Foundations of Computer Science

Layer	Description
Transistors	On the smallest scale, computers are turning transistors on and off.
Microcontroller	The Raspberry Pi Pico has millions of **transistors**.
Motherboard	Contains multiple **CPUs** all trying access shared resources (RAM).
Devices	The Apple Vision Pro contains lots of **sensors** and **chips** fit in a small box.
Supercomputer	Thousands of **CPUs**/**GPUs** working together, connected to internet and storage.
The user	Computers in **data centres** are rented out to users doign all sorts of stuff, e.g. doing AI research, watching Netflix, etc.

Abstraction

There's no way of understanding the whole tower at once: you cannot understand agentic AI in terms of transistors.

- With abstraction, you only understand the layer below.
- This is the *"What operations do I need to do the task?"* mentality of a programmer.

Definition
Abstraction barrier allows one layer to be changed without affecting levels above.

Representing Data

Definition
The concept of a **data type** involves
- How a value is represented inside the computer.
- The suite of operations (services) provided to the programmer.

How the data is represented may produce undesired results.
- The Y2K crisis.
- Floating point precision error.

Programming in OCaml

The goals of programming is to **describe a computation** so it can be done mechanically.
- Be efficient and correct.
- Allow easy modification: the effect of changes can be easily predicted.

Definitions

Keyword	What they do
Expressions	Compute values, *may* cause side effects.
Commands	Cause *only* side effects

Why OCaml?

- Interactive evaluation in **Jupyter notebooks** and in a **REPL**.
- Flexible and powerful notion of a data type.
- Hides underlying complexity - it throws an exception but never crashes, manages memory for us.
- Programs written can be understood/reasoned mathematically. (as there is no side effect)

Basics in OCaml

```
(* = Variable declaration *)
let pi = 3.1415926                  (* val pi : float = 3.1415926 *)

(* = Function declaration *)
let area r = pi *. r *. r           (* val area : float -> float *)

(* = Function invocation *)
area 2.0                            (* - : float = 12.556 *)

(* = Recursive functions *)
let rec npower x n =                (* npower : float -> int -> float = <fun> *)
    if n = 0 then 1.0
    else x *. power x (n - 1)

(* Type hints for the compiler *)
let side : float = 1.0              (* side : float *)
let square (x : float) = x *. x     (* square : float -> float *)

(* If-Else *)
let rec power x n =                 (* power : float -> int -> float *)
    if n = 1 then x
    else if (n mod 2) == 0 then
        power (x *. x) (n / 2)
    else
        x *. power (x *. x) (n / 2)
(* this is a more efficient power function than npower *)
```

OCaml automatically infers types, but it does **not** implicitly convert types.

- Type inference by looking at the operations values, `*.` for float multplication and `*` for integers.
- All branches of an if-else block must return value of the same type.

Type hints are useful to prevent OCaml from inferring all the wrong types when you make one small mistake.

Recursion and Complexity

Definition
Separating expressions from side effect is known as **functional programming**.

We can trace an expression, for example `power` function defined above.

```
⇒ power 2.0 12
⇒ power 4.0 6
⇒ power 16.0 3
⇒ 16 *. power 256.0 1
⇒ 16 *. 256.0
⇒ 4096.0
```

```
(* sums the first n integers *)
let rec nsum n =
  if n = 0 then 0
  else
    n + nsum (n - 1)
```

```
⇒ nsum 3
⇒ 3 + nsum 2
⇒ 3 + (2 + nsum 1)
⇒ 3 + (2 + (1 + nsum 0))
⇒ 3 + (2 + (1 + 0))
```

$\Rightarrow$ `6`

Nothing can progress until the innermost sum is calculated. All the intermediate values have to be stashed onto the **program stack**. Evaluating `nsum 10000000` can cause a stack overflow.

Alternative Approach: Iterative Summing

```
let rec sum n total =
  if n = 0 then total
  else
    summing (n - 1) (n + total)
```

$\Rightarrow$ `sum 3 0`
$\Rightarrow$ `sum 2 3`
$\Rightarrow$ `sum 1 5`
$\Rightarrow$ `sum 0 6`
$\Rightarrow$ `6`

The trace looks quite different.
- The total is known as an **accumulator**.
- Functions like this is called **tail recursive**.

Definition
In a **tail recursive** function, the recursive function call is the last thing the function does.

`nsum` is not tail recursive because it has to do the *add* operation after calling the function.
- `sum` won't stack overflow only if the compiler knows the function is tail recursive and optimises it.
- OCaml pops the function call off the stack before it finishes executing.

Downsides of Tail Recursion
- Extra variable needed, so easier to call the function incorrectly.
- Function is more complicated.

Don't use tail recursion with accumulator unless gain is significant.

Analysing Efficiency

```
let rec sillySum n =
  if n = 0 then 0
  else
    n + (sillySum (n - 1) + sillySum (n - 1)) / 2
```

`sillySum` is ran twice, as there may be side effects and the two functions may give different results.
- This is why pure functional evaluation is much simpler.
- Assign the value to a variable to avoid it being evaluated twice.

```
let rec sillySum n =
  if n = 0 then 0
  else
    let previousSum = sillySum (n - 1) in
      n + (previousSum + previousSum) / 2
```

Asymptotic complexity

Definition
Asymptotic complexity refers to how programs costs grow with increasing inputs.

E.g. space and time, the latter usually being larger than the former.

Definition
The **Big-O** notation is defined as $f(n) = O(g(n))$ provided that $|f(n)| \leq c|g(n)|$ for large n.

Intuitively, consider the most significant term and ignore the constant coefficient or smaller factors.

Here are some interesting results:
- $O(\log n) = O(\ln n)$
- $O(\log n)$ is contained in everything, including $O(\sqrt{n})$
- An exponential algorithm can be faster than a linear algorithm for a particular input size interval.
- $O(n \log n)$ is called **quasi-linear**.

Simple Recurrence Relation

Set the time cost of base case $T(1) = 1$.

Recurrence relation	Time complexity
$T(n+1) = T(n) + 1$	$O(n)$
$T(n+1) = T(n) + n$	$O(n^2)$
$T(n) = T(n/2) + 1$	$O(\log n)$
$T(n) = T(n/2) + n$	$O(n \log n)$

Some examples in analysing time complexity.

```
let rec nsum =
  if n = 0 then 0
  else
    n + nsum (n - 1)
```

- $T(0) = 1$
- $T(n+1) = T(n) + 1$
- So $O(n)$

```
let rec nsumsum n =
  if n = 0 then 0
  else
    nsum (n - 1) + nsumsum(n -
1)
```

- $T(0) = 1$
- $T(n+1) = T(n) + n$
- So $O(n^2)$

```
let rec power x n =
  if n = 1 then x
  else if even n then
    power (x *. x) (n / 2)
  else
    x *. power (x *. x) (n / 2)
```

- $T(0) = 1$
- $T(n) = T(\frac{n}{2}) + 1$
- So $O(\log n)$

At each call n is halved, and add 1 as there is always some extra work (e.g. calling the function, if branch).

Lists

Definition
A **list** is a finite, ordered sequence of elements, all elements must have the same type.

List Primitives

There are only 2 kinds of lists, the 2 operations covers all possible lists.

```
[] (* nil : the empty list *)
x :: xs (* cons : put one element in front of the list *)
```

`[3; 5; 9]` is syntactical sugar for `3 :: (5 :: (9 :: []))`.

```
[3; 5; 9]
(* - : int list = [3; 5; 9] *)

[[3; 1]; [2]]
(* - : int list list = [[3; 1]; [2]] *)

(* concatenate two lists *)
[3; 5; 9] @ [2; 4]
(* - : [3; 5; 9; 2; 4] *)

(* the List library contains useful functions *)
List.rev [1; 2; 3]
(* - : [3; 2; 1] *)
```

Tuples

Definition
Tuples are fixed size and hetrogeneous sequences.

```
let pair = (1, true)
(* val pair : int * bool = (1, true) *)

(* you can do it without the brackets *)
let another_pair = 1, true, 3.2
(* val another_pair : int * bool * float = (1, true, 3.2) *)

(* take care not to use commas instead of semicolons *)
let list = [1, 2, 3]
(* val list : int * int * int list *)
```

Pattern Matching

All possible values that can be matched must be matched.

```
let null = function
  | [] -> true
  | _ :: _ -> false

let is_zero = function
  | 0 -> true
  | _ -> false
```

You can also pattern match parameters.

```
let hd = (x :: _) = x

hd [1] (* 1 *)
hd [] (* match error *)
```

In this case is better to use an option type.

Polymorphic Functions

The `List.tl` function returns the tail of a list

```
List.tl
(* - : 'a list -> 'a list = <fun> *)
```

An `'a` type (read: alpha type) means it can be of any type, but all elements of the list must be the same type.

More List Functions

```
let rec append = function
    | [], ys -> ys
    | x :: xs, ys -> x :: append xs ys
(* val append : a' list * a' list -> a' list *)
```

The match keyword keeps the reference to the original value.

```
let rec append xs ys =
  match xs, ys with
    | [], ys -> ys
    | x :: xs, ys -> x :: append xs ys
(* val append : a' list -> a' list -> a' list *)
```

Take and Drop

- `take` takes the first `i` items of a list.
- `drop` returns all the items that are not included in `take`

```
let rec take = function
  | [], _ => []
  | x :: xs, i =>
    if i > 0 then
      x :: take (xs, i - 1)
    else
      []

let rec drop = function
  | [], _ -> []
  | x :: xs, i ->
    if i > 0 then
      drop (xs, i - 1)
    else
      x :: xs (* we could do this better using a match *)
```

In the `drop` function:
- We advance the pointer as we go through the list.
- Then just returns the pointer where we stop.
- No memory is used.

In the `take` function has to construct a list from scratch.

Searching

Goal is to find x in a list $[x_1; ...; x_n]$

Name	Description	Cost
Linear search	Compare each element	$O(n)$
Oredred search	The list is bisected every time	$O(\log n)$
Indexed search	Create an index, e.g. a hash map	$O(1)$

Equality Test

The polymorphic equality operator = to compare integers, bools, floats but not functions.

Do not use ==

List Membership

```
let rec member x = function
  | [] -> false
  | y :: ys -> x = y || member x ys
```

The `||` is not a normal function, if the first case evaluates to true, it will not bother to evaluate the 2nd bit.

Zip and Unzip

```
let D in E
```

- Embeds declaration `D` within expression `E`
- Useful for performing intermediate computations within a function.

```
let rec zip = function
  | (x :: xs, y :: ys) ->
    (x, y) :: zip (xs, ys)
  | _ -> []

let rec unzip pairs = function
  | [] -> []
  | (x, y) :: pairs ->
    let xs, ys = unzip pairs in
    (x :: xs, y :: ys)
```

If we redo that in an iterative algorithm.

```
let rec unzipRev pairs = function
 | [], xs, ys -> xs, ys
 | (x, y) :: pairs, xs, ys ->
  unzipRev (pairs, x :: xs, y :: ys)
```

- In `unzip`, we traverse to the end then build up the list.
- In `unzipRev` we start building up the list right away.

That's why their order is different.

Sorting

Sorting is a key part of many other algorithms:

- **Searching** is much easier in a sorted list.
- **Merging** is also much easier.
- **Finding duplicates** is much easier as they would be next to each other in a sorted list.
- **Inverting tables** (??)
- **Graphics algorithms**: don't need to check for collision between every object. If the objects are sorted by distance, objects far away don't need to be checked.

Time Complexity of Sorting

Definition

In a **comparison sort**, the only way we can sort is by taking 2 times and comparing them: bigger/smaller than or equal.

We are limiting ourselves to comparison sort.

- There are $n!$ permutations of n elements.
- Each comparison eliminates half of the permutations $n^{C(n)} = n!$
- Therefore at best $C(n) \geq \log n! \sim n \log n + 1.44n$

Insertion Sort

```
let rec ins x = function
  | [] -> [x]
  | y :: ys ->
      if x <= y then
        x :: y :: ys
      else
        y :: ins x ys

let rec insort = function
  | [] -> []
  | x :: xs ->
      ins x (insort xs)
```

- The helper function inserts an item to a sorted list, on average the item is inserted to the middle of the list, so $O(n)$.
- `insort` has cost $O(n^2)$, much worse than $O(n \log n)$.

Tracing

We can trace a **monomorphic function** with

```
#trace insort
```

This prints out the function's arguments and return values.

Quicksort

1. Choose a pivot a, e.g. the head of list.
2. **Divide**: Partition the input into 2 sublists.
   - Those $\leq a$
   - Those $> a$
3. **Conquer**: recursively sort both sublists.
4. **Combine**: append the two lists together.

Ideally the two lists should be the same length, where we have $O(n \log n)$. If a sorted list is used, then the worst case $O(n^2)$. Since real data is often unsorted, we have average case $O(n \log n)$.

```
let rec quick = function
  | [] -> []
  | x :: xs ->
      let rec part l r = function
        | [] -> (quick l) @ (quick r)
        | y :: ys ->
            if y <= x then
              part (y :: l) r ys
            else
              part l (y :: r) ys
      in part [] [] xs
```

Merge Sort

```
let rec merge = function
  | [], ys -> ys
```

```
  | xs, [] -> xs
  | x :: xs, y :: ys ->
      if x < y then
        x :: merge xs (y :: ys)
      else
        y :: merge (x :: xs) ys

let rec mergesort = function
  | [] -> []
  | xs ->
    let k = (List.length xs) / 2 in
    let l = take k xs in
    let r = drop k xs in
    merge (mergesort l) (mergesort r)
```

Merge sort has a worst case of $O(n \log n)$, but have a space complexity of $O(n \log n)$ due to the extra function calling we have to do.

Datatypes and Trees

Using custom data types improves abstraction of data away from the details of representation.

- We can pack values into a tuple, but we can easily create **invalid values**.
- We want to create abstraction that is impossible to make mistakes like that.

```
let wheels = function
  | "bike" -> 2
  | "motorcar" -> 2
  | "car" -> 4
  | "lorry" -> 4
```

We want to make **illegal state** unrepresentable: a program with invalid states is rejected at compile time.

Enumeration Types

```
type vehicle =
  | Bike
  | Motorbike
  | Car
  | Lorry
```

Instead of representing any string, it can only contain the 4 constants defined.

- The constants are the **constructors** of the `vehicle` type.
- This is more **memory efficient** than using strings.
- Adding new vehicle types is straightforward.

Even though the num is still represented as integers internally, we cannot compare it with any other data types. We have created a type that is *disjoint from any other types.*

Generalisation

Data can be store alongside each variant.

```
type vehicle =
  | Bike
  | Motorbike of int
  | Car of bool
  | Lorry of int
```

```
let b = Motorbike 123
```

The `Motorbike` constructor has type `int -> vehicle`.

Pattern Matching

```
let wheels = function
  | Bike -> 2
  | Motorbike _ -> 2
  | Car true -> 3
  | Car false -> 4
  | Lorry w -> w
```

Polymorphic Types

Types can have type parameters.

```
type 'a option =
  | None
  | Some of 'a

type ('a, 'b) result =
  | Ok of 'a
  | Error of 'b
```

Exceptions

Definition
An exception is raised when something we weren't expecting happened.

- An exception is handled by doing an alternative computation.
- If not handled, the exception will go up to the top of the program and it will crash.

Note the alternative expression must return the same type as the original.

```
exception Fail
exception NoChange of int

raise Fail
raise NoChange 123
```

- `Fail` and `NoChange` are constructors for the `exc` type. They have type `exc` and `int -> exc` respectively.
- Since exceptions are values, we can pattern match on them.

```
try
  (* do stuff *)
with
  | NoChange -> (* alternative computation *)
```

`raise` dynamically jumps to the nearest `try/with` handler that mathches the exception, it is can be used to jump to another part of the code.

Recursive Datatypes

`list` is built into the language because the list syntax is helpful, but underneath it is just using another data type.

```
type 'a list =
  | Nil
  | Cons of 'a * 'a list
```

Types are recursive by default, because we often want our types to be recursive. In fact, we need to use `nonrec` to disable this behaviour specifically.

Binary Tree

If the list has 2 tails, we have a binary tree.

```
type 'a tree =
  | Lf
  | Br of 'a * 'a tree * 'a tree
```

Basic functions on trees includes:

```
let rec count = function
  | Lf -> 0
  | Br (_, l, r) -> 1 + count l + count r

let rec depth = function
  | Lf -> 0
  | Br (_, l, r) -> 1 + max (depth l) (depth r)
```

Option or Exception?

Using the option type for fallible functions instead of using exceptions

- Pros: the detail that it can fail is included in the type signature of the function.
- Cons: if the exception is very rare, it is annoying to have a deeply nested match structure.

Dictionaries

Definition
Dictionary attaches values to identities (keys).

Operation	Description
`lookup`	Find an item in the dictionary.
`update/insert`	Replace/store an item in the dictionary.
`delete`	Remove an item from the dictionary.
`empty`	Creates the null dictionary with no keys.

Implementation with Association List

```
exception Missing

let rec lookup a = function
  | [] -> raise Missing
  | (k, v) :: ps when x = a -> v
  | _ :: ps -> lookup a ps

let rec update l k v = (k, v) :: l
```

- `lookup` is $O(n)$ because we have to go through each entry to find the key.
- `update` is $O(1)$ as we are only shadowing the previous value.

Note
If the `remove` function only removes the first match in the list, it will uncover the previous value.

The old version of the association list still exist, we can look at it if we have a pointer to that list. This is known as a **persistent data structure** as we are not modifying the original data structure.

Implementation with Binary Search Tree

- If the tree if balanced then lookup is $O(\log n)$.
- If unbalanced then lookup can be $O(n)$.

```
let rect lookup a = function
  | Lf -> raise Missing
  | Br ((k, v), l, r) when k = a -> v
  | Br ((k, v), l, r) when k < a -> lookup a l
  | Br ((k, v), l, r) -> lookup a r

let rect update a b = function
  | Lf -> Br ((a, b), Lf, Lf)
  | Br ((k, v), l, r) when k = a -> Br ((a, b), l, r)
  | Br ((k, v), l, r) when k < a -> update a l
  | Br ((k, v), l, r) -> update a r
```

Now removing a node does not uncover the previous value. If we have a pointer to the old tree, we can still see previous values in the dictionary.

Tree Traversal

Definition
The goal of **tree traversal** is to visit every node.

- Pre-order: visits the label first.
  - Gives a way to copy the elements of a tree.
  - Conversely post-order is a good way to deconstruct the tree.
- In-order: visits the label midway.
  - Using it on a binary search tree is know as **treesort**.
- Pos-order: visits the label last.
  - Linearise an **expression tree** into RPN.

The other possibilities are breadth-first and depth-first traversal, but they are quite hard to implement as they are not recursive algorithms.

Functional Arrays

Arrays update in-place, they are **imparative** and **mutable** data strcutures.

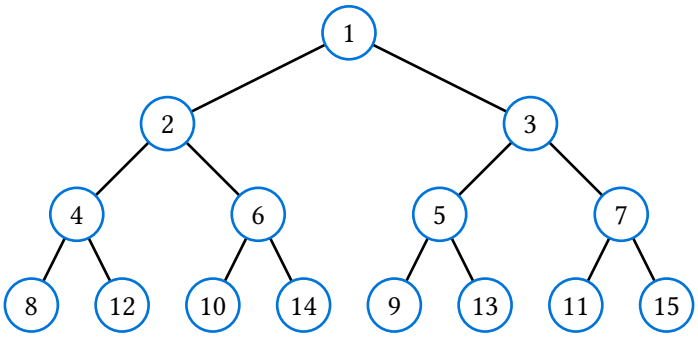
We want to keep the core of the system mutation free, so it is easier to analyse. E.g.
- It is much easier for a multi-core system to work on the same piece of immutable data.
- But they are less efficient than e.g. quicksort which sorts in-place.

