

Ordinary Differential Equations

Definition

An **n th order ODE** includes the n th derivative but no higher derivatives.

To solve an ODE we find the dependent variable as a function of the independent variables.

- In the real world: this is done numerically with computers.
- Here we do it analytically to demonstrate principles.

Some ODEs don't have analytical solutions.

First Order ODEs

Definition

Integrable ODEs have form

$$\frac{dy}{dx} = f(x)$$

where the RHS does not depend on y .

$$y = \int f(x) dx$$

If $F'(x) = f(x)$, then $y = F(x) + C$. y solves the ODE whatever C is - there are infinitely many solutions.

(*) is called the **general solution** if it contains all possible solutions.

If we choose any value for C , we get a particular solution.

Definition

Separable ODEs have form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y) \frac{dy}{dx} = f(x)$$

$$\int g(y) \frac{dy}{dx} dx = \int f(x) dx$$

$$\int g(y) dy = \int f(x) dx$$

$$G(y) = F(x) + C$$

Geometric Interpretation of First Order ODEs

Consider $\frac{dy}{dx} = F(x, y)$

For every point (x, y) :

- There is a particular solution passing through the point
- Its gradient is $F(x, y)$

Note

2 particular solutions don't have to be the same shape (shifted versions) of each other.

Linear ODE

$$\frac{dy}{dx} + y \cdot p(x) = f(x)$$

When $f(x) = 0$, the ODE is homogeneous, two ways to solve it

$$\begin{aligned}\frac{dy}{dx} &= -y \cdot p(x) & e^{P(x)} \cdot \frac{dy}{dx} + p(x) \cdot e^{P(x)} \cdot y &= 0 \\ \int \frac{1}{y} dy &= - \int p(x) dx & \frac{d}{dx} (e^{P(x)} \cdot y) &= 0 \\ y &= Ae^{-P(x)} & y &= Ae^{-P(x)}\end{aligned}$$

Note

$e^{\int p(x) dx}$ is called the integrating factor.

For inhomogenous cases, multiply both sides by the integrating factor.

$$\begin{aligned}e^{P(x)} \cdot \frac{dy}{dx} + p(x) \cdot e^{P(x)} \cdot y &= f(x) \\ \frac{d}{dx} (e^{P(x)} \cdot y) &= f(x) \\ e^{P(x)} \cdot y &= F(x) + C \\ y &= (F(x) + C)e^{-P(x)}\end{aligned}$$

Definition

n th order ODE contains n arbitrary constants: n pieces of informations to fix them. The extra information are called **boundary conditions**.

Note

A **particular solution** has no unknown constants or \pm signs.

Substitutions

$$\begin{aligned}\frac{dy}{dx} &= f\left(\frac{y}{x}\right) \\ \text{let } u(x) &= \frac{y(x)}{x} \\ x \cdot u(x) &= y(x) \\ u + x \cdot \frac{du}{dx} &= \frac{dy}{dx}\end{aligned}$$

For example

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2 + y^2}{xy} \\ &= \frac{(x/y)^2 + 1}{x/y} \\ u + x \cdot \frac{du}{dx} &= \frac{u^2 + 1}{u} \\ x \cdot \frac{du}{dx} &= \frac{1}{u} \\ &\vdots\end{aligned}$$

Benoulli Equations

$$\frac{dy}{dx} + p(x) \cdot y = q(x) \cdot y^n$$

If $y = 0$ or $y = 1$, then the equation is homogenous, otherwise

$$\text{let } z(x) = y(x)^{1-n}$$

$$\frac{dz}{dx} = (1-n) \cdot y(x)^{-n} \cdot \frac{dy}{dx}$$

$$\frac{1}{1-n} \cdot \frac{dz}{dx} \cdot y(x)^n = \frac{dy}{dx}$$

$$\frac{1}{1-n} \cdot \frac{dz}{dx} \cdot y^n + p(x) \cdot y = q(x) \cdot y^n$$

$$\frac{1}{1-n} \cdot \frac{dz}{dx} + p(x) \cdot y^{1-n} = q(x)$$

$$\frac{dz}{dx} + (1-n) \cdot p(x) \cdot z = (1-n) \cdot q(x)$$

\vdots

2nd Order ODE

We consider linear cases only: the ODE can be written as $Ly = f$.

L is the **differential operator**

$$L = \left[\frac{d^2}{dx^2} + p(x) \frac{d}{dx} + q(x) \right]$$

Similar to how d/dx is an operator on functions.

