

Infinite Series

Convergence

Given an infinite sequence of numbers, define the **partial sum** to be

$$S_n = u_1 + u_2 + \cdots + u_n$$

We want to know if the infinite series

- Have a well defined limit
- Diverges
- Oscillates

$$\lim_{n \rightarrow \infty} S_n = S \text{ iff } (\forall \varepsilon > 0)(\exists N) n > N \implies |S - S_n| < \varepsilon$$

Conditions for Convergence

Requires $u_n \rightarrow 0$ as $n \rightarrow \infty$.

The sum of two diverging series may converge, e.g. when one diverges to $+\infty$ and the other to $-\infty$, and the two cancels each other out.

Definitions

- A series is **absolutely convergent** iff $\sum |a_n|$ converges.
- A series is **conditionally convergent** iff $\sum a_n$ converges but not $\sum |a_n|$.

If we rearrange the terms in a series, a converging series can diverge, or the other way round.

- If we have an absolutely convergent series, it doesn't matter if we change the order.
- If we have a conditionally convergent series, changing the order may cause it to diverge.

Geometric Progression

Let $u_k = r^k$, we already know the sum formula for the geometric progression, here's the proof.

$$(1 - r)(1 + r + r^2 + \cdots + r^k) = 1 - r^{k+1} \quad (\text{by simplifying the expression})$$

$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}$$

For the sum of infinite series

$$\lim_{k \rightarrow \infty} \frac{1 - r^{k+1}}{1 - r} \text{ exists if } |r| < 1.$$

The Harmonic Series

The harmonic series is

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

We can group terms **without reordering**.

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

The sequence $\sum \frac{1}{2}$ diverges, so the harmonic series also diverges, this is called a **comparison test**.

$$\gamma = \lim_{k \rightarrow \infty} (S_k - \ln k) = 0.57721566$$

Which is the **Euler-Mascheroni constant**.

Convergence Tests

Comparison Tests

The comparison test can be used on two nonnegative series: $(\forall k) u_k, v_k \geq 0$

$$(\forall k \geq K) u_k \leq v_k \text{ and } \sum_{k=0}^{\infty} v_k \text{ convergent} \Rightarrow \sum_{k=0}^{\infty} u_k \text{ convergent}$$

$$(\forall k \geq K) u_k \geq v_k \text{ and } \sum_{k=0}^{\infty} v_k \text{ divergent} \Rightarrow \sum_{k=0}^{\infty} u_k \text{ divergent}$$

Ratio Tests

For a positive series: $(\forall k) u_k > 0$

$$\lim_{k \rightarrow \infty} \left(\frac{u_{k+1}}{u_k} \right) < 1 \Rightarrow \text{converges}$$

$$\lim_{k \rightarrow \infty} \left(\frac{u_{k+1}}{u_k} \right) > 1 \Rightarrow \text{diverges}$$

Proof

If $\lim_{k \rightarrow \infty} \frac{u_{k+1}}{u_k} = \alpha < \infty$, then for large enough k :

$$\sum u = u_1 + u_2 + \dots + u_k(1 + \alpha + \alpha^2 + \dots)$$

Which converges.

Leibniz Criterion

Definition

Alternating series have terms with alternating signs.

If $a_k > 0$ and a_k is monotonically decreasing for large enough k , and $\lim_{k \rightarrow \infty} a_k = 0$.

$$S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges}$$

Proof

We know it is true that

$$\sum_{k=1}^{2n} = (a_1 - a_2) + (a_3 - a_4) + \dots \text{ is a monotonically increasing series}$$

$$\sum_{k=1}^{2n+1} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots \text{ is a monotonically decreasing series}$$

$$\lim_{k \rightarrow \infty} \left(\sum_{k=1}^{2n+1} a_k - \sum_{k=1}^{2n} a_k \right) = 0$$

So we write the inequality

$$a_1 - a_2 = S_2 < S_4 < \dots < S_{2n} < S_{2n+1} < \dots < S_3 < S_1 < a_1$$

Power Series

A power series have form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \dots$$

The **domain of convergence** is either

- Only $x = 0$
- All finite x
- Only for some $|x| < R$

We can find out which case it is with the **ratio test**, which gives the shortcut if $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$ exists.

$$|x| < \frac{1}{L} \implies \text{converges}$$

$$|x| > \frac{1}{L} \implies \text{diverges}$$

$$|x| = \frac{1}{L} \implies \text{indeterminate}$$

Note the endpoints at $|x| = 1/L$ may behave differently.

