

Algorithms I

Definition

An **algorithm** is a *well defined* computation procedure that takes a set of values as input and produces a set of values as output.

Note: the term *well defined* is itself, not well defined.

Definitions

- **Problems** have specific inputs and outputs, input must be finite and not a stream of data.
- **Problem instances** is a specific set of inputs for a problem. A problem can have a Big-O but not a problem instance.
- A program is **correct** if for every input instance, it terminates with the correct output.

Note

- Randomised algorithms is a branch of incorrect algorithms.
- Some algorithms produce incorrect outputs with a probability (e.g. quantum computing)
- Some algorithms loops infinitely for some inputs, but runs a lot faster than an algorithm that guarantees termination for cases where it terminates. It might be possible to determine whether it will terminate for a specific input before running it.
- Some algorithms gives an output within a margin of error (e.g. A* vs Dijkstra)

Notation

Arrays

- $A[1]$ is the first item
- $A[1..n]$ is an array of length n
- $A.length$ is the number of items in the array

We write pseudocode that is

- Imperative
- Block structured
- Fixed form (indentation matters)
- Parameters are passed by values, objects are passed by pointers
- Loop induction (for loops) increments after the final loop

```
for i=1 to 10
    // do stuff
```

After this loop, consider $i=11$

Sorting

Each **key** may have attached payloads.

Insertion Sort

```
for j = 2 to A.length
    Key = A[j]
    i = j - 1
    while i > 0 && A[i] > Key
        A[i + 1] = A[i]
        i = i - 1
    A[i + 1] = Key
```

Use proof by induction for algorithms:

- **Initialisation:** find a property that is true at the start of the program

P : at the start of each loop, $A[1\dots j - 1]$ contains the $1\dots j - 1$ items in sorted order.

At the start of the first loop, that is just $[a_1]$, true.

Note

Define “the start of the loop” as: after assigning the value of j , but before running the first line of code in the loop.

- **Maintenance:** show that the property is maintained as the program is running.
- **Termination:** when the program terminates, show the output is correct.

After the last loop, $A[1\dots A.length]$ would have been containing all the items $1\dots A.length$ in order.

And then we can also show the program terminates as it only needs to complete the loop $A.length$ items.

Note

Which is the same as the following Hoare logic proof.

Let P, Q be pre and post-conditions, B be body of the loop, C be condition for the loop.

Given:

1. $\{P\} B \{P\}$
2. $P \wedge \neg C \implies Q$

Then $\{P\}$ while C do $B \{Q\}$

Analysis

Definition

Analysis is about predicting the resources (CPU, memory, disk operations) for input instances we haven't ran our algorithm on.

Input measurement	Description
$A.length$	Common for every day scenarios, but may be incorrect if each item in array can have variable size (e.g. big integer)
no. of bits/bytes	Useful for algorithm that operates on some bit/byte value.
$2^{A.length}$	Overestimates the size in most cases, but can be used for search lists.

Definition

The **running time** of a program is the number of **basic operations**. (as they all cost 1)

Basic operation	Cost
Indexing an array $A[i]$	1
Arithmetic operation	1
Comparisons	1

Basic operation	Cost
Assignment to variables	1

One basic operation might not be equal to one clock cycle, if you change the cost of the basic operations, the running time changes.

Note

Comparisons (numbers) is usually done by subtracting one from another, then compare with 0.

Order of Growth

- $\Theta(g(n))$ is the **asymptotic tight bound** for $g(n)$

$$f(n) \in \Theta(g(n)) \implies \exists c_1, c_2, n_0 \in \mathbb{R}^+ : (\forall n \geq n_0 : c_1 g(n) \leq f(n) \leq c_2 g(n))$$

- $O(g(n))$ is the **asymptotic tight upper bound** for $g(n)$

$$f(n) \in O(g(n)) \implies \exists c, n_0 \in \mathbb{R}^+ : (\forall n \geq n_0 : f(n) \leq cg(n))$$

- $\Omega(g(n))$ is the **asymptotic tight lower bound** for $g(n)$

$$f(n) \in \Omega(g(n)) \implies \exists c, n_0 \in \mathbb{R}^+ : (\forall n \geq n_0 : cg(n) \leq f(n))$$

- $o(g(n))$ is the **asymptotic non-tight upper bound** for $g(n)$

$$f(n) \in o(g(n)) \implies \forall c \in \mathbb{R}^+ : (\exists n_0 \in \mathbb{R}^+ : (\forall n \geq n_0 : f(n) < cg(n)))$$

