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The Maximum Deng Entropy

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ABSTRACT Deng entropy has been proposed to measure the uncertainty degree of basic probability assignment in evidence theory. In this paper, the condition of the maximum of Deng entropy is discussed. According to the proposed theorem of the maximum Deng entropy, we obtain the analytic solution of the maximum Deng entropy, which yields that the most information volume of Deng entropy is bigger than that of the previous belief entropy functions. Some numerical examples are used to illustrate the basic probability assignment with the maximum Deng entropy.

INDEX TERMS Entropy, Deng entropy, Shannon entropy, maximum entropy, Dempster-Shafer evidence theory, maximum belief entropy.

I. INTRODUCTION

How to measure the uncertainty has attracted much attention [1]–[4]. A lot of theories have been developed, such as probability theory [5], fuzzy set theory [6], Dempster-Shafer evidence theory [7], [8], rough sets [9], generalized evidence theory [10] and D numbers [11]–[15].

Since firstly proposed by Clausius in 1865 for thermodynamics [16], various types of entropies are presented, such as information entropy [17], Tsallis entropy [18], and nonadditive entropy [19]. Information entropy [17], derived from the Boltzmann-Gibbs (BG) entropy [20] in thermodynamics and statistical mechanics, has been an indicator to measures uncertainty which is associated with the probability density function (PDF). Entropy function can be seen as an uncertainty measure and is applied to real engineering [21], [22].

In the classical entropy theory, assume a system described by variable x has N different states, i.e. $x_i, i = 1, \dots N$, the classical maximum entropy of the system is $log|N|$. This indicates the variable x is uniformly distributed, and the system is the most uncertain. The most critical indication is that the different states are not only mutually EXCLUSIVE but also of SINGLE element. Unfortunately, if the states of the system are uncertain, the maximum entropy may be larger than classical entropy [23]. As the Ref. [23] describes, if the states of x extend to $\{x_1\}, \{x_2\}, \dots, \{x_N\}, \{x_1, x_2\}, \{x_1, x_3\}, \dots, \{x_1, x_2, \dots, x_N\}$, and $\emptyset$, the most uncertainty of the systems becomes N.

However, the idea of Ref. [23] still has some limitations because it assume the states x are mutually EXCLUSIVE, e.g. for the state $\{x_1, x_2\}$, the logic relation of x_1, x_2 is OR. This means the states can finally only happen one of them, either x_1, or x_2. What if the logic relation of x_1, x_2 is AND? Just as in the quantum world, the microcosmic particles can be entangled and simultaneously existing [24], [25].

Recently, a new entropy, Deng entropy is proposed to measure the uncertainty degree of basic probability assignment [26], which is possible to deal with the simultaneous events in the probable field [27], e.g. for the state $\{x_1, x_2\}$, the logic relation of x_1, x_2 is including AND.

Some properties of Deng entropy has been discussed in Ref. [28]. In Ref. [28] Prof. Abellán denotes some shortcomings of Deng entropy, especially the unsatisfying additivity law and subadditivity law for Deng entropy. In Ref. [29], Prof. Abellán also notes: "*The additivity and subadditivity properties guaranty that the total information is preserved. In the first property, it states that we do not add information in situations where a decomposition of the problem can be done, i.e., that decomposition should not imply an increase in information. The second one states that the total information*

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obtained from two independent sources is preserved; when we join two independent problems, the total information is preserved''.

In Ref. [19], Prof. Tsallis emphasized the concept of nonadditive entropy. He presented that ''*If the system is constituted by N equal elements which are not strictly independent, but quasi-independent instead (i.e., not too strongly correlated, in some nonlocal sense to be further clarified later on; typically for a Hamiltonian many-body system whose elements interact through short-range interactions, or which are weakly quantum entangled), the additivity of entropy guarantees its extensivity in the thermodynamical sense*''

If, on the contrary, the correlations between the N elements are strong enough (a feature which might typically occur for nonergodic states, e.g., in Hamiltonian many body systems with long-range interactions, or which are strongly quantum entangled), then the extensivity of entropy might be lost (at least at the level of a large subsystem of a much larger system), being therefore incompatible with classical thermodynamics [19].

