بنام خدا

رادمهر کریمیان

سوال ۱

$$B = UDU^T \Rightarrow E(e^{\lambda Q}) = U^T (e^{\lambda \gamma D}) U$$

$$\preccurlyeq U^T e^{\frac{1}{2}\lambda^2 \sigma^2 D^2} U$$

$$= e^{(\frac{1}{2}\lambda^2 \sigma^2)} U^T D^2 U$$

$$= e^{\frac{1}{2}\lambda^2 \sigma^2 B^2} \checkmark$$

$$\|B\|_2 \leq b \Rightarrow B \preccurlyeq b I_{S\times S} \Rightarrow e^{\frac{1}{2}\lambda^2\sigma^2 B^2} \preccurlyeq e^{\frac{1}{2}\lambda^2\sigma^2 b^2 I_{d\times d}} \checkmark$$

سوال ۲

از یکی از قضیه‌های کتاب وینرایت فصل ۶ استفاده می‌کنیم:

Theorem 6.15:

$$E(e^{\lambda_{max}(S_n)}) = E[\lambda_{max}(e^{\gamma S})] \leq tr(E(e^{\gamma S})) \qquad S_n = \sum \tfrac{1}{n} Q_i$$

$$= tr(\Phi_S(\tfrac{\gamma}{n})) \underset{\text{Lemma } 6.13}{\preccurlyeq} tr(\exp(\sum \log \Phi_Q(\tfrac{\gamma}{n})))$$

$$\underset{6.25}{\preccurlyeq} tr(\exp(\frac{\gamma^2}{2n^2} \sum V_i))$$

$$\preccurlyeq d e^{\frac{\gamma^2 \sigma^2}{2n^2}} \quad \|\tfrac{1}{n}\sum V_i\|_2$$

$$\rightarrow \gamma = \sqrt{\frac{2n}{\sigma^2}\log d} \rightarrow E(\lambda_{max}(\tfrac{1}{n}\sum Q_i)) \leq \sqrt{\frac{2\sigma^2}{n}\log d} \checkmark$$

سوال ۲ - ب)

$$E\left(e^{\lambda \|\frac{1}{n}\sum Q_i\|_2}\right) = E\left(e^{\lambda \max\{\lambda_{max}(\underbrace{\frac{1}{n}\sum Q_i}_{S_n}),\, |\lambda_{min}\sum Q_i/n|\}}\right)$$

$$\le E\left(e^{\gamma\lambda_{max}(S_n)} + e^{-\gamma\lambda_{min}(S_n)}\right)$$

$$\le tr\left(E(e^{\gamma\lambda_{max}(S)})\right) + tr\left(E(e^{-\gamma\lambda_{min}(S)})\right)$$

$$\le d e^{\frac{\gamma\sigma^2}{2n}} + d e^{\frac{\lambda^2\sigma^2}{2n}} = 2d e^{\frac{\gamma^2\sigma^2}{2n}}$$

$$\Rightarrow \gamma = \sqrt{\frac{2\sigma^2}{n} 2\log 2d} \Rightarrow E\|\frac{1}{n}\sum Q_i\|_2 \le \sqrt{\frac{2\log 2d}{n}}\, \|\frac{1}{n}\sum Q_i\|_2$$

سوال ۳

$$Q_i = \begin{pmatrix} 0 & A_i \\ A_i^T & 0 \end{pmatrix} \Rightarrow Q_i^2 = \begin{pmatrix} A_iA_i^T & 0 \\ 0 & A_i^TA_i \end{pmatrix}$$

$$\det(Q_i^2 - \lambda I) = (A_iA_i^T - \lambda I)(A_i^TA_i - \lambda I) \Rightarrow \text{eval } Q_i^2 = \text{eval } A_i^TA_i \pm \text{eval } A_iA_i^T$$

$$\|Q_i\|_2 = \sqrt{\|A_iA_i^T\|_2} = \sqrt{A_i^TA_i} = \|A_i\|_2 \rightarrow$$ شرایط فرض اثباتش
قضیه ۱۷.۶ -
برقرار است.

$$E(Q_i^k) \le \frac{k!}{2} b^{k-2}\, var(Q)$$

$$\Downarrow$$

$$E((g_iQ_i)^k) \le \frac{k!}{2} (b_ib_2)^{k-2}\, var(g_iQ_i) \rightarrow ?$$

$$var(g_iQ_i) = \|\sum V(Q_i)\|_2 = \max\{\|E(g_iA_iA_i^Tg_i^T)\|_2, \|E(g_i^TA_i^TA_ig_i)\|_2\} \cong \sigma^2$$

$$= \sigma^2 b_2^2 I$$

$$\Rightarrow E\left((y_i Q_i)^p\right) \leq \frac{p!}{2} (b_1 b_2)^{p-2} \underbrace{\sigma^2}_{\max\{\cdots,\cdots\}} b_2^2 I$$

چون از Q_i r.e.v ها symmetric اند $\Leftarrow$ $y_i Q_i$ e.v λ $\Leftarrow$ $y_i Q_i$ e.v $-\lambda$

$\xrightarrow{\text{[illegible]}}$ $$P\left(\left\|\frac{1}{n}\sum Q_i\right\| \geq \delta\right) \leq \underbrace{\operatorname{rank}\left(\sum V_i\right)}_{= d_1 + d_2} e^{\frac{-n\delta^2}{2(\sigma^2 b_2^2 + b_1 b_2 \delta)}}$$ ✓

