HIGH DIMENSION PROBABILITY

HomeWork 2

Radmehr Karimian

Contents

1 Problem #1

1.1 Part A

First for $N^{ext}(k,d,\epsilon) \leq N(k,d,\epsilon)$: For the external set we have bigger set fot center so it's obvious.
For the $N(k,d,\epsilon) \leq N^{ext}(k,d,\frac{\epsilon}{2})$: Consider $a_1,\ldots,a_m$. also consider that : $K \subseteq_{i=1}^{m} B_\epsilon(b_i)$.

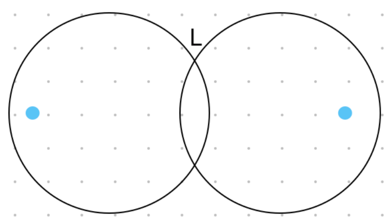
$$b_1,\ldots,b_m \ni b_i \in B_{\frac{\epsilon}{2}}(a_i) \bigcap K$$

$$\forall k \in K \ni \exists j \ni k \in B_{\frac{\epsilon}{2}}(a_j)$$

$$d(k,b_j) \leq d(k,a_j) + d(b_j,a_j) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \rightarrow N(k,d,\epsilon) \leq N^{ext}(k,d,\frac{\epsilon}{2})\square$$

1.2 Part B

We have: For the inequality we have: Consider $\frac{\epsilon}{2}$ points like $a_1,\ldots,a_m$. also consider the



$b_1,\ldots,b_{m-t}$:

$$t \geq 0, b_i \in B_{\frac{\epsilon}{2}}(a_i) \bigcap L \rightarrow d(l,b_j) \leq d(l,a_j) + d(b_j,a_j)\epsilon$$

So Because $t \geq 0$ and also we have bigger space we cleary have
$N(k,d,\epsilon) \leq N^{ext}(k,d,\frac{\epsilon}{2})$ $\square$

1.3 Part C

-I've got this part from Mr.Bagheri- The total number of all points in set is 2n every point in covering with distance m has distance of $1,\ldots,m$ m with the points in covering so we we calculate number of points with distance $1,\ldots,m$ we will have $\sum_{i=1}^{m}\begin{pmatrix} n \\ i \end{pmatrix}$.the N should cover all this points so number of covering set is at least the number of points that can cover points with distance of $1,\ldots,m$ with each other so we have : By applying union

$$total \leq B_m \rightarrow N \times S : \frac{2^n}{S} N(K,d_H,m)$$

$$\frac{2^n}{\sum_{i=1}^{m}\begin{pmatrix} n \\ i \end{pmatrix}} \leq N(K,d_H,m)$$

So we can say that every $\epsilon - packing$ is an $\epsilon - covering$
Otherwise, there exists some x such that $d(x,x_i) \geq \epsilon$,So we will have N(K,d$_H,m) \leq M(K,d_H,m)$.

$ConsiderMballsoineachballthereexist\sum_{i=1}^{[\frac{m}{2}]}$points which by definition are separated from other balls so:

$$M \times pointsineachball \leq allballs$$

$$M \times (\sum_{i=1}^{[\frac{m}{2}]}) \leq 2^n \to M \leq \frac{2^n}{\sum_{i=1}^{[\frac{m}{2}]})}\square$$

1.4 Part D

For this part we use Problem #9 from HW1.

$$P[Z_n \leq \delta n] \geq \frac{1}{n+1}e^{-nD(\alpha,\delta}$$

Lets say $m := \#ofones\&n := \#ofzeros$:

$$P[Z_n \leq \delta n] \geq \binom{n}{m}\alpha^m(1-\alpha)^{n-m}$$

Lets $\delta = \frac{m}{n}$:
we can find that the extrema is when : $\delta = 0.5$, Therefore,

$$\frac{1}{n}P[Z_n \leq \delta n] \geq \frac{1}{n}\log(\binom{n}{m}) + \delta\log(\alpha) + (1-\delta)\log(\frac{1-\alpha}{1-\delta} \to P[Z_n \leq \delta n] \geq \frac{1}{n+1}e^{-nD(\alpha,\delta)}$$

