Probability and Statistics
Seminar 11: Parameter estimations

Problem 1 (4 pts + 1 pt R task)**.** Assume that $X_1, X_2, \dots$ are **i.i.d. r.v.**'s with **p.d.f.**

$$f(x \mid \theta) = c \exp(-x^2/\theta),$$

where c is a constant independent of x that might depend on θ.

(a) Identify the distribution family and find the value of c

(b) Find the *moment method* estimator $\widehat{\Theta}$ of θ

(c) Find the *maximum likelihood estimator* $\widetilde{\Theta}$ of θ

(d) Is the estimator $\widehat{\Theta}$ *(asymptotically) unbiased*? Hint: $X_1^2 + \dots + X_n^2$ has *Gamma* distribution $\Gamma(\frac{n}{2}, \frac{1}{\theta})$

(e) Calculate the *mean squared error* of $\widehat{\Theta}$.

(f) Is the estimator $\widehat{\Theta}$ *consistent*? Justify your answer by quoting the corresponding theorems/properties.

(g) Perform R simulation (see the R notebook)

Problem 2 (4 pts + 1 pt R task)**.** Assume that $X_1, X_2, \dots$ are non-negative **i.i.d. r.v.**'s with **p.d.f.**

$$f(x \mid \theta) = c \exp(-2\theta x),$$

where $x > 0$ and c is a constant independent of x that might depend on θ.

(a) Identify the distribution family and find the value of c

(b) Find the *moment method* estimator $\widehat{\Theta}$ of θ

(c) Find the *maximum likelihood estimator* $\widetilde{\Theta}$ of θ

(d) Is the estimator $\widehat{\Theta}$ *(asymptotically) unbiased*? Hint: $X_1 + \dots + X_n$ has *Gamma* distribution $\Gamma(n, 2\theta)$

(e) Calculate the *mean squared error* of $\widehat{\Theta}$.

(f) Is the estimator $\widehat{\Theta}$ *consistent*? Justify your answer by quoting the corresponding theorems/properties.

(g) Perform R simulation (see the R notebook)

Problem 3 (4 pts + 1 pt R task)**.** SHORT INTRO, NOT REQUIRED TO SOLVE THE PROBLEM:

Particle swarm optimization **(PSO)** algorithm is a population-based optimization method, which was originally introduced by Eberhart and Kennedy in 1995. In **PSO**, the position of a particle is represented by a vector in search space, and the movement of the particle is determined by an assigned velocity vector. Each particle updates the velocity based on its current velocity, the best previous position of the particle, and the global best position of the population. In other words, **PSO** is a computational method that optimizes a problem by iteratively trying to improve a candidate solution with regard to a given measure of quality. An improved version of **PSO**, was suggested recently. In **QPSO**, each particle has a target point, which is defined as a linear combination of the best previous position of the particle and the global best position. The particle appears around the target point following a double exponential distribution.

Assume that $X_1, X_2, \dots$ are non-negative **i.i.d.**, **r.v.**'s with **p.d.f.** denoting the positions of the particles and the probability density function of the k^{th} particle is

$$f(x_k \mid \theta) = c \exp(-\theta|x_k|),$$

where $x_k \in \mathbb{R}$ and c is a constant independent of x_k that might depend on θ.

(a) Identify the distribution family and find the value of c

(b) Find the *moment method* estimator $\widehat{\Theta}$ of θ

(c) Find the *maximum likelihood estimator* $\widetilde{\Theta}$ of θ

(d) Determine p.d.f. of the MLE $\widetilde{\Theta}$.
Hint: $\widetilde{\Theta} = n/T(X)$ for $T(X) = |X_1| + \cdots + |X_n|$. Identify the distribution of $|X_k|$ and thus of $T(X)$.

(e) Is the estimator $\widehat{\Theta}$ *(asymptotically) unbiased*?

(f) Calculate the *mean squared error* of $\widehat{\Theta}$.

(g) Is the estimator $\widehat{\Theta}$ *consistent*? Justify your answer by quoting the corresponding theorems/properties.

(h) Perform R simulation (see the R notebook)

Problem 4 (4 pts + 1 pt R task)**.** Short intro, not required to solve the problem:
The muon is an elementary particle similar to the electron, with an electric charge of $-1e$ and a spin of $1/2$, but with a much greater mass. Muons are unstable elementary particles and are heavier than electrons and neutrinos but lighter than all other matter particles. They decay via the weak interaction. The dominant muon decay mode is the simplest possible: the muon decays to an electron, an electron antineutrino, and a muon neutrino. The angle α at which electrons are emitted in muon decay has a distribution with the density $f(x \mid \theta) = \frac{1+\theta x}{2}$, where $\theta \in [-1, 1]$ and x is the cosine of the emission angle, i.e. $x = \cos(\alpha)$.

Assume that $X_1, X_2, \dots$ are continuous **i.i.d. r.v.**'s describing the electron decay emission angle; thir **p.d.f.** is

$$f(x \mid \theta) = c\frac{1+\theta x}{2},$$

where $x \in [-1, 1]$, $\theta \in [-1, 1]$ and c is a constant independent of x that might depend on θ.

(a) Find the value of c.

(b) Find the *moment method* estimator $\widehat{\Theta}$ of θ.

(c) Find the *maximum likelihood estimator* $\widetilde{\Theta}$ of θ.

(d) Is the estimator $\widehat{\Theta}$ *(asymptotically) unbiased*?

(e) Calculate the *mean squared error* of $\widehat{\Theta}$.

(f) Is the estimator $\widehat{\Theta}$ *consistent*? Justify your answer by quoting the corresponding theorems/properties.

(g) Use the CLT to deduce a normal approximation to the sampling distribution of $\widehat{\Theta}$. According to this approximation, if $n = 25$ and $\theta = 0$, what is the $P(|\widehat{\Theta}| > .5)$?

(h) Perform R simulation (see the R notebook)