The Generalized Maximum Belief Entropy Model

A. Siran Li[1, a)] and B. Rui Cai[2, b)]

[1)] *Business College of Southwest University.*

[2)] *Business College of Southwest University*

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In evidence theory, the generalized belief entropy model unifies Renyi entropy, Tsallis entropy, and Deng entropy. In order to further unify the maximum values of Renyi entropy, Tsallis entropy, and Deng entropy, this paper proposes a maximum model of generalized belief entropy by analyzing the generalized belief entropy model, this model shows that the size of the maximum generalized belief entropy is not related to the specific mass value, but is related to the size of each propositional space, and the maximum values of R-D entropy and T-D entropy are obtained through this model. In addition, the applicability of the proposed model is obtained through verification tests and sensitivity analysis of the model.

In this paper, we will discuss the conditions when the generalized belief entropy reaches a maximum value, and give the maximum generalized belief entropy based on the results of the discussion. Through the work of this paper, the maximum Shannon entropy, the maximum Renyi-Deng entropy, and the maximum Tsallis-Deng entropy can be unified.

I. INTRODUCTION

Entropy can be a measure of uncertainty, and many studies have focused on the essence of entropy (Ref. 1). When the classical thermodynamic entropy is evolving (Ref. 2 and 3), the concept of entropy is also widely used in various disciplines (Ref. 4–10) and industries (Ref. 11–13). In the field of information theory, Shannon proposed information entropy (Ref. 14), classical information entropy is often used to compare with the newly proposed entropy (Ref. 15 and 16), Ref. 17 followed up on the aforementioned Ref. 15 and made some notes.

For further processing of uncertain information, Dempster and Shafar proposed the Dempster-Shafer evidence theory (Ref. 18). Based on Dempster-Shafer evidence theory, scholars such as Mehran Khalaj extended the measure of cross entropy to uncertain situations, and thus defined a new measure of cross entropy between two belief sets (Ref. 19) and Anjaria Kusha presented a framework to generate negation(Ref. 20). In evidence theory, the main factor describing uncertainty is BPA (basic probability assignment function), one important operation for BPA in evidence theory is the combination of evidence and the corresponding rule is also called the evidence combination rule. Evidence combination rules, as an important part of D-S evidence theory, are also constantly developing. Many researchers have improved the evidence combination rules (Ref. 21–23). And there are also many scholars who apply the combination of evidence to technical practice (Ref. 24–26). Evidence theory itself has a wide range of applications such as information fusion (Ref. 27–31), decision making (Ref. 19, 32–34), prediction (Ref. 35) and pattern recognition (Ref. 36 and 37), while developing (Ref. 38), it also maintains a good integration with other subjects (Ref. 39 and 40). There are many theories developed based on evidence theory, the generalized evidence theory (Ref. 41), which extended the carrier that defining the discernment framework to the open world, was proposed. It better solves the problems of classical evidence theory in the case of incomplete discernment framework.

Both evidence theory and information entropy can be used to process information. How to apply classic information entropy to evidence theory has been the work of some scholars. In this process, many entropies has been proposed (Ref. 42–51), Renyi proposed the Renyi entropy (Ref. 52), which is widely used in statistics (Ref. 53). As a kind of extension of Boltzmann entropy, Tsallis entropy (Ref. 54) is a non-scalable entropy that also has a wide range of applications (Ref. 55). Recently, Deng entropy (Ref. 56) was proposed as a new kind of entropy used to measure the uncertainty of BPA, Deng entropy better solves the problem of uncertainty measurement of multi-element sets. Based on Deng entropy, the Deng extropy(a dual measure of uncertainty, Ref. 57) and the Generalized Ordered Propositions Fusion is presented (Ref. 58). The former defined the Deng extropy and studied its relation with Deng entropy.The latter obtains the generalized ordered proposition by extending the basic support function on the power set of the ordered proposition, and proposes a fusion method of the generalized ordered proposition. Some scholars have proposed an improved belief entropy based on Shannon entropy and Deng entropy (Ref. 59), the improved belief entropy fully considers the relationship between the subsets, which makes it have a better effect in the measurement of uncertainty. Different with Deng entropy, a new belief entropy is proposed in (Ref. 60), which consists of two parts, the first part is the Shannon entropy with plausibility transform, the second part is Dubois-Prade's definition of entropy of basic probability assignments in the DS theory. What is important is that it also satisfies the six expected properties of the entropy of the DS belief function theory. We often discuss entropy based on the principle of maximum entropy, the principle of maximum retains all the uncertainty and minimizes the unknown. Applying the principle of maximum entropy and Generalized Belief Propagation, Ref. 61 presents an al-

[a)] Electronic mail: lisiran1999@email.swu.edu.cn(or lisiranswu@163.com);

