Domácí cvičení 5

(derivace funkcí)

5/ 1) Podle definice spočtěte $f'(-1)$ pro $f(x) = x + (x+1)\arcsin\sqrt{\dfrac{x}{x-1}}$.

5/ 2) Podle definice i s použitím vět o derivacích spočtěte $f'(1)$ pro $f(x) = (x-1)(x-2)^2(x-3)^3$ a najděte tečnu t a normálu n grafu funkce v bodě $x = 1$ (tj. procházející bodem $[1, f(1)]$).

5/ 3) Najděte derivaci funkce f a požadovanou tečnu t nebo normálu n:

a) $f(x) = 2\,\mathrm{arctg}\,(x-1)$; $\quad t \perp p$, $\quad p:\ x + 2y + 4 = 0$,

b) $f(x) = \dfrac{2x}{1-x^2}$; $\quad n \| q$, $\quad q:\ x + 2y - 8 = 0$,

c) $f(x) = \ln(5-x)^2$; $\quad t$ prochází bodem $A = [5, -2]$.

5/ 4) Najděte derivaci funkce f:

a) $f(x) = 3x^2 - 7\sqrt[5]{x} + 1$,

b) $f(x) = \left(\dfrac{1+x^2}{1+x}\right)^5$,

c) $f(x) = \sqrt[3]{\dfrac{1+x^3}{1-x^3}}$,

d) $f(x) = \dfrac{1}{\sqrt{1+x^2}\,(x+\sqrt{1+x^2})}$,

e) $f(x) = \left(\dfrac{1-x^2}{2}\sin x - \dfrac{1+x^2}{2}\cos x\right)\mathrm{e}^{-x}$,

f) $f(x) = \ln(x + \sqrt{x^2+1})$,

g) $f(x) = \left(\dfrac{a}{b}\right)^x \left(\dfrac{b}{x}\right)^a \left(\dfrac{x}{a}\right)^b$; $\quad a,\ b > 0$,

h) $f(x) = \arcsin\dfrac{\sqrt{x^2+1}}{3}$.

5/ 5) Najděte derivaci funkce f:

a) $f(x) = (\arcsin x)^{\ln(1-2x)}$,

b) $f(x) = (\sinh(x+1))^{2^x}$.

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Výsledky:

($k_p,\ k_q,\ k_t,\ k_n$ jsou postupně směrnice přímky p, přímky q, tečny, normály)

5/ 1) $1 + \dfrac{\pi}{4}$ $\quad \left(f'(-1) = \lim\limits_{x\to-1} \dfrac{\left(x + (x+1)\arcsin\sqrt{\frac{x}{x-1}}\right) - \left(-1 + 0\cdot\arcsin\sqrt{\frac{1}{2}}\right)}{x+1} = \right.$

$\left. = \lim\limits_{x\to-1}\left(1 + \arcsin\sqrt{\dfrac{x}{x-1}}\right) = 1 + \arcsin\sqrt{\dfrac{1}{2}} = 1 + \arcsin\dfrac{\sqrt{2}}{2} \right)$

5/ 2) $f'(1) = -8$, $\quad t:\ 8x + y - 8 = 0$, $\quad n:\ x - 8y - 1 = 0$

$\Big(f(1) = 0,\ k_t = f'(1) = -8,\ k_n = -\dfrac{1}{k_t} = \dfrac{1}{8},\ t:\ y - 0 = -8(x-1),\ n:\ y - 0 = \dfrac{1}{8}(x-1);$

výpočet $f'(1)$: $\ f'(1) = \lim\limits_{x\to1}\dfrac{(x-1)(x-2)^2(x-3)^3 - 0}{x-1} = \lim\limits_{x\to1}(x-2)^2(x-3)^3$ nebo

$f'(x) = 1(x-2)^2(x-3)^3 + (x-1)2(x-2)(x-3)^3 + (x-1)(x-2)^2 3(x-3)^2,\ f'(1) = -8 + 0 + 0 \Big)$

