Hyperbolic Functions

Definitions

Similar to how

$$\sin x = \frac{\exp(ix) - \exp(-ix)}{2i}$$
$$\cos x = \frac{\exp(ix) + \exp(-ix)}{2}$$

We have

$$\sinh x = \frac{\exp(x) - \exp(-x)}{2}$$
$$\cosh x = \frac{\exp(x) + \exp(-x)}{2}$$

Also define the other hyperbolic functions:

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$
$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

Note

We can see the relation between the circular trig and hyperbolic functions:

$$\sin ix = i \sinh x$$
$$\cos ix = \cosh x$$

Identities

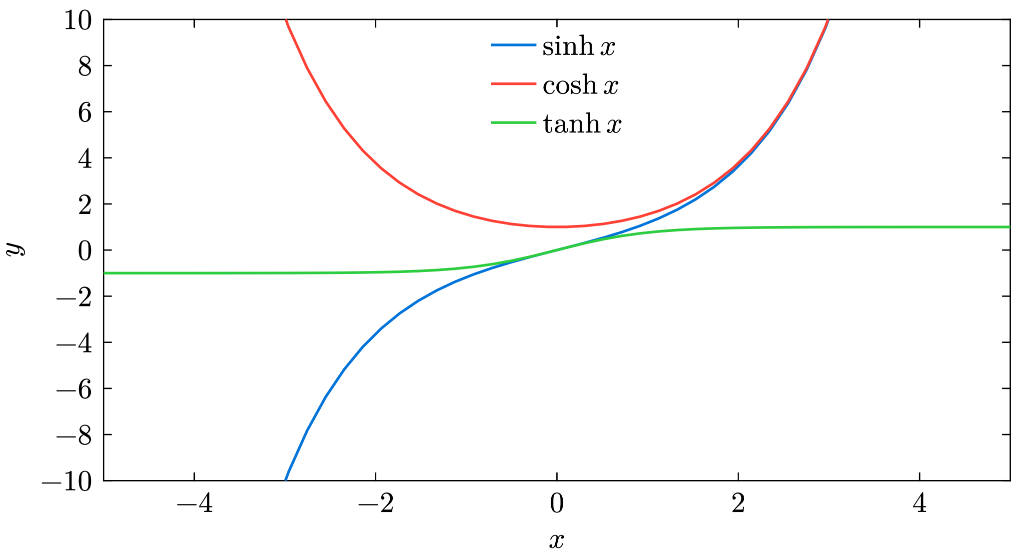
$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$$
$$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$$
$$\tanh(A \pm B) = \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}$$

These facts can be deplus.minuscircular trig identies using the relation mentioned above.

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\coth^2 x - 1 = \operatorname{csch}^2 x$$

Graphing Hyperbolic Functions



- cosh is symmetric, sinh and tanh are odd, this is similar to their circular trig equivalent.
- $\cosh x > \sinh x$ for all values of x because $\exp(-x)$ is always positive.

Inverse Hyperbolic Functions

With the example $y = \sinh^{-1} x$, we can solve for $\sinh y = x$ from definiton.

$$\begin{aligned}
\sinh^{-1} x &= \ln\left(x + \sqrt{x^2 + 1}\right) \\
\cosh^{-1} x &= \ln\left(x \pm \sqrt{x^2 - 1}\right) \\
\tanh^{-1} x &= \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)
\end{aligned}$$

- sinh is one-to-one, so its inverse is single valued.
- It is immediately obvious that cosh has two roots.

Circular Trig Roots from Inverse Hyperbolic Functions

The log of a complex number has infinitely many solutions.
- sinh has 1 solution.
- sin has infinitely many solutins.

To find all roots of $\cos z = 2$ where $z = x + iy$.

$$\begin{aligned}
\cos(x + iy) &= \cos x \cos iy - \sin x \sin iy \\
&= \cos x \cosh y - i \sin x \sinh y
\end{aligned}$$

For $\cos z = 2$, we need $\cos x \cosh y = 2$ and $\sin x \sinh y = 0$.
- Either $y = 0$, but then requires $\cos x = 2$ which is impossible.
- Or $x = n\pi$, then $\cos n\pi \cosh y = 2$. Since cosh is positive, n must be even so $\cos n\pi = 1$.

$$\begin{aligned}
x &= n\pi \\
y &= \ln\left(2 \pm \sqrt{x^2 - 1}\right)
\end{aligned}$$

Where roots z are all $z = x + iy$.

Circles

Definition
A **circle** is the set of all points equal distance r from the origin.

$$x^2 + y^2 = r^2$$

Moving the origin to (x_0, y_0):

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

We can also express in an alternate form, we can find the centre of a circle in that form by completing the square. A circle requires $r^2 > 0$.

$$\begin{aligned}
x^2 + ax + y^2 + by + c = 0 \\
\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 - c
\end{aligned}$$

We can also express as parameters which are functions of $\theta \in [0, 2\pi]$.

$$x = x_0 + r\cos\theta$$
$$y = y_0 + r\sin\theta$$

Replacing the circular trig functions with their hyperbolic equivalent will give a hyperbola.

Elipses

Generalise the circle formula:

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

So an elipse with semiaxes a and b:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

- The larger of a and b is the **semi major axis**.
- The smaller of a and b is the **semi minor axis**.

The general form of an elipse where the axis are not vertical or horizontal is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Eliptical Area

Since an elipse is a stretched circle

$$A = \pi ab$$

Note

The circumference of the elipse is not $\pi(a + b)$, it would require the epliptical integral which is pretty nasty.

The Hyperbola

Swiching the sign in the equation of a circle:

$$x^2 - y^2 = s^2$$
$$y = \pm\sqrt{x^2 - s^2}$$

Would be real if $|x| > |s|$.

The parameters of the hyperbola are the hyperbolic functions, similar to how a circle can be parametised using circular trig functions.

$$x^2 - y^2 = s^2\cosh^2\theta - s^2\sinh^2\theta = s^2$$

Identifying Shapes

To identify what is an equation in form $ax^2 + bx + cy^2 + dy + e = 0$.

First complete the square:

$$a(x - x_0)^2 + c(y - y_0)^2 = f$$

- Circle if $a = c$ and a has the same sign as f. ($a/f > 0$)
- Elipse if $a \neq c$.
- Hyperbola if a different sign as c.

`END Hyperbolic Functions`