Vectors

Basic Definitions

Definitions

- A **scalar** is a real value: value $\in \mathbb{R}$
- A **vector** has a magnitude and direction: magnitude $\times$ direction
- A **position vector** gives position relative to origin.
- A **displacement vector** gives the relation between two points.
- **Euclidean space** is where the shortest path between any two points is a straight line, and parallel lines are possible.

Vectors can have up to ∞ dimensions (used in quantum mechanics), this course focuses on vectors in 3 dimensions.

Notation

Vector type	Notation
Displacement vector	$\overrightarrow{AB}$
Position vector	$\overrightarrow{OA}$
Magnitude vector	$\lvert\boldsymbol{v}\rvert$
Unit vector	$\hat{\boldsymbol{v}}$

Definition

The unit vector $\hat{\boldsymbol{v}}$ is a vector of unit length in the direction of $\boldsymbol{v}$.

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

The 3D Euclidean space has 3 components, so 3 numbers are required to specify the vector.

$$\boldsymbol{v} = \left(v_x, v_y, v_z\right)$$

The components also depends on the axes chosen.

Basic Vector Operations

- $\boldsymbol{a} + \boldsymbol{b}$ adds their geometric displacement.
- $\lambda \cdot \boldsymbol{v}$ where $\lambda \in \mathbb{R}$ gives a vector that is:
  - Parallel to the original vector.
  - The length is scaled by λ

Vector Properties

- Vector addition is **commutative**: $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$
- Vector addition is **associative**: $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$
- Vector multiplication is **distributive**: $\lambda \cdot (\boldsymbol{a} + \boldsymbol{b}) = \lambda \cdot \boldsymbol{a} + \lambda \cdot \boldsymbol{b}$

This are non-trivial properties! Group theory studies these properties. Vector subtraction for example, is not associative.

Coordinate Systems

For vector components, we need to know the orientation of axes, but not the location of the origin.

Definition
A **coordinate system** is a selection of terminal axes:
- Axes of unit length
- An origin

The Cartesian Coordinate System
Axes are **mutually perpendicular**.

By convension we use a **right-handed coordinate system**.
- There are two ways of defining a Cartesian coordinate system.
- We use the right hand as a convension so the coordinate is uniquely defined.
- $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ are in the direction of the 1st, 2nd and 3rd finger of the right hand.
- Emanate for fixed origin O.

The coordinate of a point P relative to the axes are denoted by the **length of the projections** of the vector $\overrightarrow{OP}$ onto the 3 axes, written in form $\overrightarrow{OP} = (x, y, z)$.

The **magnitude** is given by Pythagoras' theorem:

$$|\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$$

The distance between two points is given by $|\boldsymbol{r_1} - \boldsymbol{r_2}|$

Basis Vectors
The basis vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ **spans** the space because it provides a way of accessing every point.

Equations of a Line
Our goal is to parameterise all the points on a line.
- Take $\boldsymbol{a}$ as a **reference vector**.
- The point is a scalar multiple of the direction vector $\hat{\boldsymbol{u}} = \frac{\boldsymbol{b}-\boldsymbol{a}}{|\boldsymbol{b}-\boldsymbol{a}|}$

$$\boldsymbol{r} = \boldsymbol{a} + \lambda|\boldsymbol{b} - \boldsymbol{a}|\frac{\boldsymbol{b} - \boldsymbol{a}}{|\boldsymbol{b} - \boldsymbol{a}|} = \boldsymbol{a} + \lambda\hat{\boldsymbol{t}}$$

where $\hat{\boldsymbol{t}}$ is the unit vector in the direction of $\boldsymbol{b} - \boldsymbol{a}$.