Hence, Tsallis entropy is a type of nonadditive entropy under certain conditions, for it consider the influence of the correlations of the states in a system. If the correlations are weak or none, Tsallis entropy degenerates into Shannon entropy, additivity law is satisfied. If the correlations are strong, additivity law is unsatisfied.

Similarly, Deng entropy also considered the interactions of the states, which has discussed above. It can be easily proofed if the interactions are weak or none, Deng entropy degenerates into Shannon entropy, additivity law is satisfied. In addition, if the interactions are strong, additivity law of Deng entropy is unsatisfied. Deng entropy has been used in applications [30]–[33].

The maximum entropy principle is widely used in many applications [34]–[36]. However, the maximum belief entropy has not been presented. In this paper, the maximum Deng entropy will be proposed.

The paper is organized as follows. The preliminaries briefly introduce some concepts about Dempster-Shafer evidence theory, Deng entropy in Section 2. In Section 3, the maximum belief entropy is proposed, the analytic solution of the belief entropy is discussed, and the belief assignment with max belief entropy under certain focal elements is discussed. Finally, this paper is concluded in Section 4.

II. PRELIMINARIES

In this section, some preliminaries are briefly introduced.

A. DEMPSTER-SHAFER EVIDENCE THEORY

Dempster-Shafer theory (short for D-S theory) is presented by Dempster [7] and Shafer [8]. This theory is widely applied to uncertainty modeling [37]–[39], decision making [40]–[43], fault diagnosis [44], [45], reliability analysis [46]–[50], target recognition [51], [52], conflicting management [53]–[58] and information fusion [59]–[61]. D-S theory has many advantages to handle uncertain information [62]–[64]. First, D-S theory can handle more uncertainty in real world. In contrast to the probability theory in which probability masses can be only assigned to singleton subsets, in D-S theory the belief can be assigned to both singletons and compound sets. Second, in D-S theory, prior distribution is not needed before the combination of information from individual information sources. Third, D-S theory allows one to specify a degree of ignorance in some situations instead of being forced to be assigned for probabilities. Some basic concepts in D-S theory are introduced.

Let X be a set of mutually exclusive and collectively exhaustive events, indicated by

$$X = \{\theta_1, \theta_2, \cdots, \theta_i, \cdots, \theta_{|X|}\} \tag{1}$$

where set X is called a frame of discernment (FOD). The power set of X is indicated by 2^X, namely

$$2^X = \{\emptyset, \{\theta_1\}, \cdots, \{\theta_{|X|}\}, \{\theta_1, \theta_2\}, \cdots, \{\theta_1, \theta_2, \cdots, \theta_i\}, \cdots, X\}$$

For a frame of discernment $X = \{\theta_1, \theta_2, \cdots, \theta_{|X|}\}$, a mass function, or basic probability assignment (BPA), is a mapping m from 2^X to $[0, 1]$, formally defined by:

$$m: \quad 2^X \rightarrow [0, 1] \tag{2}$$

which satisfies the following condition:

$$m(\emptyset) = 0 \quad and \quad \sum_{A \in 2^X} m(A) = 1 \tag{3}$$

where A is a focal element if $m(A)$ is not 0 [7], [8]. Recently, some operations on BPA are presented such as negation [65] and correlation [66].

There are plenty of methods to study Dempster-Shafer theory to expand its applications [67]–[69]. However, entropy plays an essential role in uncertainty. The paper mainly discusses the property of entropy.

The follow part reviews the main ideas of dealing with the uncertainty that a BPA. Yager [70] makes the distinction between two types of uncertainty called discord (or randomness or conflict) and non-specificity. The first one has been related to entropy and the second one to imprecision. One is associated with cases where the information is focused on sets with empty intersections and the other one is associated with cases where the information is focused on sets with cardinality greater than one [71], [72].

Assume that X is FOD, A and B are focal elements of the mass function, and $|A|$ denotes the cardinality of A. Then, definitions of some main uncertainties measures of BPA in DST framework are briefly introduced as follows. In this paper, we mainly conclude the most uncertainty and its conditions of these measures of BPA.

