Integration

Integration as Area

To find the area under the curve $y = f(x)$ in range $a \leq x \leq b$.

1. **Partition** $[a, b]$ into N subintervals with endpoints $x_0, x_1, ..., x_N$, the width of the partitions does not have to be equal.
2. Choose N points $\xi_1, \xi_2, ..., \xi_N$ in each of the partitions. The rectangle of each subpartition has area $A_i = (x_i - x_{i-1})f(\xi_i)$.
3. The total area is the **Riemann sum**

$$S_N = \sum_{i=1}^{N} (x - x_{i-1})f(\xi_i)$$

4. Take the **Riemann integral** where $N \to \infty$ such that all $x_i - x_{i-1} \to 0$.

$$\int_a^b f(x)dx = \lim_{N \to \infty} S_N$$

The integral in this form is also called a **definite integral**.

Properties of an Integral

From the geometric interpretation.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

But in the definition we assume $a \leq x \leq b$, to make the above property true for all c, we can define

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Integration and differentiation are **linear operations**, this means

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

In $\infty + (-\infty)$ senarios, $\int f + gdx$ may be finite, but $\int fdx + \int gdx$ are both undefined.

Fundamental Theorem of Calculus

$$F(x) = \int_a^x f(u)du \iff \frac{dF}{dx} = f(x)$$

Or

$$\frac{d}{dx}\int_a^x f(u)du = f(x)$$

Proof

$$\begin{aligned}
\frac{dF}{dx} &= \lim_{\delta x \to 0} \frac{F(x + \delta x) - F(x)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\int_a^{x+\delta x} f(u)du + \int_a^x f(u)du}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\int_x^{x+\delta x} f(u)du}{\delta x} \\
&= f(x)
\end{aligned}$$

> **Definitions**
> In $F(x) = \int f(x)dx$
> - $f(x)$ is the **integrand**.
> - $F(x)$ is the **primitive**.

Definite and Indefinite Integrals

The primitive is also called the **indefinite integral** of $f(x)$.
- If $F(x)$ is the indefinite integral, then so is $F(x) + C$.
- The definite integral equals the difference between the primitives evaluated at the endpoins.

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note that the constant disappears.

Improper Integrals

> **Definition**
> An **improper integral** is one which the integrand is *singular* (not well behaved) within the range of integration.

$$\int_a^\infty f(x)dx = \lim_{b \to \infty} \left(\int_a^b f(x)dx \right)$$

Discontinuous Integrals

> **Definition**
> A **discontinuous integrand** contains a finite number of discontinuities over the range of integration.

If $f(x)$ is discontinuous at $x = x_0$

$$\int_a^b f(x)dx = \int_a^{x_0} f(x)dx + \int_{x_0}^b f(x)dx$$

Note that the primitive of a discontinuous function is continuous.

Methods of Integration

Common Results

Since the indefinite integral is the reverse of differentiation.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad\qquad \int \cos(ax)dx = \frac{1}{a}\sin(ax) + C$$

$$\int \frac{1}{x}dx = \ln|x| + C \qquad\qquad \int \sin(ax)dx = -\frac{1}{a}\cos(ax) + C$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C \qquad\qquad \int \sec^2(ax)dx = \frac{1}{a}\tan(ax) + C$$

Using results from inverse hyperbolic functions.

$$\int \frac{1}{\sqrt{x^2 - a^2}}dx = \operatorname{arcosh}\left(\frac{x}{a}\right) + C \qquad\qquad \int \frac{1}{\sqrt{x^2 + a^2}}dx = \operatorname{arsinh}\left(\frac{x}{a}\right) + C$$

From the chain rule

$$\int (f(x))^n f'(x)dx = \frac{1}{n+1}f^{n+1}(x) + C$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

Powers of Trig Functions

Use the result $\sin^2(x) = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2(x) = \frac{1}{2}(1 + \cos 2x)$

Since we don't know how to integrate higher power trig functions.
- If the power is even, e.g. $\cos^4 x = \frac{1}{4}(1 + \cos 2x)^2$, expand and repeat until the expression can be integrated.
- If the power is odd, e.g. $\sin^3 x = \sin x(1 - \cos^2 x)$, expand and use the reverse chain rule.

Similar rules can be used for stuff related to $\sec^2 x = 1 + \tan^2 x$ and the $\csc^2 x$ equivalent.

