Infinite Series

Convergence

Given an infinite sequence of numbers, define the **partial sum** to be

$$S_n = u_1 + u_2 + \cdots + u_n$$

We want to know if the infinite series
- Have a well defined limit
- Diverges
- Oscillates

$$\lim_{n \to \infty} S_n = S \text{ iff } (\forall \varepsilon > 0)(\exists N) \; n > N \implies |S - S_n| < \varepsilon$$

Conditions for Convergence

Requires $u_n \to 0$ as $n \to \infty$.

The sum of two diverging series may converge, e.g. when one diverges to $+\infty$ and the other to $-\infty$, and the two cancels each other out.

> **Definitions**
> - A series is **absolutely convergent** iff $\sum |a_n|$ converges.
> - A series is **conditionally convergent** iff $\sum a_n$ converges but not $\sum |a_n|$.

If we rearrange the terms in a series, a converging series can diverge, or the other way round.
- If we have an absolutely convergent series, it doesn't matter if we change the order.
- If we have a conditionally convergent series, changing the order may cause it to diverge.

Geometric Progression

Let $u_k = r^k$, we already know the sum formula for the geometric progression, here's the proof.

$$(1 - r)(1 + r + r^2 + \cdots + r^k) = 1 - r^k \quad \text{(by simplifying the expression)}$$

$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^k}{1 - r}$$

For the sum of infinite series

$$\lim_{k \to \infty} \frac{1 - r^k}{1 - r} \text{ exists if } |r| < 1.$$

The Harmonic Series

The harmonic series is

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

We can group terms **without reordering**.

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{7}\right) + \cdots \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

The sequence $\sum \frac{1}{2}$ diverges, so the harmonic series also diverges, this is called a **comparison test**.

$$\gamma = \lim_{k \to \infty} (S_k - \ln k) = 0.57721566$$

Which is the **Euler-Mascheroni constant**.

Convergence Tests

Comparison Tests

The comparison test can be used on two nonnegative series: $(\forall k) u_k, v_k \geq 0$

$$(\forall k \geq K) \ u_k \leq v_k \text{ and } \sum_{k=0}^{\infty} v_k \text{ convergent} \implies \sum_{k=0}^{\infty} u_k \text{ convergent}$$

$$(\forall k \geq K) \ u_k \geq v_k \text{ and } \sum_{k=0}^{\infty} v_k \text{ divergent} \implies \sum_{k=0}^{\infty} u_k \text{ divergent}$$

Ratio Tests

For a positive series: $(\forall k) \ u_k > 0$

$$\lim_{k \to \infty} \left(\frac{u_{k+1}}{u_k} \right) < 1 \implies \text{converges}$$

$$\lim_{k \to \infty} \left(\frac{u_{k+1}}{u_k} \right) > 1 \implies \text{diverges}$$

Proof

If $\lim_{k \to \infty} \frac{u_{k+1}}{u_k} = \alpha < \infty$, then for large enough k:

$$\sum u = u_1 + u_2 + \cdots + u_k (1 + \alpha + \alpha^2 + \cdots)$$

Which converges.

Leibniz Criterion

> **Definition**
> **Alternating series** have terms with alternating signs.

If $a_k > 0$ and a_k is monotonically decreasing for large enough k, and $\lim_{k \to \infty} a_k = 0$.

$$S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges}$$

Proof

We know it is true that

$$\sum_{k=1}^{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots \text{ is a monotonically increasing series}$$

$$\sum_{k=1}^{2n+1} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots \text{ is a monotonically decreasing series}$$

$$\lim_{k \to \infty} \left(\sum_{k=1}^{2n+1} a_k - \sum_{k=1}^{2n} a_k \right) = 0$$

So we write the inequality

$$a_1 - a_2 = S_2 < S_4 < \cdots < S_{2n} < S_{2n+1} < \cdots < S_3 < S_1 < a_1$$

Power Series

A power series have form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

The **domain of convergence** is either
- Only $x = 0$
- All finite x
- Only for some $|x| < R$

We can find out which case it is with the **ratio test**, which gives the shortcut if $\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$ exists.

$$|x| < \frac{1}{L} \implies \text{converges}$$

$$|x| > \frac{1}{L} \implies \text{diverges}$$

$$|x| = \frac{1}{L} \implies \text{indeterminate}$$

Note the endpoints at $|x| = 1L$ may behave differently.

