Definition: odd number
$n \in \mathbb{N}$ is odd if $(\exists i \in \mathbb{N})\ n = 2i + 1$.

Proposition 8: product of odd integers is odd

Goal: $(\forall m,n \in \mathbb{N})$ m and n odd $\implies m \times n$ odd

Proof

Assume:
1. $m, n \in \mathbb{N}$
2. m and n odd

New goal: $m \times n$ odd

$$\begin{aligned}
&(\exists i,j \in \mathbb{N})\ m = 2i + 1 \land n = 2j + 1 \\
\implies &(\exists i,j \in \mathbb{N})\ m \times n = (2i + 1) \times (2j + 1) \\
\implies &(\exists i,j \in \mathbb{N})\ m \times n = 2(2ij + i + j) + 1 \\
\implies &(\exists k \in \mathbb{N})\ m \times n = 2k + 1 \\
\implies &m \times n \text{ odd}
\end{aligned}$$

Definition: real numbers
$(\forall x \in \mathbb{R})$
- $(\exists m,n \in \mathbb{Z})\ x = m/n \iff x$ rational
- $\neg(x \text{ rational}) \iff x$ irrational
- $x > 0 \iff x$ positive
- $x < 0 \iff x$ negative
- $\neg(x \text{ positive}) \iff x$ nonpositive
- $\neg(x \text{ negative}) \iff x$ nonnegative
- x nonnegative $\land\ x \in \mathbb{Z} \iff x \in \mathbb{N}$

Proposition 10: rational square root

Goal: $(\forall x \text{ positive})$ $\sqrt{x}$ rational $\implies x$ rational

Proof

Assume:
1. x positive
2. $\sqrt{x}$ rational

New goal: x rational

$$\begin{aligned}
&(\exists p,q \in \mathbb{Z})\ \sqrt{x} = p/q \\
\implies &(\exists p,q \in \mathbb{Z})\ x = \left(\sqrt{x}\right)^2 = p^2/q^2 \\
\implies &(\exists p',q' \in \mathbb{Z})\ x = p'/q' \\
\iff &x \text{ rational}
\end{aligned}$$

Definition: modus ponens
$P \land (P \implies Q) \implies Q$

Theorem 11: implication transitivity

$(\forall P_1,P_2,P_3 \text{ statement})\ (P_1 \implies P_2 \land P_2 \implies P_3) \implies (P_1 \implies P_3)$

Proof

Assume:
1. $P_1 \implies P_2$
2. $P_2 \implies P_3$
3. P_1

New goal: P_3

$$\begin{aligned} & P_2 \text{ as (4) by (1) and (3)} \\ \implies & P_3 \text{ by (2) and (4)} \end{aligned}$$

Definition: bi-implication
$(P \iff Q) \iff (P \implies Q \land P \impliedby Q)$

Definition: divisibility
$d|n \iff (\exists k \in \mathbb{Z})\ n = k \times d$

Definition: congruence
$(\forall m \in \mathbb{Z}^+,a,b \in \mathbb{Z})\ a \equiv b \pmod m \iff m|(a-b)$

Proposition 16: parity as congruence

Goal: $(n \text{ even} \iff n \equiv 0 \pmod 2) \land (n \text{ odd} \iff n \equiv 1 \pmod 2)$

Subgoal: n even $\iff n \equiv 0 \pmod 2$

Assume:
1. n even

New goal: $n \equiv 0 \pmod 2$

$$\begin{aligned} n \text{ even} &\iff (\exists k \in \mathbb{Z})\ n = 2 \times k \\ &\iff (\exists k \in \mathbb{Z})\ (n-0) = 2 \times k \\ &\iff n \equiv 0 \pmod 2 \end{aligned}$$

Subgoal: n odd $\iff n \equiv 1 \pmod 2$

Assume:
1. n odd

New goal: $n \equiv 1 \pmod 2$

$$\begin{aligned} n \text{ odd} &\iff (\exists k \in \mathbb{Z})\ n = 2 \times k + 1 \\ &\iff (\exists k \in \mathbb{Z})\ (n-1) = 2 \times k \\ &\iff n \equiv 1 \pmod 2 \end{aligned}$$

Proposition 18: linearity of congruence

Goal: $(\forall m \in \mathbb{Z}^+,a,b \in \mathbb{Z})\ a \equiv b \pmod m \iff ((\forall n \in \mathbb{Z}^+)\ n \times a \equiv n \times b \pmod{n \times m})$

Assume:
1. $m \in \mathbb{Z}^+$
2. $a, b \in \mathbb{Z}$

Subgoal: $a \equiv b \pmod m \implies (\forall n \in \mathbb{Z}^+)\ n \times a \equiv n \times b \pmod{n \times m}$

Assume:
3. $a \equiv b \pmod m$
4. $n \in \mathbb{Z}^+$

New goal: $n \times a \equiv n \times b \pmod{n \times m}$

$$\begin{aligned} & (\exists i \in \mathbb{Z})\ a - b = m \times i \text{ by } (3) \\ \implies & (\exists i \in \mathbb{Z})\ n \times a - n \times b = (n \times m) \times i \\ \implies & (\exists i \in \mathbb{Z})\ n \times a \equiv n \times b \pmod{n \times m} \\ \implies & n \times a \equiv n \times b \pmod{n \times m} \end{aligned}$$

Subgoal: $(\forall n \in \mathbb{Z}^+)\ n \times a \equiv n \times b \pmod{n \times m} \implies a \equiv b \pmod{m}$

Assume:
3. $(\forall n \in \mathbb{Z}^+)\ n \times a \equiv n \times b \pmod{n \times m}$