The Taylor Series

We can approximate the value of a function using its tangent at a point.

$$f(x) \approx f(a) + (x - a)f'(a)$$

$$f'(x) \approx f'(a) + (x - a)f''(a)$$

By the fundamental theorem of calculus, which links the derivative to the integral:

$$\int_a^{a+h} f'(x) dx = f(a+h) - f(a)$$

$$\begin{aligned} f(a+h) &= f(a) + \int_a^{a+h} f'(x) dx \\ &= f(a) + \int_a^{a+h} f'(a) + (x-a)f''(a) dx \\ &= f(a) + \left[xf'(a) + \frac{(x-a)^2}{2} f''(a) \right]_a^{a+h} \\ &= f(a) + hf'(a) + \frac{h^2}{2} f''(a) \end{aligned}$$

We can show that the **second order approximation** is better than the first order approximation. (how?)

Extend this to higher order approximations, and we get

$$f'(x) \approx f'(a) + (x-a)f''(a) + \frac{(x-a)^2}{2!} f'''(a) + \frac{(x-a)^3}{3!} f^{(4)}(a) + \dots$$

Since the factorial function grows faster than any polynomial, we know that $n!$ grows faster than $(x-a)^n$ so the series converges for a lot of approximations.

Taylor's Theorem

An exact result can be written by including a **remainder term** R_{n+1} .

$$f(a+h) = f(a) + hf'(a) + \dots + \frac{h^n}{n!} f^{(n)}(a) + R_{n+1}$$

Taylor's Theorem states if f is $n+1$ times differentiable, there exists a $\zeta : a < \zeta < a+h$ such that

$$R_{n+1} = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\zeta)$$

Proof

$$\int_a^x f'(t) dt = f(x) - f(a) \text{ by fundamental theorem of calculus}$$

$$\begin{aligned} f(x) &= f(a) + \int_a^x f'(t) dt \\ &= f(a) + [(t-x)f'(t)]_a^x - \int_a^x (t-x)f''(x) dt \text{ integrate by parts} \\ &= f(a) + (x-a)f'(a) + \int_a^x (t-x)f''(x) dt \end{aligned}$$

Integrate by parts repeatedly to find

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \int_a^x \frac{(t-x)^n}{n!} f^{(n+1)}(t) dt \\ R_{n+1} &= \int_a^x \frac{(t-x)^n}{n!} f^{(n+1)}(t) dt \\ &= \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\zeta) \text{ where } a \leq \zeta \leq x \text{ by mean value theorem} \\ \lim_{n \rightarrow \infty} R_{n+1} &= 0 \implies \text{Taylor series for } f(x) \text{ converges} \end{aligned}$$

By choosing the ζ that would give the largest error, we can calculate the worst case error for an approximation.

Note

If $f(x)$ is infinitely differentiable, then we can represent f exactly as an infinite **power series**.

We could also prove that two functions are the same if they have the same Taylor series.

Common Series Expansions

A lot of these expansion can be proved using the expansion for e^x .

Function	Taylor series	Function	Taylor series
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

Function	Taylor series	Function	Taylor series
$\cosh x$	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	$\sinh x$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

It makes sense that the Taylor series of an even function only have even powered terms, and an odd function only have odd powered terms.

