18 Simple Harmonic Motion

18.1 Oscillations

The **equilibrium position** is the position with the **least potential energy** - it is where the oscillation will eventually come to a standstill. The motion is described as **oscillating about equilibrium**.

- An **object on a spring** moves up and down repeatedly.
- A **pendulum** moving back and forth.
- A boat rocking from side to side.

An oscillating object moves repeatedly **one way then in the opposite direction** through its equilibrium position.

The **displacement** of the object is its distance and direction from equilibrium. It **continually changes** during the motion, in **one full cycle** after being **released from a non-equilibrium position**.

1. **Decreases** as it returns to equilibrium,
2. **Reverses and increases** as it moves away from equilibrium in the opposite direction.
3. Decreases as it returns to equilibrium.
4. Increases as it moves away from equilibrium towards its starting position.

- The **amplitude** of the oscillations is the **maximum displacement** of the oscillating object from equilibrium.
- If the **amplitude is constant** and **no frictional forces** are present, the oscillations are described as **free vibrations**.
- The **time period** of the oscillating motion is the time for **one complete cycle of oscillation**.
- One **full cycle** after passing through any position, the object passes through that same position in the same direction.
- The **frequency** of oscillations is the **number of cycles per second** made by an oscillating object.

  The unit of frequency is the **hertz** (Hz), which is one cycle per second.
- The **angular frequency** of the oscillating motion is defined as

$$\omega = \frac{2\pi}{T}$$

The **phase difference** of two oscillating objects stays the same if they oscillates with the same frequency.

If Δt is the time between successive instants when the two objects are at **maximum displacement in the same direction**, one object is always $\frac{\Delta t}{T}$ cycles ahead. So for two objects oscillating in the same frequency.

$$\text{Phase difference} = \frac{2\pi\Delta t}{T} = 2\pi\Delta t f$$

18.2 The Principles of Simple Harmonic Motion

The velocity of an object is given by the **gradient of the displacement-time graph**.

- The **magnitude of the velocity** is greatest when the object is at at **zero displacement when the object passes through equilibrium**.
- The **velocity is zero** when the object is at **maximum displacement** in either direction.

The acceleration of an object is given by the **gradient of the velocity-time graph**.

- The **acceleration is greatest** when the **velocity is zero** at maximum displacement in the opposite direction.
- The **acceleration is zero** when the displacement is zero and the velocity is a maximum.

Conditions for Simple Harmonic Motion

Simple harmonic motion is defined as oscillating motion in which acceleration is

- Proportional to the displacement, and
- Always in the **opposite direction to the displacement**

$$a \propto -x$$

or in other words, $a = -kx$.

The acceleration for simple harmonic motion is

$$a = -\omega^2 x$$

18.3 More about Sine Waves

Consider an object P in **uniform circular motion**.

1. From the centre of the circle, the coordinates of P is $x = r\cos\theta$ and $y\sin\theta$, where θ is the angle between the x-axis and the radial line OP.
2. Consider the **shadow of the ball** alongside the **shadow of a pendulum bob**, the two shadows keep up with each other exactly when their time periods are matched.
3. So they have the **same horizontal motion**, and the **same horizontal acceleration**.

$$\text{Acceleration of the ball} = -\omega^2 r$$

4. The **horizontal component of acceleration of the ball** is

$$a_x = a\cos\theta = -\omega^2 r\cos\theta$$

5. Since $x = r\cos\theta$

$$\textbf{Acceleration of the pendulum bob } a_x = -\omega^2 r$$

If $x = A$ and $v = 0$ when $t = 0$

$$x = A\cos\omega t$$

18.4 Applications of Simple Harmonic Motion

For any oscillating object, the resultant force is described as a **restoring force** because it always **acts towards the equilibrium**.

The frequency of an oscillating object can be reduced by

- **Adding extra mass** increases the **inertia** of the system. At a given displacement there will be **less acceleration** than if the extra mass is not added. So each cycle of oscillation would therefore be longer.
- **Use weaker springs** so the restoring force at any given displacement would be less, and the **acceleration and speed at any given displacement would be less**. So each cycle of oscillation would therefore be longer.

Mass-spring System

Consider an object of mass m attached to a spring.

1. Assuming the spring obeys **Hooke's law**, the tension in the spring

$$T = k\Delta L$$

2. When the object is at displacement x from its equilibrium position, the **change in tension** from its equilibrium position is

$$\Delta T = -kx$$

   Where the minus sign represents the fact that change of tension always tries to **restore the object to its equilibrium position**.

3. So the **restoring force** on the object $F = -kx$.
4. An acceleration

$$a = \frac{-kx}{m}$$

The equation can be written in form $a = -\omega^2 x$ where $\omega^2 = \dfrac{k}{m}$.

So time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Simple Pendulum

Consider a simple pendulum of a bob of mass m attached to a thread of length L.