New goal: $a \equiv b \pmod{m}$

$$\begin{aligned} & 1 \times a \equiv 1 \times b \pmod{1 \times m} \text{ by } (3) \\ \implies & a \equiv b \pmod{m} \end{aligned}$$

Definition
- $(\forall x)\ x = x$
- $(\forall x,y)\ x = y \implies (P(x) \implies P(y))$
- $(\forall a,b,c)\ (a = b \land b = c) \implies a = c$
- $(\forall a,b,x,y)\ (a = b \land x = y) \implies (a + x = b + x = b + y)$

Theorem 19: divisibility of prime products

Goal: $(\forall n \in \mathbb{Z})\ 6|n \iff 3|n \land 2|n$

Assume:
1. $n \in \mathbb{Z}$

New goal: $6|n \iff 3|n \land 2|n$

Subgoal: $6|n \implies 3|n \land 2|n$

Assume:
2. $6|n$

New goal: $3|n \land 2|n$

Subgoal: $3|n$

$$\begin{aligned} 6|n & \iff (\exists i \in \mathbb{Z})\ n = 6 \times i \\ & \implies (\exists j \in \mathbb{Z})\ n = 3 \times j \\ & \iff 3|n \end{aligned}$$

Subgoal: $2|n$

$$\begin{aligned} 6|n & \iff (\exists i \in \mathbb{Z})\ n = 6 \times i \\ & \implies (\exists j \in \mathbb{Z})\ n = 2 \times j \\ & \iff 2|n \end{aligned}$$

Subgoal: $3|n \land 2|n \implies 6|n$

Assume:
2. $2|n \land 3|n$

New goal: $6|n$

$$
\begin{aligned}
&(\exists i \in \mathbb{Z})\ n = 2 \times i \\
\implies &(\exists i \in \mathbb{Z})\ 3 \times n = 6 \times i \text{ as } (3) \\
&(\exists j \in \mathbb{Z})\ n = 3 \times j \\
\implies &(\exists j \in \mathbb{Z})\ 2 \times n = 6 \times j \text{ as } (4) \\
\implies &(\exists i,j \in \mathbb{Z})\ n = 6 \times (i - j) \text{ by } (3) \text{ and } (4) \\
\implies &(\exists k \in \mathbb{Z})\ n = 6 \times k \\
\implies &6|n
\end{aligned}
$$

Proposition 21: difference of squares

Goal: $(\forall k \in \mathbb{Z}^+)\ (\exists i,j \in \mathbb{N})\ 4 \times k = i^2 - j^2$

Assume:
1. $k \in \mathbb{Z}^+$

Let $i = k + 1, j = k - 1$

$$
\begin{aligned}
i^2 - j^2 &= (k+1)^2 - (k-1)^2 \\
&= 4 \times k
\end{aligned}
$$

Theorem 23: divisibility transitivity

Goal: $(\forall l,m,n \in \mathbb{Z})\ l|m \land m|n \implies l|n$

Assume:
1. $l, m, n \in \mathbb{Z}$
2. $l|m \land m|n$

New goal: $l|n$

$$
\begin{aligned}
&(\exists i \in \mathbb{Z})\ m = i \times l \\
&(\exists j \in \mathbb{Z})\ n = j \times m \\
\implies &(\exists i,j \in \mathbb{Z})\ n = (j \times i) \times l \\
\implies &(\exists k \in \mathbb{Z})\ n = k \times l \\
\implies &l|n
\end{aligned}
$$

Definition
$((\exists! x)\ P(x)) \iff ((\exists x)\ P(x) \land ((\forall y,z)\ P(y) \land P(z) \implies y = z))$

Proposition 24: unique existence of proper congruence

Lemma 24.1: congruence transitivity
Goal: $n \equiv x \pmod m \land n \equiv y \pmod m \implies x \equiv y \pmod m$

Assume:
1. $n \equiv x \pmod m : (\exists i \in \mathbb{Z})\ (x - n) = i \times m$
2. $n \equiv y \pmod m : (\exists j \in \mathbb{Z})\ (y - n) = j \times m$

New goal: $x \equiv y \pmod{m}$

$$\begin{aligned} & (\exists i,j \in \mathbb{Z})\ x - y = (i-j) \times m \text{ by (1) and (2)} \\ \implies & (\exists k \in \mathbb{Z})\ x - y = k \times m \\ \iff & x \equiv y \pmod{m} \end{aligned}$$

Let $P(z) : 0 \leq z < m \wedge n \equiv z \pmod{m}$

Goal: $(\forall m \in \mathbb{Z}^+, n \in \mathbb{Z})\ (\exists! z)\ P(z)$

New goal: $(\forall m \in \mathbb{Z}^+, n \in \mathbb{Z})\ ((\exists z)\ P(z)) \wedge ((\forall x,y)\ P(x) \wedge P(y) \implies x = y)$

Assume:
1. $m \in \mathbb{Z}^+$
2. $n \in \mathbb{Z}$

New goal: $((\exists z)\ P(z)) \wedge ((\forall x,y)\ P(x) \wedge P(y) \implies x = y)$

Subgoal: $(\exists z)\ P(z)$

Missing
This goal is not proved

Subgoal: $(\forall x,y)\ P(x) \wedge P(y) \implies x = y$

Assume:
3. x, y exists
4. $0 \leq x < m \wedge n \equiv x \pmod{m}$
5. $0 \leq y < m \wedge n \equiv y \pmod{m}$