Component Form

$$\boldsymbol{r} = (x, y, z) = \left(a_x + \lambda(b_x - a_x), a_y + \lambda\left(b_y - a_y\right), a_z + \lambda(b_z - a_z)\right)$$

Rearrange to give

$$\lambda = \frac{x - a_x}{b_x - a_x} = \frac{y - a_y}{b_y - a_y} = \frac{z - a_z}{b_z - a_z}$$

The Scalar Product
Given two vectors $\boldsymbol{a}$, $\boldsymbol{b}$, there are different ways of taking the product.
- $\boldsymbol{a} \cdot \boldsymbol{b} \in \mathbb{R}$
- $\boldsymbol{a} \cdot \boldsymbol{b} \in \text{pseudovector}$
- $\boldsymbol{a} \otimes \boldsymbol{b}$ - the tensor product: two vectors of dimension m and n gives a vector of dimension $m \times n$

Definition
$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos\theta$

Scalar Product Properties

- Commutative: $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- Distributive: $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$
- $\boldsymbol{a} \cdot \boldsymbol{0} = 0$

Proof of Cosine Rule

Consider a triangle of with sides represented by vector $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$

$$
\begin{aligned}
|\boldsymbol{c}|^2 &= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta \\
|\boldsymbol{a} - \boldsymbol{b}|^2 &= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2 \cdot \boldsymbol{a} \cdot \boldsymbol{b} \\
(\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) &= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{b} - 2 \cdot \boldsymbol{a} \cdot \boldsymbol{b} \quad \text{(true by distributivity)}
\end{aligned}
$$

Scalar Product for Cartesian Vectors

Product for vectors are 0 if they are orthogonal to each other.

- $\boldsymbol{a} = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}$
- $\boldsymbol{b} = b_x\hat{\boldsymbol{\imath}} + b_y\hat{\boldsymbol{\jmath}} + b_z\hat{\boldsymbol{k}}$

Since most terms evaluates to 0.

$$
\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{b} = {} & a_x\hat{\boldsymbol{\imath}} \cdot b_x\hat{\boldsymbol{\imath}} + a_x\hat{\boldsymbol{\imath}} \cdot b_y\hat{\boldsymbol{\jmath}} + a_x\hat{\boldsymbol{\imath}} \cdot b_z\hat{\boldsymbol{k}} \\
& + a_y\hat{\boldsymbol{\jmath}} \cdot b_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} \cdot b_y\hat{\boldsymbol{\jmath}} + a_y\hat{\boldsymbol{\jmath}} \cdot b_z\hat{\boldsymbol{k}} \\
& + a_z\hat{\boldsymbol{k}} \cdot b_x\hat{\boldsymbol{\imath}} + a_z\hat{\boldsymbol{k}} \cdot b_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}} \cdot b_z\hat{\boldsymbol{k}} \\
= {} & a_x b_x + a_y b_y + a_z c_z
\end{aligned}
$$

Component Vector in Direction

The scalar product projects a vector onto another vector: $\hat{\boldsymbol{b}} \cdot (\boldsymbol{a} \cdot \boldsymbol{b})$ projects $\boldsymbol{a}$ to $\boldsymbol{b}$.

This is useful for changing the coordinate system: *what is the component in an axis?*

Equation of a Plane

Normal + Point/Distance

We need a vector orthogonal to the plane $\boldsymbol{n}$, as there is only one direction that is orthogonal to the plane. If we know one point A on the plane, then the plane is **uniquely specified**.

Let $\boldsymbol{r}$ be a point on the plane.

$$
\begin{aligned}
r \in \text{plane} &\iff (\boldsymbol{r} - \boldsymbol{a}) \perp \boldsymbol{n} \\
&\iff (\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0 \\
&\iff \boldsymbol{r} \cdot \hat{\boldsymbol{n}} = p
\end{aligned}
$$

Three Reference Points

If we know reference points A, B and C on the plane, where they are not colinear.

- $\overrightarrow{AB}$ and $\overrightarrow{AC}$ must be entirely within the plane.
- The plane is any point that can be represented as a linear combination of the two vectors.

The plane is given by

$$\boldsymbol{r} - \boldsymbol{a} = \mu(\boldsymbol{b} - \boldsymbol{a}) + \nu(\boldsymbol{b} - \boldsymbol{a})$$

The plane is therefore entirely specified by giving 3 points.

Equation of Other Objects

The goal is the parameterise every point on the surface of the object. For each parameterisation you can change the center by setting $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{a}$.

