

# Infinite Series

## Convergence

Given an infinite sequence of numbers, define the **partial sum** to be

$$S_n = u_1 + u_2 + \cdots + u_n$$

We want to know if the infinite series

- Have a well defined limit
- Diverges
- Oscillates

$$\lim_{n \rightarrow \infty} S_n = S \text{ iff } (\forall \varepsilon > 0)(\exists N) n > N \implies |S - S_n| < \varepsilon$$

## Conditions for Convergence

Requires  $u_n \rightarrow 0$  as  $n \rightarrow \infty$ .

The sum of two diverging series may converge, e.g. when one diverges to  $+\infty$  and the other to  $-\infty$ , and the two cancels each other out.

### Definitions

- A series is **absolutely convergent** iff  $\sum |a_n|$  converges.
- A series is **conditionally convergent** iff  $\sum a_n$  converges but not  $\sum |a_n|$ .

If we rearrange the terms in a series, a converging series can diverge, or the other way round.

- If we have an absolutely convergent series, it doesn't matter if we change the order.
- If we have a conditionally convergent series, changing the order may cause it to diverge.

## Geometric Progression

Let  $u_k = r^k$ , we already know the sum formula for the geometric progression, here's the proof.

$$(1 - r)(1 + r + r^2 + \cdots + r^k) = 1 - r^{k+1} \quad (\text{by simplifying the expression})$$

$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}$$

For the sum of infinite series

$$\lim_{k \rightarrow \infty} \frac{1 - r^{k+1}}{1 - r} \text{ exists if } |r| < 1.$$

## The Harmonic Series

The harmonic series is

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

We can group terms **without reordering**.

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

The sequence  $\sum \frac{1}{2}$  diverges, so the harmonic series also diverges, this is called a **comparison test**.

$$\gamma = \lim_{k \rightarrow \infty} (S_k - \ln k) = 0.57721566$$

Which is the **Euler-Mascheroni constant**.

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## Convergence Tests

### Comparison Tests

The comparison test can be used on two nonnegative series:  $(\forall k) u_k, v_k \geq 0$

$$(\forall k \geq K) u_k \leq v_k \text{ and } \sum_{k=0}^{\infty} v_k \text{ convergent} \Rightarrow \sum_{k=0}^{\infty} u_k \text{ convergent}$$

$$(\forall k \geq K) u_k \geq v_k \text{ and } \sum_{k=0}^{\infty} v_k \text{ divergent} \Rightarrow \sum_{k=0}^{\infty} u_k \text{ divergent}$$

### Ratio Tests

For a positive series:  $(\forall k) u_k > 0$

$$\lim_{k \rightarrow \infty} \left( \frac{u_{k+1}}{u_k} \right) < 1 \Rightarrow \text{converges}$$

$$\lim_{k \rightarrow \infty} \left( \frac{u_{k+1}}{u_k} \right) > 1 \Rightarrow \text{diverges}$$

### Proof

If  $\lim_{k \rightarrow \infty} \frac{u_{k+1}}{u_k} = \alpha < \infty$ , then for large enough  $k$ :

$$\sum u = u_1 + u_2 + \dots + u_k(1 + \alpha + \alpha^2 + \dots)$$

Which converges.

### Leibniz Criterion

#### Definition

**Alternating series** have terms with alternating signs.

If  $a_k > 0$  and  $a_k$  is monotonically decreasing for large enough  $k$ , and  $\lim_{k \rightarrow \infty} a_k = 0$ .

$$S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges}$$

### Proof

We know it is true that

$$\sum_{k=1}^{2n} = (a_1 - a_2) + (a_3 - a_4) + \dots \text{ is a monotonically increasing series}$$

$$\sum_{k=1}^{2n+1} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots \text{ is a monotonically decreasing series}$$

$$\lim_{k \rightarrow \infty} \left( \sum_{k=1}^{2n+1} a_k - \sum_{k=1}^{2n} a_k \right) = 0$$

So we write the inequality

$$a_1 - a_2 = S_2 < S_4 < \dots < S_{2n} < S_{2n+1} < \dots < S_3 < S_1 < a_1$$

## Power Series

A power series have form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \dots$$

The **domain of convergence** is either

- Only  $x = 0$
- All finite  $x$
- Only for some  $|x| < R$

We can find out which case it is with the **ratio test**, which gives the shortcut if  $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$  exists.

$$|x| < \frac{1}{L} \implies \text{converges}$$

$$|x| > \frac{1}{L} \implies \text{diverges}$$

$$|x| = \frac{1}{L} \implies \text{indeterminate}$$

Note the endpoints at  $|x| = 1/L$  may behave differently.

