

## 2. Linear Motion

Symbol	Meaning	Definition
$V$	potential	$\int_{x_0}^x F(x')dx'$
$F$	force	$-\frac{dV}{dx}$
$T$	kinetic energy	$\frac{1}{2}m\dot{x}^2$

From  $T = \frac{1}{2}m\dot{x}^2$ , we find:

- $\dot{T} = m\ddot{x}\dot{x} = F(x)\dot{x}$
- $T = \int F(x)dx$

Conservation of energy states  $V + T = E$  which is constant.

### The Harmonic Oscillator

Symbol	Meaning	Definition
$f$	frequency	$\frac{\omega}{2\pi}$
$\tau$	time period	$\frac{2\pi}{\omega}$

The harmonic oscillator at equilibrium  $x = 0$  satisfies  $V'(0) = 0$ . Choose constant  $E$  so  $V(0) = 0$ .

$$V(x) = V(0) + xV'(0) + \frac{1}{2}x^2V''(0) + \dots$$

$$\approx \frac{1}{2}kx^2 \text{ where } k = V''(0)$$

Then  $F(x) = -kx$ .

- A conservative force depends only on  $x$
- A dissipative force additionally depend on variables other than  $x$ , such as velocity.
- All forces are conservative at microscopic scale.

Undamped oscillator:

$$m\ddot{x} = -kx$$

$$\ddot{x} + \frac{k}{m}x = 0$$

$$x = \frac{1}{2}Ae^{pt} + \frac{1}{2}Be^{-pt}$$

where  $p = \sqrt{-k/m}$

If  $k < 0$ , it is an unstable equilibrium.

If  $k > 0$ , it is a stable equilibrium.

$$x = \frac{1}{2}Ae^{i\omega t} + \frac{1}{2}Be^{-i\omega t} \text{ where } \omega = \sqrt{\frac{k}{m}}$$

For  $x$  to have no imaginary component, let

- $A = ae^{-i\theta}$
- $B = ae^{i\theta}$

Then

$$x = \frac{1}{2}ae^{i(\omega t - \theta)} + \frac{1}{2}ae^{-i(\omega t - \theta)}$$

$$= a \cos(\omega t - \theta)$$

Note if  $a\ddot{z} + b\dot{z} + cz = 0$  is a solution, where  $z = x + iy$ , then so is  $a\ddot{x} + b\dot{x} + cx = 0$  and the  $y$  equivalent.

### The Damped Oscillator

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

Let  $\gamma = \frac{\lambda}{2m}$  and  $\omega_0 = \sqrt{\frac{k}{m}}$ .

Large damping if  $\gamma > \omega_0$   
 • Let  $\gamma_{\pm} = \gamma \pm \sqrt{\gamma^2 - (\omega_0)^2}$

$$x = \frac{1}{2}Ae^{-\gamma_+t} + \frac{1}{2}Be^{-\gamma_-t}$$

The leading term is  $\frac{1}{2}Be^{-\gamma_-}$ , so the characteristic time is  $\frac{1}{\gamma_-}$

Small damping if  $\gamma < \omega_0$   
 • Let  $\omega = \sqrt{(\omega_0)^2 - \gamma^2}$

$$x = \frac{1}{2}Ae^{-\gamma+\omega_0t} + \frac{1}{2}Be^{-\gamma-\omega_0t}$$

$$= ae^{-\gamma t} \cos(\omega_0 t - \theta)$$

where  $A = ae^{-i\theta}$ ,  $B = ae^{i\theta}$

- The relaxation time is  $\frac{1}{\gamma}$
- The amplitude reduction in a single period is  $e^{\pi/Q}$  where  $Q = \omega_0/2\gamma$

Critical damping when  $\gamma = \omega_0$

$$x = (a + bt)e^{-\gamma t}$$

### Resonance

Under a periodic force  $F(t) = F_1 \cos(\omega_1 t)$ , solving the differential equation gives solution the equation for an undamped, forced oscillation.

$$x = a_1 \cos(\omega_1 t - \theta_1)$$

$$\text{amplitude } a_1 = \frac{F_1/m}{\sqrt{((\omega_0)^2 - (\omega_1)^2)^2 + 4\gamma^2(\omega_1)^2}}$$

$$\text{phase } \tan \theta_1 = \frac{2\gamma\omega_1}{(\omega_0)^2 - (\omega_1)^2}$$

- When  $\omega_1$  is low,  $\theta_1 \approx 0$ , and  $x$  satisfies  $F(t) - kx = 0$ .
- At resonance  $\omega_1 = \sqrt{(\omega_0)^2 - 2\gamma^2}$ ,  $\theta_1 = \frac{\pi}{2}$ ,  $a_1 = F_1/2m\gamma\omega_1$ .
- When  $\omega_1$  is very high,  $\theta_1 \approx \pi$  (out of phase),  $x \approx 0$ .

So the damping constant  $\lambda$  only matters when near resonance.

### General Periodic Forces

Any periodic force can be written as a sum of harmonics.

$$F(t) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega t}$$

$$F_n = \frac{1}{\tau} \int_0^{\tau} F(t) e^{-in\omega t} dt$$

The position is the sum of the general solutions for each harmonic.

$$x = \sum_{n=-\infty}^{\infty} A_n e^{in\omega t} + \text{transient}$$

**General Forces**

Define impulse delivered in  $\Delta t$  be  $I = \Delta p = F(t)\Delta t$ .

An oscillator at rest when given impulse  $I$  has  $\dot{x} = \frac{I}{m}$ , solving for small damping:

$$x(t) = \begin{cases} 0 & \text{when } t < 0 \\ \frac{I}{m\omega} e^{-\gamma t} \sin \omega t & \text{when } t \geq 0 \end{cases}$$

For a series of impulses  $I_r$  at  $t_r$

$$x(t) = \sum_r G(t - t_r) I_r + \text{transient}$$

where the Green's function  $G$  is defined as

$$G(t') = \begin{cases} 0 & \text{when } t' < 0 \\ \frac{I}{m\omega} e^{-\gamma t'} \sin \omega t' & \text{when } t' \geq 0 \end{cases}$$

Then for any force  $F(t)$

$$x(t) = \int_0^t G(t' - t) F(t') dt' + \text{transient}$$