Sampling Distributions

Jeffrey Woo

School of Data Science, University of Virginia

Introduction

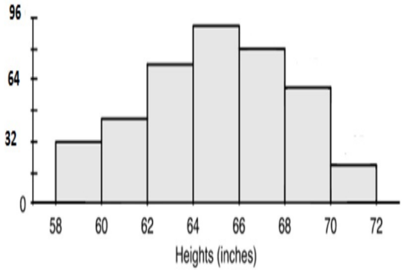
We will explore a few concepts in statistical theory that will allow us to assess how we can use data from our sample to make inferences about the larger population of interest.

1. Histograms and Probability Density Functions
2. Population and Samples
3. Sampling Distribution of Sample Mean

Histograms

Graphical tools are often used to summarize data to give us an idea about our data. For example, with quantitative data, we often use a histogram

Heights	Frequency	Relative Frequency
$58 \le x < 60$	32	0.08
$60 \le x < 62$	44	0.11
$62 \le x < 64$	72	0.18
$64 \le x < 66$	92	0.23
$66 \le x < 68$	80	0.20
$68 \le x < 70$	60	0.15
$70 \le x < 72$	20	0.05



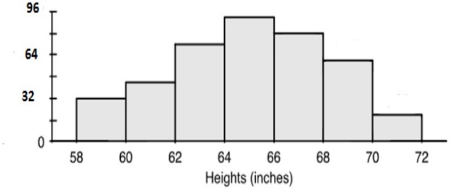
Histograms

With a histogram, we can have an idea about

- the center of the data
- the variability of the data
- the distribution of the data

The **distribution** of the histogram informs us the possible values of the variable of interest, as well as how often various values occur in our data.

Histograms

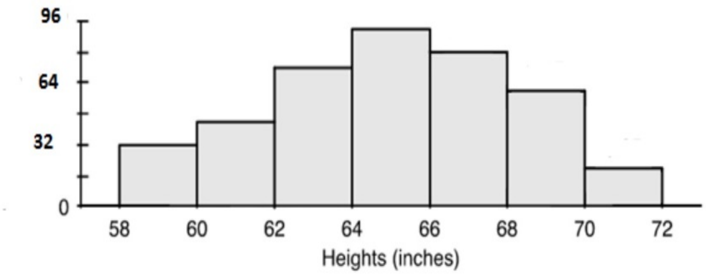


The distribution of this data

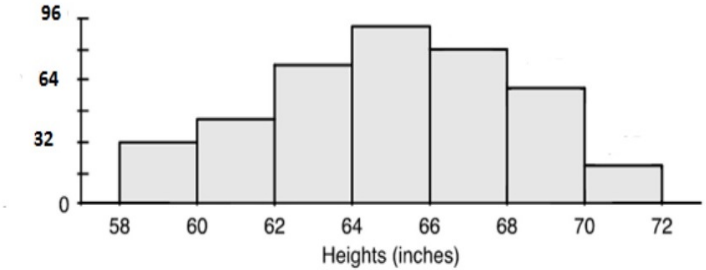
- The most common heights are around 64 to 66 inches.
- Heights are between 58 and 72 inches.
- As heights are further away 65 inches, they become less likely to occur.

Histograms

Question: Based only on this histogram, what are some ways you can estimate the proportion of girls who are at least 68 inches tall?



Histograms



We could also use a mathematical function to approximate the histogram, and use areas under the mathematical function to estimate proportions.

Probability Density Functions

These functions are called **probability density functions (pdf)**.
These functions must

- non-negative, and
- integrate to 1.

The density function is a mathematical representation of the distribution of the data.

Common Probability Density Functions

- Normal distribution.
- t distribution.
- χ^2 distribution.
- F distribution.

Normal distribution

A normal distribution is a symmetric, bell-shaped distribution. A normal distribution with mean μ and standard deviation σ is denoted by $N(\mu, \sigma)$. Its pdf is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}\exp(\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2)} \tag{1}$$

If (1) is a good approximation for the distribution of the data, we can estimate probabilities by integrating (1) over the relevant range(s).

Normal distribution

- A normal distribution with mean 0 and standard deviation 1 is called a **standard normal distribution**.
- It turns out that any normal distribution X with mean μ and standard deviation σ can be standardized by

$$Z = \frac{X - \mu}{\sigma}.$$

- Then Z follows a standard normal distribution.

1. Histograms and Probability Density Functions
2. Population and Samples
3. Sampling Distribution of Sample Mean

Motivation

In many studies, we want to get answers to questions regarding a population of interest. For example, what is the average annual income of American adults?

- Ideally, we would like to obtain the data from every single American adult.
- However, due to constraints (e.g. time and money), we are unable to obtain the data from every single American adult.
- We then typically collect data from a random sample of American adults.
- We then use the characteristics of the sample to estimate the characteristics of the population.

