

Integration

Integration as Area

To find the area under the curve $y = f(x)$ in range $a \leq x \leq b$.

1. **Partition** $[a, b]$ into N subintervals with endpoints x_0, x_1, \dots, x_N , the width of the partitions does not have to be equal.
2. Choose N points $\xi_1, \xi_2, \dots, \xi_N$ in each of the partitions. The rectangle of each subpartition has area $A_i = (x_i - x_{i-1})f(\xi_i)$.
3. The total area is the **Riemann sum**

$$S_N = \sum_{i=1}^N (x_i - x_{i-1})f(\xi_i)$$

4. Take the **Riemann integral** where $N \rightarrow \infty$ such that all $x_i - x_{i-1} \rightarrow 0$.

$$\int_a^b f(x)dx = \lim_{N \rightarrow \infty} S_N$$

The integral in this form is also called a **definite integral**.

Properties of an Integral

From the geometric interpretation.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

But in the definition we assume $a \leq x \leq b$, to make the above property true for all c , we can define

$$\int_a^b f(x)dx = - \int_b^a f(x)dx$$

Integration and differentiation are **linear operations**, this means

$$\begin{aligned} \int_a^b k f(x)dx &= k \int_a^b f(x)dx \\ \int_a^b f(x) + g(x)dx &= \int_a^b f(x)dx + \int_a^b g(x)dx \end{aligned}$$

In $\infty + (-\infty)$ scenarios, $\int f + g dx$ may be finite, but $\int f dx + \int g dx$ are both undefined.

Fundamental Theorem of Calculus

$$F(x) = \int_a^x f(u)du \iff \frac{dF}{dx} = f(x)$$

Or

$$\frac{d}{dx} \int_a^x f(u)du = f(x)$$

Proof

$$\begin{aligned}\frac{dF}{dx} &= \lim_{\delta x \rightarrow 0} \frac{F(x + \delta x) - F(x)}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \frac{\int_a^{x+\delta x} f(u)du + \int_a^x f(u)du}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \frac{\int_x^{x+\delta x} f(u)du}{\delta x} \\ &= f(x)\end{aligned}$$

Definitions

In $F(x) = \int f(x)dx$

- $f(x)$ is the **integrand**.
- $F(x)$ is the **primitive**.

Definite and Indefinite Integrals

The primitive is also called the **indefinite integral** of $f(x)$.

- If $F(x)$ is the indefinite integral, then so is $F(x) + C$.
- The definite integral equals the difference between the primitives evaluated at the endpoints.

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note that the constant disappears.

Improper Integrals

Definition

An **improper integral** is one which the integrand is *singular* (not well behaved) within the range of integration.

$$\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \left(\int_a^b f(x)dx \right)$$

Discontinuous Integrals

Definition

A **discontinuous integrand** contains a finite number of discontinuities over the range of integration.

If $f(x)$ is discontinuous at $x = x_0$

$$\int_a^b f(x)dx = \int_a^{x_0} f(x)dx + \int_{x_0}^b f(x)dx$$

Note that the primitive of a discontinuous function is continuous.

Methods of Integration

Common Results

Since the indefinite integral is the reverse of differentiation.

$$\begin{aligned}\int x^n dx &= \frac{x^{n+1}}{n+1} + C & \int \cos(ax) dx &= \frac{1}{a} \sin(ax) + C \\ \int \frac{1}{x} dx &= \ln|x| + C & \int \sin(ax) dx &= -\frac{1}{a} \cos(ax) + C \\ \int e^{ax} dx &= \frac{1}{a} e^{ax} + C & \int \sec^2(ax) dx &= \frac{1}{a} \tan(ax) + C\end{aligned}$$

Using results from inverse hyperbolic functions.

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \operatorname{arcosh}\left(\frac{x}{a}\right) + C \quad \int \frac{1}{\sqrt{x^2 + a^2}} dx = \operatorname{arsinh}\left(\frac{x}{a}\right) + C$$

From the chain rule

$$\begin{aligned}\int (f(x))^n f'(x) dx &= \frac{1}{n+1} f^{n+1}(x) + C \\ \int \frac{f'(x)}{f(x)} dx &= \ln|f(x)| + C\end{aligned}$$

Powers of Trig Functions

Use the result $\sin^2(x) = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2(x) = \frac{1}{2}(1 + \cos 2x)$

Since we don't know how to integrate higher power trig functions.

