x)=\cos x\cdot\operatorname{arccotg} x$, $x_0=+\infty;\ -\infty$, b) $f(x)=\dfrac{x^2+x-6}{x^2-5x+6}$, $x_0=2;\ 3$,

c) $f(x)=\left(\sqrt{x^2+1}-x\right)\cdot\cos\sqrt{x^2+1}$, $x_0=+\infty$, d) $f(x)=\dfrac{x^2+x-2}{\ln^2 x}$, $x_0=0^+;\ 1$.

(**Výsledky s návody jsou na dalších stránkách.**)

Výsledky:

4/1) a) $-\dfrac{1}{5}$ $\left(f(x) = \dfrac{5^x((4/5)^x+1)}{5^x(4(4/5)^x-5)} = \dfrac{(4/5)^x+1}{4(4/5)^x-5}\,;\ \text{limita typu}:\ \left\langle\!\left\langle \dfrac{0+1}{4\cdot 0-5} \right\rangle\!\right\rangle\right)$

b) $\dfrac{1}{4}$ $\left(f(x) = \dfrac{16^x+5^x}{4\cdot 16^x - 5\cdot 5^x} = \dfrac{1+(5/16)^x}{4-5(5/16)^x}\,;\ \text{limita typu}:\ \left\langle\!\left\langle \dfrac{1+0}{4-5\cdot 0} \right\rangle\!\right\rangle\right)$

4/2) a) ∞ $\left(f(x) > 2^x \quad \text{a} \quad 2^x \to \infty\right)$

b) e^4 $\left(f(x) = \left(\left(1+\dfrac{4}{x}\right)^{x/4}\right)^4 = \left(\left(1+\dfrac{1}{h}\right)^{h}\right)^4\,;\ \text{kde}:\ h = h(x) = \dfrac{x}{4} \longrightarrow \infty \ \text{ pro } x \to \infty\right)$

4/3) a)

- -1 pro $x_0 = 0$ $\left(f(0) = -1\right)$
- $-\sqrt{2}$ pro $x_0 = \dfrac{\pi}{4}$ $\left(\text{typ } \left\langle\!\left\langle \dfrac{0}{0} \right\rangle\!\right\rangle;\quad f(x) = \dfrac{\cos^2 x - \sin^2 x}{\sin x - \cos x} = -(\cos x + \sin x)\right)$

b)

- 0 pro $x_0 = \dfrac{\pi}{2}$ $\left(\text{typ } \left\langle\!\left\langle \text{neex.} - \text{neex.} \right\rangle\!\right\rangle;\ f(x) = \dfrac{\sin x}{\cos x} - \dfrac{1}{\cos x} = \dfrac{\sin x - 1}{\cos x} = \dfrac{\sin^2 x - 1}{\cos x(\sin x + 1)} = \dfrac{-\cos x}{\sin x + 1}\right)$
- $1-\sqrt{2}$ pro $x_0 = \dfrac{\pi}{4}$ $\left(f\left(\dfrac{\pi}{4}\right) = 1 - \dfrac{1}{\sqrt{2}/2}\right)$

c)

- 1 pro $x_0 = 0$ $\left(\text{typ } \left\langle\!\left\langle \dfrac{0}{0} \right\rangle\!\right\rangle;\quad f(x) = \dfrac{\sin x(\sqrt{1+\operatorname{tg} x} + \sqrt{1-\operatorname{tg} x})}{(1+\operatorname{tg} x) - (1-\operatorname{tg} x)} = \dfrac{\sin x(\sqrt{1+\operatorname{tg} x} + \sqrt{1-\operatorname{tg} x})}{2\operatorname{tg} x} = \right.$
$\left. = \dfrac{\sin x(\sqrt{1+\operatorname{tg} x} + \sqrt{1-\operatorname{tg} x})}{2\sin x/\cos x} = \dfrac{(\sqrt{1+\operatorname{tg} x} + \sqrt{1-\operatorname{tg} x})\cos x}{2}\right)$
- $\dfrac{1}{2}$ pro $x_0 = -\dfrac{\pi}{4}+$ $\left(\text{typ } \left\langle\!\left\langle \dfrac{-\sqrt{2}/2}{\sqrt{1+(-1)^+} - \sqrt{1-(-1)^+}} \right\rangle\!\right\rangle = \left\langle\!\left\langle \dfrac{-\sqrt{2}/2}{\sqrt{0^+} - \sqrt{2}} \right\rangle\!\right\rangle\right)$

4/4) a)

