Notes on Photon to Axion decoupling

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1. Photon-axion coupling in an external magnetic field

We begin by considering a photon interacting with an external magnetic field, as illustrated in the figure. The Lagrangian includes a Chern-Simons interaction term between the photon γ and the axion field a.

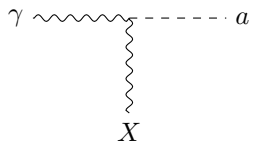


Figure 1: Photon converting into an axion in the presence of a background electromagnetic field X. We are interested in the case where X corresponds to a magnetic field. Time flows from left to right.

$$\mathscr{S} = \int \mathrm{d}x \left[-\frac{1}{2}\partial^\mu a \partial_\mu a - \frac{1}{2}m_a^2 a^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}g_{a\gamma\gamma}aF_{\mu\nu}\tilde{F}^{\mu\nu} + A_\mu J^\mu \right]. \tag{1}$$

By writing down the Euler–Lagrange equations with respect to the axion field a and the electromagnetic field A_μ, we can derive the equations of motion for the system:

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu a)} \right) = \frac{\partial \mathscr{L}}{\partial a}, \tag{2}$$

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} \right) = \frac{\partial \mathscr{L}}{\partial A_\nu}, \tag{3}$$

where the first equation leads to the modified Klein–Gordon equation and the second gives the modified Maxwell equations. Both are altered due to the presence of the Chern–Simons interaction term.

We begin with the Euler–Lagrange equation for a:

$$\frac{\partial \mathscr{L}}{\partial a} = -m_a^2 a - \frac{1}{4}g_{a\gamma\gamma}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{4}$$

and, recalling that $\partial^\mu = \eta^{\mu\nu}\partial_\nu$, we compute:

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu a)} = -\frac{1}{2}\cdot 2\partial^\mu a = -\partial^\mu a, \tag{5}$$

thus,

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu a)} \right) = -\partial_\mu \partial^\mu a = -\Box a. \tag{6}$$

Hence, the full Euler–Lagrange equation becomes:

$$-\Box a = -m_a^2 a - \frac{1}{4}g_{a\gamma\gamma}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{7}$$

or equivalently,

$$\boxed{(\Box - m_a^2)a = \frac{1}{4}g_{a\gamma\gamma}F_{\mu\nu}\tilde{F}^{\mu\nu} = g_{a\gamma\gamma}\vec{E}\cdot\vec{B}} \tag{8}$$

We now turn to the Euler–Lagrange equation for A_ν:

$$\frac{\partial \mathscr{L}}{\partial A_\nu} = \frac{\partial}{\partial A_\nu}(A_\nu J^\nu) = J^\nu, \tag{9}$$

and recall that, by definition:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{10}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{11}$$

so that

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4}\frac{\partial}{\partial(\partial_\mu A_\nu)}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4}\frac{\partial}{\partial(\partial_\mu A_\nu)}(g_{a\gamma\gamma}aF_{\mu\nu}\tilde{F}^{\mu\nu}). \tag{12}$$

From electrodynamics we know that (note that, in contrast with the Klein–Gordon equation above, both indices are raised, introducing a factor of 2, as $F^{\mu\nu} = \eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}$):

$$\frac{\partial}{\partial(\partial_\mu A_\nu)}(F_{\mu\nu}F^{\mu\nu}) = +4F^{\mu\nu}, \tag{13}$$

thus, applying this to the second term as well, we obtain:

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} - g_{a\gamma\gamma}a\tilde{F}^{\mu\nu}. \tag{14}$$

Recalling that in standard electrodynamics the Bianchi identity holds:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \tag{15}$$

we find:

$$\partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)}\right) = \partial_\mu(-F^{\mu\nu} - g_{a\gamma\gamma}a\tilde{F}^{\mu\nu}) = -\partial_\mu F^{\mu\nu} - g_{a\gamma\gamma}\tilde{F}^{\mu\nu}\partial_\mu a, \tag{16}$$

so the Euler–Lagrange equation becomes:

$$\boxed{\partial_\mu F^{\mu\nu} = -g_{a\gamma\gamma}\tilde{F}^{\mu\nu}\partial_\mu a - J^\nu} \tag{17}$$

This is essentially Maxwell's equation ($\partial_\mu F^{\mu\nu} = -J^\nu$), modified by the interaction term $-g_{a\gamma\gamma}\tilde{F}^{\mu\nu}\partial_\mu a$.

The equations of motion associated with the Lagrangian are then:

$$\begin{cases} (\Box - m_a^2)a = \frac{1}{4}g_{a\gamma\gamma}F_{\mu\nu}\tilde{F}^{\mu\nu} = g_{a\gamma\gamma}\vec{E}\cdot\vec{B} \\ \partial_\mu F^{\mu\nu} = -g_{a\gamma\gamma}\tilde{F}^{\mu\nu}\partial_\mu a - J^\nu \end{cases} . \tag{18}$$

1.1 Setting a working coordinate system

For our purposes, it is convenient to rewrite the equations of motion (18) in terms of the physical electric and magnetic fields relevant to our setup. To do this, we first define a suitable coordinate system.

We adopt a coordinate system in which the photon propagates along the z-axis, so that $\hat{z} = \hat{k}$, and the external background magnetic field points in an arbitrary direction. Given the magnetic field $\vec{B}$, we can decompose it into two components: the projection along the direction of propagation, $\vec{B}_k = B_k\hat{k}$, and the

transverse projection in the xy-plane, $\vec{B}_T$. The transverse component $\vec{B}_T$ defines a preferred direction in the xy-plane, which we label $\hat{\parallel}$, and from this we define a perpendicular direction, $\hat{\perp}$.

It is convenient to work in the rotated basis $(\hat{\perp}, \hat{\parallel}, \hat{k}) \equiv (e_\perp, e_\parallel, e_k)$, which simplifies both the discussion and the calculations.

Since the magnetic field is external[1], it may have a component along the photon's direction of propagation. In contrast, the electric field associated with the photon lies entirely in the transverse xy-plane. In this new basis, we write:

$$\vec{E} = \vec{E}_\perp + \vec{E}_\parallel, \tag{19}$$

and

$$\vec{B} = \vec{B}_{T_\perp} + \vec{B}_T = \vec{B}_k + \vec{B}_T. \tag{20}$$

Therefore, the scalar product becomes:

$$\vec{E} \cdot \vec{B} = \vec{E} \cdot \vec{B}_T = E_\parallel B_T. \tag{21}$$

Refer to the figure for a clearer illustration of the coordinate system.

ADD A FIGURE

This result is physically important: *Only the transverse component of the external magnetic field contributes to the interaction with the photon, leading to the production of axions.*

In the coordinate basis $(e_\perp, e_\parallel, e_k)$, the electric and magnetic fields take the form $\vec{E} = (E_\perp, E_\parallel, 0)$ and $\vec{B} = (0, B_T, B_k)$. Given the previous result, we are no longer interested in the B_k component. Thus, we can safely assume $\vec{B} = (0, B_T, 0)$, as the B_z component does not play a role in our analysis.

Considering the potential field $A_\mu = (\phi, \vec{A})$, in the chosen coordinate system we have $A_\mu = (\phi, A_\perp, A_\parallel, 0)$. Furthermore, we choose to work in the Lorenz gauge, $\partial_\mu A^\mu = 0$, which allows us to express the electric and (external) magnetic fields as:

$$\partial_\mu A^\mu = 0 \iff \begin{cases} \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \nabla \times \vec{A}. \end{cases} \tag{22}$$

1.2 Rewriting the equations in terms of vectors

We now return to the equations of motion (18), using the coordinate system established above. Let us rewrite the equations for the case where the external magnetic field $\vec{B}$ is transverse[2] to the photon's propagation direction. Additionally, we consider the case with no external source of photons, so we can set $J^\mu = 0$ in the equations (18).

Our goal is to express the equations in terms of $\vec{E} \cdot \vec{B}^{\text{ext}}$ and/or in terms of A^μ. In fact, the first equation in (18) is already written in the desired form, so we now focus on rewriting the second one.

First, we expand the left-hand side explicitly:

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \partial_\mu \partial^\mu A^\nu = \Box A^\nu, \tag{23}$$

[1] The photon wave also generates a magnetic field, but we are not interested in it here, and thus we do not include it.

[2] Again, it does not matter whether $\vec{B}$ has a B_k component or not, since only the transverse component relative to the photon's propagation direction contributes to the interaction.

where in the final step we used the Lorenz gauge condition.

Thus, the modified Maxwell equation in the absence of sources becomes:

$$\Box A^\nu = -g_{a\gamma\gamma}\tilde{F}^{\mu\nu}\partial_\mu a. \tag{24}$$

We now compute this equation component by component.

- **Time component (0th):**

$$\Box A^0 = -g_{a\gamma\gamma}\tilde{F}^{\mu 0}\partial_\mu a\,. \tag{25}$$

We now assume that the scalar potential ϕ is constant and that the propagation of the photon is described by waves in $\vec{A}$ only. For simplicity, we set:

$$A^0 = \phi = \text{const} = 0, \tag{26}$$

so that the 0th component equation vanishes identically.

- **Spatial component (kth):**

$$\Box A^k = -g_{a\gamma\gamma}\tilde{F}^{\mu k}\partial_\mu a. \tag{27}$$

The right-hand side expands as:

$$-g_{a\gamma\gamma}\left(\tilde{F}^{0k}\partial_0 a + \tilde{F}^{jk}\partial_j a\right).$$

From the definition of the dual field strength tensor:

$$\tilde{F}_{0k} = \frac{\varepsilon_{0k\rho\sigma}}{2}F^{\rho\sigma} = -B^k, \tag{28}$$

so the equation becomes:

$$\Box A^k = -g_{a\gamma\gamma}\left(-B^k\partial_0 a + \frac{1}{2}\varepsilon_{jk\rho\sigma}F^{\rho\sigma}\partial_j a\right). \tag{29}$$

Note that $\varepsilon_{jk\rho\sigma} \neq 0$ only when either $\rho = 0$ or $\sigma = 0$, so we can rewrite the second term explicitly as:

$$\begin{aligned}
\frac{1}{2}\varepsilon_{jk\rho\sigma}F^{\rho\sigma} &= \frac{1}{2}\varepsilon_{jk0i}F^{0i} + \frac{1}{2}\varepsilon_{jki0}F^{i0} \\
&= \frac{1}{2}\varepsilon_{jk0i}F^{0i} - \frac{1}{2}\varepsilon_{jki0}F^{i0} \\
&= \varepsilon_{jk0i}F^{0i} \\
&= \varepsilon_{jk0i}(\partial^0 A^i - \partial^i A^0) \\
&= \varepsilon_{jk0i}\partial^0 A^i = -\varepsilon_{jk0i}E^i,
\end{aligned}$$

where we used the antisymmetry of $F^{\mu\nu}$, the condition $A^0 = 0$, and the definition of the electric field $E^i = -\partial^0 A^i$.

Therefore, the equation for the spatial components becomes:

$$\Box A^k = -g_{a\gamma\gamma}\left(-B^k\partial_0 a - \varepsilon_{jk0i}E^i\partial_j a\right), \tag{30}$$

which can be rewritten using index notation as a vector product:

$$\Box A^k = -g_{a\gamma\gamma}\left(-B^k\partial_0 a + \partial_i a \times E^i\right). \tag{31}$$

Finally, the equations of motion (18) can be rewritten in vector form as:

$$\boxed{\begin{cases} (\Box - m_a^2)a = g_{a\gamma\gamma}\vec{E}\cdot\vec{B} \\ \Box\vec{A} = g_{a\gamma\gamma}\left(\vec{B}\,\partial_0 a + \nabla a \times \vec{E}\right) \end{cases}}. \tag{32}$$

1.3 Rewriting the equations in the fixed coordinate system

We now adopt the coordinate system previously defined and consider real wave solutions. Specifically, we assume:

$$\vec{A}(z,t) = (A_\perp(z), A_\parallel(z), 0)e^{i\omega_\gamma t}, \tag{33}$$

where the photon γ is polarized perpendicularly to the direction of propagation. The axion field is generated perpendicular to the photon's plane and thus also depends only on the $z = k$ direction:

$$a(z,t) = a(z)e^{i\omega_a t}. \tag{34}$$

Assuming the system is in resonance, we impose

$$\omega_\gamma = \omega_a = \omega. \tag{35}$$

Again, we take $\vec{B} = (0, B_T, 0)$, although in principle it may also have a component along $z = k$.