[b)] Electronic mail: cairui686@swu.edu.cn;

gorithm for estimating the distribution by taking into account the dependencies among variables, so that a larger distribution can be estimated from a smaller subset distribution using the principle of maximum entropy. It is often used in statistics (Ref. 62–65), especially parameter estimation (Ref. 66 and 67) and some aspects of biochemistry (Ref. 68), some theories (Ref. 69–71) are also improved based on it. The principle of maximum entropy can also be used to determine BPA. Ref. 72 proposes a new model for determining BPA. This model has certain significance for determining BPA when the available information is limited. The principle of maximum entropy is also reflected in the field of information theory: the maximum Deng entropy model (Ref. 73) gives the necessary and sufficient conditions for the maximum Deng entropy to appear and the model of the maximum Deng entropy. Given that Shannon entropy is a special case of Renyi entropy, Tsallis entropy, and Deng entropy, scholars have begun to study the relationship between the three. Masi's work (Ref. 74) unifies Renyi and Tsallis entropy. Lately, the generalized belief entropy model (Ref. 75) was proposed as a unified model of Renyi entropy, Tsallis entropy, and Deng entropy.

According to the previous ideas, a natural question will be asked: what is the maximum value of the generalized belief entropy model? And under what conditions does the generalized belief entropy reaches this maximum, this subject has not been covered by too many scholars. In this paper, we will discuss the conditions when the generalized belief entropy reaches a maximum value, and give the maximum generalized belief entropy based on the results of the discussion. Through the work of this paper, the maximum Shannon entropy, the maximum Ref. Renyi-Deng entropy, and the maximum Tsallis-Deng entropy can be unified.

The structure of this paper is as follows: The Preliminaries in the second section introduce the related concepts of evidence theory, Deng entropy, Renyi entropy, Tsallis entropy, generalized belief entropy (R-D entropy, T-D entropy). In section 3, the generalized maximum belief entropy model is proposed, and the maximum values of R-D entropy and T-D entropy are obtained according to this model. The fourth section is the conclusion of this paper.

II. PRELIMINARIES

A. Dempster Shafer Evidence Theory

D-S evidence theory (Ref. 18) is a theory that can be used to describe the uncertainty of a system. It can deal with the uncertainty caused by the unknown prior probability. Evidence theory models uncertainty by studying multiple-element subsets, so compared to probability theory, where the research field is a single element subset, evidence theory adapts to weaker axiom systems than probability theory. Some basic concepts of evidence theory are introduced below.

Let Ω be a finite set of all possible values of a variable X, and each element in Ω is mutually exclusive. It is called an frame of discernment and Ω is defined as:

$$\Omega = \{x_1, x_2, ..., x_n\}$$

2^Ω is the power set of Ω:

$$2^\Omega = \{\{\emptyset\}, \{x_1\}, ..., \{x_n\}, \{x_1, x_2\}, ...\{x_1, ..., x_i\}, \Omega\}$$

the probability assignment $m(A_i)$ of the frame of discernment Ω is a mapping from 2^Ω to $[0,1]$ defined on a power set in the form:

$$m(A_i) : 2^\Omega \rightarrow [0,1]$$

A_i is a subset of the frame of discernment. $m(A_i)$ is the probability assignment function (BPA), which satisfies the following conditions:

$$m(\emptyset) = 0, \sum_{A_i \subseteq \Omega} m(A_i) = 1$$

B. Deng Entropy

The form of Deng entropy (Ref. 56) in evidence theory is as follows:

$$E_d = -\sum_{A_i \subseteq \Omega} m(A_i) \times \log_2 \frac{m(A_i)}{2^{|A_i|} - 1}$$

where A_i is a subset of the frame of discernment Ω, and $|A_i|$ is the cardinality of the set A_i. Another form of Deng Entropy is:

$$E_d = \sum_{A_i \subseteq \Omega} m(A_i) \times \log_2(2^{|A_i|} - 1) - \sum_{A_i \subseteq \Omega} m(A_i) \times \log_2 m(A_i)$$

The formula:

$$\sum_{A_i \subseteq \Omega} m(A_i) \times \log_2(2^{|A_i|} - 1)$$

represents the non-specificity of the entire system, and the formula:

$$\sum_{A_i \subseteq \Omega} m(A_i) \times \log_2 m(A_i)$$

represents the inconsistency, that is, the conflict, of the entire system. When it is a single-element subset, Deng entropy degenerates to Shannon entropy (Ref. 14):

$$E_d = -\sum_{A_i \subseteq \Omega} m(A_i) \times \log_2 m(A_i)$$

C. Renyi Entropy

Assuming $P_\theta = \{p_i | i = 1, 2, 3, ...\}$ is the probability distribution of the discrete random variable θ, then Renyi (Ref. 52) entropy can be defined as:

$$H_\alpha(p_1, p_2, p_3, ..., p_n) = \frac{1}{1-\alpha} \times \log_2(\sum_{k=1}^{N} p_k^\alpha)$$

where $\alpha > 0$ and $\alpha \neq 1$. In particular, Renyi entropy degenerates to Shannon entropy when $\alpha > 1$:

$$\lim_{\alpha\to 1} H_\alpha(p_1, p_2, p_3, ..., p_n) = \sum_{k=1}^{n} p_k \times \log_2 \frac{1}{p_k}$$