- $\omega(g(n))$ is the **asymptotic non-tight lower bound** for $g(n)$

$$f(n) \in \omega(g(n)) \implies \forall c \in \mathbb{R}^+ : (\exists n_0 \in \mathbb{R}^+ : (\forall n \geq n_0 : cg(n) < f(n)))$$

Properties of Orders of Growth

$$\Theta(g(n)) \subseteq O(g(n))$$

$$\Theta(g(n)) \subseteq \Omega(g(n))$$

- **Transitive:** satisfied by all 5 orders

$$f(n) \in \Theta(g(n)) \wedge g(n) \in \Theta(h(n)) \implies f(n) \in \Theta(h(n))$$

- **Reflexive:** satisfied by the tight bounds Θ, O, Ω

$$f(n) \in \Theta(f(n))$$

- **Symmetric:** satisfied by Θ

$$f(n) \in \Theta(g(n)) \implies g(n) \in \Theta(f(n))$$

Analysis of Insertion Sort

```

for j = 2 to A.length          // ran (n-1)+1 times
    Key = A[j]                // ran n-1 times
    i = j - 1                 // ran n-1 times
    while i > 0 && A[i] < Key // ran sum_(j=2)^n t_j times
        A[i+1] = A[i]          // ran sum_(j=2)^n (t_j - 1) times
        i = i - 1              // ran sum_(j=2)^n (t_j - 1) times
    A[i+1] = Key              // ran n-1 times

```

Where t_j is the number of times the while loop is tested on the j th cycle.

- Best case: $t_j = 1$ then $T(n) = pn + q$
- Worst case: $t_j = j$ then $T(n) = pn^2 + qn + r$
- Average case: the claim is that on average, half of the keys in $A[1\dots j-1]$ will be less than $A[j]$

$$t_g = j/2 \text{ gives } T(n) \in O(n^2)$$

The worst case is useful because

- It gives the upper bound on resource
- Often the same as the average case

Insertion sort is an **incremental algorithm**: it builds up an output that satisfies some properties.

Divide and Conquer

1. Split into 2 or more smaller subproblems.
2. call the same algorithm on each subproblem recursively.
3. Combine solutions to the subproblems to build the solution to the original problem.

Note

Recursion will terminate because the subproblem will get smaller and smaller.

Merge Sort

```
// we are sorting A[p..r]
if p < r
    q = floor((p + r) / 2)
    MergeSort(A, p, q)
    MergeSort(A, q + 1, r)
    Merge(A, p, q, r)
```

And Merge defined as

```
n1 = q - p + 1
n2 = r - q

L = new Array(1 .. n1 + 1)
R = new Array(1 .. n2 + 1)

L[1 .. n1] = A[p .. q]
L[n1 + 1] = infinity
R[1 .. n2] = A[q + 1 .. r]
R[n2 + 1] = infinity

i = j = 1

for k = p to r
    if L[i] <= R[j]
        A[k] = L[i]
        i = i + 1
    else
        A[k] = R[j]
        j = j + 1
```

- If the length of the array is not a power of 2, pad ∞ to the end so that it is.
- After sorting, remove the added ∞ at the end of the sorted array.

The input array is modified, Merge has no return value.

Recurrence Relations

The input size is length of the region to be sorted $n = r - p + 1$

Let $T(n)$ be the cost of solving $\text{MergeSort}(A, p, r)$

- If $p = r$, $T(1) = 1$
- If $p < r$

Action	Cost
Calculate q	$\Theta(1)$
Calls itself on 2 subproblems	$T(n/2) \times 2$
Calls $\text{Merge}(A, p, q, r)$	$\Theta(n)$
Action	Cost
Creates 2 arrays of length $n + 2$	$\Theta(n)$
Loop n iterations: assign into array and increment i or j	$\Theta(n)$

$$T(1) = 1$$

$$\begin{aligned} T(n) &= \Theta(1) \text{ work} + 2 \cdot T\left(\frac{n}{2}\right) + \Theta(n) \text{ work} \\ &= k_1 + 2 \cdot T\left(\frac{n}{2}\right) + k_2 \cdot n \end{aligned}$$

Definition

A **closed form solution** is not defined in terms of itself through direct or indirect recursion.