Population Vs Sample

- **Population**: The group of all items of interest in our study.
- **Sample**: The items from which we actually collect data on.

Population Vs Sample: Example

A manufacturing company produces 5 million parts. To estimate the proportion of parts that are defective, 300 parts are randomly selected and carefully inspected for defects. What is the

- population of interest?
- sample?

Parameters Vs Statistics

- A **parameter** is a number describing a characteristic of the population. Parameters are fixed values, but in practice we do not know their numerical values.
- A **statistic** is a number describing a characteristic of a sample. Statistics vary from sample to sample.

We often use a statistic to estimate an unknown parameter.

Variability in Statistics

Each time we take a random sample from a population, we are likely to get a different set of individuals and calculate a different statistic. There is **variability** in the statistics.

Question: Can we quantify this variability without having to obtain many different random samples?

Variability in Statistics

- If we take lots of random samples of the same size from a given population, the distribution of the sample statistics, **the sampling distribution**, will follow a predictable shape.
- Under some circumstances, the sampling distribution can be well-approximated by a specific distribution and its pdf.
- The variance of statistics generally decrease as the sample size increase.

1 Histograms and Probability Density Functions

2 Population and Samples

3 Sampling Distribution of Sample Mean

Sampling Distribution of Sample Mean

When a continuous variable, X, in a population follows a $N(\mu, \sigma)$ distribution, the sampling distribution of the sample mean, $\bar{x}$, for all possible samples of size n is $N(\mu, \frac{\sigma}{\sqrt{n}})$.

Central Limit Theorem

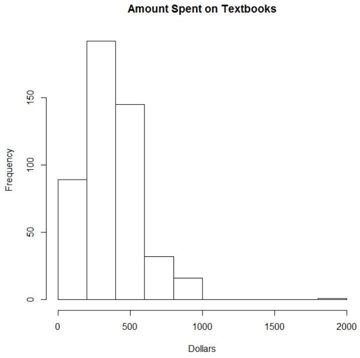
Consider a quantitative variable, X, in a population that has mean μ and standard deviation σ, and is not necessarily normally distributed. If n is **large enough**, the sampling distribution of the sample mean, $\bar{x}$, for all possible samples of size n is approximately $N(\mu, \frac{\sigma}{\sqrt{n}})$.

This is known as the **Central Limit Theorem**.

Implication: With a large enough sample size, we can use the normal distribution to find probabilities associated with sample means.

Worked Example: Textbook Spending

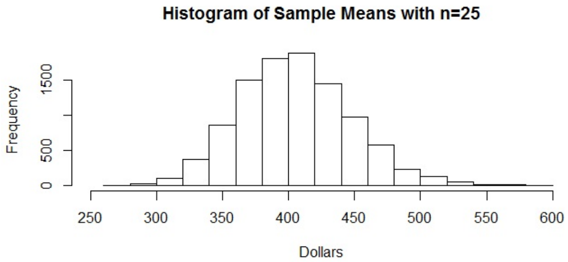
Question: Based on data from the Spring 2017 semester, the mean amount spent on textbooks for the semester is \$405.17 with standard deviation \$210.59. The histogram for the variable amount spent on textbooks that semester is displayed below. How would you describe the shape of this histogram?

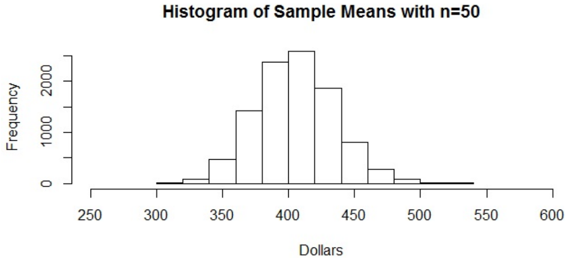


Worked Example: Textbook Spending

Question: Suppose we take repeated samples of size 25. What do we expect the sampling distribution for the sample mean to be? How about if we take repeated samples of size 50?

Worked Example: Textbook Spending





Worked Example: Textbook Spending

Question: Suppose I have a random sample of 25 students. What is the probability that the sample mean is less than $415? What if I have a random sample of 50 students instead?

Worked Example: Textbook Spending

Question: Suppose I have a random sample of 50 students. What is the probability that the sample mean is more than $400?

Where do we go from here?

- We know that the sample mean, $\bar{x}$, describes our particular sample. However, if we select another random sample, the sample mean will probably be different.
- We do know that with a large enough sample size, the distribution of the sample means can be approximated by a normal distribution.
- We also know that with a larger sample size, the sample means will be closer to the population mean, on average.

Reality: we will not know the value of the population mean, μ. So how do we use the sample mean, $\bar{x}$, to estimate μ?