D. Tsallis Entropy

Given a discrete random variable Y and its probability distribution is $\{p_i|i = 1,2,3,...\}$, then the Tsallis entropy (Ref. 54) can be expressed as:

$$S_q(p_i) = \frac{k}{q-1} \times (1 - \sum_{i=1}^{n} p_i^q)$$

where k, q are parameters, and k is often set to 1 in statistics. And q reflects the degree of non-extensiveness. In particular, when $q \to 1$, Tsallis entropy degenerates to Shannon entropy:

$$\lim_{q\to 1} S_q(p_i) = S_1(p_i) \equiv -\sum_{i}^{n} p_i \times \log_2 p_i$$

It should be noted that the literature (Ref. 76) suggests that the Tsallis entropy can be used to replace Shannon entropy when $q = 2$, so that logarithmic operations can be avoided. In the latter part of this article, the situation of $q = 2$ will also be included in the discussion, so $q = 2$ is a special case of this article.

E. Generalized Belief Entropy

Let A_i be a subset of the frame of discernment Ω, $|A_i|$ denote the cardinality of the set A_i, and $m(A_i)$ be the probability assignment function on the power set 2^Ω. Then the definition of generalized belief entropy (Ref. 75) is as follows:

$$E_{t,r}(m(A_i)) = \frac{1}{1-r} \times \left[\left[\sum_i (\frac{m(A_i)}{2^{|A_i|}-1})^t \times (2^{|A_i|}-1) \right]^{\frac{1-r}{1-t}} - 1 \right]$$

where r, t are parameters. When $r \to 1$, generalized belief entropy degenerates to R-D entropy:

$$E_\alpha(m(A_i)) = \frac{1}{1-\alpha} \times ln \left[\sum_i (\frac{m(A_i)}{2^{|A_i|}-1})^\alpha \times (2^{|A_i|}-1) \right]$$

when $r \to q$, the generalized belief entropy degenerates to T-D entropy:

$$E_q(m(A_i)) = \frac{1}{q-1} \times \left[1 - \sum_i (\frac{m(A_i)}{2^{|A_i|}-1})^q \times (2^{|A_i|}-1) \right]$$

in particular, when $q \to 1$ or $\alpha \to 1$, the T-D entropy and R-D entropy degenerate to Deng entropy:

$$\lim_{q\to 1} E_q(m(A_i)) = \lim_{\alpha\to 1} E_\alpha(m(A_i)) = \sum_i m(A_i) \times ln(\frac{m(A_i)}{2^{|A_i|}-1})$$

III. GENERALIZED MAXIMUM BELIEF ENTROPY MODEL

Let A_i be a subset of the frame of discernment Ω, $|A_i|$ denote the cardinality of the set A_i, and $m(A_i)$ be the probability assignment function on the power set 2^Ω.

A. The Necessary And Sufficient Condition Of The Maximum Generalized Belief Entropy

Theorem 1. *The maximum generalized belief entropy achieves the maximum value if and only if:* $m(A_i) = \frac{2^{|A_i|}-1}{\sum_j 2^{|A_i|}-1}$

Proof 1.

$$E_{t,r}(m(A_i)) = \frac{1}{1-r} \times \left[\left[\sum_i (\frac{m(A_i)}{2^{|A_i|}-1})^t \times (2^{|A_i|}-1) \right]^{\frac{1-r}{1-t}} - 1 \right]$$

$$\sum_{A_i \subseteq \Omega} m(A_i) = 1$$

its Lagrange function can be defined as follows:

$$F_{t,r}(m(A_i)) = \frac{1}{1-r} \times \left[\left[\sum_i (\frac{m(A_i)}{2^{|A_i|}-1})^t \times (2^{|A_i|}-1) \right]^{\frac{1-r}{1-t}} - 1 \right] + \lambda \times (\sum_i m(A_i) - 1)$$

then we can get:

$$\frac{\partial F_{t,r}(m(A_i))}{\partial m(A_i)} = \frac{1}{1-t} \times \left[\sum_i (\frac{m(A_i)}{2^{|A_i|}-1})^t \times (2^{|A_i|}-1) \right]^{\frac{1-r}{1-t}} \times t \times (\frac{m(A_i)}{2^{|A_i|}-1})^{t-1} + \lambda = 0$$

where t and λ are constants, then we can get:

