

Definition: odd number

$n \in \mathbb{N}$ is odd if $(\exists i \in \mathbb{N}) n = 2i + 1$.

Proposition 8: product of odd integers is odd

Goal: $(\forall m, n \in \mathbb{N}) m \text{ and } n \text{ odd} \implies m \times n \text{ odd}$

Proof

Assume:

1. $m, n \in \mathbb{N}$
2. m and n odd

New goal: $m \times n$ odd

$$\begin{aligned}
 & (\exists i, j \in \mathbb{N}) m = 2i + 1 \wedge n = 2j + 1 \\
 \implies & (\exists i, j \in \mathbb{N}) m \times n = (2i + 1) \times (2j + 1) \\
 \implies & (\exists i, j \in \mathbb{N}) m \times n = 2(2ij + i + j) + 1 \\
 \implies & (\exists k \in \mathbb{N}) m \times n = 2k + 1 \\
 \implies & m \times n \text{ odd}
 \end{aligned}$$

Definition: real numbers

$(\forall x \in \mathbb{R})$

- $(\exists m, n \in \mathbb{Z}) x = m/n \iff x \text{ rational}$
- $\neg(x \text{ rational}) \iff x \text{ irrational}$
- $x > 0 \iff x \text{ positive}$
- $x < 0 \iff x \text{ negative}$
- $\neg(x \text{ positive}) \iff x \text{ nonpositive}$
- $\neg(x \text{ negative}) \iff x \text{ nonnegative}$
- $x \text{ nonnegative} \wedge x \in \mathbb{Z} \iff x \in \mathbb{N}$

Proposition 10: rational square root

Goal: $(\forall x \text{ positive}) \sqrt{x} \text{ rational} \implies x \text{ rational}$

Proof

Assume:

1. x positive
2. \sqrt{x} rational

New goal: x rational

$$\begin{aligned}
 & (\exists p, q \in \mathbb{Z}) \sqrt{x} = p/q \\
 \implies & (\exists p, q \in \mathbb{Z}) x = (\sqrt{x})^2 = p^2/q^2 \\
 \implies & (\exists p', q' \in \mathbb{Z}) x = p'/q' \\
 \iff & x \text{ rational}
 \end{aligned}$$

Definition: modus ponens

$P \wedge (P \implies Q) \implies Q$

Theorem 11: implication transitivity

($\forall P_1, P_2, P_3$ statement) $(P_1 \implies P_2 \wedge P_2 \implies P_3) \implies (P_1 \implies P_3)$

Proof

Assume:

1. $P_1 \implies P_2$
2. $P_2 \implies P_3$
3. P_1

New goal: P_3

$$\begin{aligned} &P_2 \text{ as (4) by (1) and (3)} \\ &\implies P_3 \text{ by (2) and (4)} \end{aligned}$$

Definition: bi-implication

$$(P \iff Q) \iff (P \implies Q \wedge P \impliedby Q)$$

Definition: divisibility

$$d|n \iff (\exists k \in \mathbb{Z}) n = k \times d$$

Definition: congruence

$$(\forall m \in \mathbb{Z}^+, a, b \in \mathbb{Z}) a \equiv b \pmod{m} \iff m|(a - b)$$

Proposition 16: parity as congruence

Goal: $(n \text{ even} \iff n \equiv 0 \pmod{2}) \wedge (n \text{ odd} \iff n \equiv 1 \pmod{2})$

Subgoal: $n \text{ even} \iff n \equiv 0 \pmod{2}$

Assume:

1. $n \text{ even}$

New goal: $n \equiv 0 \pmod{2}$

$$\begin{aligned} n \text{ even} &\iff (\exists k \in \mathbb{Z}) n = 2 \times k \\ &\iff (\exists k \in \mathbb{Z}) (n - 0) = 2 \times k \\ &\iff n \equiv 0 \pmod{2} \end{aligned}$$

Subgoal: $n \text{ odd} \iff n \equiv 1 \pmod{2}$

Assume:

1. $n \text{ odd}$

New goal: $n \equiv 1 \pmod{2}$

$$\begin{aligned} n \text{ odd} &\iff (\exists k \in \mathbb{Z}) n = 2 \times k + 1 \\ &\iff (\exists k \in \mathbb{Z}) (n - 1) = 2 \times k \\ &\iff n \equiv 1 \pmod{2} \end{aligned}$$

Proposition 18: linearity of congruence

Goal: $(\forall m \in \mathbb{Z}^+, a, b \in \mathbb{Z}) a \equiv b \pmod{m} \iff ((\forall n \in \mathbb{Z}^+) n \times a \equiv n \times b \pmod{n \times m})$

Assume:

1. $m \in \mathbb{Z}^+$
2. $a, b \in \mathbb{Z}$

Subgoal: $a \equiv b \pmod{m} \implies (\forall n \in \mathbb{Z}^+) n \times a \equiv n \times b \pmod{n \times m}$

Assume:

3. $a \equiv b \pmod{m}$
4. $n \in \mathbb{Z}^+$

New goal: $n \times a \equiv n \times b \pmod{n \times m}$

$$\begin{aligned} & (\exists i \in \mathbb{Z}) \ a - b = m \times i \text{ by (3)} \\ \implies & (\exists i \in \mathbb{Z}) \ n \times a - n \times b = (n \times m) \times i \\ \implies & (\exists i \in \mathbb{Z}) \ n \times a \equiv n \times b \pmod{n \times m} \\ \implies & n \times a \equiv n \times b \pmod{n \times m} \end{aligned}$$

Subgoal: $(\forall n \in \mathbb{Z}^+) \ n \times a \equiv n \times b \pmod{n \times m} \implies a \equiv b \pmod{m}$

Assume:

3. $(\forall n \in \mathbb{Z}^+) \ n \times a \equiv n \times b \pmod{n \times m}$

New goal: $a \equiv b \pmod{m}$

$$\begin{aligned} & 1 \times a \equiv 1 \times b \pmod{1 \times m} \text{ by (3)} \\ \implies & a \equiv b \pmod{m} \end{aligned}$$

Definition

- $(\forall x) \ x = x$
- $(\forall x, y) \ x = y \implies (P(x) \implies P(y))$
- $(\forall a, b, c) \ (a = b \wedge b = c) \implies a = c$
- $(\forall a, b, x, y) \ (a = b \wedge x = y) \implies (a + x = b + x = b + y)$