Sphere

Defining property: all points are a fixed distance away from the center.

$$|\boldsymbol{r}| = \rho$$

Cylinder

Defining property: all points are R away from the centre line.
- A vector $\boldsymbol{r}$ is projected to $\hat{\boldsymbol{n}}$ at $(\boldsymbol{r} \cdot \hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{n}}$
- The vector from $\boldsymbol{r}$ to the projected vector is $\boldsymbol{r} - (\boldsymbol{r} \cdot \hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{n}}$

So the equation is

$$|\boldsymbol{r} - (\boldsymbol{r} \cdot \hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{n}}| = R$$

Cone

Defining property: take any point A on the cylinder given by $\boldsymbol{r}$, the angle between $\overrightarrow{OA}$ and $\boldsymbol{n}$ is θ.

$$\boldsymbol{r} \cdot \hat{\boldsymbol{n}} = |\boldsymbol{r}| \cos\theta$$

The Vector Product

The vector product is an easy formula to calculate the normal of the plane.

$$\boldsymbol{a} \wedge \boldsymbol{b} = \text{pseudovector}$$

Also denoted $\boldsymbol{a} \times \boldsymbol{b}$.

Definition
$\boldsymbol{a} \wedge \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta \cdot \widehat{\boldsymbol{m}}$

where $\widehat{\boldsymbol{m}}$ is the unit vector orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

We use the **right-handed system** so the vector is uniquely defined: $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$ are the direction of the first, second and third finger.

Properties

- Distributive: $\boldsymbol{a} \wedge (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \wedge \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{c}$
- If $\boldsymbol{a} \perp \boldsymbol{b}$, then $|\boldsymbol{a} \wedge \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|$ by definition, similar for $\boldsymbol{a} \parallel \boldsymbol{b} \implies |\boldsymbol{a} \wedge \boldsymbol{b}| = 0$

But it is **not**
- Commutative: $\boldsymbol{a} \wedge \boldsymbol{b} = -\boldsymbol{b} \wedge \boldsymbol{a}$, this is called **anti-commutative**.
- Associative: $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$ is orthogonal to $\boldsymbol{a}$, but $(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c}$ is not.

Vector Product and the Cartesian Coordinate System

We define the basis vectors to be $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{\imath}} \wedge \hat{\boldsymbol{\jmath}}$, so if we maintain the **cyclic order** or the vectors, handiness is maintained, convince yourself this is true using your hand.

- $\hat{\boldsymbol{\imath}} \wedge \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \wedge \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{k}} = \mathbf{0}$
- $\hat{\boldsymbol{\imath}} \wedge \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$
- $\hat{\boldsymbol{\jmath}} \wedge \hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}}$
- $\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}$

Component Form

$$\begin{aligned}
\boldsymbol{a} \wedge \boldsymbol{b} &= \left(a_x \hat{\boldsymbol{\imath}} + a_y \hat{\boldsymbol{\jmath}} + a_z \hat{\boldsymbol{k}}\right) \wedge \left(b_x \hat{\boldsymbol{\imath}} + b_y \hat{\boldsymbol{\jmath}} + b_z \hat{\boldsymbol{k}}\right) \\
&= a_x \hat{\boldsymbol{\imath}} \wedge b_x \hat{\boldsymbol{\imath}} + a_x \hat{\boldsymbol{\imath}} \wedge b_y \hat{\boldsymbol{\jmath}} + a_x \hat{\boldsymbol{\imath}} \wedge b_z \hat{\boldsymbol{k}} \\
&\quad + a_y \hat{\boldsymbol{\imath}} \wedge b_x \hat{\boldsymbol{\imath}} + a_y \hat{\boldsymbol{\imath}} \wedge b_y \hat{\boldsymbol{\jmath}} + a_y \hat{\boldsymbol{\imath}} \wedge b_z \hat{\boldsymbol{k}} \\
&\quad + a_z \hat{\boldsymbol{\imath}} \wedge b_x \hat{\boldsymbol{\imath}} + a_z \hat{\boldsymbol{\imath}} \wedge b_y \hat{\boldsymbol{\jmath}} + a_z \hat{\boldsymbol{\imath}} \wedge b_z \hat{\boldsymbol{k}} \\
&= \left(a_y b_z - a_z b_y\right) \hat{\boldsymbol{\imath}} + \left(a_z b_x - a_x b_z\right) \hat{\boldsymbol{\jmath}} + \left(a_x b_y - a_y b_x\right) \hat{\boldsymbol{k}}
\end{aligned}$$

