Probability Theory

Sets

Definitions

Keyword	Definition
Sample space S	The set of all possible outcomes.
Event A	An event A is a subset of S.
$A \cap B$	A and B occured.
$A \cup B$	A or B occured.
$\overline{A}$	A did not occur.

Properties of set operations.

Commutative

$$A \cap B = B \cap A$$
$$A \cup B = B \cup A$$

Associative

$$(A \cap B) \cap C = A \cap (B \cap C)$$
$$(A \cup B) \cup C = A \cup (B \cup C)$$

Definition
A and B are **mutually exclusive** iff $A \cap B = \emptyset$

The following identities are true.

$$A \cap \overline{A} = \emptyset$$
$$A \cup \overline{A} = S$$
$$S - B = \overline{B}$$
$$A - B = A \cap \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Probability

The probability $P(A)$ of A happening is defined as

$$P(A) = \lim_{N \to \infty} \frac{N_A}{N}$$

Where N_A is the number of events in N experiments.

Properties of probabilities.
- $0 \leq P(A) \leq 1$
- $P\left(A \cap \overline{A}\right) = 0$
- $P\left(A \cup \overline{A}\right) = 1$
- $P\left(\overline{A}\right) = 1 - P(A)$

The union of two events $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If A and B are **mutually exclusive**, then $P(A \cup B) = P(A) + P(B)$
- Extending for three events:

$P(A \cup B \cup C)$
$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

This can be proved using the rule for two events on $P(A \cup (B \cup C))$

Definition
Conditional probability: $P(B|A)$ is the probability of B occuring given A.

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Because $P(A \cap B) = P(A)P(B|A)$, we have

$$\begin{aligned} P(A)P(A|B)P(A|B \cap C) &= P(A \cap B)P(A|B \cap C) \\ &= P(A \cap B \cap C) \end{aligned}$$

Bayes Theorem
It is obvious that

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)P(A|B)}{P(B)} \end{aligned}$$

Provided $P(B) \neq 0$

It is also true that

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P\left(\overline{A}\right)P\left(B|\overline{A}\right)}$$