In the name of god

Pa1

a)

$$\min P_e = \tfrac{1}{2}\left(1 - d_{TV}(P_n, Q_n)\right) = 0 \rightarrow \lim_{n\to\infty} d_{TV}(P_n, Q_n) = \lim \sup P_n(E) - Q_n(E)$$

$$\rightarrow \exists E \ni P_n(E) = 1 \; , \; Q_n(E) = 0$$

چون در بی‌نهایت هر ۲ برابر صفر نمی‌شوند ← پس جداسازی امکان‌پذیر نمی‌باشد ← خطای نوع اول و نوع دوم صفر نیست!

b)

$$\|L_n\|^2 = E_{X\to Q_n}\left(\frac{dp_n}{dq_n} \cdot \frac{dp_n}{dq_n}\right) = \sum \frac{dp_n}{dQ_n} \times \frac{dp_n}{dQ_n} \times dQ_n = \sum \frac{dp_n}{dQ_n} \times dp_n \; ✓$$

$\rightarrow$

1) $P_n \to 0$, $Q_n \not\to 0$ $\rightarrow$ good!

۲) $q_n \to 0$, $P_n \not\to 0$ $\rightarrow \|L_n\|^2 \to \infty \rightarrow$?! $\Rightarrow P_n \to 0 \Rightarrow$ good! $\rightarrow$ نقض ✓

۳) $\sum \frac{dp_n}{dq_n} = E_{p_n}\left(\frac{dp_n}{dq_n}\right) < \infty \rightarrow \frac{dp_n}{dQ_n} < \infty \Rightarrow Q_n \to 0 \Rightarrow P_n \to 0$

Colab $\Rightarrow$ AA Bagheri

In the name of god

P.2

a) $P(\sum (y_i - x_i) \geq nx) \overset{\forall \theta}{\underset{\theta>}{=}} P(e^{\theta \sum (y_i - x_i)} \geq e^{n\theta x})$

$\leq e^{-nx\theta} E(e^{\theta(Y_1 - X_1)})^n = e^{-n(\theta x - \log E(e^{\theta(Y_1 - X_1)}))}$

$\rightarrow P(\sum (Y_i - X_i)) \leq \inf \{ e^{-n(\theta x - \log(E(e^{\theta Y_1}) E(e^{-\theta X_1})))} \}$

$= e^{-n \sup(\theta x - \log(E(e^{\theta Y_1}) - E(e^{-\theta X_1}))}$

$= e^{-n \sup(\theta x - \psi_q(\theta) - \psi_p(1-\theta))}$

$= e^{-n F(x)}$ □ ✓

b) $F(0) = \inf_{\theta>0} (\log(E_p((P/q)^\theta) E_q((P/q)^\theta))) = \log(\inf_{\theta>0} (E_p((P/q)^{\theta}) E_q((P/q)^\theta)))$

$\rightarrow -E_p(\log \frac{p}{q} (P/q)^{-\theta}) E_q((P/q)^\theta) + E_q(\log P/q (P/q)^\theta) E_p(\log(P/q)^{-\theta}) = 0$

$\rightarrow E_p(\log P/q (P/q)^\theta) \times E_q(P/q^\theta) = E_p((P/q)^{-\theta}) E_q(\log P/q (P/q)^\theta)$

$\rightarrow (\sum \log P/q \, q^\theta p^{1-\theta})(\sum p^\theta 1 - q^\theta) = (\sum p^{1-\theta} q^\theta)(\sum \log P/q \, p^\theta q^{1-\theta})$

$\rightarrow \theta = 1/2$

$\rightarrow F_0(0) = \log(E_p((P/q)^{1/2}))(E_q(P/q)^{1/2}) = \log \sum p^{1/2} q^{1/2} + \log \sum p^{1/2} q^{1/2}$

$= \log \sum_p \sqrt{pq} + \log \sum_q \sqrt{pq} \overset{p \ll q}{=} 2 \log \sum_q \sqrt{pq} \rightarrow F(0) = -2 \log \sum \sqrt{pq} = -2 \log B(p,q)$

