به نام خدا

سوال ۱

$$N_L \to \varepsilon_L \overset{=2^{-l}}{\leq} \min \frac{d(s,t)}{2}$$

$$N_H \to \varepsilon_H = 2^{-k} \geq diam(T) \text{ و } 2^{-k} \leq 2\, diam(T)$$

mapping $\pi_i : T \to T_i \quad d(\pi_i(t), t) \leq \varepsilon_i$

$$X_t = X_{\pi_L t} \qquad X_{\pi_k s} = X_{\pi_k t} \qquad X_s = X_{\pi_L s}$$

$$X_t - X_s = \sum_{k+1}^{L} (X_{\pi_i t} - X_{\pi_{i-1} t}) + \sum_{k+1}^{L} (X_{\pi_{i-1} s} - X_{\pi_i s})$$

$$\Rightarrow |X_t - X_s| \leq \sum \underbrace{|X_{\pi_i t} - X_{\pi_{i-1} t}|}_{subG(d^2(\pi_i t, \pi_{i-1} t)) = subG(3 \times 2^{-i})^2} + \sum_{k+1} \underbrace{|X_{\pi_{i-1} s} - X_{\pi_i s}|}_{= subG((3 \times 2^{-i})^2)}$$

$$P(\sup |X_{\pi_i t} - X_{\pi_{i-1} t}| \geq u_i) \leq P(\sup_{\substack{t \in N_i \\ t' \in N_{i-1} \\ d \leq 3 \times 2^{-i}}} |X_t - X_{t'}| > u_i)$$

$$= P(|X_t - X_{t'}| > u_i \, | \exists t_{0i})$$

$$\leq |N_i| |N_{i-1}| \exp\left(\frac{-u_i^2}{2(3 \times 2^{-i})^2}\right)$$

$$\leq |N_i|^2 \exp\left(\frac{-u_i^2}{2(3 \times 2^{-i})^2}\right)$$

$$\leq \exp\left(\frac{C \log(N_i) - u_i^2}{C'}\right)$$

$$\tilde{u}_i = \sqrt{C \log |N_i| + C' \tfrac{u_i'}{2}} \Rightarrow$$

$$P(\sup |X_{\pi_i t} - X_{\pi_{i-1} t}| \geq \tilde{u}_i) \leq 2\exp\left(\frac{-u_i'}{2}\right)$$

به همین شکل

$$P(\sup |X_{\pi_{i-1} s} - X_{\pi_i s}| \geq \tilde{u}_i \leq 2\exp\left(\frac{-u_i'}{2}\right)$$

$$P\Big(\sup |X_{\pi_i t} - X_{\pi_{i-1} t}| \geq 2\sqrt{\log|N_i|(3\times 2^{-i})} + (3\times 2^{-i})u_i',$$

$$\sup |X_{\pi_{i-1} s} - X_{\pi_i s}| \geq 2\sqrt{\log|N_i|(3\times 2^{-i})} + (3\times 2^{-i})u_i'\Big)$$

$$\leq \sum P\big(\sup |X_{\pi_i t} - X_{\pi_{i-1} t}| \geq 2\sqrt{\log|N_i|3\times 2^{-i}} + 3\times 2^{-i}u_i'\big)$$

$$+ \sum P\big(\sup |X_{\pi_{i-1} s} - X_{\pi_i s}| \geq \quad 〃 \quad\big)$$

$$\leq 2\sum 2\exp\Big(\frac{-u_i'^2}{2}\Big) \leq 8\exp\Big(\frac{g^2}{2}\Big)$$

$$u_i' = g + \sqrt{i}$$

در نتیجه با احتمال $1 - 8\exp(\frac{g^2}{2})$:

$$\sup|X_t - X_s| \leq \sum \sup|X_{\pi_i t} - X_{\pi_{i-1} t}| + \sum \sup|X_{\pi_{i-1} s} - X_{\pi_i s}|$$