Vector Product as Determinant

A **matrix** is a collection of vectors, each column (or row) contains the component of a vector.

$$\mathbf{M} = \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix}$$
$$|\mathbf{M}| = \boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c})$$

Finding Angles

By definition,

$$\frac{|\boldsymbol{a} \wedge \boldsymbol{b}|}{|\boldsymbol{a}||\boldsymbol{b}|} = \sin\theta$$

For this purpose, the scalar product is much more convenient.

Vector Product for a Line

$\boldsymbol{r} - \boldsymbol{a}$ must be parallel with $\boldsymbol{b} - \boldsymbol{a}$.

$$(\boldsymbol{r} - \boldsymbol{a}) \wedge (\boldsymbol{b} - \boldsymbol{a}) = \mathbf{0}$$

Vector Product Equation for a Plane

$(\boldsymbol{b} - \boldsymbol{a}) \wedge (\boldsymbol{c} - \boldsymbol{a})$ gives the normal to the plane. For $\boldsymbol{r}$ to lie on the plane,

$$(\boldsymbol{r} - \boldsymbol{a}) \cdot [(\boldsymbol{b} - \boldsymbol{a}) \wedge (\boldsymbol{c} - \boldsymbol{a})] = 0$$

Finding Distances

Shortest Distance from a Point to a Line

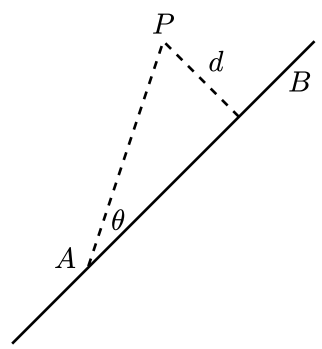
Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{p}$ be the position vectors of A, B and P respectively.

We see that $d = |\boldsymbol{p} - \boldsymbol{a}| \sin\theta$.

Using the definition of the vector product.

$$d = |(\boldsymbol{p} - \boldsymbol{a}) \wedge \hat{\boldsymbol{t}}|$$

where $\hat{\boldsymbol{t}}$ is the unit vector in the direction of $\boldsymbol{b} - \boldsymbol{a}$.



Shortest Distance from a Point to a Plane

1. Find the unit normal vector $\hat{\boldsymbol{n}}$ using the vector product.
2. $d = |(\boldsymbol{p} - \boldsymbol{a}) \cdot \hat{\boldsymbol{n}}|$

Shortest Distance from between Two Lines

Here's two lines

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{a} + \lambda\boldsymbol{t}\\ \boldsymbol{s} &= \boldsymbol{b} + \mu\boldsymbol{u}\end{aligned}$$

1. Find the unit normal vector $\hat{\boldsymbol{n}}$ of the two direction vectors using the vector product.
2. $d = (\boldsymbol{b} - \boldsymbol{a}) \cdot \hat{\boldsymbol{n}}$, this value is the same for any two points A and B on the line.

The two lines intersect if the distance is 0.

The Triple Product

Definition
The triple product of $\boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c})$ is also written as $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$.

$$\begin{aligned}[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] &= \boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c})\\ &= a_x\left(b_y c_z - b_z c_y\right) + a_y(b_z c_x - b_x c_z) + a_z\left(b_x c_y - b_y c_x\right)\end{aligned}$$

- If we maintain cyclic order, the value is the same.

$$\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} = \boldsymbol{b} \cdot \boldsymbol{c} \wedge \boldsymbol{a} = \boldsymbol{c} \cdot \boldsymbol{a} \wedge \boldsymbol{b}$$

- The value is 0 if two or more vectors are parallel. (think about volumes)

$$\mathbf{M} = \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix}$$
$$|\mathbf{M}| = \boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c}$$

By the cyclic order rule, there are many cyclic permutations of a matrix to get the same determinant.