New goal: $x = y$

Missing
$-1 < i < 1 \implies i = 0$

$$\begin{aligned} & -m < -y \leq 0 \text{ by (4) as (6)} \\ & -m < x - y < m \text{ by (4) and (6) as (7)} \\ & x \equiv y \pmod{m} \text{ by (4), (5) and (L24.1)} \\ \iff & (\exists i \in \mathbb{Z})\ x - y = i \times m \\ \iff & (\exists i \in \mathbb{Z})\ -m < i \times m < m \text{ by (7)} \\ \iff & (\exists i \in \mathbb{Z})\ -1 < i < 1 \text{ by cancellation} \\ \iff & i = 0 \text{ by magic} \\ \implies & x - y = 0 \\ \iff & x = y \end{aligned}$$

Proposition 25: parity of square

Goal: $(\forall n \in \mathbb{Z})\ n^2 \equiv 0 \pmod{4} \vee n^2 \equiv 1 \pmod{4}$

Assume:
1. $n \in \mathbb{Z}$

New goal: $n^2 \equiv 0 \pmod{4} \vee n^2 \equiv 1 \pmod{4}$

$$(\exists! z \in \mathbb{Z})\ 0 \leq z < 2 \wedge n \equiv z \pmod{2} \text{ by (P24)}$$

Missing
$0 \leq z < 2 \implies z = 0 \vee z = 1$

Assume:
2. $z = 0 \vee z = 1$

Case $z = 0$

Subgoal: $n \equiv 0 \pmod 2 \implies$
$n^2 \equiv 0 \pmod 4 \vee n^2 \equiv 1 \pmod 4$

Assume:
3. $n \equiv 0 \pmod 2$

New goal:
$n^2 \equiv 0 \pmod 4 \vee n^2 \equiv 1 \pmod 4$

$$
\begin{aligned}
& (\exists i \in \mathbb{Z})\ n = i \times 2 \\
\implies & (\exists i \in \mathbb{Z})\ n^2 = i^2 \times 4 \\
\implies & (\exists j \in \mathbb{Z})\ n^2 = j \times 4 \\
\implies & n^2 \equiv 0 \pmod 4 \\
\implies & n^2 \equiv 0 \pmod 4 \vee n^2 \equiv 1 \pmod 4
\end{aligned}
$$

Case $z = 1$

Subgoal: $n \equiv 1 \pmod 2 \implies$
$n^2 \equiv 0 \pmod 4 \vee n^2 \equiv 1 \pmod 4$

Assume:
3. $n \equiv 1 \pmod 2$

New goal:
$n^2 \equiv 0 \pmod 4 \vee n^2 \equiv 1 \pmod 4$

$$
\begin{aligned}
& (\exists i \in \mathbb{Z})\ n = i \times 2 + 1 \\
\implies & (\exists i \in \mathbb{Z})\ n^2 = 4 \times (i^2 + i) + 1 \\
\implies & (\exists j \in \mathbb{Z})\ n^2 = 4 \times j + 1 \\
\implies & n^2 \equiv 1 \pmod 4 \\
\implies & n^2 \equiv 0 \pmod 4 \vee n^2 \equiv 1 \pmod 4
\end{aligned}
$$

Lemma 27: congruence at ends of combinations

Definition: combinations
$$\binom{p}{m} = \frac{p!}{(p-m)!m!}$$

Goal: $(\forall p \in \mathbb{Z}^+, m \in \mathbb{N})\ p > 1 \wedge (m = 0 \vee m = p) \implies \binom{p}{m} \equiv 1 \pmod p$

Note
Added condition $p > 1$ for the statement to be correct.

Assume:
1. $p \in \mathbb{Z}^+, m \in \mathbb{N}$
2. $p > 1$

New goal: $m = 0 \vee m = p \implies \binom{p}{m} \equiv 1 \pmod p$

Assume:
3. $m = 0 \vee m = p$

Case $m = 0$

Subgoal: $m = 0 \implies \binom{p}{m} \equiv 1 \pmod p$

Assume:
4. $m = 0$

Case $m = p$

Subgoal: $m = p \implies \binom{p}{m} \equiv 1 \pmod p$

Assume:
4. $m = p$

New goal: $\binom{p}{m} \equiv 1 \pmod{p}$

$$\binom{p}{0} = \frac{p!}{p! \times 1} = 1$$

$$\implies \binom{p}{m} \equiv 1 \pmod{p}$$

New goal: $\binom{p}{m} \equiv 1 \pmod{p}$

$$\binom{p}{p} = \frac{p!}{1 \times p!} = 1$$

$$\implies \binom{p}{m} \equiv 1 \pmod{p}$$

Lemma 28: congruence at non-ends of combinations

Goal: $(\forall p \text{ prime}, m \in \mathbb{Z})\ 0 < m < p \implies \binom{p}{m} \equiv 0 \pmod{p}$

Assume:
1. p prime
2. $m \in \mathbb{Z}$
3. $0 < m < p$

New goal: $\binom{p}{m} \equiv 0 \pmod{p}$

Missing: Euclid's Lemma
p prime $\implies (p|(a \times b) \implies p|a \vee p|b)$

Assume:
4. $\binom{p}{m} \in \mathbb{Z}$ by magic
5. $\neg p|m!$
6. $\neg p|(p-m)!$

$$p \times (p-1)! = \binom{p}{m} \times (p-m)! \times m!$$

$$\implies p|\binom{p}{m} \times (p-m)! \times m!$$

$$\implies p|\binom{p}{m} \text{ by (5), (6) and (Euclid's Lemma)}$$

$$\implies \binom{p}{m} \equiv 0 \pmod{p}$$

Proposition 29: congruence of combinations

Goal: $(\forall p \text{ prime}, m \in \mathbb{Z})\ , 0 \leq m \leq p \implies \binom{p}{m} \equiv 0 \pmod{p} \vee \binom{p}{m} \equiv 1 \pmod{p}$