- **First equation:**

$$(\Box - m_a^2)a = g_{a\gamma\gamma}\vec{E}\cdot\vec{B}. \tag{36}$$

The left-hand side becomes:

$$(\Box - m_a^2)a(z,t) = \left(-\partial^0\partial_0 a(z,t) + \nabla^2 a(z,t) - m_a^2 a(z,t)\right), \tag{37}$$

and since

$$\nabla^2 a(z,t) = \partial_z^2 a(z,t), \tag{38}$$

and

$$-\partial^0\partial_0 a(z,t) = -a(z)\frac{\partial^2}{\partial t^2}(e^{i\omega t}) = \omega^2 a(z,t), \tag{39}$$

the full expression becomes:

$$(\Box - m_a^2)a(z,t) = (\omega^2 + \partial_z^2 - m_a^2)a(z,t). \tag{40}$$

The right-hand side evaluates to:

$$g_{a\gamma\gamma}\vec{E}\cdot\vec{B} = -g_{a\gamma\gamma}B_T\partial_t A_\parallel(z,t) = -i\omega g_{a\gamma\gamma}B_T A_\parallel(z), \tag{41}$$

so the equation becomes:

$$(\omega^2 + \partial_z^2 - m_a^2)a(z,t) = -i\omega g_{a\gamma\gamma}B_T A_\parallel(z,t). \tag{42}$$

- **Second equation:**

  We now split the second equation of motion into two components, one along $\hat{e}_\perp$ and the other along $\hat{e}_\parallel$.

  In both cases, we neglect the term $\nabla a \times \vec{E}$, as it is subdominant (the axion field varies slowly in space). Thus, we have:

$$\Box A_\perp(z,t) = g_{a\gamma\gamma} B_\perp \partial_0 a = 0, \tag{43}$$

  since $B_\perp = 0$. Making the d'Alembertian explicit:

$$(\omega^2 + \partial_z^2) A_\perp(z,t) = 0. \tag{44}$$

  For the parallel component, still neglecting the $\nabla a \times \vec{E}$ term, we obtain:

$$(\omega^2 + \partial_z^2) A_\parallel(z,t) = i\omega g_{a\gamma\gamma} B_T a(z,t). \tag{45}$$

Thus, the system (18) in the chosen coordinate system becomes:

$$\boxed{\begin{cases} (\omega^2 + \partial_z^2 - m_a^2) a(z,t) = -i\omega g_{a\gamma\gamma} B_T A_\parallel(z,t) \\ (\omega^2 + \partial_z^2) A_\perp(z,t) = 0 \\ (\omega^2 + \partial_z^2) A_\parallel(z,t) = i\omega g_{a\gamma\gamma} B_T a(z,t) \end{cases}.} \tag{46}$$

By applying the transformation $\vec{A} \to i\vec{A} = e^{i\frac{\pi}{2}}\vec{A}$, which corresponds to a phase shift in the photon field, we can further rewrite the system as:

$$\boxed{\begin{cases} (\omega^2 + \partial_z^2 - m_a^2) a(z,t) = \omega g_{a\gamma\gamma} B_T A_\parallel(z,t) \\ (\omega^2 + \partial_z^2) A_\perp(z,t) = 0 \\ (\omega^2 + \partial_z^2) A_\parallel(z,t) = \omega g_{a\gamma\gamma} B_T a(z,t) \end{cases}.} \tag{47}$$

This system can be compactly expressed by defining:

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega g_{a\gamma\gamma} B_T \\ 0 & \omega g_{a\gamma\gamma} B_T & -m_a^2 \end{pmatrix} \quad \text{and} \quad \Psi(z,t) = \begin{pmatrix} A_\perp(z,t) \\ A_\parallel(z,t) \\ a(z,t) \end{pmatrix}, \tag{48}$$

so that the system becomes:

$$\boxed{\left[\omega^2 + \partial_z^2 + \mathcal{M}\right] \Psi(z,t) = 0\ .} \tag{49}$$

We can also make another useful approximation. The first two terms can be factorized as:

$$\omega^2 + \partial_z^2 = (\omega + i\partial_z)(\omega - i\partial_z), \tag{50}$$

which suggests a possible simplification. In fact, the factor $e^{-i\omega t}$ appearing in the components of $\Psi(z,t)$ now behaves as an overall phase (i.e., a constant with respect to the differential operator) and can be factored out. We then reduce the problem to a dependence on z only:

$$\Psi(z) = \begin{pmatrix} A_\perp(z) \\ A_\parallel(z) \\ a(z) \end{pmatrix}. \tag{51}$$

For slowly varying amplitudes (as in this case), where $\partial_z|\vec{A}(z)| \ll k|\vec{A}(z)|$, we can apply the **Wentzel–Kramers–Brillouin (WKB) approximation**. We write[3]

$$\vec{A}(z) = |\vec{A}(z)|e^{-ikz}, \tag{52}$$

implying

$$\tilde{\Psi} = \Psi e^{-ikz}. \tag{53}$$

This allows us to approximate:

$$\partial_z \Psi \sim -ik\Psi + \cancel{\ldots} \quad \Longrightarrow \quad i\partial_z \Psi \sim k\Psi. \tag{54}$$

As a result, we arrive at the approximation:

$$\omega^2 + \partial_z^2 = (\omega + i\partial_z)(\omega - i\partial_z) \simeq (\omega + k)(\omega - i\partial_z). \tag{55}$$

Since $\omega = \frac{k}{n}$, and assuming we are approximately in vacuum ($n \simeq 1$), we take $\omega \simeq k$. This gives:

$$\omega^2 + \partial_z^2 \simeq 2\omega(\omega - i\partial_z). \tag{56}$$

We can then rewrite the equation of motion in the form:

$$\left[\omega - i\partial_z + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{g_{a\gamma\gamma}B_T}{2} \\ 0 & \frac{g_{a\gamma\gamma}B_T}{2} & -\frac{m_a^2}{2\omega} \end{pmatrix}\right] \tilde{\Psi} = 0, \tag{57}$$

with

$$\tilde{\Psi}(z) = \begin{pmatrix} A_\perp(z) \\ A_\parallel(z) \\ a(z) \end{pmatrix} = \begin{pmatrix} |A_\perp(z)| \\ |A_\parallel(z)| \\ |a(z)| \end{pmatrix} e^{-i\omega z} = \Psi e^{-ikz}, \tag{58}$$

where we emphasize that the phase introduced by the WKB approximation does not affect physical probabilities but helps simplify the structure of the equation.

We now define:

$$\Delta_{a\gamma} = \frac{g_{a\gamma\gamma}B_T}{2}, \qquad \Delta_a = -\frac{m_a^2}{2\omega}, \tag{59}$$

so that the mass matrix becomes:

$$\left[\omega - i\partial_z + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta_{a\gamma} \\ 0 & \Delta_{a\gamma} & \Delta_a \end{pmatrix}\right] \tilde{\Psi} = 0 \tag{60}$$

$$\Downarrow$$

$$\boxed{\left[\omega - i\partial_z + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta_{a\gamma} \\ 0 & \Delta_{a\gamma} & \Delta_a \end{pmatrix}\right] \Psi = 0 \ .} \tag{61}$$

[3]Note that the phase could also be defined with the opposite sign, i.e., e^{+ikz}.

This equation is formally equivalent to a **Schrödinger-like equation** for a three-level system described by $\tilde{\Psi}$. Indeed, by distributing the operator:

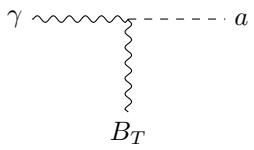
$$\omega\tilde{\Psi} - i\partial_z\tilde{\Psi} + \mathcal{M}\tilde{\Psi} = \cancel{\omega\tilde{\Psi}} - i\partial_z(\Psi)e^{-i\omega z} - \cancel{i(-i\omega)\tilde{\Psi}} + \mathcal{M}\tilde{\Psi} = 0, \tag{62}$$

and simplifying the phase factor, we obtain:

$$\boxed{i\frac{d}{dz}\Psi = \mathcal{M}\Psi \ .} \tag{63}$$

2. Photon-to-axion oscillation

We now aim to show that the evolution equations, written compactly in (63), lead to photon-to-axion (and *vice versa*) oscillations, as represented in the Feynman diagram below.



The equation for $A_\perp$ describes a simple propagation mode and does not couple to either B_T or a. Therefore, we can set it aside and focus on the coupled system described by the Klein–Gordon equation and the equation for $A_\parallel(z)$, which are the relevant components for the mixing dynamics.

2.1 Adding other interaction terms

Up to this point, we have not included quantum corrections to the Lagrangian, such as those arising from the **Euler–Heisenberg** effective Lagrangian, which introduces next-to-leading-order (NLO) corrections to Maxwell's equations. We have also neglected the **Cotton–Mouton effect**.

Including these corrections introduces two additional terms, $\Delta_\parallel = \Delta_\gamma$ and $\Delta_\perp$, modifying the mass matrix $\mathcal{M}$ to

$$\mathcal{M} = \begin{pmatrix} \Delta_\perp & 0 & 0 \\ 0 & \Delta_\parallel & \Delta_{\gamma a} \\ 0 & \Delta_{\gamma a} & \Delta_a \end{pmatrix} = \begin{pmatrix} \Delta_\perp & 0 \\ 0 & \mathcal{M}_\parallel \end{pmatrix}, \tag{64}$$

where

$$\mathcal{M}_\parallel = \begin{pmatrix} \Delta_\parallel & \Delta_{\gamma a} \\ \Delta_{\gamma a} & \Delta_a \end{pmatrix}. \tag{65}$$

The terms $\Delta_\parallel$ and $\Delta_\perp$ can be generalized as:

$$\Delta_* = \Delta_{\text{QED}} + \Delta_{\text{Cotton-Mouton}} + \Delta_{\text{Plasma}} \quad \text{where} \quad * = \{\parallel, \perp\}. \tag{66}$$

In principle, higher-order corrections could also be considered, which would render $\mathcal{M}$ fully populated. However, we will neglect such contributions. The conclusions reached earlier remain valid: the equation for $A_\perp$ continues to describe the free propagation of the photon, possibly modified by QED self-interactions, but it remains decoupled from axion dynamics and can be safely ignored in what follows.

However, the equation for $A_\parallel$ is modified due to these additional interaction terms. In particular: - Δ_{QED} arises from vacuum polarization effects, - Δ_{CM} accounts for birefringence due to the Cotton–Mouton effect, - Δ_{Plasma} describes the photon's effective mass induced by scattering with free electrons in a plasma.

Among these, only Δ_{Plasma} is dominant and essential for describing the interaction between photons and the intergalactic medium, especially in cosmological and astrophysical contexts. Photons propagating through the large-scale structure (LSS) interact with electrons in galactic halos, gaining an effective mass. We thus set

$$\Delta_{\text{QED}} = 0 \quad \text{and} \quad \Delta_{\text{CM}} = 0. \tag{67}$$

Had we included the plasma interaction from the start, an effective photon mass term would have been added to the Lagrangian, appearing in the system (47) as a modification to the second and third equations.

As for the remaining Δ terms, see also (59), we have[4]

$$\Delta_\parallel = \Delta_\gamma = -\frac{m_\gamma^2}{2\omega}. \tag{68}$$

Here, m_γ^2 depends on the electron density encountered by the photon along its trajectory:

$$m_\gamma^2(z) = e^2 \frac{n_e(z)}{m_e} = \frac{4\pi\alpha n_e(z)}{m_e}, \tag{69}$$

leading to

$$\Delta_\parallel = -\frac{1}{2\omega} \cdot \frac{4\pi\alpha n_e(z)}{m_e}, \tag{70}$$

where $n_e(z)$ is the number density of electrons, e is the elementary charge, and m_e is the electron mass.

2.2 Diagonalizing the Schrödinger equation in the case of constant parameters

We now restrict our attention to the reduced two-level Schrödinger equation:

$$\boxed{i\frac{d}{dz}\begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix} = \begin{pmatrix} \Delta_\parallel & \Delta_{\gamma a} \\ \Delta_{\gamma a} & \Delta_a \end{pmatrix}\begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix} = \mathcal{M}_\parallel \begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix}}. \tag{71}$$

In realistic settings, $B_T = B_T(z)$ and $m_\gamma^2 = m_\gamma^2(z) \sim n_e(z)$. However, we simplify the problem by assuming both B_T and n_e are constant. In this idealized case, the Hamiltonian $H = \mathcal{M}_\parallel$ is symmetric and constant, making it straightforward to diagonalize and solve the system for $A_\parallel(z)$ and $a(z)$.