$$\leq 2\sum (3\times 2^{-i})u_i' + 2\sqrt{\log|N_i|}\,(3\times 2^{-i})$$

$$\leq c\int \log\sqrt{N(T,d,\varepsilon)}\,d\varepsilon + c' g\,\mathrm{diam}(T) + \underbrace{c''}_{\leq c\int\sqrt{\log(N(T,d,\varepsilon))}\,d\varepsilon}$$

$$\leq C\int \log\sqrt{N(T,d,\varepsilon)}\,d\varepsilon + g\,\mathrm{diam}(T)$$

با احتمال $1 - 8\exp(\frac{g^2}{2})$

سوال ۲

$$N_L \to \varepsilon_L = 2^{-L} \leq \min_{s,t} \frac{d(s,t)}{2}$$

$$N_H \to \varepsilon_H = 2^{-H} \geq \delta \quad , \quad \varepsilon_H \leq 2\delta$$

$$X_t = X_{\pi_L t} \qquad X_s = X_{\pi_L s} \qquad X_{\pi_K t} = X_{\pi_K s}$$

$$X_t - X_s = \sum \left(X_{\pi_i(t)} - X_{\pi_{i-1}(t)} \right) + \sum \left(X_{\pi_{i-1}(s)} - X_{\pi_i(s)} \right)$$

$$\Rightarrow \sup (X_t - X_s) = \sum \sup \{ X_{\pi_i t} - X_{\pi_{i-1} t} \}^{A} + \sum \sup \{ X_{i-1 s} - X_{i s} \}^{B}$$

$$A \sim subG(9 \times 2^{-2i}) \qquad B \sim subG(\ldots)$$

$$\Rightarrow \begin{cases} P(\sup(X_{\pi_i t} - X_{\pi_{i-1} t}) \geq 2(3 \times 2^{-i}) \sqrt{\log(|N_i|)} + 3 \times 2^{-i} v_i \} \leq 2\exp(-\frac{v_i^2}{2}) \\ P(\sup(X_{\pi_{i-1} s} - X_{\pi_i s}) \geq 2(3 \times 2^{-i}) \sqrt{\log |N_i|} + 3 \times 2^{-i} v_i \} \leq 2\exp(\frac{-v_i^2}{2}) \end{cases}$$

حال مانند سوال قبل داریم ، با احتمال $1 - \exp(\frac{-u^2}{2})$

$$\sup \{ X_t - X_s \} \leq 2 \sum w_i \ldots$$

$$= 2 \sum \left(2(3 \times 2^{-i}) \sqrt{\log(N_i)} + 3 \times 2^{-i} v_i \right)$$

$$= \ldots$$

$$\leq c \left(\int_0^{\delta} \sqrt{\log N(T, d, \varepsilon)} \, d\varepsilon + u \delta \right)$$

$$P\left(\sup \{ X_t - X_s \} \geq c \int_0^{\delta} \sqrt{\log N(T, d, \varepsilon)} + x \right) \leq \exp\left(\frac{-x^2}{c\delta^2} \right)$$

سوال ۳

$$\log E\left(e^{\frac{\lambda(X_t - X_s)}{d(t,s)}}\right) \leq \psi(\lambda)$$

$$E\left(\sup_{t\in T}\{X_t\}\right) = E\left(\sup\left\{\sum_{i=1}^{n+1} X_{\pi_i t} - X_{\pi_{i-1} t}\right\}\right)$$ ← از سوال ۱

$$\leq \sum_{i}^{n+1} \underbrace{E\left(\sup\left\{X_{\pi_i t} - X_{\pi_{i-1} t}\right\}\right)}$$

$$= \frac{c}{\lambda} E\left(\log\left(\sup\left\{\exp\left(\frac{\lambda(X_{\pi_i} - X_{\pi_{i-1}})}{c}\right)\right\}\right)\right)$$

$$\leq \frac{c}{\lambda} E\left(\log\left(\sup\left\{\exp\left(\frac{\lambda(X_t - X_{t'})}{d(t,t')}\right)\right\}\right)\right)$$