Assume:
1. p prime, $m \in \mathbb{Z}$
2. $0 \leq m \leq p$

New goal: $\binom{p}{m} \equiv 0 \pmod{p} \vee \binom{p}{m} \equiv 1 \pmod{p}$

Assume:
3. $(m = 0 \vee m = p) \vee 0 < m < p$

Case: $m = 0 \vee m = p$

Case: $0 < m < p$

Subgoal:
$\binom{p}{m} \equiv 0 \pmod p \vee \binom{p}{m} \equiv 1 \pmod p$

$\binom{p}{m} \equiv 1 \pmod p$ by (L27)

$\implies \binom{p}{m} \equiv 0 \pmod p \vee \binom{p}{m} \equiv 1 \pmod p$

Subgoal:
$\binom{p}{m} \equiv 0 \pmod p \vee \binom{p}{m} \equiv 1 \pmod p$

$\binom{p}{m} \equiv 0 \pmod p$ by (L28)

$\implies \binom{p}{m} \equiv 0 \pmod p \vee \binom{p}{m} \equiv 1 \pmod p$

Definition: binomial theorem

$$(\forall p \in \mathbb{N}, m, n)\ (m+n)^p = \sum_{k=0}^{p} \binom{p}{k} m^{p-k} n^k$$

Corollary 33: the freshman's dream

Goal: $(\forall m, n \in \mathbb{N}, p \text{ prime})\ (m+n)^p \equiv m^p + n^p \pmod p$

Assume:
1. $m, n \in \mathbb{N}, p$ prime

New goal: $(m+n)^p \equiv m^p + n^p \pmod p$

Informal

$$(m+n)^p = \binom{p}{0} m^p + \sum_{k=1}^{p-1} m^{p-k} n^k + \binom{p}{p} n^p$$

$$\implies (m^p + n^p) + \sum_{k=1}^{p-1} m^{p-k} n^k \equiv m^p + n^p + 0 \pmod p \text{ by (L28)}$$

$$\implies (m+n)^p \equiv m^p + n^p \pmod p$$

Formal proof requires induction.

Corollary 34: the dropout lemma

Goal: $(\forall m \in \mathbb{N}, p \text{ prime})\ (m+1)^p \equiv m^p + 1 \pmod p$

Assume:
1. $m \in \mathbb{N}, p$ prime

New goal: $(m+1)^p \equiv m^p + 1 \pmod p$

$$(\forall m \in \mathbb{N})\ (m+1)^p \equiv m^p + 1^p \pmod p \text{ by (C33)}$$
$$\implies (\forall m \in \mathbb{N})\ (m+1)^p \equiv m^p + 1 \pmod p$$

Proposition 35: the many dropout lemma

Goal: $(\forall m, i \in \mathbb{N}, p \text{ prime})\ (m+i)^p \equiv m^p + i \pmod p$

Assume:
1. $m, i \in \mathbb{N}, p$ prime

New goal: $(m+i)^p \equiv m^p + i \pmod p$

Informal

$$
\begin{aligned}
&(m+i)^p = \left(m + \underbrace{1+1+1+1}_{i \text{ times}}\right)^p \\
&\implies ((m+i-1)+1)^p \equiv (m+i-1)^p + 1 \pmod p \\
&\implies ((m+i-2)+1)^p \equiv (m+i-2)^p + 1 + 1 \pmod p) \\
&\vdots \implies (m+1)^p \equiv m^p + \underbrace{1+1+1+1}_{i \text{ times}} \pmod p \\
&\implies (m+i)^p \equiv m^p + i \pmod p \text{ by transitivity}
\end{aligned}
$$

The formal proof requires induction.

Theorem 36: Fermat's little theorem, clause 1

Goal: $(\forall i \in \mathbb{N}, p \text{ prime})\ i^p \equiv i \pmod p$

Assume:
1. $i \in \mathbb{N}, p$ prime

New goal: $i^p \equiv i \pmod p$

$$
\begin{aligned}
&(0+i)^p \equiv 0^p + i \pmod p \text{ by (P35)} \\
\implies &i^p \equiv i \pmod p
\end{aligned}
$$

Logical equivalences

$$
\begin{aligned}
\neg(P \implies Q) &\iff P \wedge \neg Q \\
\neg(P \iff Q) &\iff P \iff \neg Q \\
\neg(P \wedge Q) &\iff \neg P \vee \neg Q \\
\neg(P \vee Q) &\iff \neg P \wedge \neg Q \\
\neg((\forall x)\ P(x)) &\iff (\exists x)\ \neg P(x) \\
\neg((\exists x)\ P(x)) &\iff (\forall x)\ P(x) \\
\neg(\neg P) &\iff P \\
\neg P &\iff (P \implies \text{false})
\end{aligned}
$$

Theorem 37: contrapositive forward

Goal: $(P \implies Q) \implies (\neg Q \implies \neg P)$

Assume:
1. P, Q statement
2. $P \implies Q$
3. $\neg Q$
4. P

New goal: false

$$
\begin{aligned}
&Q \text{ by (2) and (4) as (5)} \\
\implies &\text{false by (3) and (5)}
\end{aligned}
$$

Theorem 39: contrapositive reverse

Goal: $(\neg Q \implies \neg P) \implies (P \implies Q)$

Assume:
1. $\neg Q \implies \neg P$
2. P

New goal: Q

Assume:
3. $\neg Q$ by contradiction

New goal: false

$$\begin{aligned} & \neg P \text{ by } (1) \text{ and } (3) \text{ as } (4) \\ \implies & \text{false by } (2) \text{ and } (4) \end{aligned}$$

