2. Linear Motion

Symbol	**Meaning**	**Definition**
V	potential	$\int_{x_0}^{x} F(x')dx'$
F	force	$-\frac{dV}{dx}$
T	kinetic energy	$\frac{1}{2}m\dot{x}^2$

From $T = \frac{1}{2}m\dot{x}^2$, we find:
- $\dot{T} = m\ddot{x}\dot{x} = F(x)\dot{x}$
- $T = \int F(x)dx$

Conservation of energy states $V + T = E$ which is constant.

The Harmonic Oscillator

Symbol	**Meaning**	**Definition**
f	frequency	$\frac{\omega}{2\pi}$
τ	time period	$\frac{2\pi}{\omega}$

The harmonic oscillator at equilibrium $x = 0$ satisfies $V'(0) = 0$. Choose constant E so $V(0) = 0$.

$$\begin{aligned} V(x) &= V(0) + xV'(0) + \frac{1}{2}x^2V''(0) + \ldots \\ &\approx \frac{1}{2}kx^2 \text{ where } k = V''(0) \end{aligned}$$

Then $F(x) = -kx$.
- A conservative force depends only on x
- A dissipative force additionally depend on variables other than x, such as velocity.
- All forces are conservative at microscopic scale.

Undamped oscillator:

$$\begin{aligned} m\ddot{x} &= -kx \\ \ddot{x} + \frac{k}{m}x &= 0 \\ x &= \frac{1}{2}Ae^{pt} + \frac{1}{2}Be^{-pt} \end{aligned}$$

where $p = \sqrt{-k/m}$

If $k < 0$, it is an unstable equilibrium.

If $k > 0$, it is a stable equilibrium.

$$x = \frac{1}{2}Ae^{i\omega t} + \frac{1}{2}Be^{-i\omega t} \text{ where } \omega = \sqrt{\frac{k}{m}}$$

For x to have no imagniary component, let
- $A = ae^{-i\theta}$
- $B = ae^{i\theta}$

Then

$$\begin{aligned} x &= \frac{1}{2}ae^{i(\omega t - \theta)} + \frac{1}{2}ae^{-i(\omega t - \theta)} \\ &= a\cos(\omega t - \theta) \end{aligned}$$

Note if $a\ddot{z} + b\dot{z} + cz = 0$ is a solution, where $z = x + iy$, then so is $a\ddot{x} + b\dot{x} + cx = 0$ and the y equivalent.

The Damped Oscillator

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

Let $\gamma = \frac{\lambda}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$.

Large damping if $\gamma > \omega_0$

- Let $\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - (\omega_0)^2}$

$$x = \frac{1}{2}Ae^{-\gamma_+ t} + \frac{1}{2}Be^{-\gamma_- t}$$

The leading term is $\frac{1}{2}Be^{-\gamma_-}$, so the characteristic time is $\frac{1}{\gamma_-}$

Small damping if $\gamma < \omega_0$

- Let $\omega = \sqrt{(\omega_0)^2 - \gamma^2}$

$$\begin{aligned} x &= \frac{1}{2}Ae^{-\gamma + \omega_0 t} + \frac{1}{2}Be^{-\gamma - \omega_0 t} \\ &= ae^{-\gamma t}\cos(\omega_0 t - \theta) \end{aligned}$$

where $A = ae^{-i\theta t}$, $B = ae^{i\theta t}$

- The relaxation time is $\frac{1}{\gamma}$
- The amplitude reduction in a single period is $e^{\pi/Q}$ where $Q = \omega_0/2\gamma$

Critical damping when $\gamma = \omega_0$

$$x = (a + bt)e^{-\gamma t}$$

Resonance

Under a periodic force $F(t) = F_1\cos(\omega_1 t)$, solving the differential equation gives solution the equation for an undamped, forced oscillation.

$$\begin{aligned} x &= a_1\cos(\omega_1 t - \theta_1) \\ amplitude\ a_1 &= \frac{F_1/m}{\sqrt{\left((\omega_0)^2 - (\omega_1)^2\right)^2 + 4\gamma^2(\omega_1)^2}} \\ phase \tan\theta_1 &= \frac{2\gamma\omega_1}{(\omega_0)^2 - (\omega_1)^2} \end{aligned}$$

- When ω_1 is low, $\theta_1 \approx 0$, and x satisfies $F(t) - kx = 0$.
- At resonance $\omega_1 = \sqrt{(\omega_0)^2 - 2\gamma^2}$, $\theta_1 = \frac{\pi}{2}$, $a_1 = F_1/2m\gamma\omega_1$.
- When ω_1 is very high, $\theta_1 \approx \pi$ (out of phase), $x \approx 0$.

So the damping constant λ only matters when near resonance.