Definition

A **linear operator** has properties:

- $L(\alpha u) = \alpha L(u)$
- $L(u + v) = L(u) + L(v)$

Note

$\frac{d}{dx}$ is a linear operator.

The differential operator is a linear operator that includes $\frac{d^n}{dx^n}$.

Definition

A **linear ODE** can be written as $Ly = f$ where L is a linear operator.

- $f = 0$: homogeneous
- $f \neq 0$: inhomogenous

The Principle of Superposition

If y_1 and y_2 are particular solutions of a homogeneous ODE

- Then so is $y_1 + y_2$
- And so is $\alpha y_1 + \beta y_2$ for any $\alpha, \beta \in \mathbb{C}$

If:

- The **particular integral** y_p solves an inhomogenous linear ODE, and
- The **complementary function** y_c is the general solution of the homogeneous equation of the complementary ODE

Then the general solution is $y = y_p + y_c$

2nd Order ODE with Constant Coefficients

Is the restricted case where $p(x), q(x)$ are restricted to constants.

Homogenous Equations

$$\frac{d^2 y}{dx^2} + 2p \frac{dy}{dx} + qy = 0$$

Try $y(x) = e^{\lambda x}$

$$e^{\lambda x}(\lambda^2 + 2p\lambda + q) = 0$$

$$\lambda = -p \pm \sqrt{p^2 - q}$$

- Case $p^2 > q$:

$$\lambda_1 = -p + \sqrt{p^2 - q}$$

$$\lambda_2 = -p - \sqrt{p^2 - q}$$

Then $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$ is the general solution, as the general solution has 2 solutions.

- Case $p^2 < q$: let $\Omega = \sqrt{q - p^2}$

$$\lambda_1 = -p + i\Omega$$

$$\lambda_2 = -p - i\Omega$$

Then

$$y = \frac{1}{2}(e^{\lambda_1 x} + e^{\lambda_2 x}) = \Re(e^{\lambda_1 x}) = e^{-px} \cos \Omega x$$

$$y = \frac{1}{2i}(e^{\lambda_1 x} - e^{\lambda_2 x}) = \Im(e^{\lambda_1 x}) = e^{-px} \sin \Omega x$$

are both solutions. So the general solution is

$$y = e^{-px}(A \cos \Omega x + B \sin \Omega x)$$

If $A = R \cos \varphi$ and $B = R \sin \varphi$

$$y = Re^{-px} \cos(\Omega x - \varphi) \text{ by double angle formula}$$

- Case $p^2 = q$: cannot use principle of solution because we only have one solution.

Observe xe^{-px} is also a solution, then

$$y = e^{-px}(A + Bx)$$

is the general solution.

Inhomogenous Equations

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = f(x)$$

Then the general solution is $y = y_c + y_p$ (complementary equation and particular integral)

To find y_p , guess the functional form and solve for some parameters.

- If f is a polynomial of degree n , try a general polynomial of degree n
 - If f is an exponential, try a multiple of the same exponential.
 - If f is in form $u \sin kx + v \cos kx$, try $a \sin kx + b \cos kx$
 - If f is a sum of two known cases, take the y_p of each of them and add them.
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Resonance, Transients and Damping

A damped oscillator is described by

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + (\omega_0)^2x = 0$$

where $x(t)$ is the time dependent position, γ, ω_0 are positive constants.

- The parameter γ describes friction/damping.
- If $\gamma = 0$ then **simple harmonic oscillator**.
- The equilibrium position is $x = 0$

Solving the homogeneous equation

$$\lambda = -\gamma \pm \sqrt{\gamma^2 - (\omega_0)^2}$$

- Case $\gamma > \omega_0$:

$$x(t) = Ae^{(-\gamma + \sqrt{\gamma^2 - (\omega_0)^2})t} + Be^{(-\gamma - \sqrt{\gamma^2 - (\omega_0)^2})t}$$

This is called **overdamping**: the bigger the γ , the slower it returns to equilibrium.

- Case $\gamma < \omega_0$: let $\Omega = \sqrt{(\omega_0)^2 - \gamma^2}$

$$x(t) = e^{-\gamma t}(A \cos \Omega t + B \sin \Omega t)$$

This is the **underdamped** case.

- Case $\gamma = \omega_0$: repeated roots give solution

$$y = e^{-\gamma t}(A + Bx)$$

This is the boundary between overdamping and underdamping, the graph looks like the overdamped case.
