بسم الله

تمرین ۲ / ITSL / رامین ابراهیمیان ۹۸۱۰۳۵۵۴

۱)

الف/

$$\chi^2(P_n \| Q) = E_Q\left(\frac{P_n^2}{Q^2}\right) - 1 = E_Q\left(\frac{1}{Q^2}\int p_\theta \, \pi(d\theta) \int p_{\theta'} \, \pi(d\theta')\right) - 1$$

$$= \int E_Q\left(\frac{p_\theta p_{\theta'}}{Q^2}\right) \pi(d\theta)\, \pi(d\theta') - 1$$

$$= E_{\theta,\theta'}\left(G(\theta,\theta')\right) - 1 \checkmark$$

ب/

$$D_{KL}(P_n \| Q_n) < \chi^2(P \| Q) \Rightarrow D_{KL}(P_n \| Q_n) < O(1) \rightarrow \int p_n \log \frac{p_n}{q_n} < O(1)$$

$$\forall q_n = 0 \Rightarrow p_n = 0 \Rightarrow P \ll Q \checkmark$$

۲)

$$P_\sigma(A_{ij}) = \begin{cases} \sigma_i \sigma_j = 1 : P(A_{ij}) \\ \sigma_i \sigma_j = -1 : q(A_{ij}) \end{cases} \rightarrow P_\sigma(A) = \prod_{\sigma_i\sigma_j = 1} p(A_{ij}) \prod_{\sigma_i\sigma_j = -1} q(A_{ij})$$

$$= \prod\left(\frac{1+\sigma_i\sigma_j}{2} p(A_{ij}) + \frac{1-\sigma_i\sigma_j}{2} p(A_{ij})\right)$$

$$= \prod\left(\tfrac{1}{2}\left(p(A_{ij}) + q(A_{ij}) + \sigma_i\sigma_j\left(p(A_{ij}) - q(A_{ij})\right)\right)\right)$$

$$= \prod_{i<j} \tfrac{1}{2}\left(p(A_{ij}) + q(A_{ij})\right)$$

$$\rightarrow \frac{P_\sigma(A)\, P_{\tilde\sigma}(A)}{P_0} = \prod_{i<j} \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2}} \times \left(p(A_{ij}) + q(A_{ij}) - \sigma_i\sigma_j\left(p(A_{ij}) - q(A_{ij})\right)\right) \times \left(p(A_{ij}) + q(A_{ij}) - \tilde\sigma_i\tilde\sigma_j\left(p(A_{ij}) - q(A_{ij})\right)\right)$$

سادگی $$= \prod_{i<j} \tfrac{1}{2}\beta \rightarrow \frac{P_\sigma(A)\, P_{\tilde\sigma}}{P_0} = \prod_{ij} \tfrac{1}{2}\beta \Rightarrow \int \tfrac{1}{2}\beta = \tfrac{1}{2}\int p(A) + p(A) + \sigma_i\sigma_j\tilde\sigma_i\tilde\sigma_j \int p(A) - q(A)$$

$$+ \sigma_i\sigma_j\tilde\sigma_i\tilde\sigma_j \int \frac{(p(A) - q(A))^2}{p(A) - q(A)} = \tfrac{1}{2}\left(2 + 0 + 2\rho\, \sigma_i\sigma_j\tilde\sigma_i\tilde\sigma_j\right) \leq e^{\rho\sigma_i\sigma_j\tilde\sigma_i\tilde\sigma_j} \rightarrow P_\sigma < \prod e^{\rho\sigma_i\sigma_j\tilde\sigma_i\tilde\sigma_j}$$

$$e^{\rho\langle\sigma,\tilde\sigma\rangle^2} \checkmark$$

b)

$$P = \frac{1}{2}\sum \frac{(p-q)^2}{(p+q)^2} = \frac{1}{2}\sum \frac{(p-q)^2}{(p+q)^2} + \frac{((1-p)-(1-q))^2}{(1-p+1-q)^2} =$$

$$\frac{1}{2}\,\frac{(a/n - b/n)^2}{\left(\frac{a+b}{n}\right)} + \frac{1}{2}\,\frac{(b/n - a/n)^2}{2 - \frac{a+b}{n}} = \frac{(a-b)^2}{2(a+b)}\cdot\frac{1}{n} + \frac{(b-a)^2}{(2n-a-b)2n}$$

$$= \frac{I}{n} + \frac{O(1)}{n} = \frac{I+O(1)}{n}$$

(a

a) $\hat{\theta} = \arg\min \sum |y_i - \theta_i|^2 + \lambda 1(\theta_i \neq 0)$

$\Rightarrow \hat{\theta}_i = \arg\min |y_i - \theta_i|^2 + \lambda 1(\theta_i \neq 0)$

$\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_p \end{pmatrix} \in \mathbb{R}^p$ $\qquad \theta_i = 0 \Rightarrow y_i^2$

$\theta_i \neq 0 = y_i \Rightarrow \lambda$ $\qquad \Rightarrow y_i^2 = \lambda \Rightarrow |y_i| < \lambda$.

