21 Gravitational Fields

21.1 Gravitational Field Strength

Any two masses exerts a gravitational pull on each other, but usually the force is too weak to be noticed unless at least one of the masses is very large.

The force field around a mass is called a **gravitational field strength**.

1. The mass of an object **creates a force field** around itself.
2. Any other mass placed in the field is **attracted towards the object**.
3. The second mass also has a force field around itself that **pulls on the first object** with an equal force in the opposite direction.

The **magnetic field strength** g is the force per unit mass on a small test mass placed in the field. The path which the smaller mass would follow is called a **field line** or **line of force**.

$$g = \frac{F}{m}$$

The unit of gravitational field strength is the **newton per kilogram**.

The test mass needs to be small, otherwise it might pull so much on the other object that it changes its position and alters the field.

Free Fall

The weight of an object is the force of gravity on it, $F = mg$ for an object of mass m in a gravitational field.

$$a = \frac{F}{m} = \frac{mg}{m} = g$$

Therefore g is the acceleration of a freely falling object.

The object is described as **unsupported** because it is acted on by the force of gravity alone.

Field Patterns

- A **radial field** is where the field lines are **always directed to the centre**, since the force of gravity on a small mass near a much bigger spherical mass is always directed to the centre of the larger mass, regardless of position.

  The magnitude of g in a radial field decreases with increasing distance from the massive body.

- A **uniform field** is where the gravitational field strength is the **same in magnitude and direction throughout the field**.

  The field lines are therefore **parallel** to one another and **equally spaced**.

The gravitational field strength of the Earth is radial because it falls with increasing distance. But **over small distances** compared to the Earth's radius, the change in gravitational field strength is insignificant so the field can be **considered uniform**.

21.2 Gravitational Potential

Gravitational potential energy is the energy of an object due to its position in a gravitational field.

- The position for zero GPE is at infinity - where the object is so far away that that **gravitational force on it is negligible**.
- At the surface, the GPE is negative as **work needs to be done** to escape from the field completely.

The **gravitational potential** at a point is the **work done per unit mass** to move a small test mass from infinity to that point.

$$V = \frac{W}{m}$$

The unit of gravitational potential is $\mathrm{J\,kg^{-1}}$.

The work done to move a mass from V_1 to V_2 is equal to its **change of gravitational potential energy**.

$$\Delta W = m\Delta V$$

- $\Delta E_p = mg\Delta h$ can only be applied for values of Δh that are very small compared with the Earth's radius.
- $\Delta E_p = m\Delta V$ can always be applied.

Potential Gradients

Equipotentials are surfaces of **constant potential**, so **no work needs to be done** to move along an equipotential surface.

- The equipotentials near the Earth are **circles**.
- At increasing distance from the surface, the gravitational field becomes weaker, so the **gain of GPE per metre height** becomes less.
- Away from the Earth's surface, the equipotentials for equal increases of potential are **spaced further apart**.

But near the surface **over a small region**, the equipotentials are horizontal. This is because the gravitational field over a small region is uniform.

The **potential gradient** at a point in a gravitational field is the **change of potential per metre** at that point.

For a change of potential ΔV over a small distance Δr

$$\text{Potential gradient} = \frac{\Delta V}{\Delta r}$$

Consider a test mass m being moved away from a planet. To move m by a small distance Δr in the opposite direction to the gravitational force F_{grav} on it, its gravitational potential energy is increased

- By an equal and opposite force F acting through the distance Δr.
- By an equal amount of energy equal to the work done by F

$$\Delta W = F\Delta r$$

$$\begin{aligned}
\Delta V &= \frac{F\Delta r}{m} \\
F &= \frac{m\Delta V}{\Delta r} \\
F_{\text{grav}} &= -F \\
g &= \frac{F_{\text{grav}}}{m} = -\frac{\Delta V}{\Delta r}
\end{aligned}$$

So gravitational field strength is the **negative of the potential gradient**. Where the minus sign shows that g acts in the opposite direction to the potential gradient.

- The closer the equipotentials are, the greater the potential gradient and the **stronger the field** is.
- Where the equipotentials show equal changes of potential for equal changes of spacing, the **potential gradient is constant**, so the gravitational field strength is constant and the field is **uniform**.
- The gradient is always at **right angles to the equipotentials**, so the field lines are always perpendicular to equipotentials.