$$t \times \left(\frac{m(A_i)}{2^{|A_i|}-1} \right)^{t-1} \times \sum_i \left(\frac{m(A_i)}{2^{|A_i|}-1} \right)^t \times (2^{|A_i|}-1) = \mu$$

in the formula above, the value of the term $\sum_i \left(\frac{m(A_i)}{2^{|A_i|}-1} \right)^t$ *is fixed. So we can get:*

$$\frac{m(A_1)}{2^{|A_1|}-1} = \frac{m(A_2)}{2^{|A_2|}-1} = ... = \frac{m(A_i)}{2^{|A_i|}-1}$$

and according to the ratio property, we can get:

$$\frac{m(A_1) + m(A_2) + ... + m(A_n)}{(2^{|A_1|}-1) + (2^{|A_2|}-1) + ... + (2^{|A_n|}-1)} = \frac{m(A_i)}{(2^{|A_i|}-1)}$$

then we can get:

$$\frac{1}{\sum_j 2^{|A_j|}-1} = \frac{m(A_i)}{2^{|A_i|}-1}$$

so we can finally get:

$$m(A_i)=\frac{2^{|A_i|}-1}{\sum_j 2^{|A_j|}-1}$$

So based on the process of inference above, the necessary and sufficient condition for the generalized belief entropy to reach the maximum is:

$$m(A_i)=\frac{2^{|A_i|}-1}{\sum_j 2^{|A_j|}-1}$$

B. The Generalized Maximum Belief Entropy Model

We substitute the obtained necessary and sufficient condition:

$$m(A_i)=\frac{2^{|A_i|}-1}{\sum_j 2^{|A_j|}-1}$$

into the generalized belief entropy model:

$$E_{t,r}(m(A_i))=\frac{1}{1-r}\times\left[\left[\sum_i\left(\frac{m(A_i)}{2^{|A_i|}-1}\right)^t\times(2^{|A_i|}-1)\right]^{\frac{1-r}{1-t}}-1\right]$$

then we can get the generalized maximum belief entropy model:

$$E_r(A_i)_{max}=\frac{1}{1-r}\times\left[\left(\sum_j 2^{|A_j|}-1\right)^{1-r}-1\right]$$

C. The Maximum R-D entropy and Maximum T-D entropy

The proposition A_i in the generalized maximum belief entropy model proposed above is obejective, so we mainly discuss the value of parameter r to obtain the maximum value of the generalized maximum belief entropy model. We find the partial derivative about r for the generalized maximum belief entropy model obtained, that is:

$$\frac{\partial E_r(A_i)_{max}}{\partial r}=$$
$$\frac{\left(\sum_j 2^{|A_j|}-1\right)^{1-r}}{(1-r)^2}\times\left[-\ln\left(\sum_j 2^{|A_j|}-1\right)(1-r)-1\right]=0$$

then we can get two values of r:

$$r=1+\frac{1}{\ln\left(\sum_j 2^{|A_j|}-1\right)},r=1$$

we first substitute the first r value into the generalized maximum belief entropy model and we can get:

$$E_r(A_i)_{max}=$$
$$\ln\left(\sum_j 2^{|A_j|}-1\right)\times\left[1-\left(\sum_j 2^{|A_j|}-1\right)^{-\frac{1}{\ln(\sum_j 2^{|A_j|}-1)}}\right] \qquad (1)$$

we simplify the equation (1) to get:

$$E_r(A_i)_{max}=\ln\left(\sum_j 2^{|A_j|}-1\right)\times(1-\frac{1}{e})<\ln\left(\sum_j 2^{|A_j|}-1\right)$$

while $r\to 1$, we can get the maximum of the the generalized maximum belief entropy model in the form of limit:

$$\begin{aligned}&\lim_{r\to 1}\frac{1}{1-r}\left[\left(\sum_j 2^{|A_j|}-1\right)^{1-r}-1\right]\\&=\lim_{r\to 1}\frac{\left(\sum_j 2^{|A_j|}-1\right)^{1-r}\times\ln\left(\sum_j 2^{|A_j|}-1\right)}{1}\\&=\ln\left(\sum_j 2^{|A_j|}-1\right)\end{aligned}$$

Therefore, the maximum value of the generalized maximum belief entropy model can be described as follows:

$$E_r(A_i)_{max}=\ln\left(\sum_j 2^{|A_j|}-1\right)$$

Since our model is based on the generalized belief entropy model (Ref. 75), the generalized belief entropy shows that when $r\to 1$, the generalized belief entropy degenerates to R-D entropy, when $r\to q$,the generalized belief entropy degenerates to T-D entropy. In order to prove that our model is equally applicable to these two cases, we discuss the corresponding two cases separately: When $r\to 1$, this situation is the same as what we have discussed above, so we will not go into details.