Volume of a Parallelepiped

Find the volume of the parallelepiped formed by 3 vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

1. Find area spanned by $\boldsymbol{b}$ and $\boldsymbol{c} = |\boldsymbol{b} \wedge \boldsymbol{c}|$
2. Height is $\boldsymbol{a} \cdot \hat{\boldsymbol{n}}$ where $\boldsymbol{n} = \boldsymbol{b} \wedge \boldsymbol{c}$.

If $\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} = 0$, the vectors are **coplanar** and spans onl ya 2D space.

The Vector Triple Product

$$\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$$

Note the vector product is not associative.

$$\begin{aligned}
\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) &= \left(a_x \hat{\boldsymbol{\imath}} + a_y \hat{\boldsymbol{\jmath}} + a_z \hat{\boldsymbol{k}}\right) \wedge \left[\left(b_y c_z - b_z c_y\right)\hat{\boldsymbol{\imath}} + (b_z c_x - b_x c_z)\hat{\boldsymbol{\jmath}} + \left(b_x c_y - b_y c_x\right)\hat{\boldsymbol{k}}\right] \\
&= \left(-a_y \hat{\boldsymbol{k}} + a_z \hat{\boldsymbol{\jmath}}\right)\left(b_y c_z - b_z c_y\right) \\
&\quad + \left(-a_z \hat{\boldsymbol{\imath}} + a_x \hat{\boldsymbol{k}}\right)(b_z c_x - b_x c_z) \\
&\quad + \left(-a_x \hat{\boldsymbol{\jmath}} + a_y \hat{\boldsymbol{\imath}}\right)\left(b_x c_y - b_y c_x\right) \\
&= (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}
\end{aligned}$$

Similarly , $(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c} = (\boldsymbol{c} \cdot \boldsymbol{a})\boldsymbol{b} - (\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{a}$.

Basis Vectors

Basis vectors are linearly independent so no basis vector is redundant.

Definition

We say a set of basis vectors $e_1, e_2, ..., e_n$ are **linearly independent** if

$$\sum_i \lambda_i e_i = 0 \Longleftrightarrow \text{all } \lambda_i = 0$$

There are no solutions to the equation where $\lambda_i \neq 0$.

You can also test for linear independence by creating an $N \times N$ matrix $\mathbf{M}$. The basis vectors are linearly independent if $|\mathbf{M}| \neq 0$. (Think of the volume of a parallelepiped spanned by those vectors)

Reciprocal Basis

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be the basis vector, then the **reciprocal basis** is

$$\boldsymbol{A} = \frac{\boldsymbol{b} \wedge \boldsymbol{c}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \qquad \boldsymbol{B} = \frac{\boldsymbol{c} \wedge \boldsymbol{a}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \qquad \boldsymbol{C} = \frac{\boldsymbol{a} \wedge \boldsymbol{b}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}$$

- $\boldsymbol{a} \cdot \boldsymbol{A} = \boldsymbol{b} \cdot \boldsymbol{B} = \boldsymbol{c} \cdot \boldsymbol{C} = 1$
- 0 for all other products.

Let vector $\boldsymbol{r} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$, convince yourself that
- $\boldsymbol{A} \cdot \boldsymbol{r} = \alpha$
- $\boldsymbol{B} \cdot \boldsymbol{r} = \beta$
- $\boldsymbol{C} \cdot \boldsymbol{r} = \gamma$

This is useful for changing basis for a vector.

Orthonormal Basis

Definition

The basis is said to be **orthonormal** if
- They are orthogonal to each other.
- They have unit length.

For orthonormal basis $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, $\boldsymbol{A} = \boldsymbol{a}$, $\boldsymbol{B} = \boldsymbol{b}$ and $\boldsymbol{C} = \boldsymbol{c}$.

- Let $\boldsymbol{r} = r_1\boldsymbol{e_1} + r_2\boldsymbol{e_2} + r_3\boldsymbol{e_3} + \ldots$
- Let $\boldsymbol{s} = s_1\boldsymbol{e_1} + s_2\boldsymbol{e_2} + s_3\boldsymbol{e_3} + \ldots$

If $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}, \ldots$ is orthonormal.