Partial Fractions

$$f(x) = \frac{p(x)}{q(x)}$$

Where $p(x)$ and $q(x)$ are polynomials, then $f(x) = P(x) + Q(x)$

The fundamental theorem of algebra says that we can write $q(x)$ as . But we don't actually want to deal with complex numbers, so we have

$$q(x) = (x - a_1)^{j_1}(x - a_2)^{j_2}\cdots(r_{m-1}(x))^{j_{m-1}}(r_m(x))^{j_m}$$

Where $r_{i(x)}$ are the terms with no real roots. So

$$Q(x) = \sum_{i=1}^{m}\sum_{r=1}^{j_i} \frac{A_{ir}}{(x - a_i)^r}$$

Once you write down these partial fractions, it is easy to integrate the terms.

Cover-up Rule

$$f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots}{(x - r_0)(x - r_1)(x - r_2) \cdot \cdots}$$

$$= \frac{b_0}{x - r_0} + \frac{b_1}{x - r_1} + \frac{b_2}{x - r_2} + \cdots$$

$$b_0 + (x - r_0)\left(\frac{b_1}{x - r_1} + \frac{b_2}{x - r_2} + \cdots\right) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots}{(x - r_1)(x - r_2) \cdot \cdots}$$

Substitute $x = r_0$ to get

$$b_0 = \frac{a_0 + a_1 x + a_2 x^2 + \cdots}{(x - r_1)(x - r_2) \cdot \cdots}$$

> **Note**
> When there is a repeated root, the cover-up method only gives the coefficient of higest power.

Substitution

Substitution simplifies an integral by changing variables. Take example

$$\int \frac{1}{1 + x^2} dx$$

Let $x = \tan u$, then $dx = \sec^2 u \ du$.
- We are saying that making a small test in dx is the same as making a small step in $\sec^2 u \ du$.
- Look at the graph of $\tan x$ and this does make sense.

$$= \int \frac{1}{1 + \tan^2 x} \sec^2 x \ dx$$
$$= u + C$$
$$= \arctan x + C$$

Half-angle Formula

Using the substitution $\tan\left(\frac{x}{2}\right) = t$, we can show

$$\sin x = \frac{2t}{1 + t^2}$$

$$\cos x = \frac{1 - t^2}{1 + t^2}$$

$$\tan x = \frac{2t}{1 - t^2}$$

Common Substitutions

Denominator	Substitution
$a^2 + x^2$	$x = a \tan \theta$
$\sqrt{a^2 - x^2}$	$x = a \cos \theta$ or $x = a \sin \theta$
$\sqrt{x^2 - a^2}$	$x = a \cosh^2 \theta$
$\sqrt{x^2 + a^2}$	$x = a \sinh^2 \theta$

Denominator	Substitution
$a^2 - x^2$	$x = a \tanh^2 \theta$ if $\lvert x \rvert < \lvert a \rvert$
	$x = a \cosh^2 \theta$ if $\lvert x \rvert > \lvert a \rvert$

Use completing the square to deal with general quadratic denominators.

Integration by Parts

From the product rule.

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g$$

$$f\frac{dg}{dx} = \frac{d}{dx}(fg) - \frac{df}{dx}g$$

$$\int f\frac{dg}{dx}dx = fg - \int \frac{df}{dx}g \; dx$$

Integration with Complex Numbers

The integral of complex valued functions has the same rules as integrate with real valued function.

$$\int \Re(f) \; dx = \Re\left(\int f(x) \; dx \right)$$

Odd and Even Functions

Definitions
- $f(x) = f(-x) \iff$ even function
- $f(x) = -f(-x) \iff$ odd function

Using the area interpretation of an integral, we have

$$\int_{-a}^{a} f(x) \; dx = 0 \qquad\qquad \text{if } f \text{ is odd}$$

$$\int_{-a}^{a} f(x) \; dx = 2\int_{0}^{a} f(x) \; dx \quad \text{if } f \text{ is even}$$

Reduction Formulae

We could sometimes write an integral as a recurrence relation.

