بنامحرا

$$E(e^{\lambda X}) = \sum \frac{\lambda^n}{n!} E(|X|^n) = \sum \frac{(\delta\lambda)^n}{n!} \times \underbrace{\frac{1}{\delta^n} E(|X|^n)}_{\geq \frac{1}{\delta^k} E(|X|^k) \geq \inf_k} \qquad \wedge$$

$$\rightarrow E(e^{\lambda X}) \geq \sum \frac{(\delta\lambda)^n}{n!} \inf_k (E(|X|^k))$$

$$E(e^{\lambda X}) \geq e^{\lambda\delta} \inf_k (E(|X|^k)) \Rightarrow \frac{E(e^{\lambda X})}{e^{\lambda\delta}} \geq \inf_k (E(|X|^k)) \quad \checkmark$$

منام سرا

۲

a/ $P(|X|\geq t) = F(-\lambda) + 1 - F(x)$ $*$

$$E(\phi(X)) = \int_{-\infty}^{\infty}\phi(x)\,d(F(x)) = -\int_{-\infty}^{0} -\phi(|x|)\,dF(-x) + \int_{0}^{\infty} -\phi(|x|)\,d(1-F(x))$$

$\Rightarrow \phi(0)$

$$= \int_0^\infty -\phi(x)\,d(1-F(x)+F(-x)) = -\phi(x)(1-F(x)+F(-x))\Big|_0^\infty$$

$$+ \int_0^\infty \phi'(x)(1-F(x)+F(-x))\,dx \xrightarrow{*} + \int_0^\infty \phi'(x)\,P(|X|\geq t)\,dt$$

$$= \phi(0) + \int_0^\infty \phi'(x)\,P(|X|\geq t)\,dt \checkmark$$

b/ $E(e^{X^2/6\sigma^2}) = 1 + \int_0^\infty \frac{2x}{6\sigma^2} e^{x^2/6\sigma^2}\cdot e^{-x^2/2\sigma^2}\,dx = 1+1 = 2$

$\Rightarrow E(e^{X^2/6\sigma^2}) \leq 2$ ✓

c/ $E(e^{\lambda X}) \leq \exp(\frac{\sigma^2\lambda^2}{2})$, $P(X\geq t) \leq e^{-t^2/2\sigma^2}$

λ large: $E(e^{\lambda X}) \leq E(e^{a\lambda^2/2 + X^2/2a}) \rightarrow a = 3\sigma^2 \Rightarrow e^{3\lambda^2\sigma^2/2}\, E(e^{X^2/6\sigma^2}) \leq 2\cdot e^{3\lambda^2\sigma^2/2}$

$2e^{3\lambda^2\sigma^2/2} \leq e^{18\frac{\sigma^2\lambda^2}{2}} \Rightarrow e^{\log 2 + 3\lambda^2\sigma^2/2} \leq e^{18\sigma^2\lambda^2/2} \Rightarrow \log 2 + 3\lambda^2\sigma^2/2 \leq 9\sigma^2\lambda^2$

$\Rightarrow 7.5\,\sigma^2\lambda^2 \geq \log 2 \Rightarrow \lambda^2 \geq \frac{\log 2}{7.5\,\sigma^2}$ ✓

λ small: $E(e^{\lambda X}) \leq 1 + \lambda\underbrace{E(X)}_{a} + E(\frac{\lambda^2 e^{\lambda a} X^2}{2!})$, $e^{\lambda x}\uparrow$: $x\uparrow \Rightarrow e^{\lambda a} \leq e^{\lambda X}$

بنام خدا

$\Rightarrow ۱-۴\sigma^۲\lambda > ۰ \Rightarrow \lambda < \frac{۱}{۴\sigma^۲}$

$\Rightarrow \frac{۱+۱۲\sigma^۴\lambda^۲}{۱-۴\sigma^۲\lambda} \leq \exp(۳۲\sigma^۴\lambda^۲) \Rightarrow \text{sub-exp}(۸\sigma^۲, ۱۸\sigma^۲)$ ✓

به نام خدا

۳

a/ $\lambda > 0 \Rightarrow$

$$\frac{1}{\varepsilon}\left(E(e^{\varepsilon X})-1\right) \leq \frac{1}{\varepsilon}\left(1+\frac{1}{2}\varepsilon^2\sigma^2+\mu\varepsilon+O(\varepsilon)+1\right)$$

$$= \frac{1}{2}\sigma^2\varepsilon+\mu+o(1)$$

$\Rightarrow \varepsilon \to 0 : E(X) \leq \mu \Rightarrow$ حال ε را $-\varepsilon$ می‌کنیم و نامساوی را برای $1-E(e^{\varepsilon X})$ می‌نویسیم $\Rightarrow E(X) \geq \mu \Rightarrow E(X)=\mu$

b/

$$E(e^{\lambda(X-\mu)}) = E\left[\sum\frac{(\lambda(x-\mu))^n}{n!}\right] = \sum\frac{E(\lambda(x-\mu))^n}{n!} = 1+\frac{1}{2}\lambda^2E((x-\mu)^2)+o(\lambda^2)$$

$$\leq 1+\frac{1}{2}\lambda^2\sigma^2+o(\lambda^2) \Rightarrow \lambda\to 0 \Rightarrow var(X)\leq\sigma^2$$