B. HOHLE'S CONFUSION MEASURE

Hohle's confusion measure is one of earlier confusion measures for D-S theory was due to Hohle [73].

$$C_H(m) = -\sum_{A \subseteq X} m(A) \log_2 Bel(A) \tag{4}$$

The maximum uncertainty of $C_H(m)$ is $log_2(|X|)$ such that $m(\{\theta_i\}) = \frac{1}{|X|}, \forall\theta_i \in X$. The constraints mean the beliefs are only assigned to the single focal sets, and uniformly distributed [73].

C. YAGER'S DISSONANCE MEASURE

Dissonance measure of BPA was defined by Yager as follow [70]:

$$E_Y(m) = -\sum_{A\subseteq X} m(A)\log_2 Pl(A) \tag{5}$$

The maximum uncertainty of $E_Y(m)$ is $log_2(|X|)$ such that $A_i \cap A_j = \emptyset$, and $m(A) = \frac{1}{K}, \forall A \subseteq X$. The constraints mean that the focal elements are disjoint and uniformly distributed. As in Ref. [70], Yager denotes the maximal number of disjoint subsets of X consist of the $|X|$ disjoint sets of singletons and has a value of $log_2|X|$ when the belief mass is equally divided.

D. HIGASHI & KLIR'S WEIGHTED HARTLEY ENTROPY

Higashi & Klir's entropy is shown as follow [74]:

$$E_{DP}(m) = \sum_{A\subseteq X} m(A)\log_2|A| \tag{6}$$

The maximum uncertainty of E_{DP} is $log_2(|X|)$ such that $m(X) = 1$, and $m(A) = 0, \forall A \subset X$ [74]. The constraints mean that Higashi & Klir's entropy obtains its maximal value when the belief are all assigned to the full set X where the unknown information is the most.

E. KLIR & RAMER'S DISCORD MEASURE

Klir & Ramer's discord measure is denoted as follow [75]:

$$S_{KP}(m) = -\sum_{A\subseteq X} m(A)\log_2\sum_{B\subseteq X} m(B)\frac{|A\cap B|}{|B|} \tag{7}$$

The maximum uncertainty of $S_{KP}(m)$ is $log_2(|X|)$ such that $m(\{\theta_i\}) = \frac{1}{|X|}, \forall\theta_i \in X$. The constraints mean the beliefs are only assigned to the single focal sets, and uniformly distributed [75].

F. KLIR & PARVIZ"S STRIFE MEASURE

Another strife measure of BPA was defined by Klir and Ramer, as follow [76]:

$$D_{KR}(m) = -\sum_{A\subseteq X} m(A)\log_2\sum_{B\subseteq X} m(B)\frac{|A\cap B|}{|A|} \tag{8}$$

The maximum uncertainty of $S_{KP}(m)$ is $log_2(|X|)$ such that $m(\{\theta_i\}) = \frac{1}{|X|}, \forall\theta_i \in X$. The constraints mean the beliefs are only assigned to the single focal sets, and uniformly distributed [76].

G. GEORGE & PAL'S CONFLICT MEASURE

The total conflict measure prospered by George & Pal,denoted as H_{GP}, is defined as follow [77]:

$$H_{GP}(m) = \sum_{A\subseteq X} m(A)\sum_{B\subseteq X} m(B)\left(1-\frac{|A\cap B|}{|A\cup B|}\right) \tag{9}$$

The maximum uncertainty of $S_{KP}(m)$ is $log_2(|X|)$ such that $m(\{\theta_i\}) = \frac{1}{|X|}, \forall\theta_i \in X$. The constraints mean the beliefs are only assigned to the single focal sets, and uniformly distributed [77].

H. WANG & SONG'S INTERVAL MEASURE

Wang & Song's interval measure denoted by SU_{WS} is defined as follows [37],

$$SU_{WS}(m) = \sum_{i=1}^{n}\left[-\frac{Bel(\theta_i)+Pl(\theta_i)}{2}\log_2\frac{Bel(\theta_i)+Pl(\theta_i)}{2} + \frac{Pl(\theta_i)-Bel(\theta_i)}{2}\right] \tag{10}$$

The maximum uncertainty of SU_{WS} is $|X|$ such that $Bel(\theta_i) = 0, Pl(\theta_i) = 1, \forall\theta_i \in X$. The constraints mean the beliefs are only assigned to the single focal sets with the most unknown information [37].