So consider d_H as sum of d Bernoulli random variable:

$$\frac{1}{d+1}e^{-dD(\frac{\delta}{2}||\frac{1}{2})} \leq \frac{1}{M} \to P[Z_n \leq \frac{\delta d}{2}] \leq \frac{1}{M}\blacksquare$$

2 Problem #2

2.1 Part A

We have from Union bound and the fact that $\sum A_k = 1$

$$P(\bigcup_{k=1}^{N} A_k) \leq \sum_{k=1}^{N} A_k \leq \min(1, P(\bigcup_{k=1}^{N} A_k)$$

for other side from help & taylor series we have:

$$P(\bigcup_{k=1}^{N} A_k) = 1 - \prod_{k=1}^{n}(1 - P(A_k)) \geq 1 - e^{\sum_{k=1}^{N} P(A_k)} \geq (1 - \frac{1}{e}) \min(1, P(\sum_{k=1}^{N} A_k)\square$$

2.2 Part B

If you consider $X_t \geq \eta^{*-1}\log|T| + u$, From Part A we have:

$$P(\bigcup_{k=1}^{N} A_k) \geq (1 - \frac{1}{e}) \min(1, P(\bigcup_{k=1}^{N} A_k) = (1 - \frac{1}{e}) \min(1, P(\sum_{k=1}^{|T|} e^{-\log|T|-u}) = (1 - \frac{1}{e})e^{-u}$$

On the other hand

$$P[\sup_{t\in T} X_t \geq \eta^{*-1}(\log|T| + u)] \geq P(\bigcup_{k=1}^{N} A_k)$$

$$P[\sup_{t\in T} X_t \geq \eta^{*-1}(\log|T| + u)] \geq (1 - \frac{1}{e})e^{-u}\square$$

2.3 Part C

For this part we use last part results:

$$P[X_t \geq \frac{\eta^{*-1}(2\log|T|)}{2} + x] \geq e^{-\eta^*(\eta^{*-1}(2\log|T|+x))}$$

$$\eta^*(\eta^{*-1}(2\log|T| + x)) \geq \frac{1}{2}\eta^*(\eta^{*-1}(2\log|T|)) + \frac{1}{2}\eta^*(2x) \geq \log|T| + \frac{1}{2}\eta^*(2x)$$

so if we concern $A_k \geq \frac{\eta^{*-1}(2\log|T|)}{2} + x$ from part B we have:

$$P[\sup_{t\in T} X_t \geq \frac{\eta^{*-1}(2\log|T|)}{2} + x] \geq (1 - \frac{1}{e})e^{\frac{1}{2}\eta^*(2x)}\square$$

2.4 Part D

For the Lower bound we have :

$$E[\sup_{t\in T} X_t] = E[\sup_{t\in T} \min(X_t,0)] + E[\sup_{t\in T} \max(X_t,0)]$$

$$\geq \sup_{t\in T} E[\min(X_t,0)] + \int_0^\infty P[\sup_{t\in T} \max(X_t,0) \geq x]dx$$

$$\geq \sup_{t\in T} E[\min(X_t,0)] + \int_0^{\eta^{*-1}(\log|T|)} P[\sup_{t\in T} \max(X_t,0) \geq x]dx$$

now because we know for $t \leq \eta^{*-1}(\log|T|) \rightarrow P[\sup_{t\in T} \max(X_t,0) \geq \eta^{*-1}(\log|T|)] = 0$

$$\geq \sup_{t\in T} E[\min(X_t,0)] + \int_0^{\eta^{*-1}(\log|T|)} P[\sup_{t\in T} \max(X_t,0) \geq \eta^{*-1}(\log|T|)]dx$$

now from Part B we have:

$$\geq \sup_{t\in T} E[\min(X_t,0)] + \frac{e-1}{e}\eta^{*-1}(\log|T|)\Box$$

For the Upper bound we have:

$$E[\sup_{t\in T} X_t] \leq E[\max(0,\sup_{t\in T} X_t)]$$

$$= \int_0^\infty P(\sup_{t\in T} X_t \geq x)dx = \int_0^{\psi^{*-1}(\log|T|)} P(\sup_{t\in T} X_t \geq x)dx + \int_{\psi^{*-1}(\log|T|)}^\infty P(\sup_{t\in T} X_t \geq x)dx$$