The Hamiltonian can be diagonalized as:

$$D = P^{-1}\mathcal{M}_\parallel P = P^T \mathcal{M}_\parallel P = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \tag{72}$$

where P is a unitary matrix ($\det P = 1$ and $P^T = P^{-1}$) representing a basis rotation:

$$P = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}. \tag{73}$$

[4] We use $\Delta_\parallel$ and Δ_γ interchangeably, following standard conventions in the literature.

To find the eigenvalues $\lambda_\pm$, we solve the characteristic equation:

$$\det\left(\mathcal{M}_\parallel - \lambda \not{K}\right) = 0 \quad \Rightarrow \quad (\Delta_\parallel - \lambda)(\Delta_a - \lambda) - \Delta_{\gamma a}^2 = 0, \tag{74}$$

which yields:

$$\lambda^2 - (\Delta_a + \Delta_\parallel)\lambda + (\Delta_\parallel \Delta_a - \Delta_{\gamma a}^2) = 0. \tag{75}$$

Solving this quadratic equation gives:

$$\lambda_\pm = \frac{\Delta_a + \Delta_\parallel \pm \sqrt{(\Delta_a - \Delta_\parallel)^2 + 4\Delta_{\gamma a}^2}}{2} = \frac{\Delta_a + \Delta_\parallel \pm \Delta_{\rm osc}}{2}, \tag{76}$$

where we define:

$$\Delta_{\rm osc} = \sqrt{(\Delta_a - \Delta_\parallel)^2 + 4\Delta_{\gamma a}^2}. \tag{77}$$

The mixing angle ϑ satisfies:

$$\sin(2\vartheta) = \frac{2\Delta_{\gamma a}}{\Delta_{\rm osc}}, \qquad \cos(2\vartheta) = \frac{\Delta_\parallel - \Delta_a}{\Delta_{\rm osc}}, \tag{78}$$

which implies:

$$\tan(2\vartheta) = \frac{2\Delta_{\gamma a}}{\Delta_\parallel - \Delta_a}. \tag{79}$$

[5]

This result is particularly relevant in vacuum, i.e., in the absence of plasma ($\Delta_\parallel = 0$) and under a constant magnetic field:

$$\sin(2\vartheta) = \frac{2g_{a\gamma\gamma}B_T\omega}{\sqrt{m_a^4 + (2g_{a\gamma\gamma}B_T\omega)^2}}, \qquad \cos(2\vartheta) = \frac{m_a^2}{\sqrt{m_a^4 + (2g_{a\gamma\gamma}B_T\omega)^2}}, \tag{80}$$

$$\Downarrow$$

$$\tan(2\vartheta) = \frac{2g_{a\gamma\gamma}B_T\omega}{m_a^2}. \tag{81}$$

2.3 Solving the Schrödinger equation in the case of constant quantities

Once the Hamiltonian $H = \mathcal{M}_\parallel$ is diagonalized in the simple case of z-independent quantities, it becomes possible to solve the Schrödinger equation explicitly. The solution to (98) follows from solving the corresponding diagonalized equation (with components denoted by tildes):

$$i\frac{d}{dz}\begin{pmatrix}\tilde{A}_\parallel(z)\\ \tilde{a}(z)\end{pmatrix} = \begin{pmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}\begin{pmatrix}\tilde{A}_\parallel(z)\\ \tilde{a}(z)\end{pmatrix}. \tag{82}$$

In general, the solution $\Psi(z) = \begin{pmatrix}\Psi_1(z)\\ \Psi_2(z)\end{pmatrix}$ of the non-diagonalized system is recovered from the solution $\tilde{\Psi}(z) = \begin{pmatrix}\tilde{\Psi}_1(z)\\ \tilde{\Psi}_2(z)\end{pmatrix}$ of the diagonalized one via the transformation:

$$\Psi = P\tilde{\Psi}. \tag{83}$$

[5] This is the standard diagonalization procedure for a symmetric 2×2 matrix. For a matrix $A = \begin{pmatrix}a & c\\ c & b\end{pmatrix}$, the mixing angle satisfies $\tan(2\vartheta) = \frac{2c}{a-b}$.

Since the Schrödinger equation describes wave-like propagation, the time evolution (in this context, along z) of the eigenstates is:

$$\tilde{\Psi}_i(z) = \tilde{\Psi}_i(z_0) e^{-i\lambda_i z}, \tag{84}$$

with $\lambda_1 \equiv \lambda_+$ and $\lambda_2 \equiv \lambda_-$. Substituting into (83), the solution becomes:

$$\Psi_i(z) = \sum_{j=1}^{2} P_{ij}^{-1} \tilde{\Psi}_j(z) = \sum_{j=1}^{2} P_{ij}^{-1} P_{kj} \Psi_j(z_0) e^{-i\lambda_j z} = \sum_{j=1}^{2} P_{ij}^{-1} [P\Psi(z_0)]_j e^{-i\lambda_j z}. \tag{85}$$

In our case:

$$\begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix} = P \begin{pmatrix} e^{-i\lambda_+ z} & 0 \\ 0 & e^{-i\lambda_- z} \end{pmatrix} P^{-1} \begin{pmatrix} A_\parallel(z_0) \\ a(z_0) \end{pmatrix}. \tag{86}$$

By specifying initial and final states appropriately, we describe the desired physical process. For example, taking

$$\Psi_0 = \begin{pmatrix} A_\parallel(z_0) \\ a(z_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Psi(z) = \begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix}, \tag{87}$$

we model a scenario in which the initial state at z_0 contains only a photon, and we compute the probability of conversion into an axion at position z.

Carrying out the matrix multiplications yields:

$$\begin{cases} A_\parallel(z) = \cos\vartheta \sin\vartheta \left(e^{-i\lambda_+ z} - e^{-i\lambda_- z}\right) a(z_0) + \left(\sin^2\vartheta\, e^{-i\lambda_+ z} + \cos^2\vartheta\, e^{-i\lambda_- z}\right) A_\parallel(z_0), \\ a(z) = \left(\cos^2\vartheta\, e^{-i\lambda_+ z} + \sin^2\vartheta\, e^{-i\lambda_- z}\right) a(z_0) + \cos\vartheta \sin\vartheta \left(e^{-i\lambda_+ z} - e^{-i\lambda_- z}\right) A_\parallel(z_0), \end{cases} \tag{88}$$

which in the case $\Psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ reduces to:

$$\begin{cases} A_\parallel(z) = \sin^2\vartheta\, e^{-i\lambda_+ z} + \cos^2\vartheta\, e^{-i\lambda_- z}, \\ a(z) = \cos\vartheta \sin\vartheta \left(e^{-i\lambda_+ z} - e^{-i\lambda_- z}\right). \end{cases} \tag{89}$$

We are now in a position to compute the conversion probability. Denoting $|\gamma\rangle \equiv |\Psi_0\rangle$ and $|a\rangle \equiv |\Psi\rangle$, we obtain:

$$P_{\gamma\to a}(z) = |\langle a|\gamma\rangle|^2 = |a(z)|^2 \tag{90}$$

$$= 2\cos^2\vartheta \sin^2\vartheta\, \sin^2\left(\frac{\lambda_- - \lambda_+}{2} z\right) \tag{91}$$

$$= \sin^2(2\vartheta)\, \sin^2\left(\frac{\Delta_{\rm osc}}{2} z\right) \tag{92}$$

$$= \frac{(\Delta_{\gamma a} z)^2}{\left(\frac{\Delta_{\rm osc}}{2} z\right)^2} \sin^2\left(\frac{\Delta_{\rm osc}}{2} z\right), \tag{93}$$

where in the last step we used the expression for $\sin(2\vartheta)$ in terms of $\Delta_{\gamma a}$ and $\Delta_{\rm osc}$.

This result assumes a constant transverse magnetic field B_T and a constant effective photon mass m_γ^2 (i.e., constant electron density n_e) over a domain of size $L_{\rm dom} = z$. The conversion probability across this domain is therefore:

$$P_{\gamma\to a}(L_{\rm dom}) = \frac{(g_{a\gamma\gamma} B_T L_{\rm dom})^2}{(\Delta_{\rm osc} L_{\rm dom})^2} \sin^2\left(\frac{\Delta_{\rm osc}}{2} L_{\rm dom}\right). \tag{94}$$

A particularly important case is that of **resonance**, where

$$\text{Resonance}: \quad m_a = m_\gamma, \tag{95}$$

which implies $\vartheta = \frac{\pi}{4}$, and the probability simplifies to:

$$P^{\text{res}}_{\gamma\to a} = \sin^2\left(\Delta_{\gamma a} L_{\text{dom}}\right) = \sin^2\left(\frac{g_{a\gamma\gamma} B_T}{2} L_{\text{dom}}\right), \tag{96}$$

a particularly elegant result as it is independent of the frequency ω.

Substituting the relevant parameters confirms the qualitative insight:

$$P^{\text{res}}_{\gamma\to a} \gg P_{\gamma\to a}, \tag{97}$$

highlighting that **when resonance occurs, the conversion probability is significantly enhanced**. This emphasizes the relevance of the resonant regime in both theoretical analyses and observational searches for photon–axion conversion.

3. The real two-level photon-to-axion system

In the previous section, we derived the probability of photon-to-axion conversion in the case of constant quantities—namely, constant $m_\gamma^2 \sim n_e$ and B_T.

In the realistic case, however, these quantities depend on the spatial coordinate z: $B_T = B_T(z)$ and $m_\gamma^2 = m_\gamma^2(z) \sim n_e(z)$. Consequently, the Hamiltonian $H = \mathcal{M}_\parallel$ is also z-dependent.[6]

The Schrödinger equation (98) is thus more accurately expressed as:

$$\boxed{i\frac{d}{dz}\begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix} = \begin{pmatrix} \Delta_\parallel(z) & \Delta_{\gamma a}(z) \\ \Delta_{\gamma a}(z) & \Delta_a \end{pmatrix}\begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix} = \mathcal{M}_\parallel(z)\begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix}}, \tag{98}$$

where

$$\begin{cases} \Delta_\parallel(z) = -\frac{1}{2\omega}\left(\frac{4\pi\alpha n_e(z)}{m_e}\right) = -\frac{m_\gamma^2(z)}{2\omega}, \\ \Delta_{\gamma a}(z) = \frac{g_{a\gamma\gamma}}{2} B_T(z), \\ \Delta_a = -\frac{m_a^2}{2\omega}. \end{cases} \tag{99}$$

In this case, it is no longer possible to diagonalize the Hamiltonian globally, due to its dependence on z (or analogously, on t). Therefore, the problem must be treated differently.

As we will show in the next sections, it is still possible to reformulate the problem in terms of a second-order differential equation. Under certain simplifying assumptions, this equation yields an approximate but tractable solution for the conversion probability.

The appropriate framework for analyzing such a situation is provided by the **Landau-Zener approximation**, developed independently by Landau and Zener in 1938. The probability formula derived under this approximation is known as the **Landau-Zener formula**.

This section follows the discussion in [1], together with the original, pedagogically clear treatment by Zener [2].

[6] An identical treatment can be carried out for the full three-level system, but we continue focusing on the two-level case.

3.1 General case and Landau-Zener approximation

Let us consider a general two-level system described by a time-dependent Hamiltonian $H_0(t)$, characterized by instantaneous energy levels $E_i(t)$:

$$H_0(t) = \begin{pmatrix} E_1(t) & 0 \\ 0 & E_2(t) \end{pmatrix}. \tag{100}$$

Suppose the system is initially prepared in the state $|E_2\rangle$. At a given time t_0, the energy levels cross:

$$E_1(t_0) = E_2(t_0). \tag{101}$$

At this point, a transition $\psi_2 \to \psi_1$ becomes possible.

Now consider adding a perturbation $V(t)$ to the system, such that the full Hamiltonian becomes $H(t) = H_0(t) + V(t)$. This perturbation prevents an exact level crossing, transforming it into an *avoided crossing*. The system, initially in ψ_2 with energy E_2, ends up in a superposition:

$$\psi(t \to \infty) = \alpha\psi_1 + \beta\psi_2,$$

with transition probability $P = |\alpha|^2$ and survival probability $|\beta|^2$, subject to normalization: $|\alpha|^2 + |\beta|^2 = 1$.

