

ج:  $A = \{p, q\}$ ,  $\begin{cases} p' = (p(A), 1 - p(A)) \text{ ber} \\ q' = (q(A), 1 - q(A)) \text{ ber} \end{cases}$

لنظروا  $q(A) < p(A)$   $\rightarrow$   
 $\frac{p(A)}{q(A)} > 1$   $\rightarrow$   $\frac{1-p(A)}{1-q(A)} < 1$   $\textcircled{+}$

$D_{KL}(P, Q) \geq D_{KL}(P', Q')$   $\leftarrow$  طبق المتباينة

$\hookrightarrow I(P, Q) \geq p(A) \log \frac{p(A)}{q(A)} + (1-p(A)) \log \frac{1-p(A)}{1-q(A)}$

$\geq D_{KL}(P(A) \parallel YTV(P, Q))$

$\inf_{x \in (0, 1-TV)} D_{KL}(x, YTV) = D_{KL}(x(YTV), YTV) \quad TV \in [0, 1]$

$\rightarrow D_{KL}(x, YTV) \geq D_{KL}(x_0, YTV) + D'_{KL}(x_0, YTV)(x - x_0)$

$\frac{\partial D_{KL}}{\partial x} \Big|_{x=x_0}$

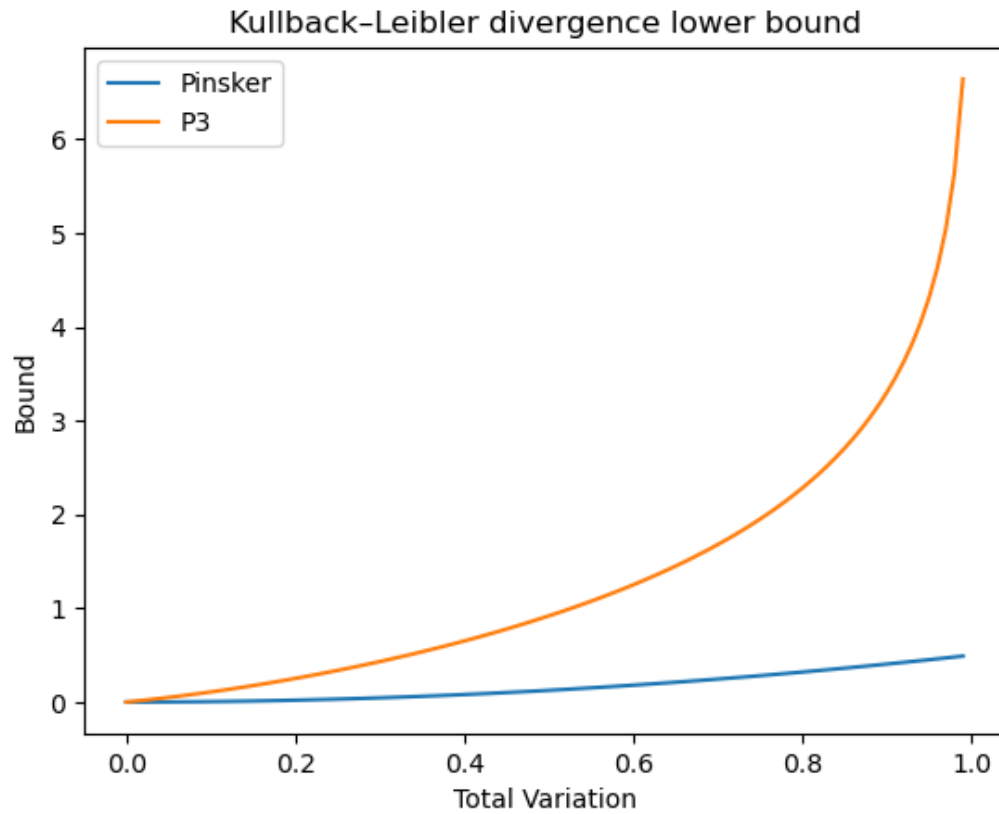
$\hookrightarrow D'_{KL}(x(YTV), YTV) = 0 \Rightarrow D'_{KL}(x_0, YTV) \geq 0 \quad \text{لأن } x_0 \geq x(YTV)$

$\Rightarrow D_{KL}(x(YTV), YTV) \geq D_{KL}(x_0, YTV) - D'_{KL}(x_0, YTV)x_0$

لذا:  $x_0 = \frac{1-TV}{1-TV}$   $\Rightarrow D'_{KL} = \frac{1-TV}{1-TV} - x_0 \log \frac{x_0 + YTV}{x_0 - YTV} \geq 0$

$\rightarrow D_{KL}(x_0, YTV) = D'_{KL}(x_0, YTV)x_0 = \log \frac{x_0 + YTV}{x_0 - YTV} - \frac{YTV}{x_0 + YTV}$   
 $= \log \frac{1+TV}{1-TV} - \frac{YTV}{1+TV}$

النتيجة:  $D_{KL}(P \parallel Q) \geq \log \frac{1+TV}{1-TV} - \frac{YTV}{1+TV} \checkmark$



```
import numpy as np
import matplotlib.pyplot as plt
TV = np.arange(0, 1, 0.01)
y = TV*TV
y = y/2
z = np.log2(1+TV) - np.log2(1-TV) - (2*TV)/(1+TV)
plt.title("Kullback-Leibler divergence lower bound")
plt.ylabel('Bound')
plt.xlabel("Total Variation")
plt.plot(TV,y)
plt.plot(TV,z)
plt.legend(['Pinsker', 'P3'])
```