And when $r\to q$,we can get:

$$\begin{aligned}&\lim_{r\to q}\frac{1}{1-r}\times\left[\left(\sum_j 2^{|A_j|}-1\right)^{1-r}-1\right]\\&=\frac{1}{1-q}\times\left[\left(\sum_j 2^{|A_i|}-1\right)^{1-q}-1\right]\\&=\frac{1}{1-q}\times\left[exp\left\{(1-q)\times\ln\left(\sum_j 2^{|A_j|}-1\right)\right\}-1\right]\end{aligned} \qquad (2)$$

then we perform power series expansion on equation above to get:

$$(2)=\frac{1}{1-q}\times\left[1+(1-q)\times\ln\left(\sum_j 2^{|A_j|}-1\right)+\sum_{i=2}^{n}\frac{(1-q)^n\times\left[\ln\left(\sum_j 2^{|A_j|}-1\right)\right]^n}{n!}-1\right]$$

next we can get:

$$(2)=\ln\left(\sum_j 2^{|A_j|}-1\right)+\frac{1}{1-q}\times\sum_{i=2}^{n}\frac{(1-q)^n\times\left[\ln\left(\sum_j 2^{|A_j|}-1\right)\right]^n}{n!} \qquad (3)$$

for the Eq.3 we discuss the scope of q in two cases: we discuss the corresponding two cases separately:

1) when $0 \le q < 2$, that is $1-q \in [-1,1)$, and for a definite frame of discernment Ω, $\left(\sum_j 2^{|A_j|}-1\right)$ is also definite, so we can get:

$$\lim_{n\to\infty}\sum_{i=2}^{n}\frac{(1-q)^n \times \left[\ln\left(\sum_j 2^{|A_j|}-1\right)\right]^n}{n!}=0$$

so Eq.3 can be rewritten as : $\ln\left(\sum_j 2^{|A_j|}-1\right)+0$, therefore, in this situation, the maximum value of T-D entropy is: $\ln\left(\sum_j 2^{|A_j|}-1\right)$.

2) when $q \ge 2$, we can transform Eq.3 and get the following expression:

$$\begin{aligned}&\ln\left(\sum_j 2^{|A_j|}-1\right)+\frac{1}{1-q}\times\Bigg[\\ &exp\left\{(1-q)\times\ln\left(\sum_j 2^{|A_j|}-1\right)\right\}\\ &-(1-q)\times\ln\left(\sum_j 2^{|A_j|}-1\right)-1\Bigg]\end{aligned}\tag{4}$$

and let us consider the expression :

$$\begin{aligned}&exp\left\{(1-q)\times\ln\left(\sum_j 2^{|A_j|}-1\right)\right\}\\ &-(1-q)\times\ln\left(\sum_j 2^{|A_j|}-1\right)-1\end{aligned}\tag{5}$$

let :

$$x=(1-q)\times\ln\left(\sum_j 2^{|A_j|}-1\right)(x<0)$$

then we can rewrite the expression above to get a function about x:

$$f(x)=e^x-x-1(x<0)\tag{6}$$

now we consider the monotonicity of Eq.6:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}=e^x-1=0$$

then we can get $x=0$, that is, the Eq.6 decreases monotonically when $x<0$, so we can get:

$$f(x)>f(0)=0$$

so $\frac{f(x)}{1-q}<0$, and for Eq.3, we can get:

$$(3)<\ln\left(\sum_j 2^{|A_j|}-1\right)$$

so, when $r \to q$, the maximum value of T-D entropy demostrates as follows:

$$T-D_{max}=\ln\left(\sum_j 2^{|A_j|}-1\right)$$

In summary, the maximum value of both entropies expressed as:

$$E_{max}=\ln\left(\sum_j 2^{|A_j|}-1\right)$$

We compared the entropy mentioned in the article, as shown in Table.I. It can be seen that the conditions for the maximum value of Renyi entropy and Tsallis entropy proposed earlier are the same, that is, at this time, each p_i occupies the same possibility, and the system is most uncertain. The maximum value of Tsallis entropy is consistent with the maximum value of Renyi entropy when $q \to 1$. Deng entropy generalizes the individual in question to multi-element subsets. The condition for Deng entropy to obtain the maximum value can be understood as that for each multi-element subset, all its possible values (power set) account for all possible elements of all multi-element subsets. Value ratio, this idea is in line with Renyi entropy and Tsallis entropy. Our model is based on a multi-element subset, and the condition and maximum form of our maximum value are consistent with Deng entropy, R-D entropy and T-D entropy.

In the framework of D-S evidence theory, there are many ways to measure uncertainty. We can degenerate our model into other measures of uncertainty by changing the parameters of the maximum generalized entropy model, thus adapting our model to situations that are met by other measures:

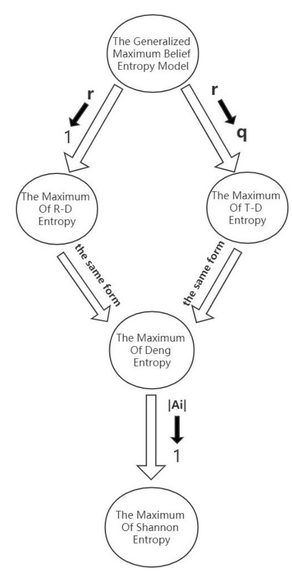

FIG. 1. Relationship between several maximum entropy models.

