Diploma Programme

Syllabus

Syllabus outline

Syllabus component	Suggested teaching hours	
	SL	**HL**
Topic 1—Number and algebra	19	39
Topic 2—Functions	21	32
Topic 3— Geometry and trigonometry	25	51
Topic 4—Statistics and probability	27	33
Topic 5 —Calculus	28	55
The toolkit and the mathematical exploration Investigative, problem-solving and modelling skills development leading to an individual exploration. The exploration is a piece of written work that involves investigating an area of mathematics.	30	30
Total teaching hours	**150**	**240**

All topics are compulsory. Students must study all the sub-topics in each of the topics in the syllabus as listed in this guide. Students are also required to be familiar with the topics listed as prior learning.

Prior learning topics

Prior to starting a DP mathematics course students have extensive previous mathematical experiences, but these will vary. It is expected that mathematics students will be familiar with the following topics before they take the examinations because questions assume knowledge of them. Teachers must therefore ensure that any topics listed here that are unknown to their students at the start of the course are included at an early stage. Teachers should also take into account the existing mathematical knowledge of their students to design an appropriate course of study for mathematics. This table lists the knowledge, together with the syllabus content, that is essential for successful completion of the mathematics course.

Number and algebra

- Number systems: natural numbers N; integers, Z; rationals, Θ, and irrationals; real numbers, P
- SI (Système International) units for mass, time, length and their derived units, eg. speed, area and volume
- Rounding, decimal approximations and significant figures, including appreciation of errors
- Definition and elementary treatment of absolute value (modulus), $|a|$
- Use of addition, subtraction, multiplication and division using integers, decimals and fractions, including order of operations
- Prime numbers, factors (divisors) and multiples
- Greatest common factor (divisor) and least common multiples (HL only)
- Simple applications of ratio, percentage and proportion
- Manipulation of algebraic expressions, including factorization and expansion
- Rearranging formulae
- Calculating the numerical value of expressions by substitution
- Evaluating exponential expressions with simple positive exponents

- Evaluating exponential expressions with rational exponents (HL only)
- Use of inequalities, $<$, $\leq$, $>$, $\geq$, intervals on the real number line
- Simplification of simple expressions involving roots (surds or radicals)
- Rationalising the denominator (HL only)
- Expression of numbers in the form $a \times 10^k$, $1 \leq a < 10$, $k \in \mathbb{Z}$
- Familiarity with commonly accepted world currencies
- Solution of linear equations and inequalities
- Solution of quadratic equations and inequalities with rational coefficients (HL only)
- Solving systems of linear equations in two variables
- Concept and basic notation of sets. Operations on sets: union and intersection
- Addition and subtraction of algebraic fractions (HL only).

Functions

- Graphing linear and quadratic functions using technology
- Mappings of the elements of one set to another. Illustration by means of sets of ordered pairs, tables, diagrams and graphs.

Geometry and trigonometry

- Pythagoras' theorem and its converse
- Mid-point of a line segment and the distance between two points in the Cartesian plane
- Geometric concepts: point, line, plane, angle
- Angle measurement in degrees, compass directions
- The triangle sum theorem
- Right-angle trigonometry, including simple applications for solving triangles
- Three-figure bearings
- Simple geometric transformations: translation, reflection, rotation, enlargement
- The circle, its centre and radius, area and circumference. The terms diameter, arc, sector, chord, tangent and segment

- Perimeter and area of plane figures. Properties of triangles and quadrilaterals, including parallelograms, rhombuses, rectangles, squares, kites and trapezoids; compound shapes
- Familiarity with three-dimensional shapes (prisms, pyramids, spheres, cylinders and cones)
- Volumes and surface areas of cuboids, prisms, cylinders, and compound three-dimensional shapes

Statistics and probability

- The collection of data and its representation in bar charts, pie charts, pictograms, and line graphs
- Obtaining simple statistics from discrete data, including mean, median, mode, range
- Calculating probabilities of simple events
- Venn diagrams for sorting data
- Tree diagrams

Calculus

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

Syllabus content

Topic 1: Number and algebra

Concepts

Essential understandings:

Number and algebra allow us to represent patterns, show equivalencies and make generalizations which enable us to model real-world situations. Algebra is an abstraction of numerical concepts and employs variables which allow us to solve mathematical problems.

Suggested concepts embedded in this topic:

Generalization, representation, modelling, equivalence, patterns, quantity

AHL: Validity, systems.

Content-specific conceptual understandings:

- Modelling real-life situations with the structure of arithmetic and geometric sequences and series allows for prediction, analysis and interpretation.
- Different representations of numbers enable equivalent quantities to be compared and used in calculations with ease to an appropriate degree of accuracy.
- Numbers and formulae can appear in different, but equivalent, forms, or representations, which can help us to establish identities.
- Formulae are a generalization made on the basis of specific examples, which can then be extended to new examples.
- Logarithm laws provide the means to find inverses of exponential functions which model real-life situations.
- Patterns in numbers inform the development of algebraic tools that can be applied to find unknowns.
- The binomial theorem is a generalization which provides an efficient method for expanding binomial expressions.

AHL

- Proof serves to validate mathematical formulae and the equivalence of identities.
- Representing partial fractions and complex numbers in different forms allows us to easily carry out seemingly difficult calculations.
- The solution for systems of equations can be carried out by a variety of equivalent algebraic and graphical methods.

SL content

Recommended teaching hours: 19

The aim of the SL content of the number and algebra topic is to introduce students to numerical concepts and techniques which, combined with an introduction to arithmetic and geometric sequences and series, can be used for financial and other applications. Students will also be introduced to the formal concept of proof.

Sections SL1.1 to SL1.5 are content common to Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 1.1

Content	Guidance, clarification and syllabus links
Operations with numbers in the form $a \times 10^k$ where $1 \le a < 10$ and k is an integer.	Calculator or computer notation is **not** acceptable. For example, 5.2E30 is **not** acceptable and should be written as 5.2×10^{30}.

Connections

Other contexts: Very large and very small numbers, for example astronomical distances, sub-atomic particles in physics, global financial figures

Links to other subjects: Chemistry (Avogadro's number); physics (order of magnitude); biology (microscopic measurements); sciences group subjects (uncertainty and precision of measurement)

International-mindedness: The history of number from Sumerians and its development to the present Arabic system

TOK: Do the names that we give things impact how we understand them? For instance, what is the impact of the fact that some large numbers are named, such as the googol and the googolplex, while others are represented in this form?

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SL 1.2

Content	Guidance, clarification and syllabus links
Arithmetic sequences and series. Use of the formulae for the n^{th} term and the sum of the first n terms of the sequence. Use of sigma notation for sums of arithmetic sequences.	Spreadsheets, GDCs and graphing software may be used to generate and display sequences in several ways. If technology is used in examinations, students will be expected to identify the first term and the common difference.
Applications.	Examples include simple interest over a number of years.
Analysis, interpretation and prediction where a model is not perfectly arithmetic in real life.	Students will need to approximate common differences.

Connections

International-mindedness: Aryabhatta is sometimes considered the “father of algebra”–compare with alKhawarizmi; the use of several alphabets in mathematical notation (for example the use of capital sigma for the sum).

TOK: Is all knowledge concerned with identification and use of patterns? Consider Fibonacci numbers and connections with the golden ratio.

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SL 1.3

Content	Guidance, clarification and syllabus links
Geometric sequences and series. Use of the formulae for the n th term and the sum of the first n terms of the sequence. Use of sigma notation for the sums of geometric sequences.	Spreadsheets, GDCs and graphing software may be used to generate and display sequences in several ways. If technology is used in examinations, students will be expected to identify the first term and the ratio. **Link to:** models/functions in topic 2 and regression in topic 4.
Applications.	Examples include the spread of disease, salary increase and decrease and population growth.

Connections

Links to other subjects: Radioactive decay, nuclear physics, charging and discharging capacitors (physics).

International-mindedness: The chess legend (Sissa ibn Dahir).

TOK: How do mathematicians reconcile the fact that some conclusions seem to conflict with our intuitions? Consider for instance that a finite area can be bounded by an infinite perimeter.

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SL 1.4

Content	Guidance, clarification and syllabus links
Financial applications of geometric sequences and series: • compound interest • annual depreciation.	Examination questions may require the use of technology, including built-in financial packages. The concept of simple interest may be used as an introduction to compound interest. Calculate the real value of an investment with an interest rate and an inflation rate. In examinations, questions that ask students to derive the formula will not be set. Compound interest can be calculated yearly, half-yearly, quarterly or monthly. **Link to**: exponential models/functions in topic 2.

Connections

Other contexts: Loans.

Links to other subjects: Loans and repayments (economics and business management).