$$\underset{\text{Union}}{\leq} \frac{c}{\lambda} \log \sum e^{\psi(\lambda)}$$

$$\longrightarrow 2\log|N_i| \geq \sup_{\lambda\geq 0}\left\{\frac{\lambda}{c} \cdot \frac{E(\sup\{X_{\pi_i} - X_{\pi_{i-1}}\})}{c} - \psi(\lambda)\right\}$$

$$\sup \lambda x - \psi(\lambda) = \psi^*\left(\frac{E(\sup\{X_{\pi_i t} - X_{\pi_{i-1} t}\})}{c}\right)$$ از تمرین ۲ و مباحث ۳٫۵ (مثل) کتاب ون‌هندل می‌دانیم ،

$$2\log(N_i) = \psi^*(\psi^{*-1}(2\log N_i)) \geq \psi^*\left(\frac{E(\sup\{X_{\pi_i t} - X_{\pi_{i-1} t}\})}{3\cdot 2^{-i}}\right)$$

$$\frac{\partial}{\partial\lambda}(\lambda x - \psi(\lambda)) = 0 \Rightarrow \lambda^* = \psi'^{-1}(x), \quad d\psi^*/dx = \psi'^{-1}(x) \checkmark$$

$$\psi^{*\prime}(x) \geq 0 \Rightarrow$$ صعودی است ✓

سوال ۲

a)

$w(T) = E(\sup\{t^T x\})$

$$E\left(\sup\left\{ \frac{X_k}{\sqrt{1+\log(k)}} \right\}\right)$$

$$\leq \sum E\left(\frac{X_k}{\sqrt{1+\log k}} \right) = E(X_k)_{max} \sum \frac{1}{\sqrt{1+\log k}}$$

$$= \sqrt{2/n} \sum_{k=1}^{n} \frac{1}{\sqrt{1+\log k}} \leq \sqrt{2/n} \sum \frac{1}{1} = \sqrt{2/n}\, n \leq \infty \checkmark$$

b)

$$E(e^{\lambda(Y_t - Y_s)}) \doteq \exp\left(\frac{\lambda^2 \|t-s\|_2^2}{2} \right)$$

$$\|t_k - t_l\| = \sqrt{\frac{1}{1+\log k} + \frac{1}{1+\log l}} \geq \sqrt{\frac{1}{1+\log m} + \frac{1}{1+\log m}} \geq \sqrt{\frac{1}{\log m}}$$

$$\Rightarrow m \leq P(T, d, \varepsilon) \leq N(T, d, \varepsilon)$$

$$\exp\left(\frac{1}{2\varepsilon^2}\right) \leq N \Rightarrow \int_0^\infty \sqrt{\log N}\, d\varepsilon \geq \int \sqrt{\log(\exp(\frac{1}{2\varepsilon^2}))}\, d\varepsilon = \infty \checkmark$$

۵) اثبات ون ماندل

ثابت کنیم $\delta > 0, E(X_t) = 0$

$$\{N_i\}_{i\in\mathbb{Z}} \quad \varepsilon_i = \frac{1}{2^i} \begin{cases} \varepsilon_L = 2^{-L} = \max\limits_{l} 2^{-l} \leq \min\left(\frac{d(s,t)}{2}\right) \\ \varepsilon_H = 2^{-H} = \min\limits_{H} 2^{-H} \geq \operatorname{diam}(T) \end{cases}$$

$$X_t = X_{\pi_L}(t), \quad X_{\pi_H}(t) = X_{\pi_H}(s)$$
$$X_s = X_{\pi_S}(S)$$

$$\longrightarrow X_t = X_t - X_{\pi_L t} + \sum \left(X_{\pi_i(t)} - X_{\pi_{i-1}(t)} \right) + X_{\pi_H t}$$

$$d(\pi_i(t), \pi_{i-1}(t)) \leq d(\pi_i(t), t) + d(t, \pi_{i-1}(t)) \leq 2^{-i} + 2^{-i+1} \leq 3 \times 2^{-i}$$

