

Multivariable Differential Equations

Partial Derivatives

For $f = f(x, y)$

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$
$$\frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Are the gradients of f if we travel along the x or y direction.

For functions of more variables, you can calculate $\partial f / \partial x$ by treating all other variables as constants.

Higher Derivatives

There are four 2nd order partial derivatives for a function $f(x, y)$

$$\frac{\partial^2 f}{(\partial x)^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$
$$\frac{\partial^2 f}{(\partial y)^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right)$$

For any well defined function

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

For $f(x_1, x_2, x_3, \dots, x_n)$, the gradient of f is a vector.

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right)$$

Partial Differentials

Integration is the opposite of differentiation.

$$\frac{\partial f}{\partial x} = 2xy^2 \quad \text{holding } y \text{ constant}$$
$$f(x, y) = x^2 y^2 + G(y)$$
$$\frac{\partial f}{\partial y} = 2x^2 y + 2y \quad \text{holding } x \text{ constant}$$
$$f(x, y) = x^2 y^2 + y^2 + H(x)$$
$$\therefore f(x, y) = x^2 y^2 + y^2 + C$$

There may be no $f(x, y)$ satisfying $\partial f / \partial x$ and $\partial f / \partial y$, for examples those that leads to

$$\frac{\partial^2 f}{\partial y \partial x} \neq \frac{\partial^2 f}{\partial x \partial y}$$

Taylor series of f centred at (x, y) , all partial derivatives are evaluated at (x_0, y_0)

$$\begin{aligned} f(x_0 + k, y_0 + k) &= f(x_0, y_0) + \left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} \right) \\ &+ \frac{1}{2!} \left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \right) + \dots \end{aligned}$$

The **linear approximation** of f

$$f(x_0 + h, y_0 + k) \approx f(x_0, y_0) + h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}$$

For very small change in x and y

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Chain Rule

Consider

$$df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

Then

$$df = \left(\frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \right) dx + \left(\frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} \right) dy$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y}$$

Reciprocity and Cyclic Relations

Let $z(x, y)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \quad (1)$$

$$dx = \frac{\partial x}{\partial y} dy + \frac{\partial x}{\partial z} dz \quad (2)$$

$$dy = \frac{\partial y}{\partial z} dz + \frac{\partial y}{\partial x} dx \quad (3)$$

Rewrite (2)

$$dz = \frac{1}{\left(\frac{\partial x}{\partial z}\right)} dx - \frac{\left(\frac{\partial x}{\partial y}\right)}{\left(\frac{\partial x}{\partial z}\right)} dy$$

Comparing coefficients of dx and dy with (1)

$$\frac{\partial z}{\partial x} = \frac{1}{\left(\frac{\partial x}{\partial z}\right)}$$

$$\frac{\partial z}{\partial y} = -\frac{\left(\frac{\partial x}{\partial y}\right)}{\left(\frac{\partial x}{\partial z}\right)}$$

$$\frac{\partial z}{\partial y} \frac{\partial x}{\partial z} = -\frac{\partial x}{\partial y}$$

$$\frac{\partial z}{\partial y} \frac{\partial x}{\partial z} \frac{\partial y}{\partial x} = -1$$

The last three lines are called the **cyclic relation** for partial derivatives.