$\rightarrow F(0) = -\psi_p(1/2) - \psi_q(1/2) = \alpha$ ✓

c) $F(x) = \sup_{\theta > 0} \left(\theta x - \psi_P(-\theta) - \psi_Q(\theta)\right) \underset{\theta = 1/2}{\geq} F(x) = \frac{1}{2} x - \psi_P(-\frac{1}{2}) - \psi_Q(\frac{1}{2})$

[illegible] $= \frac{1}{2} - \alpha$

$\rightarrow F(x) \geq x/2 + \alpha \rightarrow e^{-nF(x)} \leq e^{-n(x/2 + \alpha)}$

$\rightarrow P(\sum (Y_i - X_i) \geq nx) \leq e^{-n(x/2 + \alpha)}$ ✓

$\xrightarrow{\text{Colab}}$ Ali Bagheri

In the name of god

P.3

a)

استقرا

$$f_n^*(x) = f(\sum x_i) = \frac{\lambda^n x^{n-1} e^{-\lambda n}}{(n-1)!} \rightarrow f_{n+1}^*(x) = f_n^*(x) * f_{x_{n+1}}$$

$$\rightarrow f_{n+1}^*(x) = \int_0^n \frac{\lambda^n t^{n-1} e^{-\lambda n}}{(n-1)!} \lambda e^{-\lambda(x-t)} dt = \frac{\lambda^{n+1} e^{-\lambda x}}{n!} x^n \checkmark$$

پایه استقرا

$$\rightarrow n=2 \rightarrow f_2^*(x) = f_{x_1} * f_{x_2} = \int_0^x \lambda e^{-\lambda t} \lambda e^{-\lambda(x-t)} dt$$

$$= \lambda^2 x e^{-\lambda x} = \frac{\lambda^2 x^{2-1} e^{-\lambda x}}{(2-1)!} \checkmark \rightarrow \text{تمام است}$$

b)

$$P(\sum x_i \geq n\xi) \xrightarrow{\text{چرنف}} P(\sum x_i \geq n\xi) \leq \frac{E(e^{t\sum x_i})}{e^{n\xi t}}$$

$$\rightarrow E(e^{t\sum x_i}) = (1-t)^{-n} \rightarrow P(\sum x_i \geq n\xi) \leq \underbrace{(1-t)^{-n} e^{-n\xi t}}_{m(t)}$$

$$\frac{\partial m(t)}{\partial t} = -n\xi(1-t)^{-n} e^{-n\xi t} + n e^{-n\xi t}(1-t)^{-n-1} = 0 \rightarrow t = \frac{\xi-1}{\xi} \checkmark$$

$$\rightarrow m^* = e^{-n(\xi - \log\xi - 1)} \checkmark$$

c) $P(\sum x_i \leqslant n\xi) \leqslant \underbrace{e^{nt\xi} E(e^{-t\sum x_i})}_{M(t)}$

$M(t) \leqslant e^{-n \sup(-t\xi + \underbrace{\log(1+t)}_{N(t)})} \rightarrow \sup N(t) \rightarrow t^{*} = \frac{1}{\xi} - 1$

$\rightarrow P(\sum x_i \leqslant n\xi) \leqslant e^{-n(\xi - 1 - \log\xi)}$ ✓

In the name of god

P.4

a)

$$\frac{1}{2}\left(P(\hat{H}=1)+Q(\hat{H}=0)\right)=\frac{1}{2}\left(\sum z(1|x)p(x)+\sum z(0|x)q(x)\right)$$

$$=\frac{1}{2}\sum \min(p(x),q(x))$$

$$\frac{1}{2}\left(P(\hat{H}=1)+Q(\hat{H}=0)\right)\leq \Pi\sum \min\left(\frac{p(x)}{2},\frac{q(x)}{2}\right)\leq \Pi\sum p(x)^{\lambda}q(x)^{1-\lambda}$$

$$=\left(\sum p(x)^{\lambda}q(x)^{1-\lambda}\right)\leq \inf_{\lambda}\left(\sum\left(\frac{p(x)}{q(x)}\right)^{\lambda}q(1-x)\right)^{n}$$