Aim 8: Ethical perceptions of borrowing and lending money.

International-mindedness: Do all societies view investment and interest in the same way?

TOK: How have technological advances affected the nature and practice of mathematics? Consider the use of financial packages for instance.

Enrichment: The concept of e can be introduced through continuous compounding, $\left(1+\frac{1}{n}\right)^n \to \mathrm{e}$, as $n \to \infty$, however this will not be examined.

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SL 1.5

Content	Guidance, clarification and syllabus links
Laws of exponents with integer exponents.	**Examples:** $5^3 \times 5^{-6} = 5^{-3}$, $6^4 \div 6^3 = 6$, $(2^3)^{-4} = 2^{-12}$, $(2x)^4 = 16x^4$, $2x^{-3} = \frac{2}{x^3}$.
Introduction to logarithms with base 10 and e. Numerical evaluation of logarithms using technology.	Awareness that $a^x = b$ is equivalent to $\log_a b = x$, that $b > 0$, and $\log_e x = \ln x$.

Connections

Other contexts: Richter scale and decibel scale.

Links to other subjects: Calculation of pH and buffer solutions (chemistry)

TOK: Is mathematics invented or discovered? For instance, consider the number e or logarithms–did they already exist before man defined them? (This topic is an opportunity for teachers to generate reflection on “the nature of mathematics”).

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SL 1.6

Content	Guidance, clarification and syllabus links
Simple deductive proof, numerical and algebraic; how to lay out a left-hand side to right-hand side (LHS to RHS) proof. The symbols and notation for equality and identity.	**Example:** Show that $\frac{1}{4}+\frac{1}{12}=\frac{1}{3}$. Show that the algebraic generalisation of this is $\frac{1}{m+1}+\frac{1}{m^2+m}\equiv\frac{1}{m}$ LHS to RHS proofs require students to begin with the left-hand side expression and transform this using known algebraic steps into the expression on the right-hand side (or vice versa). **Example:** Show that $(x-3)^2+5\equiv x^2-6x+14$. Students will be expected to show how they can check a result including a check of their own results.

Connections

TOK: Is mathematical reasoning different from scientific reasoning, or reasoning in other Areas of Knowledge?

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SL 1.7

Content	Guidance, clarification and syllabus links
Laws of exponents with rational exponents.	$a^{\frac{1}{m}} = \sqrt[m]{a}$, if m is even this refers to the positive root. For example: $16^{\frac{3}{4}} = 8$.
Laws of logarithms. $\log_a xy = \log_a x + \log_a y$ $\log_a \frac{x}{y} = \log_a x - \log_a y$ $\log_a x^m = m\log_a x$ for $a, \quad x, \quad y > 0$	$y = a^x \Leftrightarrow x = \log_a y$; $\log_a a = 1$, $\log_a 1 = 0$, $a, \quad y \in N, \quad x \in Z$ **Link to:** introduction to logarithms (SL1.5) **Examples:** $\frac{3}{4} = \log_{16} 8$, $\log 32 = 5\log 2$ $\log 24 = \log 8 + \log 3$ $\log_3 \frac{10}{4} = \log_3 10 - \log_3 4$ $\log_4 3^5 = 5\log_4 3$ **Link to:** logarithmic and exponential graphs (SL2.9)
Change of base of a logarithm. $\log_a x = \frac{\log_b x}{\log_b a}$, for $a, \quad b, \quad x > 0$	**Examples:** $\log_4 7 = \frac{\ln 7}{\ln 4}$ $\log_{25} 125 = \frac{\log_5 125}{\log_5 25} \quad \left(= \frac{3}{2}\right)$
Solving exponential equations, including using logarithms.	**Examples:** $\left(\frac{1}{3}\right)^x = 9^{x+1}$, $2^{x-1} = 10$. **Link to:** using logarithmic and exponential graphs (SL2.9).

Connections

Links to other subjects: pH, buffer calculations and finding activation energy from experimental data (chemistry).

TOK: How have seminal advances, such as the development of logarithms, changed the way in which mathematicians understand the world and the nature of mathematics?

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SL 1.8

Content	Guidance, clarification and syllabus links
Sum of infinite convergent geometric sequences.	Use of $\lvert r \rvert < 1$ and modulus notation. **Link to**: geometric sequences and series (SL1.3).

Connections

TOK: Is it possible to know about things of which we can have no experience, such as infinity?

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SL 1.9

Content	Guidance, clarification and syllabus links
The binomial theorem: expansion of $(a+b)^n$, $n \in N$.	Counting principles may be used in the development of the theorem.
Use of Pascal's triangle and ${}^nC_r$.	${}^nC_r$ should be found using **both** the formula and technology. **Example:** Find r when ${}^6C_r = 20$, using a table of values generated with technology.

Connections

Aim 8: Ethics in mathematics–Pascal's triangle. Attributing the origin of a mathematical discovery to the wrong mathematician.

International-mindedness: The properties of "Pascal's triangle" have been known in a number of different cultures long before Pascal. (for example the Chinese mathematician Yang Hui).

TOK: How have notable individuals shaped the development of mathematics as an area of knowledge? Consider Pascal and "his" triangle.

Download connections template

AHL content

Recommended teaching hours: 20

The aim of the AHL content in the number and algebra topic is to extend and build upon the aims, concepts and skills from the SL content. It introduces students to some important techniques for expansion, simplification and solution of equations. Complex numbers are introduced and students will extend their knowledge of formal proof to proof by mathematical induction, proof by contradiction and proof by counterexample.

AHL 1.10

Content	Guidance, clarification and syllabus links
Counting principles, including permutations and combinations.	**Not required:** Permutations where some objects are identical. Circular arrangements.
Extension of the binomial theorem to fractional and negative indices, ie $(a+b)^n, \quad n \in \Theta$.	$(a+b)^n = \left(a\left(1+\frac{b}{a}\right)\right)^n = a^n\left(1+\frac{b}{a}\right)^n, n \in \Theta$ **Link to**: power series expansions (AHL5.19) **Not required:** Proof of binomial theorem.

Connections

Other contexts: Finding approximations to $\sqrt{2}$

Aim 8: How many different tickets are possible in a lottery? What does this tell us about the ethics of selling lottery tickets to those who do not understand the implications of these large numbers?

International-mindedness: The properties of "Pascal's triangle" have been known in a number of different cultures long before Pascal (for example the Chinese mathematician Yang Hui).

TOK: What counts as understanding in mathematics? Is it more than just getting the right answer?

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AHL 1.11

Content	Guidance, clarification and syllabus links
Partial fractions.	Maximum of two distinct linear terms in the denominator, with degree of numerator less than the degree of the denominator. **Example:** $\frac{2x+1}{x^2+x-2} \equiv \frac{1}{(x-1)} + \frac{1}{(x+2)}$. **Link to:** use of partial fractions to rearrange the integrand (AHL5.15).

Connections

Download connections template

AHL 1.12

Content	Guidance, clarification and syllabus links
Complex numbers: the number i, where $\mathrm{i}^2 = -1$. Cartesian form $z = a + b\mathrm{i}$; the terms real part, imaginary part, conjugate, modulus and argument.	
The complex plane.	The complex plane is also known as the Argand diagram. **Link to:** vectors (AHL3.12).

Connections

Other contexts: Concepts in electrical engineering—impedance as a combination of resistance and reactance, also apparent power as a combination of real and reactive powers. These combinations take the form $a + b\mathrm{i}$.

TOK: How does language shape knowledge? For example, do the words “imaginary” and “complex” make the concepts more difficult than if they had different names?

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AHL 1.13

Content	Guidance, clarification and syllabus links
Modulus–argument (polar) form: $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{cis}\theta$. Euler form: $z = re^{\mathrm{i}\theta}$ Sums, products and quotients in Cartesian, polar or Euler forms and their geometric interpretation.	The ability to convert between Cartesian, modulus-argument (polar) and Euler form is expected.

Connections

Other contexts: Concepts in electrical engineering–phase angle/shift, power factor and apparent power as a complex quantity in polar form.

TOK: Why might it be said that $e^{\mathrm{i}\pi} + 1 = 0$ is beautiful? What is the place of beauty and elegance in mathematics? What about the place of creativity?

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AHL 1.14

Content	Guidance, clarification and syllabus links
Complex conjugate roots of quadratic and polynomial equations with real coefficients.	Complex roots occur in conjugate pairs.
De Moivre's theorem and its extension to rational exponents. Powers and roots of complex numbers.	Includes proof by induction for the case where $n \in Z^+$. **Link to:** sum and product of roots of polynomial equations (AHL 2.12), compound angle identities (AHL 3.10).

