Infinite Series

Convergence

Given an infinite sequence of numbers, define the **partial sum** to be

$$S_n = u_1 + u_2 + \cdots + u_n$$

We want to know if the infinite series
- Have a well defined limit
- Diverges
- Oscillates

$$\lim_{n\to\infty} S_n = S \text{ iff } (\forall \varepsilon > 0)(\exists N)\ n > N \Longrightarrow |S - S_n| < \varepsilon$$

Conditions for Convergence

Requires $u_n \to 0$ as $n \to \infty$.

The sum of two diverging series may converge, e.g. when one diverges to $+\infty$ and the other to $-\infty$, and the two cancels each other out.

Definitions
- A series is **absolutely convergent** iff $\sum |a_n|$ converges.
- A series is **conditionally convergent** iff $\sum a_n$ converges but not $\sum |a_n|$.

If we rearrange the terms in a series, a converging series can diverge, or the other way round.
- If we have an absolutely convergent series, it doesn't matter if we change the order.
- If we have a conditionally convergent series, changing the order may cause it to diverge.

Geometric Progression

Let $u_k = r^k$, we already know the sum formula for the geometric progression, here's the proof.

$$(1-r)\left(1 + r + r^2 + \cdots + r^k\right) = 1 - r^k \quad \text{(by simplifying the expression)}$$
$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^k}{1 - r}$$

For the sum of infinite series

$$\lim_{k\to\infty} \frac{1 - r^k}{1 - r} \text{ exists if } \ |r| < 1.$$

The Harmonic Series

The harmonic series is

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

We can group terms **without reordering**.

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{7}\right) + \cdots \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

The sequence $\sum \frac{1}{2}$ diverges, so the harmonic series also diverges, this is called a **comparison test**.

The harmonic series is the slowest diverging sequence, to show this series diverges as slow as $\log x$

$$\gamma = \lim_{k \to \infty} (S_k - \ln k) = 0.57721566$$

Which is the **Euler-Mascheroni constant**.