$$-\frac{1}{n}\log P_e\geq -\frac{1}{n}\inf_{x} n\log\left(\sum \frac{p(x)}{q(x)}^{\lambda}q(x)\right.=-\inf\left\{\log E_Q\left(\frac{p(x)}{q(x)}\right)^{\lambda}\right.$$

$$\geq \max\left\{-\log E_Q\left(\frac{p(x)}{q(x)}\right)^{\lambda}\right\} \checkmark$$

b)

$$P(\hat{H}=1)=P\left(\sum \log \tfrac{p}{q}\geq n\gamma\right)\leq \exp(-n\varkappa_P^{*}(\gamma))$$

$$P(\hat{H}=0)=Q\left(\sum \log \tfrac{p}{q}\geq n\gamma\right)\leq \exp(-n\varkappa_Q^{*}(\gamma))$$

$$\varkappa_Q^{*}(\gamma)=\sup \lambda\gamma-\varkappa_Q(\lambda)$$

$$\varkappa_P^{*}(\gamma)=\sup \lambda\gamma-\varkappa_P(\lambda)$$ $\Rightarrow$ داریم: $\varkappa_Q^{*}(\gamma)-\varkappa_P^{*}(\gamma)=\gamma \rightarrow$ فرض [illegible] ی نسبت‌ها [illegible]

$(\varkappa_Q(\lambda)=\varkappa_P(\lambda+1))$ و

$$\rightarrow P(\hat{H}=1)=\exp(-nE_0)\leq \exp(-n\varkappa_P^{*}(\gamma))\xrightarrow{n\to\infty} E_0=\varkappa_P^{*}(\gamma)=\varkappa_Q^{*}(\gamma)-\gamma$$

$$\rightarrow P(\hat{H}=0)=\exp(-nE_1)\leq \exp(-n\varkappa_Q^{*}(\gamma))\xrightarrow{n\to\infty} E_1=\varkappa_Q^{*}(\gamma)$$

$$\rightarrow -\frac{1}{2}\lim \frac{1}{n}\log\left(P(\hat{H}=1)+q(\hat{H}=0)\right)\leq -\lim \frac{1}{n}\log\left\{\sup_{\gamma} e^{-n\varkappa_P^{*}(\gamma)}\right\}$$

→ [illegible] = [illegible] ✓

P.5

b) $D_f(P_x/Q_x) = \int f\left(\frac{dp(x)}{dQ(x)}\right) dQ_x = \int f\left(\frac{p(x)}{q(x)}\right) q(x)\,dx$

$\rightarrow D_f(P_{g(x)}, Q_{g(x)}) = \int f\left(\frac{p(g(x))}{q(g(x))}\right) q(g(x))\,dg(x)$

$\rightarrow g(x) = x' \qquad = \int f\left(\frac{p(x')}{q(x')}\right) q(x')\,dx'$

$\rightarrow$ بنابراین $: D_f(P_x/Q_x) = D_f(P_{g(x)}/Q_{g(x)})$ ✓

c) $d_{TV}(P_0 \otimes Q, P_1 \otimes Q) = \frac{1}{2}\int |dp_0 dq - dp_1 dq| = \frac{1}{2}\int |dp_0 - dp_1| dq$

$= \frac{1}{2}\int |dp_0 - dp_1| = d_{TV}(P_0, P_1)$ ✓

a) قسمت ۱

قسمت ۱ [illegible]

$d_{TV}(\mu, \nu) \leq p(X \neq Y)$

$\mu(E) - \nu(E) = p(X \in E, y \in E) + p(X \in E, y \notin E)$

$- (p(X \notin E), y \in E)) + p(X \in E, y \in E))$

$\rightarrow p(X \in E, y \notin E) - p(y \in E, x \notin E) \leq p(x \in E, y \notin E)$

$d_{TV}(\mu, \nu) = \sup_E(\mu(E) - \nu(E)) \leq \inf_{x \in E, Y \notin E} p(X \neq Y)$