Corollary 40: contrapositive bi-implication

Goal: $(P \implies Q) \iff (\neg Q \implies \neg P)$

Subgoal: $(P \implies Q) \implies (\neg Q \implies \neg P)$	Subgoal: $(P \implies Q) \impliedby (\neg Q \implies \neg P)$
Exact (T37)	Exact (T38)

Corollary 41: square root irrational

Goal: $(\forall x)$ x irrational $\land\, x > 0 \implies \sqrt{x}$ irrational

Assume:
1. x irrational $\land\, x > 0$

New goal: $\sqrt{x}$ irrational

Assume:
2. $\sqrt{x}$ rational by contradiction

New goal: false

$$\begin{aligned} & (\exists p,q \in \mathbb{Z})\ \sqrt{x} = p/q \\ \implies & (\exists p,q, \in \mathbb{Z})\ x = \left(\sqrt{x}\right)^2 = p^2/q^2 \\ \implies & (\exists p',q' \in \mathbb{Z})\ x = p'/q' \\ \implies & x \text{ rational} \\ \implies & \text{false by } (1) \end{aligned}$$

Lemma 42: rational lowest terms

Lemma 42.1: positive integers cannot be a product of infinitely many primes

Informal

Goal: $(\forall a \in \mathbb{Z}^+, p_1, p_2, ..., p_\infty \text{ prime})\ a \neq p_1 \times p_2 \times \cdots \times p_\infty$

Assume:
1. $a \in Z^+, p_1, p_2, ..., p_\infty$ prime
2. $a = p_1 \times p_2 \times \cdots \times p_\infty$ by contradiction

New goal: false

$$\begin{aligned} & a \geq 2^{\infty} \text{ by 2 is the smallest prime} \\ \implies & a \notin \mathbb{Z}^+ \text{ by all integers are finite} \\ \implies & \text{false by (1)} \end{aligned}$$

Goal: $(\forall x \in \mathbb{R} \wedge x > 0)\ x$ rational $\iff (\exists m,n \in \mathbb{Z}^+)\ x = m/n \wedge (\neg(\exists p \text{ prime})\ p|m \wedge p|n)$

Assume:
1. $x \in \mathbb{R} \wedge x > 0$

New goal: x rational $\iff (\exists m,n \in \mathbb{Z}^+)\ x = m/n \wedge (\neg(\exists p \text{ prime})\ p|m \wedge p|n)$

Subgoal: $(\exists m,n \in \mathbb{Z}^+)\ x = m/n \wedge (\neg(\exists p \text{ prime})\ p|m \wedge p|n) \implies x$ rational

$$\begin{aligned} & (\exists m,n \in \mathbb{Z}^+)\ x = m/n \wedge (\neg(\exists p \text{ prime})\ p|m \wedge p|n) \\ \implies & (\exists m,n \in \mathbb{Z})\ x = m/n \\ \iff & x \text{ rational} \end{aligned}$$

Subgoal: x rational $\implies (\exists m,n \in \mathbb{Z}^+)\ x = m/n \wedge (\neg(\exists p \text{ prime})\ p|m \wedge p|n)$

Assume:
2. x rational : $(\exists a,b \in \mathbb{Z}^+)\ x = a/b$

New goal: $(\exists m,n \in \mathbb{Z}^+)\ x = m/n \wedge (\neg(\exists p \text{ prime})\ p|m \wedge p|n)$

Assume:
3. $(\forall m,n \in \mathbb{Z}^+)\ x \neq m/n \vee (\exists p \text{ prime})\ p|m \wedge p|n$ by contradiction

New goal: false

$$\begin{aligned} & (\forall m,n \in \mathbb{Z}^+)\ x = m/n \implies (\exists p \text{ prime})\ p|m \wedge p|n \text{ by (3)} \\ \implies & (\exists p_0 \text{ prime})\ p_0|a \wedge p_0|b \text{ by (2)} \\ \implies & (\exists a_0,b_0 \in \mathbb{Z}^+)\ a = p_0 \times a_0 \wedge b = p_0 \times b_0 \\ \implies & (\exists p_1 \text{ prime})\ p_1|a_0 \wedge p_1|b_0 \text{ by setting } x = a_0/b_0 \\ \implies & (\exists a_1,b_1 \in \mathbb{Z}^+)\ a = p_0 \times p_1 \times a_1 \wedge b = p_0 \times p_1 \times b_1 \\ & \vdots \end{aligned}$$

false, positive integers cannot be written as a product of infinitely many primes by (L42.1).

Definition: commutative laws
$(A, e, \otimes)$ satisfies commutative laws iff

$$a \otimes b = b \otimes a$$

Definition: monoid laws
$(A, e, \otimes)$ satisfies monoid laws iff

$$0 \otimes e = e = e \otimes 0$$
$$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$

A monoid is commutative if $(A, e, \otimes)$ satisfies the commutative laws.

Definition: semiring

$(A, e_1, \oplus, e_2, \otimes)$ is a semiring if

- $(A, e_1, \oplus)$ is a commutative monoid.
- $(A, e_1, \otimes)$ is a monoid.
- $\otimes$ is distributive over $\oplus : a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

A semiring is commutative if $\otimes$ is.

Definition: cancellation

- $\oplus$ allows cancellation by c on the left if $c \oplus x = c \oplus y \implies x = y$
- $\oplus$ allows cancellation by c on the right if $x \oplus c = y \oplus c \implies x = y$
- $\oplus$ allows cancellation by c if it allows cancellation both on the left and on the right.