$$\int_0^{\psi^{*-1}(\log|T|)} P(\sup_{t\in T} X_t \geq x)dx \leq \int_0^{\psi^{*-1}(\log|T|)} dx = \psi^{*-1}(\log|T|)$$

on the other hand we know:

$$\int_{\psi^{*-1}(\log|T|)}^\infty P(\sup_{t\in T} X_t \geq x)dx = \int_0^\infty P(\sup_{t\in T} X_t \geq \psi^{*-1}(\log|T|)+x)dx$$

$$\leq \int_0^\infty |T|e^{-\psi^*(-\psi^{*-1}(\log|T|)+x))}dx$$

From all ov above:

$$E[\sup_{t\in T} X_t] \leq \psi^{*-1}(\log|T|) + \int_0^\infty |T|e^{-\psi^*(-\psi^{*-1}(\log|T|)+x))}dx$$

$$= \psi^{*-1}(\log|T|) + \int_0^\infty e^{-\psi^*(-\psi^{*-1}(\log|T|)+x))--\psi^*(-\psi^{*-1}(\log|T|)}dx$$

$$\leq \psi^{*-1}(\log|T|) + \int_0^\infty e^{-\psi^{*'}(-\psi^{*-1}(\log|T|))x}dx$$

$$\leq \psi^{*-1}(\log|T|) + \frac{1}{-\psi^{*'}(-\psi^{*-1}(\log|T|))}$$

We know that $|T| \geq 1$ and ψ is increasing convex function witch mean its derivative is also increasing:

$$E[\sup_{t\in T} X_t] \leq \psi^{*-1}(\log|T|) + \frac{1}{-\psi^{*\prime}(-\psi^{*-1}(0))}\square$$

2.5 Part E

$$P(X \geq x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt = \int_0^\infty e^{\frac{-(t+x)^2}{2}} dt \geq \int_0^\infty e^{\frac{-(t^2+x^2)}{2}} dt = \frac{e^{-x^2}}{2\sqrt{2}}\square$$

2.6 Part F

We know that

$$P(X \geq x) \leq e^{\frac{-x^2}{2}} \rightarrow \eta^*(x) = x^2 + \log(2\sqrt{2}) and \psi^*(x) = \frac{x^2}{2}$$

now we can say:

$$\psi^{*\prime(x)=x}$$

$$\psi^{*-1}(x) = \sqrt{2x}$$

$$\eta^{*-1}(x) = \sqrt{x - \log(2\sqrt{2})}$$

For Upper Bound:

$$E[\max X_i] = \frac{1}{k} E[\max \log(e^{kx_i})] = \frac{1}{k} E[\log \max(e^{kx_i})] \leq E[\log \sum (e^{kx_i})] \leq \log \sum E[(e^{kx_i}))] = \frac{\log(\text{n})}{k} + \frac{k}{2}$$

set $k = \sqrt{2\log(n)}$.
For Lower Bound:

$$E[\max X_i] \geq \frac{e-1}{e}(\sup E[\min(X_i, 0)] + \sqrt{\log n - \log 2\sqrt{2}})$$

on the other hand we have:

$$\sup E[\min(X_i, 0)] = E[\min(X, 0)] = \int_{-\infty}^0 e^{\frac{-x^2}{2}} = -\frac{1}{\sqrt{2\pi}}$$

So we have:

$$E[\max X_i] \geq \frac{e-1}{e}(-\frac{1}{\sqrt{2\pi}} + \sqrt{\log n - \log 2\sqrt{2}})$$

■

3 Probelm #3

3.1 Part A

For the Lower bound we have:

$$L_n \geq \sum_{k=1}^{n} \min |X_k - X_l| \rightarrow E[L_n] \geq \sum_{k=1}^{n} E[\min_{l\neq k} |X_k - X_l|] = \int_0^{\infty} P(\min_{t\neq k} |X_k - X_l| \geq t)dt$$

