Multivariable Differential Equations

Partial Derivatives

For $f = f(x, y)$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$
$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Are the gradients of f if we travel along the x or y direction.

For functions of more variables, you can calculate $\partial f / \partial x$ by treating all other variables as constants.

Higher Derivatives

There are four 2nd order partial derivatives for a function $f(x, y)$

$$\frac{\partial^2 f}{(\partial x)^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$
$$\frac{\partial^2 f}{(\partial y)^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

For any well defined function

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

For $f(x_1, x_2, x_3, ..., x_n)$, the gradient of f is a vector.

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, ..., \frac{\partial f}{\partial x_n}\right)$$

Partial Differentials

Integration is the opposite of differentiation.

$$\frac{\partial f}{\partial x} = 2xy^2 \quad \text{holding } y \text{ constant}$$
$$f(x, y) = x^2y^2 + G(y)$$
$$\frac{\partial f}{\partial y} = 2x^2y + 2y \quad \text{holding } x \text{ constant}$$
$$f(x, y) = x^2y^2 + y^2 + H(x)$$
$$\therefore f(x, y) = x^2y^2 + y^2 + C$$

There may be no $f(x, y)$ satisfying $\partial f / \partial x$ and $\partial f / \partial y$, for examples those that leads to

$$\frac{\partial^2 f}{\partial y \partial x} \neq \frac{\partial^2 f}{\partial x \partial y}$$

Taylor series of f centred at (x, y), all partial derivatives are evaluated at (x_0, y_0)

$$\begin{aligned} f(x_0 + k, y_0 + k) = f(x_0, y_0) + \left(h\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) \\ + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots \end{aligned}$$

The **linear approximation** of f

$$f(x_0 + h, y_0 + k) \approx f(x_0, y_0) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

For very small change in x and y

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Chain Rule

Consider

$$\begin{aligned} df &= \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv \\ du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \end{aligned}$$

Then

$$\begin{aligned} df &= \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\right)dy \\ \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \end{aligned}$$

Reciprocity and Cyclic Relations

Let $z(x, y)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad (1)$$

$$dx = \frac{\partial x}{\partial y}dy + \frac{\partial x}{\partial z}dz \qquad (2)$$

$$dy = \frac{\partial y}{\partial z}dz + \frac{\partial y}{\partial x}dx \qquad (3)$$

Rewrite (2)

$$dz = \frac{1}{\left(\frac{\partial x}{\partial z}\right)} dx - \frac{\left(\frac{\partial x}{\partial y}\right)}{\left(\frac{\partial x}{\partial z}\right)} dy$$

Comparing coefficients of dx and dy with (1)

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{1}{\left(\frac{\partial x}{\partial z}\right)} \\
\frac{\partial z}{\partial y} &= -\frac{\left(\frac{\partial x}{\partial y}\right)}{\left(\frac{\partial x}{\partial z}\right)} \\
\frac{\partial z}{\partial y}\frac{\partial x}{\partial z} &= -\frac{\partial x}{\partial y} \\
\frac{\partial z}{\partial y}\frac{\partial x}{\partial z}\frac{\partial y}{\partial x} &= -1
\end{aligned}$$

The last three lines are called the **cyclic relation** for partial derivatives.

Exact Differentials

Definition

If there exist a function $f(x, y)$ such that

$$df = P(x, y)dx + Q(x, y)dy$$

then $P(x, y)dx + Q(x, y)dy$ is an **exact differential**.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Since $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

$$P(x, y)dx + Q(x, y)dy \text{ is an exact differential} \Longleftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Solving ODEs with Exact Differentials

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{P(x, y)}{Q(x, y)} \\
df &= P(x, y)dx + Q(x, y)dy = 0 \\
f(x, y) &= C
\end{aligned}$$

You will need to solve for $f(x, y)$ yourself, using the fact that

$$P = \frac{\partial f}{\partial x} \text{ and } Q = \frac{\partial f}{\partial y}$$

Integrating Factors for Inexact Differentials

In case $P(x, y)dx + Q(x, y)dy$ is not an exact differential, there may be a function $\mu(x, y)$ such that

$$\mu(x, y)[P(x, y)dx + Q(x, y)dy]$$

is an exact differential.

This requires

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$$

Stationary Points

For a function of two variables $f(x, y)$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Let $\boldsymbol{x} = (x, y)$

$$\begin{aligned} f(\boldsymbol{x}) &= f(\boldsymbol{x}_0) + (\boldsymbol{x} - \boldsymbol{x}_0) \cdot \nabla f(x) + \cdots \\ df &= (\nabla f) \cdot d\boldsymbol{x} \end{aligned}$$

So $\boldsymbol{x}$ is a stationary point if any of the equivalent statement is true at $\boldsymbol{x}$

- $\nabla f(\boldsymbol{x}) = 0$, or
- $df = 0$, or
- $\partial f / \partial x = 0 \land \partial f / \partial y = 0$

Form	**Type of stationary point**	**Property**
$f = (x - x_0)^2 + (y - y_0)^2$	Minimum	Curvature positive in all directions.
$f = -(x - x_0)^2 - (y - y_0)^2$	Maximum	Curvature negative in all directions.
$f = (x - x_0)^2 - (y - y_0)^2$	Saddle point	Curvature sign depends on direction.

Note
Contour lines form ellipses around maximum/minimum, and cross at saddle point.

Hessian Matrix

Define the symbols $(\delta x, \delta y) = (x - x_0, y - y_0)$

$$H_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad H_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$

$$H_{yx} = \frac{\partial^2 f}{\partial x \partial y} \qquad H_{yy} = \frac{\partial^2 f}{\partial y^2}$$

Let $D = H_{xx} H_{yy} < \left(H_{xy}\right)^2$

At a stationary point

$$\begin{aligned} f(x, y) - f(x_0, y_0) &= \frac{1}{2}\left(H_{xx} \delta x^2 + 2 H_{xy} \delta x \delta y + H_{yy} \delta y^2\right) \\ &= \frac{1}{2 H_{xx}}\left((H_{xx})^2 \delta x^2 + 2 H_{xx} H_{xy} \delta x \delta y + H_{xx} H_{yy} \delta y^2\right) \\ &= \frac{1}{2 H_{xx}}\left(\left(H_{xx} \delta x + H_{xy} \delta y\right)^2 + \left(H_{xx} H_{yy} - \left(H_{xy}\right)^2\right) \delta y^2\right) \end{aligned}$$

- Case $H_{xx} > 0$ and $D > 0$, RHS positive so is a minimum.
- Case $H_{xx} < 0$ and $D > 0$, RHS negative so is a maximum.

> **Note**
> When $D > 0 : \left(H_{xx} > 0 \Longleftrightarrow H_{yy} > 0\right)$

- Case $D < 0$
  - Moving in direction of y makes the 1st term disappear
  - Moving in direction of x makes the 2nd term disappear

  So it is a **saddle point**.