The full time-dependent Hamiltonian is then:

$$H(t) = \begin{pmatrix} \varepsilon_1(t) & \varepsilon_{12}(t) \\ \varepsilon_{12}(t) & \varepsilon_2(t) \end{pmatrix}, \tag{102}$$

and the evolution is governed by the Schrödinger equation:

$$i\frac{d}{dt}\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = H(t)\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}. \tag{103}$$

This is formally equivalent to (98).

Following Zener [2], it is convenient to define:

$$\Psi_1(t) = e^{-i\int \varepsilon_1(t)dt}\psi_1(t), \qquad \Psi_2(t) = e^{-i\int \varepsilon_2(t)dt}\psi_2(t). \tag{104}$$

This redefinition only introduces a global phase and does not affect transition probabilities.

The transformed system reads:

$$i\frac{d}{dt}\begin{pmatrix} \Psi_1(t) \\ \Psi_2(t) \end{pmatrix} = H(t)\begin{pmatrix} \Psi_1(t) \\ \Psi_2(t) \end{pmatrix}, \tag{105}$$

which gives:

$$\begin{cases} i\dot{\psi}_1(t) = \varepsilon_{12}(t)\psi_2(t)e^{-i\int \Delta\varepsilon(t)dt}, \\ i\dot{\psi}_2(t) = \varepsilon_{12}(t)\psi_1(t)e^{+i\int \Delta\varepsilon(t)dt}, \end{cases} \tag{106}$$

with $\Delta\varepsilon(t) = \varepsilon_2(t) - \varepsilon_1(t)$.

Differentiating again and substituting yields the second-order equation:

$$\boxed{\ddot{\psi}_1 + \left[i\Delta\varepsilon - \frac{\dot{\varepsilon}_{12}}{\varepsilon_{12}}\right]\dot{\psi}_1 + \varepsilon_{12}^2\psi_1 = 0.} \tag{107}$$

This equation is exact but difficult to solve. The **Landau-Zener approximation** introduces two simplifications:

1. **Linear crossing:** Near t_0, the energy difference is approximated as linear:

$$\Delta\varepsilon(t) \approx \varepsilon_0 \frac{t}{t_0}.$$

2. **Slow coupling variation:** ε_{12} varies slowly compared to $\Delta\varepsilon(t)$ and is treated as constant.

With these assumptions, (107) reduces to Weber's equation:

$$\frac{d^2\Psi_1}{dw^2} + \left(1 - \frac{1}{2n} - \frac{w^2}{4n^2}\right)\Psi_1 = 0, \tag{108}$$

with the definitions:

$$\Psi_1 = e^{\frac{i}{2}\int \Delta\varepsilon dt}\psi_1, \quad w = \sqrt{\frac{\varepsilon_0}{t_0}}t, \quad \gamma = \frac{\varepsilon_{12}^2}{|\frac{d}{dt}\Delta\varepsilon|}, \quad n = i\gamma. \tag{109}$$

Assuming initial conditions:

$$\psi_1(-\infty) = 0, \qquad |\psi_2(-\infty)|^2 = 1, \tag{110}$$

the solution is expressed in terms of parabolic cylinder functions:

$$\Psi_1(w) = \sqrt{\gamma}e^{-\frac{\pi}{4}\gamma}\mathcal{D}_{-1-n}\left(\frac{iw}{\sqrt{n}}\right). \tag{111}$$

The Landau-Zener transition probability is then:

$$\boxed{P_{\psi_2\to\psi_1} = \lim_{w\to+\infty}|\Psi_1(w)|^2 = 1 - e^{-2\pi\gamma}.} \tag{112}$$

3.2 Solving the photon-to-axion system

We now apply this formalism to the photon-to-axion system (98), using the redefinition:

$$\begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix} = \begin{pmatrix} e^{-i\int_{z_i}^{z}\Delta_\parallel(z')dz'}\tilde{A}_\parallel(z) \\ e^{-i\int_{z_i}^{z}\Delta_a\, dz'}\tilde{a}(z) \end{pmatrix}. \tag{113}$$

This yields:

$$\begin{cases} iA'_\parallel(z) = \Delta_{\gamma a}(z)\, a(z)\, e^{-i\int_{z_i}^{z}(\Delta_a - \Delta_\parallel(z'))dz'}, \\ ia'(z) = \Delta_{\gamma a}(z)\, A_\parallel(z)\, e^{+i\int_{z_i}^{z}(\Delta_a - \Delta_\parallel(z'))dz'}. \end{cases} \tag{114}$$

Defining $\Delta\varepsilon(z) = \Delta_a - \Delta_\parallel(z)$, and using prime to denote z-derivatives, we derive a second-order ODE for $A_\parallel(z)$:

$$\boxed{A''_\parallel(z) + \left[i(\Delta_a - \Delta_\parallel(z)) - \frac{\Delta'_{\gamma a}}{\Delta_{\gamma a}}\right]A'_\parallel(z) + \Delta^2_{\gamma a}(z)A_\parallel(z) = 0.} \tag{115}$$

Alternatively, working with $a(z)$, we find:

$$\boxed{a''(z) + \left[i(\Delta_\parallel(z) - \Delta_a) - \frac{\Delta'_{\gamma a}(z)}{\Delta_{\gamma a}(z)}\right]a'(z) + \Delta^2_{\gamma a}(z)a(z) = 0.} \tag{116}$$

With initial conditions:

$$\begin{cases} A_\parallel(z_i) = 1, \\ a(z_i) = 0, \\ a'(z_i) = -i\Delta_{\gamma a}(z_i). \end{cases} \tag{117}$$

Equations (115) and (116) are the **master equations** governing photon-to-axion conversion. They are equivalent and can be used interchangeably. Importantly, they are derived without any approximation.

4. Approaches to the master equation

In this section, we discuss various approaches to solving the master equation (116) (or equivalently (115)).

4.1 Constant magnetic field and electron density

In this case, both the transverse magnetic field and the electron number density are constant: $B_T(z) = B_T$ and $n_e(z) = n_e$. This reduces the problem to the constant-coefficient case studied analytically in Section 2.3.

The master equation (116) becomes:

$$a''(z) + i(\Delta_\parallel - \Delta_a)a'(z) + \Delta_{a\gamma}^2 a(z) = 0, \tag{118}$$

where all the Δ parameters are constant. The initial conditions are given in (117).

The solution to this equation is given in (89), and the corresponding probability is provided in (96).

4.2 Complete Landau-Zener approach

In the general case governed by the master equation (107), we showed that, under the **Landau-Zener approximation**, one can derive an analytic expression for the transition probability, known as the **Landau-Zener formula** (112).

In our specific case, the two simplifying assumptions of the Landau-Zener approximation are:

1. **Simplification 1:** The resonance at z_0 occurs over a short enough interval such that the energy gap

$$\Delta\varepsilon(z) = \Delta_a - \Delta_\parallel(z) \tag{119}$$

can be approximated as linear:

$$\Delta\varepsilon(z) \sim \Delta_0 \frac{z}{z_0}. \tag{120}$$

Since $\Delta_\parallel(z) \sim m_\gamma^2(z) \sim n_e(z)$, this implies a linear electron density:

$$n_e(z) \sim n_e(z_0)\frac{z}{z_0}. \tag{121}$$

2. **Simplification 2:** The photon-axion coupling term $\Delta_{\gamma a}(z)$ varies slowly compared to $\Delta\varepsilon(z)$, so it can be approximated as constant:

$$\Delta'_{\gamma a}(z) \simeq 0 \quad \Longleftrightarrow \quad \Delta_{\gamma a} \sim \text{const.} \tag{122}$$

That is,

$$B_T = \text{const.} \tag{123}$$

Under these assumptions, the master equation becomes:

$$\boxed{a''(z) + i(\Delta_\parallel(z) - \Delta_a)a'(z) + \Delta_{a\gamma}^2 a(z) = 0,} \tag{124}$$

where $\Delta_{a\gamma}^2$ is constant and $\Delta_\parallel(z)$ is linear. The initial conditions remain those in (117).

A known issue is that the original Landau-Zener formula assumes asymptotic boundary conditions. Nevertheless, if we assume that our initial and final conditions effectively approximate asymptotic behavior, we can adopt the Landau-Zener formula for the $\gamma \to a$ transition:

$$P_{\gamma\to a} = 1 - e^{-2\pi\gamma}, \quad \text{with} \quad \gamma = \frac{\Delta_{a\gamma}^2}{\left|\frac{d\Delta_\parallel(z)}{dz}\right|}. \tag{125}$$

In most realistic cases, we expect $\gamma \ll 1$, so we can Taylor-expand the exponential to first order:

$$P \simeq 2\pi\gamma. \tag{126}$$

This can be recast in a more convenient form widely used in cosmology and astroparticle physics (see, e.g., [3, 4]).

We recall that:

$$\frac{d\Delta_\parallel}{dz} = -\frac{1}{2\omega}\frac{4\pi\alpha}{m_e}\frac{dn_e(z)}{dz} = -\frac{1}{2\omega}\frac{dm_\gamma^2(z)}{dz}. \tag{127}$$

At resonance, we impose $\Delta_\parallel(z_{\rm res}) = \Delta_a$, and for simplicity we can set $z_{\rm res} = 0$. This leads to:

$$\Delta_\varepsilon(z_{\rm res}) = 0 \quad \Rightarrow \quad n_e(z_{\rm res}) = \frac{m_a^2 m_e}{4\pi\alpha}. \tag{128}$$

Substituting into the formula for γ and the probability:

$$\begin{aligned} P_{\gamma\to a} &= \frac{2\pi\,\frac{g_{a\gamma\gamma}^2}{4}B_T^2}{\frac{1}{2\omega}\frac{dm_\gamma^2}{dz}} = \frac{2\pi\,\frac{g_{a\gamma\gamma}^2}{4}B_T^2}{\frac{1}{2\omega}\,m_\gamma^2(z_{\rm res})\left|\frac{d\ln m_\gamma^2}{dz}\right|_{z_{\rm res}}} \\ &= \frac{2\pi\,\frac{g_{a\gamma\gamma}^2}{4}B_T^2}{\frac{1}{2\omega}\frac{4\pi\alpha}{m_e}\,n_e(z_{\rm res})}\left|\frac{d\ln m_\gamma^2}{dz}\right|_{z_{\rm res}}^{-1} \\ &= \frac{2\pi\,\frac{g_{a\gamma\gamma}^2}{4}B_T^2}{\frac{1}{2\omega}\frac{4\pi\alpha}{m_e}\cdot\frac{m_a^2 m_e}{4\pi\alpha}}\left|\frac{d\ln m_\gamma^2}{dz}\right|_{z_{\rm res}}^{-1}. \end{aligned}$$

$$\Downarrow$$

$$\boxed{P_{\gamma\to a}^{\rm res} \sim \frac{\omega\pi g_{a\gamma\gamma}^2 B_T^2}{m_a^2}\left|\frac{d\ln m_\gamma^2}{dz}\right|_{z_{\rm res}}^{-1}}. \tag{129}$$

This is the most widely used expression for resonant photon-to-axion conversion in the cosmology and astroparticle physics literature. While this specific form appears in [3] for axion conversion, a similar structure arises in photon-to-massive dark photon oscillations [4].

4.3 Non-asymptotic Landau-Zener approach

One may still apply the Landau-Zener approach to derive a solution, but now explicitly taking into account that the initial conditions are not asymptotic. While the form of the differential equation (a Weber equation) remains the same, the different boundary conditions lead to a different solution.

This refined treatment was developed by Carenza and Marsh in [1] for the case of $a \to \gamma$. As they write:

"The only difference between the equation in this case and (107) are the initial conditions. In (107), we considered asymptotic initial conditions, while those here are set at a finite distance from the would-be level crossing. This has a technical but important consequence: the solution, (111), oscillates asymptotically and does not have a well-defined first derivative as $w \to \infty$, while the new Weber equation sets $w = -i$ at the initial point. [...] The conversion probability is now more complicated, and does not reduce to the familiar form of the LZ formula."

The authors derive the resulting solution and compute the transition probability by squaring the amplitude. Compared to the asymptotic case, the probability now exhibits oscillatory behavior and deviates from the standard Landau-Zener result.

