Integration

Integration as Area

To find the area under the curve $y = f(x)$ in range $a \leq x \leq b$.

1. **Partition** $[a, b]$ into N subintervals with endpoints $x_0, x_1, ..., x_N$, the width of the partitions does not have to be equal.
2. Choose N points $\xi_1, \xi_2, ..., \xi_N$ in each of the partitions. The rectangle of each subpartition has area $A_i = (x_i - x_{i-1}) f(\xi_i)$.
3. The total area is the **Riemann sum**

$$S_N = \sum_{i=1}^{N} (x - x_{i-1}) f(\xi_i)$$

4. Take the **Riemann integral** where $N \to \infty$ such that all $x_i - x_{i-1} \to 0$.

$$\int_a^b f(x) dx = \lim_{N \to \infty} S_N$$

   The integral in this form is also called a **definite integral**.

Properties of an Integral

From the geometric interpretation.

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

But in the definition we assume $a \leq x \leq b$, to make the above property true for all c, we can define

$$\int_a^b f(x) dx = -\int_b^a f(x) dx$$

Integration and differentiation are **linear operations**, this means

$$\int_a^b k f(x) dx = k \int_a^b f(x) dx$$

$$\int_a^b f(x) + g(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

In $\infty + (-\infty)$ senarios, $\int f + g dx$ may be finite, but $\int f dx + \int g dx$ are both undefined.

Fundamental Theorem of Calculus

$$F(x) = \int_a^x f(u) du \Longleftrightarrow \frac{dF}{dx} = f(x)$$

Or

$$\frac{d}{dx} \int_a^x f(u) du = f(x)$$

Proof

$$\begin{aligned}\frac{dF}{dx} &= \lim_{\delta x \to 0} \frac{F(x+\delta x) - F(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{\int_a^{x+\delta x} f(u)du + \int_a^x f(u)du}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{\int_x^{x+\delta x} f(u)du}{\delta x} \\ &= f(x)\end{aligned}$$

Definitions
In $F(x) = \int f(x)dx$
- $f(x)$ is the **integrand**.
- $F(x)$ is the **primitive**.

Definite and Indefinite Integrals
The primitive is also called the **indefinite integral** of $f(x)$.
- If $F(x)$ is the indefinite integral, then so is $F(x) + C$.
- The definite integral equals the difference between the primitives evaluated at the endpoins.

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note that the constant disappears.

Improper Integrals

Definition
An **improper integral** is one which the integrand is *singular* (not well behaved) within the range of integration.

$$\int_a^\infty f(x)dx = \lim_{b \to \infty} \left(\int_a^b f(x)dx \right)$$

Discontinuous Integrals

Definition
A **discontinuous integrand** contains a finite number of discontinuities over the range of integration.

If $f(x)$ is discontinuous at $x = x_0$

$$\int_a^b f(x)dx = \int_a^{x_0} f(x)dx + \int_{x_0}^b f(x)dx$$

Note that the primitive of a discontinuous function is continuous.

Methods of Integration

Common Results

Since the indefinite integral is the reverse of differentiation.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int \cos(ax)dx = \frac{1}{a}\sin(ax) + C$$

$$\int \frac{1}{x}dx = \ln|x| + C \qquad \int \sin(ax)dx = -\frac{1}{a}\cos(ax) + C$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C \qquad \int \sec^2(ax)dx = \frac{1}{a}\tan(ax) + C$$

Using results from inverse hyperbolic functions.

$$\int \frac{1}{\sqrt{x^2 - a^2}}dx = \operatorname{arcosh}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2 + a^2}}dx = \operatorname{arsinh}\left(\frac{x}{a}\right) + C$$

From the chain rule

$$\int (f(x))^n f'(x)dx = \frac{1}{n+1}f^{n+1}(x) + C$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

Powers of Trig Functions

Use the result $\sin^2(x) = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2(x) = \frac{1}{2}(1 + \cos 2x)$

Since we don't know how to integrate higher power trig functions.

- If the power is even, e.g. $\cos^4 x = \frac{1}{4}(1 + \cos 2x)^2$, expand and repeat until the expression can be integrated.
- If the power is odd, e.g. $\sin^3 x = \sin x(1 - \cos^2 x)$, expand and use the reverse chain rule.

Similar rules can be used for stuff related to $\sec^2 x = 1 + \tan^2 x$ and the $\csc^2 x$ equivalent.