Theorem 19: divisibility of prime products

Goal: $(\forall n \in \mathbb{Z}) \ 6|n \iff 3|n \wedge 2|n$

Assume:

1. $n \in \mathbb{Z}$

New goal: $6|n \iff 3|n \wedge 2|n$

Subgoal: $6|n \implies 3|n \wedge 2|n$

Assume:

2. $6|n$

New goal: $3|n \wedge 2|n$

Subgoal: $3|n$

$$\begin{aligned} 6|n & \iff (\exists i \in \mathbb{Z}) \ n = 6 \times i \\ & \implies (\exists j \in \mathbb{Z}) \ n = 3 \times j \\ & \iff 3|n \end{aligned}$$

Subgoal: $2|n$

$$\begin{aligned} 6|n & \iff (\exists i \in \mathbb{Z}) \ n = 6 \times i \\ & \implies (\exists j \in \mathbb{Z}) \ n = 2 \times j \\ & \iff 2|n \end{aligned}$$

Subgoal: $3|n \wedge 2|n \implies 6|n$

Assume:

2. $2|n \wedge 3|n$

New goal: $6|n$

$$(\exists i \in \mathbb{Z}) \ n = 2 \times i$$

$$\implies (\exists i \in \mathbb{Z}) \ 3 \times n = 6 \times i \text{ as (3)}$$

$$(\exists j \in \mathbb{Z}) \ n = 3 \times j$$

$$\implies (\exists j \in \mathbb{Z}) \ 2 \times n = 6 \times j \text{ as (4)}$$

$$\implies (\exists i, j \in \mathbb{Z}) \ n = 6 \times (i - j) \text{ by (3) and (4)}$$

$$\implies (\exists k \in \mathbb{Z}) \ n = 6 \times k$$

$$\implies 6|n$$

Proposition 21: difference of squares

Goal: $(\forall k \in \mathbb{Z}^+) (\exists i, j \in \mathbb{N}) \ 4 \times k = i^2 - j^2$

Assume:

1. $k \in \mathbb{Z}^+$

Let $i = k + 1, j = k - 1$

$$\begin{aligned} i^2 - j^2 &= (k + 1)^2 - (k - 1)^2 \\ &= 4 \times k \end{aligned}$$

Theorem 23: divisibility transitivity

Goal: $(\forall l, m, n \in \mathbb{Z}) \ l|m \wedge m|n \implies l|n$

Assume:

1. $l, m, n \in \mathbb{Z}$
2. $l|m \wedge m|n$

New goal: $l|n$

$$(\exists i \in \mathbb{Z}) \ m = i \times l$$

$$(\exists j \in \mathbb{Z}) \ n = j \times m$$

$$\implies (\exists i, j \in \mathbb{Z}) \ n = (j \times i) \times l$$

$$\implies (\exists k \in \mathbb{Z}) \ n = k \times l$$

$$\implies l|n$$

Definition

$$((\exists! x) P(x)) \iff ((\exists x) P(x) \wedge ((\forall y, z) P(y) \wedge P(z) \implies y = z))$$

Proposition 24: unique existence of proper congruence

Lemma 24.1: congruence transitivity

Goal: $n \equiv x \pmod{m} \wedge n \equiv y \pmod{m} \implies x \equiv y \pmod{m}$

Assume:

1. $n \equiv x \pmod{m} : (\exists i \in \mathbb{Z}) (x - n) = i \times m$
2. $n \equiv y \pmod{m} : (\exists j \in \mathbb{Z}) (y - n) = j \times m$

New goal: $x \equiv y \pmod{m}$

$$(\exists i, j \in \mathbb{Z}) \ x - y = (i - j) \times m \text{ by (1) and (2)}$$

$$\implies (\exists k \in \mathbb{Z}) \ x - y = k \times m$$

$$\iff x \equiv y \pmod{m}$$

Let $P(z) : 0 \leq z < m \wedge n \equiv z \pmod{m}$

Goal: $(\forall m \in \mathbb{Z}^+, n \in \mathbb{Z}) \ (\exists! z) \ P(z)$

New goal: $(\forall m \in \mathbb{Z}^+, n \in \mathbb{Z}) \ ((\exists z) \ P(z)) \wedge ((\forall x, y) \ P(x) \wedge P(y) \implies x = y)$

Assume:

1. $m \in \mathbb{Z}^+$

2. $n \in \mathbb{Z}$

New goal: $((\exists z) \ P(z)) \wedge ((\forall x, y) \ P(x) \wedge P(y) \implies x = y)$

Subgoal: $(\exists z) \ P(z)$

Missing

This goal is not proved

Subgoal: $(\forall x, y) \ P(x) \wedge P(y) \implies x = y$

Assume:

3. x, y exists

4. $0 \leq x < m \wedge n \equiv x \pmod{m}$

5. $0 \leq y < m \wedge n \equiv y \pmod{m}$

New goal: $x = y$

Missing

$$-1 < i < 1 \implies i = 0$$

$$-m < -y \leq 0 \text{ by (4) as (6)}$$

$$-m < x - y < m \text{ by (4) and (6) as (7)}$$

$$x \equiv y \pmod{m} \text{ by (4), (5) and (L24.1)}$$

$$\iff (\exists i \in \mathbb{Z}) \ x - y = i \times m$$

$$\iff (\exists i \in \mathbb{Z}) \ -m < i \times m < m \text{ by (7)}$$

$$\iff (\exists i \in \mathbb{Z}) \ -1 < i < 1 \text{ by cancellation}$$

$$\iff i = 0 \text{ by magic}$$

$$\implies x - y = 0$$

$$\iff x = y$$

Proposition 25: parity of square

Goal: $(\forall n \in \mathbb{Z}) \ n^2 \equiv 0 \pmod{4} \vee n^2 \equiv 1 \pmod{4}$

Assume:

1. $n \in \mathbb{Z}$

New goal: $n^2 \equiv 0 \pmod{4} \vee n^2 \equiv 1 \pmod{4}$

$$(\exists! z \in \mathbb{Z}) \ 0 \leq z < 2 \wedge n \equiv z \pmod{2} \text{ by (P24)}$$

Missing

$$0 \leq z < 2 \implies z = 0 \vee z = 1$$

Assume:

$$2. \ z = 0 \vee z = 1$$

Case $z = 0$

Subgoal: $n \equiv 0 \pmod{2} \implies$
 $n^2 \equiv 0 \pmod{4} \vee n^2 \equiv 1 \pmod{4}$

