Multivariable Differential Equations

Partial Derivatives

For $f = f(x, y)$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$
$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Are the gradients of f if we travel along the x or y direction.

For functions of more variables, you can calculate $\partial f / \partial x$ by treating all other variables as constants.

Higher Derivatives

There are four 2nd order partial derivatives for a function $f(x, y)$

$$\frac{\partial^2 f}{(\partial x)^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$
$$\frac{\partial^2 f}{(\partial y)^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

For any well defined function

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

For $f(x_1, x_2, x_3, ..., x_n)$, the gradient of f is a vector.

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, ..., \frac{\partial f}{\partial x_n}\right)$$