$\Rightarrow \theta_i = y_i 1(|y_i| > \sqrt{\lambda}) \Rightarrow \vec{\theta} = \vec{y}\, 1(|y| \geq \sqrt{\lambda}) \Rightarrow \tau \geq \sqrt{\lambda}$

(b

$$\hat{\theta} = \underset{\theta \in \mathbb{R}^p}{\arg\min} \sum_{i=1}^{p} |y_i - \theta_i|^2 + \lambda |\theta_i|$$

$$\Rightarrow \begin{cases} \theta_i < 0 \\ \theta_i > 0 \end{cases} \Rightarrow \partial/\partial\theta_i := \begin{cases} -2(y_i - \theta_i) - \lambda = 0 \\ -2(y_i - \theta_i) + \lambda = 0 \end{cases} \Rightarrow \begin{cases} \theta_i = y_i + \lambda/2 \\ \theta_i = y_i - \lambda/2 \end{cases}$$

$\Rightarrow \theta_i = 0 \quad \Rightarrow -2(y_i - \theta_i) \in [-\lambda, \lambda] \Rightarrow \theta_i = 0$

$$\rightarrow \hat{\theta}_i^{ST} = \begin{cases} y_i - \lambda/2 & y_i \geq \lambda/2 \\ 0 & |y_i| < \lambda/2 \\ y_i + \lambda/2 & y_i \leq -\lambda/2 \end{cases}$$

(c چون $\|\theta\|_0$ تعداد عناصر غیرصفر θ می‌باشد بنابراین اگر $\|y\|_F \leq \tau$ برقرار است دیگر این مورد ...

if $\|y\|_F > \tau \rightarrow \tau$ را برابر $|y_i|$ ها قرار داد θ برداری که ... برمی‌گرداند ... جواب است $\rightarrow [y_i, y_{i+1}]$ هر مقداری بین ... جواب است

(d اگر $\tau \geq \|y\|$ باشد بردار صفر کمترین مقدار ممکن را دارد ... $\|\theta - y\|$ شده $\hat{\theta}_{ST}$ که ... کند

اگر $\|y\|_F \leq \tau \Rightarrow \vec{\theta} = 0$

و $\|y\|_F > \tau \Rightarrow \theta_i = \min\{|y_i + \tau|, |y_i - \tau|\}$

پاسخ:

۹)

الف/

$$\hat{\theta}(x) := \begin{cases} \hat{\theta}_1(x) = x_1 \\ \hat{\theta}_{/1}(x) = 0 \end{cases} \rightarrow E_\theta(\|\theta - \hat{\theta}\|_2^2) = E_\theta(\underbrace{(x_1 - \theta_1)^2}_{N(0,1)} + \|\theta_{/1}\|^2)$$

$$= 1 + 2^2 = 5$$

$$\inf_{\hat{\theta}} \sup_{\theta \in \Theta} E_\theta(\|\hat{\theta} - \theta\|_2^2) \leq 1 \checkmark$$

ب/

۷)

a)

پاسخ:

$$\begin{cases} H_0 := \theta = 0 \\ H_1 := \theta \sim \mathrm{uni}(\{\tau \vec{e}_1, \dots, \tau \vec{e}_p\}) \end{cases} \rightarrow$$ $e_1, \dots, e_p$ are vector in $\mathbb{R}^n$ and $\tau > 0$, sepreation condition is $\Delta = \tau^2/2$.

$$\chi^2(P_0 \| P_1) = E(e^{\langle\theta^T \theta'\rangle} - 1) = \frac{\exp(\tau^2) - 1}{p}$$

$$\xrightarrow{H_1} \theta' \text{ iid of } \theta: \quad \underset{\varepsilon \in [0,1]}{\tau = \sqrt{(1-\varepsilon)\log p}} \rightarrow \chi^2(P_1 \| P_0) \rightarrow 0$$

$$\Rightarrow TV(\|P_1 - P_0\|) \leq \sqrt{\frac{D_{KL}(P_1\|P_0)}{2}} \leq \sqrt{\frac{\chi^2(P_1\|P_0)}{2}} \Rightarrow p \rightarrow \infty \Rightarrow TV(P_1 - P_0) = 0$$

$$\rightarrow \inf_{T} \sup_{\theta} E_\theta((T - \theta_{max})^2) \leq \frac{(1-\varepsilon)\log p}{2} = C \log p$$

b)