$$\begin{aligned} T(n) &= k_1 + k_2 \cdot n + 2 \cdot T\left(\frac{n}{2}\right) \\ &= k_1 + k_2 \cdot n + 2 \cdot \left(k_1 + k_2 \cdot \frac{n}{2} + 2 \cdot T\left(\frac{n}{4}\right) \right) \\ &= k_1 + k_2 \cdot n + 2 \cdot \left(k_1 + k_2 \cdot \frac{n}{2} + 2 \cdot \left(k_1 + k_2 \cdot \frac{n}{4} + 2 \cdot T\left(\frac{n}{4}\right) \right) \right) \\ &\vdots \\ &= k_1 \cdot \underbrace{(1 + 2 + 4 + \dots)}_{\log n \text{ terms}} + k_2 \cdot n \cdot \underbrace{(1 + 1 + 1 + \dots)}_{\log n \text{ times}} + 2^{\log n} \cdot T(1) \\ &= k_1 \cdot (n - 1) + k_2 \cdot n \log n + n \\ &\in \Theta(n \log n) \end{aligned}$$

Note

We preserved the equal signs instead of saying “this term dominates” so we know $T(n) \in \Theta(f(n))$ instead of just $O(f(n))$

If the array length is not a power of 2

$$T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + k_1 + k_2 \cdot n$$

Which gives the same solution.

The Master Theorem

Let $a \geq 1$ and $b > 1$ be constants.

- $T(1) = 1$
- $T(n) = a \cdot T(n/b) + f(n)$

Note

n/b can be interpreted as ceil or floor, it doesn't matter.

$$f(n) \in O(n^{-\varepsilon + \log_b a}) \text{ for some } \varepsilon > 0 \implies T(n) \in \Theta(n^{\log_b a})$$

$$f(n) \in \Theta(n^{\log_b a}) \implies T(n) \in \Theta(n^{\log_b a} \cdot \lg n)$$

$$f(n) \in \Omega(n^{\varepsilon + \log_b a}) \text{ for some } \varepsilon > 0 \text{ and } f(n/b) \leq c f(n)$$

$$\text{for some } c > 1 \text{ for all sufficiently large } n \implies T(n) \in \Theta(f(n))$$

Note

There is an extended master theorem for conditions between case 2 and 3.

Quicksort

```
QuickSort(A, p, r)
  if p < r
    q = partition(A, p, r)
    QuickSort(A, p, q - 1)
    QuickSort(A, q + 1, r)
```

```
Partition(A, p, r)
  x = A[r]
  i = p - 1
  for j = p to r - 1
    if A[j] <= x
      i = i + 1
      swap(A[i], A[j])
  swap(A[i + 1], A[r])
  return i + 1
```

The strategy: **divide** (partition array into 3 parts), **conquer** (recurse on L and G), **combine** (no-op).

Requirements for **Partition(A, p, r)** is

- Pick any element as pivot
- Rearrange so array looks like $[items \leq \text{pivot}, \text{pivot}, items \geq \text{pivot}]$

$$[L, x, G]$$

Proof

Let P be the statement:

1. If $p \leq k \leq i$ then $A[k] \leq x$
2. If $i + 1 \leq k \leq j - 1$ then $A[k] > x$
3. If $k = r$ then $A[k] = x$

Note

No statements made for $j \leq k \leq r - 1$ which is the unprocessed region.

- **Initialisation:** (1) and (2) vacuously true, (3) is true by definition.
- **Maintenance**

- Case does not enter if branch: (1) and (3) remains true, (2) is true as the new item $> x$
- Case enters if branch: (3) remains true, (1) and (2) are true as their new items both satisfies constraint.
- The final swap ensures post-condition.

Performance

The number of comparisons depends on how Partition splits the array.

- Best case:** partition splits array in half

$$T(1) = 1$$

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + k \cdot n$$

$$T(n) \in \Theta(n \log n)$$

by master theorem.

- Unbalanced partition**, e.g.

$$T(1) = 1$$

$$T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{3n}{4}\right) + kn$$

The **computation tree** is unbalanced.

- The shallowest node has depth $\log_4 n$
- The deepest node has depth $\log_{4/3} n$

Both gives $\Theta(n \log n)$, so any ratio split gives $\Theta(n \log n)$

- Worst case** when the pivot is the largest/smallest item in array.

$$\begin{aligned} T(n) &= T(n-1) + T(0) + kn \\ &= T(n-1) + kn \\ &\in \Theta(n^2) \end{aligned}$$

The probability of worst case on each split is $2^n/n!$, when n is small, this is quite significant.

- Constant case** where a constant number of items are partitioned to one side. Still $\Theta(n^2)$

Order Statistic

Definition

The i th order statistic is the i th smallest item in a set.

- Input: a set A , and an integer i
- Output: $x \in A$ so it is larger than exactly $i - 1$ other elements.

- Selecting the minimum and maximum item is simple at $O(n)$
- Selecting the i th item can be done in $\Theta(n \lg n)$
 - Sort the array
 - Get the i th element

```
QuickSort(A, p, r, i)
  if p = r
    return A[p]
```

```

q = Partition(A, p, r)
k = q - p + 1
if i == k
    return A[q]
else if i < k
    return QuickSelect(A, p, q - 1, i)
else
    return QuickSelect(A, q + 1, r, i - k)

```

Worst case:

$$\begin{aligned}
T(1) &= 1 \\
T(n) &= T(n-1) + kn \\
&\in \Theta(n^2)
\end{aligned}$$

Optimisations

- Randomise input data before starting
- Choose pivot at random
- Median of 3: choose the median of 3 selected items as pivot.

Hitting worst case if:

- One of the 3 element is max or minimum element
- The other median element is right next to the max/minimum element.

This has probability $2/n^2$

- Median of medians:
 1. Consider array as groups of 5 elements.
 2. Pick the median of each group with insertion sort.
 3. Use quick select to select the median of the medians as pivot.

The final median is a median of $\lceil n/5 \rceil$ “medians”

- Half the “medians” are greater than the median.
- Number of elements greater/smaller than the pivot is

$$\text{number of elements} \geq 3 \cdot \left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil - 2 \right\rceil \geq \frac{3n}{10} - 6$$

Note

-2 removes 2 groups for worst case, includes the last group that is possibly incomplete, and the group that the pivot is in.

Worst case is $3n/10 - 6$ on one side and $7n/10 + 6$ on the other.

Suppose

$$T(n) = k \quad \text{if } n < 140$$

$$T(n) = T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 6\right) + kn \quad \text{as we are considering the worst case}$$

If we guess that $T(n) \in \Theta(n)$, then

$$\begin{aligned}
 T(n) &\leq c \cdot \left\lceil \frac{n}{5} \right\rceil + c \cdot \left(\frac{7n}{10} + 6 \right) + kn \\
 &\leq \frac{cn}{5} + c + \frac{7cn}{10} + 6c + kn \\
 &= cn + \left(-\frac{cn}{10} + 7c + kn \right)
 \end{aligned}$$

If $(-cn/10 + 7c + kn) \leq 0$, then $T(n) \in O(n)$

- (Not in handout) Also need to check lower bound is in $\Omega(n)$ to show $T(n) \in \Theta(n)$

HeapSort

Definition

- A **heap** is a full tree, except the lowest level, which is filled from left to right.
- The **ordering property**: for a **min-heap** the key of every node is less than its children.
Similar for a **max heap**.

Heaps can be stored as a tree or in an array:

- Root node at $A[1]$
- Left/right child of node at $2i$ and $2i + 1$
- Parent of a node at $\lfloor i/2 \rfloor$
- A child exist only if $2i$ and/or $2i + 1$ is \leq the array's length
- A node has no parent if $\lfloor i/2 \rfloor = 0$

Storing in array uses less memory as no pointers need to be stored.

Heaps are **semi-structures** - it only maintain **partial order**.

- Smallest item at the root
- 2nd smallest item in 2 possible places
- 3rd smallest item in 3 places, etc.

A semi-structure is cheaper to build than fully structured data structures.

Consider operations on a **max-heap**.

Operation	$O(n)$	Description
MaxPeek	$O(1)$	Just return the root node
MaxReheapify	$O(\lg n)$	Call on heaps where the root node is larger than one of its children to repair the heap. 1. Swap root node with child if needed, this damages the child heap 2. Call reheapify on the child heap.
MaxFullHeapify	$O(n)$	Perform swaps on an array so it is a valid heap, by calling reheapify on all nodes from $\lfloor n/2 \rfloor$ down to 1.
MaxExtract	$O(\lg n)$	1. Swap the root with bottom right leaf. 2. Remove the now bottom right leaf. 3. Call reheapify to repair the heap. This is better than removing the root node then merge the heaps, because the ordering property is easier to repair than the structural property.