$p(X \neq Y) = 1 - p(X = Y) = 1 - p(X_1 = X_0)\, p(Y_1 = Y_0) \rightarrow p(X_1 = Y_1)(1 - p(X_2 = Y_2))$

$\leq 1 - p(X_2, Y_2$

$$\rightarrow -P(X_1=Y_1)\,P(X_2=Y_2) \leq 1-\big(P(X_1=Y_1)+P(X_2=Y_2)\big)$$

$$\rightarrow 1-P(X_1=Y_1)\,P(X_2=Y_2) \leq 1-P(X_1=Y_1)+1-P(X_2=Y_2)$$

$$\leq P(X_1\neq Y_1)+P(X_2\neq Y_2) \rightarrow d_{TV}(\mu_1\mu_2,\nu_1\nu_2)$$

$$\leq P(X_1\neq Y_1)+P(X_2\neq Y_2)$$

$$\leq d_{TV}(\mu_1,\nu_1)+d_{TV}(\mu_2,\nu_2)$$

→ ثابت شد

Calab Ali Bagheri

d)

In the name of god

P.6

a)

قانون : $p(z|x) = \begin{cases} 1 & p/q \geq \pi_1/\pi_0 \\ 0 & p/q < \pi_0/\pi_1 \end{cases}$

داریم :

$$P_e = \pi_0 \int p(x)\,dx + \pi_1 \int q(x)\,dx \geq \int \min\{\pi_0 p(x), \pi_1 q(x)\}\,dx$$

$\tau = \frac{\pi_1}{\pi_0}$ ثابت :

$$\int \min\{\pi_0 p(x), \pi_1 q(x)\}\,dx = \int_{\pi_0 p(x) < \pi_1 q(x)} \min\{\pi_0 p(x), \pi_1 q(x)\}\,dx$$

$$+ \int_{\pi_1 q(x) < \pi_0 p(x)} \min\{\pi_0 p(x), \pi_1 q(x)\}\,dx = \int \pi_0 p(x)\,dx + \int \pi_1 q(x)\,dx = P_e$$

$$\rightarrow \tau = \pi_1/\pi_0$$

b)

$$\alpha - \gamma\beta = c \rightarrow \alpha - \frac{\pi_1}{\pi_0}\beta = c \rightarrow \frac{\partial \beta}{\partial \alpha} = \frac{\partial \alpha^2}{\partial} = 2\alpha$$

$$\rightarrow 1 - 2\alpha\frac{\pi_1}{\pi_0} = 0 \Rightarrow 2\alpha\frac{\pi_1}{\pi_0} = 1 \Rightarrow \alpha = \frac{\pi_0}{2\pi_1} \rightarrow \beta = \left(\frac{\pi_0}{2\pi_1}\right)^2$$

c)

شرایط : $0 < \alpha < 1 \rightarrow 0 < \frac{\pi_0}{\pi_1} < 2 \checkmark \rightarrow$ $\pi_0 + \pi_1 = 1 \Rightarrow \pi_0 \leq 2(1-\pi_0)$

$\Rightarrow \pi_0 \leq 2/3 \rightarrow \pi_1 \geq 1/3 \checkmark$

In the name of god

P7 (from text book \ & other stanford lecture notes)

a)

$\pi_{0|1} \leq 2^{-n E_1}$

فرض شد $\min(E_0, E_1)$, $\max(E_0, E_1)$

$\to E_2 \leq \lambda E_1 + (1-\lambda) E_2 \leq E_1 \to -n E_1 \leq -n(\lambda E_1 + (1-\lambda) E_2) \leq -n E_2$

$\pi_{0|0} \leq 2^{-n_0 E_2}$

$\to \pi_{0|1} \leq 2^{-n(\lambda E_1 + (1-\lambda) E_2)} \to$ تحدب ✓

به همین شکل باید نشان دهیم که $\pi_{1|0}$ نیز محدب است با شرط $\lambda' = 1-\lambda$, $(\lambda = 1-\lambda')$

b)