5. Photon-to-dark photon conversion

Up to this point, we have focused on the case of photon-to-axion conversion. However, the same physical formalism applies to the conversion of photons into massive dark photons. In this context, the relevant Lagrangian is given by:

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - \frac{m_{\mathcal{A}}^2}{2}\mathcal{A}_\mu\mathcal{A}^\mu - \frac{\varepsilon}{2}F_{\mu\nu}F'^{\mu\nu}, \tag{130}$$

where $\mathcal{A}_\mu$ denotes the dark photon field, and ε is the kinetic mixing parameter.

Unlike the axion case, where the interaction is mediated by an external magnetic field through a Chern-Simons-like term, the coupling here is direct via the kinetic mixing term $\frac{\varepsilon}{2}F_{\mu\nu}F'^{\mu\nu}$. Therefore, an external magnetic field is not required to trigger the transition.

By deriving the equations of motion, one finds that the dynamics of the photon–dark photon system are governed by a Schrödinger-like equation structurally analogous to the axion case:

$$i\frac{d}{dz}\begin{pmatrix} A(z) \\ \mathcal{A}(z) \end{pmatrix} = \begin{pmatrix} \Delta(z) & \varepsilon\Delta_{\mathcal{A}} \\ \varepsilon\Delta_{\mathcal{A}} & \Delta_{\mathcal{A}} \end{pmatrix}\begin{pmatrix} A(z) \\ \mathcal{A}(z) \end{pmatrix}, \tag{131}$$

where

$$\Delta = \Delta_\parallel = -\frac{m_{\text{eff}}^2}{2\omega}, \qquad \Delta_{\mathcal{A}} = -\frac{m_{\mathcal{A}}^2}{2\omega}. \tag{132}$$

As a result, the equations governing dark photon dynamics can be obtained directly from those of the axion case via the substitution:

$$\Delta_{a\gamma} \quad \longrightarrow \quad \varepsilon\Delta_{\mathcal{A}}. \tag{133}$$

For instance, within the Landau-Zener framework, the adiabaticity parameter γ for photon-to-axion conversion is:

$$\gamma_{\gamma\to a} = \frac{\Delta_{a\gamma}^2}{\left|\frac{d\Delta_\parallel}{dz}\right|}, \tag{134}$$

which becomes, in the dark photon case:

$$\gamma_{\gamma\to\mathcal{A}} = \frac{\varepsilon^2\Delta_{\mathcal{A}}^2}{\left|\frac{d\Delta}{dz}\right|}. \tag{135}$$

This correspondence allows the full machinery developed for photon–axion mixing—including exact solutions, approximations, and numerical methods—to be applied directly to the case of dark photon production.

6. Interference and phase terms for multiple level crossings

In [5], a novel method for treating the Landau-Zener approximation is introduced. This approach not only recovers the standard Landau-Zener transition probability but also accounts for possible interference effects between the two states $|\gamma\rangle$ and $|a\rangle$.

The authors begin (and clarify the derivation) from the standard perturbed two-level Hamiltonian:

$$H = H_0 + H_1 = \begin{pmatrix} \Delta_\parallel(z) & \Delta_{\gamma a}(z) \\ \Delta_{\gamma a}(z) & \Delta_a \end{pmatrix} = \begin{pmatrix} \Delta_\parallel(z) & 0 \\ 0 & \Delta_a \end{pmatrix} + \begin{pmatrix} 0 & \Delta_{\gamma a}(z) \\ \Delta_{\gamma a}(z) & 0 \end{pmatrix}, \tag{136}$$

where H_1 is treated as a perturbation. Since $\Delta_{a\gamma} \ll \Delta_\parallel, \Delta_a$, and thus $H_1 \ll H_0$, perturbation theory is applicable.

The analysis proceeds in the interaction picture, where the evolution equation becomes:

$$H_{\rm int} \begin{pmatrix} A \\ a \end{pmatrix} = i\frac{d}{dz} \begin{pmatrix} A_{\rm int} \\ a_{\rm int} \end{pmatrix}, \tag{137}$$

with the field redefinition:

$$\begin{pmatrix} A_{\rm int} \\ a_{\rm int} \end{pmatrix} = \mathcal{U}^\dagger(z) \begin{pmatrix} A \\ a \end{pmatrix}, \tag{138}$$

and the interaction Hamiltonian defined by:

$$H_{\rm int} = \mathcal{U}^\dagger(z) H_1 \mathcal{U}(z), \quad \mathcal{U}(z) = e^{-i\int_{z_i}^{z} dz'\, H_0(z')}. \tag{139}$$

This is equivalent to the phase redefinitions performed in the works of Zener and Carenza–Marsh [1, 2] (cf. equations (104) and (113)).

The initial condition is set by:

$$\begin{pmatrix} A_i \\ a_i \end{pmatrix} = \mathcal{U}(z_i) \begin{pmatrix} A_{{\rm int},i} \\ a_{{\rm int},i} \end{pmatrix} = \begin{pmatrix} A_{{\rm int},i} \\ a_{{\rm int},i} \end{pmatrix}, \tag{140}$$

where we assume $\mathcal{U}(z_i) = \mathbb{1}$ for simplicity.

The time evolution in the interaction picture is then governed by:

$$\mathcal{U}_{\rm int} = e^{-i\int_{z_i}^{z} dz'\, H_{\rm int}(z')}, \tag{141}$$

so the state evolves as:

$$\begin{pmatrix} A_{\rm int}(z) \\ a_{\rm int}(z) \end{pmatrix} = \mathcal{U}_{\rm int} \begin{pmatrix} A(z_i) \\ a(z_i) \end{pmatrix}. \tag{142}$$

Switching back to the Schrödinger picture:

$$\begin{pmatrix} A \\ a \end{pmatrix} = \mathcal{U}(z) \begin{pmatrix} A_{\text{int}} \\ a_{\text{int}} \end{pmatrix}, \tag{143}$$

we arrive at:

$$\begin{pmatrix} A(z) \\ a(z) \end{pmatrix} = e^{-i\int_{z_i}^{z} dz'\, H_0(z')} e^{-i\int_{z_i}^{z} dz'\, H_{\text{int}}(z')} \begin{pmatrix} A(z_i) \\ a(z_i) \end{pmatrix}. \tag{144}$$

Using the Baker–Campbell–Hausdorff formula and assuming first-order perturbation in $\Delta_{a\gamma}$, the interaction Hamiltonian becomes:

$$H_{\text{int}}(z) = \Delta_{a\gamma}(z) \begin{pmatrix} 0 & e^{i\Phi(z)} \\ e^{-i\Phi(z)} & 0 \end{pmatrix}, \tag{145}$$

where the phase is defined as:

$$\Phi(z) = \int_{z_i}^{z} dz' \, [\Delta_a - \Delta_{\parallel}(z')]. \tag{146}$$

This leads to a system equivalent to the integrated form of (**??**):

$$\begin{pmatrix} A(z) \\ a(z) \end{pmatrix} = \begin{pmatrix} 1 & -ic_+ \\ ic_- e^{i\Phi(z)} & e^{i\Phi(z)} \end{pmatrix} \begin{pmatrix} A(z_i) \\ a(z_i) \end{pmatrix} + \mathcal{O}(\Delta_{a\gamma}^2), \tag{147}$$

where:

$$c_{\pm} = \int_{z_i}^{z} dz' \, \Delta_{a\gamma}(z') e^{\pm i\Phi(z')}. \tag{148}$$

From this, the transition probability is directly extracted from the modulus square of the off-diagonal term:

$$\boxed{P_{\gamma\to a} = \left| \int_{z_i}^{z} dz' \, \Delta_{a\gamma}(z') e^{i\Phi(z')} \right|^2 + \mathcal{O}(\Delta_{a\gamma}^2).} \tag{149}$$

This is an exact result, obtained without any approximations.

If there is only a single resonance point, then applying the Landau-Zener approximation yields:

$$P_{\gamma\to a} \simeq 2\pi\gamma. \tag{150}$$

However, if multiple resonance points are present, the total probability is not simply the sum of the individual Landau-Zener probabilities:

$$P_{\gamma\to a} \not\simeq \sum_n 2\pi\gamma_n. \tag{151}$$

This discrepancy arises due to interference between wave packets, which can either enhance or suppress the transition probability (constructive or destructive interference).

The resonance points correspond to stationary points of the phase $\Phi(z)$, satisfying $\Phi'(z_n) = 0$, i.e., where $\Delta_{\parallel}(z_n) = \Delta_a$.

To evaluate (149), we use the **stationary phase approximation**, expanding:

$$\Phi(z') \approx \Phi(z_n) + \frac{1}{2}\Phi''(z_n)(z' - z_n)^2, \tag{152}$$

$$\Delta_{a\gamma}(z') \approx \Delta_{a\gamma}(z_n). \tag{153}$$

This gives:

$$P_{\gamma\to a} = \left| \int dz' \, \Delta_{a\gamma}(z_n) e^{i\Phi(z_n) + \frac{i}{2}\Phi''(z_n)(z'-z_n)^2} \right|^2 \tag{154}$$

$$= \left| \Delta_{a\gamma} e^{i\Phi(z_n)} \int dz' \, e^{\frac{i}{2}\Phi''(z_n)(z'-z_n)^2} \right|^2 . \tag{155}$$

The integral is a Fresnel-type Gaussian:

$$\int_{\mathbb{R}} e^{\frac{i}{2}cx^2} dx = \sqrt{\frac{2\pi i}{c}} = \sqrt{\frac{2\pi}{|c|}} e^{i\frac{\pi}{4}\mathrm{sign}(c)}. \tag{156}$$

Thus, the probability becomes:

$$P_{\gamma\to a} = \left| \Delta_{a\gamma} \sqrt{\frac{2\pi}{|\Phi''(z_n)|}} e^{i\Phi(z_n) + i\frac{\pi}{4}\mathrm{sign}(\Phi''(z_n))} \right|^2 . \tag{157}$$

Defining:

$$\gamma_n = \frac{\Delta_{a\gamma}^2(z_n)}{|\Phi''(z_n)|}, \quad A_n = 2\pi\gamma_n, \tag{158}$$

we recover:

$$P_{\gamma\to a} = 2\pi\gamma_n, \tag{159}$$

in the single-resonance case.

In the case of two resonance points z_1 and z_2, we find:

$$P_{\gamma\to a} = \left| \sum_{n=1}^{2} \sqrt{2\pi\gamma_n} e^{i\Phi(z_n) + i\frac{\pi}{4}\mathrm{sign}(\Phi''(z_n))} \right|^2 \tag{160}$$

$$= 2\pi(\gamma_1 + \gamma_2) + 2\sqrt{(2\pi\gamma_1)(2\pi\gamma_2)} \cos(\Phi_{12}), \tag{161}$$

where:

$$\Phi_{12} = \Phi(z_1) - \Phi(z_2) + \frac{\pi}{4} \left[\mathrm{sign}(\Phi''(z_1)) - \mathrm{sign}(\Phi''(z_2)) \right]. \tag{162}$$

Therefore, in general, the total transition probability is not additive.

In the general case of N resonance points:

$$\boxed{P_{\gamma\to a} = \left| \sum_n \sqrt{2\pi\gamma_n} e^{i\Phi(z_n) + i\frac{\pi}{4}\mathrm{sign}(\Phi''(z_n))} \right|^2 = \sum_n 2\pi\gamma_n + 2\sum_{n<m} \sqrt{4\pi^2\gamma_n\gamma_m} \cos(\Phi_{nm}).} \tag{163}$$

7. Numerical approach

Solving the exact master equation analytically, without any approximation, is extremely difficult—if not impossible. However, a numerical approach can be developed. Obtaining numerical results is crucial, as it allows us to incorporate any functional form of $B_T(z)$ and $n_e(z)$, and from the solution, extract the corresponding probability.