Assume:

$$3. \ n \equiv 0 \pmod{2}$$

New goal:

$$n^2 \equiv 0 \pmod{4} \vee n^2 \equiv 1 \pmod{4}$$

$$(\exists i \in \mathbb{Z}) \ n = i \times 2$$

$$\implies (\exists i \in \mathbb{Z}) \ n^2 = i^2 \times 4$$

$$\implies (\exists j \in \mathbb{Z}) \ n^2 = j \times 4$$

$$\implies n^2 \equiv 0 \pmod{4}$$

$$\implies n^2 \equiv 0 \pmod{4} \vee n^2 \equiv 1 \pmod{4}$$

Case $z = 1$

Subgoal: $n \equiv 1 \pmod{2} \implies$
 $n^2 \equiv 0 \pmod{4} \vee n^2 \equiv 1 \pmod{4}$

Assume:

$$3. \ n \equiv 1 \pmod{2}$$

New goal:

$$n^2 \equiv 0 \pmod{4} \vee n^2 \equiv 1 \pmod{4}$$

$$(\exists i \in \mathbb{Z}) \ n = i \times 2 + 1$$

$$\implies (\exists i \in \mathbb{Z}) \ n^2 = 4 \times (i^2 + i) + 1$$

$$\implies (\exists j \in \mathbb{Z}) \ n^2 = 4 \times j + 1$$

$$\implies n^2 \equiv 1 \pmod{4}$$

$$\implies n^2 \equiv 0 \pmod{4} \vee n^2 \equiv 1 \pmod{4}$$

Lemma 27: congruence at ends of combinations**Definition: combinations**

$$\binom{p}{m} = \frac{p!}{(p-m)!m!}$$

$$\text{Goal: } (\forall p \in \mathbb{Z}^+, m \in \mathbb{N}) \ p > 1 \wedge (m = 0 \vee m = p) \implies \binom{p}{m} \equiv 1 \pmod{p}$$

Note

Added condition $p > 1$ for the statement to be correct.

Assume:

$$1. \ p \in \mathbb{Z}^+, m \in \mathbb{N}$$

$$2. \ p > 1$$

$$\text{New goal: } m = 0 \vee m = p \implies \binom{p}{m} \equiv 1 \pmod{p}$$

Assume:

$$3. \ m = 0 \vee m = p$$

Case $m = 0$

Subgoal: $m = 0 \implies \binom{p}{m} \equiv 1 \pmod{p}$

Assume:

$$4. \ m = 0$$

Case $m = p$

Subgoal: $m = p \implies \binom{p}{m} \equiv 1 \pmod{p}$

Assume:

$$4. \ m = p$$

New goal: $\binom{p}{m} \equiv 1 \pmod{p}$

$$\binom{p}{0} = \frac{p!}{p! \times 1} = 1$$

$$\Rightarrow \binom{p}{m} \equiv 1 \pmod{p}$$

New goal: $\binom{p}{m} \equiv 1 \pmod{p}$

$$\binom{p}{p} = \frac{p!}{1 \times p!} = 1$$

$$\Rightarrow \binom{p}{m} \equiv 1 \pmod{p}$$

Lemma 28: congruence at non-ends of combinations

Goal: $(\forall p \text{ prime}, m \in \mathbb{Z}) \ 0 < m < p \Rightarrow \binom{p}{m} \equiv 0 \pmod{p}$

Assume:

1. p prime
2. $m \in \mathbb{Z}$
3. $0 < m < p$

New goal: $\binom{p}{m} \equiv 0 \pmod{p}$

Missing: Euclid's Lemma

$p \text{ prime} \Rightarrow (p|(a \times b) \Rightarrow p|a \vee p|b)$

Assume:

4. $\binom{p}{m} \in \mathbb{Z}$ by magic
5. $\neg p|m!$
6. $\neg p|(p-m)!$

$$p \times (p-1)! = \binom{p}{m} \times (p-m)! \times m!$$

$$\Rightarrow p|\binom{p}{m} \times (p-m)! \times m!$$

$$\Rightarrow p|\binom{p}{m} \text{ by (5), (6) and (Euclid's Lemma)}$$

$$\Rightarrow \binom{p}{m} \equiv 0 \pmod{p}$$

Proposition 29: congruence of combinations

Goal: $(\forall p \text{ prime}, m \in \mathbb{Z}) \ , 0 \leq m \leq p \Rightarrow \binom{p}{m} \equiv 0 \pmod{p} \vee \binom{p}{m} \equiv 1 \pmod{p}$

Assume:

1. p prime, $m \in \mathbb{Z}$
2. $0 \leq m \leq p$

New goal: $\binom{p}{m} \equiv 0 \pmod{p} \vee \binom{p}{m} \equiv 1 \pmod{p}$

Assume:

3. $(m = 0 \vee m = p) \vee 0 < m < p$

Case: $m = 0 \vee m = p$

Case: $0 < m < p$

Subgoal:

$$\binom{p}{m} \equiv 0 \pmod{p} \vee \binom{p}{m} \equiv 1 \pmod{p}$$

$$\binom{p}{m} \equiv 1 \pmod{p} \text{ by (L27)}$$

$$\Rightarrow \binom{p}{m} \equiv 0 \pmod{p} \vee \binom{p}{m} \equiv 1 \pmod{p}$$

Subgoal:

$$\binom{p}{m} \equiv 0 \pmod{p} \vee \binom{p}{m} \equiv 1 \pmod{p}$$

$$\binom{p}{m} \equiv 0 \pmod{p} \text{ by (L28)}$$

$$\Rightarrow \binom{p}{m} \equiv 0 \pmod{p} \vee \binom{p}{m} \equiv 1 \pmod{p}$$

Definition: binomial theorem

$$(\forall p \in \mathbb{N}, m, n) (m + n)^p = \sum_{k=0}^p \binom{p}{k} m^{p-k} n^k$$

Corollary 33: the freshman's dream

$$\text{Goal: } (\forall m, n \in \mathbb{N}, p \text{ prime}) (m + n)^p \equiv m^p + n^p \pmod{p}$$

Assume:

1. $m, n \in \mathbb{N}, p \text{ prime}$

$$\text{New goal: } (m + n)^p \equiv m^p + n^p \pmod{p}$$

Informal

$$(m + n)^p = \binom{p}{0} m^p + \sum_{k=1}^{p-1} m^{p-k} n^k + \binom{p}{p} n^p$$

$$\Rightarrow (m^p + n^p) + \sum_{k=1}^{p-1} m^{p-k} n^k \equiv m^p + n^p + 0 \pmod{p} \text{ by (L28)}$$

$$\Rightarrow (m + n)^p \equiv m^p + n^p \pmod{p}$$

Formal proof requires induction.