We simply from Area have:

$$E[\min_{l\neq k} |X_k - X_l|] = \int_0^{\frac{1}{\sqrt{\pi}}} (1 - \pi t^2)^{n-1}$$

Hence we have $E[\min_{l\neq k} |X_k - X_l|] \leq E[\sqrt{\min_{l\neq k} |X_k - X_l|^2}] \leq \sqrt{E[\min^2_{l\neq k} |X_k - X_l|]}$:

$$E[\min_{l\neq k}^{2} |X_k - X_l|] = \int_0^{\infty} P(\min_{t\neq k} |X_k - X_l|^2 \geq t)dt = \int_0^{\infty} P(\min_{t\neq k} |X_k - X_l| \geq \sqrt{t})dt = \int_0^{\frac{1}{\sqrt{\pi}}} (1 - \pi x^2)^{n-1} 2x dx = \mathrm{O}(\frac{1}{n})$$

$$E[L_n] \geq \sum_{k=1}^{n} E[\min_{l\neq k} |X_k - X_l| \geq n\mathrm{O}(\frac{1}{\sqrt{n}}) = \mathrm{O}(\frac{1}{\sqrt{n}})$$

For upper bound(from triangular inequality):

$$L_n \leq L_{n-1} + 2\min |X_n - X_k| \rightarrow E[L_n] \leq 2E[\sum_{k=1}^{n} \min_{l\leq k} |X_k - X_l|]$$

$$E[\min_{l\leq k} |X_k - X_l|] \leq \int_0^{\infty} P(\min_{l\leq k} |X_k - X_l| \geq t)dt \leq \int_0^{\frac{2}{\sqrt{\pi}}} (1 - \frac{\pi t^2}{4})^{k-1} dt$$

$$E[L_n] \leq 2E[\sum_{k=1}^{n} \min_{l\leq k} |X_k - X_l|] \leq 2\sum_{k=1}^{n} \int_0^{\frac{2}{\sqrt{\pi}}} (1 - \frac{\pi t^2}{4})^{k-1} dt = \frac{C}{\sqrt{n}} \leq \frac{C'}{\sqrt{n}}$$

Last part came from mathematical induction. $\square$

3.2 Part B

for this part we know changing a one point in direction would cause at most $2r$different.Therefore from McDiarmid we have:

$$|f(x_1, ..., x_n) - f(x_1, ..., x'_i, ..., x_n)| \leq 2\sqrt{2} \rightarrow subg((2\sqrt{2})^2 \frac{n}{4}) = subg(2n)\square$$

3.3 Part C

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$\gamma \geq \frac{\pi}{2} \rightarrow \cos\gamma \leq 0$

$$c^2 \geq a^2 + b^2\square$$

3.4 Part D

mathematical induction:
For the base we had last part $n = 1$. consider this: both triangles have a point at least. So

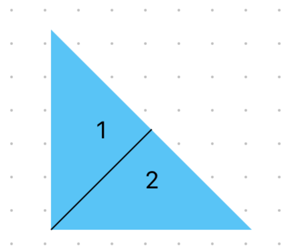


Figure 1: Part D

what we can say is:

$$||w - x_{\sigma_{(1)}}||^2 + \sum_{i=1}^{m-1} + ||x_{\sigma_{(i)}} - x_{\sigma_{(i+1)}}||^2 + ||x_{\sigma_{(m)}}||^2 \leq a^2$$

$$||v - x_{\sigma_{(m+1)}}||^2 + \sum_{i=m+1}^{n} + ||x_{\sigma_{(i)}} - x_{\sigma_{(i+1)}}||^2 + ||x_{\sigma_{(m+1)}}||^2 \leq b^2$$

With summation of two equation and the fact $a^2 + b^2 = ||v - w||^2$ the result had been proved.
a important point is what if all points were in one triangle for that we have(e.g in number 1):

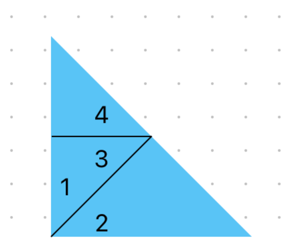


Figure 2: Part D

So we will use triangle 3,4 if that doesn't work we continue this fractal until we have 2 point in 2 different triangles. $\square$