- If the power is even, e.g. $\cos^4 x = \frac{1}{4}(1 + \cos 2x)^2$, expand and repeat until the expression can be integrated.
- If the power is odd, e.g. $\sin^3 x = \sin x(1 - \cos^2 x)$, expand and use the reverse chain rule.

Similar rules can be used for stuff related to $\sec^2 x = 1 + \tan^2 x$ and the $\csc^2 x$ equivalent.

Partial Fractions

$$f(x) = \frac{p(x)}{q(x)}$$

Where $p(x)$ and $q(x)$ are polynomials, then $f(x) = P(x) + Q(x)$

The fundamental theorem of algebra says that we can write $q(x)$ as . But we don't actually want to deal with complex numbers, so we have

$$q(x) = (x - a_1)^{j_1} (x - a_2)^{j_2} \dots (r_{m-1}(x))^{j_{m-1}} (r_m(x))^{j_m}$$

Where $r_{i(x)}$ are the terms with no real roots. So

$$Q(x) = \sum_{i=1}^m \sum_{r=1}^{j_i} \frac{A_{ir}}{(x - a_i)^r}$$

Once you write down these partial fractions, it is easy to integrate the terms.

Cover-up Rule

$$\begin{aligned}
 f(x) &= \frac{a_0 + a_1x + a_2x^2 + \dots}{(x - r_0)(x - r_1)(x - r_2) \dots} \\
 &= \frac{b_0}{x - r_0} + \frac{b_1}{x - r_1} + \frac{b_2}{x - r_2} + \dots \\
 b_0 + (x - r_0) \left(\frac{b_1}{x - r_1} + \frac{b_2}{x - r_2} + \dots \right) &= \frac{a_0 + a_1x + a_2x^2 + \dots}{(x - r_1)(x - r_2) \dots}
 \end{aligned}$$

Substitute $x = r_0$ to get

$$b_0 = \frac{a_0 + a_1x + a_2x^2 + \dots}{(x - r_1)(x - r_2) \dots}$$

Note

When there is a repeated root, the cover-up method only gives the coefficient of highest power.

Substitution

Substitution simplifies an integral by changing variables. Take example

$$\int \frac{1}{1 + x^2} dx$$

Let $x = \tan u$, then $dx = \sec^2 u \, du$.

- We are saying that making a small test in dx is the same as making a small step in $\sec^2 u \, du$.
- Look at the graph of $\tan x$ and this does make sense.

$$\begin{aligned}
 &= \int \frac{1}{1 + \tan^2 x} \sec^2 x \, dx \\
 &= u + C \\
 &= \arctan x + C
 \end{aligned}$$

Half-angle Formula

Using the substitution $\tan\left(\frac{x}{2}\right) = t$, we can show

$$\begin{aligned}
 \sin x &= \frac{2t}{1 + t^2} \\
 \cos x &= \frac{1 - t^2}{1 + t^2} \\
 \tan x &= \frac{2t}{1 - t^2}
 \end{aligned}$$

Common Substitutions

Denominator	Substitution
$a^2 + x^2$	$x = a \tan \theta$
$\sqrt{a^2 - x^2}$	$x = a \cos \theta$ or $x = a \sin \theta$
$\sqrt{x^2 - a^2}$	$x = a \cosh^2 \theta$
$\sqrt{x^2 + a^2}$	$x = a \sinh^2 \theta$

Denominator	Substitution
$a^2 - x^2$	$x = a \tanh^2 \theta$ if $ x < a $ $x = a \cosh^2 \theta$ if $ x > a $

Use completing the square to deal with general quadratic denominators.

Integration by Parts

From the product rule.

$$\begin{aligned}\frac{d}{dx}(fg) &= f \frac{dg}{dx} + \frac{df}{dx}g \\ f \frac{dg}{dx} &= \frac{d}{dx}(fg) - \frac{df}{dx}g \\ \int f \frac{dg}{dx} dx &= fg - \int \frac{df}{dx}g dx\end{aligned}$$

Integration with Complex Numbers

The integral of complex valued functions has the same rules as integrate with real valued function.

$$\int \Re(f) dx = \Re\left(\int f(x) dx\right)$$

Odd and Even Functions

Definitions

- $f(x) = f(-x) \iff$ even function
- $f(x) = -f(-x) \iff$ odd function

Using the area interpretation of an integral, we have

$$\begin{aligned}\int_{-a}^a f(x) dx &= 0 && \text{if } f \text{ is odd} \\ \int_{-a}^a f(x) dx &= 2 \int_0^a f(x) dx && \text{if } f \text{ is even}\end{aligned}$$

