

# Integration

## Integration as Area

To find the area under the curve  $y = f(x)$  in range  $a \leq x \leq b$ .

1. **Partition**  $[a, b]$  into  $N$  subintervals with endpoints  $x_0, x_1, \dots, x_N$ , the width of the partitions does not have to be equal.
2. Choose  $N$  points  $\xi_1, \xi_2, \dots, \xi_N$  in each of the partitions. The rectangle of each subpartition has area  $A_i = (x_i - x_{i-1})f(\xi_i)$ .
3. The total area is the **Riemann sum**

$$S_N = \sum_{i=1}^N (x_i - x_{i-1})f(\xi_i)$$

4. Take the **Riemann integral** where  $N \rightarrow \infty$  such that all  $x_i - x_{i-1} \rightarrow 0$ .

$$\int_a^b f(x)dx = \lim_{N \rightarrow \infty} S_N$$

The integral in this form is also called a **definite integral**.

## Properties of an Integral

From the geometric interpretation.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

But in the definition we assume  $a \leq x \leq b$ , to make the above property true for all  $c$ , we can define

$$\int_a^b f(x)dx = - \int_b^a f(x)dx$$

Integration and differentiation are **linear operations**, this means

$$\begin{aligned} \int_a^b k f(x)dx &= k \int_a^b f(x)dx \\ \int_a^b f(x) + g(x)dx &= \int_a^b f(x)dx + \int_a^b g(x)dx \end{aligned}$$

In  $\infty + (-\infty)$  scenarios,  $\int f + g dx$  may be finite, but  $\int f dx + \int g dx$  are both undefined.

## Fundamental Theorem of Calculus

$$F(x) = \int_a^x f(u)du \iff \frac{dF}{dx} = f(x)$$

Or

$$\frac{d}{dx} \int_a^x f(u)du = f(x)$$

## Proof

$$\begin{aligned}\frac{dF}{dx} &= \lim_{\delta x \rightarrow 0} \frac{F(x + \delta x) - F(x)}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \frac{\int_a^{x+\delta x} f(u)du + \int_a^x f(u)du}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \frac{\int_x^{x+\delta x} f(u)du}{\delta x} \\ &= f(x)\end{aligned}$$

### Definitions

In  $F(x) = \int f(x)dx$

- $f(x)$  is the **integrand**.
- $F(x)$  is the **primitive**.

## Definite and Indefinite Integrals

The primitive is also called the **indefinite integral** of  $f(x)$ .

- If  $F(x)$  is the indefinite integral, then so is  $F(x) + C$ .
- The definite integral equals the difference between the primitives evaluated at the endpoints.

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note that the constant disappears.

## Improper Integrals

### Definition

An **improper integral** is one which the integrand is *singular* (not well behaved) within the range of integration.

$$\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \left( \int_a^b f(x)dx \right)$$

## Discontinuous Integrals

### Definition

A **discontinuous integrand** contains a finite number of discontinuities over the range of integration.

If  $f(x)$  is discontinuous at  $x = x_0$

$$\int_a^b f(x)dx = \int_a^{x_0} f(x)dx + \int_{x_0}^b f(x)dx$$

Note that the primitive of a discontinuous function is continuous.

## Methods of Integration

### Common Results

Since the indefinite integral is the reverse of differentiation.

$$\begin{aligned}\int x^n dx &= \frac{x^{n+1}}{n+1} + C & \int \cos(ax) dx &= \frac{1}{a} \sin(ax) + C \\ \int \frac{1}{x} dx &= \ln|x| + C & \int \sin(ax) dx &= -\frac{1}{a} \cos(ax) + C \\ \int e^{ax} dx &= \frac{1}{a} e^{ax} + C & \int \sec^2(ax) dx &= \frac{1}{a} \tan(ax) + C\end{aligned}$$

Using results from inverse hyperbolic functions.

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \operatorname{arcosh}\left(\frac{x}{a}\right) + C \quad \int \frac{1}{\sqrt{x^2 + a^2}} dx = \operatorname{arsinh}\left(\frac{x}{a}\right) + C$$

From the chain rule

$$\begin{aligned}\int (f(x))^n f'(x) dx &= \frac{1}{n+1} f^{n+1}(x) + C \\ \int \frac{f'(x)}{f(x)} dx &= \ln|f(x)| + C\end{aligned}$$

### Powers of Trig Functions

Use the result  $\sin^2(x) = \frac{1}{2}(1 - \cos 2x)$  and  $\cos^2(x) = \frac{1}{2}(1 + \cos 2x)$

Since we don't know how to integrate higher power trig functions.

- If the power is even, e.g.  $\cos^4 x = \frac{1}{4}(1 + \cos 2x)^2$ , expand and repeat until the expression can be integrated.
- If the power is odd, e.g.  $\sin^3 x = \sin x(1 - \cos^2 x)$ , expand and use the reverse chain rule.

