

2. Linear Motion

Symbol	Meaning	Definition
V	potential	$\int_{x_0}^x F(x')dx'$
F	force	$-\frac{dV}{dx}$
T	kinetic energy	$\frac{1}{2}m\dot{x}^2$

From $T = \frac{1}{2}m\dot{x}^2$, we find:

- $\dot{T} = m\ddot{x}\dot{x} = F(x)\dot{x}$
- $T = \int F(x)dx$

Conservation of energy states $V + T = E$ which is constant.

The Harmonic Oscillator

Symbol	Meaning	Definition
f	frequency	$\frac{\omega}{2\pi}$
τ	time period	$\frac{2\pi}{\omega}$

The harmonic oscillator at equilibrium $x = 0$ satisfies $V'(0) = 0$. Choose constant E so $V(0) = 0$.

$$\begin{aligned} V(x) &= V(0) + xV'(0) + \frac{1}{2}x^2V''(0) + \dots \\ &\approx \frac{1}{2}kx^2 \text{ where } k = V''(0) \end{aligned}$$

Then $F(x) = -kx$.

- A conservative force depends only on x
- A dissipative force additionally depend on variables other than x , such as velocity.
- All forces are conservative at microscopic scale.

Undamped oscillator:

$$\begin{aligned} m\ddot{x} &= -kx \\ \ddot{x} + \frac{k}{m}x &= 0 \\ x &= \frac{1}{2}Ae^{pt} + \frac{1}{2}Be^{-pt} \end{aligned}$$

where $p = \sqrt{-k/m}$

If $k < 0$, it is an unstable equilibrium.

If $k > 0$, it is a stable equilibrium.

$$x = \frac{1}{2}Ae^{i\omega t} + \frac{1}{2}Be^{-i\omega t} \text{ where } \omega = \sqrt{\frac{k}{m}}$$

For x to have no imaginary component, let

- $A = ae^{-i\theta}$
- $B = ae^{i\theta}$

Then

$$\begin{aligned} x &= \frac{1}{2}ae^{i(\omega t - \theta)} + \frac{1}{2}ae^{-i(\omega t - \theta)} \\ &= a \cos(\omega t - \theta) \end{aligned}$$

Note if $a\ddot{z} + b\dot{z} + cz = 0$ is a solution, where $z = x + iy$, then so is $a\ddot{x} + b\dot{x} + cx = 0$ and the y equivalent.

The Damped Oscillator

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

Let $\gamma = \frac{\lambda}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$.

Large damping if $\gamma > \omega_0$

- Let $\gamma_{\pm} = \gamma \pm \sqrt{\gamma^2 - (\omega_0)^2}$

$$x = \frac{1}{2}Ae^{-\gamma_+ t} + \frac{1}{2}Be^{-\gamma_- t}$$

The leading term is $\frac{1}{2}Be^{-\gamma_- t}$, so the characteristic time is $\frac{1}{\gamma_-}$

Small damping if $\gamma < \omega_0$

- Let $\omega = \sqrt{(\omega_0)^2 - \gamma^2}$

$$\begin{aligned} x &= \frac{1}{2}Ae^{-\gamma+\omega_0 t} + \frac{1}{2}Be^{-\gamma-\omega_0 t} \\ &= ae^{-\gamma t} \cos(\omega_0 t - \theta) \end{aligned}$$

where $A = ae^{-i\theta t}$, $B = ae^{i\theta t}$

- The relaxation time is $\frac{1}{\gamma}$
- The amplitude reduction in a single period is $e^{\pi/Q}$ where $Q = \omega_0/2\gamma$

Critical damping when $\gamma = \omega_0$

$$x = (a + bt)e^{-\gamma t}$$

Resonance

Under a periodic force $F(t) = F_1 \cos(\omega_1 t)$, solving the differential equation gives solution the equation for an undamped, forced oscillation.

$$\begin{aligned} x &= a_1 \cos(\omega_1 t - \theta_1) \\ \text{amplitude } a_1 &= \frac{F_1/m}{\sqrt{((\omega_0)^2 - (\omega_1)^2)^2 + 4\gamma^2(\omega_1)^2}} \\ \text{phase } \tan \theta_1 &= \frac{2\gamma\omega_1}{(\omega_0)^2 - (\omega_1)^2} \end{aligned}$$

- When ω_1 is low, $\theta_1 \approx 0$, and x satisfies $F(t) - kx = 0$.
- At resonance $\omega_1 = \sqrt{(\omega_0)^2 - 2\gamma^2}$, $\theta_1 = \frac{\pi}{2}$, $a_1 = F_1/2m\gamma\omega_1$.
- When ω_1 is very high, $\theta_1 \approx \pi$ (out of phase), $x \approx 0$.

So the damping constant λ only matters when near resonance.

General Periodic Forces

Any periodic force can be written as a sum of harmonics.

$$\begin{aligned} F(t) &= \sum_{n=-\infty}^{\infty} F_n e^{in\omega t} \\ F_m &= \frac{1}{\tau} \int_0^{\tau} F(t) e^{-im\omega t} dt \end{aligned}$$

The position is the sum of the general solutions for each harmonic.

$$x = \sum_{n=-\infty}^{\infty} A_n e^{in\omega t} + \text{transient}$$

General Forces

Define impulse delivered in Δt be $I = \Delta p = F(t)\Delta t$.

An oscillator at rest when given impulse I has $\dot{x} = \frac{I}{m}$, solving for small damping:

$$x(t) = \begin{cases} 0 & \text{when } t < 0 \\ \frac{I}{m\omega} e^{-\gamma t} \sin \omega t & \text{when } t \geq 0 \end{cases}$$

For a series of impulses I_r at t_r

$$x(t) = \sum_r G(t - t_r) I_r + \text{transient}$$

where the Green's function G is defined as

$$G(t') = \begin{cases} 0 & \text{when } t' < 0 \\ \frac{I}{m\omega} e^{-\gamma t'} \sin \omega t' & \text{when } t' \geq 0 \end{cases}$$

Then for any force $F(t)$

$$x(t) = \int_0^t G(t' - t) F(t') dt' + \text{transient}$$