

A LEAST SQUARE ESTIMATION OF THREE PARAMETERS OF A WEIBULL DISTRIBUTION

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Abstract—The present paper applies a least square method to estimate parameters of a Weibull distribution, with the shape parameter lying in the range 0–3, where other methods like the maximum likelihood method are generally not applicable. Further, Fisher's *F*-test is employed to find the goodness of fit of a straight line. Two approximate methods have also been developed for the remaining range of the shape parameter. Illustrations are provided wherever necessary.

NOTATION

c shape parameter of Weibull distribution
b scale parameter of Weibull distribution
 μ location parameter of Weibull distribution
n sample size
 $\hat{c}, \hat{b}, \hat{\mu}$ estimated values of shape, scale and location parameters respectively
F Fisher's *F*-test ratio
 t_i *i*th ordered observation from a random sample of size *n*

1. INTRODUCTION

The Weibull distribution is one of the most extensively used distributions in life testing and reliability studies. Weibull [1] proposed it for the first time in 1939 in connection with studies of strength of materials. Later he showed that the distribution is also useful in describing wear out and fatigue failures. Kao [2] used it as a model for electronic valve failures. Mann [3] provided a variety of situations in which this distribution can be used for other types of failure data.

A three-parameter Weibull distribution has its probability density function (pdf) given by

$$f(t/c, b, \mu) = (c/b) \left[\frac{t - \mu}{b} \right]^{c-1} \exp \left[-\left(\frac{t - \mu}{b} \right)^c \right], \quad t > \mu, b, c > 0. \quad (1)$$

The maximum likelihood estimation of parameters *c*, *b* and μ , which are known as shape, scale and location parameters respectively is available in standard textbooks on life testing. However, since a relative maximum may not always exist, in some cases a solution cannot be found using this method. It is also worthwhile mentioning that the Weibull pdf presents a discontinuity at $c = 1$ (shape parameter), and for $c < 1$ and $c > 1$; this family of distributions shows a different behaviour at the lower finite end. For $c < 1$, the likelihood function can be made infinite if μ is made equal to the minimum of the

sample values. For this reason and for the reason that some regularity problems [4] occur for $1 < c < 2$, the maximum likelihood method of estimation is not appropriate for a Weibull population with $c < 2$.

In this paper, we propose a least square method of estimation which provides very good simulation results in the range $0 < c < 3$. The method presented here is a simple extension of White's [5] method for a two-parameter Weibull distribution. The method is also applicable to complete as well as to censored samples.

2. THE LEAST SQUARE PROCEDURE

For two parameters, the Weibull distribution function can be written as

$$F(t_i) = 1 - \exp \left[-\left(\frac{t_i}{b} \right)^c \right]. \quad (2)$$

On taking logarithms, we obtain

$$\log(t_i) = \log(b) + (1/c) \log[-\log\{1 - F(t_i)\}]. \quad (3)$$

Equation (3) can be expressed as

$$Y_i = \alpha + \beta X_i \quad (4)$$

where $Y_i = \log(t_i)$, $\alpha = \log(b)$, $\beta = 1/c$ and $X_i = \log[-\log\{1 - F(t_i)\}]$. Here, X_i are not known; however, they are order statistics from the distribution with a density given by

$$h(w) = \exp(w - \exp(w)), \quad -\alpha < w < \alpha. \quad (5)$$

White [5] suggested estimating X_i by the expected values of the *i*th order statistics (in a sample of size *n*) from (5), which is defined as the reduced Log-Weibull distribution. Then α and β are obtained through a least square method, with estimators obtained as

$$\hat{b} = \exp(\hat{\alpha}) \quad \text{and} \quad \hat{c} = 1/\hat{\beta}. \quad (6)$$

For a three-parameter Weibull distribution the authors propose the following iterative process.

Procedure

Step 1. Take $\hat{\alpha} = t_i - 0.01$, subtract $\hat{\mu}$ from each observation and find White's least square estimators of \hat{b} and \hat{c} .

Step 2. Compute Fisher's F -test ratio, $F = S_y^2/S_{res}^2$ where

$$S_y^2 = \frac{1}{n-1} \sum (Y_i - \bar{Y})^2$$

$$S_{res}^2 = \frac{1}{n-2} \sum (Y_i - \hat{Y}_i)^2$$

$$\hat{Y}_i = \hat{\alpha} + \hat{\beta}X_i.$$

Step 3. Reduce $\hat{\mu}$ and repeat steps (1) and (2).

Step 4. Choose $\hat{\mu}$, \hat{c} and \hat{b} corresponding to which F is maximum.

The procedure is fast and simple to program, if the tables of first order moments of reduced Log-Weibull order statistics are available. Otherwise the approximate value of these can be obtained using the orthogonal inverse expansion method [6]. We illustrate the above iteration process through the following two numerical examples.

Example 1

A sample of 30, with Weibull pdf having $\theta = 500$, $c = 1.7$ and $b = 40$ is generated and is given as follows:

552.0525 548.6083 506.3785 565.5352 511.7620
515.2365 516.1401 514.7593 520.6400 513.8919
543.7966 556.5106 517.9078 562.4553 531.9044
539.6028 518.5398 565.7348 541.8414 554.3255
506.0710 547.0142 571.4103 523.5997 541.8120
519.8716 556.1709 548.5325 544.3080 538.5575

Following the steps of the least square iterative procedure outlined earlier, we obtain the results shown in Table 1.