I. DENG ENTROPY

With the range of uncertainty mentioned above, Deng entropy [26] can be presented as follows

$$E_d = -\sum_i m(F_i)\log\frac{m(F_i)}{2^{|F_i|}-1} \tag{11}$$

where, F_i is a proposition in mass function m, and $|F_i|$ is the cardinality of F_i. As shown in the above definition, Deng entropy, formally, is similar with the classical Shannon entropy, but the belief for each proposition F_i is divided by a term $(2^{|F_i|}-1)$ which represents the potential number of states in F_i (of course, the empty set is not included) [78].

Specially, Deng entropy can definitely degenerate to the Shannon entropy if the belief is only assigned to single elements. Namely,

$$E_d = -\sum_i m(\theta_i)\log\frac{m(\theta_i)}{2^{|\theta_i|}-1} = -\sum_i m(\theta_i)\log m(\theta_i)$$

In Dempster-Shafer theory, there are many methods to measure the uncertainty [79]. Deng entropy has attracted a lot of attention [80]–[83] because its measure of total non-specificity and discord. Hence, analysing the maximum Deng entropy is essential, which can get Deng entropy to more applications.

In this section, we summarize the main different uncertainty measures. The maximal values and their conditions of the different uncertainty measures are concluded in Table 1. From the Table 1, most of the previous uncertainty measures support the maximal value of BPA as $log_2(|X|)$ from two types of uncertainty called discord (or randomness or conflict) and non-specificity. Then Deng and Deng [23] think if the states of the system are uncertain, the maximum entropy may be larger than classical entropy. As the Ref. [23] describes, if the states of x extend to $\{x_1\}$, $\{x_2\}, \ldots, \{x_N\}, \{x_1, x_2\}, \{x_1, x_3\}, \ldots, \{x_1, x_2, \ldots, x_N\}$, and $\emptyset$, the most uncertainty of the systems becomes $|X|$. The work of Wang and Song [37] support the idea of Deng and Deng [23].

TABLE 1. Uncertainty measures (UM) of BPA.

Item	Uncertainty expression	Maximum value	Conditions	Remarks
Hohle's UM [74] (Confusion)	$C_H(m) = -\sum\limits_{A\subseteq X} m(A)\log_2 Bel(A)$	$log_2(\|X\|)$	$m(\{\theta_i\}) = \frac{1}{\|X\|}$, $\forall\theta_i \in X$	single focal sets, and uniformly distributed belief.
Yager's UM [71] (Dissonance)	$E_Y(m) = -\sum\limits_{A\subseteq X} m(A)\log_2 Pl(A)$	$log_2(\|X\|)$	$A_i \cap A_j = \emptyset$, AND $m(A) = \frac{1}{K}$, $\forall A \subseteq X$	Focal sets are disjoint, and belief mass is equally divided.
Higashi & Klir's UM [75] (Non-specificity)	$E_{DP}(m) = \sum\limits_{A\subseteq X} m(A)\log_2 \|A\|$	$log_2(\|X\|)$	$m(X) = 1$, AND $m(A) = 0, \forall A \subset X$	Unknown is the most.
Klir & Ramer's UM [76] (Discord)	$S_{KP}(m) = -\sum\limits_{A\subseteq X} m(A)\log_2 \sum\limits_{B\subseteq X} m(B)\frac{\|A\cap B\|}{\|B\|}$	$log_2(\|X\|)$	$m(\{\theta_i\}) = \frac{1}{\|X\|}$, $\forall\theta_i \in X$	single focal sets, and uniformly distributed belief.
Klir & Parviz's UM [77] (Strife)	$D_{KR}(m) = -\sum\limits_{A\subseteq X} m(A)\log_2 \sum\limits_{B\subseteq X} m(B)\frac{\|A\cap B\|}{\|A\|}$	$log_2(\|X\|)$	$m(\{\theta_i\}) = \frac{1}{\|X\|}$, $\forall\theta_i \in X$	single focal sets, and uniformly distributed belief.
George & Pal's UM [78] (Conflict)	$H_{GP}(m) = \sum\limits_{A\subseteq X} m(A)\sum\limits_{B\subseteq X} m(B)\left(1 - \frac{\|A\cap B\|}{\|A\cup B\|}\right)$	$log_2(\|X\|)$	$m(\{\theta_i\}) = \frac{1}{\|X\|}$ $\forall\theta_i \in X$	single focal sets, and uniformly distributed belief.
Wang & Song's UM [79]	$SU(m) = \sum\limits_{i=1}^{n}\left[-\frac{Bel(\theta_i)+Pl(\theta_i)}{2}\log_2\frac{Bel(\theta_i)+Pl(\theta_i)}{2} + \frac{Pl(\theta_i)-Bel(\theta_i)}{2}\right]$	$\|X\|$	$Bel(\theta_i) = 0$, $Pl(\theta_i) = 1$ $\forall\theta_i \in X$	single focal sets with the most uncertainty
Deng's UM [26]	$E_D = -\sum\limits_i m(F_i)\log\frac{m(F_i)}{2^{\|F_i\|}-1}$	$\log_2\sum\limits_i\left(2^{\|F_i\|}-1\right)$	$m(F_i) = \frac{2^{\|F_i\|}-1}{\sum_i 2^{\|F_i\|}-1}$	proportional to scal of sub-focal sets