The Taylor Series

We can approximate the value of a function using its tangent at a point.

$$f(x) \approx f(a) + (x - a)f'(a)$$
$$f'(x) \approx f'(a) + (x - a)f''(a)$$

By the fundamental theorem of calculus, which links the derivative to the integral:

$$\int_a^{a+h} f'(x)dx = f(a + h) - f(a)$$

$$f(a + h) = f(a) + \int_a^{a+h} f'(x)dx$$

$$= f(a) + \int_a^{a+h} f'(a) + (x - a)f''(a)dx$$

$$= f(a) + \left[xf'(a) + \frac{(x - a)^2}{2} f''(a) \right]_a^{a+h}$$

$$= f(a) + hf'(a) + \frac{h^2}{2} f''(a)$$

We can show that the **second order approximation** is better than the first order approximation. (how?)

Extend this to higher order approximations, and we get

$$f'(x) \approx f'(a) + (x - a)f''(a) + \frac{(x - a)^2}{2!} f'''(a) + \frac{(x - a)^3}{3!} f''''(a) + \cdots$$

Since the factorial function grows faster than any polynomial, we know that $n!$ grows faster than $(x - a)^n$ so the series converges for a lot of approximations.

Taylor's Theorem

An exact result can be written by including a **remainder term** R_{n+1}.

$$f(a + h) = f(a) + hf'(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + R_{n+1}$$

Taylor's Theorem states if f is $n + 1$ times differentiable, there exists a $\zeta : a < \zeta < a + h$ such that

$$R_{n+1} = \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\zeta)$$

Proof

$$\int_a^x f'(t)dt = f(x) - f(a) \text{ by fundamental theorem of calculus}$$

$$f(x) = f(a) + \int_a^x f'(t)dt$$

$$= f(a) + [(t - x)f'(t)]_a^x - \int_a^x (t - x)f''(x)dt \text{ integrate by parts}$$

$$= f(a) + (x - a)f'(a) + \int_a^x (t - x)f''(x)dt$$

Integrate by parts repeatedly to find

$$f(x) = f(a) + (x - a)f'(a) + \cdots + \frac{(x - a)^n}{n!}f^{(n)}(a) + \int_a^x \frac{(t - x)^n}{n!}f^{(n+1)}(t)dt$$

$$R_{n+1} = \int_a^x \frac{(t - x)^n}{n!}f^{(n+1)}(t)dt$$

$$= \frac{(x - a)^{n+1}}{(n+1)!}f^{(n+1)}(\zeta) \text{ where } a \leq \zeta \leq x \text{ by mean value theorem}$$

$$\lim_{n \to \infty} R_{n+1} = 0 \implies \text{Taylor series for } f(x) \text{ converges}$$

By choosing the ζ that would give the largest error, we can calculate the worst case error for an approximation.

> **Note**
> If $f(x)$ is infinitely differentiable, then we can represent f exactly as an infinite **power series**.

We could also prove that two functions are the same if they have the same Taylor series.

Common Series Expansions

A lot of these expansion can be proved using the expansion for e^x.

Function	Taylor series
e^x	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Function	Taylor series
$\ln(1 + x)$	$\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$

Function	Taylor series	Function	Taylor series
$\cosh x$	$\displaystyle\sum_{n=0} \frac{x^{2n}}{(2n)!}$	$\sinh x$	$\displaystyle\sum_{n=0} \frac{x^{2n+1}}{(2n+1i)!}$
$\cos x$	$\displaystyle\sum_{n=0} (-1)^n \frac{x^{2n}}{(2n)!}$	$\sin x$	$\displaystyle\sum_{n=0} (-1)^n \frac{x^{2n+1}}{(2n+1i)!}$

It makes sense that the Taylor series of an even function only have even powered terms, and an odd function only have odd powered terms.

The series expansion for *tanh* has a radius of convergence of $|x| < \frac{\pi}{2}$, this makes sense because
- Polynomials cannot capture the flat asymptote.
- *tan* is only continuous over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.