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_j a_j b_j$$

Cylindrical Polar Coordinates

A point P in cylindrical polar coordinates is written as (r, θ, z).

- r is the length of the projection of $\overrightarrow{OP}$ onto the xy-plane.
- θ is the anticlockwise angle between the projection of $\overrightarrow{OP}$ and the x-axis.
- z is the same as for Cartesian coordinates.

$$\boldsymbol{p} = r\cos\theta\hat{\boldsymbol{\imath}} + r\sin\theta\hat{\boldsymbol{\jmath}} + z$$

To define the **cylindrical polar basis vectors**, set

$$\begin{aligned}
\widehat{\boldsymbol{e_r}} &= \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}} \\
\widehat{\boldsymbol{e_\theta}} &= -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}} \\
\widehat{\boldsymbol{e_z}} &= \hat{\boldsymbol{k}}
\end{aligned}$$

So the basis vectors are orthonormal and forms a right-handed coordinate system. $\widehat{\boldsymbol{e_r}}, \widehat{\boldsymbol{e_\theta}}, \widehat{\boldsymbol{e_z}}$ are in the direction where P would move if r, θ, z are increased respectively.

The oritentation of the basis depends on θ, if $\boldsymbol{a}$ and $\boldsymbol{b}$ uses different orthonormal basis, generally

$$\boldsymbol{a} \cdot \boldsymbol{b} \neq a_r b_r + a_\theta b_\theta + a_z b_z$$

We can find the orientation θ of a vector using the tangent, note that arctan only maps to value between $-\pi/2$ and $\pi/2$.

Definition
Plane polar coordinates only works on the xy-plane.

Spherical Coordinates

Start with (x, y, z), we want to work out (r, θ, φ)

- r is the distance from the origin to that point, $r \geq 0$
- θ is the angle between the z-axis to the vector, $0 \leq \theta \leq \pi$
- φ is the angle the projection of the vector to the xy-plane makes with the x-axis, $0 \leq \varphi \leq 2\pi$

If $r = 0$, we don't need information about θ or φ as there is no angle with either the axes.

- The distance from the z-axis to P is $r\sin\theta$

$$\begin{aligned}
x &= r\sin\theta\cos\varphi \\
y &= r\sin\theta\sin\varphi \\
z &= r\cos\theta
\end{aligned}$$

Since we restrict $0 \leq \theta \leq \pi$, $\sin\theta$ is always positive.

Again define the orthonormal basis.

$$
\begin{aligned}
\widehat{\boldsymbol{e}_{\boldsymbol{r}}} &= \sin\theta(\cos\varphi\hat{\boldsymbol{\imath}} + \sin\varphi\hat{\boldsymbol{\jmath}}) + \cos\theta\hat{\boldsymbol{k}} \\
\widehat{\boldsymbol{e}_{\boldsymbol{\theta}}} &= \cos\theta(\cos\varphi\hat{\boldsymbol{\imath}} + \sin\varphi\hat{\boldsymbol{\jmath}}) - \sin\theta\hat{\boldsymbol{k}} \\
\widehat{\boldsymbol{e}_{\boldsymbol{\varphi}}} &= -\sin\varphi\hat{\boldsymbol{\imath}} + \cos\varphi\hat{\boldsymbol{\jmath}}
\end{aligned}
$$

Vector Area

A surface on a 3-dimensional space have a normal vector $\hat{\boldsymbol{n}}$.

- If the surface has area A, we can write a vector about it $\boldsymbol{s} = A\widehat{\boldsymbol{m}}$.
- The length of $\boldsymbol{s}$ is proportional to the area of the surface.

$$\boldsymbol{s} = \left(s_x, s_y, s_z\right)$$

If we project the area to the yz-plane, it is an image with area s_x.

Vector Area of Closed Surfaces

Imagine an object (e.g. that of a cube) where the vector on each surface points outwards.

$$\sum_i S_i = \mathbf{0}$$

for any object.

If we have a curved surface in 3D, if the surface is **closed** then

- Take the surface vector at every infitesimally small point and sum them together.
- It will be 0 for any **closed surface**.

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END Vectors
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