Connections

TOK: Could we ever reach a point where everything important in a mathematical sense is known? Reflect on the creation of complex numbers before their applications were known.

Enrichment: Can De Moivre's theorem be extended to all n?

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AHL 1.15

Content	Guidance, clarification and syllabus links
Proof by mathematical induction.	Proof should be incorporated throughout the course where appropriate. Mathematical induction links specifically to a wide variety of topics, for example complex numbers, differentiation, sums of sequences and divisibility.
Proof by contradiction.	**Examples:** Irrationality of $\sqrt{3}$; irrationality of the cube root of 5; Euclid's proof of an infinite number of prime numbers; if a is a rational number and b is an irrational number, then $a+b$ is an irrational number.
Use of a counterexample to show that a statement is not always true.	**Example:** Consider the set P of numbers of the form $n^2+41n+41,\ n \in N$, show that not all elements of P are prime. **Example:** Show that the following statement is not always true: there are no positive integer solutions to the equation $x^2+y^2=10$. It is not sufficient to state the counterexample alone. Students must explain why their example is a counterexample.

Connections

Other contexts: The Four-colour theorem

International-mindedness: How did the Pythagoreans find out that $\sqrt{2}$ is irrational?

TOK: What is the role of the mathematical community in determining the validity of a mathematical proof? Do proofs provide us with completely certain knowledge? What is the difference between the inductive method in science and proof by induction in mathematics?

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AHL 1.16

Content	**Guidance, clarification and syllabus links**
Solutions of systems of linear equations (a maximum of three equations in three unknowns), including cases where there is a unique solution, an infinite number of solutions or no solution.	These systems should be solved using both algebraic and technological methods, for example row reduction or matrices. Systems which have no solution(s) are inconsistent. Finding a general solution for a system with an infinite number of solutions. **Link to:** intersection of lines and planes (AHL 3.18).

Connections

TOK: Mathematics, Sense, Perception and Reason: If we can find solutions in higher dimensions can we reason that these spaces exist beyond our sense perception?

Download connections template

Topic 2: Functions

Concepts

Essential understandings

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.

Suggested concepts embedded in this topic:

Representation, relationships, space, quantity, equivalence.

AHL: Systems, patterns.

Content-specific conceptual understandings:

- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Our spatial frame of reference affects the visible part of a function and by changing this “window” can show more or less of the function to best suit our needs.
- Equivalent representations of quadratic functions can reveal different characteristics of the same relationship.
- Functions represent mappings that assign to each value of the independent variable (input) one and only one dependent variable (output).

AHL

- Extending results from a specific case to a general form can allow us to apply them to a larger system.

- Patterns can be identified in behaviours which can give us insight into appropriate strategies to model or solve them.
- The intersection of a system of equations may be represented graphically and algebraically and represents the solution that satisfies the equations.

SL content

Recommended teaching hours: 21

The aim of the SL content in the functions topic is to introduce students to the important unifying theme of a function in mathematics and to apply functional methods to a variety of mathematical situations.

Throughout this topic students should be given the opportunity to use technology, such as graphing packages and graphing calculators to develop and apply their knowledge of functions, rather than using elaborate analytic techniques.

On examination papers:

- questions may be set requiring the graphing of functions that do not explicitly appear on the syllabus
- the domain will be the largest possible domain for which a function is defined unless otherwise stated; this will usually be the real numbers

Sections SL2.1 to SL2.4 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 2.1

Content	Guidance, clarification and syllabus links
Different forms of the equation of a straight line. Gradient; intercepts. Lines with gradients m_1 and m_2 Parallel lines $m_1 = m_2$. Perpendicular lines $m_1 \times m_2 = -1$.	$y = mx + c$ (gradient-intercept form). $ax + by + d = 0$ (general form). $y - y_1 = m(x - x_1)$ (point-gradient form). Calculate gradients of inclines such as mountain roads, bridges, etc.

Connections

Other contexts: Gradients of mountain roads, gradients of access ramps.

Links to other subjects: Exchange rates and price and income elasticity, demand and supply curves (economics); graphical analysis in experimental work (sciences group subjects).

TOK: Descartes showed that geometric problems could be solved algebraically and vice versa. What does this tell us about mathematical representation and mathematical knowledge?

Download connections template

SL 2.2

Content	Guidance, clarification and syllabus links
Concept of a function, domain, range and graph. Function notation, for example $f(x)$, $v(t)$, $C(n)$. The concept of a function as a mathematical model.	**Example:** $f(x)=\sqrt{2-x}$, the domain is $x\leq 2$, range is $f(x)\geq 0$. A graph is helpful in visualizing the range.
Informal concept that an inverse function reverses or undoes the effect of a function. Inverse function as a reflection in the line $y=x$, and the notation $f^{-1}(x)$.	**Example:** Solving $f(x)=10$ is equivalent to finding $f^{-1}(10)$. Students should be aware that inverse functions exist for one to one functions; the domain of $f^{-1}(x)$ is equal to the range of $f(x)$.

Connections

Other contexts: Temperature and currency conversions**.**

Links to other subjects: Currency conversions and cost functions (economics and business management); projectile motion (physics).

Aim 8: What is the relationship between real-world problems and mathematical models?

International-mindedness: The development of functions by Rene Descartes (France), Gottfried Wilhelm Leibnitz (Germany) and Leonhard Euler (Switzerland); the notation for functions was developed by a number of different mathematicians in the 17th and 18th centuries–how did the notation we use today become internationally accepted?

TOK: Do you think mathematics or logic should be classified as a language?

Download connections template

SL 2.3

Content	Guidance, clarification and syllabus links
The graph of a function; its equation $y = f(x)$.	Students should be aware of the difference between the command terms “draw” and “sketch”.
Creating a sketch from information given or a context, including transferring a graph from screen to paper. Using technology to graph functions including their sums and differences.	All axes and key features should be labelled. This may include functions not specifically mentioned in topic 2.

Connections

Links to other subjects: Sketching and interpreting graphs (sciences group subjects, geography, economics).

TOK: Does studying the graph of a function contain the same level of mathematical rigour as studying the function algebraically? What are the advantages and disadvantages of having different forms and symbolic language in mathematics?

Download connections template

SL 2.4

Content	Guidance, clarification and syllabus links
Determine key features of graphs.	Maximum and minimum values; intercepts; symmetry; vertex; zeros of functions or roots of equations; vertical and horizontal asymptotes using graphing technology.
Finding the points of intersection of two curves or lines using technology.	

Connections

Links to other subjects: Identification and interpretation of key features of graphs (sciences group subjects, geography, economics); production possibilities curve model, market equilibrium (economics).

International-mindedness: Bourbaki group analytical approach versus the Mandelbrot visual approach.

Use of technology: Graphing technology with sliders to determine the effects of altering parameters and variables.

Download connections template

SL 2.5

Content	Guidance, clarification and syllabus links
Composite functions.	$(f \circ g)(x) = f(g(x))$
Identity function. Finding the inverse function $f^{-1}(x)$.	$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$ The existence of an inverse for one-to-one functions. **Link to**: concept of inverse function as a reflection in the line $y = x$ (SL 2.2).

Connections

TOK: Do you think mathematics or logic should be classified as a language?

Download connections template

SL 2.6

Content	Guidance, clarification and syllabus links
The quadratic function $f(x) = ax^2 + bx + c$: its graph, y -intercept $(0, c)$. Axis of symmetry. The form $f(x) = a(x - p)(x - q)$, x- intercepts $(p, 0)$ and $(q, 0)$. The form $f(x) = a(x - h)^2 + k$, vertex (h, k).	A quadratic graph is also called a parabola. **Link to**: transformations (SL 2.11). Candidates are expected to be able to change from one form to another.

Connections

Links to other subjects: Kinematics, projectile motion and simple harmonic motion (physics).

TOK: Are there fundamental differences between mathematics and other areas of knowledge? If so, are these differences more than just methodological differences?

Download connections template

SL 2.7

Content	Guidance, clarification and syllabus links
Solution of quadratic equations and inequalities. The quadratic formula.	Using factorization, completing the square (vertex form), and the quadratic formula. Solutions may be referred to as roots or zeros.
The discriminant $\Delta = b^2 - 4ac$ and the nature of the roots, that is, two distinct real roots, two equal real roots, no real roots.	**Example**: For the equation $3kx^2 + 2x + k = 0$, find the possible values of k, which will give two distinct real roots, two equal real roots or no real roots.

Connections

Links to other subjects: Projectile motion and energy changes in simple harmonic motion (physics); equilibrium equations (chemistry).

International-mindedness: The Babylonian method of multiplication: $ab = \dfrac{(a+b)^2 - a^2 - b^2}{2}$. Sulba Sutras in ancient India and the Bakhshali Manuscript contained an algebraic formula for solving quadratic equations.