- $-\dfrac{1}{\sin 1}$ pro $x_0 = 0$ $\left(f(0) = -\dfrac{1}{\sin 1}\right)$
- -2 pro $x_0 = -1$ $\left(\text{typ } \left\langle\!\left\langle \dfrac{0}{0} \right\rangle\!\right\rangle;\quad f(x) = (x-1)\dfrac{x+1}{\sin(x+1)} = (x-1)\dfrac{1}{\sin(x+1)/(x+1)} = \left\langle\!\left\langle \dfrac{-2}{1} \right\rangle\!\right\rangle,\right.$
protože $(x+1) \to 0$ pro $x \to -1$ a $\left.\dfrac{\sin y}{y} \to 1 \ \text{pro}\ y \to 0\right)$

b)

- 4 pro $x_0 = 0$ $\left(\text{typ } \left\langle\!\left\langle \dfrac{0}{0} \right\rangle\!\right\rangle;\quad f(x) = \dfrac{x^2}{\left(\sin^2(\frac{x}{2})\right)/\left(\cos^2(\frac{x}{2})\right)} = \dfrac{(\frac{x}{2})^2\cdot 4\cdot\cos^2(\frac{x}{2})}{\sin^2(\frac{x}{2})} = \right.$
$\left. = \left(\dfrac{\frac{x}{2}}{\sin(\frac{x}{2})}\right)^2 \cdot 4 \cdot \cos^2\left(\dfrac{x}{2}\right) \to 1^2\cdot 4\cdot 1^2\right)$
- 0 pro $x_0 = \pi$ $\left(\text{typ } \left\langle\!\left\langle \dfrac{\pi^2}{(\pm\infty)^2} \right\rangle\!\right\rangle = \left\langle\!\left\langle \dfrac{\pi^2}{\infty} \right\rangle\!\right\rangle\right)$

c)

- $\dfrac{3}{5}$ pro $x_0 = 0$ $\left(\text{typ } \left\langle\!\left\langle 0\cdot(\pm\infty) \right\rangle\!\right\rangle;\quad f(x) = \sin 3x \cdot \dfrac{\cos 5x}{\sin 5x} = \left(\dfrac{\sin 3x}{3x}\cdot 3\right)\left(\dfrac{5x}{\sin 5x}\cdot\dfrac{1}{5}\right)\cdot\cos 5x \to\right.$
$\left.\to 1\cdot 3\cdot 1\cdot\dfrac{1}{5}\cdot 1\right)$
- 0 pro $x_0 = \dfrac{\pi}{2}$ $\left(f\left(\dfrac{\pi}{2}\right) = -1\cdot 0\right)$

4/5) a) e^3 $\left(f(x) = \exp(\operatorname{cotg} x \cdot \ln(1+3\operatorname{tg} x)) = \exp(h(x)) = \mathrm{e}^{h(x)},\ \text{přitom} \lim\limits_{x\to\pi^\pm} h(x) = \left\langle\!\left\langle (\pm\infty)\cdot 0 \right\rangle\!\right\rangle = \right.$
$= \lim\limits_{x\to\pi^\pm} \dfrac{\ln(1+3\operatorname{tg} x)}{\operatorname{tg} x} = \lim\limits_{x\to\pi^\pm} \dfrac{\ln(1+3\operatorname{tg} x)}{3\operatorname{tg} x}\cdot 3 = 3,$ protože $3\operatorname{tg} x \to 0$ pro $x \to \pi$ a $\dfrac{\ln(1+t)}{t} \to 1$
pro $\left. t \to 0\right)$

b) e^{-2} $\left(f(x) = \exp\left(\left(\dfrac{x+1}{x}\right)\cdot\ln(1-2x)\right) = \mathrm{e}^{h(x)},\ \text{přitom} \lim\limits_{x\to 0} h(x) = \left\langle\!\left\langle \dfrac{1}{0}\cdot 0 \right\rangle\!\right\rangle = \right.$
$= \lim\limits_{x\to 0}\left(\dfrac{\ln(1-2x)}{(1-2x)-1}\cdot((1-2x)-1)\cdot\dfrac{x+1}{x}\right) = \lim\limits_{x\to 0}\left(\dfrac{\ln(1-2x)}{(1-2x)-1}\cdot(-2)\cdot(x+1)\right) = 1\cdot(-2)\cdot 1,$
protože $(1-2x) \to 1$ pro $x \to 0$ a $\left.\dfrac{\ln t}{t-1} \to 1 \text{ pro } t \to 1\right)$

4/6) a) $M = (-\infty;-5) \cup (-5;-3\rangle \cup \langle 3;\infty)$ $\Big(= \big((-\infty;-3\rangle \cup \langle 3;\infty)\big) \setminus \{-5\} \Big)$

