Complex Numbers

The Chain of Generalisation

Number set	Description
Counting numbers	For counting objects.
Natural numbers	Counting numbers and 0, took 3000 years the realise that it is useful.
Integers	Natural numbers with negative integers. The negation of any two integer is also an integer.
Rational numbers	Any number that can be written as a ratio of two integers. It is not continuous, but there are infinitely many of them.
Irrational numbers	Numbers that cannot be expressed as a ratio of two numbers, some of them are solutions to equations.
Real number	Union of rational and irrationals.
Complex numbers	Many calculations are much easier in complex numbers. Many functions in physics are functions over complex numbers.

Negative Square Roots in Equations

The formula for $t^3 + pt + q = 0$ is given by

$$t = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

- If the term inside the square root < 0, it will give a negative square root.
- But we know the cubic function has at least 1 real root, negative square roots are needed to find the real root of a cubic.

Definition

The **fundamental theorem of algebra**:

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m = 0$$

Always have m roots.

And you can factor like $c(x - x_0)(x - x_1)\ldots(x - x_m) = 0$, which requires complex numbers.

Properties of Complex Numbers

Define $i^2 = -1$. There are two roots for i, it doesn't matter which one you pick they will work the same, you just have to be consistent with that choice.

Definition

$$i = \sqrt{-1}$$

Complex Numbers

We can write $z = x + iy \in \mathbb{C}$, where $x, y \in \mathbb{R}$. Every complex number has a tuple attached to it.

- $\mathrm{Re}(z) = \Re(z) = x$
- $\mathrm{Im}(z) = \Im(z) = y$

Properties of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$.
- $z_1 = z_2 \iff a = c$ and $b = d$

We can represent complex numbers as points in 2D space in an **Argand diagram**. We can work with them the same as we worked with vectors. See the vector properties:
- Add commutative: $z_1 + z_2 = z_2 + z_1$
- Add associative: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

And multiplication is also commutative, associative and distributive over addition. It does not always have an inverse (e.g. when $z = 0$).

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Modulus and Argument
- $r = |z|$ be the distance from the origin.
- $\theta = \arg(z)$ be the angle between z and the x-axis.

Definition
The **principal argument** is $\arg(z)$ restricted to $[-\pi, \pi]$.

Note tan does not uniquely define $\arg(z)$.

Multiplication in Modulus and Argument Form

$$z = |z|(\cos\theta + i\sin\theta)$$
$$|z_1 z_2| = |z_1||z_2|$$
$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

Definition
The **complex conjugate** $z^* = x - iy$ when $z = x + iy$.

This gives an easy way to calculate the modulus $zz^* = |z|^2$.

Division
To express $z_1 \div z_2$ in format $a + ib$

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{z_1}{z_2} \cdot \frac{z_2^*}{z_2^*} \\ &= \frac{z_1 z_2^*}{|z_2|^2}\end{aligned}$$

Exponential Form
We have not define the exponential function yet, for now we will use this as definition

Definition
$e^{i\theta} = \cos\theta + i\sin\theta$

$$e^a + e^b = e^{a+b}$$
$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$

This is why complex numbers are so often used, they are really convenient to multiply.

Note that the exponential form is not unique, $e^{i\theta} = e^{i(\theta+2\pi)}$

Roots of Unity

Solve for $z^4 = 1$

$$\begin{aligned} z &= re^{i\theta} \\ z^4 &= r^4 e^{4i\theta} \end{aligned}$$

So $4\theta = 2\pi m$ where $m \in \mathbb{Z}$, so $\theta = \frac{\pi m}{2}$

$$z = e^{\frac{i\pi m}{2}}$$

Which gives 4 solutions according to the fundamental theorem of algebra.

DeMoivre's Theorem

Using the exponent form of complex numbers:

$$\begin{aligned} z^n &= \exp(i\theta)^n \\ &= \exp(ni\theta) \\ \cos n\theta + i \sin n\theta &= (\cos\theta + i\sin\theta)^n \end{aligned}$$

Complex Conjugate using DeMoivre's

We can also use DeMoivre's theorem to take the complex conjugate of z.

$$\exp(-i\theta) = \cos\theta - i\sin\theta$$

Yielding identities:

$$\begin{aligned} \cos\theta &= \frac{1}{2}(\exp(i\theta) + \exp(-i\theta)) \\ \sin\theta &= \frac{1}{2i}(\exp(i\theta) - \exp(-i\theta)) \end{aligned}$$

We can also express $\cos^3\theta$ in terms of $\cos 3\theta$ and $\cos 3\theta$, or $\cos 4\theta$ in terms of $\cos^4\theta$ and $\cos^2\theta$.

Sum Series

We can work out the sum of trigonometric functions.

$$\sum_{N=0}^{N-1} \cos k\theta = \Re\left[\sum_{N=0}^{N-1} \exp(ki\theta)\right]$$

Then we can use the geometric sum formula.

Complex Logarithms

Definition

ln is the inverse of the **exp** function.

$$\exp(\ln z) = z$$

$$\begin{aligned} \ln z &= \ln(|z| \exp(i(\theta + 2\pi n))) \\ &= \ln(|z|) + i(\theta + 2\pi n) \end{aligned}$$

The log of a complex number is **multivalued**, there are infinitely many solutions. This is similar to how taking the root of natural nubmers give 2 solutions.

> **Definition**
> The **principal value** is the root closest to the x-axis.

General Power of $z_1^{z_2}$
- Let $z_1 = |z_1| \exp(i\theta)$
- Let $z_2 = x + iy$

$$\begin{aligned} z_1 &= \exp(z_2 \ln z_1) \\ &= \exp(z_2(\ln|z_1| + i(\theta + 2\pi n))) \\ &= \exp((x + iy)(\ln|z_1| + i(\theta + 2\pi n))) \\ &= \exp(x \ln|z_1| - y(\theta + 2\pi n) + i(y \ln|z_1| + x(\theta + 2\pi n))) \\ &= \frac{|z_1|^x}{\exp(y(\theta + 2\pi n))} \cdot \exp(i(y \ln|z_1| + x(\theta + 2\pi n))) \end{aligned}$$

We can substitute any $z_2 \in \mathbb{Q}$ to show it is the expected behaviour.

Applications of Complex Numbers

Used in problems that involve oscillatory/periodic motion.

E.g. a pendulum about the vertical

$$\begin{aligned} x(t) &= a \cos \omega t + b \sin \omega t \\ &= \Re(A \exp i\omega t) \end{aligned}$$

The big advantage is that taking derivatives of the exponential function is very easy.

$$\begin{aligned} v(t) &= \frac{d}{dx} \Re(\exp i\omega t) \\ &= \Re\left(\frac{d}{dx} \exp i\omega t\right) \end{aligned}$$

We can easily fix it to an initial condition to find a particular solution.

Fundamental Theorem of Algebra (The Sequel)

> **Theorem**
> A polynomial of n degree where $a_i \in \mathbb{C}$ has n complex roots (possibly repeated).

If $P(z)$ is a function of n degrees, then $P(z) = (z - z_1)Q(z)$ where Q a function of $n - 1$ degrees.

We can prove by induction (?) that there is at least one route $(z - z_1)(z - z_2)...R(z) = 0$.

`END Complex Numbers`