TOK: What are the key concepts that provide the building blocks for mathematical knowledge?

Use of technology: Dynamic graphing software with a slider.

Enrichment: Deriving the quadratic formula by completing the square.

Download connections template

SL 2.8

Content	Guidance, clarification and syllabus links
The reciprocal function $f(x)=\frac{1}{x},\ x\neq 0$: its graph and self-inverse nature.	
Rational functions of the form $f(x)=\frac{ax+b}{cx+d}$ and their graphs. Equations of vertical and horizontal asymptotes.	Sketches should include all horizontal and vertical asymptotes and any intercepts with the axes. **Link to:** transformations (SL2.11). Vertical asymptote: $x=-\frac{d}{c}$; Horizontal asymptote: $y=\frac{a}{c}$.

Connections

International-mindedness: The development of functions, Rene Descartes (France), Gottfried Wilhelm Leibniz (Germany) and Leonhard Euler (Switzerland).

TOK: What are the implications of accepting that mathematical knowledge changes over time?

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SL 2.9

Content	Guidance, clarification and syllabus links
Exponential functions and their graphs: $f(x)=a^x,\ a>0,\ f(x)=e^x$ Logarithmic functions and their graphs: $f(x)=\log_a x,\ x>0,\ f(x)=\ln x,\ x>0.$	**Link to**: financial applications of geometric sequences and series (SL 1.4). Relationships between these functions: $a^x=e^{x\ln a};\ \log_a a^x=x,\ a,\quad x>0,\quad a\neq 1$ Exponential and logarithmic functions as inverses of each other.

Connections

Links to other subjects: Radioactive decay, charging and discharging capacitors (physics); first order reactions and activation energy (chemistry); growth curves (biology).

Aim 8: The phrase “exponential growth” is used popularly to describe a number of phenomena. Is ths a misleading use of a mathematical term?

TOK: What role do “models” play in mathematics? Do they play a different role in mathematics compared to their role in other areas of knowledge?

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SL 2.10

Content	Guidance, clarification and syllabus links
Solving equations, both graphically and analytically.	**Example:** $e^{2x} - 5e^{x} + 4 = 0$. **Link to:** function graphing skills (SL 2.3).
Use of technology to solve a variety of equations, including those where there is no appropriate analytic approach.	**Examples:** $e^{x} = \sin x$ $x^{4} + 5x - 6 = 0$
Applications of graphing skills and solving equations that relate to real-life situations.	**Link to:** exponential growth (SL 2.9)

Connections

Other contexts: Radioactive decay and population growth and decay, compound interest, projectile motion, braking distances.

Links to other subjects: Radioactive decay (physics); modelling (sciences group subjects); production possibilities curve model (economics).

TOK: What assumptions do mathematicians make when they apply mathematics to real-life situations?

Download connections template

SL 2.11

Content	Guidance, clarification and syllabus links
Transformations of graphs. Translations: $y = f(x) + b$; $y = f(x - a)$. Reflections (in both axes): $y = -f(x)$; $y = f(-x)$. Vertical stretch with scale factor p: $y = pf(x)$. Horizontal stretch with scale factor $\frac{1}{q}$: $y = f(qx)$.	Students should be aware of the relevance of the order in which transformations are performed. Dynamic graphing packages could be used to investigate these transformations.
Composite transformations.	**Example:** Using $y = x^2$ to sketch $y = 3x^2 + 2$ **Link to:** composite functions (SL2.5). **Not required at SL**: transformations of the form $f(ax + b)$.

Connections

Links to other subjects: Shift in supply and demand curves (Economics); induced emf and simple harmonic motion (physics).

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AHL content

Recommended teaching hours: 11

The aim of the AHL functions topic is to extend and build upon the aims, concepts and skills from the SL content. It introduces students to useful techniques for finding and using roots of polynomials, graphing and interpreting rational functions, additional ways to classify functions, solving inequations and solving equations involving modulus notation.

HL students may be required to use technology to solve equations where there is no appropriate analytic approach.

AHL 2.12

Content	Guidance, clarification and syllabus links
Polynomial functions, their graphs and equations; zeros, roots and factors. The factor and remainder theorems.	
Sum and product of the roots of polynomial equations.	For the polynomial equation: $\sum_{r=0}^{n} a_r x^r = 0$, the sum is $\frac{-a_{n-1}}{a_n}$ the product is $\frac{(-1)^n a_0}{a_n}$ **Link to:** complex roots of quadratic and polynomial equations (AHL 1.14).

Connections

Links to other subjects: Modelling (sciences group subjects)

TOK: Is it an oversimplification to say that some areas of knowledge give us facts whereas other areas of knowledge give us interpretations?

Enrichment: Viete's theorem in full, "The equation that couldn't be solved" quadratic formula reducing a quadratic to a linear, Cardano and Bombelli.

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AHL 2.13

Content	Guidance, clarification and syllabus links
Rational functions of the form $f(x)=\dfrac{ax+b}{cx^2+dx+e}$, and $f(x)=\dfrac{ax^2+bx+c}{dx+e}$	The reciprocal function is a particular case. Graphs should include all asymptotes (horizontal, vertical and oblique) and any intercepts with axes. Dynamic graphing packages could be used to investigate these functions. **Link to:** rational functions (SL 2.8).

Connections

International mindedness: Bourbaki group analytical approach versus Mandelbrot visual approach.

TOK: Does studying the graph of a function contain the same level of mathematical rigour as studying the function algebraically? What are the advantages and disadvantages of having different forms and symbolic language in mathematics?

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AHL 2.14

Content	Guidance, clarification and syllabus links
Odd and even functions.	Even: $f(-x)=f(x)$ Odd: $f(-x)=-f(x)$ Includes periodic functions.
Finding the inverse function, $f^{-1}(x)$, including domain restriction.	
Self-inverse functions.	

Connections

International-mindedness: The notation for functions was developed by a number of different mathematicians in the 17th and 18th centuries. How did the notation we use today become internationally accepted?

TOK: If systems of notation and measurement are culturally and historically situated, does this mean mathematics cannot be seen as independent of culture?

Download connections template

AHL 2.15

Content	Guidance, clarification and syllabus links
Solutions of $g(x) \geq f(x)$, both graphically and analytically.	Graphical or algebraic methods for simple polynomials up to degree 3. Use of technology for these and other functions.

Connections

TOK: Are there differences in terms of value that different cultures ascribe to mathematics, or to the relative value that they ascribe to different areas of knowledge?

Download connections template

AHL 2.16

Content	Guidance, clarification and syllabus links
The graphs of the functions, $y = \|f(x)\|$ and $y = f(\|x\|)$, $y = \frac{1}{f(x)}$, $y = f(ax+b)$, $y = [f(x)]^2$.	Dynamic graphing packages could be used to investigate these transformations.
Solution of modulus equations and inequalities.	**Example:** $\|3x\arccos(x)\| > 1$

Connections

International-mindedness: The Bourbaki group analytic approach versus Mandlebrot visual approach.

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Topic 3: Geometry and trigonometry

Concepts

Essential understandings:

Geometry and trigonometry allows us to quantify the physical world, enhancing our spatial awareness in two and three dimensions. This topic provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

Suggested concepts embedded in this topic:

Generalization, space, relationships, equivalence, representation,

AHL: Quantity, Modelling.

Content-specific conceptual understandings:

- The properties of shapes depend on the dimension they occupy in space.
- Volume and surface area of shapes are determined by formulae, or general mathematical relationships or rules expressed using symbols or variables.
- The relationships between the length of the sides and the size of the angles in a triangle can be used to solve many problems involving position, distance, angles and area.
- Equivalent measurement systems, such as degrees and radians, can be used for angles to facilitate ease of calculation.
- Different representations of the values of trigonometric relationships, such as exact or approximate, may not be equivalent to one another.
- The trigonometric functions of angles may be defined on the unit circle, which can visually and algebraically represent the periodic or symmetric nature of their values.

AHL

- Position and movement can be modelled in three-dimensional space using vectors.
- The relationships between algebraic, geometric and vector methods can help us to solve problems and quantify those positions and movements.

SL content

Recommended teaching hours: 25

The aim of the SL content of the geometry and trigonometry topic is to introduce students to geometry in three dimensions and to non right-angled trigonometry. Students will explore the circular functions and use properties and identities to solve problems in abstract and real-life contexts.

Throughout this topic students should be given the opportunity to use technology such as graphing packages, graphing calculators and dynamic geometry software to develop and apply their knowledge of geometry and trigonometry.

On examination papers, radian measure should be assumed unless otherwise indicated.