$$\Rightarrow X_t - X_{\pi_L t} \leq C d(t, \pi_L t) \leq 2C\delta$$

$$\sup\{X_t\} \leq \sup_{t\in T} \{X_t - X_{\pi_L t}\} + \sum \sup\left(X_{\pi_i(t)} - X_{\pi_{i+1} t}\right) + \sup X_{\pi_H t}$$

$$\leq 2C\delta + \sum_{i=H+1}^{L} \sup_{\substack{t\in N_i \\ t'\in N_{i-1}}} (X_t - X_{t'}) + X_{\pi_H t}$$

$$\Rightarrow E(\sup\{X_t\}) \leq 2\delta E(C) + \sum \underbrace{E(\sup(X_{\pi_i t} - X_{\pi_{i-1} t}))}_{} + \cancel{E(\sup X_{\pi_H})}^{0}$$

$$= \sum \frac{1}{\lambda} E(\log(\sup(\exp(\lambda(X_t - X_{t'})))$$

$$\underset{\text{anion}}{\leq} \frac{1}{\lambda} E(\log(\sum \cdots)) \underset{\text{Jensen}}{\leq} \frac{1}{\lambda} \log(\sum E(\exp(\lambda(X_t - X_{t'})))$$

$$\leq \frac{1}{\lambda} \log\left(N_i^2 \exp\left(\frac{\lambda^2 d^2}{2}\right)\right) \leq \frac{1}{\lambda} \log\left(N_i^2 \exp\left(\frac{\lambda^2\, 3^2 \times 2^{-2i}}{2}\right)\right)$$

۵۴۱۲۱

$$\leq \frac{۲}{\lambda}\log(۲N_i) + \frac{\lambda^۲ (۳\cdot ۲^{-i})^۲}{۲} \leq ۲\; ۶\times ۲^{-i}\sqrt{\log(N_i)} \Rightarrow \lambda = \checkmark$$

$$E(\sup(X_t)) \leq ۲\delta E(C) + \sum ۶\times ۲^{-i}\sqrt{\log(N_i)}$$

$$\leq ۲\delta E(C) + \int_\delta^\infty ۱۲\sqrt{\log(N,T,d,\varepsilon)}\, d\varepsilon$$

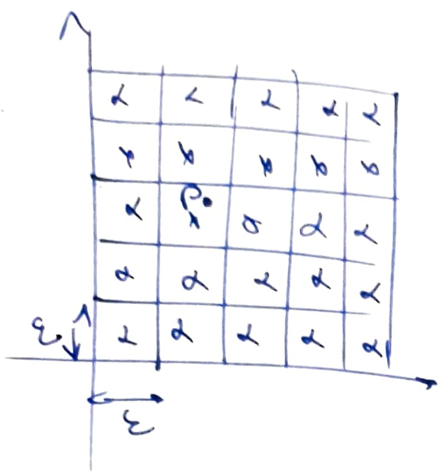
$$\Rightarrow \leq \inf\{ ۲\delta E(C) + \int_\delta^\infty ۱۲\sqrt{\log(N,T,d,\varepsilon)}\, d\varepsilon \checkmark$$

۶)

سوال ۵.۷ ونرشاین

a)

[illegible]



$$f(x) - (\text{near } c.p) \le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$N(F_0, \|\cdot\|_\infty, \varepsilon) \le 3^{(\frac{2}{\varepsilon})(\frac{2}{\varepsilon})}$$

$$\le 3^{(2/\varepsilon + 1)^2}$$

$$\le 3^{c/\varepsilon^2} = \exp(c/\varepsilon^2)$$

b) لم ماشیگ $X_f - X_g$ ساب‌گاوسی است با پارامتر $c\frac{\|f-g\|_2}{n}$ و $X_f = \sup$

$$E(\sup X_f) \le c\int_0^\infty \sqrt{\log N}\, d\varepsilon = c'\int_0^\infty \sqrt{\log(N(F_0, \|\cdot\|_\infty, \tfrac{\varepsilon\sqrt n}{c}))}\, d\varepsilon$$