Defintion: natural numbers

```
type N =
| zero
| succ of N
```

- $(\mathbb{N}, 0 +, 1, \times)$ is a commutative semiring.
- $+$ supports cancellation by any c
- $\times$ supports cancellation by any $c \neq 0$

Definition: inverse

$$(\exists l \in A)\ l \oplus x = e \implies x \text{ admits an inverse to the left}$$
$$(\exists l \in A)\ x \oplus r = e \implies x \text{ admits an inverse to the right}$$

x admits an inverse to the left and right $\implies x$ admits an inverse

Proposition 47: equality of inverses

Goal: $(\forall (A, e, \oplus)$ monoid$)\ x$ admits an inverse $\implies l = r$

Assume:

1. $(A, e, \oplus)$ monoid
2. $(\exists l)\ l \oplus x = e$
3. $(\exists r)\ x \oplus r = e$

New goal: $l = r$

$$\begin{aligned} l &= l \oplus e \\ &= l \oplus (x \oplus r) \\ &= (l \oplus x) \oplus r \text{ commutative by monoid property} \\ &= e \oplus r \\ &= r \end{aligned}$$

Definition: group
- A group is a monoid which every element has an inverse.
- The group is commutative if the monoid is commutative.

Definition: additive and multiplicative inverses
- x admits an additive inverse if $(\exists y)\ x + y = 0$
- x admits a multiplicative inverse if $(\exists y)\ x \times y = 1$

Definition: ring
A ring $(A, e_1, \oplus, e_2, \otimes)$ is a semiring where $(A, e_1, \oplus)$ is a group.

The ring is commutative if $(A, e_2, \otimes)$ is.

Definition: field
A field $(A, e_1, \oplus, e_2, \otimes)$ is a commutative ring where every element besides 0 has an inverse with respect to $\otimes$.

Definition: integers and rationals
- The integers $\mathbb{Z}$ admits all additive inverses, $(\mathbb{Z}, 0, +, 1, \times)$ is a commutative ring.
- The rationals $\mathbb{Q}$ also admits all multiplicative inverses, $(\mathbb{Q}, 0 +, 1, \times)$ is a field.

Theorem 53: division theorem

Let $P(q, r) : q \geq 0 \land 0 \leq r \leq n \land q \times n = r$

Goal: $(\forall m,n \in \mathbb{N})\ (\exists! q,! r \in \mathbb{Z})\ P(q, r)$

Assume:
1. $m, n \in \mathbb{N}$

New goal: $((\exists q,r \in \mathbb{Z})\ P(q, r)) \land (P(q, r) \land P(q', r') \implies q = q' \land r = r')$

Subgoal: $(\exists q,r \in \mathbb{Z})\ P(q, r)$

Missing
This goal is not proved.

Subgoal: $P(q, r) \land P(q', r') \implies q = q' \land r = r'$

Assume:
2. $q \geq 0 \land 0 \leq r < n \land q \times n + r = m$
3. $q' \geq 0 \land 0 \leq r' < n \land q' \times n + r' = m$

Goal: $q = q' \land r = r'$

$$
\begin{aligned}
& 0 \leq r < n \land -n < r \leq 0 \\
\implies & -n < r - r' < n \text{ by (2) and (3) as (4)} \\
m - r = q \times n \implies & m \equiv r \pmod n \text{ by (2)} \\
m - r' = q' \times n \implies & m \equiv r' \pmod n \text{ by (3)} \\
\implies & r \equiv r' \pmod n \text{ by (L24.1)} \\
\implies & (\exists i \in \mathbb{Z})\ r - r' = i \times n \\
\implies & i = 0 \land \boldsymbol{r = r'} \text{ by (4)} \\
\implies & \boldsymbol{q = q'} \text{ by cancellation}
\end{aligned}
$$

Definition: quo and rem

For the q and r associated by $m = q \times n + r$

- $\text{quo}(m, n) = q$
- $\text{rem}(m, n) = r$

And satisfies the property $m = \text{quo}(m, n) \times n + \text{rem}(m, n)$

Theorem 56: existence of quotient and remainder

```
let divalg m n =
  let diviter q r =
    if r < n then (q, r)
    else diviter (q + 1) (r - n)
  in diviter 0 m
```

Goal: `divalg` terminates $\wedge$ (`divalg` terminates with $(q_0, r_0) \implies r_0 < n \wedge m = q_0 \times n + r_0$)

Subgoal: `divalg` terminates

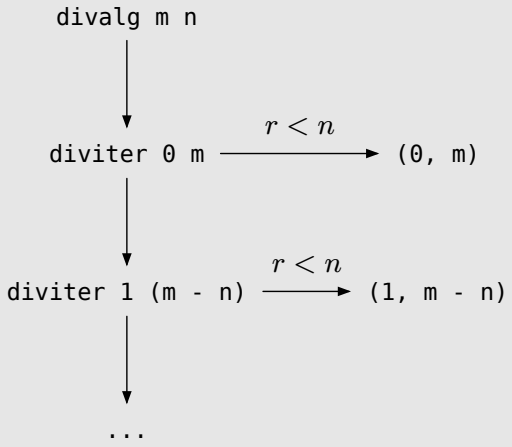
Note

This goal is also called **total correctness**.

Informal

$r \in \mathbb{N}$ decreases in each step, which is bounded below. This cannot continue forever.

It must terminate.



Subgoal:
`divalg` terminates with $(q_0, r_0) \implies r_0 < n \wedge m = q_0 \times n + r_0$

Note

This goal is also called **partial correctness**.

Assume:

1. `divalg` terminates with (q_0, r_0)

New goal: $r_0 < n \wedge m = q_0 \times n + r_0$

Informal

In the nth call of `diviter`

1. `diviter 0 m` : $m = q_0 \times n + r_0$ by trivial, if $r_0 < n$ terminate. Otherwise
2. `diviter q1 r1` $m = q_1 \times n + r_1$ by trivial, if $r_1 < n$ terminate. Otherwise
3. ...

$$m = (q + 1) \times n + (r - n)$$

If the property holds for one point of the computation, then it holds for the next point of computation, this is called **proof by invariant**.