So the HeapSort algorithm.

```
MaxFullHeapify(A) # O(n)

for i = A.length downto 2: # O(n lg n)
    # ASSERT: A[i..] is sorted
    swap(A[1], A[i]) # essentially a MaxExtract
    A.length = A.length - 1
    MaxReheapify(A)
```

Cost of MaxFullHeapify

- Best case: array satisfies heap properties, no swaps, 2 comparisons per key on $n/2$ keys. $\Theta(n)$
- Worst case: every recursive call of MaxReheapify results in a swap. $\Theta(n)$

HeapSort MaxFullHeapify once then calls MaxReheapify $n - 1$ times

$$\begin{aligned} T(n) &= k_1 n + k_2 \lceil \lg n \rceil + k_2 \lceil \lg(n-1) \rceil + k_2 \lceil \lg(n-2) \rceil + \dots \\ &\leq k_1 n + k_2 \lg n + k_2 \lg(n-1) + \dots + k_2 n \\ &= (k_1 + k_2)n + k_2 \lg(n!) \\ &\leq (k_1 + k_2)n + k_2(n \lg n - n) \\ &\in O(n \lg n) \end{aligned}$$

$O(n)$ Sorting Algorithms

Number of required comparisons for comparison sorts are $\in \Omega(n \lg n)$

CountingSort(A, B, k)

Where A is input, B is output, k is top limit on range of values.

```
C = new Array[0..k]

for i = 0 to k:
    C[i] = 0

for j = 1 to A.length:
    C[A[j]] = C[A[j]] + 1

for i = 1 to k:
    C[i] = C[i] + C[i - 1]

# ASSERT: C[n] is the ending index for
# runs of `n`

for j = A.length downto 1:
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j] - 1]
```

The time complexity is $\Theta(n + k)$

RadixSort(A, d)

```
for i = 1 to d:
    # sort array A on digit i with any stable sort
```

Where 1 is the least significant digit.

Definition

A **stable sort** preserves the order of inputs when their keys are equal.

- Time complexity is $\in \Theta(d(n + k))$ where k is the number of values a digit can take.
- Essentially run CountingSort once on each digit.

BucketSort(A)

Used for sorting values distributed uniformly in a range.

1. Put values into n buckets
2. Sort values inside each bucket
3. Merge

```

n = A.length
B = new Array[0..n-1]

for i = 0 to n-1:
    B[i] = empty_list

for i = 1 to n:
    # insert A[i] into list B[floor(n * A[i])]

for i = 0 to n-1:
    InsertionSort(B[i])

# concatenate B[0], B[1], .. B[n-1] in ordered

```

$$\begin{aligned}
T(n) &= \Theta(n) + \sum_{i=0}^{n-1} O((n_i)^2) \\
&\vdots \\
&= \Theta(n)
\end{aligned}$$

Ending note

The two main types of strategies studied are:

- Incremental
- Divide and conquer

Dynamic Programming

- **Divide and conquer** splits into subproblems that don't overlap.
- **Dynamic programming** is useful when the subproblems do overlap.

Definition

Optimal substructure problems involves maximising or minimising something.

We want any optimal solution if there are many.

Top-down:

1. Start with the problem
2. Split into subproblems
3. Continue until base case is solved

Bottom-up:

1. Start with base case
2. Solve every problem that combines solved subproblems
3. Continue until the original problem is encountered

Avoid solving same subproblem twice by memorising results in a table.

Virtual Machine Hosting Problem (Rod Cutting Problem)

Top-Down

Subdivide a server with n cores into virtual machines, a server machine with i cores has value $v[i]$.

```
def VMify(v, n):
    if n == 0:
        return 0
    q = -1
    for i = 1 to n:
        q = max(q, v[i] + VMify(v, n - i))
    return q
```

$$T(1) = 1$$

$$T(n) = 1 + \sum_{i=0}^{n-1} T(i) \in O(2^n)$$

If we add caching for the results

$$\begin{aligned} T(n) &= 1 + n \\ &= O(n^2) \end{aligned}$$

Bottom-Up

```
m[0 .. n] = new Array()
m[0] = 0
for j = 1 to n:
    q = -1
    for i = 1 to j:
        q = max(q, v[i] + m[n - i])
    m[j] = q
return m[n]
```