Corollary 34: the dropout lemma

$$\text{Goal: } (\forall m \in \mathbb{N}, p \text{ prime}) (m + 1)^p \equiv m^p + 1 \pmod{p}$$

Assume:

1. $m \in \mathbb{N}, p \text{ prime}$

$$\text{New goal: } (m + 1)^p \equiv m^p + 1 \pmod{p}$$

$$(\forall m \in \mathbb{N}) (m + 1)^p \equiv m^p + 1^p \pmod{p} \text{ by (C33)}$$

$$\Rightarrow (\forall m \in \mathbb{N}) (m + 1)^p \equiv m^p + 1 \pmod{p}$$

Proposition 35: the many dropout lemma

$$\text{Goal: } (\forall m, i \in \mathbb{N}, p \text{ prime}) (m + i)^p \equiv m^p + i \pmod{p}$$

Assume:

1. $m, i \in \mathbb{N}, p \text{ prime}$

$$\text{New goal: } (m + i)^p \equiv m^p + i \pmod{p}$$

Informal

$$\begin{aligned}
(m+i)^p &= \left(m + \underbrace{1+1+1+1}_{i \text{ times}} \right)^p \\
&\Rightarrow ((m+i-1)+1)^p \equiv (m+i-1)^p + 1 \pmod{p} \\
&\Rightarrow ((m+i-2)+1)^p \equiv (m+i-2)^p + 1 + 1 \pmod{p} \\
&\vdots \Rightarrow (m+1)^p \equiv m^p + \underbrace{1+1+1+1}_{i \text{ times}} \pmod{p} \\
&\Rightarrow (m+i)^p \equiv m^p + i \pmod{p} \text{ by transitivity}
\end{aligned}$$

The formal proof requires induction.

Theorem 36: Fermat's little theorem, clause 1

Goal: $(\forall i \in \mathbb{N}, p \text{ prime}) \ i^p \equiv i \pmod{p}$

Assume:

1. $i \in \mathbb{N}, p \text{ prime}$

New goal: $i^p \equiv i \pmod{p}$

$$\begin{aligned}
(0+i)^p &\equiv 0^p + i \pmod{p} \text{ by (P35)} \\
&\Rightarrow i^p \equiv i \pmod{p}
\end{aligned}$$

Logical equivalences

$$\begin{aligned}
\neg(P \Rightarrow Q) &\Leftrightarrow P \wedge \neg Q \\
\neg(P \Leftrightarrow Q) &\Leftrightarrow P \Leftrightarrow \neg Q \\
\neg(P \wedge Q) &\Leftrightarrow \neg P \vee \neg Q \\
\neg(P \vee Q) &\Leftrightarrow \neg P \wedge \neg Q \\
\neg((\forall x) P(x)) &\Leftrightarrow (\exists x) \neg P(x) \\
\neg((\exists x) P(x)) &\Leftrightarrow (\forall x) \neg P(x) \\
\neg(\neg P) &\Leftrightarrow P \\
\neg P &\Leftrightarrow (P \Rightarrow \text{false})
\end{aligned}$$

Theorem 37: contrapositive forward

Goal: $(P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P)$

Assume:

1. P, Q statement
2. $P \Rightarrow Q$
3. $\neg Q$
4. P

New goal: false

$$\begin{aligned}
&Q \text{ by (2) and (4) as (5)} \\
&\Rightarrow \text{false by (3) and (5)}
\end{aligned}$$

Theorem 39: contrapositive reverseGoal: $(\neg Q \Rightarrow \neg P) \Rightarrow (P \Rightarrow Q)$

Assume:

1. $\neg Q \Rightarrow \neg P$
2. P

New goal: Q

Assume:

3. $\neg Q$ by contradiction

New goal: false

$$\begin{aligned} & \neg P \text{ by (1) and (3) as (4)} \\ & \Rightarrow \text{false by (2) and (4)} \end{aligned}$$

Corollary 40: contrapositive bi-implicationGoal: $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$

Subgoal: $(P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P)$
 Exact (T37)

Subgoal: $(P \Rightarrow Q) \Leftarrow (\neg Q \Rightarrow \neg P)$
 Exact (T38)

Corollary 41: square root irrationalGoal: $(\forall x) x \text{ irrational} \wedge x > 0 \Rightarrow \sqrt{x} \text{ irrational}$

Assume:

1. $x \text{ irrational} \wedge x > 0$

New goal: \sqrt{x} irrational

Assume:

2. \sqrt{x} rational by contradiction

New goal: false

$$\begin{aligned} & (\exists p, q \in \mathbb{Z}) \sqrt{x} = p/q \\ & \Rightarrow (\exists p, q \in \mathbb{Z}) x = (\sqrt{x})^2 = p^2/q^2 \\ & \Rightarrow (\exists p', q' \in \mathbb{Z}) x = p'/q' \\ & \Rightarrow x \text{ rational} \\ & \Rightarrow \text{false by (1)} \end{aligned}$$

Lemma 42: rational lowest terms**Lemma 42.1: positive integers cannot be a product of infinitely many primes****Informal**Goal: $(\forall a \in \mathbb{Z}^+, p_1, p_2, \dots, p_\infty \text{ prime}) a \neq p_1 \times p_2 \times \dots \times p_\infty$

Assume:

1. $a \in \mathbb{Z}^+, p_1, p_2, \dots, p_\infty \text{ prime}$
2. $a = p_1 \times p_2 \times \dots \times p_\infty$ by contradiction

New goal: false

$$\begin{aligned} & a \geq 2^\infty \text{ by } 2 \text{ is the smallest prime} \\ \Rightarrow & a \notin \mathbb{Z}^+ \text{ by all integers are finite} \\ \Rightarrow & \text{false by (1)} \end{aligned}$$

Goal: $(\forall x \in \mathbb{R} \wedge x > 0) \ x \text{ rational} \Leftrightarrow (\exists m, n \in \mathbb{Z}^+) \ x = m/n \wedge (\neg(\exists p \text{ prime}) \ p|m \wedge p|n)$

Assume:

1. $x \in \mathbb{R} \wedge x > 0$

New goal: $x \text{ rational} \Leftrightarrow (\exists m, n \in \mathbb{Z}^+) \ x = m/n \wedge (\neg(\exists p \text{ prime}) \ p|m \wedge p|n)$

Subgoal: $(\exists m, n \in \mathbb{Z}^+) \ x = m/n \wedge (\neg(\exists p \text{ prime}) \ p|m \wedge p|n) \Rightarrow x \text{ rational}$

$$\begin{aligned} & (\exists m, n \in \mathbb{Z}^+) \ x = m/n \wedge (\neg(\exists p \text{ prime}) \ p|m \wedge p|n) \\ \Rightarrow & (\exists m, n \in \mathbb{Z}) \ x = m/n \\ \Leftrightarrow & x \text{ rational} \end{aligned}$$

Subgoal: $x \text{ rational} \Rightarrow (\exists m, n \in \mathbb{Z}^+) \ x = m/n \wedge (\neg(\exists p \text{ prime}) \ p|m \wedge p|n)$

Assume:

2. $x \text{ rational} : (\exists a, b \in \mathbb{Z}^+) \ x = a/b$

New goal: $(\exists m, n \in \mathbb{Z}^+) \ x = m/n \wedge (\neg(\exists p \text{ prime}) \ p|m \wedge p|n)$

Assume:

3. $(\forall m, n \in \mathbb{Z}^+) \ x \neq m/n \vee (\exists p \text{ prime}) \ p|m \wedge p|n$ by contradiction

New goal: false

$$\begin{aligned} & (\forall m, n \in \mathbb{Z}^+) \ x = m/n \Rightarrow (\exists p \text{ prime}) \ p|m \wedge p|n \text{ by (3)} \\ \Rightarrow & (\exists p_0 \text{ prime}) \ p_0|a \wedge p_0|b \text{ by (2)} \\ \Rightarrow & (\exists a_0, b_0 \in \mathbb{Z}^+) \ a = p_0 \times a_0 \wedge b = p_0 \times b_0 \\ \Rightarrow & (\exists p_1 \text{ prime}) \ p_1|a_0 \wedge p_1|b_0 \text{ by setting } x = a_0/b_0 \\ \Rightarrow & (\exists a_1, b_1 \in \mathbb{Z}^+) \ a = p_0 \times p_1 \times a_1 \wedge b = p_0 \times p_1 \times b_1 \\ & \vdots \end{aligned}$$

false, positive integers cannot be written as a product of infinitely many primes by (L42.1).

Definition: commutative laws

(A, e, \otimes) satisfies commutative laws iff

$$a \otimes b = b \otimes a$$

Definition: monoid laws

(A, e, \otimes) satisfies monoid laws iff

$$0 \otimes e = e = e \otimes 0$$

$$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$

A monoid is commutative if (A, e, \otimes) satisfies the commutative laws.

Definition: semiring

$(A, e_1, \oplus, e_2, \otimes)$ is a semiring if

- (A, e_1, \oplus) is a commutative monoid.
- (A, e_2, \otimes) is a monoid.
- \otimes is distributive over \oplus : $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

A semiring is commutative if \otimes is.

Definition: cancellation

- \oplus allows cancellation by c on the left if $c \oplus x = c \oplus y \implies x = y$
- \oplus allows cancellation by c on the right if $x \oplus c = y \oplus c \implies x = y$
- \oplus allows cancellation by c if it allows cancellation both on the left and on the right.

Definition: natural numbers

```
type N =
| zero
| succ of N
```

- $(\mathbb{N}, 0, +, 1, \times)$ is a commutative semiring.
- $+$ supports cancellation by any c
- \times supports cancellation by any $c \neq 0$

Definition: inverse

$(\exists l \in A) l \oplus x = e \implies x$ admits an inverse to the left

$(\exists r \in A) x \oplus r = e \implies x$ admits an inverse to the right

x admits an inverse to the left and right $\implies x$ admits an inverse

Proposition 47: equality of inverses

Goal: $(\forall (A, e, \oplus) \text{ monoid}) x \text{ admits an inverse} \implies l = r$

Assume:

1. (A, e, \oplus) monoid
2. $(\exists l) l \oplus x = e$
3. $(\exists r) x \oplus r = e$

New goal: $l = r$

$$\begin{aligned}
 l &= l \oplus e \\
 &= l \oplus (x \oplus r) \\
 &= (l \oplus x) \oplus r \text{ commutative by monoid property} \\
 &= e \oplus r \\
 &= r
 \end{aligned}$$

Definition: group

- A group is a monoid which every element has an inverse.
- The group is commutative if the monoid is commutative.

Definition: additive and multiplicative inverses

- x admits an additive inverse if $(\exists y) x + y = 0$
- x admits a multiplicative inverse if $(\exists y) x \times y = 1$

Definition: ring

A ring $(A, e_1, \oplus, e_2, \otimes)$ is a semiring where (A, e_1, \oplus) is a group.

The ring is commutative if (A, e_2, \otimes) is.

Definition: field

A field $(A, e_1, \oplus, e_2, \otimes)$ is a commutative ring where every element besides 0 has an inverse with respect to \otimes .

Definition: integers and rationals

- The integers \mathbb{Z} admits all additive inverses, $(\mathbb{Z}, 0, +, 1, \times)$ is a commutative ring.
- The rationals \mathbb{Q} also admits all multiplicative inverses, $(\mathbb{Q}, 0, +, 1, \times)$ is a field.

Theorem 53: division theorem

Let $P(q, r) : q \geq 0 \wedge 0 \leq r < n \wedge q \times n + r = m$

Goal: $(\forall m, n \in \mathbb{N}) (\exists! q, r \in \mathbb{Z}) P(q, r)$

Assume:

1. $m, n \in \mathbb{N}$

New goal: $((\exists q, r \in \mathbb{Z}) P(q, r)) \wedge (P(q, r) \wedge P(q', r')) \implies q = q' \wedge r = r'$

Subgoal: $(\exists q, r \in \mathbb{Z}) P(q, r)$

Missing

This goal is not proved.