$$= \frac{\tilde c}{\sqrt n}\int_0^\infty \sqrt{\log N} \le \frac{\tilde c}{\sqrt n}\int_0^\infty \sqrt{\log e^{c/\varepsilon^2}} = \infty$$

c) ادامه سوال b

$$E(\sup|X_f|) \le \inf_{\delta > 0}\left\{2\delta\sqrt n + 32\int_\delta^\infty \sqrt{\log N(F_0, d, \varepsilon)}\, d\varepsilon\right\}$$

$$= \inf_{\delta > 0}\left\{2\delta\sqrt n + \frac{c}{\sqrt n}\int_\delta^{2} \sqrt{\log N}\, d\varepsilon\right\} = \inf_{\delta > 0}\left\{2\delta\sqrt n, \frac{\tilde c}{\sqrt n}\int_\delta^2 \frac{d\varepsilon}{\varepsilon}\right\}$$

$$= \inf_{\delta \ge 0}\left\{2\delta\sqrt n + \frac{\tilde c}{\sqrt n}\log\left(\frac{2}{\delta}\right)\right\} \Rightarrow \delta = \frac{\tilde c}{n} \Rightarrow$$ δ بهینه $O\left(\frac{\log n}{\sqrt n}\right)$ ✓

سوال ۷ (انیسه_مارشل)

(a

$$w(aT) = |a|\,w(T) \rightarrow$$

اثبات:

$$\begin{cases} a)\ a \geq 0: \\ w(aT) = E(\sup(a t^T X)) \\ \quad = |a| E(\sup(t^T X)) \\ \quad = a\,w(T) \\ a < 0 \rightarrow \tilde{X} = -X \end{cases}$$

متقارن ✓

$$w(S+T) = w(S) + w(T)$$

$$\hookrightarrow w(S+T) = E(\sup\{u^T X\})$$
$$= E(\sup((t+s)^T X)) = E(\sup(s^T X)) + E(\sup(t^T X))$$
$$= w(S) + w(T) \checkmark$$

$$\rightarrow w(T-T) = w(T) + w(-T) = 2w(T) \checkmark$$

$$w(T) = \tfrac{1}{2} w(T-T) = \tfrac{1}{2} E(\sup (t_1 - t_0)^T X)$$
$$= \tfrac{1}{2} E(\overset{\max}{\cancel{\sup}}\{(u-v)^T X, (v-u)^T X\})$$
$$= \tfrac{1}{2} E(\underbrace{|(u-v)^T X|}_{Y \sim N(0, \|u-v\|^2)})$$

$$E(Y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}\, 2\, e^{-\frac{z^2}{2\sigma^2}} dz = \sigma \Rightarrow w(T) \geq \frac{\sigma}{2} \cdot \sqrt{\frac{2}{\pi}}$$
$$= \frac{\|u-v\|}{\sqrt{2\pi}} \checkmark$$

$$\Rightarrow w(T) \geq \sup \frac{\|u-v\|}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \operatorname{diam}(T)$$

$$w(T) = \tfrac{1}{2} w(T-T) = \tfrac{1}{2} E(\sup(\|u-v\|_2 \|X\|)) = \tfrac{1}{2} E(\|X\|) \sup(\|u-v\|)$$
$$= \tfrac{1}{2} \operatorname{diam}(T)\, E(\sqrt{\|X\|_2^2}) \underset{\text{Jensen}}{\leq} \tfrac{1}{2} \sqrt{n} \operatorname{diam}(T) \checkmark$$

(b

(c

$$w(AT) = E(\sup\langle g, At\rangle) = E(\sup(t^T A^T x))$$

$$\begin{cases} \ell_t = t^T A^T x \\ \tilde{\ell}_t = \langle \|A\|_{op} t, z\rangle \end{cases}$$