The changes happen where the non-specificities are different. We have briefly give the explanations of the changes for non-specificity in the part of introduction. In the following section, we focus the maximum Deng entropy and its conditions.

In section III, the condition of the maximum belief entropy is discussed. A simple example with two-scale frame of discernment is used to illustrate the computation of the maximum belief entropy in this section.

III. THE MAXIMUM DENG ENTROPY

Assume F_i is the focal element and $m(F_i)$ is the basic probability assignment for F_i, then the maximum Deng entropy for a belief function happens when the basic probability assignment satisfy the condition $m(F_i) = \frac{2^{|F_i|}-1}{\sum_i 2^{|F_i|}-1}$, where $i = 1, 2, \ldots, 2^X - 1$, and X is the scale of the frame of discernment.

Theorem 1 (The maximum Deng entropy): The maximum Deng entropy: $E_d = -\sum_i m(F_i)\log\frac{m(F_i)}{2^{|F_i|}-1}$ *if and only if* $m(F_i) = \frac{2^{|F_i|}-1}{\sum_i 2^{|F_i|}-1}$

Proof: Let

$$D = -\sum_i m(F_i)\log\frac{m(F_i)}{2^{|F_i|}-1} \tag{12}$$

$$\sum_i m(F_i) = 1 \tag{13}$$

Then the Lagrange function can be defined as

$$D_0 = -\sum_i m(F_i)\log\frac{m(F_i)}{2^{|F_i|}-1} + \lambda\left(\sum_i m(F_i) - 1\right) \tag{14}$$

Now we can calculate the gradient,

$$\frac{\partial D_0}{\partial m(F_i)} = -\log\frac{m(F_i)}{2^{|F_i|}-1} - m(F_i)\frac{1}{\frac{m(F_i)}{2^{|F_i|}-1}\ln a}\cdot\frac{1}{2^{|F_i|}-1} + \lambda = 0 \tag{15}$$

Then Eq. (15) can be simplified as

$$-\log\frac{m(F_i)}{2^{|F_i|}-1} - \frac{1}{\ln a} + \lambda = 0 \tag{16}$$

From Eq. (16), we can get

$$\frac{m(F_1)}{2^{|F_1|}-1} = \frac{m(F_2)}{2^{|F_2|}-1} = \cdots = \frac{m(F_n)}{2^{|F_n|}-1} \tag{17}$$

Let

$$\frac{m(F_1)}{2^{|F_1|}-1} = \frac{m(F_2)}{2^{|F_2|}-1} = \cdots = \frac{m(F_n)}{2^{|F_n|}-1} = k \tag{18}$$

