

**Definition 7**

$n \in \mathbb{N}$  is odd if  $(\exists i \in \mathbb{N}) n = 2i + 1$ .

**Proposition 8**

Goal:  $(\forall m, n \in \mathbb{N}) m \text{ and } n \text{ odd} \implies m \times n \text{ odd}$

**Proof**

Assume:

1.  $m, n \in \mathbb{N}$
2.  $m$  and  $n$  odd

New goal:  $m \times n$  odd

$$\begin{aligned}
 & (\exists i, j \in \mathbb{N}) m = 2i + 1 \text{ and } n = 2j + 1 \\
 \implies & (\exists i, j \in \mathbb{N}) m \times n = (2i + 1) \times (2j + 1) \\
 \implies & (\exists i, j \in \mathbb{N}) m \times n = 2(2ij + i + j) + 1 \\
 \implies & (\exists k \in \mathbb{N}) m \times n = 2k + 1 \\
 \implies & m \times n \text{ odd}
 \end{aligned}$$

**Definition 9**

$(\forall x \in \mathbb{R})$

- $(\exists m, n \in \mathbb{Z}) x = m/n \iff x$  rational
- $\neg(x \text{ rational}) \iff x$  irrational
- $x > 0 \iff x$  positive
- $x < 0 \iff x$  negative
- $\neg(x \text{ positive}) \iff x$  nonpositive
- $\neg(x \text{ negative}) \iff x$  nonnegative
- $x$  nonnegative and  $x \in \mathbb{Z} \iff x \in \mathbb{N}$

**Proposition 10**

Goal:  $(\forall x \text{ positive}) \sqrt{x}$  rational  $\implies x$  rational

**Proof**

Assume:

1.  $x$  positive
2.  $\sqrt{x}$  rational

New goal:  $x$  rational

$$\begin{aligned}
 & (\exists p, q \in \mathbb{Z}) \sqrt{x} = p/q \\
 \implies & (\exists p, q \in \mathbb{Z}) x = (\sqrt{x})^2 = p^2/q^2 \\
 \implies & (\exists p', q' \in \mathbb{Z}) x = p'/q' \\
 \iff & x \text{ rational}
 \end{aligned}$$

**Theorem 11**

Goal: Let  $P_1, P_2, P_3$  be statements,  $(P_1 \implies P_2 \text{ and } P_2 \implies P_3) \implies (P_1 \implies P_3)$

**Proof**

Assume:

1.  $P_1 \implies P_2$

2.  $P_2 \Rightarrow P_3$
3.  $P_1$

New goal:  $P_3$

$$\begin{aligned} &P_2 \text{ as (4) by (1) and (3)} \\ &\Rightarrow P_3 \text{ by (2) and (4)} \end{aligned}$$

**Definition 12**