$x, z = N(0, I)$

$$\Rightarrow E(\ell_t - \ell_s) = E((t-s)^T A^T x x^T A (t-s))$$

$$= (t-s)^T A^T E(xx^T) A(t-s) = (t-s)^T A^T A (t-s)$$

$$E(\tilde{\ell}_t - \tilde{\ell}_s) = \|A\|_{op}^2 \|t-s\|_2^2 \longrightarrow \text{so: } E(\ell_t - \ell_s) \leq E(\tilde{\ell}_t - \tilde{\ell}_s)$$

7.2 version $\hookrightarrow E(\sup_{t\in T}\{X_t\}) \leq E(\sup_{t\in T}\{Y_t\})$

$$\Rightarrow w(AT) = E(\sup(\ell_t)) \leq E(\sup(\tilde{\ell}_t)) = \|A\|_{op} w(T) \checkmark$$

(c

$p > 1$: $B_p^n \in [-1, 1] \Rightarrow E(\sup(t^T x)) \leq E(\max|x_i|)$

$$\max|x_i| \leq \sqrt{r}\cdot\sqrt{\log n} \longrightarrow w(B) \leq \sqrt{r \log n} \checkmark$$

$$\mathcal{L}(t, \lambda) = -t^T x - \lambda(1 - \|t\|_p^p) = -\sum t_i x_i - \lambda(1 - \sum t_i^p)$$

$$\Rightarrow \frac{\partial}{\partial t_i} = 0 \Rightarrow t_i^* = \left(\frac{x_i}{\lambda p}\right)^{\frac{1}{p-1}}$$

$$\frac{\partial}{\partial \lambda} = 0 \Rightarrow 1 - \sum t_i^p = 0 \rightarrow \lambda = \frac{1}{p}\left(\sum_i x_i^{\frac{p}{p-1}}\right)^{\frac{1}{p-1}}$$

$$\rightarrow t_i^* = \left(\frac{x_i}{\lambda p}\right)^{\frac{1}{p-1}}, \lambda = \dots \Rightarrow t^{*T} x = \left(\sum x_i^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

$$\rightarrow w(B) = E\left(\sum x_i^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \leq \sum \left(E(x_i)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \text{ also: } \sqrt{\frac{2}{\pi}}\, 2^{\frac{\frac{p}{p-1}-1}{2}}\, \Gamma\left(\frac{\frac{p}{p-1}+1}{2}\right)$$

Jensen

$$\Gamma(ax+by) \leq \Gamma(x)^{a} \times \Gamma(y)^{b} \quad , \quad \Gamma(x) \leq C\sqrt{2\pi(x-1)}\left(\frac{x-1}{e}\right)^{x-1}$$

$(a+b=1)$

$$E|X_i|^{\frac{p}{p-1}} \leq \sqrt{\frac{2}{\pi}}\, 2^{\frac{\frac{p}{p-1}-1}{2}} \left(C\sqrt{2\pi\left(\frac{p}{p-1}-1\right)} \left(\frac{\frac{p}{p-1}-1}{e}\right)^{\frac{p}{p-1}-1} \right)^{\frac{1}{2}}$$

$$\Rightarrow w(B_p^n) \leq \left(\sum \left(\sqrt{\frac{2}{\pi}}\, 2^{\frac{\frac{p}{p-1}-1}{2}} \left(C\sqrt{2\pi\left(\frac{p}{p-1}-1\right)} \left(\frac{\frac{p}{p-1}-1}{e}\right)^{\frac{p}{p-1}-1} \right)^{\frac{1}{2}} \right) \right)^{\frac{p-1}{p}}$$

$$\leq \frac{2\sqrt{C}}{\sqrt[4]{2\pi}}\, n^{\frac{p-1}{p}} \sqrt{\frac{p}{p-1}}$$

ازطرف دیگر داریم:

$$w(B_p^n) \leq E(\sup t^{T} x) \leq E(\max(|X_i|)) \leq \sqrt{2\log n}$$

$$w(B_p^n) \leq \text{const.} \min\left(n^{\frac{p-1}{p}} \sqrt{\frac{p}{p-1}} \,,\, \log(n) \right) \checkmark$$