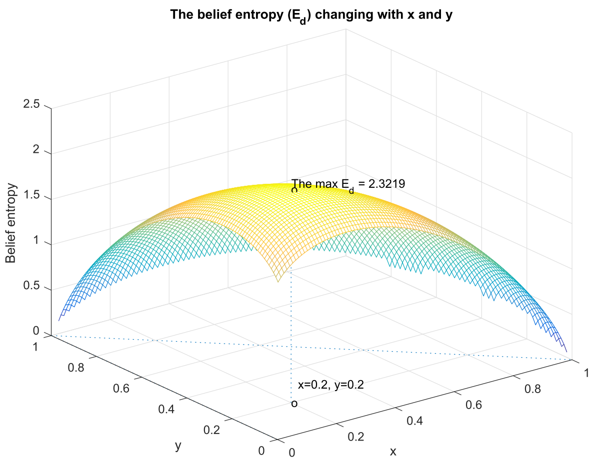

FIGURE 1. **Deng entropy when frame of discernment is $\{a, b\}$.**

Then

$$m(F_i) = k\left(2^{|F_i|} - 1\right) \tag{19}$$

According to Eq. (13), we can get

$$k = \frac{1}{\sum_i 2^{|F_i|} - 1} \tag{20}$$

According to Eq. (18), we can get

$$m(F_i) = \frac{2^{|F_i|} - 1}{\sum_i 2^{|F_i|} - 1} \tag{21}$$

Hence, the maximum belief entropy

$$E_d = -\sum_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|} - 1}$$

if and only if

$$m(F_i) = \frac{2^{|F_i|} - 1}{\sum_i 2^{|F_i|} - 1}$$

□

Let $FOD = X = \{a, b\}$, $m(a) = x$, $m(b) = y$, $m(a, b) = 1 - x - y$, which is under the condition of $0 \le x \le 1$, $0 \le y \le 1$, $0 \le 1 - x - y \le 1$, then relation between Deng entropy and (x, y) can be denoted as the Figure 1.

A. ANALYTIC SOLUTION OF THE MAXIMUM DENG ENTROPY

Assume F_i is the focal element and $m(F_i)$ is the basic probability assignment for F_i, where $i = 1, 2, \ldots, 2^X - 1$, and X is the scale of the frame of discernment. Then the analytic solution of the maximum Deng entropy for a belief function is denoted as

Lemma 1: (The analytic solution of the maximum Deng entropy)

$$D_{\max} = \log \sum_i \left(2^{|F_i|} - 1\right) \tag{22}$$

Proof: According to Theorem 1, the Deng entropy obtains its maximum value when the basic probability assignment satisfy the condition $m(F_i) = \frac{2^{|F_i|}-1}{\sum_i 2^{|F_i|}-1}$, where $i = 1, 2, \ldots, 2^X - 1$, and X is the scale of the frame of discernment. Hence,

$$D_{\max} = -\sum_i m(F_i) \log \frac{m(F_i)}{2^{|F_i|} - 1} \tag{23}$$

$$= -\sum_i \frac{2^{|F_i|} - 1}{\sum_i \left(2^{|F_i|} - 1\right)} \log \frac{\frac{2^{|F_i|}-1}{\sum_i \left(2^{|F_i|}-1\right)}}{2^{|F_i|} - 1} \tag{24}$$

$$= -\sum_i \frac{2^{|F_i|} - 1}{\sum_i \left(2^{|F_i|} - 1\right)} \log \frac{1}{\sum_i \left(2^{|F_i|} - 1\right)} \tag{25}$$

Because $\sum_i \left(2^{|F_i|} - 1\right)$ is constant for a sure frame of discernment, then

$$-\sum_i \frac{2^{|F_i|} - 1}{\sum_i \left(2^{|F_i|} - 1\right)} \log \frac{1}{\sum_i \left(2^{|F_i|} - 1\right)} \tag{26}$$

$$= -\frac{1}{\sum_i \left(2^{|F_i|} - 1\right)} \cdot \sum_i \left(2^{|F_i|} - 1\right) \cdot \log \frac{1}{\sum_i \left(2^{|F_i|} - 1\right)} \tag{27}$$

$$= -\log \frac{1}{\sum_i \left(2^{|F_i|} - 1\right)} = \log \sum_i \left(2^{|F_i|} - 1\right) \tag{28}$$

Hence,

$$D_{\max} = \log \sum_i \left(2^{|F_i|} - 1\right) \tag{29}$$

□

Example 1: Assume $FOD = X = \{a, b\}$, then the power set of X is $F = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, the most possible state amount of F is denoted by

$$\sum_i \left(2^{|F_i|} - 1\right) = 0 + 1 + 1 + 3 = 5 \tag{30}$$

where the number 3 for $\{a, b\}$ refers to the state of $\{a\}$, $\{b\}$, and the part of entanglement of $\{a\}$ and $\{b\}$.