- -1 pro $x_0 = -\infty$ $\left(\text{typ } \left\langle\!\!\left\langle \dfrac{\infty}{-\infty} \right\rangle\!\!\right\rangle; \quad f(x) = \dfrac{|x|(\sqrt{1-9/x^2} - 4/|x|)}{x(1+5/x)} = \operatorname{sgn} x \cdot \dfrac{\sqrt{1-9/x^2} - 4/|x|}{1+5/x} \right)$
- $-\dfrac{5}{4}$ pro $x_0 = -5$ $\left(\text{typ } \left\langle\!\!\left\langle \dfrac{0}{0} \right\rangle\!\!\right\rangle; \quad f(x) = \dfrac{(\sqrt{x^2-9}-4)(\sqrt{x^2-9}+4)}{(x+5)(\sqrt{x^2-9}+4)} = \dfrac{(x^2-9)-16}{(x+5)(\sqrt{x^2-9}+4)} = \right.$
 $\left. = \dfrac{x^2-25}{(x+5)(\sqrt{x^2-9}+4)} = \dfrac{x-5}{\sqrt{x^2-9}+4} \right)$
- -2 pro $x_0 = -3^-$ $\big(f(-3) = -2\big)$
- $-\dfrac{1}{2}$ pro $x_0 = 3^+$ $\left(f(3) = -\dfrac{1}{2}\right)$
- 1 pro $x_0 = \infty$ (jako pro $x_0 = -\infty$, není tu ale nutné psát absolutní hodnotu)

b) $M = \langle -\frac{\pi}{4}; 0) \cup (0; \frac{\pi}{2})$ $\Big(= \langle -\frac{\pi}{4}; \frac{\pi}{2}) \setminus \{0\}$ (musí byt $\operatorname{tg} x \geq -1$) $\Big)$

- $\dfrac{\sqrt{2-\sqrt{2}}}{\sqrt{2}\,(\frac{\pi}{4})^3}$ pro $x_0 = -\dfrac{\pi}{4}^+$ $\left(f\left(-\dfrac{\pi}{4}\right) = \dfrac{0 - \sqrt{1-\sqrt{2}/2}}{(-\frac{\pi}{4})^3}\right)$
- $\dfrac{1}{4}$ pro $x_0 = 0$ $\left(\text{typ } \left\langle\!\!\left\langle \dfrac{1-1}{0} \right\rangle\!\!\right\rangle; \quad f(x) = \dfrac{(1+\operatorname{tg} x) - (1+\sin x)}{x^3(\sqrt{1+\operatorname{tg} x} + \sqrt{1+\sin x})} = \right.$
 $= \dfrac{\operatorname{tg} x - \sin x}{x^3(\sqrt{1+\operatorname{tg} x} + \sqrt{1+\sin x})} = \dfrac{\sin x(1-\cos x)}{x^3 \cos x(\sqrt{1+\operatorname{tg} x} + \sqrt{1+\sin x})} =$
 $\left. = \dfrac{\sin x(1-\cos^2 x)}{x^3 \cos x(1+\cos x)(\sqrt{1+\operatorname{tg} x} + \sqrt{1+\sin x})} = \left(\dfrac{\sin x}{x}\right)^3 \dfrac{1}{\cos x(1+\cos x)(\sqrt{1+\operatorname{tg} x} + \sqrt{1+\sin x})} \right)$
- $+\infty$ pro $x_0 = \dfrac{\pi}{2}^-$ $\left(\text{typ } \left\langle\!\!\left\langle \dfrac{\sqrt{1+\infty} - \sqrt{2}}{(\pi/2)^3} \right\rangle\!\!\right\rangle \right)$

c) $M = (-\infty;-1) \cup (0;\infty)$ $\Big(f(x) = \exp\big(\cos \pi x \ \ln(\ln \frac{2x+1}{x})\big) = \exp(h(x))$, tedy musí být $\ln \frac{2x+1}{x} > 0$,
 tj. $\frac{2x+1}{x} > 1$, tj. $\frac{1}{x} > -1 \Big)$