5/ 3) (tečny nebo normály jsou v bodě x_0)

a) $f'(x) = \dfrac{2}{1+(x-1)^2}$, $x \in \mathbb{R}$, $\quad t:\ 2x - y - 2 = 0 \quad (x_0 = 1)$ $\quad \Big(p:\ y = -\dfrac{1}{2}(x+4),\ k_p = -\dfrac{1}{2} = -\dfrac{1}{k_t},$

$k_t = 2$, tedy řešíme $\dfrac{2}{1+(x_0-1)^2} = 2$, vyhovuje $x_0 = 1,\ f(1) = 0,\ t:\ y - 0 = 2(x-1) \Big)$

b) $f'(x) = \dfrac{2(x^2+1)}{(1-x^2)^2}, \quad x \neq \pm 1, \qquad n_1: \ x + 2y = 0 \quad (x_0 = 0), \qquad n_2: \ x + 2y + \sqrt{3} = 0 \quad (x_0 = \sqrt{3}),$
$n_3: \ x + 2y - \sqrt{3} = 0 \quad (x_0 = -\sqrt{3})$

$\Big(f'(x) = \dfrac{2(1-x^2) - 2x(-2x)}{(1-x^2)^2}\,; \ \ q: \ y = -\dfrac{1}{2}(x-8), \ k_q = -\dfrac{1}{2} = k_n, \ k_t = -\dfrac{1}{k_n} = 2,$ tedy řešíme $\dfrac{2(x_0^2+1)}{(1-x_0^2)^2} = 2$, vyhovuje $x_0 = 0 \ \Big(f(0) = 0, \ n: \ y - 0 = -\dfrac{1}{2}(x-0) \Big), \ x_0 = \sqrt{3} \ \Big(f(\sqrt{3}) = -\sqrt{3},$
$n: \ y + \sqrt{3} = -\dfrac{1}{2}(x - \sqrt{3}) \Big)$ a $x_0 = -\sqrt{3} \ \Big(f(-\sqrt{3}) = \sqrt{3}, \ n: \ y - \sqrt{3} = -\dfrac{1}{2}(x + \sqrt{3}) \Big) \Big)$

c) $f'(x) = \dfrac{2}{x-5}, \quad x \neq 5, \qquad t_1: \ 2x + y - 8 = 0 \quad (x_0 = 4), \qquad t_2: \ 2x - y - 12 = 0 \quad (x_0 = 6).$
$\Big(f'(x) = \dfrac{1}{(5-x)^2} \cdot 2(5-x)(-1); \ \ k_t = f'(x_0) = \dfrac{2}{x_0 - 5}\,, \ \ t: \ y - f(x_0) = f'(x_0)(x - x_0),$
tj. $t: \ y - \ln(5-x_0)^2 = \dfrac{2}{x_0-5}(x - x_0);$ musí být $A \in t$, tedy $-2 - \ln(5-x_0)^2 = \dfrac{2}{x_0-5}(5 - x_0),$
po úpravě $\ln(5-x_0)^2 = 0, \ |5 - x_0| = 1, \ x_0 = 4$ nebo $x_0 = 6;$ $\underline{x_0 = 4}: \ f(4) = 0, \ f'(4) = -2,$
$t: \ y - 0 = -2(x-4);$ $\underline{x_0 = 6}: \ f(6) = 0, \ f'(6) = 2, \ t: \ y - 0 = 2(x-6) \Big)$

5/ 4) a) $f'(x) = 6x - \dfrac{7}{5} x^{-\frac{4}{5}} = 6x - \dfrac{7}{5} \dfrac{1}{\sqrt[5]{x^4}}\,; \quad x \neq 0\,,$

b) $f'(x) = 5 \left(\dfrac{1+x^2}{1+x} \right)^4 \cdot \dfrac{2x\,(1+x) - (1+x^2)\cdot 1}{(1+x)^2} = 5 \left(\dfrac{1+x^2}{1+x} \right)^4 \cdot \dfrac{x^2 + 2x - 1}{(1+x)^2}\,; \quad x \neq -1\,,$

