بنام خدا

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(۵)

$$P(\hat{S} \neq S) \geq 1 - \frac{I(y,S) + \log 2}{\log |S_k|} \Rightarrow \frac{I(y,S) + \log(2)}{\log |S_k|} > \frac{1}{2}$$

$I(y,S) = H(y) - H(y|S)$:

$$H(y|S) = H(X\theta^{S} + \varepsilon | S) = H(\varepsilon | \theta X S) = H(\varepsilon)$$

$$\rightarrow H(\varepsilon) = H(\varepsilon_1, \dots, \varepsilon_n) = \sum_{i=1}^{n} H(\varepsilon_i) = n H(\varepsilon)$$

$$= \frac{n}{2} \log(2\pi\sigma^2 e) \text{ (طبق تعریف)}$$

$\max H(y) \rightarrow y =$ گوسی ،

$$H(y) = n/2 \left(\log(2\pi e) \right) + 1/2 \log\left(\mathrm{cov}(y,y) \right)$$

$$\mathrm{cov}(y,y) = E(yy^T) - E(y)E(y)^T$$

$$\hookrightarrow E(yy^T) = E\left((X\theta + \varepsilon)(X\theta + \varepsilon)^T \right)$$

$$= E\left((X\theta + \varepsilon)(\theta^T X^T + \varepsilon^T) \right)$$

$$= E(X\theta\theta^T X^T) + 2\cancel{E(X\theta\varepsilon^T)} + E(\varepsilon\varepsilon^T) = X E(\theta\theta^T) X^T + \sigma^2 I$$

$$\Rightarrow E(\theta_i \theta_j) \begin{cases} 0 & i \neq j \\ \frac{k}{d}\theta_{min}^2 & i = j \end{cases} \Rightarrow E(yy^T) = k/d\, \theta_{min}^2 X X^T + \sigma^2 I$$

$$\Rightarrow \mathrm{cov}(y,y) = \frac{k}{d} \theta_{min}^2 X X^T + \sigma^2 I \checkmark$$

$$E(y) = E(X\theta + \varepsilon) = E(\theta_{min} X S + \varepsilon) = A\cancel{E(S)} + \cancel{E(\varepsilon)} = 0$$

$$\rightarrow H(y) \leq \frac{n}{2} \log(2\pi e) \det(\mathrm{cov}(y,y)) \checkmark$$

$$\Rightarrow H(y) \leq \frac{n}{2} \log(2\pi e) + 1/2 \det(\mathrm{cov}(y,y)) \leq n/2 \left(\log(2\pi e) + \log\left(\frac{k}{d} \theta_{min}^2 \| n^{-1/2} X \|_F^2 + \sigma^2 \right) \right)$$

$$\rightarrow I(y,S) = H(y) - H(y|S) \xrightarrow{\text{مستقل}} \leq \frac{n}{2} \log\left(1 + k/d \frac{\theta_{min}^2}{\sigma^2} \| n^{-1/2} X \|_F^2 \right) \leq \frac{n}{2} \cdot \frac{k}{d} \cdot \frac{\theta_{min}^2}{\sigma^2} \| n^{-1/2} X \|_F^2$$

$$\to |S_k| = \binom{d}{k} 2^k \to \frac{\frac{n}{2}\cdot\frac{k}{d}\cdot\frac{\theta_{min}^2}{\sigma^2}\|n^{-1/2}X\|_F^2 + \log(2)}{\log\binom{d}{k} + k\log(2)} \geq \frac{1}{2}$$

$$\Rightarrow n \geq \frac{\log\binom{d}{k} + k\log(2)}{\frac{k}{d}\cdot\frac{\theta_{min}^2}{\sigma^2}\|n^{-1/2}X\|_F^2 + \log 2}$$

$$\Rightarrow n \geq \frac{d\cdot\sigma^2\log\binom{d}{k}}{k\cdot\theta_{min}^2\|n^{-1/2}X\|_F^2} \times C$$ ($k \geq 4$ [illegible])

ب)

$$\|n^{-1/2}X\|^2 = \frac{1}{n}\|X\|^2 = \frac{1}{n}\times n\times\sum^{d}|\pm 1| = d$$

$$\Rightarrow n \geq C\cdot\frac{d\times\log\binom{d}{k}}{k\times d}\cdot\frac{\sigma^2}{\theta_{min}^2} \geq 1\times\frac{\log\binom{d}{k}}{k}\times\frac{\sigma^2}{\theta_{min}^2}$$ ✓