Sections SL3.1 to SL3.3 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 3.1

Content	**Guidance, clarification and syllabus links**
The distance between two points in three-dimensional space, and their midpoint. Volume and surface area of three-dimensional solids including right-pyramid, right cone, sphere, hemisphere and combinations of these solids. The size of an angle between two intersecting lines or between a line and a plane.	In SL examinations, only right-angled trigonometry questions will be set in reference to three-dimensional shapes. In problems related to these topics, students should be able to identify relevant right-angled triangles in three-dimensional objects and use them to find unknown lengths and angles.

Connections

Other contexts: Architecture and design.

Links to other subjects: Design technology; volumes of stars and inverse square law (physics).

TOK: What is an axiomatic system? Are axioms self evident to everybody?

Download connections template

SL 3.2

Content	Guidance, clarification and syllabus links
Use of sine, cosine and tangent ratios to find the sides and angles of right-angled triangles.	In all areas of this topic, students should be encouraged to sketch well-labelled diagrams to support their solutions. **Link to:** inverse functions (SL2.2) when finding angles.
The sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$; $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$. Area of a triangle as $\frac{1}{2}ab\sin C$.	This section does not include the ambiguous case of the sine rule.

Connections

Other contexts: Triangulation, map-making.

Links to other subjects: Vectors (physics).

International-mindedness: Diagrams of Pythagoras' theorem occur in early Chinese and Indian manuscripts. The earliest references to trigonometry are in Indian mathematics; the use of triangulation to find the curvature of the Earth in order to settle a dispute between England and France over Newton's gravity.

TOK: Is it ethical that Pythagoras gave his name to a theorem that may not have been his own creation? What criteria might we use to make such a judgment?

Download connections template

SL 3.3

Content	Guidance, clarification and syllabus links
Applications of right and non-right angled trigonometry, including Pythagoras's theorem. Angles of elevation and depression. Construction of labelled diagrams from written statements.	Contexts may include use of bearings.

Connections

Other contexts: Triangulation, map-making, navigation and radio transmissions. Use of parallax for navigation.

Links to other subjects: Vectors, scalars, forces and dynamics (physics); field studies (sciences group subjects)

Aim 8: Who really invented Pythagoras's theorem?

Aim 9: In how many ways can you prove Pythagoras's theorem?

International-mindedness: The use of triangulation to find the curvature of the Earth in order to settle a dispute between England and France over Newton's gravity.

TOK: If the angles of a triangle can add up to less than 180°, 180° or more than 180°, what does this tell us about the nature of mathematical knowledge?

Download connections template

SL 3.4

Content	Guidance, clarification and syllabus links
The circle: radian measure of angles; length of an arc; area of a sector.	Radian measure may be expressed as exact multiples of π, or decimals.

Connections

Links to other subjects: Diffraction patterns and circular motion (physics).

International-mindedness: Seki Takakazu calculating π to ten decimal places; Hipparchus, Menelaus and Ptolemy; Why are there 360 degrees in a complete turn? Links to Babylonian mathematics.

TOK: Which is a better measure of angle: radian or degree? What criteria can/do/should mathematicians use to make such decisions?

Download connections template

SL 3.5

Content	Guidance, clarification and syllabus links
Definition of $\cos\theta$, $\sin\theta$ in terms of the unit circle.	Includes the relationship between angles in different quadrants. **Examples:** $\cos x = \cos(-x)$ $\tan(3\pi - x) = -\tan x$ $\sin(\pi + x) = -\sin x$
Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$.	The equation of a straight line through the origin is $y = x\tan\theta$, where θ is the angle formed between the line and positive x-axis.
Exact values of trigonometric ratios of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ and their multiples.	$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$, $\tan 210° = \frac{\sqrt{3}}{3}$
Extension of the sine rule to the ambiguous case.	

Connections

International-mindedness: The first work to refer explicitly to the sine as a function of an angle is the Aryabhatiya of Aryabhata (ca 510).

TOK: Trigonometry was developed by successive civilizations and cultures. To what extent is mathematical knowledge embedded in particular traditions or bound to particular cultures? How have key events in the history of mathematics shaped its current form and methods?

Enrichment: The proof of Pythagoras' theorem in three dimensions**.**

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SL 3.6

Content	Guidance, clarification and syllabus links
The Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$. Double angle identities for sine and cosine.	Simple geometrical diagrams and dynamic graphing packages may be used to illustrate the double angle identities (and other trigonometric identities).
The relationship between trigonometric ratios.	**Examples:** Given $\sin\theta$, find possible values of $\tan\theta$, (without finding θ). Given $\cos x = \frac{3}{4}$ and x is acute, find $\sin 2x$, (without finding x).

Connections

Download connections template

SL 3.7

Content	Guidance, clarification and syllabus links
The circular functions $\sin x$, $\cos x$, and $\tan x$; amplitude, their periodic nature, and their graphs Composite functions of the form $f(x) = a\sin(b(x+c)) + d$.	Trigonometric functions may have domains given in degrees or radians. **Examples:** $f(x) = \tan\left(x - \frac{\pi}{4}\right)$, $f(x) = 2\cos(3(x-4)) + 1$.
Transformations.	**Example:** $y = \sin x$ used to obtain $y = 3\sin 2x$ by a stretch of scale factor 3 in the y direction and a stretch of scale factor $\frac{1}{2}$ in the x direction. **Link to:** transformations of graphs (SL2.11).
Real-life contexts.	**Examples:** height of tide, motion of a Ferris wheel. Students should be aware that not all regression technology produces trigonometric functions in the form $f(x) = a\sin(b(x+c)) + d$.

Connections

Links to other subjects: Simple harmonic motion (physics).

TOK: Music can be expressed using mathematics. What does this tell us about the relationship between music and mathematics?

Download connections template

SL 3.8

Content	Guidance, clarification and syllabus links
Solving trigonometric equations in a finite interval, both graphically and analytically.	**Examples:** $2\sin x = 1,\quad 0 \le x \le 2\pi$ $2\sin 2x = 3\cos x,\quad 0° \le x \le 180°$ $2\tan(3(x-4)) = 1,\quad -\pi \le x \le 3\pi$
Equations leading to quadratic equations in $\sin x,\ \cos x$ or $\tan x$.	**Examples:** $2\sin^2 x + 5\cos x + 1 = 0$ for $0 \le x \le 4\pi$, $2\sin x = \cos 2x,\quad -\pi \le x \le \pi$ **Not required:** The general solution of trigonometric equations.

Connections

Download connections template

AHL content

Recommended teaching hours: 26

The aim of the AHL content in the geometry and trigonometry topic is to extend and build upon the aims, concepts and skills from the SL content. It further explores the circular functions, introduces some important trigonometric identities, and introduces vectors in two and three dimensions. This will facilitate problem-solving involving points, lines and planes.

On examination papers radian measure should be assumed unless otherwise indicated.

AHL 3.9

Content	Guidance, clarification and syllabus links
Definition of the reciprocal trigonometric ratios $\sec\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$. Pythagorean identities: $1+\tan^2\theta=\sec^2\theta$ $1+\cot^2\theta=\operatorname{cosec}^2\theta$ The inverse functions $f(x)=\arcsin x$, $f(x)=\arccos x$, $f(x)=\arctan x$; their domains and ranges; their graphs.	

Connections

International-mindedness: The origin of degrees in the mathematics of Mesopotamia and why we use minutes and seconds for time; the origin of the word sine.

TOK: What is the relationship between concepts and facts? To what extent do the concepts that we use shape the conclusions that we reach?

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AHL 3.10

Content	Guidance, clarification and syllabus links
Compound angle identities. Double angle identity for tan.	Derivation of double angle identities from compound angle identities. **Link to:** De Moivre's theorem (AHL1.14).

Connections

Other contexts: Triangulation used by GPSs (global positioning systems); concepts in electrical engineering including generation of sinusoidal voltage.

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AHL 3.11

Content	Guidance, clarification and syllabus links
Relationships between trigonometric functions and the symmetry properties of their graphs.	$\sin(\pi - \theta) = \sin\theta \cos(\pi - \theta) = -\cos\theta \tan(\pi - \theta) = -\tan\theta$ **Link to**: the unit circle (SL3.5), odd and even functions (AHL2.14), compound angles (AHL3.10).

Connections

Links to other subjects: Simple harmonic motion graphs (physics)

TOK: Mathematics and knowledge claims: how can there be an infinite number of discrete solutions to an equation?