$$I_{2n} = \int_{0}^{\frac{\pi}{2}} \sin^{2n} x \; dx$$

$$= \left[\sin^{2n-1} x \cdot (-\cos x)\right]_{0}^{\frac{\pi}{2}} + \int_{0}^{\frac{\pi}{2}} (2n-1)\sin^{2n-2} x \cos^2 x \; dx$$

$$= (2n-1)\int_{0}^{\frac{\pi}{2}} \sin^{2n-2} x(1 - \sin^2 x)dx$$

$$= (2n-1)(I_{2n-n} - I_{2n})$$

$$I_{2n} = \frac{2n-1}{2n}I_{2n-2}$$

$$I_{2n} = \frac{(2n-1)!!}{2n!!}I_0 = \frac{(2n-1)!!}{2n!!}\frac{\pi}{2}$$

> **Definition**
> **Double factorials** $n!!$ multiplies only the even or odd terms less than or equal to n.
> $$n!! = n(n-2)(n-4)\cdots$$

Differentiation of Integrals

When we want to differentiate an integral where
- Boundaries depends on q, or
- The function depends on q.

Such as

$$I(q) = \int_{a(q)}^{b(q)} f(x,q)dx$$

Since there are 3 objects that depends on $q : a(q), b(q)$ and $f(x,q)$, so $I'(q)$ have at least 3 terms.

$$\frac{dI}{dq} = \underbrace{\int_{a(q)}^{b(q)} \frac{\partial}{\partial q}f(x,q)dx}_{\substack{\text{area gained}\\\text{from curve changing}}} + \underbrace{f(b(q),q)\frac{db}{dq}}_{\substack{\text{area gained from}\\b(q)\text{ increasing}}} - \underbrace{f(a(q),q)\frac{da}{dq}}_{\substack{\text{area lost from}\\a(q)\text{ increasing}}}$$

Partial Differentiation

Let $h(x,y)$, by treating y as a constant

$$\frac{\partial h}{\partial x} = \frac{h(x+\delta x, y) - h(x,y)}{\delta x}$$

We also define $\frac{\partial h}{\partial y}$, because there's no reason for the two derivatives to be related.

Chain Rule

Let $\tilde{h}(s) = h(x(s), y(s))$, the change in $\tilde{h}$ will be the combined effect of change in x and y.

$$\delta h = \frac{\partial h}{\partial x}\delta x + \frac{\partial h}{\partial y}\delta y$$

And we know

$$\delta x = \frac{dx}{ds}\delta s \qquad \delta y = \frac{dy}{ds}\delta s$$

The chain rule allows us to take the derivative without expanding out all terms.

$$\frac{dh}{ds} = \frac{\partial h}{\partial x}\frac{dx}{ds} + \frac{\partial h}{\partial y}\frac{dy}{ds}$$

Example: Gamma Function

$$\int_0^\infty x^n e^{-x}\ dx$$

Let $I(a) = \int_0^\infty e^{-\alpha x}dx = \frac{1}{a}$

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$$\frac{dI}{d\alpha} = \int_0^\infty \frac{\partial}{\partial\alpha} e^{-\alpha x} dx \qquad\qquad \frac{d^2 I}{d\alpha^2} = \int_0^\infty \frac{\partial}{\partial\alpha}(-xe^{-\alpha x}) dx$$

$$= \int_0^\infty -xe^{-\alpha x} dx \qquad\qquad = \int_0^\infty \frac{\partial}{\partial\alpha}(x^2 e^{-\alpha x}) dx$$

$$= -\frac{1}{\alpha^2} \qquad\qquad\qquad = \frac{2}{\alpha^3}$$

Therefore $\int_0^\infty x^n e^{-\alpha x} dx = n!/\alpha^n$. Set $\alpha = 1$ to get the factorial function.

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-\alpha x} dx = (n-1)!$$

Example: Stirling's Approximation
Use integrals to approximate summation.

$$\ln n! = \sum_{k=1}^n \ln k$$

If $k \le x \le k+1$, then $\ln k \le \ln x \le \ln(k+1)$.

$$\sum_{k=1}^n \ln k \le \int_1^n \ln x \ dx \le \sum_{k=1}^{n-1} \ln(k+1)$$

$$= \sum_{k=1}^n \ln k$$

$$\le \int_1^{n+1} \ln x \ dx$$

Boundnig $\sum_{k=1}^n \ln k$ with the two integrals.

$$n \ln n - n + 1 \le \ln n! \le (n+1)\ln(n+1) - (n+1) + 1$$

Taking the leading terms of the expressions

$$\ln n! \approx n \ln n - n$$

We can show the fractional error $\dfrac{\ln n! - n\ln n - n}{n!} = O\!\left(\dfrac{1}{n}\right)$ as $n \to \infty$.