Reduction Formulae

We could sometimes write an integral as a recurrence relation.

$$\begin{aligned}I_{2n} &= \int_0^{\frac{\pi}{2}} \sin^{2n} x dx \\ &= [\sin^{2n-1} x \cdot (-\cos x)]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (2n-1) \sin^{2n-2} x \cos^2 x dx \\ &= (2n-1) \int_0^{\frac{\pi}{2}} \sin^{2n-2} x (1 - \sin^2 x) dx \\ &= (2n-1)(I_{2n-2} - I_{2n}) \\ I_{2n} &= \frac{2n-1}{2n} I_{2n-2}\end{aligned}$$

$$I_{2n} = \frac{(2n-1)!!}{2n!!} I_0 = \frac{(2n-1)!!}{2n!!} \frac{\pi}{2}$$

Definition

Double factorials $n!!$ multiplies only the even or odd terms less than or equal to n .

$$n!! = n(n-2)(n-4)\dots$$

Differentiation of Integrals

When we want to differentiate an integral where

- Boundaries depends on q , or
- The function depends on q .

Such as

$$I(q) = \int_{a(q)}^{b(q)} f(x, q) dx$$

Since there are 3 objects that depends on q : $a(q)$, $b(q)$ and $f(x, q)$, so $I'(q)$ have at least 3 terms.

$$\frac{dI}{dq} = \underbrace{\int_{a(q)}^{b(q)} \frac{\partial}{\partial q} f(x, q) dx}_{\text{area gained from curve changing}} + \underbrace{f(b(q), q) \frac{db}{dq}}_{\text{area gained from } b(q) \text{ increasing}} - \underbrace{f(a(q), q) \frac{da}{dq}}_{\text{area lost from } a(q) \text{ increasing}}$$

Partial Differentiation

Let $h(x, y)$, by treating y as a constant

$$\frac{\partial h}{\partial x} = \frac{h(x + \delta x, y) - h(x, y)}{\delta x}$$

We also define $\frac{\partial h}{\partial y}$, because there's no reason for the two derivatives to be related.

Chain Rule

Let $\tilde{h}(s) = h(x(s), y(s))$, the change in \tilde{h} will be the combined effect of change in x and y .

$$\delta h = \frac{\partial h}{\partial x} \delta x + \frac{\partial h}{\partial y} \delta y$$

And we know

$$\delta x = \frac{dx}{ds} \delta s \quad \delta y = \frac{dy}{ds} \delta s$$

The chain rule allows us to take the derivative without expanding out all terms.

$$\frac{dh}{ds} = \frac{\partial h}{\partial x} \frac{dx}{ds} + \frac{\partial h}{\partial y} \frac{dy}{ds}$$

Example: Gamma Function

$$\int_0^\infty x^n e^{-x} dx$$

$$\text{Let } I(a) = \int_0^\infty e^{-ax} dx = \frac{1}{a}$$

$$\begin{aligned}\frac{dI}{d\alpha} &= \int_0^\infty \frac{\partial}{\partial \alpha} e^{-\alpha x} dx & \frac{d^2 I}{d\alpha^2} &= \int_0^\infty \frac{\partial}{\partial \alpha} (-x e^{-\alpha x}) dx \\ &= \int_0^\infty -x e^{-\alpha x} dx & &= \int_0^\infty \frac{\partial}{\partial \alpha} (x^2 e^{-\alpha x}) dx \\ &= -\frac{1}{\alpha^2} & &= \frac{2}{\alpha^3}\end{aligned}$$

Therefore $\int_0^\infty x^n e^{-\alpha x} dx = n!/\alpha^n$. Set $\alpha = 1$ to get the factorial function.

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-\alpha x} dx = (n-1)!$$

Example: Stirling's Approximation

Use integrals to approximate summation.

$$\ln n! = \sum_{k=1}^n \ln k$$

If $k \leq x \leq k+1$, then $\ln k \leq \ln x \leq \ln(k+1)$.

$$\begin{aligned}\sum_{k=1}^n \ln k &\leq \int_1^n \ln x \, dx \leq \sum_{k=1}^{n-1} \ln(k+1) \\ &= \sum_{k=1}^n \ln k \\ &\leq \int_1^{n+1} \ln x \, dx\end{aligned}$$

Bound $\sum_{k=1}^n \ln k$ with the two integrals.

$$n \ln n - n + 1 \leq \ln n! \leq (n+1) \ln(n+1) - (n+1) + 1$$

Taking the leading terms of the expressions

$$\ln n! \approx n \ln n - n$$

We can show the fractional error $\frac{\ln n! - n \ln n - n}{n!} = O\left(\frac{1}{n}\right)$ as $n \rightarrow \infty$.