سوال ۴

برای سادگی فرض شده: $\boxed{\sigma=1}$

$$x_{ij} = subg(1),\quad x_{ij}^2 = subg(2,4) \Rightarrow \sum x_{ij}^2 \; subg(2\sqrt{n},4)$$

$$\Rightarrow \frac{1}{n}\sum x_{ij}^2 = subg\left(\frac{2\sqrt{n}}{n}, \frac{4}{n}\right)$$

$\hookrightarrow$ tail bound: $P(|\hat{D}_{ii} - D_{ii}| \geq t) \leq 2e^{-\frac{n}{8}\min(t,t^2)}$

union bound: $P(\|\hat{D} - D\|_2 \geq t) \leq 2de^{-\frac{n}{8}\min(t,t^2)}$

$$P(\|\hat{D} - D\|_2 \geq t+\epsilon) \leq 2de^{-\frac{n}{8}\min(t,t^2) - \frac{n}{8}\epsilon^2}$$

$\epsilon = \sqrt{\frac{8\log(d)}{n}}$, $\xrightarrow{n \geq 8\log(d)}$ $P\left(\|\hat{D} - D\|_2 \geq \sqrt{\frac{8\log d}{t}} + t\right) \leq 2e^{-\frac{n}{8}\min(t,t^2)}$ ✓

$n < 8\log d$, $P\left(\|\hat{D} - D\|_2 \geq \frac{8\log d}{n} + t\right) \leq 2e^{-\frac{n}{8}\min(t,t^2)}$ ✓

حال برای قسمت دوم از نامساوی مارکوف و رزنتال داریم:

$$P\left(\|\hat{D} - D\|_2 \geq 4\delta\sqrt{\frac{d^{2m}}{t}}\right) \leq \frac{1}{(4\delta)^m} * \frac{n^{m/2}}{d^{2m}} E(\max|\hat{D}_{jj} - D_{jj}|^m)$$

$$\leq \frac{1}{(4\delta)^m} \cdot \frac{n^{m/2}}{2^m} \|\hat{D}_{11} - D_{11}\|_m^m$$

$$\leq \frac{C_m^m}{n^m}\left(\left(\sum E((x_{11}^2 - D_{11})^2)\right)^{1/2} + \sum E(|x_{11}^2 - D_{11}|^m)^{\frac{1}{m}}\right)^m$$

$$\leq \frac{C_m^m}{h^m} \cdot \left(\sqrt{nc_2} + \sqrt[m]{nc_m}\right)^m \leq C_m^m \times \frac{2^m}{n^{m/2}} \max(c_2, c_m)$$

$$\Rightarrow P\left(\|\hat{D} - D\|_2 \geq 4\delta \sqrt{\frac{d^{2/m}}{n}}\right) \leq \frac{C_m^m \max(c_2, c_m)}{(2\delta)^m}$$

سوال ۵ (برداشته شده از کتاب)

$$Z = \bar{R}_n - B = \frac{1}{n}\sum (R_k - E(R)) = \sum \underbrace{S_k}_{R_k - E(R)}$$

$$\|S_k\| \leq \frac{1}{n}(\|R_k\| + \|E R\|) \underset{\text{Jensen}}{\leq} \frac{1}{n}(\|R_k\| + E\|R\|) \leq \frac{2L}{n} \quad (\|R\| < L)$$

$$\nu(Z) = \max\left\{ \left\|\sum E(S_k S_k^*)\right\|, \left\|\sum E(S_k^* S_k)\right\| \right\} = n \cdot \max\left\{ E\|S_1 S_1^*\|, E(\|S_1^* S_1\|) \right\}$$

$$0 \leq E(S_1 S_1^*) = \frac{1}{n^2} E\left((R - E(R))(R - E(R))^*\right) \leq n^{-2}\Big(E(RR^*) - \underbrace{E(R)E(R)^*}_{\geq 0}\Big)$$

$$\leq \frac{1}{n^2} E(RR^*)$$

$$\Rightarrow E(S_1 S_1^*) \leq \frac{1}{n^2}\|E(RR^*)\|$$

$$E(S_1^* S_1) \leq \frac{1}{n^2}\|E(R^* R)\| \quad \rightarrow \nu(Z) = \frac{1}{n}\max\left\{ \|E(RR^*)\|, \|E(R^* R)\| \right\}$$

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matrix Bernstien:

$$E\|Z\| \leq \sqrt{2\nu(Z)\log(d_1+d_2)} + \frac{1}{3}\log(d_1+d_2) L \quad \checkmark$$

$$P(\|Z\| \geq t) \leq (d_1+d_2)\exp\left(\frac{-t^2/2}{\nu(Z) + Lt/3}\right)$$

$$\Rightarrow \begin{cases} E\|R_n - B\| \leq \sqrt{\dfrac{2\log(d_1+d_2)}{n}\max\{E(RR^*), E(R^*R)\}} + \dfrac{2L\log(d_1+d_2)}{3n} \quad \checkmark \\ P(\|R_n - B\|) \leq (d_1+d_2)\exp\left(\dfrac{-nt^2/2}{2Lt/3 + \max\{E\|RR^*\|, E\|R^*R\|\}}\right) \end{cases}$$