Download connections template

AHL 3.12

Content	Guidance, clarification and syllabus links
Concept of a vector; position vectors; displacement vectors. Representation of vectors using directed line segments. Base vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. Components of a vector: $\boldsymbol{v}=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k}$.	
Algebraic and geometric approaches to the following: • the sum and difference of two vectors • the zero vector $\boldsymbol{0}$, the vector $-\boldsymbol{v}$ • multiplication by a scalar, $k\boldsymbol{v}$, parallel vectors • magnitude of a vector, $\lvert\boldsymbol{v}\rvert$; unit vectors, $\frac{\boldsymbol{v}}{\lvert\boldsymbol{v}\rvert}$ • position vectors $\overrightarrow{\mathrm{OA}}=\boldsymbol{a}$, $\overrightarrow{\mathrm{OB}}=\boldsymbol{b}$ • displacement vector $\overrightarrow{\mathrm{AB}}=\boldsymbol{b}-\boldsymbol{a}$ Proofs of geometrical properties using vectors.	Distance between points A and B is the magnitude of $\overrightarrow{\mathrm{AB}}$

Connections

Links to other subjects: Vectors, scalars, forces and dynamics (physics).

Aim 8: Vectors are used to solve many problems in position location. This can be used to save a lost sailor or destroy a building with a laser-guided bomb.

TOK: Vectors are used to solve many problems in position location. This can be used to save a lost sailor or destroy a building with a laser-guided bomb. To what extent does possession of knowledge carry with it an ethical obligation?

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AHL 3.13

Content	Guidance, clarification and syllabus links
The definition of the scalar product of two vectors. The angle between two vectors. Perpendicular vectors; parallel vectors.	Applications of the properties of the scalar product $\boldsymbol{v}\cdot\boldsymbol{w}=\boldsymbol{w}\cdot\boldsymbol{v}$; $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{w}$; $(k\boldsymbol{v})\cdot\boldsymbol{w}=k(\boldsymbol{v}\cdot\boldsymbol{w})$; $\boldsymbol{v}\cdot\boldsymbol{v}=\lvert\boldsymbol{v}\rvert^2$. $\boldsymbol{v}\cdot\boldsymbol{w}=\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. For non-zero vectors $\boldsymbol{v}\cdot\boldsymbol{w}=0$ is equivalent to the vectors being perpendicular; for parallel vectors $\lvert\boldsymbol{v}\cdot\boldsymbol{w}\rvert=\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert$.

Connections

Links to other subjects: Forces and dynamics (physics).

TOK: The nature of mathematics: why this definition of scalar product?

Enrichment: Proof of the cosine rule using the dot product.

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AHL 3.14

Content	Guidance, clarification and syllabus links
Vector equation of a line in two and three dimensions: $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$.	Relevance of $\boldsymbol{a}$ (position) and $\boldsymbol{b}$ (direction). Knowledge of the following forms for equations of lines: Parametric form: $x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$. Cartesian form: $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$.
The angle between two lines.	Using the scalar product of the two direction vectors.
Simple applications to kinematics.	Interpretation of λ as time and $\boldsymbol{b}$ as velocity, with $\lvert\boldsymbol{b}\rvert$ representing speed.

Connections

Other contexts: Modelling linear motion in three dimensions; navigational devices, for example GPS.

TOK: Why might it be argued that one form of representation is superior to another? What criteria might a mathematician use in making such an argument?

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AHL 3.15

Content	Guidance, clarification and syllabus links
Coincident, parallel, intersecting and skew lines, distinguishing between these cases. Points of intersection.	Skew lines are non-parallel lines that do not intersect in three-dimensional space.

Connections

TOK: How can there be an infinite number of discrete solutions to an equation? What does this suggest about the nature of mathematical knowledge and how it compares to knowledge in other disciplines?

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AHL 3.16

Content	Guidance, clarification and syllabus links
The definition of the vector product of two vectors.	The vector product is also known as the “cross product”. $\boldsymbol{v}\times\boldsymbol{w}=\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\sin\theta\boldsymbol{n}$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$, and $\boldsymbol{n}$ is the unit normal vector whose direction is given by the right-hand screw rule.
Properties of the vector product.	$\boldsymbol{v}\times\boldsymbol{w}=-\boldsymbol{w}\times\boldsymbol{v}$; $\boldsymbol{u}\times(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{u}\times\boldsymbol{v}+\boldsymbol{u}\times\boldsymbol{w}$; $(k\boldsymbol{v})\times\boldsymbol{w}=k(\boldsymbol{v}\times\boldsymbol{w})$; $\boldsymbol{v}\times\boldsymbol{v}=\boldsymbol{0}$. For non-zero vectors $\boldsymbol{v}\times\boldsymbol{w}=\boldsymbol{0}$ is equivalent to the vectors being parallel.
Geometric interpretation of $\lvert\boldsymbol{v}\times\boldsymbol{w}\rvert$	Use of $\lvert\boldsymbol{v}\times\boldsymbol{w}\rvert$ to find the area of a parallelogram (and hence a triangle).

Connections

Links to other subjects: Magnetic forces and fields (physics).

TOK: To what extent is certainty attainable in mathematics? Is certainty attainable, or desirable, in other areas of knowledge?

Download connections template

AHL 3.17

Content	Guidance, clarification and syllabus links
Vector equations of a plane: $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$, where $\boldsymbol{b}$ and $\boldsymbol{c}$ are non-parallel vectors within the plane. $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is a normal to the plane and $\boldsymbol{a}$ is the position vector of a point on the plane. Cartesian equation of a plane $ax + by + cz = d$.	

Connections

Download connections template

AHL 3.18

Content	Guidance, clarification and syllabus links
Intersections of: a line with a plane; two planes; three planes. Angle between: a line and a plane; two planes.	Finding intersections by solving equations; geometrical interpretation of solutions. **Link to**: solutions of systems of linear equations (AHL 1.16).

Connections

TOK: Mathematics and the knower: are symbolic representations of three-dimensional objects easier to deal with than visual representations? What does this tell us about our knowledge of mathematics in other dimensions?

Download connections template

Topic 4: Statistics and probability

Concepts

Essential understandings:

Statistics is concerned with the collection, analysis and interpretation of data and the theory of probability can be used to estimate parameters, discover empirical laws, test hypotheses and predict the occurrence of events. Statistical representations and measures allow us to represent data in many different forms to aid interpretation.

Probability enables us to quantify the likelihood of events occurring and so evaluate risk. Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned to differentiate between the theoretical and the empirical/observed. Probability theory allows us to make informed choices, to evaluate risk, and to make predictions about seemingly random events.

Suggested concepts embedded in this topic:

Quantity, validity, approximation, generalization.

AHL: Change, systems.

Content-specific conceptual understandings:

- Organizing, representing, analysing and interpreting data and utilizing different statistical tools facilitates prediction and drawing of conclusions.
- Different statistical techniques require justification and the identification of their limitations and validity.
- Approximation in data can approach the truth but may not always achieve it.
- Some techniques of statistical analysis, such as regression, standardization or formulae, can be applied in a practical context to apply to general cases.
- Modelling through statistics can be reliable, but may have limitations.

AHL

- Properties of probability density functions can be used to identify measure of central tendency such as mean, mode and median.
- Probability methods such as Bayes theorem can be applied to real-world systems, such as medical studies or economics, to inform decisions and to better understand outcomes.

SL content

Recommended teaching hours: 27

The aim of the SL content in the statistics and probability topic is to introduce students to the important concepts, techniques and representations used in statistics and probability. Students should be given the opportunity to approach this topic in a practical way, to understand why certain techniques are used and to interpret the results. The use of technology such as simulations, spreadsheets, statistics software and statistics apps can greatly enhance this topic.

It is expected that most of the calculations required will be carried out using technology, but explanations of calculations by hand may enhance understanding. The emphasis is on understanding and interpreting the results obtained, in context.

In examinations students should be familiar with how to use the statistics functionality of allowed technology.

At SL the data set will be considered to be the population unless otherwise stated.

Sections SL4.1 to SL4.9 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 4.1

Content	Guidance, clarification and syllabus links
Concepts of population, sample, random sample, discrete and continuous data.	This is designed to cover the key questions that students should ask when they see a data set/ analysis.
Reliability of data sources and bias in sampling.	Dealing with missing data, errors in the recording of data.
Interpretation of outliers.	Outlier is defined as a data item which is more than $1.5 \times$ interquartile range (IQR) from the nearest quartile. Awareness that, in context, some outliers are a valid part of the sample but some outlying data items may be an error in the sample. **Link to**: box and whisker diagrams (SL4.2) and measures of dispersion (SL4.3).
Sampling techniques and their effectiveness.	Simple random, convenience, systematic, quota and stratified sampling methods.

Connections

Links to other subjects: Descriptive statistics and random samples (biology, psychology, sports exercise and health science, environmental systems and societies, geography, economics; business management); research methodologies (psychology).

