

Infinite Series

Convergence

Given an infinite sequence of numbers, define the **partial sum** to be

$$S_n = u_1 + u_2 + \cdots + u_n$$

We want to know if the infinite series

- Have a well defined limit
- Diverges
- Oscillates

$$\lim_{n \rightarrow \infty} S_n = S \text{ iff } (\forall \varepsilon > 0)(\exists N) n > N \implies |S - S_n| < \varepsilon$$

Conditions for Convergence

Requires $u_n \rightarrow 0$ as $n \rightarrow \infty$.

The sum of two diverging series may converge, e.g. when one diverges to $+\infty$ and the other to $-\infty$, and the two cancels each other out.

Definitions

- A series is **absolutely convergent** iff $\sum |a_n|$ converges.
- A series is **conditionally convergent** iff $\sum a_n$ converges but not $\sum |a_n|$.

If we rearrange the terms in a series, a converging series can diverge, or the other way round.

- If we have an absolutely convergent series, it doesn't matter if we change the order.
- If we have a conditionally convergent series, changing the order may cause it to diverge.

Geometric Progression

Let $u_k = r^k$, we already know the sum formula for the geometric progression, here's the proof.

$$(1 - r)(1 + r + r^2 + \cdots + r^k) = 1 - r^{k+1} \quad (\text{by simplifying the expression})$$

$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}$$

For the sum of infinite series

$$\lim_{k \rightarrow \infty} \frac{1 - r^{k+1}}{1 - r} \text{ exists if } |r| < 1.$$

The Harmonic Series

The harmonic series is

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

We can group terms **without reordering**.

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

The sequence $\sum \frac{1}{2}$ diverges, so the harmonic series also diverges, this is called a **comparison test**.

$$\gamma = \lim_{k \rightarrow \infty} (S_k - \ln k) = 0.57721566$$

Which is the **Euler-Mascheroni constant**.

Convergence Tests

Comparison Tests

The comparison test can be used on two nonnegative series: $(\forall k) u_k, v_k \geq 0$

$$(\forall k \geq K) u_k \leq v_k \text{ and } \sum_{k=0}^{\infty} v_k \text{ convergent} \Rightarrow \sum_{k=0}^{\infty} u_k \text{ convergent}$$

$$(\forall k \geq K) u_k \geq v_k \text{ and } \sum_{k=0}^{\infty} v_k \text{ divergent} \Rightarrow \sum_{k=0}^{\infty} u_k \text{ divergent}$$

Ratio Tests

For a positive series: $(\forall k) u_k > 0$

$$\lim_{k \rightarrow \infty} \left(\frac{u_{k+1}}{u_k} \right) < 1 \Rightarrow \text{converges}$$

$$\lim_{k \rightarrow \infty} \left(\frac{u_{k+1}}{u_k} \right) > 1 \Rightarrow \text{diverges}$$

Proof

If $\lim_{k \rightarrow \infty} \frac{u_{k+1}}{u_k} = \alpha < \infty$, then for large enough k :

$$\sum u = u_1 + u_2 + \dots + u_k(1 + \alpha + \alpha^2 + \dots)$$

Which converges.

Leibniz Criterion

Definition

Alternating series have terms with alternating signs.

If $a_k > 0$ and a_k is monotonically decreasing for large enough k , and $\lim_{k \rightarrow \infty} a_k = 0$.

$$S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges}$$

Proof

We know it is true that

$$\sum_{k=1}^{2n} = (a_1 - a_2) + (a_3 - a_4) + \dots \text{ is a monotonically increasing series}$$

$$\sum_{k=1}^{2n+1} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots \text{ is a monotonically decreasing series}$$

$$\lim_{k \rightarrow \infty} \left(\sum_{k=1}^{2n+1} a_k - \sum_{k=1}^{2n} a_k \right) = 0$$

So we write the inequality

$$a_1 - a_2 = S_2 < S_4 < \dots < S_{2n} < S_{2n+1} < \dots < S_3 < S_1 < a_1$$

Power Series

A power series have form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \dots$$

The **domain of convergence** is either

- Only $x = 0$
- All finite x
- Only for some $|x| < R$

We can find out which case it is with the **ratio test**, which gives the shortcut if $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$ exists.

$$|x| < \frac{1}{L} \implies \text{converges}$$

$$|x| > \frac{1}{L} \implies \text{diverges}$$

$$|x| = \frac{1}{L} \implies \text{indeterminate}$$

Note the endpoints at $|x| = 1/L$ may behave differently.

The Taylor Series

We can approximate the value of a function using its tangent at a point.

$$f(x) \approx f(a) + (x - a)f'(a)$$

$$f'(x) \approx f'(a) + (x - a)f''(a)$$

By the fundamental theorem of calculus, which links the derivative to the integral:

$$\int_a^{a+h} f'(x) dx = f(a+h) - f(a)$$

$$\begin{aligned} f(a+h) &= f(a) + \int_a^{a+h} f'(x) dx \\ &= f(a) + \int_a^{a+h} f'(a) + (x-a)f''(a) dx \\ &= f(a) + \left[xf'(a) + \frac{(x-a)^2}{2} f''(a) \right]_a^{a+h} \\ &= f(a) + hf'(a) + \frac{h^2}{2} f''(a) \end{aligned}$$

We can show that the **second order approximation** is better than the first order approximation. (how?)

Extend this to higher order approximations, and we get

$$f'(x) \approx f'(a) + (x-a)f''(a) + \frac{(x-a)^2}{2!} f'''(a) + \frac{(x-a)^3}{3!} f^{(4)}(a) + \dots$$

Since the factorial function grows faster than any polynomial, we know that $n!$ grows faster than $(x-a)^n$ so the series converges for a lot of approximations.