According to the Theorem 1, the belief distribution of the BPA (m_1) with the maximum Deng entropy in $FOD = X = \{a, b\}$ is

$$m_1(\emptyset) = 0$$
$$m_1(\{a\}) = 1/5$$
$$m_1(\{b\}) = 1/5$$

TABLE 2. The maximum deng entropy.

Scale of FOD	the max E_d	$\sum_i \left(2^{\lvert F_i \rvert} - 1\right)$	$\log \sum_i \left(2^{\lvert F_i \rvert} - 1\right)$
1	1	1	1
2	2.3219	5	2.3219
3	4.2479	19	4.2479
4	6.0224	65	6.0224
5	7.7211	211	7.7211
6	9.3772	665	9.3772
7	11.0077	2059	11.0077
8	12.6223	6305	12.6223
9	14.2266	19171	14.2266
10	15.8244	58025	15.8244

[a] the logarithm is based on 2.

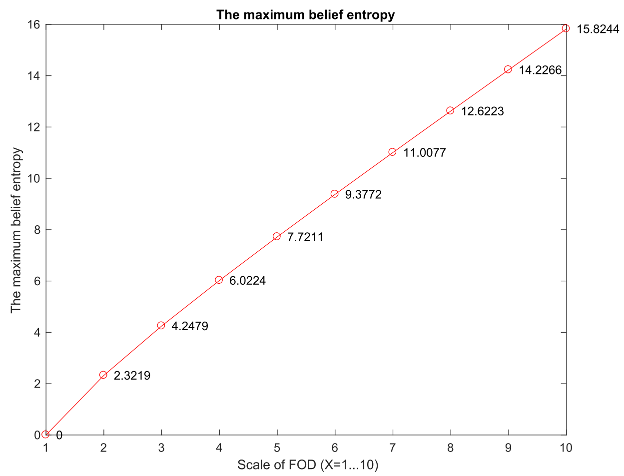

FIGURE 2. The maximum deng entropy changing with different scales of FOD.

$$m_1(\{a, b\}) = 3/5$$

According to the Theorem 1 or Lemma III-A, the value of the maximum Deng entropy of m_1 is

$$\max E_d(m_1) = \begin{cases} \log_2 5 = 2.3219 \\ -\frac{1}{5}\log_2\frac{1}{5} - \frac{1}{5}\log_2\frac{1}{5} - \frac{3}{5}\log_2\frac{3/5}{3} = 2.3219 \end{cases}$$

The maximum Deng entropy changing with different scale of FODs can be shown in Table 2 and Figure 2.

B. THE MAXIMUM DENG ENTROPY UNDER CERTAIN FOCAL ELEMENTS

Example 2: Assume FOD $= X = \{a, b\}$, then the power set of X is $F = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, $m_2(\{a\}) = 0$, which means there are only two focal elements $\{b\}, \{a, b\}$, the most possible state amount of F is denoted by

$$\sum_i \left(2^{|F_i|} - 1\right) = 0 + 1 + 3 = 4 \tag{31}$$

where the number 3 for $\{a, b\}$ refers to the state of $\{a\}$, $\{b\}$, and the part of entanglement of $\{a\}$ and $\{b\}$.

According to the Theorem 1, the belief distribution of the BPA (m_2) with the maximum Deng entropy in FOD $= X = \{a, b\}$ is

$$m_2(\emptyset) = 0$$
$$m_2(\{a\}) = 0$$
$$m_2(\{b\}) = 1/4$$
$$m_2(\{a, b\}) = 3/4$$

According to the Theorem 1 or Lemma III-A, the value of the maximum Deng entropy of m_1 is

$$\max E_d(m_2) = \begin{cases} \log_2 4 = 2 \\ -\frac{1}{4}\log_2\frac{1}{4} - \frac{3}{4}\log_2\frac{3/4}{3} = 2 \end{cases}$$

IV. CONCLUSION AND FUTURE WORK

In this paper, we proposed the maximum Deng entropy and got the analytic solution of the maximum Deng entropy. Some numerical examples is used to illustrate the basic probability assignment and the analytic solution of the maximum Deng entropy. In the future work, we plan to try the maximum Deng entropy in the applications of fractals and quantum physics, e.g. self-similarity of the complex systems, and measurement of quantum entanglement, etc.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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