21.3 Newton's Law of Gravitation

Kepler's third law states that the value of $\frac{r^3}{T^2}$ is the same for all planets.

Newton's law of gravitation states that the gravitational force between any two **point objects** is

- Always an **attractive** force.
- Proportional to the mass of each object.
- Proportional to $\frac{1}{r^2}$, where r is their distance apart.

The last two requirements can be summarised as

$$F_g = \frac{Gm_1m_2}{r^2}$$

Where G is the **universal constant of gravitation**.

$$G = \frac{Fr^2}{m_1m_2}$$

The value of G is $6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$

21.4 Planetary Fields

Newton's law of gravitation can be used to determine the gravitational field strength at any point in the field of **any spherical mass -** the field of a spherical mass is the same as if the mass were concentrated at its centre.

This is because the field lines of a spherical mass are **always directed towards the centre**, so the field pattern is just the same as for a point mass.

- For a **point mass** M, the magnitude of the force of attraction on a test mass m at distance r from M is given by Newton's law of gravitation.

  $$F = \frac{GMm}{r^2}$$

  The magnitude of the **gravitational field strength** at distance r is

  $$g = \frac{F}{m} = \frac{GM}{r^2}$$

- For a **spherical mass** M **and radius** R, the force of attraction on a test mass at distance r from M is the same as if mass M is concentrated at its centre.

  $$F = \frac{GMm}{r^2}$$

  So the magnitude of gravitational field strength

  $$g = \frac{F}{m} = \frac{GM}{r^2}$$

  Provided $r \geq R$

The **surface gravitational field strength** $g_s = \dfrac{GM}{R^2}$, so $g = \dfrac{g_s R^2}{r^2}$, for $r \geq R$ the curve is an **inverse-square law** curve because g decreases in inverse proportion to r^2.

For $r < R$, $g \propto r$, more specifically

$$g = \frac{4\pi G\rho r}{3}$$

Escape Velocity

The gravitational potential on Earth's surface is $-63\text{MJ}\,\text{kg}^{-1}$, meaning 63MJ of work needs to be done to remove a 1kg mass from the surface of the Earth to infinity.

The **escape velocity** from a planet is the minimum velocity an object must be given to escape from the planet when projected vertically from the surface.

1. The potential at the surface of a planet is $V = -\dfrac{GM}{r}$
2. To move an object of mass m from the surface to infinity, the work that must be done is $\Delta W = m\Delta V = \dfrac{GMm}{R}$

3. If the object is projected at speed v, for it to be able to escape from the planet, its **initial kinetic energy**

$$\frac{1}{2}mv^2 \geq m\Delta V$$

Therefore

$$v^2 \geq \frac{2GM}{R}$$
$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

Substituting $g = \dfrac{GM}{R^2}$

$$v_{\text{esc}} = \sqrt{2gR}$$

The work under the curve of g against r represents the work done to move a 1kg mass from infinity to the surface. Therefore the area under the curve gives the value of **gravitational potential** at the surface.

21.5 Satellite Motion

Any small mass that orbits a larger mass is a **satellite**.

1. The **gravitational attraction** between each planet and the Sun is the **centripetal force** that keeps the planet on its orbit.
2. The gravitational force is given by $\dfrac{GMm}{r^2}$.
3. The **gravitational field strength** therefore equals **centripetal acceleration**.

$$\frac{v^2}{r} = \frac{GM}{r^2}$$
$$v^2 = \frac{GM}{r}$$

4. Since $v = \dfrac{2\pi r}{T}$, we have

$$\left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r}$$
$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

Because $\dfrac{GM}{4\pi^2}$ is the same for all the planets, then $\dfrac{r^3}{T^2}$ is the same for all the planets.

Therefore Kepler's third law can be proved by assuming the force of attraction varies with distance according to the **inverse-square law**.

Geostationary Satellites

A geostationary satellite orbits the Earth **directly above the equator** and has a **time period of 24h**. It therefore remains in a **fixed position above the equator** because it has the same time period as the Earth's rotation.

The radius of orbit of a geostationary satellite can be calculated using the equation

$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

Energy of Satellite

Since the speed of a satellite is given by $v^2 = \dfrac{GM}{r}$

$$E_k = \frac{1}{2}mv^2 = \frac{GMm}{2r}$$
$$E_p = mV = -\frac{GMm}{r}$$

Therefore the **total energy** of the satellite

$$E = E_p + E_k = -\frac{GMm}{2r}$$