As shown in Fig.1, we can see that:

1) When the value of r tends to 1, the maximum generalized belief entropy model degenerates into the maximum R-D entropy model.

TABLE I. Uncertainty measures

Name	Expression	Conditions	Maximum value
Renyi Entropy	$H_\alpha(p_i) = \frac{1}{1-\alpha} \times \log_2(\sum_{k=1}^{N} p_k^\alpha)$	$p_k = \frac{1}{N}$	$\log_2(N)$
Tsallis Entropy	$S_q(p_i) = \frac{k}{q-1} \times (1 - \sum_{i=1}^{n} p_i^q)$	$p_i = \frac{1}{N}$	$\frac{1-N^{1-q}}{q-1}$
Deng Entropy	$E_d = -\sum_{A_i \subseteq \Omega} m(A_i) \times \log_2 \frac{m(A_i)}{2^{\|A_i\|}-1}$	$m(A_i) = \frac{2^{\|A_i\|}-1}{\sum_j 2^{\|A_j\|}-1}$	$\log_2\left(\sum_j 2^{\|A_j\|} - 1\right)$
R-D Entropy	$E_\alpha(m(A_i)) =$ $\frac{1}{1-\alpha} \times \ln\left[\sum_i (\frac{m(A_i)}{2^{\|A_i\|}-1})^\alpha \times (2^{\|A_i\|}-1)\right]$	$m(A_i) = \frac{2^{\|A_i\|}-1}{\sum_j 2^{\|A_j\|}-1}$	$\log_2\left(\sum_j 2^{\|A_j\|} - 1\right)$
T-D Entropy	$E_q(m(A_i)) =$ $\frac{1}{q-1} \times \left[1 - \sum_i (\frac{m(A_i)}{2^{\|A_i\|}-1})^q \times (2^{\|A_i\|}-1)\right]$	$m(A_i) = \frac{2^{\|A_i\|}-1}{\sum_j 2^{\|A_j\|}-1}$	$\log_2\left(\sum_j 2^{\|A_j\|} - 1\right)$
Gen Entropy	$E_{t,r}(m(A_i)) =$ $\frac{1}{1-r} \times \left[\left[\sum_i (\frac{m(A_i)}{2^{\|A_i\|}-1})^t \times (2^{\|A_i\|}-1)\right]^{\frac{1-r}{1-t}} - 1\right]$	$m(A_i) = \frac{2^{\|A_i\|}-1}{\sum_j 2^{\|A_j\|}-1}$	$\log_2\left(\sum_j 2^{\|A_j\|} - 1\right)$

2) When the value of r tends to q, the maximum generalized belief entropy model degenerates into the maximum T-D entropy model.

3) The maximum R-D entropy model and the maximum T-D entropy model are consistent in expression (Ref. 75) with the maximum Deng entropy.

4) The maximum Deng entropy degenerates to the maximum Shannon entropy in the case of a single element subset. (Ref. 73)

IV. NUMERICAL EXPERIMENTS

A. Experiment 1

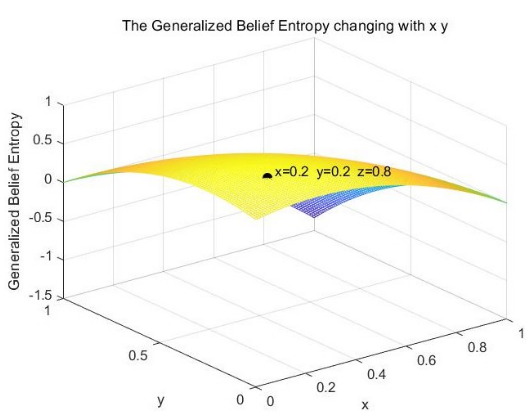

FIG. 2. The Generalized Belief Entropy changing with x y

Based on the model proposed in Ref. 75, which is the basis of our work, we select the discernment frame with the elements $\{a,b\}$, and the parameter r,t in the Ref. 75 model is set to 2. Then we obtained the figure above.

From the above figure, we can see that the maximum value of the generalized belief entropy in the this case is obtained when $x = y = 0.2$, and the maximum value is 0.8. We apply the above conditions to our proposed model, namely:

$$E(A_i) = 1 - \frac{1}{\sum_j 2|A_j| - 1}$$

under the premise that the power set is $\{\{a\},\{b\},\{a,b\},\emptyset\}$, the value of formula $\sum_j 2^|A_j| - 1$ is 5, therefore, we can calculate the maximum value of generalized belief entropy under our model: $E(A_i)_{max} = 0.8$. And according to the conditions when it reaches the maximum value, it can be obtained: $m(\{a\}) = m(\{b\}) = 1/5 = 0.2$ therefore, our model is consistent with 75 in the actual situation.