$$T = \max x_i \rightarrow E_\theta((T - \theta_{max})^2) \leq E_\theta(\|x - \theta\|_\infty^2) = E(\|z\|^2) \leq (2 + O_p(1)) \log p \checkmark$$

a)

$$L(f(x), Y) = 1\{f(x)Y < 0\}$$

$$\phi(f(x), y) = \phi(f(x)y) \Rightarrow R_\phi(f) = E(\phi(f(x)y))$$

$$= \int P(Y=1, X=x)\, \phi(f(x)) + P(y=-1, X=x)\, \phi(-f(x))\, dx^n$$

$$= E_x\Big(\underbrace{\eta(x)\, \phi(f(x)) + (1-\eta(x))\, \phi(-f(x))}_{\ell_\phi(f(x), \eta(x))}\Big) \checkmark$$

b) $B_{\phi,\pi} = R^*_\phi$; $R_{\phi^*} = B_{\phi,\pi}(P_1, P_{-1}) = \inf_\gamma \int \pi\phi(\gamma(x))\, p_1(x) + (1-\pi)\phi(-\gamma(x))\, p_{-1}(x)$

$P_{\pm 1} = P(X \mid Y = \pm 1)$ $\quad = \int \inf_\alpha (\phi(\alpha)\, P_1(x) + \phi(-\alpha)\, P_{-1}(x)\,(1-\pi))\, dx$

$B_{\phi,\pi} = B_{\phi,\pi}(P_{1,-1})$ $\quad = \int (\ell^*_\phi(\pi) - \ell^*_\phi(\eta(x)))\, p(x)\, dx$

$$= \int \sup_\alpha \left(\ell^*_\phi(\pi) - \phi(\alpha) \frac{P_1(x)}{p(x)} \pi - \phi(-\alpha) \frac{P_{-1}(x)}{p_x}(1-\pi) \right) p(x)\, dx$$

$$= \int \sup_\alpha \left(\ell^*_\phi(\pi) - \frac{\phi(\alpha) \frac{P_1(x)}{P_{-1}(x)} \pi + \phi(-\alpha)(1-\pi)}{1-\pi+\pi \frac{P_1(x)}{P_{-1}(x)}} \right) \left(\frac{P_1(x)}{P_{-1}(x)} \pi + 1 - \pi \right) P_{-1}(x)\, dx)$$

$$\rightarrow f(a) = \sup_\alpha \left(\ell^*_\phi(\pi) - \frac{\pi\phi(\alpha) t + (1-\pi)\phi(-\alpha)}{\pi t + (1-\pi)} \right)(a\pi + 1 - \pi)$$

$$\longrightarrow R^*_{f,\phi} - R^*_\phi = D_f(P_1 \,\|\, P_{-1}) \checkmark$$

(C

$$\ell_\phi^d(\pi) = \inf_a \pi\phi(a) + (1-\pi)\phi(-a)$$

$$= \inf_a \pi \log(1+e^{-a}) + (1-\pi)\log(1+e^{a})$$

$$\rightarrow \frac{-\pi e^{-\alpha}}{1+e^{-\alpha}} + \frac{(1-\pi)\, e^{\alpha}}{1+e^{\alpha}} = 0$$

$$\hookrightarrow e^{\alpha} = \pi/_{1-\pi} \rightarrow \ell_\phi^d(\pi) = \pi\log\left(1+\frac{1-\pi}{\pi}\right) + (1-\pi)\log\left(1+\frac{\pi}{1-\pi}\right)$$

$$= H(Y)$$

$$\frac{-\pi a e^{-\alpha}}{1+e^{-\alpha}} + \frac{(1-\pi)e^{\alpha}}{1+e^{\alpha}} = 0 \rightarrow e^{\alpha} = \frac{\pi a}{1-\pi}$$

$$\hookrightarrow f(a) = \left(\frac{\ell_\phi^d(\pi) - \pi a\log(\pi a) - (1-\pi)\log(1-\pi)}{\pi a + 1-\pi} - \log\frac{1}{\pi a + (1-\pi)}\right)$$

$$\times(\pi a + 1-\pi)$$

$$H(Y|X=x) = p(Y=1|x)\log\frac{1}{p(Y=1|x)} + p(Y=-1|x)\log\frac{1}{p(Y=-1|x)}$$

$$\downarrow$$

$$\frac{\pi p_1(x)}{\pi p_1(x) + p_{-1}(x)(1-\pi)}$$

$$\Rightarrow D_f(P_1, u_{P_{-1}}) = \int \left(\ell_\phi^d(\pi) - H(Y|X=x)\right) p(x)dx$$

$$= \int \left(H(Y) - H(Y|X=x)\right) p(x)dx$$

$$= I(X,Y)$$ ✓