Aim 8: Misleading statistics; examples of problems caused by absence of representative samples, for example Google flu predictor, US presidential elections in 1936, Literary Digest v George Gallup, Boston “pot-hole” app.

International-mindedness: The Kinsey report–famous sampling techniques.

TOK: Why have mathematics and statistics sometimes been treated as separate subjects? How easy is it to be misled by statistics? Is it ever justifiable to purposely use statistics to mislead others?

Download connections template

SL 4.2

Content	Guidance, clarification and syllabus links
Presentation of data (discrete and continuous): frequency distributions (tables).	Class intervals will be given as inequalities, without gaps.
Histograms. Cumulative frequency; cumulative frequency graphs; use to find median, quartiles, percentiles, range and interquartile range (IQR).	Frequency histograms with equal class intervals. **Not required:** Frequency density histograms.
Production and understanding of box and whisker diagrams.	Use of box and whisker diagrams to compare two distributions, using symmetry, median, interquartile range or range. Outliers should be indicated with a cross. Determining whether the data may be normally distributed by consideration of the symmetry of the box and whiskers.

Connections

Links to other subjects: Presentation of data (sciences, individuals and societies).

International-mindedness: Discussion of the different formulae for the same statistical measure (for example, variance).

TOK: What is the difference between information and data? Does “data” mean the same thing in different areas of knowledge?

Download connections template

SL 4.3

Content	Guidance, clarification and syllabus links
Measures of central tendency (mean, median and mode). Estimation of mean from grouped data.	Calculation of mean using formula and technology. Students should use mid-interval values to estimate the mean of grouped data.
Modal class.	For equal class intervals only.
Measures of dispersion (interquartile range, standard deviation and variance).	Calculation of standard deviation and variance of the sample using only technology, however hand calculations may enhance understanding. Variance is the square of the standard deviation.
Effect of constant changes on the original data.	**Examples:** If three is subtracted from the data items, then the mean is decreased by three, but the standard deviation is unchanged. If all the data items are doubled, the mean is doubled and the standard deviation is also doubled.
Quartiles of discrete data.	Using technology. Awareness that different methods for finding quartiles exist and therefore the values obtained using technology and by hand may differ.

Connections

Other contexts: Comparing variation and spread in populations, human or natural, for example agricultural crop data, social indicators, reliability and maintenance.

Links to other subjects: Descriptive statistics (sciences and individuals and societies); consumer price index (economics).

International-mindedness: The benefits of sharing and analysing data from different countries; discussion of the different formulae for variance.

TOK: Could mathematics make alternative, equally true, formulae? What does this tell us about mathematical truths? Does the use of statistics lead to an over-emphasis on attributes that can be easily measured over those that cannot?

Download connections template

SL 4.4

Content	Guidance, clarification and syllabus links
Linear correlation of bivariate data. Pearson's product-moment correlation coefficient, *r*.	Technology should be used to calculate *r*. However, hand calculations of *r* may enhance understanding. Critical values of *r* will be given where appropriate. Students should be aware that Pearson's product moment correlation coefficient (*r*) is only meaningful for linear relationships.
Scatter diagrams; lines of best fit, by eye, passing through the mean point.	Positive, zero, negative; strong, weak, no correlation. Students should be able to make the distinction between correlation and causation and know that correlation does not imply causation.
Equation of the regression line of y on x. Use of the equation of the regression line for prediction purposes. Interpret the meaning of the parameters, a and b, in a linear regression $y = ax + b$.	Technology should be used to find the equation. Students should be aware: • of the dangers of extrapolation • that they cannot always reliably make a prediction of x from a value of y, when using a y on x line.

Connections

Other contexts: Linear regressions where correlation exists between two variables. Exploring cause and dependence for categorical variables, for example, on what factors might political persuasion depend?

Links to other subjects: Curves of best fit, correlation and causation (sciences group subjects); scatter graphs (geography).

Aim 8: The correlation between smoking and lung cancer was "discovered" using mathematics. Science had to justify the cause.

TOK: Correlation and causation–can we have knowledge of cause and effect relationships given that we can only observe correlation? What factors affect the reliability and validity of mathematical models in describing real-life phenomena?

Download connections template

SL 4.5

Content	Guidance, clarification and syllabus links
Concepts of trial, outcome, equally likely outcomes, relative frequency, sample space (U) and event. The probability of an event A is $\mathrm{P}(A)=\frac{n(A)}{n(U)}$. The complementary events A and A' (not A).	Sample spaces can be represented in many ways, for example as a table or a list. Experiments using coins, dice, cards and so on, can enhance understanding of the distinction between experimental (relative frequency) and theoretical probability. Simulations may be used to enhance this topic.
Expected number of occurrences.	**Example:** If there are 128 students in a class and the probability of being absent is 0.1, the expected number of absent students is 12.8.

Connections

Other contexts: Actuarial studies and the link between probability of life spans and insurance premiums, government planning based on likely projected figures, Monte Carlo methods.

Links to other subjects: Theoretical genetics and Punnett squares (biology); the position of a particle (physics).

Aim 8: The ethics of gambling.

International-mindedness: The St Petersburg paradox; Chebyshev and Pavlovsky (Russian).

TOK: To what extent are theoretical and experimental probabilities linked? What is the role of emotion in our perception of risk, for example in business, medicine and travel safety?

Use of technology: Computer simulations may be useful to enhance this topic.

Download connections template

SL 4.6

Content	Guidance, clarification and syllabus links
Use of Venn diagrams, tree diagrams, sample space diagrams and tables of outcomes to calculate probabilities.	
Combined events: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$. Mutually exclusive events: $\mathrm{P}(A \cap B) = 0$.	The non-exclusivity of "or".
Conditional probability: $\mathrm{P}(A\vert B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.	An alternate form of this is: $\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A\vert B)$. Problems can be solved with the aid of a Venn diagram, tree diagram, sample space diagram or table of outcomes without explicit use of formulae. Probabilities with and without replacement.
Independent events: $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.	

Connections

Aim 8: The gambling issue: use of probability in casinos. Could or should mathematics help increase incomes in gambling?

TOK: Can calculation of gambling probabilities be considered an ethical application of mathematics? Should mathematicians be held responsible for unethical applications of their work?

Download connections template

SL 4.7

Content	Guidance, clarification and syllabus links
Concept of discrete random variables and their probability distributions. Expected value (mean), for discrete data. Applications.	Probability distributions will be given in the following ways: $\begin{array}{cccccc} X & 1 & 2 & 3 & 4 & 5 \\ \mathrm{P}(X=x) & 0.1 & 0.2 & 0.15 & 0.05 & 0.5 \end{array}$ $\mathrm{P}(X=x)=\frac{1}{18}(4+x)$ for $x \in \{1, 2, 3\}$ $\mathrm{E}(X)=0$ indicates a fair game where X represents the gain of a player.

Connections

Other contexts: Games of chance.

Aim 8: Why has it been argued that theories based on the calculable probabilities found in casinos are pernicious when applied to everyday life (for example, economics)?

TOK: What do we mean by a “fair” game? Is it fair that casinos should make a profit?

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SL 4.8

Content	Guidance, clarification and syllabus links
Binomial distribution. Mean and variance of the binomial distribution.	Situations where the binomial distribution is an appropriate model. In examinations, binomial probabilities should be found using available technology. **Not required:** Formal proof of mean and variance. **Link to:** expected number of occurrences (SL4.5).

Connections

Aim 8: Pascal's triangle, attributing the origin of a mathematical discovery to the wrong mathematician.

International-mindedness: The so-called "Pascal's triangle" was known to the Chinese mathematician Yang Hui much earlier than Pascal.

TOK: What criteria can we use to decide between different models?

Enrichment: Hypothesis testing using the binomial distribution.

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SL 4.9

Content	Guidance, clarification and syllabus links
The normal distribution and curve. Properties of the normal distribution. Diagrammatic representation.	Awareness of the natural occurrence of the normal distribution. Students should be aware that approximately 68% of the data lies between $\mu \pm \sigma$, 95% lies between $\mu \pm 2\sigma$ and 99.7% of the data lies between $\mu \pm 3\sigma$.
Normal probability calculations.	Probabilities and values of the variable must be found using technology.
Inverse normal calculations	For inverse normal calculations mean and standard deviation will be given. This does not involve transformation to the standardized normal variable *z*.

Connections

Links to other subjects: Normally distributed real-life measurements and descriptive statistics (sciences group subjects, psychology, environmental systems and societies)

Aim 8: Why might the misuse of the normal distribution lead to dangerous inferences and conclusions?