Schwarz's Inequality

As we have seen for vectors, $\vec{p} \cdot \vec{q} = |\vec{p}||\vec{q}| \cos \theta$.

$$\begin{aligned}\Rightarrow (\vec{p} \cdot \vec{q})^2 &\leq \vec{p}^2 \vec{q}^2 \\ \Rightarrow \left(\sum_i p_i \times q_i \right)^2 &\leq \left(\sum_i (p_i)^2 \right) \left(\sum_i (q_i)^2 \right) \text{ where } p, q \text{ have components } p_1, p_2, \dots\end{aligned}$$

The general proof for vectors more than 2 components:

$$\begin{aligned} \implies (\forall \lambda \in \mathbb{R}) \quad \sum_i^n (\lambda p_i + q_i)^2 &\geq 0 \\ \iff \lambda^2 \left(\sum_i^n (p_i)^2 \right) + 2\lambda \left(\sum_i^n p_i q_i \right) + \left(\sum_i^n (q_i)^2 \right) &\geq 0 \end{aligned}$$

This is a quadratic in λ . Since λ has 0 or 1 root, the discriminant $b^2 \leq 4ac$

$$\begin{aligned} 4 \left(\sum_i^n p_i q_i \right)^2 &\leq 4 \left(\sum_i^n (p_i)^2 \right) \left(\sum_i^n (q_i)^2 \right) \\ \left(\sum_i^n p_i q_i \delta s \right)^2 &\leq \left(\sum_i^n (p_i)^2 \delta s \right) \left(\sum_i^n (q_i)^2 \delta s \right) \end{aligned}$$

This looks like the Riemann sum, we could write a proof for integrals only, or if we take $n \rightarrow \infty$.

$$\left(\int_a^b p(x)q(x)dx \right)^2 \leq \left(\int_a^b p(x)^2 dx \right) \left(\int_a^b q(x)^2 dx \right)$$

This is an inequality iff $p(x) \propto q(x)$ - the vectors are in the same direction as each other.

Multiple Integrals

We want to calculate the value under the surface $z = h(x, y)$

Similar to Riemann sums, divide the region into little boxes, the i th box has area δA_i .

$$\begin{aligned} V_n &= \sum_{i=1}^n h(x_i, y_i) \delta A_i \\ V &= \iint_A h(x, y) dA = \lim_{n \rightarrow \infty} V_n \end{aligned}$$

2D Regions

For a rectangular region:

$$\begin{aligned} \iint_A h(x, y) dA &= \int_{x=a}^{x=b} \int_{y=c}^{y=d} h(x, y) dy dx \\ &= \int_{y=c}^{y=d} \int_{x=a}^{x=b} h(x, y) dx dy \end{aligned}$$

From the geometric interpretation of the integrals, the order does not matter.

For a region bounded by $y = f(x)$ below and $y = g(x)$ above.

$$V = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=f(x)} h(x, y) dy dx$$

We can change the order at which x, y are if the new integral is easier to evaluate.

If we set $h(x, y) = 1$, then we get the area of the region by

$$A = \iint_A dA$$

Separating Integrand

If we can separate $f(x, y)$ as $g(x)h(y)$, where the bounds don't depend on x or y (a rectangle).

$$\int_a^b \int_c^d g(x)h(y) dy dx = \left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right)$$

We can do this because $f(x)dx$ is a constant with respect to y , we can take it out using the constant multiple rule.

$$\int k f(y) dy = k \int f(y) dy$$

Polar Coordinates

Many regions have symmetry, it may be easier to write $h(r, \theta)$ than $h(x, y)$.

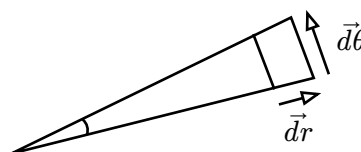
$$x = r \cos \theta$$

$$y = r \sin \theta$$

For a small change in r and θ , a point moves by vector

$$\frac{d}{dr} \begin{pmatrix} x \\ y \end{pmatrix} dr = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} dr$$

$$\frac{d}{d\theta} \begin{pmatrix} x \\ y \end{pmatrix} d\theta = \begin{pmatrix} -r \sin \theta \\ r \cos \theta \end{pmatrix} d\theta$$



dA is the area spanned by the two vectors. We can find dA using the determinant.

$$dA = \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} dr d\theta = r dr d\theta$$

So we have

$$\iint_A f(r, \theta) dA = \iint_A f(r, \theta) r dr d\theta$$

Then multiple integration can continue as normal.