Similar rules can be used for stuff related to  $\sec^2 x = 1 + \tan^2 x$  and the  $\csc^2 x$  equivalent.

### Partial Fractions

$$f(x) = \frac{p(x)}{q(x)}$$

Where  $p(x)$  and  $q(x)$  are polynomials, then  $f(x) = P(x) + Q(x)$

The fundamental theorem of algebra says that we can write  $q(x)$  as . But we don't actually want to deal with complex numbers, so we have

$$q(x) = (x - a_1)^{j_1} (x - a_2)^{j_2} \dots (r_{m-1}(x))^{j_{m-1}} (r_m(x))^{j_m}$$

Where  $r_{i(x)}$  are the terms with no real roots. So

$$Q(x) = \sum_{i=1}^m \sum_{r=1}^{j_i} \frac{A_{ir}}{(x - a_i)^r}$$

Once you write down these partial fractions, it is easy to integrate the terms.

## Cover-up Rule

$$\begin{aligned}
 f(x) &= \frac{a_0 + a_1x + a_2x^2 + \dots}{(x - r_0)(x - r_1)(x - r_2) \dots} \\
 &= \frac{b_0}{x - r_0} + \frac{b_1}{x - r_1} + \frac{b_2}{x - r_2} + \dots \\
 b_0 + (x - r_0) \left( \frac{b_1}{x - r_1} + \frac{b_2}{x - r_2} + \dots \right) &= \frac{a_0 + a_1x + a_2x^2 + \dots}{(x - r_1)(x - r_2) \dots}
 \end{aligned}$$

Substitute  $x = r_0$  to get

$$b_0 = \frac{a_0 + a_1x + a_2x^2 + \dots}{(x - r_1)(x - r_2) \dots}$$

### Note

When there is a repeated root, the cover-up method only gives the coefficient of highest power.

## Substitution

Substitution simplifies an integral by changing variables. Take example

$$\int \frac{1}{1 + x^2} dx$$

Let  $x = \tan u$ , then  $dx = \sec^2 u \, du$ .

- We are saying that making a small test in  $dx$  is the same as making a small step in  $\sec^2 u \, du$ .
- Look at the graph of  $\tan x$  and this does make sense.

$$\begin{aligned}
 &= \int \frac{1}{1 + \tan^2 x} \sec^2 x \, dx \\
 &= u + C \\
 &= \arctan x + C
 \end{aligned}$$

## Half-angle Formula

Using the substitution  $\tan\left(\frac{x}{2}\right) = t$ , we can show

$$\begin{aligned}
 \sin x &= \frac{2t}{1 + t^2} \\
 \cos x &= \frac{1 - t^2}{1 + t^2} \\
 \tan x &= \frac{2t}{1 - t^2}
 \end{aligned}$$

## Common Substitutions

Denominator	Substitution
$a^2 + x^2$	$x = a \tan \theta$
$\sqrt{a^2 - x^2}$	$x = a \cos \theta$ or $x = a \sin \theta$
$\sqrt{x^2 - a^2}$	$x = a \cosh^2 \theta$
$\sqrt{x^2 + a^2}$	$x = a \sinh^2 \theta$

Denominator	Substitution
$a^2 - x^2$	$x = a \tanh^2 \theta$ if $ x  <  a $ $x = a \cosh^2 \theta$ if $ x  >  a $

Use completing the square to deal with general quadratic denominators.

### Integration by Parts

From the product rule.

$$\begin{aligned}\frac{d}{dx}(fg) &= f \frac{dg}{dx} + \frac{df}{dx}g \\ f \frac{dg}{dx} &= \frac{d}{dx}(fg) - \frac{df}{dx}g \\ \int f \frac{dg}{dx} dx &= fg - \int \frac{df}{dx}g dx\end{aligned}$$

### Integration with Complex Numbers

The integral of complex valued functions has the same rules as integrate with real valued function.

$$\int \Re(f) dx = \Re\left(\int f(x) dx\right)$$

### Odd and Even Functions

#### Definitions

- $f(x) = f(-x) \iff$  even function
- $f(x) = -f(-x) \iff$  odd function

Using the area interpretation of an integral, we have

$$\begin{aligned}\int_{-a}^a f(x) dx &= 0 && \text{if } f \text{ is odd} \\ \int_{-a}^a f(x) dx &= 2 \int_0^a f(x) dx && \text{if } f \text{ is even}\end{aligned}$$

### Reduction Formulae

We could sometimes write an integral as a recurrence relation.