3.5 Part E

By dividing square into two triangles and last part result, we will have:

$$|X_{\sigma_{(1)}} - [1,1]^T|^2 + \sum_{i=1}^{m-1} |X_{\sigma_i} - X_{\sigma_{i+1}}|^2 + |X_{\sigma_{(m)}} - [0,0]^T|^2 \leq 2$$

$$|X_{\sigma_{(m+1)}} - [0,0]^T|^2 + \sum_{i=m+1}^{n} |X_{\sigma_i} - X_{\sigma_{i+1}}|^2 + |X_{\sigma_{(n)}} - [1,1]^T|^2 \leq 2$$

There for we have(angel of the connection):

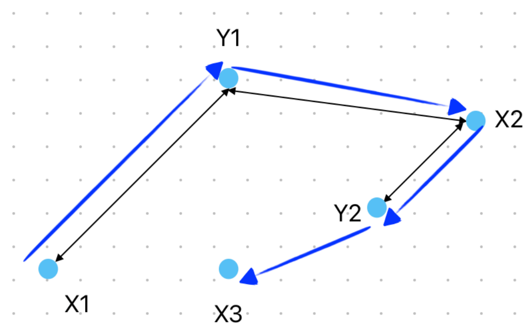
$$|X_{\sigma_1} - X_{\sigma_n}|^2 \leq |X_{\sigma_1} - [1,1]^T|^2 + |X_{\sigma_n} - [1,1]^T|^2$$

so we can say:

$$\sum_{i=1}^{n} |X_{\sigma_{i+1}} - X_{\sigma_i}|^2 \leq 4\square$$

3.6 Part F

AS you can see using hint we have: In the back path we can directly go to Y_i. $\square$



3.7 Part G

We can say if τ is back path of σ we clearly can say: there is a path on $A := X \bigcup Y$:

$$l_{2n}(A) \geq \min_p l_{2n}(A,p)$$

$$\min_p l_{2n}(A,p) \geq \min_\sigma l_n(x,\sigma)$$

So It's obvious that we have:

$$\min_\sigma l_n(x,\sigma) \leq \min_\sigma l_n(y,\sigma) + \sum_{i=1}^{n} 2d_i(x,\sigma_x)\mathbf{1}_{x_i \neq y_i}\square$$

3.8 Part H

We use talagrand inequality:

$$\min_\sigma l_n(x,\sigma) - \min_\sigma l_n(y,\sigma) \leq \sum_{i=1}^{n} 2d_i(x,\sigma_x)\mathbf{1}_{x_i \neq y_i}$$

$$c_i(x) = 2d_i(x,\sigma(x)) \to \sum_{i=1}^{n} c_i^2 \leq \sum_{i=1}^{n} 4d_i(x,\sigma(x))^2 \leq 16$$

so thats $subg(16)$ $\blacksquare$

4 Problem #4

we use the same method in vershynin book:

Step 1: Approximation. Using Corollary 4.2.13, we can find a 1/4-net $\mathcal{N}$ of the unit sphere S^{n-1} with cardinality

$$|\mathcal{N}| \leq 9^n.$$

Using Lemma 4.4.1, we can evaluate the operator norm in (4.22) on $\mathcal{N}$:

$$\left\| \frac{1}{m} A^\top A - I_n \right\| \leq 2 \max_{x \in \mathcal{N}} \left| \left\langle \left(\frac{1}{m} A^\top A - I_n \right) x \right\rangle x \right| = 2 \max_{x \in \mathcal{N}} \left| \frac{1}{m} \|Ax\|_2^2 - 1 \right|.$$

To complete the proof of (4.22) it suffices to show that, with the required probability,

$$\max_{x \in \mathcal{N}} \left| \frac{1}{m} \|Ax\|_2^2 - 1 \right| \leq \frac{\varepsilon}{2} \quad \text{where} \quad \varepsilon := K^2 \max(\delta, \delta^2).$$

