Ordinary Differential Equations

Definition
An **nth order ODE** includes the nth derivate but no higher derivatives.

To solve an ODE we find the dependent variable as a function of the independent variables.
- In the real world: this is done numerically with computers.
- Here we do it analytically to demonstrate principles.

Some ODEs don't have analytical solutions.

First Order ODEs

Definition
Integrable ODEs have form

$$\frac{dy}{dx} = f(x)$$

where the RHS does not depend on y.

$$y = \int f(x)dx$$

If $F'(x) = f(x)$, then $y = F(x) + C$. y solves the ODE whatever C is - threr are infinitely many solutions.

$(*)$ is called the **general solution** if it contains all possible solutions.

If we choose any value for C, we get a particular solution.

Definition
Separable ODEs have form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y)\ \frac{dy}{dx} = f(x)$$

$$\int g(y)\ \frac{dy}{dx}dx = \int f(x)dx$$

$$\int g(y)dy = \int f(x)dx$$

$$G(y) = F(x) + C$$

Geometric Interpretation of First Order ODEs

Consider $\frac{dy}{dx} = F(x, y)$

For every point (x, y):
- There is a particular solution passing through the point
- Its gradient is $F(x, y)$

Note
2 particular solutions don't have to be the same shape (shifted versions) of each other.

Linear ODE

$$\frac{dy}{dx} + y \cdot p(x) = f(x)$$

When $f(x) = 0$, the ODE is homogenous, two ways to solve it

$$\begin{aligned} \frac{dy}{dx} &= -y \cdot p(x) \\ \int \frac{1}{y} dy &= -\int p(x) dx \\ y &= Ae^{-P(x)} \end{aligned}$$

$$\begin{aligned} e^{P(x)} \cdot \frac{dy}{dx} + p(x) \cdot e^{P(x)} \cdot y &= 0 \\ \frac{d}{dx}\left(e^{P(x)} \cdot y\right) &= 0 \\ y &= Ae^{-P(x)} \end{aligned}$$

Note
$e^{\int p(x)\ dx}$ is called the integrating factor.

For inhomogenous cases, multiply both sides by the integrating factor.

$$\begin{aligned} e^{P(x)} \cdot \frac{dy}{dx} + p(x) \cdot e^{P(x)} \cdot y &= f(x) \\ \frac{d}{dx}\left(e^{P(x)} \cdot y\right) &= f(x) \\ e^{P(x)} \cdot y &= F(x) + C \\ y &= (F(x) + C)e^{-P(x)} \end{aligned}$$

Definition
nth order ODE contains n arbitrary constants: n pieces of informations to fix them. The extra information are called **boundary conditions**.

Note
A **particular solution** has no unknown constants or $\pm$ signs.

Substitutions

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

$$\begin{aligned} \text{let } u(x) &= \frac{y(x)}{x} \\ x \cdot u(x) &= y(x) \\ u + x \cdot \frac{du}{dx} &= \frac{dy}{dx} \end{aligned}$$

For example

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2 + y^2}{xy} \\ &= \frac{(x/y)^2 + 1}{x/y} \\ u + x \cdot \frac{du}{dx} &= \frac{u^2 + 1}{u} \\ x \cdot \frac{du}{dx} &= \frac{1}{u} \\ &\vdots \end{aligned}$$

Benoulli Equations

$$\frac{dy}{dx} + p(x) \cdot y = q(x) \cdot y^n$$

If $y = 0$ or $y = 1$, then the equation is homogenous, otherwise

$$\begin{aligned}
\text{let } z(x) &= y(x)^{1-n} \\
\frac{dz}{dx} &= (1-n) \cdot y(x)^{-n} \cdot \frac{dy}{dx} \\
\frac{1}{1-n} \cdot \frac{dz}{dx} \cdot y(x)^n &= \frac{dy}{dx} \\
\frac{1}{1-n} \cdot \frac{dz}{dx} \cdot y^n + p(x) \cdot y &= q(x) \cdot y^n \\
\frac{1}{1-n} \cdot \frac{dz}{dx} + p(x) \cdot y^{1-n} &= q(x) \\
\frac{dz}{dx} + (1-n) \cdot p(x) \cdot z &= (1-n) \cdot q(x) \\
&\vdots
\end{aligned}$$