7.1 Numerical approach in the case of Landau-Zener approximation

In the Landau-Zener approximation, the key parameter is the factor γ. In this context, Carenza and Marsh [1] used a reparametrization involving a phase of the state and expressed the master equation in units of $\Delta_{a\gamma}$. Starting from the master equation (124), one can define the following quantity:

$$u(z) = e^{\frac{i}{2}\int_{z_i}^{z} dz'(\Delta_\parallel - \Delta_a)} a(z), \tag{164}$$

and, under the Landau-Zener approximation, we can write (cf. (119)):

$$\Delta_\parallel - \Delta_a = \Delta\varepsilon = \frac{\Delta_0}{z_0} z = \frac{\Delta_{a\gamma}^2}{\gamma} z, \tag{165}$$

This allows us to express $a''(z)$, $a'(z)$, and $a(z)$ in terms of $u''(z)$, $u'(z)$, $u(z)$, and γ, and rewrite the master equation accordingly. Defining the dimensionless quantity

$$\tilde{z} = \Delta_{a\gamma} z, \tag{166}$$

after a straightforward (albeit algebraically intensive) derivation, we arrive at

$$\boxed{u''(\tilde{z}) + \left(1 + \frac{1}{2n} - \frac{\tilde{z}^2}{(2n)^2}\right) u(\tilde{z}) = 0,} \tag{167}$$

where

$$n = i\gamma \tag{168}$$

and the initial condition is

$$u'(\tilde{z}_i) = -i. \tag{169}$$

In this formulation, the entire equation is expressed in terms of γ. This is the equation used by Carenza and Marsh, and we refer to it as the **Carenza and Marsh equation**.

It is also possible, specifically within the Landau-Zener approximation, to work with the master equation expressed in terms of γ without redefining the field $a \to u$. This can be done in two ways:

1. Starting from (165) (valid only under the Landau-Zener approximation), we substitute this into the master equation (124) to obtain:

$$a''(z) + \left[i\frac{\Delta_{a\gamma}^2}{\gamma} z\right] a'(z) + \Delta_{a\gamma}^2 a = 0. \tag{170}$$

   To relate this equation to the one by Carenza and Marsh, we switch to $\tilde{z} = \Delta_{a\gamma} z$:

$$\frac{d^2 a(z)}{d\tilde{z}^2} \Delta_{a\gamma}^2 + i\frac{\Delta_{a\gamma}^2}{\gamma} z \Delta_{a\gamma} \frac{da(z)}{d\tilde{z}} + \Delta_{a\gamma}^2 a(z) = 0, \tag{171}$$

   which simplifies to:

$$\frac{d^2 a(z)}{d\tilde{z}^2} + \frac{i}{\gamma}\tilde{z}\frac{da(z)}{d\tilde{z}} + a(z) = 0, \tag{172}$$

   where the initial condition is:

$$\left.\frac{da(z)}{dz}\right|_{z_i} = \left.\frac{da(z)}{d\tilde{z}}\right|_{\tilde{z}_i} \Delta_{a\gamma} = -i\Delta_{a\gamma},$$

   hence:

$$a'(\tilde{z}_i) = \left.\frac{da(z)}{d\tilde{z}}\right|_{\tilde{z}_i} = -i. \tag{173}$$

7.2 Numerical approach in the general case

The equations above assume a linear form of $n_e(z)$ and do not account for possible changes in its slope, even if γ remains unchanged. For instance, $n_e(z)$ could increase linearly up to a certain point and then decrease linearly with the same magnitude but opposite sign. Since γ depends on the absolute value of the slope of $n_e(z)$—i.e., the absolute derivative of $\Delta_\parallel$ with respect to z—γ remains constant in this case. However, both the Carenza and Marsh equation and the previous formulation fail to capture this change in the behavior of $n_e(z)$.

Moreover, we are interested in solving the equation in a more general case where γ is not constant, while the Landau-Zener approximation still holds at first order.

Starting from the master equation, we define the dimensionless variable

$$\tilde{z} = \Delta_{a\gamma} z, \tag{174}$$

and using $d\tilde{z} = \Delta_{a\gamma} dz$, we obtain:

$$\frac{d^2 a(\tilde{z})}{d\tilde{z}^2} + i\frac{\Delta_\parallel(\tilde{z}) - \Delta_a}{\Delta_{a\gamma}(\tilde{z})}\frac{da(\tilde{z})}{d\tilde{z}} + a = 0, \tag{175}$$

with the initial condition:

$$\left.\frac{da}{d\tilde{z}}\right|_{\tilde{z}_i} = a'(\tilde{z}_i) = -i. \tag{176}$$

By defining

$$f \equiv \frac{da}{d\tilde{z}}, \tag{177}$$

the equation becomes:

$$f'(\tilde{z}) + i\frac{\Delta_\parallel(\tilde{z}) - \Delta_a}{\Delta_{a\gamma}(\tilde{z})} f(\tilde{z}) + a = 0, \tag{178}$$

and the initial conditions $a(z_i) = 0$ and $a'(z_i) = -i\Delta_{a\gamma}$ become:

$$\text{IC} := \begin{cases} a(\tilde{z}_i) = 0 \\ f(\tilde{z}_i) = -i \end{cases} . \tag{179}$$

Naturally, if we solve the equation numerically using a dimensionless quantity z_A, and we express $1\text{Mpc} = 1.564 \times 10^{29}\,\text{eV}^{-1}$, with $z = \text{Mpc} \times z_A$, we must account for this when restoring physical units:

$$z_{\text{Physical}} = \frac{\tilde{z}}{\Delta_{a\gamma} \times \text{Mpc}}. \tag{180}$$

Working with $\tilde{z}$ leads to the definition of γ with respect to it:

$$\gamma = \frac{\Delta_{a\gamma}(\tilde{z})}{\left|\frac{d\Delta_\parallel(\tilde{z})}{d\tilde{z}}\right|}. \tag{181}$$

8. Halo model

9. Global and cut integrations

In the $\tilde{z}$ parametrization, typical halo distances of the order of several Mpc are transformed into extremely large values, and the solution oscillates thousands of times between the two points of resonance. For this reason, the solution over the whole interval becomes computationally expensive, even when using numerical acceleration tools such as `Numba`[7]. At first glance, one might consider further reducing the interval by selecting a smaller one that contains both resonance points. However, despite this reduction, the computational cost remains excessive, making it impractical to analyze (many) multiple trajectories.

Most of the oscillations occur far from the two resonance points, whereas all the relevant physical processes take place near them. Therefore, we can consider a *cut integration*, defining two intervals I_1 and I_2 around the two resonance points, and integrating only over these two regions, while propagating the solution from the endpoint of the first interval to the starting point of the second. This propagation can be performed by introducing an oscillation wavelength $\lambda_{\rm osc}$ that depends on z,

$$\lambda(z) = \frac{2\pi}{\Delta_{\parallel}(z) - \Delta_a}, \tag{182}$$

and assuming the phase evolves as

$$\frac{d\varphi(z)}{dz} = \frac{2\pi}{\lambda(z)}, \tag{183}$$

so that

$$\varphi_{\rm cumulative}(z) = \int_{z_1^{\rm max}}^{z} \frac{2\pi}{\lambda(z)} dz = \int_{z_1^{\rm max}}^{z} \left[\Delta_{\parallel}(z) - \Delta_a\right] dz. \tag{184}$$

We aim to set the initial conditions (ICs) for the integration over I_2 in such a way as to match the global solution perfectly and correctly capture the interference between the two wave packets $|a\rangle$ and $|\gamma\rangle$. This can be done by starting from $z_1^{\rm max}$ and defining the point $z_2^{\rm min}$ as a location after N oscillations. That is, we define $z_2^{\rm min}$ from $\varphi_{\rm cumulative}(z)$ such that:

$$\varphi_{\rm cumulative}(z_2^{\rm min}) = \int_{z_1^{\rm max}}^{z_2^{\rm min}} \left[\Delta_{\parallel}(z) - \Delta_a\right] dz = 2\pi N. \tag{185}$$

We adopt the following algorithm to achieve this:

1. Compute $\varphi_{\rm cumulative}(z)$ over a certain interval I_{12} between I_1 and I_2.
2. Define N. From $\varphi_{\rm cumulative}$, we can also compute its value over I, and then:
$$\frac{\varphi_{\rm cumulative}(I)}{2\pi} = N. \tag{186}$$
3. Interpolate the inverse function $(z, \varphi_{\rm cumulative})$ to obtain $z(\varphi_{\rm cumulative})$.
4. Set $z_2^{\rm min} = z(\varphi_{\rm cumulative} = 2\pi N)$.
5. Set the ICs at $z_2^{\rm min}$ as:
$$\begin{cases} a(z_1^{\rm max}) = a(z_2^{\rm min}) \\ \frac{da}{dz}\Big|_{z_1^{\rm max}} = \frac{da}{dz}\Big|_{z_2^{\rm min}} \end{cases}. \tag{187}$$

[7] https://numba.pydata.org

This approach seems to work in *most cases*, but in several others we observe an error at the second resonance point, due to an incorrect setting of the ICs on I_2. The error in the probability of the cut solution with respect to the global integration remains below $\sim 10\%$, but our goal is to achieve high precision.

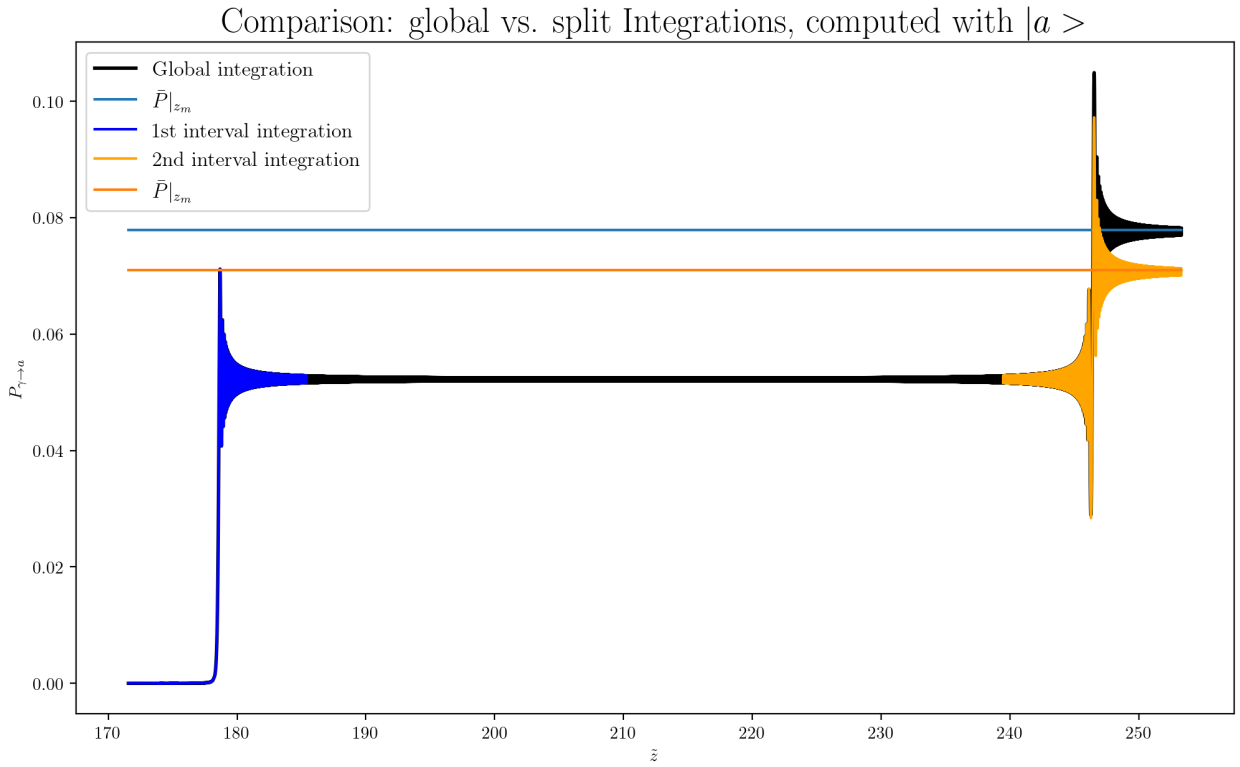


Figure 2: Figure showing the comparison between the global numerical solution and the cut one, for $P_{\gamma\mapsto a} = |a|^2$. The parameters are fixed to: $m_a = 3 \times 10^{-13}\,\text{eV}$, $g_{a\gamma\gamma} = 10^{-10}\,\text{GeV}^{-1}$, $|\vec{B}_\perp| = 1\,\mu\text{G}$, and the electron number density profile is taken from a Battaglia profile for a halo with mass $M = 10^{13.5} M_\odot$, redshift $z = 0$, for a trajectory crossing 1.435 Mpc from the center.