Subgoal: $P(q, r) \wedge P(q', r') \implies q = q' \wedge r = r'$

Assume:

2. $q \geq 0 \wedge 0 \leq r < n \wedge q \times n + r = m$
3. $q' \geq 0 \wedge 0 \leq r' < n \wedge q' \times n + r' = m$

Goal: $q = q' \wedge r = r'$

$$0 \leq r < n \wedge -n < r \leq 0$$

$$\implies -n < r - r' < n \text{ by (2) and (3) as (4)}$$

$$m - r = q \times n \implies m \equiv r \pmod{n} \text{ by (2)}$$

$$m - r' = q' \times n \implies m \equiv r' \pmod{n} \text{ by (3)}$$

$$\implies r \equiv r' \pmod{n} \text{ by (L24.1)}$$

$$\implies (\exists i \in \mathbb{Z}) r - r' = i \times n$$

$$\implies i = 0 \wedge r = r' \text{ by (4)}$$

$$\implies q = q' \text{ by cancellation}$$

Definition: quo and rem

For the q and r associated by $m = q \times n + r$

- $\text{quo}(m, n) = q$
- $\text{rem}(m, n) = r$

And satisfies the property $m = \text{quo}(m, n) \times n + \text{rem}(m, n)$

Theorem 56: existence of quotient and remainder

```

let divalg m n =
  let diviter q r =
    if r < n then (q, r)
    else diviter (q + 1) (r - n)
  in diviter 0 m

```

Goal: divalg terminates \wedge (divalg terminates with $(q_0, r_0) \implies r_0 < n \wedge m = q_0 \times n + r_0$)

Subgoal: divalg terminates

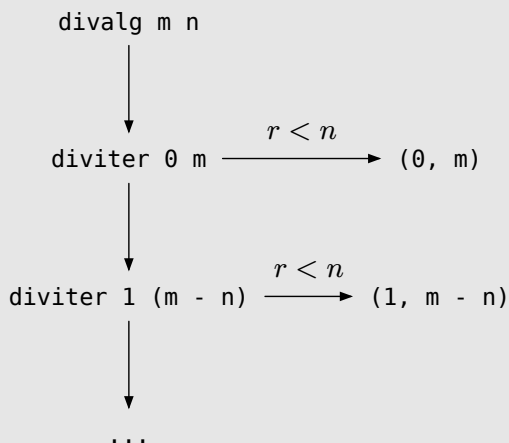
Note

This goal is also called **total correctness**.

Informal

$r \in \mathbb{N}$ decreases in each step, which is bounded below. This cannot continue forever.

It must terminate.



Subgoal:

divalg terminates with $(q_0, r_0) \implies r_0 < n \wedge m = q_0 \times n + r_0$

Note

This goal is also called **partial correctness**.

Assume:

1. divalg terminates with (q_0, r_0)

New goal: $r_0 < n \wedge m = q_0 \times n + r_0$

Informal

In the n th call of diviter

1. $\text{diviter } 0 \ m : m = q_0 \times n + r_0$ by trivial, if $r_0 < n$ terminate. Otherwise
2. $\text{diviter } q_1 \ r_1 \ m = q_1 \times n + r_1$ by trivial, if $r_1 < n$ terminate. Otherwise
3. ...

$$m = (q + 1) \times n + (r - n)$$

If the property holds for one point of the computation, then it holds for the next point of computation, this is called **proof by invariant**.

Proposition 57: uniqueness of remainder

Goal: $k \equiv l \pmod{m} \iff \text{rem}(k, m) = \text{rem}(l, m)$

Subgoal: $\text{rem}(k, m) = \text{rem}(l, m) \implies k \equiv l \pmod{m}$

Assume:

$$1. \text{rem}(k, m) = \text{rem}(l, m)$$

New goal: $k \equiv l \pmod{m}$

$$(\exists q, r \in \mathbb{N}) \ k = q \times m + r \text{ by (1)}$$

$$(\exists q', r' \in \mathbb{N}) \ l = q' \times m + r' \text{ by (1)}$$

$$\implies k - l = (q - q') \times m + (r - r')$$

$$\implies (\exists a \in \mathbb{N}) \ k - l = a \times m$$

$$\implies k \equiv l \pmod{m}$$

Subgoal: $k \equiv l \pmod{m} \implies \text{rem}(k, m) \wedge \text{rem}(l, m)$

Assume:

$$1. \ k \equiv l \pmod{m}$$

New goal: $\text{rem}(k, m) = \text{rem}(l, m)$

$$(\exists a \in \mathbb{Z}) \ k - l = a \times m$$

$$\implies (\exists q, r, q', r' \in \mathbb{Z}) \ (q \times m + r) - (q' \times m + r') = a \times m \text{ by (T56)}$$

$$\implies r - r' = (a - q + q') \times m$$

$$\implies r - r' = 0 \times m \text{ by bound } (-m) < r - r' < m$$

$$\implies r = r'$$

Corollary 58.1: congruence with remainder

Goal: $(\forall m \in \mathbb{N}) \ n \equiv \text{rem}(n, m) \pmod{m}$

Assume:

$$1. \ m \in \mathbb{N}$$

New goal: $n \equiv \text{rem}(n, m) \pmod{m}$

$$(\exists a \in \mathbb{Z}) \ n = m \times \text{quo}(n, m) + \text{rem}(n, m)$$

$$\implies n - \text{rem}(n, m) = m \times \text{quo}(n, m)$$

$$\implies n \equiv \text{rem}(n, m) \pmod{m}$$

Corollary 58.2: existence of modulus integer

Goal: $(\forall k \in \mathbb{Z}) \ (\exists! [k]_m) \ 0 \leq [k]_m < m \wedge k \equiv [k]_m \pmod{m}$

Subgoal: $(\forall k \in \mathbb{N}) \ (\exists! [k]_m) \ 0 \leq [k]_m < m \wedge k \equiv [k]_m \pmod{m}$

Assume:

$$1. \ k \in \mathbb{N}$$

New goal: $(\exists! [k]_m) \ 0 \leq [k]_m < m \wedge k \equiv [k]_m \pmod{m}$

Let $[k]_m = \text{rem}(k, m)$.

Let $P(k) : 0 \leq [k]_m < m \wedge k \equiv [k]_m \pmod{m}$

New goal: $P([k]_m) \wedge (P(a) \wedge P(b) \implies a = b)$

Subgoal:

$$0 \leq [k]_m < m \wedge k \equiv [k]_m \pmod{m}$$

Exact by (T56) and (P57).