In the table, the values shown in bold face correspond to the maximum value of F and are therefore the least square estimates of the parameters.

Example 2

A sample of size 30 having Weibull pdf with $\mu = 100$, $c = 0.8$ and $b = 10$ is generated. The values are:

102.4378 114.7585 103.5102 101.3378 141.7785
102.5250 102.5244 124.9970 146.9202 117.0452
103.5730 113.6165 102.2618 110.0926 107.1926
125.1443 100.3264 102.9202 100.0017 107.7962
101.3272 101.3620 102.5391 100.0935 104.8785
125.1759 105.1076 101.6966 102.4999 130.1677

Once again following the procedure outlined above, we obtain the values shown in Table 2.

Through a large number of simulations, it is found that sometimes widely different sets of c ($c > 3$), b and μ give almost the same sample values. This can be the reason why in certain instances MLE fails to provide

Table 1. Partial print out of the result

$\hat{\mu}$	\hat{c}	\hat{b}	F -ratio
496.6	2.08	44.98	20.55
499.6	1.844	41.98	23.02
501.1	1.699	40.29	24.30
502.1	1.603	39.26	24.65
502.7	1.5365	38.58	24.44
503.1	1.488	38.12	23.97

The values in bold face are the least square estimates.

a definite estimate. However, in most of the engineering applications the shape parameter ranges from 1 to 6 only. We now describe two approximate methods, which offer estimates very close to the true values from which the data is simulated (for c ranging from 3 to 6). These methods are very fast compared to the MLE. Moreover, even for small sample sizes they give close estimates and in this respect these estimates are better than those provided by Dubey's method [7], which requires more than 80 observations. If one is interested in MLE, then this estimate can be used as an initial estimate and can search for a relative maxima in its neighbourhood. It will reduce the computational time considerably. Both the proposed methods are based on the fact that the mean minus 3.3 times the standard deviation of first order statistics gives a good estimate of the location parameter.

Estimation based on White's method

White's method [5] for two-parameter estimation, and its extension to the case of three-parameter estimation when $c < 3$, has already been discussed. For $3 < c < 6$, the following procedure is proposed.

Step 1. Assuming the location parameter to be zero, find the shape and the scale parameter using White's least square method. Let them be \hat{c}_0 and \hat{b}_0 .

Step 2. Calculate the approximate value of the mean minus 3.3 times the standard deviation of first order statistics using the expression

$$\hat{\mu}_i = \exp(\log(t_i - \mu_{i-1}) - 3.3(1.644/\hat{c}_{i-1})^{1/2}). \quad (7)$$

Here $\mu_0 = 0$.

Step 3. Subtract $\hat{\mu}_i$ from data (data after subtracting μ_1 for $i = 2$) and estimate shape and scale parameters. Let these be \hat{c}_i and \hat{b}_i .

Step 4. Execute steps (2) and (3) for $i = 1, 2$ and obtain $\hat{\mu}_1, \hat{\mu}_2, \hat{c}_2$ and \hat{b}_2 . The final estimate is given by $\hat{\mu} = \hat{\mu}_1 + \hat{\mu}_2, \hat{c} = \hat{c}_2$ and $\hat{b} = \hat{b}_2$.

Table 2. Partial print out of the results

$\hat{\mu}$	\hat{c}	\hat{b}	F -ratio
99.8	0.9047	8.9792	20.1853
99.85	0.8735	8.9115	23.1260
99.9	0.8361	8.8521	26.7412
99.99	0.7089	8.8663	20.2310
100.0	0.6399	9.0185	10.2045

The values in bold face are the least square estimates.

Table 3. Results of estimations

Parameter values used for simulation			Estimated parameter values					
			$\hat{\mu}$		\hat{c}		\hat{b}	
μ	c	b	w	s	w	s	w	s
100	5	20	97.5	97.7	4.0	3.82	23.4	22.6
100	4	75	106.2	103.4	3.31	3.60	82.0	86.48
100	3	75	105.4	93.0	3.66	3.02	75.6	83.50
50	3	50	37.7	38.0	3.19	3.01	60.45	65.62
100	4	50	99.3	99.9	3.71	3.78	54.24	56.52
100	3	50	88.4	91.9	3.72	3.34	59.50	58.6
100	4	25	101.6	99.1	3.94	3.78	25.9	30.4
100	5	25	101.3	98.1	4.10	4.03	21.30	25.86

Estimation based on failure rate estimator

This method is based on a failure rate estimator suggested by Sinha [8]. This is based on two fundamental concepts, namely:

- (i) conditional failure rate;
- (ii) the least square principle.

The method of Ref. [8] has been modified for a three-parameter Weibull distribution with c ranging from 3 to 6. The procedure is the same as that used earlier except that the estimation of b and c is done by failure rate estimator. The main advantage of this method is that it does not require any tables for estimation.

For different combinations of location, shape and scale parameters, data is simulated. In all cases the sample size is taken as 20. The results of estimations using both methods are given in Table 3. In Table 3, the values under w and s represent the estimates based on White's method and those based on the failure rate estimator, respectively.

3. CONCLUSION

The approach presented in this paper is very simple and fast computationally. The extensive simulation attempted indicates that the estimated values are very

close to the true values even for a small sample size. The estimates are a good alternative when the MLE fails or can at least serve as a good initial estimate for ML estimation.

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