$$\begin{aligned}I_{2n} &= \int_0^{\frac{\pi}{2}} \sin^{2n} x dx \\ &= [\sin^{2n-1} x \cdot (-\cos x)]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (2n-1) \sin^{2n-2} x \cos^2 x dx \\ &= (2n-1) \int_0^{\frac{\pi}{2}} \sin^{2n-2} x (1 - \sin^2 x) dx \\ &= (2n-1)(I_{2n-2} - I_{2n}) \\ I_{2n} &= \frac{2n-1}{2n} I_{2n-2}\end{aligned}$$

$$I_{2n} = \frac{(2n-1)!!}{2n!!} I_0 = \frac{(2n-1)!!}{2n!!} \frac{\pi}{2}$$

### Definition

**Double factorials**  $n!!$  multiplies only the even or odd terms less than or equal to  $n$ .

$$n!! = n(n-2)(n-4)\dots$$

## Differentiation of Integrals

When we want to differentiate an integral where

- Boundaries depends on  $q$ , or
- The function depends on  $q$ .

Such as

$$I(q) = \int_{a(q)}^{b(q)} f(x, q) dx$$

Since there are 3 objects that depends on  $q$  :  $a(q)$ ,  $b(q)$  and  $f(x, q)$ , so  $I'(q)$  have at least 3 terms.

$$\frac{dI}{dq} = \underbrace{\int_{a(q)}^{b(q)} \frac{\partial}{\partial q} f(x, q) dx}_{\text{area gained from curve changing}} + \underbrace{f(b(q), q) \frac{db}{dq}}_{\text{area gained from } b(q) \text{ increasing}} - \underbrace{f(a(q), q) \frac{da}{dq}}_{\text{area lost from } a(q) \text{ increasing}}$$

## Partial Differentiation

Let  $h(x, y)$ , by treating  $y$  as a constant

$$\frac{\partial h}{\partial x} = \frac{h(x + \delta x, y) - h(x, y)}{\delta x}$$

We also define  $\frac{\partial h}{\partial y}$ , because there's no reason for the two derivatives to be related.

## Chain Rule

Let  $\tilde{h}(s) = h(x(s), y(s))$ , the change in  $\tilde{h}$  will be the combined effect of change in  $x$  and  $y$ .

$$\delta h = \frac{\partial h}{\partial x} \delta x + \frac{\partial h}{\partial y} \delta y$$

And we know

$$\delta x = \frac{dx}{ds} \delta s \quad \delta y = \frac{dy}{ds} \delta s$$

The chain rule allows us to take the derivative without expanding out all terms.

$$\frac{dh}{ds} = \frac{\partial h}{\partial x} \frac{dx}{ds} + \frac{\partial h}{\partial y} \frac{dy}{ds}$$

## Example: Gamma Function

$$\int_0^\infty x^n e^{-x} dx$$

$$\text{Let } I(a) = \int_0^\infty e^{-ax} dx = \frac{1}{a}$$

$$\begin{aligned}\frac{dI}{d\alpha} &= \int_0^\infty \frac{\partial}{\partial \alpha} e^{-\alpha x} dx & \frac{d^2 I}{d\alpha^2} &= \int_0^\infty \frac{\partial}{\partial \alpha} (-x e^{-\alpha x}) dx \\ &= \int_0^\infty -x e^{-\alpha x} dx & &= \int_0^\infty \frac{\partial}{\partial \alpha} (x^2 e^{-\alpha x}) dx \\ &= -\frac{1}{\alpha^2} & &= \frac{2}{\alpha^3}\end{aligned}$$

Therefore  $\int_0^\infty x^n e^{-\alpha x} dx = n!/\alpha^n$ . Set  $\alpha = 1$  to get the factorial function.

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-\alpha x} dx = (n-1)!$$

### Example: Stirling's Approximation

Use integrals to approximate summation.

$$\ln n! = \sum_{k=1}^n \ln k$$

If  $k \leq x \leq k+1$ , then  $\ln k \leq \ln x \leq \ln(k+1)$ .

$$\begin{aligned}\sum_{k=1}^n \ln k &\leq \int_1^n \ln x \, dx \leq \sum_{k=1}^{n-1} \ln(k+1) \\ &= \sum_{k=1}^n \ln k \\ &\leq \int_1^{n+1} \ln x \, dx\end{aligned}$$

Bound  $\sum_{k=1}^n \ln k$  with the two integrals.

$$n \ln n - n + 1 \leq \ln n! \leq (n+1) \ln(n+1) - (n+1) + 1$$

Taking the leading terms of the expressions

$$\ln n! \approx n \ln n - n$$

We can show the fractional error  $\frac{\ln n! - n \ln n - n}{n!} = O\left(\frac{1}{n}\right)$  as  $n \rightarrow \infty$ .