Proposition 57: uniqueness of remainder

Goal: $k \equiv l \pmod m \iff \text{rem}(k, m) = \text{rem}(l, m)$

Subgoal: $\text{rem}(k, m) = \text{rem}(l, m) \implies k \equiv l \pmod m$

Assume:

1. $\text{rem}(k, m) = \text{rem}(l, m)$

New goal: $k \equiv l \pmod m$

$$
\begin{aligned}
&(\exists q,r \in \mathbb{N})\ k = q \times m + r \text{ by } (1)\\
&(\exists q',r' \in \mathbb{N})\ l = q' \times m + r' \text{ by } (1)\\
\implies & k - l = (q - q') \times m + (r - r')\\
\implies & (\exists a \in \mathbb{N})\ k - l = a \times m\\
\implies & k \equiv l \pmod m
\end{aligned}
$$

Subgoal: $k \equiv l \pmod m \implies \text{rem}(k, m) \land \text{rem}(l, m)$

Assume:
1. $k \equiv l \pmod m$

New goal: $\text{rem}(k, m) = \text{rem}(l, m)$

$$
\begin{aligned}
&(\exists a \in \mathbb{Z})\ k - l = a \times m\\
\implies & (\exists q,r,q',r' \in \mathbb{Z})\ (q \times m + r) - (q' \times m + r') = a \times m \text{ by (T56)}\\
\implies & r - r' = (a - q + q') \times m\\
\implies & r - r' = 0 \times m \text{ by bound } (-m) < r - r' < m\\
\implies & r = r'
\end{aligned}
$$

Corollary 58.1: congruence with remainder

Goal: $(\forall m \in \mathbb{N})\ n \equiv \text{rem}(n, m) \pmod m$

Assume:
1. $m \in \mathbb{N}$

New goal: $n \equiv \text{rem}(n, m) \pmod m$

$$
\begin{aligned}
&(\exists a \in \mathbb{Z})\ n = m \times \text{quo}(n, m) + \text{rem}(n, m)\\
\implies & n - \text{rem}(n, m) = m \times \text{quo}(n, m)\\
\implies & n \equiv \text{rem}(n, m) \pmod m
\end{aligned}
$$

Corollary 58.2: existence of modulus integer

Goal: $(\forall k \in \mathbb{Z})\ (\exists! [k]_m)\ 0 \leq [k]_m < m \land k \equiv [k]_m \pmod m$

Subgoal: $(\forall k \in \mathbb{N})\ (\exists! [k]_m)\ 0 \leq [k]_m < m \land k \equiv [k]_m \pmod m$

Assume:
1. $k \in \mathbb{N}$

New goal: $(\exists! [k]_m)\ 0 \leq [k]_m < m \land k \equiv [k]_m \pmod m$

Let $[k]_m = \text{rem}(k, m)$.

Let $P(k) : 0 \leq [k]_m < m \land k \equiv [k]_m \pmod m$

New goal: $P([k]_m) \land (P(a) \land P(b) \implies a = b)$

Subgoal:
$0 \leq [k]_m < m \wedge k \equiv [k]_m \pmod m$

Exact by (T56) and (P57).

Subgoal: $P(a) \wedge P(b) \implies a = b$

Assume:
2. $0 \leq a < m \wedge k \equiv a \pmod m$
3. $0 \leq b < m \wedge k \equiv b \pmod m$

New goal: $a = b$

$-m < a - b < m$ by (2) and (3)
$(\exists i \in \mathbb{Z})\ a - b = i \times m$ by (2) and (3)
$\implies i = 0 \wedge \boldsymbol{a = b}$ by bounds

,

Subgoal: $(\forall k \in \mathbb{Z}^-)\ (\exists![k]_m)\ 0 \leq [k]_m < m \wedge k \equiv [k]_m \pmod m$

Assume:
1. $k \in \mathbb{Z}^-$

New goal: $(\exists![k]_m)\ 0 \leq [k]_m < m \wedge k \equiv [k]_m \pmod m$

$(\exists i \in \mathbb{Z})\ k + i \times m \geq 0$ by magic
$k + i \times m \equiv k \pmod m$ by trivial

Then get the goal by setting $k' = k + i \times m$ using the previous subgoal.

Definition: modular arithmetic
$k +_m l = [k + l]_m = \text{rem}(k + l, m)$
$k \times_m l = [k \times l]_m = \text{rem}(k + l, m)$

Proposition 62: modular arithmetic is a commutative ring

Goal: $(\mathbb{Z}_m, 0, +_m, 1, \times_m)$ is a commutative ring

Subgoal: $(\mathbb{Z}_m, 0, +_m)$ is a commutative monoid

Property	**Proof**
Commutative	$a_m +_m b_m = \text{rem}(a_m + b_m) = b_m +_m a_m$
Neutral element	$0_m +_m a_m = a_m = a_m + 0_m$
Associativity	$(a_m +_m b_m) +_m c_m = \text{rem}(a_m + b_m + c_m) = a_m +_m (b_m +_m c_m)$

Subgoal: $(\mathbb{Z}_m, 1, \times_m)$ is a commutative monoid

Property	**Proof**
Commutative	$a_m \times_m b_m = \text{rem}(a_m \times b_m) = b_m \times_m a_m$
Neutral element	$1_m \times_m a_m = a_m = a_m \times 1_m$
Associativity	$(a_m \times_m b_m) \times_m c_m = \text{rem}(a_m \times b_m \times c_m) = a_m \times_m (b_m \times_m c_m)$

Subgoal: $(\mathbb{Z}_m, 0, +_m, 1, \times_m)$ is a semiring

Apply subgoals (1) and (2).