Step 2: Concentration. Fix $x \in S^{n-1}$ and express $\|Ax\|_2^2$ as a sum of independent random variables:

$$\|Ax\|_2^2 = \sum_{i=1}^m \langle A_i, x \rangle^2 =: \sum_{i=1}^m X_i^2 \tag{4.23}$$

where the A_i denote the rows of A. By assumption, the A_i are independent, isotropic, and sub-gaussian random vectors with $\|A_i\|_{\psi_2} \leq K$. Thus the $X_i = \langle A_i, x \rangle$ are independent sub-gaussian random variables with $\mathbb{E} X_i^2 = 1$ and $\|X_i\|_{\psi_2} \leq K$. Therefore the $X_i^2 - 1$ are independent, mean-zero, and sub-exponential random variables, with

$$\|X_i^2 - 1\|_{\psi_1} \leq C K^2.$$

(Check this; we did a similar computation in the proof of Theorem 3.1.1.) Thus we can use Bernstein's inequality (Corollary 2.8.3) and obtain

$$\begin{aligned}
\mathbb{P}\left\{ \left| \frac{1}{m} \|Ax\|_2^2 - 1 \right| \geq \frac{\varepsilon}{2} \right\} &= \mathbb{P}\left\{ \left| \frac{1}{m} \sum_{i=1}^m X_i^2 - 1 \right| \geq \frac{\varepsilon}{2} \right\} \\
&\leq 2 \exp\left(-c_1 \min\left(\frac{\varepsilon^2}{K^4}, \frac{\varepsilon}{K^2} \right) m \right) \\
&= 2 \exp\left(-c_1 \delta^2 m \right) \quad \left(\text{since } \frac{\varepsilon}{K^2} = \max(\delta, \delta^2)\right) \\
&\leq 2 \exp\left(-c_1 C^2 (n + t^2) \right).
\end{aligned}$$

The last bound follows from the definition of δ in (4.22) and using the inequality $(a+b)^2 \geq a^2 + b^2$ for $a, b \geq 0$.

Step 3: Union bound. Now we can unfix $x \in \mathcal{N}$ using a union bound. Recalling that $\mathcal{N}$ has cardinality bounded by 9^n, we obtain

$$\mathbb{P}\left\{ \max_{x \in \mathcal{N}} \left| \frac{1}{m} \|Ax\|_2^2 - 1 \right| \geq \frac{\varepsilon}{2} \right\} \leq 9^n \, 2 \exp\left(-c_1 C^2 (n + t^2) \right) \leq 2 \exp(-t^2),$$

if we choose the absolute constant C in (4.22) large enough. As we noted in step 1, this completes the proof of the theorem. ■

5 Problem #5

From Poincaré We have:

$$Var(f(Y)) \leq E[|\nabla f|^2]$$

So we also know that $f(Y) = \max_i \Sigma^{0.5} Y$ we also know that :

$$E[|\nabla f|^2] = E[\sum_{i=0}^{n} (\Sigma^{0.5}_{jmax,i})^2] = E[\sum_{i=0}^{n} \Sigma_{jmax,i}]$$

it's cleat that we can say $E(X) \leq max(X)$:

$$E[|\nabla f|^2] = E[\sum_{i=0}^{n} (\Sigma^{0.5}_{jmax,i})^2] = E[\sum_{i=0}^{n} \Sigma_{jmax,i}] \leq \max_j \sum_{i=0}^{n} \Sigma_{jmax,i} = \max_j Var(X_i)$$

$$\longrightarrow Var(\max_i X_i) \leq \max_j Var(X_i) \blacksquare$$

6 Problem #7

6.1 Part A

Consider $\underline{X} = (X_1, ..., X_n)$ and $\underline{X}^i = (X_1, ..., X_i', ..., X_n)$ So We can say by Efron-Stein inequality we have:

$$\text{Var}(f(\underline{X})) \leq \frac{1}{2}\sum_{i=1}^{n} E[(f(\underline{X}) - f(\underline{X}^i))^2]$$

We can also say that:

$$\frac{1}{2}\sum_{i=1}^{n} E[(f(\underline{X}) - f(\underline{X}^i))^2] = \frac{1}{2}\sum_{i=1}^{n} 2\text{var}_i(f)$$

$$\Longrightarrow \text{Var}(f(\underline{X})) \leq \sum_{i=1}^{n} \text{Var}_i(f)$$