### Schwarz's Inequality

As we have seen for vectors,  $\vec{p} \cdot \vec{q} = |\vec{p}||\vec{q}| \cos \theta$ .

$$\begin{aligned}\Rightarrow (\vec{p} \cdot \vec{q})^2 &\leq \vec{p}^2 \vec{q}^2 \\ \Rightarrow \left( \sum_i p_i \times q_i \right)^2 &\leq \left( \sum_i (p_i)^2 \right) \left( \sum_i (q_i)^2 \right) \text{ where } p, q \text{ have components } p_1, p_2, \dots\end{aligned}$$

The general proof for vectors more than 2 components:

$$\begin{aligned} \implies (\forall \lambda \in \mathbb{R}) \quad \sum_i^n (\lambda p_i + q_i)^2 &\geq 0 \\ \iff \lambda^2 \left( \sum_i^n (p_i)^2 \right) + 2\lambda \left( \sum_i^n p_i q_i \right) + \left( \sum_i^n (q_i)^2 \right) &\geq 0 \end{aligned}$$

This is a quadratic in  $\lambda$ . Since  $\lambda$  has 0 or 1 root, the discriminant  $b^2 \leq 4ac$

$$\begin{aligned} 4 \left( \sum_i^n p_i q_i \right)^2 &\leq 4 \left( \sum_i^n (p_i)^2 \right) \left( \sum_i^n (q_i)^2 \right) \\ \left( \sum_i^n p_i q_i \delta s \right)^2 &\leq \left( \sum_i^n (p_i)^2 \delta s \right) \left( \sum_i^n (q_i)^2 \delta s \right) \end{aligned}$$

This looks like the Riemann sum, we could write a proof for integrals only, or if we take  $n \rightarrow \infty$ .

$$\left( \int_a^b p(x)q(x)dx \right)^2 \leq \left( \int_a^b p(x)^2 dx \right) \left( \int_a^b q(x)^2 dx \right)$$

This is an inequality iff  $p(x) \propto q(x)$  - the vectors are in the same direction as each other.

## Multiple Integrals

We want to calculate the value under the surface  $z = h(x, y)$

Similar to Riemann sums, divide the region into little boxes, the  $i$ th box has area  $\delta A_i$ .

$$\begin{aligned} V_n &= \sum_{i=1}^n h(x_i, y_i) \delta A_i \\ V &= \iint_A h(x, y) dA = \lim_{n \rightarrow \infty} V_n \end{aligned}$$

## 2D Regions

For a rectangular region:

$$\begin{aligned} \iint_A h(x, y) dA &= \int_{x=a}^{x=b} \int_{y=c}^{y=d} h(x, y) dy dx \\ &= \int_{y=c}^{y=d} \int_{x=a}^{x=b} h(x, y) dx dy \end{aligned}$$

From the geometric interpretation of the integrals, the order does not matter.

For a region bounded by  $y = f(x)$  below and  $y = g(x)$  above.

$$V = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=f(x)} h(x, y) dy dx$$

We can change the order at which  $x, y$  are if the new integral is easier to evaluate.

If we set  $h(x, y) = 1$ , then we get the area of the region by

$$A = \iint_A dA$$



## Separating Integrand

If we can separate  $f(x, y)$  as  $g(x)h(y)$ , where the bounds don't depend on  $x$  or  $y$  (a rectangle).

$$\int_a^b \int_c^d g(x)h(y) dy dx = \left( \int_a^b g(x) dx \right) \left( \int_c^d h(y) dy \right)$$

We can do this because  $f(x)dx$  is a constant with respect to  $y$ , we can take it out using the constant multiple rule.

$$\int k f(y) dy = k \int f(y) dy$$

## Polar Coordinates

Many regions have symmetry, it may be easier to write  $h(r, \theta)$  than  $h(x, y)$ .

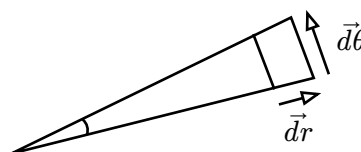
$$x = r \cos \theta$$

$$y = r \sin \theta$$

For a small change in  $r$  and  $\theta$ , a point moves by vector

$$\frac{d}{dr} \begin{pmatrix} x \\ y \end{pmatrix} dr = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} dr$$

$$\frac{d}{d\theta} \begin{pmatrix} x \\ y \end{pmatrix} d\theta = \begin{pmatrix} -r \sin \theta \\ r \cos \theta \end{pmatrix} d\theta$$



$dA$  is the area spanned by the two vectors. We can find  $dA$  using the determinant.

$$dA = \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} dr d\theta = r dr d\theta$$

So we have

$$\iint_A f(r, \theta) dA = \iint_A f(r, \theta) r dr d\theta$$

Then multiple integration can continue as normal.

## Triple Integrals

By finding  $dV$  using the matrices technique.

Coordinate system	$dV$
Cartesian coordinates	$dx dy dz$
Cylindrical polar coordinates	$r dr d\theta dz$
Spherical polar coordinates	$r^2 \sin \theta dr d\theta d\varphi$