

# Multivariable Differential Equations

## Partial Derivatives

For  $f = f(x, y)$

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}$$
$$\frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Are the gradients of  $f$  if we travel along the  $x$  or  $y$  direction.

For functions of more variables, you can calculate  $\partial f / \partial x$  by treating all other variables as constants.

## Higher Derivatives

There are four 2nd order partial derivatives for a function  $f(x, y)$

$$\frac{\partial^2 f}{(\partial x)^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$$
$$\frac{\partial^2 f}{(\partial y)^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right)$$

For any well defined function

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

For  $f(x_1, x_2, x_3, \dots, x_n)$ , the gradient of  $f$  is a vector.

$$\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right)$$

## Partial Differentials

**Integration** is the opposite of differentiation.

$$\frac{\partial f}{\partial x} = 2xy^2 \quad \text{holding } y \text{ constant}$$

$$f(x, y) = x^2y^2 + G(y)$$

$$\frac{\partial f}{\partial y} = 2x^2y + 2y \quad \text{holding } x \text{ constant}$$

$$f(x, y) = x^2y^2 + y^2 + H(x)$$

$$\therefore f(x, y) = x^2y^2 + y^2 + C$$

There may be no  $f(x, y)$  satisfying  $\partial f / \partial x$  and  $\partial f / \partial y$ , for examples those that leads to

$$\frac{\partial^2 f}{\partial y \partial x} \neq \frac{\partial^2 f}{\partial x \partial y}$$

**Taylor series** of  $f$  centred at  $(x, y)$ , all partial derivatives are evaluated at  $(x_0, y_0)$

$$f(x_0 + k, y_0 + k) = f(x_0, y_0) + \left( h \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) + \frac{1}{2!} \left( h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \right) + \dots$$

The **linear approximation** of  $f$

$$f(x_0 + h, y_0 + k) \approx f(x_0, y_0) + h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}$$

For very small change in  $x$  and  $y$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$


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## Chain Rule

Consider

$$\begin{aligned} df &= \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv \\ du &= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \\ dv &= \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \end{aligned}$$

Then

$$\begin{aligned} df &= \left( \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \right) dx + \left( \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} \right) dy \\ \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} \end{aligned}$$

## Reciprocity and Cyclic Relations

Let  $z(x, y)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \quad (1)$$

$$dx = \frac{\partial x}{\partial y} dy + \frac{\partial x}{\partial z} dz \quad (2)$$

$$dy = \frac{\partial y}{\partial z} dz + \frac{\partial y}{\partial x} dx \quad (3)$$

Rewrite (2)

$$dz = \frac{1}{\left(\frac{\partial x}{\partial z}\right)} dx - \frac{\left(\frac{\partial x}{\partial y}\right)}{\left(\frac{\partial x}{\partial z}\right)} dy$$

Comparing coefficients of  $dx$  and  $dy$  with (1)

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{1}{\left(\frac{\partial x}{\partial z}\right)} \\ \frac{\partial z}{\partial y} &= -\frac{\left(\frac{\partial x}{\partial y}\right)}{\left(\frac{\partial x}{\partial z}\right)} \\ \frac{\partial z}{\partial y} \frac{\partial x}{\partial z} &= -\frac{\partial x}{\partial y} \\ \frac{\partial z}{\partial y} \frac{\partial x}{\partial z} \frac{\partial y}{\partial x} &= -1\end{aligned}$$

The last three lines are called the **cyclic relation** for partial derivatives.

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