It is possible to identify that the issue lies in how the ICs on I_2 are set. By inspecting the solutions $|a\rangle$ and $\frac{d|a\rangle}{dz}$, one notices that—beyond the expected amplitude modulation $A(z)$—in some cases (such as the one shown) a slope appears in the curves, as shown in the plots for $\Re\,|a\rangle$ and $\Im\,|a\rangle$.

Because of this issue, the naive approach of setting $|a\rangle$ and its derivative at the starting point of I_2 equal to their values at the end of I_1, under the assumption that only a phase evolution occurs in between, does not hold.

9.1 Cut and WKB implementation

Performing a cut remains an optimal choice, as it allows for reducing the computational time from minutes or hours to just a few seconds. However, a more precise mathematical implementation is needed.

If the ODE is rewritten in an alternative form, valid far from the resonance points—i.e., in the region we wish to skip—the Wentzel–Kramers–Brillouin (WKB) approximation holds.

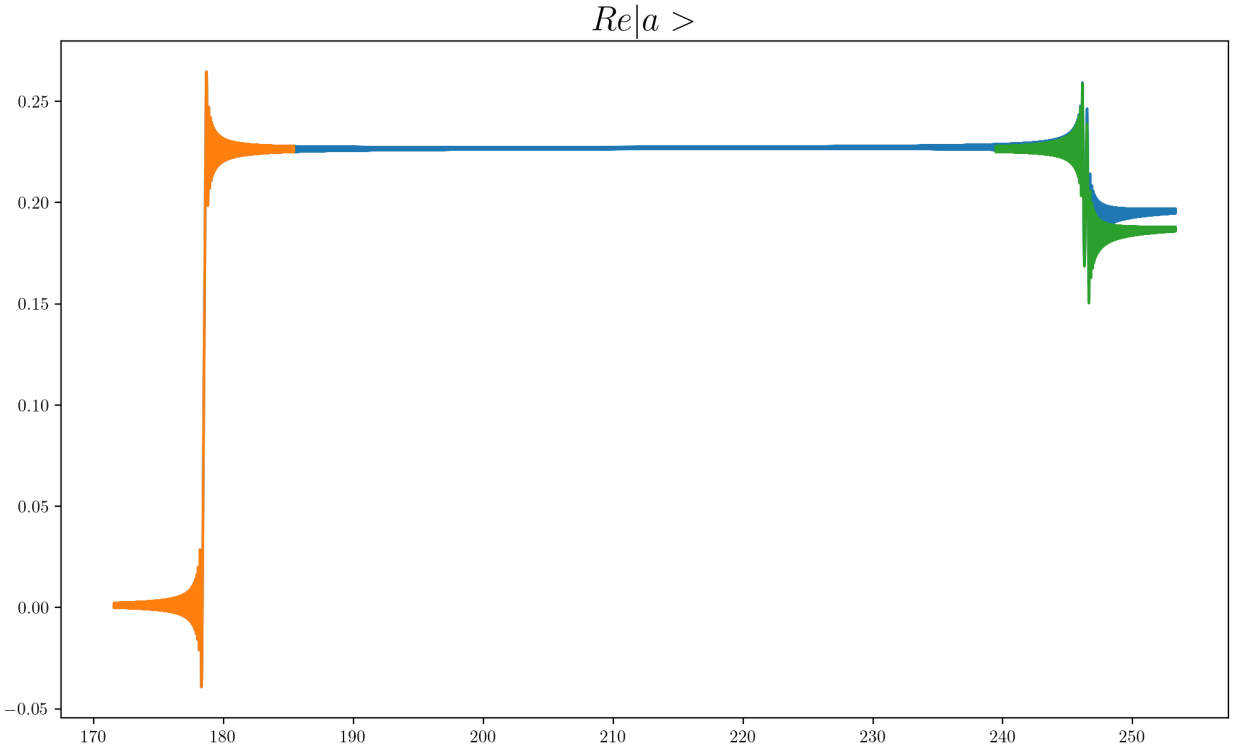


Figure 3:

By redefining $|a\rangle$ with a phase, it becomes possible to eliminate the derivative term in the ODE (116), obtaining a Schrödinger-like equation. This is particularly useful because, far from resonance, we can express the solution analytically using the WKB method. In this way, we can propagate the solution from $z_1^{\max}$ to $z_2^{\min}$ and impose highly accurate ICs on I_2.

Let us define

$$u(z) = e^{\frac{i}{2}\Phi(z)} a(z) \quad \Leftrightarrow \quad a(z) = e^{-\frac{i}{2}\Phi(z)} u(z), \tag{188}$$

where

$$\Phi(z) = \int_{z_i}^{z} \left[\Delta_{\parallel}(z) - \Delta_a\right] dz \quad \Leftrightarrow \quad \Phi(\tilde{z}) = \int_{\tilde{z}_i}^{\tilde{z}} \frac{\Delta_{\parallel}(\tilde{z}) - \Delta_a}{\Delta_{a\gamma}} d\tilde{z}. \tag{189}$$

From the expression of $a(z)$, we can compute both the first and second derivatives appearing in (116) and rewrite the equation in terms of u. Carrying out the full computation, we arrive at the **master equation in terms of** u:

$$\boxed{u''(z) + \left[1 + \left(\frac{\Phi'(z)}{2}\right)^2 - \frac{i}{2}\Phi''(z)\right] u(z) = 0,} \tag{190}$$

which is exactly the desired WKB form, where

$$\Phi'(z) = \frac{d\Phi}{dz} = \Delta_{\parallel} - \Delta_a, \tag{191}$$

$$\Phi''(z) = \frac{d\Phi'}{dz} = \frac{d\Delta_{\parallel}}{dz}, \tag{192}$$

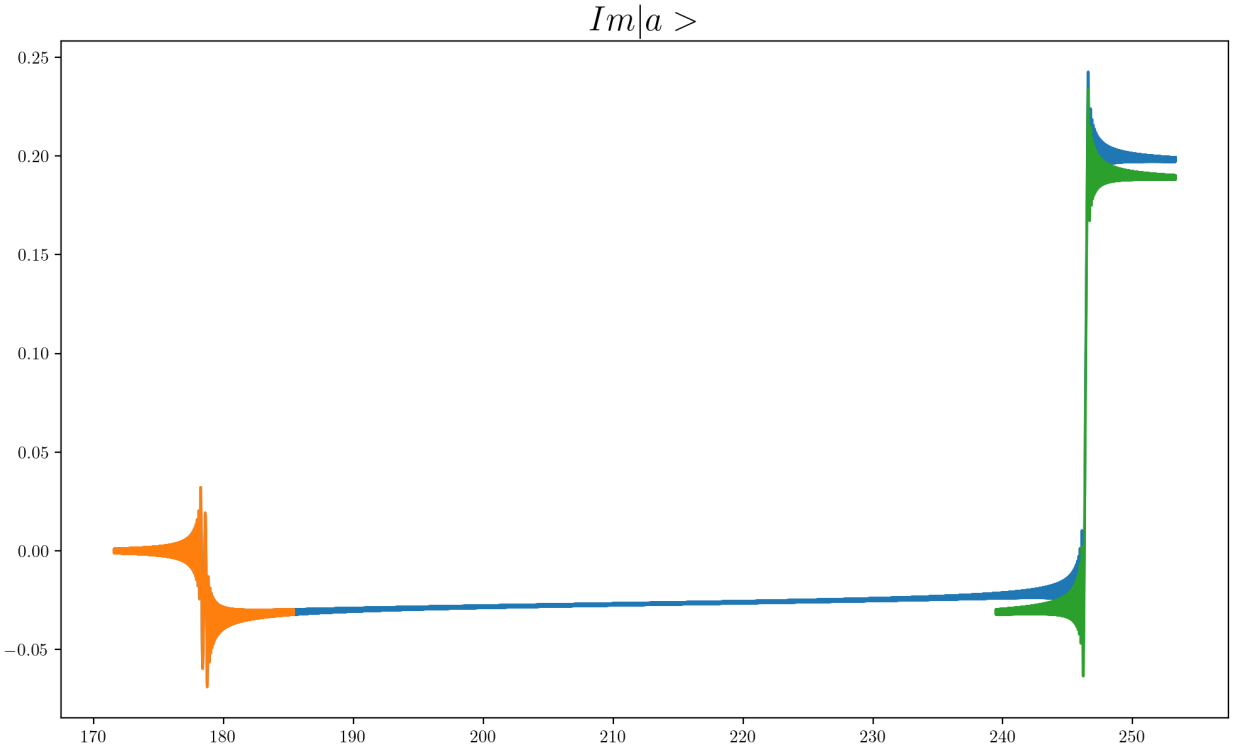


Figure 4:

and $\Phi'(\tilde{z})$ and $\Phi''(\tilde{z})$ are obtained by rescaling with $\Delta_{a\gamma}$. *The form of the equation remains identical in the* $\tilde{z}$ *parametrization.*

In the WKB approximation, starting from the Schrödinger equation

$$\frac{d^2\Psi(x)}{dx^2} = \frac{2m}{\hbar^2}\left(V(x) - E\right)\Psi(x), \tag{193}$$

we obtain:

$$\frac{d^2\Psi(x)}{dx^2} = \frac{p^2(x)}{\hbar^2}\Psi(x), \tag{194}$$

where $p(x) = \sqrt{2m(E - V(x))}$, leading to the following solution:

$$\Psi(x) \simeq \frac{C_+ e^{\frac{i}{\hbar}\int p(x)dx} + C_- e^{-\frac{i}{\hbar}\int p(x)dx}}{\sqrt{\frac{|p(x)|}{\hbar}}} = \frac{C_+ e^{\frac{i}{\hbar}\int \sqrt{2m(E-V(x))}dx} + C_- e^{-\frac{i}{\hbar}\int \sqrt{2m(E-V(x))}dx}}{\frac{1}{\hbar}[2m|E - V(x)|]^{1/4}}. \tag{195}$$

In our case, we have

$$\Omega = \sqrt{1 + \left(\frac{\Phi'(z)}{2}\right)^2 - \frac{i}{2}\Phi''(z)}, \tag{196}$$

$$|\Omega| = \left[\left(1 + \frac{\Phi'^2}{4}\right)^2 + \frac{\Phi''^2}{4}\right]^{1/4}, \tag{197}$$

so

$$\frac{|p|}{\hbar} = \left[\left(1 + \frac{\Phi'^2}{4}\right) + \frac{\Phi''^2}{4}\right]^{1/4}, \tag{198}$$

and therefore, far from the resonance point where the WKB approximation holds, we can write:

$$u_{\mathrm{WKB}} = \frac{C_+ e^{i\int \Omega(z)\,dz} + C_- e^{-i\int \Omega(z)\,dz}}{\left[\left(1 + \frac{\Phi'^2}{4}\right) + \frac{\Phi''^2}{4}\right]^{1/8}}. \tag{199}$$

Defining $\Xi(z) = \int \Omega(z)dz$, we can rewrite this in a more manageable form:

$$u_{\mathrm{WKB}}(z) = \frac{C_+ e^{i\Xi(z)} + C_- e^{-i\Xi(z)}}{\sqrt{|\Omega(z)|}}. \tag{200}$$

Our goal is now to determine $C_\pm$, which we can achieve by integrating over I_1 and matching u_{WKB} at $z_1^{\max}$ with the numerical solution. Once $C_\pm$ are known, we can propagate u_{WKB} to $z_2^{\min}$ to set the initial conditions. To determine $C_\pm$, we need two equations; the second comes from the derivative u'_{WKB}, which we derive from the expression above.