B. Experiment 2

In the third section, we derive the maximum values of R-D entropy and T-D entropy through the obtained model, and here we verify it.

In Ref. 75, it is mentioned that the condition for R-D entropy and T-D entropy to achieve the maximum value is:

$$m(A_i) = \frac{2^{|A_i|} - 1}{\sum_j 2^{|A_j|} - 1} \quad (7)$$

We substitute Eq.7 into R-D entropy (Ref. 75) and we can get the maximum model of R-D as:

$$R - D_{max} = \ln\left(\sum_j 2^{|A_j|} - 1\right)$$

this is consistent with the conclusion drawn by our model on the issue of maximum R-D. Then we substitute Eq.7 into T-D entropy 75 too, and what we can actually get is:

$$T-D_{max}=\frac{1}{q-1}\left[1-\frac{1}{\left[\sum_j\left(2^{|A_i|}-1\right)\right]^{q-1}}\right]$$

in fact, the equation above and Eq.2 are essentially the same. Therefore, the maximum R-D entropy and T-D entropy have been verified.

C. Experiment 3

In this example, we mainly discuss the motivation of maximum generalized belief entropy. We will explain how to adapt this model to the maximum R-D entropy model, the maximum T-D entropy model, and the maximum Deng entropy model by modifying the parameters in the maximum generalized entropy model.

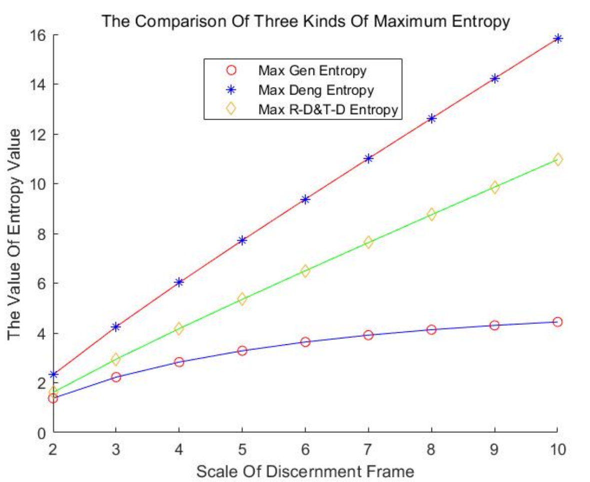

FIG. 3. The Comparison Of Three Kinds Of Maximum Entropy

In order to discuss the relationship between the maximum values of the three entropies, we fixed the value of the parameter r so that our model was determined. Furthermore, we discuss the difference in the maximum values of the three types of entropy under different discernment frames, and draw the figure above. From the figure above, we can see that although our model and the maximum entropy value under different discernment frames are the same in the overall trend, the specific value is not very satisfactory. Therefore, we consider the effect of the value of parameter r on this experiment.

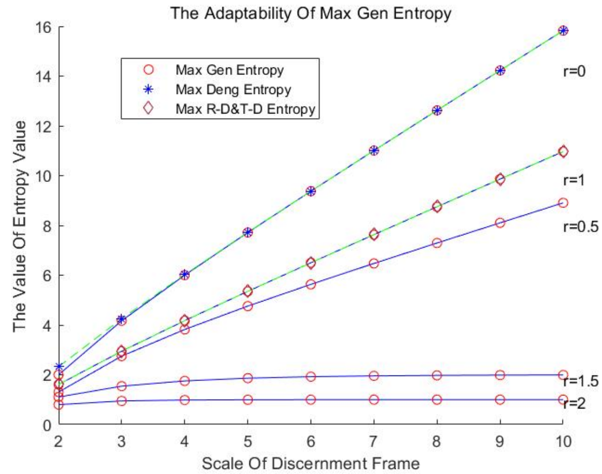

FIG. 4. The Adaptability Of Max Gen Entropy

Under the change of r, our model also changes accordingly. From Fig.4, we can see that the value of r is between 0-2. When r equals 2, the value calculated by our model under each discernment frame tends to be gentle. When the value of r is within 1-2, we can see that our model is approaching the maximum R-D entropy and the maximum T-D entropy. In particular, when r = 1, our model coincides with the maximum R-D entropy and maximum T-D entropy models, which is consistent with what we described in Section III. It is worth mentioning that r = 1 is a singularity of our model, so at this point the values under different discernment frames are obtained by the limit. When the value of r is within 0-1, our model is approaching the maximum Deng entropy. In particular, when r = 0, our model basically coincides with the maximum Deng entropy model. What needs to be explained here is that in the experiment with r<1, we performed log processing on the obtained data.

