# The Mechanics of n-Player Differentiable Games

ICML 2018 best paper, summarized by

## Highlight

The authors propose a new gradient descent method and algorithm called Symplectic Gradient Adjustment (SGA) to find the stable fixed point in n-player games e.g. generative adversarial net (GAN). The key to SGA is to decompose the second-order dynamics, which is also knnown as Hessian, into two components. The first is related to potential games and the second is related to Hamiltonian games. The potential function of potential games and Hamiltonian function of Hamiltonian games are easy to solve and can converge to Nash equilibrium by gradient descent. Therefore they build up mechanics to solve general n-players differentiable games. The experimental result shows that SGA has a wider converged learning rate threshold and a faster converge speed than optimistic mirror decent (OMD).

## Summary

Today's machine learning method strongly dependent on gradient descent method, to optimize the parameter of model to reduce a single loss (objective) function. However, there is more and more model which have more than one objective function, e.g. GAN. This model trains generative model and discriminate model at the same time, who compete to each other via a minimax game. Simple gradient descent failed to cope with these cases, they provide no guarantee to converge to equilibrium or any stable point, which means the parameter of the model will never stabilize. Paper has shown that finding the equilibrium of a game, in general, is a PPAD hard problem.

When using gradient descent in games, there are three main difficulties of the problem. Firstly, the potential existence of cycles implies that there are no convergence guarantees. Secondly, even if converged, the rate may be slow due to rotational forces. This problem leads to long training time and small learning rate. Finally, there is no way to measure progress since there are more than one parameters. Current method strongly depends on looking at samples.

To introduce the construction progress of SGA, we start with the formulate the n-player game. Let denote to the parameters of players, and players has twice continuously losses . Player i control . Let denote to simultaneous gradient descent of the respective players. Then, use Helmholtz decomposition to divide the Hessian matrix of into a symmetric matrix and anti-symmetric matrix .

Now introducing the concept of stable fixed point: A fix point with is stable if for is a neighborhood of . Note that a stable fixed point is also a local Nash equilibrium.

If , the game is a potential game. It follows the gradient descent on to converge to fixed point. If , the game is a Hamiltonian game. It follows the gradient descent on to converge to fixed point, where .

With all the previous definitions and theorems, we can draw a adjustment gradient descent instead of the original , which has to satisfy five desiderata:

1. compatible with game dynamic: , where
2. compatible with potential game: , where
3. compatible with Hamiltonian game: , where
4. attracted to stable fixed point (Nash equilibrium): in neighborhood where , require
5. repelled by unstable fixed point: in neighborhood where , require

To satisfy all the five desiderata, they build adjustment gradient descent:

and its aligned version

The key to SGA is generalized Helmholtz decomposition. In classical Helmholtz decomposition, any vector field in 3-dimensions is the sum of curl-free(gradient) and divergence-free(infinitesimal rotation) component:

Remind that the rotation component is the antisymmetric part of the game Hessian. And thus the relation between classical Helmholtz decomposition and generalized Helmholtz decomposition follows.

After construct the SGA, the authors design three experiments to show the effect.

In the first experiment, they compare the performance of between gradient descent and SGA in a simple 2-players zero-sum game using different learning rate. When learning rate is 0.01, gradient descent slowly converges to Nash equilibrium (0,0), while SGA converges rapidly. When learning rate increase to 0.32, gradient descent fails to converge while SGA converges even faster. When learning rate increase to 0.1, gradient descent fails to converge again while SGA still converges.

In the second experiment, they compare gradient descent, OMD, SGA in a basic adversarial game. Although OMD's peak performance is better than SGA, they found that SGA converges for a wider range in learning rate, and does so faster. Gradient descent does not converge.

In the third experiment, they use gradient descent, SGA without alignment and SGA with alignment in GAN. This GAN will learn data that samples from a highly multimodal distribution. The distribution is a mixture of 16 Gaussians arranged in a four by four grid. The networks are trained under RMSProp and the generative and discriminator networks both have 6 ReLU layers of 384 neurons. After 8000 iterations, both SGA converge to correct distribution while gradient descent fails. SGA with alignment speeds up convergence slightly.

SGA has many advantages in solving n-player games. It uses Helmholtz decomposition to find the rotation momentum and apply adjustment to neutralize the rotation momentum's influence to gradient descent. Furthermore, SGA keeps the simplicity and feasibility of gradient descent, avoiding the large computational cost of the traditional mathematical method. To sum up, SGA brings a generic to solve n-players differentiable games.

I have to admit that SGA add extra computing complexity to traditional gradient descent, especially the aligned SGA. In traditional gradient descent, all we need is to calculate the first order derivative of the game, which is , while in aligned SGA, we first calculate the Hessian matrix , then calculate the dot product of and the , which are both . It will be a great cost if the size of the game increase. The authors cleverly avoid the discussion of algorithm complexity and training time. I guess what I mentioned above is one of their concerns.

As I analyzed above, the SGA is more than a brutal application of Helmholtz decomposition without any algorithm and mathematical optimization, which leads to great computation cost. The first idea is to avoid calculating the dot product of the Hessian matrix and the first-order derivative vector. Secondly, the authors imply that SGA will make players act against their own interests because of the adjustment in rotation momentum. This phenomenon shows that SGA is actually against the nature of game theory, which leads to my doubt of its versatility.