Subgoal: $P(a) \wedge P(b) \Rightarrow a = b$

Assume:

$$2. 0 \leq a < m \wedge k \equiv a \pmod{m}$$

$$3. 0 \leq b < m \wedge k \equiv b \pmod{m}$$

New goal: $a = b$

$$-m < a - b < m \text{ by (2) and (3)}$$

$$(\exists i \in \mathbb{Z}) a - b = i \times m \text{ by (2) and (3)}$$

$$\Rightarrow i = 0 \wedge a = b \text{ by bounds}$$

Subgoal: $(\forall k \in \mathbb{Z}^-) (\exists! [k]_m) 0 \leq [k]_m < m \wedge k \equiv [k]_m \pmod{m}$

Assume:

$$1. k \in \mathbb{Z}^-$$

New goal: $(\exists! [k]_m) 0 \leq [k]_m < m \wedge k \equiv [k]_m \pmod{m}$

$$(\exists i \in \mathbb{Z}) k + i \times m \geq 0 \text{ by magic}$$

$$k + i \times m \equiv k \pmod{m} \text{ by trivial}$$

Then get the goal by setting $k' = k + i \times m$ using the previous subgoal.

Definition: modular arithmetic

$$k +_m l = [k + l]_m = \text{rem}(k + l, m)$$

$$k \times_m l = [k \times l]_m = \text{rem}(k \times l, m)$$

Proposition 62: modular arithmetic is a commutative ring

Goal: $(\mathbb{Z}_m, 0, +_m, 1, \times_m)$ is a commutative ring

Subgoal: $(\mathbb{Z}_m, 0, +_m)$ is a commutative monoid

Property	Proof
Commutative	$a_m +_m b_m = \text{rem}(a_m + b_m) = b_m +_m a_m$
Neutral element	$0_m +_m a_m = a_m = a_m +_m 0_m$
Associativity	$(a_m +_m b_m) +_m c_m = \text{rem}(a_m + b_m + c_m) = a_m +_m (b_m +_m c_m)$

Subgoal: $(\mathbb{Z}_m, 1, \times_m)$ is a commutative monoid

Property	Proof
Commutative	$a_m \times_m b_m = \text{rem}(a_m \times b_m) = b_m \times_m a_m$
Neutral element	$1_m \times_m a_m = a_m = a_m \times_m 1_m$
Associativity	$(a_m \times_m b_m) \times_m c_m = \text{rem}(a_m \times b_m \times c_m) = a_m \times_m (b_m \times_m c_m)$

Subgoal: $(\mathbb{Z}_m, 0, +_m, 1, \times_m)$ is a semiring

Apply subgoals (1) and (2).

New goal: distributivity $a_m \times (b_m +_m c_m) = a_m \times_m b_m +_m a_m \times_m c_m$

$$\begin{aligned} a_m \times (b_m +_m c_m) &\equiv a_m \times (b_m +_m c_m) \pmod{m} \\ &\equiv a_m \times (b_m + c_m) \pmod{m} \\ &\equiv a_m \times b_m + a_m \times c_m \pmod{m} \\ &\equiv a_m \times_m b_m +_m a_m \times_m c_m \pmod{m} \end{aligned}$$

Subgoal: $(\mathbb{Z}_m, 0, +)$ is a group

Apply subgoal (1).

New goal: $(\forall x_m \in \mathbb{Z}_m) (\exists y_m \in \mathbb{Z}_m) [x_m +_m y_m] = 0$

Assume:

1. $x_m \in \mathbb{Z}_m$

Let $y_m = [-x_m]_m$

New goal: $x_m +_m y_m = 0$

$$\begin{aligned} x_m +_m y_m &\equiv x_m + [-x]_m \pmod{m} \\ &\equiv x_m + (m - x_m) \pmod{m} \text{ by trivial} \equiv m \pmod{m} \\ &\equiv 0 \pmod{m} \end{aligned}$$

Sugboal: $(\mathbb{Z}_m, 0, +_m, 1, \times_m)$ is a commutative ring.

Exact by subgoal (2) and (4).

Proposition 63: condition for reciprocal

Goal: $(\forall m \in \mathbb{Z}^+, k \in \mathbb{Z}_m) ((\exists l \in \mathbb{Z}_m) k \times_m l = 1) \iff ((\exists i, j \in \mathbb{Z}) k \times i + m \times j = 1)$

Assume:

1. $m \in \mathbb{Z}^+, k \in \mathbb{Z}_m$

Subgoal: $((\exists l \in \mathbb{Z}_m) k \times_m l = 1) \implies ((\exists i, j \in \mathbb{Z}) k \times i + m \times j = 1)$

Assume:

2. $(\exists l \in \mathbb{Z}_m) k \times_m l = 1$

New goal: $(\exists i, j \in \mathbb{Z}) k \times i + m \times j = 1$

Let $i = k$

New goal: $(\exists j \in \mathbb{Z}) k \times l + m \times j = 1$

$$\begin{aligned} &(\exists l \in \mathbb{Z}_m) k \times l \equiv 1 \pmod{m} \\ &\implies (\exists a \in \mathbb{Z}) k \times l - 1 = m \times a \\ &\implies (\exists j \in \mathbb{Z}) k \times l + m \times j = 1 \end{aligned}$$

Subgoal: $((\exists i, j \in \mathbb{Z}) \ k \times i + m \times j = 1) \implies ((\exists l \in \mathbb{Z}_m) \ k \times_m l = 1)$

Assume:

$$2. (\exists i, j \in \mathbb{Z}) \ k \times i + m \times j = 1$$

New goal: $(\exists l \in \mathbb{Z}_m) \ k \times_m l = 1$

$$(\exists i, j \in \mathbb{Z}) \ k \times i - 1 = m \times (-j)$$

$$(\exists i, a \in \mathbb{Z}) \ k \times i - 1 = m \times a$$

$$k \times i \equiv 1 \pmod{m}$$

$$k \times_m [i]_m = 1$$

Definition: linear combination

$r \in \mathbb{Z}$ is a linear combination of $m, n \in \mathbb{Z}$ if

$$(\exists s, t \in \mathbb{Z}) \ s \times m + t \times n = r$$

Proposition 65: condition for reciprocal (linear combination)

Goal: $(\forall k \in \mathbb{Z}_m) \ ((\exists l \in \mathbb{Z}_m) \ k \times_m l = 1) \iff 1 \text{ is a linear combination of } m \text{ and } k$

Exact by (P63).

