Elementary Analysis

The Limit

This section gives a more rigorous understanding of the limit.

The function $\operatorname{sinc} x = \frac{\sin x}{x}$ is well behaved for $x \neq 0$, but $\operatorname{sinc} 0 = \frac{0}{0}$ is undefined.
- The intuitive understanding of the limit tells us to take some step away from the limit.
- Then get closer to the value and see what happens to the function value:
  - Does it get closer and closer?
  - Or does it diverges.

Tangent

A **monster function** is a function everywhere continuous, but nowhere differentiable.

$$f(x) = \sum_{k=0}^{\infty} a^k \cos\left(b^k \pi x\right)$$

Where $0 < a < 1$, $b \in \mathbb{Z}_{\text{odd}}$, $ab > 1 + \frac{3\pi}{2}$

Epsilon Delta Definition of a Limit

Let f be defined and continuous on an open interval containing x_0, but not necessarily on x_0.

$$\lim_{x \to x_0} f(x) = K$$

Iff $(\forall \varepsilon > 0)(\exists \delta > 0)\ 0 < |x - x_0| < \delta \Longrightarrow |f(x) - K| < \varepsilon$

We can prove a function has a limit iff this has a value.

Definition

A function does not have a limit at $x = x_0$ iff

$(\forall L)(\exists \varepsilon > 0)(\forall \delta > 0)\ 0 < |x - x_0| < \delta \Longrightarrow |f(x) - L| \geq \varepsilon$

One-Sided Limits

$$\lim_{x \to x_0} f(x) = L \Longleftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0)\ 0 < x - x_a < \delta \Longrightarrow |f(x) - L| < \varepsilon$$

Limits at Infinity

If we define a box that have finite height and extend to infinity, check if the function is still in the box.

$$\lim_{x \to \infty} f(x) = L \Longleftrightarrow (\forall \varepsilon > 0)(\exists X < \infty)\ x > X \Longrightarrow |f(x) - L| < \varepsilon$$

Algebra of Limits

$$\lim_{x \to x_0} f(x) = A$$

$$\lim_{x \to x_0} g(x) = B$$

$$\lim_{x \to x_0} f(x) + g(x) = A + B$$

$$\lim_{x \to x_0} f(x)g(x) = AB \quad (\text{unless } A = 0, B = \infty)$$

These also applies to $x \to \infty$.

l'Hopital's Rule

Used in cases where the limit is undefined: $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty$.

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

- Requires f and g be differentiable in an open interval.
- $f'(0)$ and $g'(x)$ not equal to 0 in the open interval and exists.

The way to understand this is with the Taylor series.