Domácí cvičení 11

(Riemannův integrál)

11/ 1) Vypočtěte:

a) $\displaystyle\int_a^b k\,\mathrm{d}x \quad (k\ -\ \text{konstanta})$, b) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} (x^3-3x^2+6x-8)\,\mathrm{d}x$, c) $\displaystyle\int_0^{\frac{1}{2}} \frac{2}{\sqrt{1-x^2}}\,\mathrm{d}x$,

d) $\displaystyle\int_{-\frac{\pi}{2}}^{\pi} \cos 5x\,\mathrm{d}x$, e) $\displaystyle\int_{\pi}^{3\pi} \frac{1}{\sin^2 \frac{x}{4}}\,\mathrm{d}x$, f) $\displaystyle\int_3^5 \frac{1}{(x-2)^3}\,\mathrm{d}x$.

11/ 2) Vypočtěte:

a) $\displaystyle\int_{-1}^{1} |2x+1|\,\mathrm{d}x$, b) $\displaystyle\int_0^3 |x^2-3x+2|\,\mathrm{d}x$, c) $\displaystyle\int_0^{2\pi} |\sin x|\,\mathrm{d}x$,

d) $\displaystyle\int_{-\pi}^{\pi} |\cos x|\,\mathrm{d}x$, e) $\displaystyle\int_{-2}^{1} \mathrm{e}^{|x|-3}\,\mathrm{d}x$.

11/ 3) Vypočtěte:

a) $\displaystyle\int_0^{\frac{\pi}{6}} (x+2)\sin 3x\,\mathrm{d}x$, b) $\displaystyle\int_1^{\mathrm{e}} \ln x\,\mathrm{d}x$, c) $\displaystyle\int_{-2}^{2} (x^2+1)\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{d}x$.

11/ 4) Vypočtěte:

a) $\displaystyle\int_0^{\frac{1}{2}} \frac{x}{2x^2+3x+1}\,\mathrm{d}x$, b) $\displaystyle\int_{-3}^{-2} \frac{2}{x^4-x^2}\,\mathrm{d}x$, c) $\displaystyle\int_1^2 \frac{4}{x^3+x}\,\mathrm{d}x$, d) $\displaystyle\int_{-2}^{0} \frac{3x^3+14x-2}{(x-1)(x^2+4)}\,\mathrm{d}x$.

11/ 5) Vypočtěte:

a) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\mathrm{d}x}{\cos x}$, b) $\displaystyle\int_{\ln 2}^{\ln 5} \frac{\mathrm{d}x}{\mathrm{e}^x-1}$, c) $\displaystyle\int_{\mathrm{e}^{-1}}^{\mathrm{e}} \frac{\ln x+1}{x(\ln^2 x+1)}\,\mathrm{d}x$.

11/ 6) Vypočtěte:

a) $\displaystyle\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin^4 2x\,\mathrm{d}x$, b) $\displaystyle\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{4\arcsin x}{\sqrt{1-x^2}}\,\mathrm{d}x$, c) $\displaystyle\int_0^1 \frac{3x^2+4x+2}{\sqrt{x^3+2x^2+2x+4}}\,\mathrm{d}x$.

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Výsledky:

V každém příkladu je potřeba ověřit existenci hledaného Riemannova integrálu. K tomu stačí, je-li integrovaná funkce spojitá na (uzavřeném) intervalu, přes který integrujeme. Zde toto platí v každém příkladu. Nebudu to tedy již uvádět u každého příkladu zvlášť (i když v písemce byste to uvedené mít měli).

I je opět hledaný integrál. Jako dříve také neuvádím úplný popis substituce, ale jen jakou soubstituci jsem použila.

11/ 1) a) $I = k(b-a)$ (nakreslete si obrázek a integrály z konstanty příště nepočítejte přes primitivní funkci!),

b) $I = \left[\frac{x^4}{4} - x^3 + 3x^2 - 8x\right]_{-\sqrt{2}}^{\sqrt{2}} = -20\sqrt{2}$, c) $I = [2\arcsin x]_0^{\frac{1}{2}} = \frac{\pi}{3}$, d) $I = \left[\frac{\sin 5x}{5}\right]_{-\frac{\pi}{2}}^{\pi} = \frac{1}{5}$,

e) $I = \left[-4\mathrm{cotg}\,\frac{x}{4}\right]_{\pi}^{3\pi} = 8$, f) $I = \left[-\frac{1}{2}\,\frac{1}{(x-2)^2}\right]_3^5 = \frac{4}{9}$.