Partial Fractions

$$f(x) = \frac{p(x)}{q(x)}$$

Where $p(x)$ and $q(x)$ are polynomials, then $f(x) = P(x) + Q(x)$

The fundamental theorem of algebra says that we can write $q(x)$ as . But we don't actually want to deal with complex numbers, so we have

$$q(x) = (x - a_1)^{j_1}(x - a_2)^{j_2} \cdots (r_{m-1}(x))^{j_{m-1}}(r_m(x))^{j_m}$$

Where $r_{i(x)}$ are the terms with no real roots. So

$$Q(x) = \sum_{i=1}^{m}\sum_{r=1}^{j_i}\frac{A_{ir}}{(x - a_i)^r}$$

Once you write down these partial fractions, it is easy to integrate the terms.

Cover-up Rule

$$\begin{aligned} f(x) &= \frac{a_0 + a_1 x + a_2 x^2 + \cdots}{(x - r_0)(x - r_1)(x - r_2) \cdot \cdots} \\ &= \frac{b_0}{x - r_0} + \frac{b_1}{x - r_1} + \frac{b_2}{x - r_2} + \cdots \end{aligned}$$

$$b_0 + (x - r_0)\left(\frac{b_1}{x - r_1} + \frac{b_2}{x - r_2} + \cdots\right) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots}{(x - r_1)(x - r_2) \cdot \cdots}$$

Substitute $x = r_0$ to get

$$b_0 = \frac{a_0 + a_1 x + a_2 x^2 + \cdots}{(x - r_1)(x - r_2) \cdot \cdots}$$

Note
When there is a repeated root, the cover-up method only gives the coefficient of higest power.

Substitution

Substitution simplifies an integral by changing variables. Take example

$$\int \frac{1}{1 + x^2} dx$$

Let $x = \tan u$, then $dx = \sec^2 u \ du$.

- We are saying that making a small test in dx is the same as making a small step in $\sec^2 u \ du$.
- Look at the graph of $\tan x$ and this does make sense.

$$\begin{aligned} &= \int \frac{1}{1 + \tan^2 x} \sec^2 x \ dx \\ &= u + C \\ &= \arctan x + C \end{aligned}$$

Half-angle Formula

Using the substitution $\tan\left(\frac{x}{2}\right) = t$, we can show

$$\begin{aligned} \sin x &= \frac{2t}{1 + t^2} \\ \cos x &= \frac{1 - t^2}{1 + t^2} \\ \tan x &= \frac{2t}{1 - t^2} \end{aligned}$$

Common Substitutions

Denominator	**Substitution**
$a^2 + x^2$	$x = a \tan \theta$
$\sqrt{a^2 - x^2}$	$x = a \cos \theta$ or $x = a \sin \theta$
$\sqrt{x^2 - a^2}$	$x = a \cosh^2 \theta$
$\sqrt{x^2 + a^2}$	$x = a \sinh^2 \theta$

Denominator	**Substitution**
$a^2 - x^2$	$x = a\tanh^2\theta$ if $\lvert x\rvert < \lvert a\rvert$ $x = a\cosh^2\theta$ if $\lvert x\rvert > \lvert a\rvert$

Use completing the square to deal with general quadratic denominators.

Integration by Parts

From the product rule.

$$\begin{aligned}\frac{d}{dx}(fg) &= f\frac{dg}{dx} + \frac{df}{dx}g \\ f\frac{dg}{dx} &= \frac{d}{dx}(fg) - \frac{df}{dx}g \\ \int f\frac{dg}{dx}dx &= fg - \int \frac{df}{dx}g\ dx\end{aligned}$$

Integration with Complex Numbers

The integral of complex valued functions has the same rules as integrate with real valued function.

$$\int \Re(f)\ dx = \Re\left(\int f(x)\ dx\right)$$

Odd and Even Functions

Definitions
- $f(x) = f(-x) \iff$ even function
- $f(x) = -f(-x) \iff$ odd function

Using the area interpretation of an integral, we have

$$\begin{aligned}\int_{-a}^{a} f(x)\ dx &= 0 && \text{if } f \text{ is odd} \\ \int_{-a}^{a} f(x)\ dx &= 2\int_{0}^{a} f(x)\ dx && \text{if } f \text{ is even}\end{aligned}$$