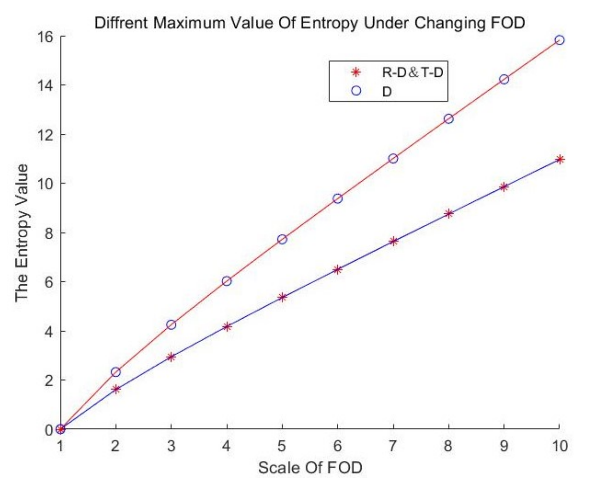

FIG. 5. Diffrent Maximum Value Of Entropy Under Changing FOD

In view of the change in the value of r will cause changes in our model, we discuss the relationship between the maximum

R-D entropy, maximum T-D entropy and maximum Deng entropy model. Their relationship is shown in Fig.5. From the figure above, we can see that the maximum R-D entropy, maximum T-D entropy, and maximum Deng entropy are consistent under different discernment frames. But the maximum Deng entropy value is higher than the maximum R-D entropy and maximum T-D entropy.

From the data in the table at page 9, we can see that the mutiple between maximum R-D entropy, the maximum T-D entropy and the maximum Deng entropy is 1.4427. This is due to the calculation method of the two entropies. The maximum R-D entropy and maximum T-D entropy calculated by our model apply the logarithm of the Euler number at the end, and the maximum Deng entropy is based on the logarithm of 2 bases, so the difference between the two will be $log_2(e)$ times.

D. Experiment 4

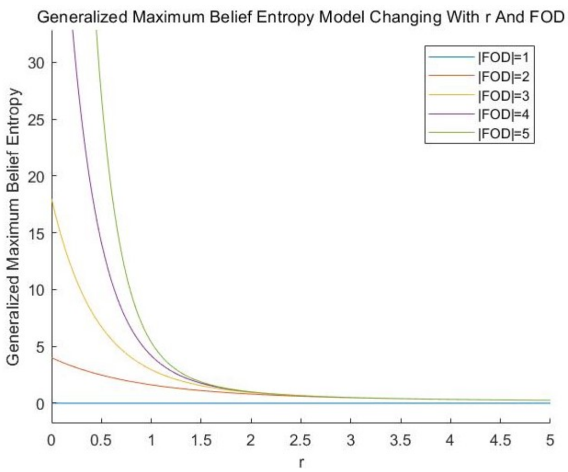

FIG. 6. Generalized Maximum Belief Entropy Model Changing With r And FOD

From the figure above, we can see that our model performs the same under different discernment frame and different r, from this perspective, our model is insensitive to r. At the same time, we can see that when $r = 1$, the convergence rate of the model slows down rapidly, and at r = 2, for the same model, the difference between the maximum values calculated by different discernment frame through this model is not particularly obvious. Therefore, we determined the value of r as 2 in Example 1, which helps the generality of Example 1. In fact, from Example.3, we recommend that the value of r is between $0-2$.

V. CONCLUSION

In this article, we propose the generalized maximum belief entropy model, and use this model to derive the analytical solutions of maximum R-D entropy and maximum T-D entropy. At the same time, the relationship between the generalized maximum belief entropy model, maximum R-D entropy, maximum T-D entropy, maximum Deng entropy, and maximum Shannon entropy is revealed.

The novelty of our work lies in the unification of the maximum R-D entropy model, the maximum T-D entropy model and the maximum Deng entropy model through the maximum generalized entropy model we have obtained. The unification process is demostrated in Section 4.3.

We have not discussed the practical significance of r in the generalized maximum belief entropy model, which can become the direction of future work, and we will also try to apply the maximum generalized entropy model to information conflict handling.

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AIP PUBLISHING DATA SHARING POLICY

The data that support the findings of this study are available from the author upon reasonable request.

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TABLE II. The quantitative relationship between entropy

Scale of FOD	R-D&T-D Entropy	Deng Entropy	R-D&T-D/Deng
2	1.6094	2.3219	1.4427
3	2.9444	4.2479	1.4427
4	4.1744	6.0224	1.4427
5	5.3519	7.7211	1.4427
6	6.4998	9.3772	1.4427
7	7.6300	11.0077	1.4427
8	8.7491	12.6223	1.4427
9	9.8612	14.2266	1.4427
10	10.9686	15.8244	1.4427

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