Definition: sets

- $x \in A$ if x is an element in A
- $A = B \iff ((\forall x) \ x \in A \iff x \in B)$
- $\{x \in A \mid P(x)\} = \{x \in A \mid Q(x)\} \iff ((\forall x \in A) \ P(x) \iff Q(x))$

Definition: common divisors

$$D(m) = \{d : d|m\}$$

$$CD(m, n) = \{d : d|m \wedge d|n\}$$

Lemma 71: key lemma

Lemma 71.1: divisibility of linear combinations

Goal: $(\forall d, i, j, m, n \in \mathbb{Z}) \ d|m \wedge d|n \implies d|(i \times m + j \times n)$

Assume:

$$1. \ d, i, j, m, n \in \mathbb{Z}$$

$$2. \ d|m \wedge d|n$$

New goal: $d|(i \times m + j \times n)$

$$(\exists x, y \in \mathbb{Z}) \ m = x \times d \wedge n = y \times d \text{ by (2)}$$

$$\implies i \times m + j \times n = i \times x \times d + j \times y \times d$$

$$\implies i \times m + j \times n = (i \times x + j \times y) \times d$$

$$\implies (\exists z \in \mathbb{Z}) \ i \times m + j \times n = z \times d$$

$$\implies d|(i \times m + j \times n)$$

Goal: $(\forall m, m' \in \mathbb{N}, n \in \mathbb{Z}^+) m \equiv m' \pmod{n} \implies \text{CD}(m, n) = \text{CD}(m', n)$

Assume:

1. $m, m' \in \mathbb{N}, n \in \mathbb{Z}^+$
2. $m \equiv m' \pmod{n}$

New goal: $d|m \wedge d|n \iff d|m' \wedge d|n$ (equality of predicates)

$$(\exists k \in \mathbb{Z}) m' = m + k \times n \text{ by (1)}$$

Subgoal: $d|m \wedge d|n \implies d|m' \wedge d|n$

Assume:

3. $d|m \wedge d|n$

New goal: $d|m' \wedge d|n$

$d|n$ by (3)

$(\exists i \in \mathbb{Z}) m - m' = i \times n$ by (2)

$$\implies m' = i \times n - m$$

$$\implies (\exists j, k \in \mathbb{Z}) m' = d \times j \times i - d \times k$$

$$\implies (\exists l \in \mathbb{Z}) m' = d \times l$$

$$\implies d|m'$$

Subgoal: $d|m' \wedge d|n \implies d|m \wedge d|n$

Note

This is true by symmetry of the first subgoal.

Lemma 73: common divisor reduction

Goal:

$$\text{CD}(m, n) = \begin{cases} D(n) & n|m \\ \text{CD}(n, \text{rem}(m, n)) & \text{otherwise} \end{cases}$$

Case:

1. $n|m$

Subgoal: $\text{CD}(m, n) = D(n)$

$$(\forall x \in \mathbb{Z}) x|n \implies x|m \text{ by (1) and (T23) as (2)}$$

New goal: $(\forall d \in \mathbb{Z}) d|m \wedge d|n \iff d|n$ by equality of predicates

Assume:

3. $d \in \mathbb{Z}$

New goal: $d|m \wedge d|n \iff d|n$

Goal: $d|m \wedge d|n \implies d|n$

Trivial.

Goal: $d|n \implies d|m \wedge d|n$

Assume:

4. $d|n$

New goal: $d|m$

Exact by (2).

Goal: $\text{CD}(m, n) = \text{CD}(n, \text{rem}(m, n))$

$m \equiv \text{rem}(m, n) \pmod{n}$ by (C58.1)

$\Rightarrow \text{CD}(m, n) = \text{CD}(n, \text{rem}(m, n))$ by (L71)

Definition: Euclid's Algorithm

$$\text{GCD}(m, n) = \begin{cases} n & \text{if } n|m \\ \text{GCD}(n, \text{rem}(m, n)) & \text{otherwise} \end{cases}$$

Proposition 75: uniqueness of common divisor

Goal: $(\forall m, n, a, b \in \mathbb{N}) \text{CD}(m, n) = D(a) \wedge \text{CD}(m, n) = D(b) \Rightarrow a = b$

Assume:

1. $m, n, a, b \in \mathbb{N}$
2. $\text{CD}(m, n) = D(a) \wedge \text{CD}(m, n) = D(b)$

New goal: $a = b$

$$\begin{aligned} D(a) &= D(b) \text{ by (2)} \\ \Rightarrow (\forall d) d|a &\iff d|b \text{ by equality of predicates} \\ \Rightarrow a|b \wedge b|a &\text{ by setting } d = a \text{ and } d = b \\ \Rightarrow a &= b \end{aligned}$$

Proposition 76: defining property of the greatest common divisor

Goal: $(\forall m, n, k \in \mathbb{Z}) \text{CD}(m, n) = D(k) \iff (k|m \wedge k|n \wedge (\forall d \in \mathbb{N}) d|m \wedge d|n \Rightarrow d|k)$

Assume:

1. $m, n, k \in \mathbb{Z}$

Subgoal: $\text{CD}(m, n) = D(k) \Rightarrow (k|m \wedge k|n \wedge (\forall d \in \mathbb{N}) d|m \wedge d|n \Rightarrow d|k)$

Assume:

2. $\text{CD}(m, n) = D(k)$

New goal: $k|m \wedge k|n \wedge (\forall d \in \mathbb{N}) d|m \wedge d|n \Rightarrow d|k$

$$\begin{aligned} (\forall d \in \mathbb{Z}) d|k &\Rightarrow d|m \wedge d|n \text{ by (2)} \\ \Rightarrow k|m \wedge k|n &\text{ by setting } d = k \end{aligned}$$

New goal: $(\forall d \in \mathbb{N}) d|m \wedge d|n \Rightarrow d|k$

Assume:

3. $d \in \mathbb{N}$
4. $d|m \wedge d|n$

New goal: $d|k$

Exact by (2) and (4)

Subgoal: $(k|m \wedge k|n \wedge (\forall d \in \mathbb{N}) d|m \wedge d|n \Rightarrow d|k) \Rightarrow \text{CD}(m, n) = D(k)$

Assume:

2. $k|m \wedge k|n$

3. $(\forall d \in \mathbb{N}) d|m \wedge d|n \implies d|k$

New goal: $(\forall d \in \mathbb{N}) d|m \wedge d|n \iff d|k$

Assume:

4. $d \in \mathbb{N}$

Subgoal: $d|m \wedge d|n \implies d|k$

Exact by (3)

Subgoal: $d|k \implies d|m \wedge d|n$

Exact by (2)