

Definition 7

$n \in \mathbb{N}$ is odd if $(\exists i \in \mathbb{N}) n = 2i + 1$.

Proposition 8

Goal: $(\forall m, n \in \mathbb{N}) m \text{ and } n \text{ odd} \implies m \times n \text{ odd}$

Proof

Assume:

1. $m, n \in \mathbb{N}$
2. m and n odd

New goal: $m \times n$ odd

$$\begin{aligned}
 & (\exists i, j \in \mathbb{N}) m = 2i + 1 \text{ and } n = 2j + 1 \\
 \implies & (\exists i, j \in \mathbb{N}) m \times n = (2i + 1) \times (2j + 1) \\
 \implies & (\exists i, j \in \mathbb{N}) m \times n = 2(2ij + i + j) + 1 \\
 \implies & (\exists k \in \mathbb{N}) m \times n = 2k + 1 \\
 \implies & m \times n \text{ odd}
 \end{aligned}$$

Definition 9

$(\forall x \in \mathbb{R})$

- $(\exists m, n \in \mathbb{Z}) x = m/n \iff x \text{ rational}$
- $\neg(x \text{ rational}) \iff x \text{ irrational}$
- $x > 0 \iff x \text{ positive}$
- $x < 0 \iff x \text{ negative}$
- $\neg(x \text{ positive}) \iff x \text{ nonpositive}$
- $\neg(x \text{ negative}) \iff x \text{ nonnegative}$
- $x \text{ nonnegative and } x \in \mathbb{Z} \iff x \in \mathbb{N}$

Proposition 10

Goal: $(\forall x \text{ positive}) \sqrt{x} \text{ rational} \implies x \text{ rational}$

Proof

Assume:

1. x positive
2. \sqrt{x} rational

New goal: x rational

$$\begin{aligned}
 & (\exists p, q \in \mathbb{Z}) \sqrt{x} = p/q \\
 \implies & (\exists p, q \in \mathbb{Z}) x = (\sqrt{x})^2 = p^2/q^2 \\
 \implies & (\exists p', q' \in \mathbb{Z}) x = p'/q' \\
 \iff & x \text{ rational}
 \end{aligned}$$

Theorem 11

Goal: Let P_1, P_2, P_3 be statements, $(P_1 \implies P_2 \text{ and } P_2 \implies P_3) \implies (P_1 \implies P_3)$

Proof

Assume:

1. $P_1 \implies P_2$

$$2. P_2 \implies P_3$$

$$3. P_1$$

New goal: P_3

$$\begin{aligned} &P_2 \text{ as (4) by (1) and (3)} \\ \implies &P_3 \text{ by (2) and (4)} \end{aligned}$$

Definition 12