مثال نقض

c/ $X \sim Ber(p) \to var(X)=p(1-p)$

نامساوی : $(1-p)+pe^{\lambda} \leq exp(p\lambda+\frac{1}{2}\sigma^2\lambda^2) \Rightarrow$ فرض $\lambda=1 : ۰٫۲ \leq \sigma^2$

$\Rightarrow var(X) = ۳/۱۶ \neq ۰٫۲$ ✓

تمرین
۲

a/

$E(\lambda X^n) \le E(\lambda X^2)(\lambda b)^{n-2} = \lambda^2 E(X^2)(\lambda b)^{n-2} = \lambda^2\sigma^2(\lambda b)^{n-2}$ ✓ ε

$\to E(e^{\lambda X}) = \sum \frac{1}{n!}E((\lambda X)^n) = 1+\sum_{n=2} \frac{1}{n!}E((\lambda X)^n) \le 1+\lambda^2\sigma^2\sum\frac{1}{n!}(\lambda b)^{n-2}$

$= 1+\frac{\sigma^2}{b^2}\underbrace{\left(\sum_{n=0}\frac{1}{n!}(\lambda b)^n - 1 - \lambda b\right)}_{\hat X} \Rightarrow e^x = \sum \frac{x^n}{n!} \ge 1+x$

$\Rightarrow E(e^{\lambda X}) \le \exp(\hat X)$, $f = \frac{1}{x^2}(e^x-1-x) \Rightarrow \hat X = (\lambda\sigma)^2 f(\lambda b)$

$\Rightarrow E(e^{\lambda X}) \le e^{(\lambda\sigma)^2 f(\lambda b)}$ ∴ ✓

$\Rightarrow \log E(e^{\lambda X}) \le \lambda\sigma^2 f(\lambda b)$ ✓

b/

$X = \sum X_i \Rightarrow \sigma^2 = \frac{1}{n}\sum\sigma_i^2$

$\forall \lambda > 0: \log P(X \ge n\delta) \le -\lambda n\delta + \underbrace{\sum \log E(e^{\lambda X_i})}_{\text{مستقل}} \le -\lambda n\delta + n\lambda^2\bar\sigma^2 f(\lambda b)$

$\hookrightarrow \log P(X\ge n\delta) \le -\frac{n\bar\sigma^2}{b^2}\underbrace{\left(\frac{b\delta}{\sigma^2}(\lambda b) - e^{\lambda b} + \lambda b + 1\right)}_{\tilde h(u)}$

$\tilde h(x) = \inf \frac{b\delta}{\sigma^2}x - e^x + x + 1 \Rightarrow \frac{b\delta}{\sigma^2}+1-e^x = 0 \to x = \left(\frac{b\delta}{\sigma^2}+1\right)\log\left(\frac{b\delta}{\sigma^2}+1\right) - \frac{b\delta}{\sigma^2}$

$\Rightarrow P(X\ge n\delta) \le \exp\left(-\frac{n\bar\sigma^2}{b^2}h\left(\frac{b\delta}{\bar\sigma^2}\right)\right)$ ∴ ✓

c/

ادامه

استفاده کنیم

$$\rightarrow -\frac{a^2}{b^2} h\left(\frac{bx}{a^2}\right) \leq -\frac{x^2}{2(a^2+bx)} = -\frac{a^2}{b^2} g\left(\frac{bx}{a^2}\right)$$

$$\rightarrow h(x) = (x+1)\log(x+1)$$

$$g(x) = \frac{x^2}{2(1+x)}$$

$$\Rightarrow h(0)=g(0),\ h'(0)=g'(0),\ h''(x) \geq g''(x)$$

$\hookrightarrow$ ✓

بنام خدا

From a good paper :)

/۵

a

لم : اگر $X \sim N(0,1) \Rightarrow X^2$: sub-exp $(\nu,\alpha) = (2,4)$

طبق لم $\Rightarrow \lambda_i^2 x_i^2$: subexp $(2\lambda_i, 4\lambda_i)$

$$\hookrightarrow (\nu,\alpha) = \left(2\left(\sum \lambda_i^2\right)^{1/2}, 4\max \lambda_i\right)$$

$$= \left(2\|Q\|_F, 4\|Q_{op}\|\right)$$

بنا به sub-exp

$$P(\langle X, QX\rangle \geq tr\, Q + t) \leq \exp\left(-\min\left(t/4\alpha, t^2/2\nu^2\right)\right)$$

$$= \exp\left(-\min\left(\frac{t}{8\|Q\|_{op}}, \frac{t^2}{8\|Q\|_F^2}\right)\right)$$