c) $f'(x) = \dfrac{1}{3} \left(\dfrac{1+x^3}{1-x^3} \right)^{-\frac{2}{3}} \dfrac{3x^2(1-x^3) - (1+x^3)(-3x^2)}{(1-x^3)^2} =$

$$= \frac{1}{3} \sqrt[3]{\left(\frac{1-x^3}{1+x^3} \right)^2} \ \frac{3x^2(1-x^3) - (1+x^3)(-3x^2)}{(1-x^3)^2} = \frac{2x^2}{1-x^6} \sqrt[3]{\frac{1+x^3}{1-x^3}}\,; \quad x \neq \pm 1\,,$$

d) $f'(x) = \dfrac{-1}{(\sqrt{1+x^2}(x + \sqrt{1+x^2}))^2} \ \left(\dfrac{1}{2} \dfrac{1}{\sqrt{1+x^2}} 2x \left(x + \sqrt{1+x^2} \right) + \sqrt{1+x^2} \Big(1 + \dfrac{1}{2} \dfrac{1}{\sqrt{1+x^2}} 2x \Big) \right) =$

$$= -\frac{1}{\sqrt{(1+x^2)^3}}\,; \quad x \in \mathbb{R}\,,$$

e) $f'(x) = \dfrac{1}{2} \Big((-2x \sin x + (1-x^2)\cos x) - (2x \cos x - (1+x^2)\sin x) \Big) \mathrm{e}^{-x} +$

$$+ \left(\frac{1-x^2}{2} \sin x - \frac{1+x^2}{2} \cos x \right) (-\mathrm{e}^{-x}) = \mathrm{e}^{-x}(1-x)(\cos x - x \sin x)\,; \quad x \in \mathbb{R}\,,$$

f) $f'(x) = \dfrac{1}{x + \sqrt{x^2+1}} \left(1 + \dfrac{1}{2} \ \dfrac{1}{\sqrt{x^2+1}} \ 2x \right) = \dfrac{1}{\sqrt{x^2+1}}\,; \quad x \in \mathbb{R} \qquad$ (!! $f(x) = \operatorname{argsinh} x$!!)

g) $f'(x) = \left(\left(\dfrac{a}{b} \right)^x \cdot \ln \left(\dfrac{a}{b} \right) \right) \left(\dfrac{b}{x} \right)^a \left(\dfrac{x}{a} \right)^b + \left(\dfrac{a}{b} \right)^x \left(a \cdot \left(\dfrac{b}{x} \right)^{a-1} \cdot \left(-\dfrac{b}{x^2} \right) \right) \left(\dfrac{x}{a} \right)^b +$

$$+ \left(\frac{a}{b} \right)^x \left(\frac{b}{x} \right)^a \left(b \cdot \left(\frac{x}{a} \right)^{b-1} \cdot \frac{1}{a} \right) = f(x) \cdot \left(\ln \frac{a}{b} + \frac{b-a}{x} \right); \quad x > 0\,,$$

h) $f'(x) = \dfrac{1}{\sqrt{1 - \left(\frac{\sqrt{x^2+1}}{3} \right)^2}} \cdot \dfrac{1}{3} \cdot \dfrac{1}{2} \cdot \dfrac{1}{\sqrt{x^2+1}} \cdot 2x = \dfrac{x}{\sqrt{x^2+1}\,\sqrt{8-x^2}}\,; \quad |x| < 2\sqrt{2}\,.$

5/ 5) (jde tu o logaritmické derivování - viz přednášky strana [P29])

a) $f(x) = \exp\big(\ln(1-2x) \ \ln(\arcsin x) \big),$

$$f'(x) = f(x) \cdot \left(-\frac{2}{1-2x} \ln(\arcsin x) + \ln(1-2x) \frac{1}{\arcsin x} \frac{1}{\sqrt{1-x^2}} \right); \quad 0 < x < \frac{1}{2}$$

b) $f(x) = \exp\big(2^x \ \ln\big(\sinh(x+1) \big) \big),$

$$f'(x) = f(x) \cdot \left(2^x \ln 2 \ \ln(\sinh(x+1)) + 2^x \frac{\cosh(x+1)}{\sinh(x+1)} \right); \quad x > -1$$