Triple Integrals

By finding dV using the matrices technique.

Coordinate system	dV
Cartesian coordinates	$dx dy dz$
Cylindrical polar coordinates	$r dr d\theta dz$
Spherical polar coordinates	$r^2 \sin \theta dr d\theta d\varphi$

The Gaussian Integral

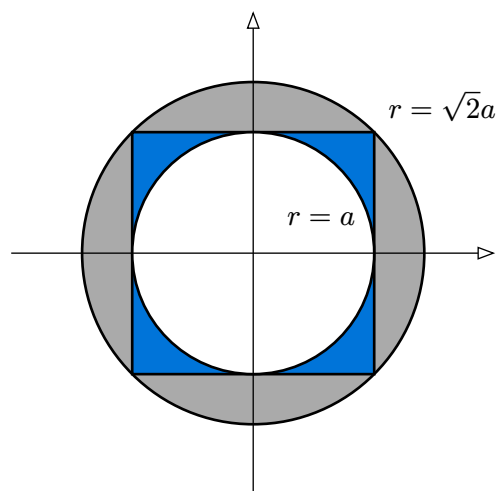
Note

$f(x) = e^{-x^2}$ gives the normal distribution.

The integral $I_a = \int_{-a}^a e^{-x^2} dx$ has no close form expression, but we can compute I_∞ .

Suppose the square has integral $(I_a)^2$

$$\begin{aligned} &= \left(\int_{-a}^a e^{-x^2} dx \right)^2 \\ &= \left(\int_{-a}^a e^{-x^2} dx \right) \left(\int_{-a}^a e^{-y^2} dy \right) \\ &= \int_{-a}^a \int_{-a}^a e^{-(x^2+y^2)} dy dx \end{aligned}$$



Consider the area of the circles, a circle with radius a has integral

$$J(a) = \int_0^{2\pi} \int_0^a e^{-r^2} r dr d\theta$$

By inspection $J(a) < (I_a)^2 < J(\sqrt{2}a)$. Let the area between the circles $K(a) = J(\sqrt{2}a) - J(a)$

$$K(a) \leq \left(\pi(\sqrt{2}a)^2 - \pi a^2 \right) e^{-a^2} = \pi a^2 e^{-a^2}$$

Which is the area between the circles multiplied by the maximum value of e^{-a^2} in that region.

$\therefore K(a) \rightarrow 0$ as $a \rightarrow \infty$, $\therefore (I_a)^2 \rightarrow J(\infty)$ as $a \rightarrow \infty$.

$$\begin{aligned} (I_a)^2 &= J(\infty) \\ &= \int_0^{2\pi} d\theta \int_0^\infty e^{-r^2} r dr \\ &= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^\infty \\ &= \pi \\ I_a &= \sqrt{\pi} \end{aligned}$$

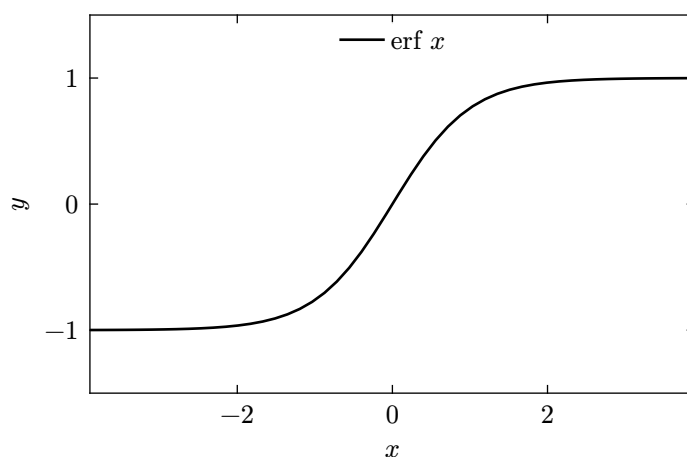
The Error Function

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx$$

erf gives the value of the integral when $z \neq \infty$.

As $z \rightarrow \infty$, $\text{erf } z \rightarrow 1$.

The value is normalised between -1 and 1 by the constant $2/\sqrt{\pi}$.



END Integration