Recalling that:

$$\frac{d}{dz}\left(\frac{1}{\sqrt{|\Omega(z)|}}\right) = -\frac{1}{2}\frac{1}{|\Omega(z)|^{3/2}}\frac{d|\Omega(z)|}{dz}, \tag{201}$$

and that

$$\frac{d\Xi}{dz} = \Omega(z), \tag{202}$$

we can write:

$$u'_{\mathrm{WKB}}(z) = A_+(z)C_+ e^{i\Xi(z)} + A_-(z)C_- e^{-i\Xi(z)}, \tag{203}$$

where

$$A_\pm(z) = -\frac{1}{2}\frac{D[|\Omega(z)|]}{|\Omega(z)|^{3/2}} \pm \frac{i\Omega(z)}{|\Omega(z)|^{1/2}}, \tag{204}$$

and $D[\cdot]$ denotes the derivative with respect to z. Developing $D[|\Omega(z)|]$, the full form of $A_\pm$ becomes:

$$A_\pm(z) = -\frac{1}{2}\frac{\Re[\Omega(z)]\Re[\Omega'(z)] + \Im[\Omega(z)]\Im[\Omega'(z)]}{|\Omega(z)|^{5/2}} \pm \frac{i\Omega(z)}{|\Omega(z)|^{1/2}}. \tag{205}$$

In our case:

$$\Omega'(z) = \frac{\Phi'(z)\Phi''(z) - i\Phi'''(z)}{4\sqrt{1 + \left(\frac{\Phi'(z)}{2}\right)^2 - \frac{i}{2}\Phi''(z)}}. \tag{206}$$

A convenient choice is to set the lower limit of the integral defining Ξ as $z_0 \equiv z_1^{\max}$, so that $\Xi(z_1^{\max}) = 0$ and the exponentials reduce to unity at $z_1^{\max}$. Therefore, the system of equations to solve for $C_\pm$ is:

$$\begin{cases} u_{\mathrm{numeric}}(z_1^{\max}) = \frac{C_+ + C_-}{\sqrt{|\Omega(z_1^{\max})|}} \\ u'_{\mathrm{numeric}}(z_1^{\max}) = C_+ A_+(z_1^{\max}) + C_- A_-(z_1^{\max}) \end{cases}. \tag{207}$$

Once $C_\pm$ are determined, we can propagate $u_{\mathrm{WKB}}(z)$ and $u'_{\mathrm{WKB}}(z)$ up to $z_2^{\min}$ and set the ICs on I_2 as $\mathrm{IC}_{\mathrm{I}_2} = [\mathrm{u_{WKB}}(\mathrm{z_2^{min}}), \mathrm{u'_{WKB}}(\mathrm{z_2^{min}})]$. In this way, the cut becomes precise and we find that the two integrations (global vs. cut) agree, as shown in the figure. Moreover, we observe that both $\Re[u]$ and $\Im[u]$ do not display any slope. By starting from $u(z)$ and transforming back to $a(z)$, we recover the eventual slope observed when working directly with a, indicating that this slope is physical and arises from the phase factor $e^{-i\Phi(z)/2}$, which depends on z.

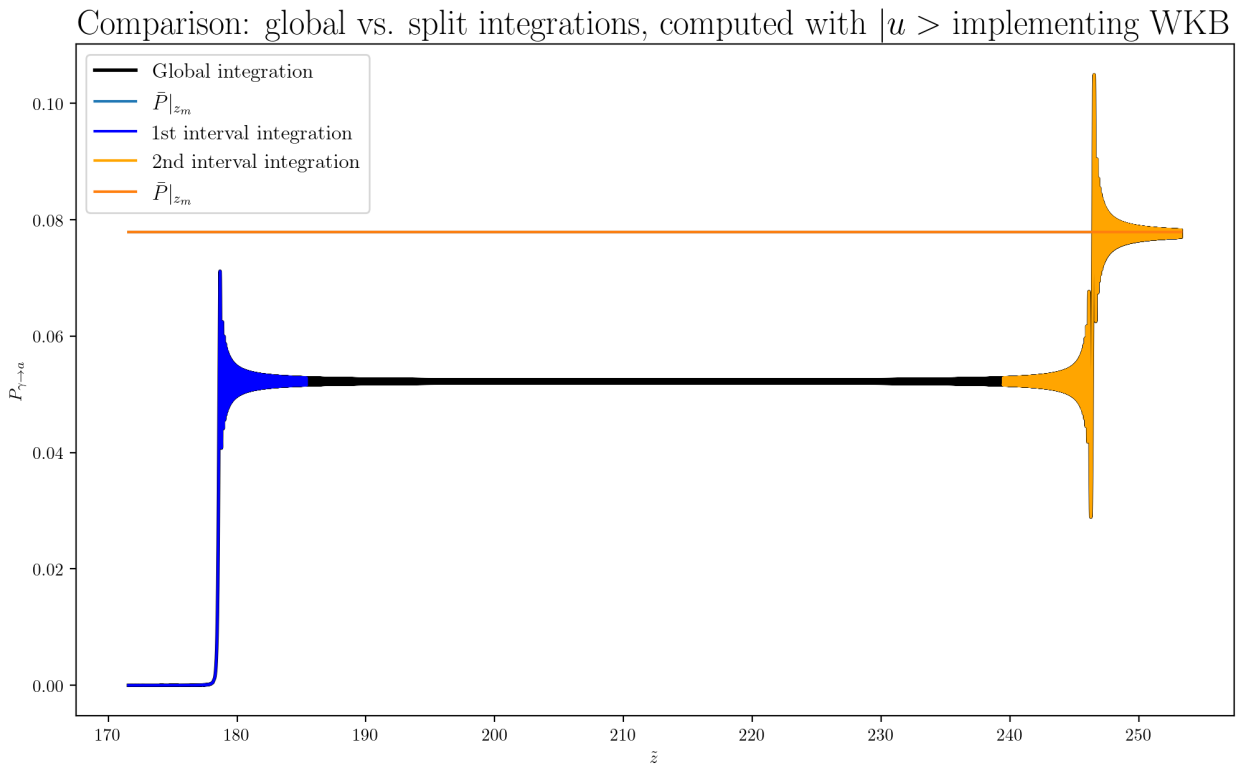


Figure 5: Same as Figure 2 but now using $|u\rangle$ and implementing the ICs with the WKB method.

References

[1] P. Carenza and M. C. D. Marsh, JCAP **04**, 021 (2023), arXiv:2302.02700 [hep-ph] .

[2] C. Zener and R. H. Fowler, Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character **137**, 696 (1932), https://royalsocietypublishing.org/doi/pdf/10.1098/rspa.1932.0165 .

[3] C. Mondino, D. Pirvu, J. Huang, and C. Johnson, JCAP **10**, 107 (2024), arXiv:2405.08059 [hep-ph] .

[4] D. Pirvu, J. Huang, and M. C. Johnson, JCAP **01**, 019 (2024), arXiv:2307.15124 [hep-ph] .

[5] N. Brahma, A. Berlin, and K. Schutz, Phys. Rev. D **108**, 095045 (2023), arXiv:2308.08586 [hep-ph] .

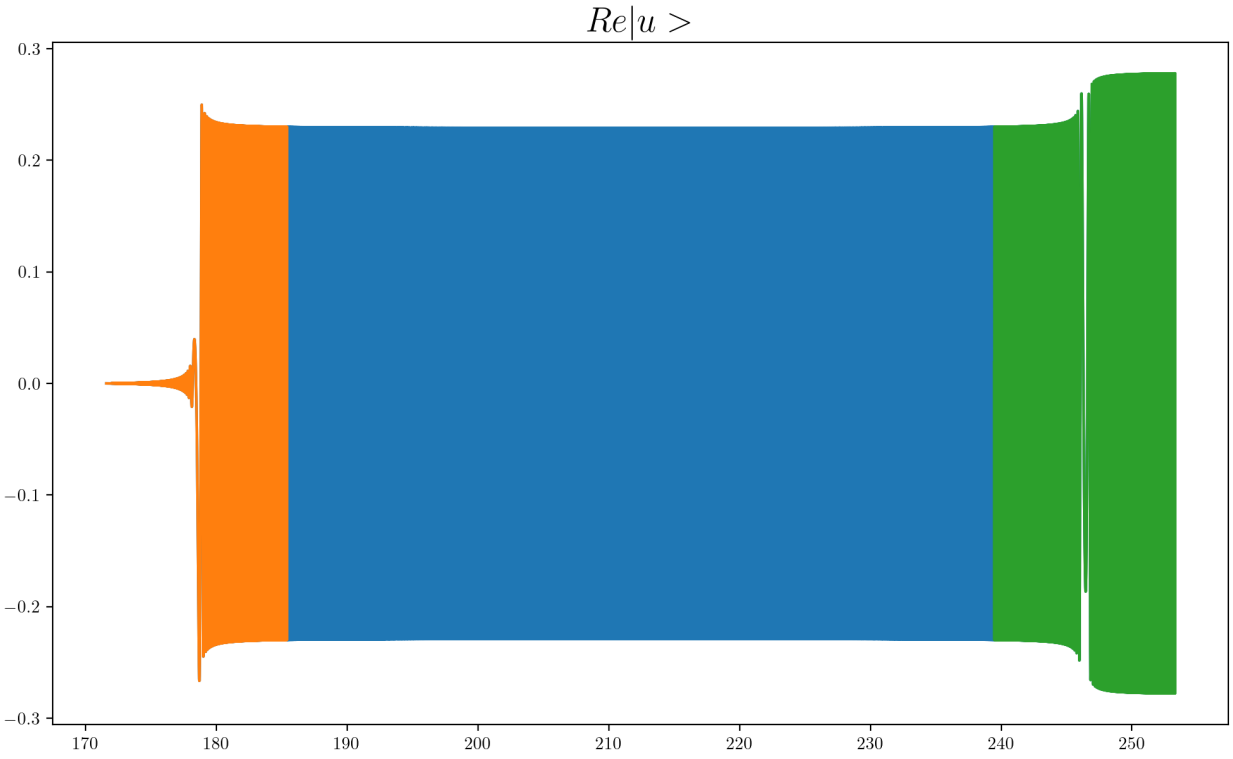


Figure 6:

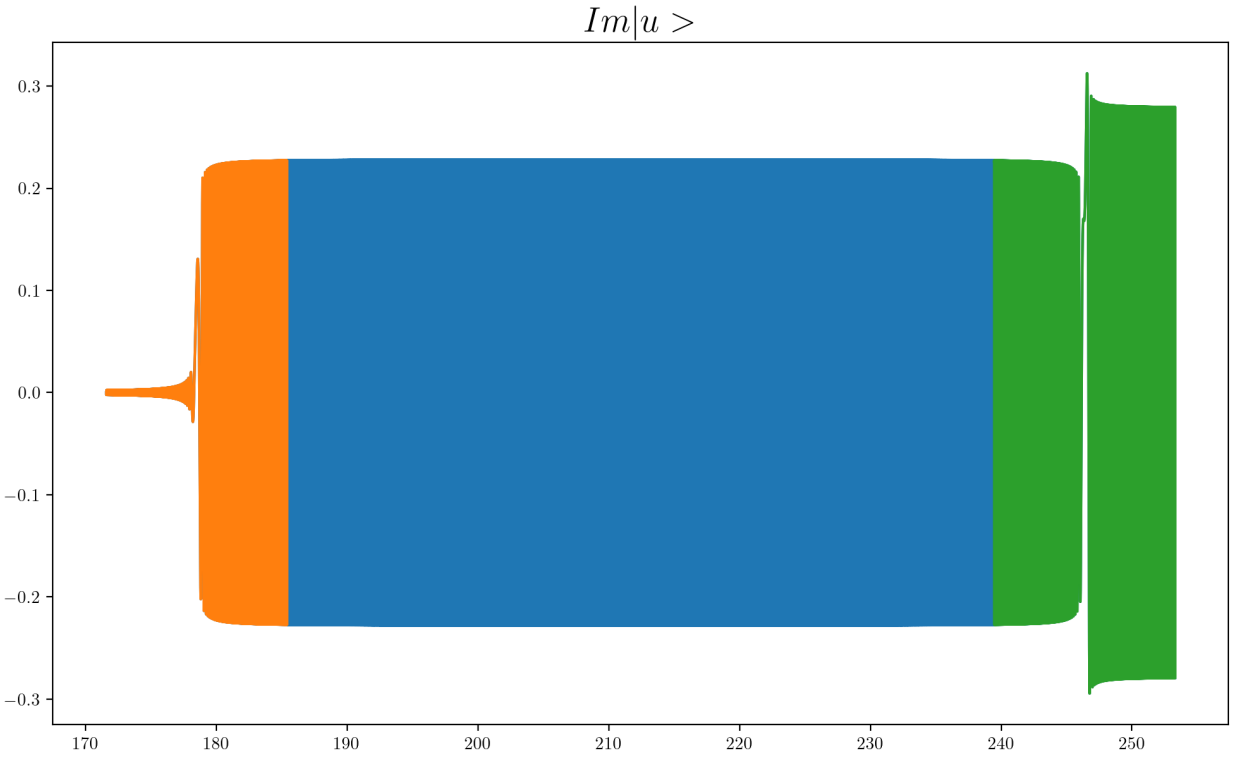


Figure 7: