Ordinary Differential Equations

Definition
An **nth order ODE** includes the nth derivate but no higher derivatives.

To solve an ODE we find the dependent variable as a function of the independent variables.
- In the real world: this is done numerically with computers.
- Here we do it analytically to demonstrate principles.

Some ODEs don't have analytical solutions.

First Order ODEs

Definition
Integrable ODEs have form

$$\frac{dy}{dx} = f(x)$$

where the RHS does not depend on y.

$$y = \int f(x) dx$$

If $F'(x) = f(x)$, then $y = F(x) + C$. y solves the ODE whatever C is - threr are infinitely many solutions.

$(*)$ is called the **general solution** if it contains all possible solutions.

If we choose any value for C, we get a particular solution.

Definition
Separable ODEs have form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y)\ \frac{dy}{dx} = f(x)$$
$$\int g(y)\ \frac{dy}{dx} dx = \int f(x) dx$$
$$\int g(y) dy = \int f(x) dx$$
$$G(y) = F(x) + C$$

Geometric Interpretation of First Order ODEs

Consider $\frac{dy}{dx} = F(x, y)$

For every point (x, y):
- There is a particular solution passing through the point
- Its gradient is $F(x, y)$

Note
2 particular solutions don't have to be the same shape (shifted versions) of each other.

Linear ODE

$$\frac{dy}{dx} + y \cdot p(x) = f(x)$$

When $f(x) = 0$, the ODE is homogeneous, two ways to solve it

$$
\begin{aligned}
\frac{dy}{dx} &= -y \cdot p(x) \\
\int \frac{1}{y} dy &= -\int p(x) dx \\
y &= Ae^{-P(x)}
\end{aligned}
\qquad
\begin{aligned}
e^{P(x)} \cdot \frac{dy}{dx} + p(x) \cdot e^{P(x)} \cdot y &= 0 \\
\frac{d}{dx}\left(e^{P(x)} \cdot y\right) &= 0 \\
y &= Ae^{-P(x)}
\end{aligned}
$$

Note
$e^{\int p(x)\, dx}$ is called the integrating factor.

For inhomogenous cases, multiply both sides by the integrating factor.

$$
\begin{aligned}
e^{P(x)} \cdot \frac{dy}{dx} + p(x) \cdot e^{P(x)} \cdot y &= f(x) \\
\frac{d}{dx}\left(e^{P(x)} \cdot y\right) &= f(x) \\
e^{P(x)} \cdot y &= F(x) + C \\
y &= (F(x) + C)e^{-P(x)}
\end{aligned}
$$

Definition
nth order ODE contains n arbitrary constants: n pieces of informations to fix them. The extra information are called **boundary conditions**.

Note
A **particular solution** has no unknown constants or $\pm$ signs.

Substitutions

$$
\frac{dy}{dx} = f\left(\frac{y}{x}\right)
$$

$$
\begin{aligned}
\text{let } u(x) &= \frac{y(x)}{x} \\
x \cdot u(x) &= y(x) \\
u + x \cdot \frac{du}{dx} &= \frac{dy}{dx}
\end{aligned}
$$

For example

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{x^2 + y^2}{xy} \\
&= \frac{(x/y)^2 + 1}{x/y} \\
u + x \cdot \frac{du}{dx} &= \frac{u^2 + 1}{u} \\
x \cdot \frac{du}{dx} &= \frac{1}{u} \\
&\vdots
\end{aligned}
$$

Benoulli Equations

$$\frac{dy}{dx} + p(x) \cdot y = q(x) \cdot y^n$$

If $y = 0$ or $y = 1$, then the equation is homogenous, otherwise

$$\begin{aligned}
\text{let } z(x) &= y(x)^{1-n} \\
\frac{dz}{dx} &= (1-n) \cdot y(x)^{-n} \cdot \frac{dy}{dx} \\
\frac{1}{1-n} \cdot \frac{dz}{dx} \cdot y(x)^n &= \frac{dy}{dx} \\
\frac{1}{1-n} \cdot \frac{dz}{dx} \cdot y^n + p(x) \cdot y &= q(x) \cdot y^n \\
\frac{1}{1-n} \cdot \frac{dz}{dx} + p(x) \cdot y^{1-n} &= q(x) \\
\frac{dz}{dx} + (1-n) \cdot p(x) \cdot z &= (1-n) \cdot q(x) \\
&\vdots
\end{aligned}$$

2nd Order ODE

We consider linear cases only: the ODE can be written as $Ly = f$.

L is the **differential operator**

$$L = \left[\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)\right]$$

Similar to how d/dx is an operator on functions.

Definition
A **linear operator** has properties:
- $L(\alpha u) = \alpha L(u)$
- $L(u + v) = L(u) + L(v)$

Note
$\frac{d}{dx}$ is a linear operator.

The differential operator is a linear operator that includes $\frac{d^n}{dx^n}$.

Definition
A **linear ODE** can be written as $Ly = f$ where L is a linear operator.
- $f = 0$: homogeneous
- $f \neq 0$: inhomogenous

The Principle of Superposition

If y_1 and y_2 are particular solutions of a homogeneous ODE
- Then so is $y_1 + y_2$
- And so is $\alpha y_1 + \beta y_2$ for any $\alpha, \beta \in \mathbb{C}$

If:

- The **particular integral** y_p solves an inhomogenous linear ODE, and
- The **complementary function** y_c is the general solution of the homogeneous equation of the complementary ODE

Then the general solution is $y = y_p + y_c$

2nd Order ODE with Constant Coefficients

Is the restricted case where $p(x), q(x)$ are restricted to constants.

Homogenous Equations

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = 0$$

Try $y(x) = e^{\lambda x}$

$$e^{\lambda x}(\lambda^2 + 2p\lambda + q) = 0$$
$$\lambda = e^{-p \pm \sqrt{p^2 - q}}$$

- Case $p^2 > q$:

$$\lambda_1 = -p + \sqrt{p^2 - q}$$
$$\lambda_2 = -p - \sqrt{p^2 - 1}$$

  Then $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$ is the general solution, as the general solution has 2 solutions.
- Case $p^2 < q$: let $\Omega = \sqrt{q - p^2}$

$$\lambda_1 = -p + i\Omega$$
$$\lambda_2 = -p - i\Omega$$

  Then

$$y = \frac{1}{2}\left(e^{\lambda_1 x} + e^{\lambda_2 x}\right) = \Re\left(e^{\lambda_1 x}\right) = e^{-px}\cos\Omega x$$
$$y = \frac{1}{2i}\left(e^{\lambda_1 x} - e^{\lambda_2 x}\right) = \Im\left(e^{\lambda_1 x}\right) = e^{-px}\sin\Omega x$$

  are both solutions. So the general solution is

$$y = e^{-px}(A\cos\Omega x + B\sin\Omega x)$$

  If $A = R\cos\varphi$ and $B = R\sin\varphi$

$$y = Re^{-px}\cos(\Omega x - \varphi) \text{ by double angle formula}$$

- Case $p^2 = q$: cannot use principle of solution because we only have one solution.

  Observe xe^{-px} is also a solution, then

$$y = e^{-px}(A + Bx)$$

  is the general solution.

Inhomogenous Equations

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = f(x)$$

Then the general solution is $y = y_c + y_p$ (complementary equation and particular integral)

To find y_p, guess the functional form and solve for some parameters.

- If f is a polynomial of degree n, try a general polynomial of degree n
- If f is an exponential, try a multiple of the same exponential.
- If f is in form $u \sin kx + v \cos kx$, try $a \sin kx + b \cos kx$
- If f is a sum of two known cases, take the y_p of each of them and add them.

Resonance, Transients and Damping

A damped oscillator is described by

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + (\omega_0)^2 x = 0$$

where $x(t)$ is the time dependent position, γ, ω_0 are positive constants.

- The parameter γ describes friction/damping.
- If $\gamma = 0$ then **simple harmonic oscillator**.
- The equilibrium position is $x = 0$

Solving the homogeneous equation

$$\lambda = -\gamma \pm \sqrt{\gamma^2 - (\omega_0)^2}$$

- Case $\gamma > \omega_0$:

  $$x(t) = Ae^{\left(-\gamma + \sqrt{\gamma^2 - (\omega_0)^2}\right)t} + Be^{\left(-\gamma - \sqrt{\gamma^2 - (\omega_0)^2}\right)t}$$

  This is called **overdamping**: the bigger the γ, the slower it returns to equilibrium.
- Case $\gamma < \omega_0$: let $\Omega = \sqrt{(\omega_0)^2 - \gamma^2}$

  $$x(t) = e^{-\gamma t}(A \cos \Omega t + B \sin \Omega t)$$

  This is the **underdamped** case.
- Case $\gamma = \omega_0$: repeated roots give solution

  $$y = e^{-\gamma t}(A + Bx)$$

  This is the boundary between overdamping and underdamping, the graph looks like the overdamped case.

Sinusoidal Forces

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + (\omega_0)^2 x = f_0 \cos \omega t$$

Try $x_{p(t)} = a \cos \omega t + b \sin \omega t$ to find

$$-a\omega^2 + 2\gamma b\omega + a(\omega_0)^2 = f_0 \quad (1)$$
$$-b\omega^2 - 2\gamma a\omega + b(\omega_0)^2 = 0$$

Solving for a

$$a = \frac{b\left((\omega_0)^2 - \omega^2\right)}{2\gamma a}$$

Substitute a into (1)

$$a\left((\omega_0)^2 - \omega^2\right) + 2\gamma b\omega = f_0$$
$$\frac{b\left((\omega_0)^2 - \omega^2\right)^2}{2\gamma\omega} + 2\gamma\omega b = f_0$$
$$b\left(\frac{\left((\omega_0)^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}{2\gamma\omega}\right) = f_0$$
$$b = \frac{2\gamma\omega f_0}{\left((\omega_0)^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}$$
$$a = \frac{\left((\omega_0)^2 - \omega^2\right) f_0}{\left((\omega_0)^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}$$

Then $x = a\cos\omega t + b\sin\omega t$

Let $\mu f_0 = \sqrt{a^2 + b^2}$, so $x = \mu f_0 \cos(\omega t - \varphi)$

$$\mu = \frac{1}{\sqrt{\left((\omega_0)^2 - \omega^2\right)^2 + 4\gamma^2\omega f_0}}$$

- The size of osciallation is proportional to μ and f_0
- The phase difference is φ

When damping is small, μ has a peak near $\omega = \omega_0$ when μ is plotted against ω_0