The series expansion for \tanh has a radius of convergence of $|x| < \frac{\pi}{2}$, this makes sense because

- Polynomials cannot capture the flat asymptote.
- \tan is only continuous over $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Binomial Expansion

Consider $f(x) = (1+x)^n$

$$f'(x) = n(1+x)^{n-1}$$

$$f''(x) = n(n-1)(1+x)^{n-2}$$

$$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$$

⋮

So we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

Which is the **binomial theorem**, when n is a positive integer, it agrees with the binomial expansion.

We can show that the series is absolutely convergent for $|x| < 1$ by the ratio test.

Stationary points with Taylor Series

If f is stationary at $x = a$, and suppose $f''(a) \neq 0$.

$$f(x) = f(a) + x f'(a) + \frac{x^2}{2} f''(a) + \dots$$

$$f(x) - f(a) \approx \frac{x^2}{2} f''(a)$$

Now we know why if $f''(a) > 0$ then f has a minimum at $x = a$, if $f''(a) < 0$ then f has a maximum at $x = a$.

If the first $n-1$ derivatives of $f(x) - f(a)$ are **vanishing**, then the function will behave like the first nonvanishing (the n th) term. This is where the idea of **L'Hopital's rule** comes from.

$$f(x) - f(a) \approx \frac{(x-a)^n}{n!} f^{(n)}(a)$$

- If n is even then $x = a$ is either a local maximum or minimum for f , as $x < a$ and $x > a$ are both either less than or greater than f at $x = a$.
- If n is odd then $x = a$ is a point of inflection for f .

Approximating Functions with Taylor Series

Functions can be approximated by replacing them with their Taylor series, for example.

$$\begin{aligned}\frac{\log(1+x)}{1-x} &= \log(1+x) \cdot (1-x)^{-1} \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) \cdot (1+x+x^2+\dots) \\ &= x + \frac{x^2}{2} + \frac{5x^3}{6} + \dots\end{aligned}$$

Newton-Rapson Method

This is an efficient way of finding roots of a function.

The first order approximation of a function at $x = a$ is, we want to find $f(x) = 0$.

$$f(x) \approx f(a) + (x-a)f'(a)$$

We want to find $f(x) = 0$.

$$\begin{aligned}0 &\approx f(a) + (x-a)f'(a) \\ &= \frac{f(a)}{f'(a)} + x - a \\ x &= a - \frac{f(a)}{f'(a)}\end{aligned}$$

So if we define the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This turns out to converge really quickly.

Estimating Error

Write a recursive relation that gives the error term ε_n , so we can find how quickly it converges.

Let x^* be the exact solution where $f(x^*) = 0$.

$$f(x) \approx f(x^*) + (x-x^*)f'(x^*) + \frac{1}{2}(x-x^*)^2 f''(x^*)$$

If f is linear, then we find x^* in one step. Otherwise, define error $\varepsilon_n = x_n - x^*$.

$$\begin{aligned}f'(x) &= f'(x^*) + (x-x^*)f''(x^*) + O(x^3) \\ \varepsilon_{n+1} &= x_{n+1} - x^* \\ &= \left(x_n - \frac{f(x_n)}{f'(x_n)}\right) - x^* \\ &= x_n - x^* - \frac{f(x^*) + \varepsilon_n f'(x^*) + \frac{1}{2}(\varepsilon_n)^2 f''(x^*)}{f'(x^*) + \varepsilon_n f''(x^*)} \\ &= \varepsilon_n - \frac{\varepsilon_n f'(x^*) + \frac{1}{2}(\varepsilon_n)^2 f''(x^*)}{f'(x^*) + \varepsilon_n f''(x^*)}\end{aligned}$$

Assuming we are close, how much closer will we get for the next step?

For very small ε

$$\varepsilon_{n+1} = -\frac{\frac{1}{2}(\varepsilon_n)^2 f''(x^*)}{f'(x^*)} + O((\varepsilon_n)^2)$$

So the first term will be a quadratic term in ε_n - the error will be squared for each term.

END Infinite Series