## The Taylor Series

We can approximate the value of a function using its tangent at a point.

$$f(x) \approx f(a) + (x - a)f'(a)$$

$$f'(x) \approx f'(a) + (x - a)f''(a)$$

By the fundamental theorem of calculus, which links the derivative to the integral:

$$\int_a^{a+h} f'(x) dx = f(a+h) - f(a)$$

$$\begin{aligned} f(a+h) &= f(a) + \int_a^{a+h} f'(x) dx \\ &= f(a) + \int_a^{a+h} f'(a) + (x-a)f''(a) dx \\ &= f(a) + \left[ xf'(a) + \frac{(x-a)^2}{2} f''(a) \right]_a^{a+h} \\ &= f(a) + hf'(a) + \frac{h^2}{2} f''(a) \end{aligned}$$

We can show that the **second order approximation** is better than the first order approximation. (how?)

Extend this to higher order approximations, and we get

$$f'(x) \approx f'(a) + (x-a)f''(a) + \frac{(x-a)^2}{2!} f'''(a) + \frac{(x-a)^3}{3!} f^{(4)}(a) + \dots$$

Since the factorial function grows faster than any polynomial, we know that  $n!$  grows faster than  $(x-a)^n$  so the series converges for a lot of approximations.

## Taylor's Theorem

An exact result can be written by including a **remainder term**  $R_{n+1}$ .

$$f(a+h) = f(a) + hf'(a) + \dots + \frac{h^n}{n!} f^{(n)}(a) + R_{n+1}$$

**Taylor's Theorem** states if  $f$  is  $n+1$  times differentiable, there exists a  $\zeta : a < \zeta < a+h$  such that

$$R_{n+1} = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\zeta)$$

## Proof

$$\int_a^x f'(t) dt = f(x) - f(a) \text{ by fundamental theorem of calculus}$$

$$\begin{aligned} f(x) &= f(a) + \int_a^x f'(t) dt \\ &= f(a) + [(t-x)f'(t)]_a^x - \int_a^x (t-x)f''(x) dt \text{ integrate by parts} \\ &= f(a) + (x-a)f'(a) + \int_a^x (t-x)f''(x) dt \end{aligned}$$

Integrate by parts repeatedly to find

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \int_a^x \frac{(t-x)^n}{n!} f^{(n+1)}(t) dt \\ R_{n+1} &= \int_a^x \frac{(t-x)^n}{n!} f^{(n+1)}(t) dt \\ &= \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\zeta) \text{ where } a \leq \zeta \leq x \text{ by mean value theorem} \\ \lim_{n \rightarrow \infty} R_{n+1} &= 0 \implies \text{Taylor series for } f(x) \text{ converges} \end{aligned}$$

By choosing the  $\zeta$  that would give the largest error, we can calculate the worst case error for an approximation.

## Note

If  $f(x)$  is infinitely differentiable, then we can represent  $f$  exactly as an infinite **power series**.

We could also prove that two functions are the same if they have the same Taylor series.

## Common Series Expansions

A lot of these expansion can be proved using the expansion for  $e^x$ .

Function	Taylor series
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

Function	Taylor series
$\sinh x$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

It makes sense that the Taylor series of an even function only have even powered terms, and an odd function only have odd powered terms.

The series expansion for  $\tanh$  has a radius of convergence of  $|x| < \frac{\pi}{2}$ , this makes sense because

- Polynomials cannot capture the flat asymptote.
- $\tan$  is only continuous over  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .

## Binomial Expansion

Consider  $f(x) = (1+x)^n$

$$f'(x) = n(1+x)^{n-1}$$

$$f''(x) = n(n-1)(1+x)^{n-2}$$

$$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$$

⋮

So we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

Which is the **binomial theorem**, when  $n$  is a positive integer, it agrees with the binomial expansion.

We can show that the series is absolutely convergent for  $|x| < 1$  by the ratio test.

## Stationary points with Taylor Series

If  $f$  is stationary at  $x = a$ , and suppose  $f''(a) \neq 0$ .

$$f(x) = f(a) + xf'(a) + \frac{x^2}{2}f''(a) + \dots$$

$$f(x) - f(a) \approx \frac{x^2}{2}f''(a)$$

Now we know why if  $f''(a) > 0$  then  $f$  has a minimum at  $x = a$ , if  $f''(a) < 0$  then  $f$  has a maximum at  $x = a$ .

If the first  $n-1$  derivatives of  $f(x) - f(a)$  are **vanishing**, then the function will behave like the first nonvanishing (the  $n$ th) term. This is where the idea of **L'Hopital's rule** comes from.

$$f(x) - f(a) \approx \frac{(x-a)^n}{n!}f^{(n)}(a)$$

- If  $n$  is even then  $x = a$  is either a local maximum or minimum for  $f$ , as  $x < a$  and  $x > a$  are both either less than or greater than  $f$  at  $x = a$ .

- If  $n$  is odd then  $x = a$  is a point of inflection for  $f$ .

### Approximating Functions with Taylor Series

Functions can be approximated by replacing them with their Taylor series, for example.

$$\begin{aligned}\frac{\log(1+x)}{1-x} &= \log(1+x) \cdot (1-x)^{-1} \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) \cdot (1 + x + x^2 + \dots) \\ &= x + \frac{x^2}{2} + \frac{5x^3}{6} + \dots\end{aligned}$$

### Newton-Rapson Method

This is an efficient way of finding roots of a function.

The first order approximation of a function at  $x = a$  is, we want to find  $f(x) = 0$ .

$$f(x) \approx f(a) + (x - a)f'(a)$$

We want to find  $f(x) = 0$ .

$$\begin{aligned}0 &\approx f(a) + (x - a)f'(a) \\ &= \frac{f(a)}{f'(a)} + x - a \\ x &= a - \frac{f(a)}{f'(a)}\end{aligned}$$

So if we define the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This turns out to converge really quickly.

### Estimating Error

Write a recursive relation that gives the error term  $\varepsilon_n$ , so we can find how quickly it converges.

Let  $x^*$  be the exact solution where  $f(x^*) = 0$ .

$$f(x) \approx f(x^*) + (x - x^*)f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*)$$

If  $f$  is linear, then we find  $x^*$  in one step. Otherwise, define error  $\varepsilon_n = x_n - x^*$ .

$$\begin{aligned}f'(x) &= f'(x^*) + (x - x^*)f''(x^*) + O(x^3) \\ \varepsilon_{n+1} &= x_{n+1} - x^* \\ &= \left(x_n - \frac{f(x_n)}{f'(x_n)}\right) - x^* \\ &= x_n - x^* - \frac{f(x^*) + \varepsilon_n f'(x^*) + \frac{1}{2}(\varepsilon_n)^2 f''(x^*)}{f'(x^*) + \varepsilon_n f''(x^*)} \\ &= \varepsilon_n - \frac{\varepsilon_n f'(x^*) + \frac{1}{2}(\varepsilon_n)^2 f''(x^*)}{f'(x^*) + \varepsilon_n f''(x^*)}\end{aligned}$$

Assuming we are close, how much closer will we get for the next step?

For very small  $\varepsilon$

$$\varepsilon_{n+1} = -\frac{\frac{1}{2}(\varepsilon_n)^2 f''(x^*)}{f'(x^*)} + O((\varepsilon_n)^2)$$

So the first term will be a quadratic term in  $\varepsilon_n$  - the error will be squared for each term.

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END Infinite Series