Quantum Mechanics as a Deformation of Classical Mechanics

UConn Math Club

Dion Mann

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Contents

Classical Mechanics

How to Describe Motion

- Suppose a particle with mass m is moving on the line $\mathbb{R}$ (or on the plane $\mathbb{R}^2$, etc.)
- Some things we can discuss about the particle:
  1. Its position $x(t)$ on the line at time t.
  2. Its velocity $v(t) = \frac{dx}{dt}(t)$ at time t.
     - In physics, we often speak of the particle's momentum $p = m\frac{dx}{dt}$ as much as its velocity.
- The path traced out by $x(t)$ is called the particle's trajectory.

Forces and Newton's Second Law

- Sometimes all we know is the *forces* acting on the object.
  - Gravity.
  - Electrostatic.
  - Friction.

Newton's Second Law of Motion

The total force on an object is the product of its mass and acceleration:

$$F = ma(t) \iff F = m\frac{dv}{dt} \iff F = m\frac{d^2x}{dt^2}$$

- Notice that $\dfrac{dp}{dt} = \dfrac{d}{dt}\left(m\dfrac{dx}{dt}\right) = m\dfrac{d^2x}{dt^2} = F$.

Newton's Second Law of Motion (with momentum)

The total force on an object is the instantaneous change of its momentum:

$$F = \frac{dp}{dt}$$

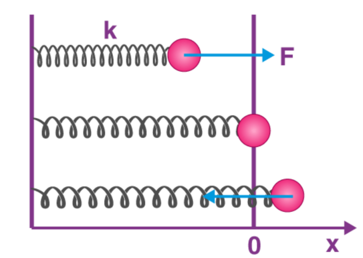
The Harmonic Oscillator: Introduction

- Our focus will be on a physical system called the harmonic oscillator.
  - This is a system where the motion is *oscillatory.*
- There are two types of harmonic oscillators:
  - The classical harmonic oscillator; we will do this now.
  - The quantum analogue... also called the harmonic oscillator.
- The values of measurements are drastically different between classical and quantum harmonic oscillators.
  - The classical harmonic oscillator has continuous values of total energy.
  - The quantum harmonic oscillator has discrete values of total energy.

Classical Harmonic Oscillator

Example: A Block Attached to a Spring

A block of mass m attached to a spring is pulled a distance of x_0 away from equilibrium, and is then released *from rest*.



- Using Newton's 2nd law $F = m\dfrac{d^2x}{dt^2}$, we show that the trajectory of the block is given by $x(t) = x_0 \cos(\omega t)$.

Conservation of Energy

- We have $x(t) = x_0 \cos(\omega t)$ when $x(0) = x_0$ and $x'(0) = 0$.

- There are two types of energy for the harmonic oscillator:
  1. Kinetic energy: the energy of motion, $K = \frac{1}{2}mv^2 = \frac{1}{2m}(mv)^2 = \frac{p^2}{2m}$
  2. Potential energy: the energy of position, $U = \frac{1}{2}kx^2$.

- Consider the total energy of the system:

$$H = K + U = \frac{1}{2}\left(mv^2 + kx^2\right) = \frac{p^2}{2m} + U(x),$$

called the Hamiltonian.

  - We treat x and p as independent variables: $H = H(x, p)$.
  - The Hamiltonian is constant throughout time, i.e. $\frac{dH}{dt} = 0$.

Hamiltonian Mechanics

- Generally, the total energy (Hamiltonian)

$$H(x,p) = \frac{p^2}{2m} + U(x)$$

for any system is *conserved*.

- Hamilton used $H(x,p)$ to shift the focus from forces to energy:

$$\frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\left(\frac{p^2}{2m} + U(x)\right) = \frac{p}{m} = \frac{dx}{dt}$$
$$\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}\left(\frac{p^2}{2m} + U(x)\right) = \frac{dU}{dx}$$

Hamilton's Equations

- What is $\dfrac{dU}{dx}$?
  - In harmonic oscillator, we had $U(x) = \frac{1}{2}kx^2$, so $\dfrac{dU}{dx} = kx$
  - This almost looks like the spring force $F = -kx$.

- In general, potential energy is a function $U(x)$ such that $F = -\dfrac{dU}{dx}$.
  - Resuming our calculation,

$$H(x,p) = \frac{p^2}{2m} + U(x) \implies \frac{\partial H}{\partial x} = \frac{dU}{dx} = -F = -\frac{dp}{dt}.$$

Hamilton's Equations

If $H = H(x,p)$ is the Hamiltonian of a system, then

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

Poisson Brackets

- Hamilton's two equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

can be expressed in a unified way.

- The Poisson bracket: for $f(x,p)$ and $g(x,p)$, their Poisson bracket is

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}.$$

- **Examples:**
  1. $f(x,p) = x,\ g(x,p) = p.$
  2. $f(x,p) = x,\ g(x,p) = H(x,p).$
  3. $f(x,p) = p,\ g(x,p) = H(x,p).$
- Hamilton's equations are both special cases of the single equation

$$\frac{df}{dt} = \{f, H\}.$$

Quantum Mechanics

Summary of Classical Setting

- What mathematical objects are used in classical mechanics?
  - Particles are described by numbers (x, p)
  - Measurements are real numbers.
    1. Position: $r(x, p) = x$.
    2. Velocity: $v(x, p) = \frac{p}{m}$.
    3. Kinetic energy: $K(x, p) = \frac{p^2}{2m}$.
    4. Total energy (Hamiltonian): $H(x, p) = \frac{p^2}{2m} + U(x)$.
  - Properties we measure (e.g. position) are called observables.
  - The time-evolution of a particle is described by

$$\frac{dx}{dt} = \{x, H\}.$$

The Quantum Setting

- What mathematical objects are used in quantum mechanics?
  - Particles are represented by functions $\psi(x)$.
  - Observables (properties we measure) are linear operators on functions.
    1. Position operator: $\hat{x}: \ \psi(x) \mapsto x\psi(x)$
    2. Momentum operator: $\hat{p}: \ \psi(x) \mapsto -i\hbar \frac{d\psi}{dx}$
       ($\hbar \approx 10^{-34}\ J \cdot s$ is called Planck's constant)
    3. Hamiltonian: $\hat{H}$. We will discuss this later in an example.
  - Measurements of a property are eigenvalues of its operator.
  - The time-evolution of a particle is given by Schrödinger's equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi,$$

where $\Psi(x,t)$ is the particle's state at time t and $\Psi(x,0) = \psi(x)$.

Differences in Algebraic Structure

- Classical observables (functions) have a commutative product:

$$fg = gf, \quad \text{so } fg - gf = 0.$$

- Quantum observables (operators) have a noncommutative product, given by composition of operators.
  - For example, using $\hat{x}\psi(x) = x\psi(x)$ and $\hat{p}\psi(x) = -i\hbar\dfrac{d\psi}{dx}$:

$$(\hat{x}\hat{p})\psi = \hat{x}\left(-i\hbar\frac{d\psi}{dx}\right) = -i\hbar x\frac{d\psi}{dx}$$

$$(\hat{p}\hat{x})\psi = \hat{p}\,(x\psi) = -i\hbar\frac{d}{dx}\,(x\psi) = -i\hbar\left(\psi + x\frac{d\psi}{dx}\right) = -i\hbar x\frac{d\psi}{dx} - i\hbar\psi$$

  - So $(\hat{x}\hat{p} - \hat{p}\hat{x})\psi = i\hbar\psi$, so $\hat{x}\hat{p}$ and $\hat{p}\hat{x}$ differ by scaling by $i\hbar$:

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar \neq 0.$$

Quantum Harmonic Oscillator: Introduction

- The classical Hamiltonian for the harmonic oscillator is

$$H(x,p) = \frac{1}{2}\left(x^2 + p^2\right)$$

(for simplicity, we won't write m or ω)

- The quantum Hamiltonian is obtained by $x \mapsto \hat{x}$ and $p \mapsto \hat{p}$ in H, where $\hat{x}\psi(x) = x\psi(x)$ and $\hat{p}\psi(x) = -i\hbar\frac{d\psi}{dx}$. Set

$$\hat{H} = \frac{1}{2}\left(\hat{x}^2 + \hat{p}^2\right)$$

where

$$\hat{x}^2(\psi(x)) = \hat{x}(\hat{x}(\psi(x))) = x^2\psi(x)$$

$$\hat{p}^2(\psi(x)) = -i\hbar\frac{d}{dx}\left(-i\hbar\frac{d\psi}{dx}\right) = (-i\hbar)^2\frac{d^2\psi}{dx^2}$$

Energy Values of the Quantum Harmonic Oscillator

- Recall the total energy of a classical harmonic oscillator is
$$H(x,p) = \frac{1}{2}\left(x^2 + p^2\right).$$
Its values are all numbers ≥ 0.

- What about the quantum harmonic oscillator?
$$\hat{H} = \frac{1}{2}\left(\hat{x}^2 + \hat{p}^2\right)$$
  - Measurements are *eigenvalues* of operators: the energy values of the quantum harmonic oscillator are the eigenvalues of $\hat{H}$.

Energy Levels of Quantum Harmonic Oscillator

The possible energies (eigenvalues) of $\hat{H}$ turn out to be

$$\left(n + \frac{1}{2}\right)\hbar \geq \frac{1}{2}\hbar \qquad n = 0, 1, 2, \ldots$$

Eigenfunctions of the Quantum Harmonic Oscillator

Eigenvalues and Eigenfunctions for $\hat{H}$

The function $\psi_n(x) = e^{-x^2/2\hbar} H_n(x/\sqrt{\hbar})$ satisfies $\hat{H}\psi_n = \left(n + \frac{1}{2}\right) \hbar \cdot \psi_n$, where H_n are the *Hermite polynomials*.

Proof (Only for $n = 0$).

- Consider $\psi_0(x) = e^{-x^2/2\hbar}$. We have $\hat{H}\psi_0 = \frac{1}{2}\left(x^2\psi_0 - \hbar^2\psi_0''\right)$.
- Check that $\hbar^2\psi_0'' = x^2\psi_0 - \hbar\psi_0$.
- Then $\hat{H}\psi_0 = \frac{1}{2}\hbar\psi_0$.

$\square$

Disjoint Theories of Physics

- Consider how different classical and quantum mechanics are.
  - Continuous vs. discrete energies: $E \geq 0$ vs. $E_n = \left(n + \frac{1}{2}\right)\hbar$.
  - The lowest classical energy level is $H(0,0) = 0$, the lowest quantum energy level is $E_0 = \frac{1}{2}\hbar > 0$.
    - ★ But, note $\frac{1}{2}\hbar \to 0$ as $\hbar \to 0$.
- This is related to fundamentally different mathematical objects.
  - Classical mechanics uses functions, quantum mechanics uses operators.
  - Products of functions commute, products of operators don't.
- Can a single theory describe both classical and quantum mechanics?
  - One attempt is known as deformation quantization, which considers only functions (no operators!) using a noncommutative product on functions involving $\hbar$ that becomes ordinary product when $\hbar \to 0$.

Deformation Quantization

The Deformation Philosophy

- New physics can be viewed as a deformation of old physics.

Example: Shape of Earth

- We (humans approximately 5-6ft tall) experience similar physics whether...
  1. The earth is flat.
  2. The earth is a big sphere.
- The sphere theory deforms to the flat theory in the limit where we zoom in very closely (spheres are locally Euclidean).

- We'll view quantum theory as a deformation of classical theory. We want formulas with $\hbar$ to become classical formulas when $\hbar \to 0$.

Deforming Products

Main Idea

If we only use functions, let's mimic the noncommutative nature of quantum operators by a noncommutative product on functions.

- We want a product $\star$ on functions $f(x,p)$ using $\hbar$ such that
  1. $\star$ is noncommutative
     - In particular: $x \star p - p \star x = i\hbar$, just like for $\hat{x}$ and $\hat{p}$ before.
  2. If $\hbar \to 0$, then $f \star g \to f \cdot g$, where $\cdot$ is the usual product.

- Viewing $\hbar$ as a small parameter, we deform $\cdot$ by having $f \star g$ be a power series in $\hbar$ with coefficients depending on f and g :

$$f \star g = fg + \mu_1(f,g)\hbar + \mu_2(f,g)\hbar^2 + \cdots$$

As $\hbar \to 0$, $f \star g \to fg$. Call this $\star$ a star-product.

Properties of Star-Products

- We want

$$f \star g = fg + \mu_1(f,g)\hbar + \mu_2(f,g)\hbar^2 + \cdots ,$$

where $f = f(x,p)$ and $g = g(x,p)$. What properties should the coefficients $\mu_n(f,g)$ have?

- We want $x \star p - p \star x = i\hbar$.
  - Then $\mu_1(x,p) - \mu_1(p,x) = i$.
  - Compare to $\{x,p\} = 1$, where $\{f,g\} = \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial p} - \dfrac{\partial f}{\partial p}\dfrac{\partial g}{\partial x}$.

Star-Product Axioms

For all functions $f(x,p)$, $g(x,p)$, and $h(x,p)$:

- Classical limit: $\mu_1(f,g) - \mu_1(g,f) = i\{f,g\}$.
- Associativity: $f \star (g \star h) = (f \star g) \star h$.

Example of a Star-Product: The Basic Star-Product

- Recall for classical harmonic oscillator $H(x, p) = \frac{1}{2}\left(x^2 + p^2\right)$.

- In $\mathbb{C}$, we factor H by letting $a = \frac{1}{\sqrt{2}}(x + ip)$ and $b = \frac{1}{\sqrt{2}}(x - ip)$:
  - $H = \frac{1}{2}\left(x^2 + p^2\right) = \frac{1}{\sqrt{2}}(x + ip) \cdot \frac{1}{\sqrt{2}}(x - ip) = ab.$
  - So we can view H as a function of a and b : $H(a, b) = ab$.

- In general, think of functions $f(x, p)$ as functions $f(a, b)$.
  - Like $(x, y) \to (r, \theta)$ in polar coordinates.

The basic star-product

Define the basic star-product by

$$f \star g = fg + \frac{\partial f}{\partial a}\frac{\partial g}{\partial b}\hbar + \frac{\partial^2 f}{\partial a^2}\frac{\partial^2 g}{\partial b^2}\frac{\hbar^2}{2} + \cdots + \frac{\partial^n f}{\partial a^n}\frac{\partial^n g}{\partial b^n}\frac{\hbar^n}{n!} + \cdots$$

Examples with basic star-product

- Let's do some examples with the basic star-product:

$$f \star g = fg + \frac{\partial f}{\partial a}\frac{\partial g}{\partial b}\hbar + \frac{\partial^2 f}{\partial a^2}\frac{\partial^2 g}{\partial b^2}\frac{\hbar^2}{2} + \cdots + \frac{\partial^n f}{\partial a^n}\frac{\partial^n g}{\partial b^n}\frac{\hbar^n}{n!} + \cdots$$

where $a = \frac{1}{\sqrt{2}}(x + ip)$ and $b = \frac{1}{\sqrt{2}}(x - ip)$.

Example 1

Note that $x = \frac{\sqrt{2}}{2}(a + b)$ and $p = \frac{\sqrt{2}}{2i}(a - b)$. Calculate $x \star p$ and $p \star x$.

Example 2

Let $H(a, b) = ab$ and $g = g(a, b)$ be any function. Show that

$$H \star g = Hg + \hbar b \frac{\partial g}{\partial b}.$$

The Star-Schrödinger Equation

- Recall quantum mechanics is controlled by Schrödinger's equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$

where an operator $\hat{H}$ acts on the function $\Psi(x,t)$.

- In deformation quantization, we have functions $\psi = \psi(a,b)$ and the $\star$-product, but no operators.

Star-Schrödinger Equation

Let $H = H(a,b)$ be the *classical* Hamiltonian function. Then the quantum time-evolution of $\psi = \psi(a,b)$ is given by the star-Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = H \star \psi$$

Deformation Quantization of the Harmonic Oscillator

- Recall the classical harmonic oscillator is described by the Hamiltonian

$$H(x,p) = \frac{1}{2}\left(x^2 + p^2\right) \quad \text{or} \quad H(a,b) = ab,$$

where $a = \frac{1}{\sqrt{2}}(x + ip)$ and $b = \frac{1}{\sqrt{2}}(x - ip)$.

- This represents the total energy of the harmonic oscillator; it takes continuous values in $[0, \infty)$.

- Using a suitable star-product, the Hamiltonian function $H(a,b)$ can give us the discrete energy levels of the quantum harmonic oscillator by a type of eigenvalue.
  - Call a function $g(a,b)$ a star-eigenfunction of $H(a,b)$ if there is a number λ, called the star-eigenvalue, such that $H \star g = \lambda g$.
  - We want the star-eigenvalues of the Hamiltonian for classical harmonic oscillator H to be the energy levels of quantum harmonic oscillator.

Energy Levels of the Harmonic Oscillator

Eigenvalues of $H(a,b) = ab$

For each $n \in \mathbb{Z}_{\geq 0}$, let $g_n(a,b) = (ab)^n e^{-ab/\hbar}$. Then

$$H \star g_n = (n\hbar) g_n,$$

where $f \star g = fg + \frac{\partial f}{\partial a}\frac{\partial g}{\partial b}\hbar + \frac{\partial^2 f}{\partial a^2}\frac{\partial^2 g}{\partial b^2}\frac{\hbar^2}{2} + \cdots$ is the basic star-product.

Proof (Outline).

- By a previous example, $H \star g = Hg + \hbar b \frac{\partial g}{\partial b}$ for any $g = g(a,b)$.
- Take $g = g_n$, and use the product rule in the second term to get

$$H \star g_n = (ab)^{n+1} e^{-ab/\hbar} + n\hbar (ab)^n e^{-ab/\hbar} - (ab)^{n+1} e^{-ab/\hbar}$$

- Simplify: $H \star g_n = n\hbar g_n$.

$\square$

Discussion: Energy Levels of the Harmonic Oscillator

- Let $\psi_n(a, b, t) = g_n(a, b)e^{-int}$.
  - Then $i\hbar\frac{\partial \psi_n}{\partial t} = i\hbar\left(-in \cdot g_n e^{-int}\right) = n\hbar\psi_n$.
  - On the other hand, $H \star \psi_n = H \star \left(g_n e^{-int}\right) = n\hbar g_n e^{-int} = n\hbar\psi_n$.
- So ψ_n *solves* the star-Schrödinger equation $i\hbar\frac{\partial \psi_n}{\partial t} = H \star \psi_n$.
- But there is a problem:
  - We found the energy levels to be $n\hbar$, for $n = 0, 1, 2, \ldots$
  - Standard quantum mechanics say $\left(n + \frac{1}{2}\right)\hbar$, for $n = 0, 1, 2, \ldots$
- A different star-product, called the Moyal star-product, corrects this:

$$\begin{aligned} f \star g = fg &+ \{f, g\}\frac{(\hbar/2)}{1!} \\ &+ \left(\frac{\partial^2 f}{\partial a^2}\frac{\partial^2 g}{\partial b^2} - 2\frac{\partial^2 f}{\partial a \partial b}\frac{\partial^2 g}{\partial a \partial b} + \frac{\partial^2 f}{\partial b^2}\frac{\partial^2 g}{\partial a^2}\right)\frac{(\hbar/2)^2}{2!} + \cdots \end{aligned}$$

Physics according to Star-Products

- Using functions like the classical Hamiltonian $H(a, b)$ and a suitable star-product, we obtain the energy levels of the quantum harmonic oscillator as star-eigenvalues of H.
  - Can match the ordinary eigenvalues of $\hat{H}$ in standard quantum mechanics.

- The basic star-product gave us $E_n = n\hbar$, which did not include the shift of $\frac{1}{2}$, but the Moyal star-product corrects this shift.

- This leads us to an interesting question:

Uniqueness of Quantum Mechanics

Different star-products determine different physics. What is the mathematical and physical significance of each one?