Optimization Assignment

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Problem Statement **- The normal to the curve $x^2 = 4y$ passing (1,2) is:**
(a)x+y=3 **(b)x-y=3**
(c)x+y=1 **(d)x-y=1**

Solution

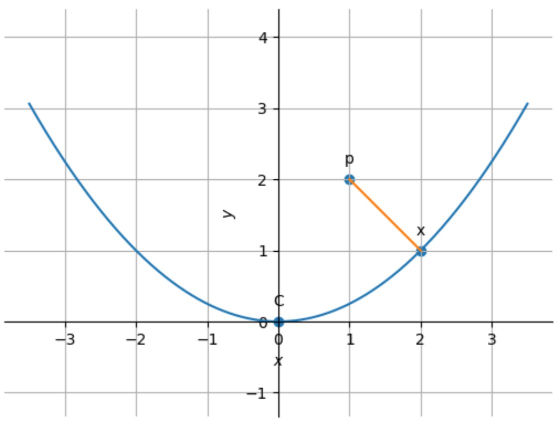

Figure 1: Normal to the curve $x^2 = 4y$

The given equation of parabola $x^2 = 4y$ can be written in the general quadratic form as

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \tag{3}$$

$$f = 0 \tag{4}$$

Any conic of the form $\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$

can be written as $\mathbf{x}^\top \mathbf{A}\mathbf{x} = 0$

$$\text{where } \mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

The distance from point $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ to the point 'x' on parabola is $\|\mathbf{x} - \mathbf{p}\|^2$

$$\implies \mathbf{x}^\top \mathbf{x} - 2\mathbf{p}^\top \mathbf{x} + \|\mathbf{p}\|^2$$

The above equation can be written as $\mathbf{x}^\top \mathbf{C}\mathbf{x}$

$$\text{where } \mathbf{C} = \begin{pmatrix} \mathbf{I} & -\mathbf{p} \\ -\mathbf{p}^\top & \|\mathbf{p}\|^2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

The shortest distance is given by, min $\mathbf{x}^\top \mathbf{C}\mathbf{x}$

such that, $\mathbf{x}^\top \mathbf{A}\mathbf{x} = 0$
Using SDR(Semi Definite Relaxation), it can be rewritten as
min $Tr\left(\mathbf{C}\mathbf{X}\right)$

Suc that, $Tr\left(\mathbf{A}\mathbf{X}\right) = 0$, $\mathbf{X} \geq 0$

Here , $\mathbf{X}$ is a 3×3 matrix of variables where
$\mathbf{X} = \mathbf{x}\mathbf{x}^\top$

Thus after solving we get the point on the given parabola as $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with the shortest distance from $\mathbf{p}$

Thus the points $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ satisfies the equation of the normal i.e. $x + y = 3$

Construction

Symbol	Value	Description
$\mathbf{p}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	Given point through which Normal is passing
$\mathbf{x}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	Foot of Normal