فرض: $E_1^*(E_0) \triangleq \sup \{ E_1 : \exists n_0, \forall n \geq n_0, \exists P_{Z|X^n} \ \alpha > 1 - 2^{-n_0 E_0}, \ \beta < 2^{-n_0 E_1} \}$

$= \lim_{n \to \infty} \inf \frac{1}{n} \log \frac{1}{\beta_{1-2^{-n_0 E_0}}(P^n, Q^n)}$

(*)

$T_k \triangleq \log \frac{dQ}{dP}(X_k) \to \log \frac{dQ^n}{dP^n}(X_n) = \sum_{i=1}^{n} T_i$

$\to \phi_P(\lambda) = \log(E_P(e^{\lambda T})), \ \phi_P^* = \sup \theta\lambda - \phi_P(\lambda)$

$\to T = -n\theta \to NP:$

$\pi_{1|0} = P(\sum T_k \geq n\theta) \leq 2^{-n \phi_P^*(\theta)}$ $\quad \theta \geq E_P(\log \frac{Q}{P}) = -D(P\|Q)$

$\pi_{0|1} = Q(\sum T_k < n\theta) \leq 2^{-n \phi_Q^*(\theta)}$ $\quad \theta \leq E_Q(\log \frac{P}{Q}) - D(Q\|P)$

c)

$\to \phi_Q(\lambda) = \log E_Q(e^{\lambda T}) = \log E_P(e^{(1+\lambda)T}) = \phi_P(\lambda + 1)$ $\longrightarrow$ [illegible]

$\to \phi_Q^*(\theta) = \sup(\theta\lambda) - \phi_P(\lambda+1) = \phi_P^*(\theta) - \theta$

d)

Converse:

$$\pi_{1|0} + \gamma\,\pi_{0|1} \geq P\left(\sum_{k=1}^{n} T_k \geq -\log\gamma\right)$$

$$\to \gamma = 2^{-n\theta} \to \pi_{1|0} + 2^{-n\theta}\pi_{0|1} \geq P\left(\sum T_k \geq n\theta\right)$$

اگر

$$\to \pi_{1|0} = 2^{-nE_0},\ \pi_{0|1} = 2^{-nE_1}:$$

$$2^{-nE_0} + 2^{-n\theta} \times 2^{-nE_1} \geq 2^{(n\phi_p^* + o(n))}$$

$$\to \min(E_0, E_1, \theta) \leq \phi_p^*(\theta) \to E_0 \leq \phi_p^*(0)\ ,\ E_1 \leq \phi_p^* - \theta$$

* تفاوت تعمیم دادن این است که در سوال و این عبارت کتاب p و q برعکس هستند اما جواب p کاملا تبدیل است!

e)

$$P_e^*(n) = \min_{P(Z|X^n)} \{\pi_0 \pi_{1|0} + \pi_1 \pi_{0|1}\} = \max_{\theta} \pi_0 e^{-nE_0(\theta)} + \pi_1 e^{-nE_1(\theta)}$$

$$\to E \triangleq \lim_{n\to\infty} \frac{1}{n}\log\frac{1}{P_e^*(n)} \to E = \max_{\theta} \min(E_0(\theta), E_1(\theta)) = \phi_p^*(0)$$

$\underline{\underline{\theta \geq 0}}$

$$\min(E_0(\theta), E_1(\theta)) \begin{cases} E_0 & \theta \geq 0 \\ E_1 & \theta < 0 \end{cases} \to E^* = \max_{\theta \geq 0} \{\underbrace{\phi_p^*(\theta)}_{E_0}\} = \max \sup \lambda\theta - \phi_0(\lambda)$$

$$\to \lambda = 0 \quad = \phi_p^*(0)$$

$\underline{\theta < 0}$

$$E^* = \max\{\underbrace{\phi_p^*(0) - \theta}_{E_1}\} = \max \sup(\lambda - 1)\theta - \phi_p(\lambda) \Rightarrow$$ مثل قبل →

$$= \phi_p^*(0) \implies$$ حکم ثابت شد ✓