Quantum Mechanics (A) - Assignment 5

October 9 , 2014

2.2 **证明：**

根据《量子力学教程》[2nd,曾谨言]-2.2.1 p31-32

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a}) \quad , \quad 0 < x < a \\ 0 \quad , \quad x < 0, x > a \end{cases}$$

则由期望值公式可得

$$\begin{aligned} \overline{x_n} &= \int_{-\infty}^{\infty} x|\psi(x)|^2 \mathrm{dx} \\ &= \int_0^a x(\frac{2}{a}sin^2(\frac{n\pi x}{a}))\mathrm{dx} \\ &= \frac{a}{2} \end{aligned}$$

$$\begin{aligned} \overline{x_n^2} &= \int_{-\infty}^{\infty} x^2|\psi_n(x)|^2 \mathrm{dx} \\ &= \int_0^a x^2(\frac{2}{a}sin^2(\frac{n\pi x}{a}))\mathrm{dx} \\ &= \frac{a^2}{3}(1 - \frac{3}{2n^2\pi^2}) \end{aligned}$$

由方差公式

$$\overline{(x - \overline{x})^2} = \overline{x^2} - \overline{x}^2 = \frac{a^2}{12}(1 - \frac{6}{n^2\pi^2})$$

***讨论$n \to \infty$的情况，并与经典力学计算结果比较.**

经典力学中，粒子在允许区内随机均匀分布，即$P(x) = \frac{1}{a}$

$$\overline{x} = \int_{-\infty}^{\infty} xP(x)\mathrm{dx} = \int_0^a x\frac{\mathrm{dx}}{a} = \frac{a}{2}$$

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2P(x)\mathrm{dx} = \int_0^a x^2\frac{\mathrm{dx}}{a} = \frac{a^2}{3}$$

$$\overline{(x-\overline{x})^2}=\overline{x^2}-\overline{x}^2=\frac{a^2}{12}$$

当$n\to\infty$时,由式(1)得

$$\lim_{n\to\infty}\overline{(x-\overline{x})^2}=\lim_{n\to\infty}\frac{a^2}{12}(1-\frac{6}{n^2\pi^2})=\frac{a^2}{12}$$

与经典力学计算结果一致

***计算Δp和$\Delta x\cdot\Delta p$.**

根据变换公式可得

$$\begin{aligned}\varphi_n(p) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi_n(x)e^{-ipx/\hbar}\mathrm{dx}\\ &= \frac{1}{\sqrt{2\pi\hbar}}\int_0^a\sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})\cos(\frac{p}{\hbar}x)\mathrm{dx}\\ &= n\sqrt{\pi a\hbar^3}\frac{1-(-1)^n\cos(\frac{ap}{\hbar})}{n^2\pi^2\hbar^2-a^2p^2}\end{aligned}$$

代入公式计算期望值

$$\overline{p_n}=\int_{-\infty}^{\infty}p|\varphi_n(p)|^2\mathrm{dp}=0$$

$$\overline{p_n^2}=\int_{-\infty}^{\infty}p^2\cdot|\varphi_p|^2\mathrm{dp}=\frac{3n^2\pi^2\hbar^2}{a^2(1-6/n^2\pi^2)}$$

$$\Delta p=\sqrt{\overline{x-\overline{x}}^2}=\sqrt{\overline{x^2}-\overline{x}^x}=\frac{n\pi\hbar}{a}\sqrt{\frac{3}{n^2\pi^2-6}}$$

$$\Delta x\cdot\Delta p=\sqrt{\frac{a^2}{12}(1-\frac{6}{n^2\pi^2})}\cdot\frac{n\pi\hbar}{a}\sqrt{\frac{3}{n^2\pi^2-6}}=\frac{\hbar}{2}$$

2.3 **解:**

根据《量子力学教程》[2nd,曾谨言]-2.2.1 p32-33

$$\text{基态}\quad\psi_1(x)=\sqrt{\frac{2}{a}}\cos(\frac{\pi x}{a})$$

概率密度分布函数

$$\begin{aligned}\varphi_1(p) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi_n(x)e^{-ipx/\hbar}\mathrm{dx}\\ &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-a/2}^{a/2}\sqrt{\frac{2}{a}}\cos(\frac{\pi x}{a})\cos(\frac{px}{\hbar})\mathrm{dp}\\ &= 2\sqrt{\pi a\hbar^3}\cdot\frac{\cos(pa/2\hbar)}{\pi^2\hbar^2-p^2a^2}\end{aligned}$$

概率分布函数

$$|\varphi_1(p)|^2 = 2\pi a\hbar^3\cdot\frac{\cos^2(pa/2\hbar)}{(\pi^2\hbar^2-p^2a^2)^2}$$

2.4 **解：**

(a) 根据归一化条件

$$\int_{-\infty}^{\infty}|\psi(x)|^2\mathrm{dx} = \int_0^a[Ax(a-x)]^2\mathrm{dx} = 1 \quad 即 \quad A=\sqrt{\frac{30}{a^5}}$$

(b) $\psi(x)$用$\psi_n(x)$展开

$$\psi(x) = \sum C_n\psi_n(x)$$

$$\begin{aligned}C_n &= \int\psi_n^*(x)\psi(x)\mathrm{dx} = \int_0^a\sin(\frac{n\pi x}{a})\sqrt{\frac{30}{a^5}}x(a-x)\mathrm{dx}\\ &= \frac{4\sqrt{15}}{n^2\pi^3}[1-(-1)^n]\\ P_n &= |C_n|^2 = \frac{240}{n^6\pi^6}[1-(-1)^n]^2\end{aligned}$$

特别的，基态$P_1 = \dfrac{960}{\pi^6}\approx 0.99855.$

(c) 作图比较$\psi_1(x)$和$\psi(x)$

由图可见，$\psi_1(x)$和$\psi(x)$曲线基本重合，
又简单计算比较有:

$$\frac{P_n}{P_1}(n\neq 1) < \frac{1-P_1}{P_1} = \frac{1-0.99855}{0.99855}\approx 1.4\times 10^{-3}\ll 1 即 P_n|_{n\neq 1}\ll P_1$$

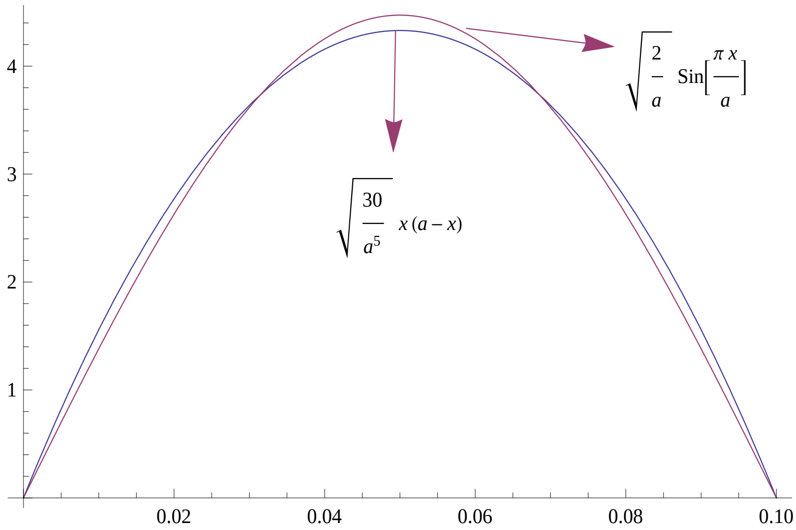

Figure 1: $\psi(x,0) vs \psi_1(x)$

由此可知$\psi_1(x)$和$\psi(x)$几乎一致。

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***计算无限深势阱中的δp**

解：

根据2.2的计算结果已知

$$\varphi_n(p) = n\sqrt{\pi a\hbar^3}\frac{1-(-1)^n\cos(\frac{ap}{\hbar})}{n^2\pi^2\hbar^2 - a^2p^2}$$