11/2) a) $I = \int_{-1}^{1} \left|2\Big(x - \Big(-\frac{1}{2}\Big)\Big)\right| \mathrm{d}x = \int_{-1}^{-\frac{1}{2}} (-2x-1)\,\mathrm{d}x + \int_{-\frac{1}{2}}^{1} (2x+1) = \frac{1}{4} + \frac{9}{4} = \frac{5}{2}\,,$

b) $I = \int_{0}^{3} |(x-1)(x-2)|\,\mathrm{d}x = \int_{0}^{1} (x^2-3x+2)\,\mathrm{d}x + \int_{1}^{2} (-x^2+3x-2)\,\mathrm{d}x + \int_{2}^{3} (x^2-3x+2)\,\mathrm{d}x = \frac{5}{6} + \frac{1}{6} + \frac{5}{6} = \frac{11}{6},$

c) $I = \int_{0}^{\pi} \sin x\,\mathrm{d}x + \int_{\pi}^{2\pi} (-\sin x)\,\mathrm{d}x = 2 + 2 = 4,$

d) $I = \int_{-\pi}^{-\frac{\pi}{2}} (-\cos x)\,\mathrm{d}x + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\,\mathrm{d}x = 1 + 2 + 1 = 4,$

e) $I = \int_{-2}^{0} \mathrm{e}^{-x-3}\,\mathrm{d}x + \int_{0}^{1} \mathrm{e}^{x-3}\,\mathrm{d}x = (-\mathrm{e}^{-3} + \mathrm{e}^{-1}) + (\mathrm{e}^{-2} - \mathrm{e}^{-3}) = \mathrm{e}^{-3}(\mathrm{e}^2 + \mathrm{e} - 2).$

11/3) a) $I = \Big[(x+2)(-\frac{1}{3}\cos 3x)\Big]_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} \frac{1}{3}\cos 3x\,\mathrm{d}x = \Big[(x+2)(-\frac{1}{3}\cos 3x)\Big]_0^{\frac{\pi}{6}} + \Big[\frac{1}{9}\sin 3x\Big]_0^{\frac{\pi}{6}} = \frac{2}{3} + \frac{1}{9} = \frac{7}{9},$

b) $I = [x\ln x]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} 1\,\mathrm{d}x = \mathrm{e} - (\mathrm{e} - 1) = 1,$

c) $I = \Big[(x^2+1)\cdot 2\,\mathrm{e}^{\frac{x}{2}}\Big]_{-2}^{2} - \int_{-2}^{2} 2\,x \cdot 2\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{d}x = \Big[(x^2+1)\cdot 2\,\mathrm{e}^{\frac{x}{2}}\Big]_{-2}^{2} - \bigg(\Big[2\,x\cdot 4\,\mathrm{e}^{\frac{x}{2}}\Big]_{-2}^{2} - \int_{-2}^{2} 2\cdot 4\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{d}x\bigg) =$

$= \Big[2(x^2+1)\,\mathrm{e}^{\frac{x}{2}}\Big]_{-2}^{2} - \bigg(\Big[8x\,\mathrm{e}^{\frac{x}{2}}\Big]_{-2}^{2} - \Big[8\cdot 2\,\mathrm{e}^{\frac{x}{2}}\Big]_{-2}^{2}\bigg) = 10(\mathrm{e} - \mathrm{e}^{-1}) - \big(16(\mathrm{e} + \mathrm{e}^{-1}) - 16(\mathrm{e} - \mathrm{e}^{-1})\big) = 10\,\mathrm{e} - 42\,\mathrm{e}^{-1}.$

11/4) a) $I = \frac{1}{2}\int_0^{\frac{1}{2}} \frac{x}{(x+1)(x+\frac{1}{2})}\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{1}{2}} \left(\frac{2}{x+1} - \frac{1}{x+\frac{1}{2}}\right)\mathrm{d}x = \frac{1}{2}\left[2\ln|x+1| - \ln\left|x+\frac{1}{2}\right|\right]_0^{\frac{1}{2}} =$