New goal: distributivity $a_m \times (b_m +_m c_n) = a_m \times_m b_m +_m a_m \times_m c_m$

$$\begin{aligned} a_m \times (b_m +_m c_n) &\equiv a_m \times (b_m +_m c_m) \pmod m \\ &\equiv a_m \times (b_m + c_m) \pmod m \\ &\equiv a_m \times b_m + a_m \times c_m \pmod m \\ &\equiv a_m \times_m b_m +_m a_m \times_m c_m \pmod m \end{aligned}$$

Subgoal: $(\mathbb{Z}_m, 0, +)$ is a group

Apply subgoal (1).

New goal: $(\forall x_m \in \mathbb{Z}_m)\ (\exists y_m \in \mathbb{Z}_m)\ [x_m +_m y_m] = 0$

Assume:
1. $x_m \in \mathbb{Z}_m$

Let $y_m = [-x_m]_m$

New goal: $x_m +_m y_m = 0$

$$\begin{aligned} x_m +_m y_m &\equiv x_m + [-x]_m \pmod m \\ &\equiv x_m + (m - x_m) \pmod m \text{ by trivial } \equiv m \pmod m \\ &\equiv 0 \pmod m \end{aligned}$$

Sugboal: $(\mathbb{Z}_m, 0, +_m, 1, \times_m)$ is a commutative ring.

Exact by subgoal (2) and (4).

Proposition 63: condition for reciprocal

Goal: $(\forall m \in \mathbb{Z}^+, k \in \mathbb{Z}_m)\ ((\exists l \in \mathbb{Z}_m)\ k \times_m l = 1) \iff ((\exists i,j \in \mathbb{Z})\ k \times i + m \times j = 1)$

Assume:
1. $m \in \mathbb{Z}^+, k \in \mathbb{Z}_m$

Subgoal: $((\exists l \in \mathbb{Z}_m)\ k \times_m l = 1) \implies ((\exists i,j \in \mathbb{Z})\ k \times i + m \times j = 1)$

Assume:
2. $(\exists l \in \mathbb{Z}_m)\ k \times_m l = 1$

New goal: $(\exists i,j \in \mathbb{Z})\ k \times i + m \times j = 1$

Let $i = k$

New goal: $(\exists j \in \mathbb{Z})\ k \times l + m \times j = 1$

$$\begin{aligned} &(\exists l \in \mathbb{Z}_m)\ k \times l \equiv 1 \pmod m \\ \implies &(\exists a \in \mathbb{Z})\ k \times l - 1 = m \times a \\ \implies &(\exists j \in \mathbb{Z})\ k \times l + m \times j = 1 \end{aligned}$$

Subgoal: $((\exists i,j \in \mathbb{Z})\ k \times i + m \times j = 1) \implies ((\exists l \in \mathbb{Z}_m)\ k \times_m l = 1)$

Assume:
2. $(\exists i,j \in \mathbb{Z})\ k \times i + m \times j = 1$

New goal: $(\exists l \in \mathbb{Z}_m)\ k \times_m l = 1$

$$(\exists i,j \in \mathbb{Z})\ k \times i - 1 = m \times (-j)$$
$$(\exists i,a \in \mathbb{Z})\ k \times i - 1 = m \times a$$
$$k \times i \equiv 1 \pmod{m}$$
$$k \times_m [i]_m = 1$$

Definition: linear combination
$r \in \mathbb{Z}$ is a linear combination of $m, n \in \mathbb{Z}$ if

$$(\exists s,t \in \mathbb{Z})\ s \times m + t \times n = r$$

Proposition 65: condition for reciprocal (linar combination)

Goal: $(\forall k \in \mathbb{Z}_m)\ ((\exists l \in \mathbb{Z}_m)\ k \times_m l = 1) \iff 1$ is a linear combination of m and k

Exact by (P63).

Definition: sets
- $x \in A$ if x is an element in A
- $A = B \iff ((\forall x)\ x \in A \iff x \in B)$
- $\{x \in A \mid P(x)\} = \{x \in A \mid Q(x)\} \iff ((\forall x \in A)\ P(x) \iff Q(x))$

Definition: common divisiors

$$D(m) = \{d : d|m\}$$
$$\mathrm{CD}(m,n) = \{d : d|m \wedge d|n\}$$

Lemma 71: key lemma

Goal: $(\forall m,m' \in \mathbb{N}, n \in \mathbb{Z}^+)\ m \equiv m' \pmod{n} \implies \mathrm{CD}(m,n) = \mathrm{CD}(m',n)$

Assume:
1. $m, m' \in \mathbb{N}, n \in \mathbb{Z}^+$
2. $m \equiv m' \pmod{n}$

New goal: $d|m \wedge d|n \iff d|m' \wedge d|n$ (equality of predicates)

$$(\exists k \in \mathbb{Z})\ m' = m + k \times n \text{ by } (1)$$

Subgoal: $d|m \wedge d|n \implies d|m' \wedge d|n$

Assume:
3. $d|m \wedge d|n$

New goal: $d|m' \wedge d|n$

Subgoal: $d|m' \wedge d|n \implies d|m \wedge d|n$

Note
This is true by symmetry of the first subgoal.

$$
\begin{aligned}
& d|n \text{ by } (3) \\
& (\exists i \in \mathbb{Z})\ m - m' = i \times n \text{ by } (2) \\
\implies & m' = i \times n - m \\
\implies & (\exists j,k \in \mathbb{Z})\ m' = d \times j \times i - d \times k \\
\implies & (\exists l \in \mathbb{Z})\ m' = d \times l \\
\implies & d|m'
\end{aligned}
$$