بنام خدا

/۴

/ الف:

$$E(Y) = E(Y \mid y \leq aE(Y)) + E(Y \mid y > aE(Y))$$

$$\leq aE(Y) + E(Y \mid y > aE(Y))$$

$$\hookrightarrow (1-a)E(Y) \leq E(Y \mid y > aE(Y)) \quad \underline{1}$$

از کوشی:

$$E(XY)^2 \leq E(X^2) \cdot E(Y^2)$$

$$\Rightarrow E(Y \mid y > aE(Y))^2 \leq E(Y)^2 \, E(Y \geq aE(Y))^2 \quad \underline{2}$$

$$1, 2 \Rightarrow (1-a)^2 E^2(Y) \leq E(Y \mid y > aE(Y))^2 \leq E(Y^2) \, E(Y \geq aE(Y))^2$$

$$\Rightarrow (1-a)^2 E^2(Y) \leq E(Y^2) \, E(Y \geq aE(Y))^2$$

$$\Rightarrow \frac{(1-a)^2 E^2(Y)}{E(Y^2)} \leq P(Y \geq aE(Y))$$

ب/ $\sum x_i = X \Rightarrow \mu(X) = \frac{n}{2}, \; \sigma^2(X) = \frac{n(n+1)}{4} \Rightarrow P(X > a \times \frac{n}{2}) \geq (1-a^2) \cdot \frac{\frac{n^2}{4}}{\frac{n + (n+1)}{4}}$

$$\Rightarrow a = \frac{1}{\sqrt{n}} \Rightarrow P(X > \frac{\sqrt{n}}{2}) \geq \left(1 - \frac{1}{\sqrt{n}}\right)^2 \cdot \frac{n}{n+1} \Rightarrow P(X > \frac{\sqrt{n}}{2}) \geq \frac{n}{n+1} \cdot \left(\frac{\sqrt{n}-1}{\sqrt{n}}\right)^2$$

$$P(X > \sqrt{n}/2) \geq \frac{(\sqrt{n}-1)^2}{n+1} = 1 - \frac{2\sqrt{n}}{n+1}$$

حال داریم

$n \geq \begin{cases} ۱ \\ ۲ \end{cases} \Rightarrow n = ۱ \Rightarrow P\left(X \geq \frac{\sqrt{۱}}{۲}\right) = \frac{۱}{۲} \geq \frac{۳}{۳۲}$ ✓

$\Rightarrow n \geq ۲ \Rightarrow P\left(X \geq \sqrt{\frac{n}{۲}}\right) \geq ۱ - \frac{۲\sqrt{n}}{n+۱} \overset{۲}{\underset{}{\overset{۱م}{\Rightarrow}}} ۱ - \frac{۲\sqrt{n}}{n+۱} \geq \frac{۳}{۳۲} \Rightarrow \frac{۲\sqrt{n}}{n+۱} \leq \frac{۲۹}{۳۲}$

$\Rightarrow n \geq ۱۲$ ✓

بنام خدا

$$g : (a_1, \dots, a_n, b_1, \dots, b_n) \to [0, 2n]$$

$\to$ اگر یک تاس عوض شود $\quad \left| g(x_1, \dots, x_k, \dots x_{2n}) - g(x_1, \dots, x_k', \dots, x_{2n}) \right| \leq 1$

McDiarmid's ineq : $\quad X_n = g(a_1, \dots, a_n, b_1, \dots, b_n) = \text{subG}\left(\frac{\sum c_i^2}{4}\right)$

$$\Rightarrow c_i = 1 \Rightarrow X_n = \text{subG}\left(2n/4\right) = \text{subG}(n/2) \checkmark$$

$$\Rightarrow P(|X_n - E(X_n)| \geq \delta) \leq 2e^{-\delta^2/2n/2} = 2e^{-\delta^2/n} \checkmark$$

به نام خدا

رادمهر کریمیان

/۸

a)

$d_i = (n-1)p \Rightarrow \exists M \Rightarrow d \le M \log n \Rightarrow$ فرض: $C \ge$ $\frac{eM}{C} \le e^{-\alpha}$ $(\alpha > 0)$

$\Rightarrow P(d_i \ge C \log n) \le e^{-d} \left(\frac{ed}{C \log n}\right)^{C \log n} \le n^{-\alpha C}$

$\hookrightarrow \mathbb{P}(\exists i\ (d_i \ge C \log n)) \le \sum P(d_i \ge C \log n) \le n^{1-\alpha C} \Rightarrow C$ به اندازه کافی بزرگ باشد

سری‌سری

$\Rightarrow n^{1-\alpha C} \to 0$ ✓

b) یک روش اثبات مانند کلاس $\Rightarrow$ استفاده از پارت $t \log(t) + o(t) \ge \log(n)$

$\hookrightarrow o(t) = o\left(\frac{\log(n)}{\log(\log(n))}\right)$ ✓