Definition 7

$n \in \mathbb{N}$ is odd if $(\exists i \in \mathbb{N})\ n = 2i + 1$.

Proposition 8

Goal: $(\forall m,n \in \mathbb{N})\ m$ and n odd $\Longrightarrow m \times n$ odd

Proof

Assume:
1. $m, n \in \mathbb{N}$
2. m and n odd

New goal: $m \times n$ odd

$$\begin{aligned}
&(\exists i,j \in \mathbb{N})\ m = 2i + 1 \wedge n = 2j + 1 \\
\Longrightarrow\ &(\exists i,j \in \mathbb{N})\ m \times n = (2i + 1) \times (2j + 1) \\
\Longrightarrow\ &(\exists i,j \in \mathbb{N})\ m \times n = 2(2ij + i + j) + 1 \\
\Longrightarrow\ &(\exists k \in \mathbb{N})\ m \times n = 2k + 1 \\
\Longrightarrow\ &m \times n \text{ odd}
\end{aligned}$$

Definition 9

$(\forall x \in \mathbb{R})$
- $(\exists m,n \in \mathbb{Z})\ x = m/n \Longleftrightarrow x$ rational
- $\neg(x$ rational$) \Longleftrightarrow x$ irrational
- $x > 0 \Longleftrightarrow x$ positive
- $x < 0 \Longleftrightarrow x$ negative
- $\neg(x$ positive$) \Longleftrightarrow x$ nonpositive
- $\neg(x$ negative$) \Longleftrightarrow x$ nonnegative
- x nonnegative $\wedge\ x \in \mathbb{Z} \Longleftrightarrow x \in \mathbb{N}$

Proposition 10

Goal: $(\forall x$ positive$)\ \sqrt{x}$ rational $\Longrightarrow x$ rational

Proof

Assume:
1. x positive
2. $\sqrt{x}$ rational

New goal: x rational

$$\begin{aligned}
&(\exists p,q \in \mathbb{Z})\ \sqrt{x} = p/q \\
\Longrightarrow\ &(\exists p,q \in \mathbb{Z})\ x = \left(\sqrt{x}\right)^2 = p^2/q^2 \\
\Longrightarrow\ &(\exists p',q' \in \mathbb{Z})\ x = p'/q' \\
\Longleftrightarrow\ &x \text{ rational}
\end{aligned}$$

Definition

$P \wedge (P \Longrightarrow Q) \Longrightarrow Q$

Theorem 11

Goal: Let P_1, P_2, P_3 be statements, $(P_1 \Longrightarrow P_2 \wedge P_2 \Longrightarrow P_3) \Longrightarrow (P_1 \Longrightarrow P_3)$

Proof

Assume:

1. $P_1 \implies P_2$
2. $P_2 \implies P_3$
3. P_1

New goal: P_3

$$\begin{aligned} & P_2 \text{ as (4) by (1) and (3)} \\ \implies & P_3 \text{ by (2) and (4)} \end{aligned}$$

Definition

$(P \iff Q) \iff (P \implies Q \wedge P \impliedby Q)$

Definition 12

$d|n \iff (\exists k \in \mathbb{Z})\ n = k \times d$

Definition 14

$(\forall m \in \mathbb{Z}^+, a, b \in \mathbb{Z})\ a \equiv b \pmod m \iff m|(a - b)$

Proposition 16

Goal: $(n \text{ even} \iff n \equiv 0 \pmod 2) \wedge (n \text{ odd} \iff n \equiv 1 \pmod 2)$

Subgoal: n even $\iff n \equiv 0 \pmod 2$

Assume:

1. n even

New goal: $n \equiv 0 \pmod 2$

$$\begin{aligned} n \text{ even} & \iff (\exists k \in \mathbb{Z})\ n = 2 \times k \\ & \iff (\exists k \in \mathbb{Z})\ (n - 0) = 2 \times k \\ & \iff n \equiv 0 \pmod 2 \end{aligned}$$

Subgoal: n odd $\iff n \equiv 1 \pmod 2$

Assume:

1. n odd

New goal: $n \equiv 1 \pmod 2$

$$\begin{aligned} n \text{ odd} & \iff (\exists k \in \mathbb{Z})\ n = 2 \times k + 1 \\ & \iff (\exists k \in \mathbb{Z})\ (n - 1) = 2 \times k \\ & \iff n \equiv 1 \pmod 2 \end{aligned}$$

Proposition 18

Goal: $(\forall m \in \mathbb{Z}^+, a, b \in \mathbb{Z})\ a \equiv b \pmod m \iff ((\forall n \in \mathbb{Z}^+)\ n \times a \equiv n \times b \pmod{n \times m})$

Assume:

1. $m \in \mathbb{Z}^+$
2. $a, b \in \mathbb{Z}$

Subgoal: $a \equiv b \pmod{m} \implies (\forall n \in \mathbb{Z}^+)\ n \times a \equiv n \times b \pmod{n \times m}$

Assume:
3. $a \equiv b \pmod{m}$
4. $n \in \mathbb{Z}^+$

New goal: $n \times a \equiv n \times b \pmod{n \times m}$

$$\begin{aligned} & (\exists i \in \mathbb{Z})\ a - b = m \times i \text{ by } (3) \\ \implies & (\exists i \in \mathbb{Z})\ n \times a - n \times b = (n \times m) \times i \\ \implies & (\exists i \in \mathbb{Z})\ n \times a \equiv n \times b \pmod{n \times m} \\ \implies & n \times a \equiv n \times b \pmod{n \times m} \end{aligned}$$

Subgoal: $(\forall n \in \mathbb{Z}^+)\ n \times a \equiv n \times b \pmod{n \times m} \implies a \equiv b \pmod{m}$

Assume:
3. $(\forall n \in \mathbb{Z}^+)\ n \times a \equiv n \times b \pmod{n \times m}$

$$\begin{aligned} & 1 \times a \equiv 1 \times b \pmod{1 \times m} \text{ by } (3) \\ \implies & a \equiv b \pmod{m} \end{aligned}$$

Definition
- $(\forall x)\ x = x$
- $(\forall x,y)\ x = y \implies (P(x) \implies P(y))$
- $(\forall a,b,c)\ (a = b \land b = c) \implies a = c$
- $(\forall a,b,x,y)\ (a = b \land x = y) \implies (a + x = b + x = b + y)$