Implementation

Functional arrays store values in a **balanced tree**, so access time to each element is $O(\log n)$.

Say we want to access the node with index 12 (`1100`), we read the binary digits from righ to left.
1. `0`, so go to the left subtree.

2. `0`, so go to the left subtree.
3. `1`, so go to the right subtree.
4. `1` is the only digit remaining, the node we are currently at is the node we are searching for.



Note the numbers represents the index of the node, not its content.

```
exception Subscript (* index out of bounds *)
let rec lookup i = function
  | Lf -> raise Subscript
  | Br (v, l, r) when i = 1 -> v
  | Br (v, l, r) when i mod 2 = 1 -> lookup (i / 2) r
  | Br (v, l, r) -> lookup (i / 2) l

let rec update i v = function
  | Lf when i = 1 -> Br (v, Lf, Lf)
  | Lf -> raise Subscript
  | Br (_, l, r) when i = 1 -> Br (v, l, r)
  | Br (_, l, r) when i mod 2 = 1 -> update (i / 2) v r
  | Br (_, l, r) -> update (i / 2) v l
```

Function Currying

Function as Values

Functions can be

- Passed as arugments into other functions
- Returned as results
- Put into lists, trees, etc

They cannot be tested for equality, it is in general really difficult to tell if two functions are equal.

Anonymous Functions

The `fun` keyword is a function that cannot use pattern matching.

```
(fun n -> n * 2) 17
```

We can assign functions to variables.

```
let double = fun n -> n * 2
(* is the same as let double n = n * 2 *)
```

The `function` keyword is a function that can be used for pattern matching.

```
(function
  | [] -> 0
  | x :: _ -> x)
```

Curried Functions

A function only have one argument, to take more arguments, we can either:

- Use a tuple
- Have a function that takes an argument, and return another function as result.

```
let add = (fun a -> (fun b -> a + b))
(* val add : int -> int -> int *)
```

The `->` operator is right associative, so the type is `int -> (int -> int)`.

Definition

A **curried function** is a function that returns another function as a result.

A function with multiple arguments is the shorthand syntax for a curried function.

When we partially apply functions, we turn a generic function into a more specific one.

Higher Order Functions

Definition

A **higher order function** is one that manipulates functions: takes a function as input or outputs a function.

Map

The `map` function applies `f` to all elements in the list.

```
let rec map f = function
  | [] -> []
  | x :: xs -> f x :: map f xs
```

List Functions for Predicates

Definition

A **predicate** is a boolean valued function.

```
let rec exists p = function
  | [] -> false
  | x :: xs -> (p x) || exists p xs

let rec all p = function
  | [] -> true
  | x :: xs -> (p x) && exists p xs

let rec filter p = function
  | [] -> []
  | x :: xs when p x -> (p x) :: filter p xs
  | x :: xs -> filter p xs
```

The type of an OCaml function is the most generic type it can be.

Definition

`a || b` is is syntactic sugar for

```
if a then true
else b
```

Data Streams

Most programs we use are in a **perception-action** loop.

- **Sequential programs** do a finite sequence of tasks, e.g. searching in a list.
- **Reactive programs** go into some never ending control loop.

Lazy Lists

Lazy lists are lists with possibly infinitely length.

- We cannot store lists with infinite length on computer.
- Instead we have a short data structure that compute items on demand.

In OCaml we can delay the evaluation of the tail of the list.

Definition
The **unit type** has only one possible value `()`, it behaves like a tuple.