$= \frac{1}{2}\left(\left(2\ln\frac{3}{2} - \ln 1\right) - \left(2\ln 1 - \ln\frac{1}{2}\right)\right) = \ln\frac{3}{2} + \frac{1}{2}\ln\frac{1}{2},$

b) $I = \int_{-3}^{-2} \left(-\frac{2}{x^2} + \frac{0}{x} + \frac{1}{x-1} - \frac{1}{x+1}\right)\mathrm{d}x = \left[\frac{2}{x} + \ln\left|\frac{x-1}{x+1}\right|\right]_{-3}^{-2} = (-1 + \ln 3) - \left(-\frac{2}{3} + \ln 2\right) = -\frac{1}{3} + \ln\frac{3}{2},$

c) $I = \int_1^2 \left(\frac{4}{x} - \frac{4x+0}{x^2+1}\right)\mathrm{d}x = \big[4\ln|x| - 2\ln(x^2+1)\big]_1^2 = (4\ln 2 - 2\ln 5) - (4\ln 1 - 2\ln 2) = 6\ln 2 - 2\ln 5,$

d) $I = \int_{-2}^{0} \left(3 + \frac{3}{x-1} + \frac{0x+2}{x^2+4}\right)\mathrm{d}x = \left[3x + 3\ln|x-1| + \operatorname{arctg}\frac{x}{2}\right]_{-2}^{0} = 0 - (-6 + 3\ln 3 + \operatorname{arctg}(-1)) =$

$= 6 - 3\ln 3 + \frac{\pi}{4}.$

11/5) a) $I = \left\langle\!\!\left\langle t = \sin x \right\rangle\!\!\right\rangle = \int_0^{\frac{\pi}{6}} \frac{\cos x}{\cos^2 x}\,\mathrm{d}x = \int_0^{\frac{1}{2}} \frac{1}{1-t^2}\,\mathrm{d}t = \int_0^{\frac{1}{2}} \frac{-1}{t^2-1}\,\mathrm{d}t = \int_0^{\frac{1}{2}} \left(\frac{-\frac{1}{2}}{t-1} + \frac{\frac{1}{2}}{t+1}\right)\mathrm{d}t =$

$= \left[-\frac{1}{2}\ln|t-1| + \frac{1}{2}\ln|t+1|\right]_0^{\frac{1}{2}} = \left(-\frac{1}{2}\ln\frac{1}{2} + \frac{1}{2}\ln\frac{3}{2}\right) - \left(-\frac{1}{2}\ln 1 + \frac{1}{2}\ln 1\right) = \frac{1}{2}\ln 3,$

b) $I = \left\langle\!\!\left\langle t = \mathrm{e}^x \right\rangle\!\!\right\rangle = \int_2^5 \left(\frac{1}{t-1} - \frac{1}{t}\right)\mathrm{d}t = [\ln|t-1| - \ln|t|\,]_2^5 = (\ln 4 - \ln 5) - (\ln 1 - \ln 2) = 3\ln 2 - \ln 5,$

c) $I = \left\langle\!\!\left\langle t = \ln x \right\rangle\!\!\right\rangle = \int_{-1}^{1} \frac{t+1}{t^2+1}\,\mathrm{d}t = \int_{-1}^{1} \left(\frac{1}{2}\,\frac{2t}{t^2+1} + \frac{1}{t^2+1}\right)\mathrm{d}t = \left[\frac{1}{2}\ln|t^2+1| + \operatorname{arctg} t\right]_{-1}^{1} =$

$= \left(\frac{1}{2}\ln 2 + \operatorname{arctg} 1\right) - \left(\frac{1}{2}\ln 2 + \operatorname{arctg}(-1)\right) = \frac{\pi}{2}.$

11/6) a) $I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \left(\frac{1-\cos 4x}{2}\right)^2\mathrm{d}x = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \left(\frac{1}{4} - \frac{1}{2}\cos 4x + \frac{1}{4}\left(\frac{1+\cos 8x}{2}\right)\right)\mathrm{d}x = \left[\frac{3}{8}x - \frac{1}{8}\sin 4x + \frac{1}{64}\sin 8x\right]_{\frac{\pi}{8}}^{\frac{3\pi}{8}} =$

$= \frac{3\pi}{32} + \frac{1}{4},$

b) $I = \left\langle\!\!\left\langle t = \arcsin x \right\rangle\!\!\right\rangle = \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} 4t\,\mathrm{d}t = \big[2t^2\big]_{-\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{7\pi^2}{72},$

c) $I = \left\langle\!\!\left\langle t = x^3 + 2x^2 + 2x + 4 \right\rangle\!\!\right\rangle = \int_4^9 \frac{1}{\sqrt{t}}\,\mathrm{d}t = \big[2\sqrt{t}\big]_4^9 = 2.$