讨论$n^2\pi^2\hbar^2 - a^2p^2 = 0$ 即$p = n\pi\hbar/a$的情况。根据L.Hospital法则可知$\lim\limits_{p\to n\pi\hbar/a}\varphi(p) = 0$，故$\varphi(p)$在$(-\infty,\infty)$上连续无奇点。

令$\varphi(p) = 0$计算的节点p_k(k为非负整数)

$$p_k = \begin{cases} 2m\pi\hbar/a, & n = 2k \\ (2m+1)\pi\hbar/a, & n = 2k+1 \end{cases} \quad \text{(m 为整数)}$$

由Figure 2可知，$\varphi(p)$最大值在$p = \pm n\pi\hbar/a$附近，故

$$\delta p = |p_k - p_{k-1}| = |p_{-k} - p_{-k+1}| = \frac{2\pi\hbar}{a}$$

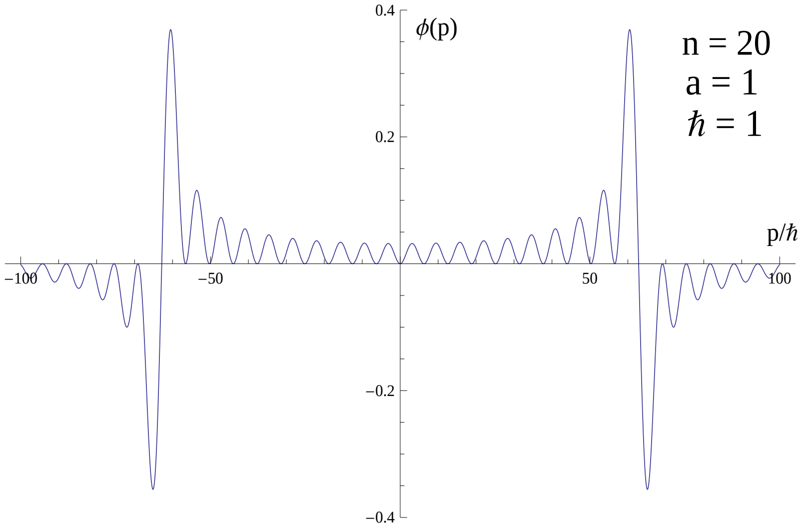

Figure 2: $\varphi(p)$

2.5 **解：**

根据2.2中结果，将$a \to 2a$可得

$$\psi_n(x) = \sqrt{\frac{1}{a}}\sin(\frac{n\pi x}{2a}) \quad , \quad E_n = \frac{n^2\pi_2\hbar^2}{8ma^2}$$

故$\psi(x,0)$不是能量本征态，但恰有$E_1 = \dfrac{\pi^2\hbar^2}{2ma^2} = \epsilon_2$，将$\psi(x,0)$用$\psi_n(x)$展开,即$\psi(x,0) = \sum C_n\psi_n(x)$,其中：

$$\begin{aligned} C_n &= \int \psi_n^*(x)\psi(x)\mathrm{dx} = \int_0^a \sqrt{\frac{2}{a}}\sin(\frac{\pi x}{a})\sqrt{\frac{1}{a}}\sin(\frac{n\pi x}{2a})\mathrm{dx} \\ &= \frac{4\sqrt{2}}{(4-n^2)\pi}\sin(\frac{n\pi}{2}) \quad (\psi(x,0) = 0, x < 0 \bigcup x > a) \end{aligned}$$

求测得粒子能量仍为E_1的概率

$$P = |C_2|^2 = |\lim_{n\to 2} C_n|^2 = \frac{1}{2}.$$

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