- It is used to delay the evaluation of a **constant function**, the value of the function is not evaluated until the aruments are provided.
- It is also used as the return type of commands (functions with only side effects).

```
type 'a seq =
  | Nil
  | Cons of 'a * (unit -> 'a seq)

let head = function
  | Cons (x, _) -> x
  | Nil -> raise (Failure "head")

let tail = function
  | Cons (_, xf) -> xf ()
  | Nil -> raise (Failure "tail")
```

To create an infinite sequence starting from k

```
let rec from k = Cons (k, fun () -> from (k + 1))
```

Operations on Sequences

```
(* take first n elements of a sequence *)
let rec get n s =
  if n = 0 then []
  else match s with
  | Nil -> []
  | Cons (x, xf) -> x :: get (n - 1) (xf ())

(* join 2 sequences, this has no effect if xq is infinite *)
let rec append xq yq =
  match xq with
  | Nil -> yq
  | Cons (x, xf) -> x :: append (xf ()) yq

(* interleaving 2 sequences *)
let rec interleave xq yq =
```

```
  match xq with
  | Nil -> yq
  | Cons (x, xf) ->
      Cons (x, fun () -> interleave yq (xf ()))
```

Higher Order Functions on Sequences

```
(* if something doesn't satisfies the predicate *)
(* continue until something satisfies the predicate *)
let rec filter p = function
  | Nil -> Nil
  | Cons (x, xf) when p x -> Cons (x, fun () -> filter p (xf()))
  | Cons (x, xf) -> filter p (xf())

(* returns x, f(x), f(f(x)), f(f(f(x))), ... *)
let rec iterates f x =
  Cons (x, fun () -> iterates f (f x))
```

Note that function application is left associative, whereas type arrows (->) are right associate.

Queues, Stacks and Tree Traversal

The pre/in/post-order tree traversals are variants of a depth-first traversal.

Queues

A queue is an FIFO structure with these function

Function	Description
qempty	Creates an empty list
qnull	Checks of an empty list
qhd	Returns head
deq	Returns tail
enq	Add item to back of queue

Suppose we have a pair of lists: the first one stores items in order, the other in reverse order.

$$\text{queue}([x_1; x_2; \cdots; x_m], [y_1; y_2; \cdots; y_m])$$

- deq remove items from the front of the first list.
- enq add items to the front of the rear list.

If the front list is empty, reverse the rear and move to front.

For a queue of length n

- There are n enq and n deq operations, cost $2n$
- 1 reverse list operation, cost n.

So the **amortised time** per operation is $O(1)$, note the operations are not spread out evenly.

```
type 'a queue =
  | Q of 'a list * 'a list

let norm = function
  | Q ([], tls) -> Q (List.rev tls, [])
  | q -> q
```

The norm function normalises the queue so it satisfies the property: if the front list is empty, the tail list is also empty.

```
exception Empty

let enq a = function
  | Q (xs, ys) -> Q (xs, x :: ys)

let deq = function
  | Q ([], _) -> raise Empty
  | Q (x :: xs, ys) -> norm (Q (xs, ys))
```

We create a data type of queue instead of just letting the user pass in a tuble pair
- So we can constraint what the lists can be (we require the list to have certain properties)
- This property is not reflected on teh type signature.

Breadth-first Traversal

```
let rec breadth q =
  if qnull q then []
  else
    match qhd q with
    | Lf -> breadth (deq q)
    | Br (x, l, r) -> x :: breadth (enq (enq (deq q) l) r)
```

Breadth-first search examines $1 + b + b^2 + \cdots + b^d$ nodes, which is in $O(b^d)$.

Breadth-first search is not suitable for infinite trees as it stores parts of the tree in memory.

Depth-first Iterative Deepening

Iterative deepening peforms repeated depth-first search with increasing depth bounds.
1. Searches to depth 1
2. Discard the previous search, and searches to depth 2.
3. Repeats until found.

The time needed for iterative deepening is only $\frac{b}{b-1}$ times of breadth-first search.

Best-first Search

Similar to breadth-first search, but uses a priority queue where items are ranked using a heuristic to approximate the distance of a node to the solution.

Algorithm	Data structure used	Time complexity	Space complexity
Depth-first	(Call) stack	$O(b^d)$	$O(d)$
Breadth-first	Queue	$O(b^d)$	$O(b^d)$
Iterative deepening	(Call) stack	$O(b^d)$	$O(d)$
Best-first	Priority queue	Depends	Depends