This part is tricky but from class we know(The proof is not very hard you can use $\sup - \inf$ instead of μ in variance definition:

$$\text{Var}_i(f(\underline{X})) \leq \frac{1}{4}(\sup f(\underline{X}) - \inf f(\underline{X}))^2$$

So we have:

$$\text{Var}(f(\underline{X})) \leq \frac{1}{4}E[\sum_{i=1}^{n}((D_i f(\underline{X})^2))]\square$$

6.2 Part B

6.2.1 B-1

from last part we have:

$$\text{Var} f(X) \leq \frac{1}{4}E(\sum_{i=1}^{n} 1^2) = \frac{n}{4}\square$$

6.2.2 B-2

it's obvious that we have more Compartment than load so we can say

$$B_n \geq \sum_{i=1}^{n} X_i \longrightarrow E(B_n) \geq E(\sum_{i=1}^{n} X_i) = \frac{n}{2}\blacksquare$$

7 Problem #8

7.1 Part A

From Jensen inequality we have for any $\lambda > 0$

$$E[\sup_{t\in T} X_t] \leq \frac{1}{\lambda} E[e^{\lambda \sup_{t\in T} X_t}] \leq \frac{1}{\lambda} log \sum_{t\in T} E[e^{\lambda X_t}] \leq \frac{1}{\lambda} log|T| e^{\psi(\lambda)} = \frac{1}{\lambda}(\psi(\lambda) + log|T|)$$

We also know that:

$$\sup(\lambda x - \psi(\lambda)) = log|T|$$

$$(\lambda' x - \psi(\lambda')) = log|T| \rightarrow x = \frac{log|T| + \lambda'}{\lambda'}$$

$$\psi^{*^{-1}}(log|T|) = \frac{log|T| + \lambda'}{\lambda'} \geq \inf \frac{1}{\lambda}(\psi(\lambda) + log|T|)$$

So the proof is complete:

$$E[\sup_{t\in T} X_t] \leq \psi^{*^{-1}}(log|T|)\square$$

7.2 Part B

from Union bound we :

$$P(\sup_{t\in T} X_t \geq x) = P[\bigcup_{t\in T}\{X_t \geq x\}]$$

$$P[X_t \geq x] \leq e^{-\psi^*(x)} \rightarrow \sum_{t\in T} P[X_t \geq x] \leq |T| e^{-\psi^*(x)}$$

But we know that $x = \frac{log|T| + \lambda'}{\lambda'}$, Therefore:

$$P[\sup_{t\in T} X_t \geq \psi^{*-1}(\log|T| + u)] \leq e^{-u}\blacksquare$$

8 Problem #10

8.1 Part A

Consider $Z(\theta) = X_k sin(\theta) + Y_k cos(\theta)$,therefore we have, $Z'(\theta) = X_k cos(\theta) - Y_k sin(\theta)$. it's simple that they are both gaussian.

$$E[Z(\theta)] = E[X_k]sin(\theta) + E[Y_k]cos(\theta) = 0$$

$$E[Z'(\theta)] = E[X_k]cos(\theta) - E[Y_k]sin(\theta) = 0$$

$$E[Z(\theta)Z^T(\theta)] = E[X_k X_k^T]sin^2(\theta) + E[Y_k Y_k^T]cos^2(\theta) = I_n$$

$$E[Z'(\theta)Z'^T(\theta)] = E[X_k X_k^T]cos^2(\theta) + E[Y_k Y_k^T]sin^2(\theta) = I_n$$

$$E[Z(\theta)Z'^T(\theta)] = E[Z'(\theta)Z^T(\theta)] = E[(E[X_k]sin(\theta)+E[Y_k]cos(\theta))(E[X_k^T]cos(\theta)-E[Y_k^T]sin(\theta))] = 0\square$$

8.2 Part B

$$E_x[\phi(f(x) - E(f(x)))] \leq E(\phi(\frac{\pi}{2}\langle((x), y\rangle)))$$

8.3 Part C

8.4 Part D