- neexistuje pro $x_0 = \pm\infty$ $\Big($typ pro h: $\left\langle\!\!\left\langle \text{neex.} \cdot \ln(\ln 2) \right\rangle\!\!\right\rangle \Big)$
- $+\infty$ pro $x_0 = -1^-$ $\Big($typ pro h: $\left\langle\!\!\left\langle (-1) \cdot \ln(\ln 1^+) \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle -1 \cdot (-\infty) \right\rangle\!\!\right\rangle$; pro f: $\left\langle\!\!\left\langle e^{+\infty} \right\rangle\!\!\right\rangle \Big)$
- $+\infty$ pro $x_0 = 0^+$ $\Big($typ pro h: $\left\langle\!\!\left\langle 1 \cdot \ln(\ln(2 + \frac{1}{0^+})) \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle 1 \cdot \ln(\ln \infty) \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle 1 \cdot \infty \right\rangle\!\!\right\rangle$; pro f: $\left\langle\!\!\left\langle e^{+\infty} \right\rangle\!\!\right\rangle \Big)$

4/7) a) 0 $\Big($ limita typu: $\left\langle\!\!\left\langle 0 \cdot \text{omez.} \right\rangle\!\!\right\rangle \Big)$

b) neexistuje $\left(\text{limita typu: } \left\langle\!\!\left\langle 0 + \text{neex.} \right\rangle\!\!\right\rangle, \text{ protože } \lim\limits_{x\to 0^\pm} \operatorname{arctg} \dfrac{1}{x} = \lim\limits_{y \to \pm\infty} \operatorname{arctg} y = \pm\dfrac{\pi}{2}\right)$

c) $+\infty$ $\left(\text{podle Věty 3.10,4b), protože } \dfrac{1}{x^2} + \operatorname{arctg}\dfrac{1}{x} \geq \dfrac{1}{x^2} - \dfrac{\pi}{2} \text{ a } \lim\limits_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{\pi}{2}\right) = +\infty \right.$
 NEBO: $\left. \lim\limits_{x\to 0^\pm}\left(\dfrac{1}{x^2} + \operatorname{arctg}\dfrac{1}{x}\right) = +\infty \pm \dfrac{\pi}{2} = +\infty, \text{ tedy } \lim\limits_{x\to 0}\left(\dfrac{1}{x^2} + \operatorname{arctg}\dfrac{1}{x}\right) = +\infty \right)$

d) neexistuje $\left(\lim\limits_{x\to 0^\pm}\left(\operatorname{cotg} x + \operatorname{arctg}\dfrac{1}{x}\right) = \pm\infty \pm \dfrac{\pi}{2} = \pm\infty\right)$

e) $+\infty$ $\left(\lim\limits_{x\to 0^\pm}\left(\operatorname{cotg} x \cdot \operatorname{arctg}\dfrac{1}{x}\right) = \pm\infty \cdot (\pm\dfrac{\pi}{2}) = +\infty\right)$

4/8) a)
- 0 pro $x_0 = +\infty$ $\Big($typ $\left\langle\!\!\left\langle \text{omez.} \cdot 0 \right\rangle\!\!\right\rangle \Big)$
- neexistuje pro $x_0 = -\infty$ $\Big($typ $\left\langle\!\!\left\langle \text{neex.} \cdot \pi \right\rangle\!\!\right\rangle \Big)$

b) $f(x) = \dfrac{(x-2)(x+3)}{(x-2)(x-3)} = \dfrac{x+3}{x-3}$ pro $x \notin \{2;3\}$

- -5 pro $x_0 = 2$
- neexistuje pro $x_0 = 3$ $\left(\lim\limits_{x\to 3^-} f(x) = \left\langle\!\!\left\langle \dfrac{6}{0^-} \right\rangle\!\!\right\rangle = -\infty, \quad \lim\limits_{x\to 3^+} f(x) = \left\langle\!\!\left\langle \dfrac{6}{0^+} \right\rangle\!\!\right\rangle = +\infty \right)$

c) • 0 $\left(\text{typ } \left\langle\!\left\langle 0 \cdot \text{omez.} \right\rangle\!\right\rangle;\ \ (\sqrt{x^2+1}\ - x) = \dfrac{(x^2+1)-x^2}{\sqrt{x^2+1}\ +x} = \dfrac{1}{\sqrt{x^2+1}\ +x}\right)$

d) • 0 pro $x_0 = 0^+$ $\left(\text{typ } \left\langle\!\left\langle \dfrac{-2}{+\infty} \right\rangle\!\right\rangle\right)$

• neexistuje pro $x_0 = 1$ $\Big($ podle Př. 3.4 v přednáškách je $\lim\limits_{x\to 1^\pm} f(x) = \left\langle\!\left\langle \dfrac{0}{0} \right\rangle\!\right\rangle = \lim\limits_{x\to 1^\pm} \dfrac{x-1}{\ln x} \cdot \dfrac{x+2}{\ln x} = \left\langle\!\left\langle 1 \cdot \dfrac{3}{0^\pm} \right\rangle\!\right\rangle = \pm\infty,\ \Big)$