1. At displacement s from the lowest point, the thread is at angle θ to the vertical.
2. Weight mg has components.
   - Perpendicular to path of bob: $mg\cos\theta$
   - Along the path towards the equilibrium position: $mg\sin\theta$, which is the **restoring force**.
3. The acceleration is
$$a = \frac{-mg\sin\theta}{m} = -g\sin\theta$$
4. As long as θ does not exceed $10°$, $\sin\theta \approx \frac{s}{L}$
$$a - \frac{g}{L}s = -\omega^2 s$$
where $\omega^2 = \dfrac{g}{L}$

So time period
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

18.5 Energy and Simple Harmonic Motion

A **freely oscillating object** oscillates with a **constant amplitude** because there is no friction acting on it - the only forces acting on it combine to provide the **restoring force**.

Velocity of Simple Harmonic Motion

Consider a small object of mass m oscillating on a spring

1. The **potential energy** changes with displacement x from equilibrium
$$E_P = \frac{1}{2}kx^2$$
2. The **total energy** of the system is equal to its maximum kinetic energy.
$$E_T = \frac{1}{2}kA^2$$
3. The total energy is $E_T = E_P + E_K$, so
$$E_K = E_T - E_P = \frac{1}{2}k(A^2 - x^2)$$

4. Using $E_K = \frac{1}{2}mv^2$.

$$\frac{1}{2}mv^2 = \frac{1}{2}k(A^2 - x^2)$$
$$v^2 = \frac{k}{m}(A^2 - x^2)$$
$$v = \pm\omega\sqrt{A^2 - x^2} \qquad \text{where } \omega = \frac{k}{m}$$

Damped Oscillations

Oscillations die away because **air resistance** gradually **reduces the total energy** of the system.

- In any oscillating system where **friction** or **air resistance** is present, the amplitude decreases.
- These forces are described as **dissipative forces** because they dissipate energy of the system to the surroundings as thermal energy.
- The motion is said to be **damped** if dissipative forces are present.

Light damping occurs when the **time period is independent of the amplitude** so each cycle takes the same length of time as the oscillations die away.

- The displacement of a lightly damped oscillating system **decreases with time**.
- The amplitude **reduced by the same fraction** each cycle.

Critical damping is just enough to stop the system from oscillating after it has been displaced from equilibrium and released.

- The oscillating system returns to equilibrium in the **shortest possible time** without overshooting.

Heavy damping is when the damping is so strong the displaced object **returns to equilibrium more slowly** than if the system is critically damped. **no oscillating motion** occurs.

18.6 Forced Vibrations and Resonance

- A **periodic force** is a force applied at regular intervals.
- When a system oscillates **without a periodic force** being applied to it, the system's frequency is called its **natural frequency**.
- The system undergoes **forced vibrations** when a periodic force is applied to it.

Investigating Forced Vibrations

A fixed mass is attached to **stretched strings**.

- The bottom end of the bottom spring is attached to a **mechanical oscillator** connected to a signal generator.
- The top end of the top spring is fixed.

The mechanical oscillator **pulls repeatedly** on the lower string, and its frequency is called the **applied frequency**. As the applied frequency increases

1. The amplitude of oscillations of the system **increases until it reaches a maximum amplitude** at a particular frequency, and then the amplitude decreases again.
2. The **phase difference** between the displacement and the periodic force increases from zero to $\frac{\pi}{2}$ at the maximum amplitude, then from $\frac{\pi}{2}$ to π as the frequency increases further.

Resonance

When the system is oscillating at the maximum frequency, the periodic force is **exactly in phase with the velocity of the oscillating system**, and the system is in **resonance**.

The frequency at the maximum amplitude is called the **resonant frequency**.

The larger the damping

- The **larger the maximum amplitude** becomes at resonance.
- The closer the resonant frequency to its **natural frequency** of the system.

As the **applied frequency becomes increasingly larger** than the resonant frequency of the mass-spring system.

- The **amplitude decreases** more and more.
- The **phase difference** between the displacement and the periodic force increases from $\frac{\pi}{2}$ until the displacement is π radians out of phase with the periodic force.

For an oscillating system with **little to no damping**.

$$\text{Resonant frequency} = \text{natural frequency of the system}$$

Barton's Pendulums

Simple pendulums of different lengths hangs from a support thread stretched between two fixed points.

1. A **driver pendulum** is displaced and released so it oscillates in a plane perpendicular to the plane of the pendulums at rest.
2. The effects of the oscillating motion of the driver is **transmitted along the support thread**.
3. Each of the other pendulums are subjected to **forced oscillations**.
4. The pendulum with the **same length as the driver** responds much more than any other pendulum.

- This is because it has the **same length** and therefore the **same time period** as the driver.
- So its natural frequency is the **same as the natural frequency** of the driver.
- Therefore it **oscillates in resonance with the driver** because it is subjected to the same frequency as its own natural frequency.

The oscillations of each other pendulum **depends on how close its length** is to the length of the driver.

Bridge Oscillations

A bridge can oscillate because of its **springiness** and its **mass**.

A bridge not fitted with dampers can be made to oscillate at resonance if the bridge span is **subjected to a suitable periodic force**.