International-mindedness: De Moivre's derivation of the normal distribution and Quetelet's use of it to describe *l'homme moyen*.

TOK: To what extent can we trust mathematical models such as the normal distribution? How can we know what to include, and what to exclude, in a model?

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SL 4.10

Content	Guidance, clarification and syllabus links
Equation of the regression line of x on y.	
Use of the equation for prediction purposes.	Students should be aware that they cannot always reliably make a prediction of y from a value of x, when using an x on y line.

Connections

TOK: Is it possible to have knowledge of the future?

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SL 4.11

Content	Guidance, clarification and syllabus links
Formal definition and use of the formulae: $\mathrm{P}(A\|B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$ for conditional probabilities, and $\mathrm{P}(A\|B)=\mathrm{P}(A)=\mathrm{P}(A\|B')$ for independent events.	An alternate form of this is: $\mathrm{P}(A\cap B)=\mathrm{P}(B)\mathrm{P}(A\|B)$. Testing for independence.

Connections

Other contexts: Use of probability methods in medical studies to assess risk factors for certain diseases.

TOK: Given the interdisciplinary nature of many real-world applications of probability, is the division of knowledge into discrete disciplines or areas of knowledge artificial and/or useful?

Download connections template

SL 4.12

Content	**Guidance, clarification and syllabus links**
Standardization of normal variables (z - values).	Probabilities and values of the variable must be found using technology. The standardized value (z) gives the number of standard deviations from the mean.
Inverse normal calculations where mean or standard deviation are unknown.	Use of z-values to calculate unknown means and standard deviations.

Connections

Links to other subjects: The normal distribution (biology); descriptive statistics (psychology).

Download connections template

AHL content

Recommended teaching hours: 6

The aim of the AHL content in the statistics and probability topic is to extend and build upon the aims, concepts and skills from the SL content. Students are introduced to further conditional probability theory in the form of Bayes Theorem and properties of discrete and continuous random variables are further explored.

AHL 4.13

Content	Guidance, clarification and syllabus links
Use of Bayes’ theorem for a maximum of three events.	**Link to:** independent events (SL4.6).

Connections

Other contexts: Use of probability methods in medical studies to assess risk factors for certain diseases.

TOK: Does the applicability of knowledge vary across the different areas of knowledge? What would the implications be if the value of all knowledge was measured solely in terms of its applicability?

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AHL 4.14

Content	Guidance, clarification and syllabus links
Variance of a discrete random variable.	**Link to:** discrete random variables (SL 4.7)
Continuous random variables and their probability density functions.	$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$ including piecewise functions.
Mode and median of continuous random variables.	For a continuous random variable, a value at which the probability density function has a maximum value is called a mode and for the median: $\int_{-\infty}^{m} f(x)\mathrm{d}x = \frac{1}{2}$.
Mean, variance and standard deviation of both discrete and continuous random variables.	Use of the notation $\mathrm{E}(X)$, $\mathrm{E}(X^2)$, $\mathrm{Var}(X)$, where $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ and related formulae. Use of $\mathrm{E}(X)$ for "fair" games.
The effect of linear transformations of X.	$\mathrm{E}(aX+b) = a\mathrm{E}(X) + b, \quad \mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$

Connections

Other contexts: Other discrete distributions, for example Poisson, may be appropriate for IA/toolkit and further investigation; expected value used in decision making in business, economics and life in general; expected gain to insurance companies.

TOK: Is mathematics more or less useful than other areas of knowledge for solving problems?

Enrichment: Is there a relationship between the interquartile range and the standard deviation for a normally distributed data set?

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Topic 5: Calculus

Concepts

Essential understandings:

Calculus describes rates of change between two variables and the accumulation of limiting areas. Understanding these rates of change and accumulations allow us to model, interpret and analyze real-world problems and situations. Calculus helps us to understand the behaviour of functions and allows us to interpret the features of their graphs.

Suggested concepts embedded in this topic:

Change, patterns, relationships, approximation, generalization, space, modelling.

AHL: Systems, quantity.

Content-specific conceptual understandings:

- The derivative may be represented physically as a rate of change and geometrically as the gradient or slope function.
- Areas under curves can be can be approximated by the sum of the areas of rectangles which may be calculated even more accurately using integration.
- Examining rates of change close to turning points helps to identify intervals where the function increases/decreases, and identify the concavity of the function.
- Numerical integration can be used to approximate areas in the physical world.
- Mathematical modelling can provide effective solutions to real-life problems in optimization by maximizing or minimizing a quantity, such as cost or profit.
- Derivatives and integrals describe real-world kinematics problems in two and three-dimensional space by examining displacement, velocity and acceleration.

AHL

- Some functions may be continuous everywhere but not differentiable everywhere.

- A finite number of terms of an infinite series can be a general approximation of a function over a limited domain.
- Limits describe the output of a function as the input approaches a certain value and can represent convergence and divergence.
- Examining limits of functions at a point can help determine continuity and differentiability at a point.

SL content

Recommended teaching hours: 28

The aim of the SL content in the calculus topic is to introduce students to the concepts and techniques of differential and integral calculus and their applications.

Throughout this topic students should be given the opportunity to use technology such as graphing packages and graphing calculators to develop and apply their knowledge of calculus.

Sections SL5.1 to SL5.5 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 5.1

Content	Guidance, clarification and syllabus links
Introduction to the concept of a limit.	Estimation of the value of a limit from a table or graph. **Not required:** Formal analytic methods of calculating limits.
Derivative interpreted as gradient function and as rate of change.	Forms of notation: $\frac{\mathrm{d}y}{\mathrm{d}x}$, $f'(x)$, $\frac{\mathrm{d}V}{\mathrm{d}r}$ or $\frac{\mathrm{d}s}{\mathrm{d}t}$ for the first derivative. Informal understanding of the gradient of a curve as a limit.

Connections

Links to other subjects: Marginal cost, marginal revenue, marginal profit, market structures (economics); kinematics, induced emf and simple harmonic motion (physics); interpreting the gradient of a curve (chemistry)

Aim 8: The debate over whether Newton or Leibnitz discovered certain calculus concepts; how the Greeks' distrust of zero meant that Archimedes' work did not lead to calculus.

International-mindedness: Attempts by Indian mathematicians (500-1000 CE) to explain division by zero.

TOK: What value does the knowledge of limits have? Is infinitesimal behaviour applicable to real life? Is intuition a valid way of knowing in mathematics?

Use of technology: Spreadsheets, dynamic graphing software and GDC should be used to explore ideas of limits, numerically and graphically. Hypotheses can be formed and then tested using technology.

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SL 5.2

Content	Guidance, clarification and syllabus links
Increasing and decreasing functions. Graphical interpretation of $f'(x) > 0,\ f'(x) = 0,\ f'(x) < 0$.	Identifying intervals on which functions are increasing $(f'(x) > 0)$ or decreasing $(f'(x) < 0)$.

Connections

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SL 5.3

Content	Guidance, clarification and syllabus links
Derivative of $f(x) = ax^n$ is $f'(x) = anx^{n-1},\ n \in Z$ The derivative of functions of the form $f(x) = ax^n + bx^{n-1} + \ldots$ where all exponents are integers.	

Connections

TOK: The seemingly abstract concept of calculus allows us to create mathematical models that permit human feats such as getting a man on the Moon. What does this tell us about the links between mathematical models and reality?

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SL 5.4

Content	Guidance, clarification and syllabus links
Tangents and normals at a given point, and their equations.	Use of both analytic approaches and technology.

Connections

Links to other subjects: Instantaneous velocity and optics, equipotential surfaces (physics); price elasticity (economics).

TOK: In what ways has technology impacted how knowledge is produced and shared in mathematics? Does technology simply allow us to arrange existing knowledge in new and different ways, or should this arrangement itself be considered knowledge?

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SL 5.5

Content	Guidance, clarification and syllabus links
Introduction to integration as anti-differentiation of functions of the form $f(x) = ax^n + bx^{n-1} + \ldots$, where $n \in Z,\quad n \neq -1$	Students should be aware of the link between anti-derivatives, definite integrals and area.
Anti-differentiation with a boundary condition to determine the constant term.	**Example:** If $\frac{dy}{dx} = 3x^2 + x$ and $y = 10$ when $x = 1$, then $y = x^3 + \frac{1}{2}x^2 + 8.5$.
Definite integrals using technology. Area of a region enclosed by a curve $y = f(x)$ and the x-axis, where $f(x) > 0$.	Students are expected to first write a correct expression before calculating the area, for example $\int_2^6 (3x^2 + 4)dx$. The use of dynamic geometry or graphing software is encouraged in the development of this concept.

Connections

Other contexts: Velocity-time graphs

Links to other subjects: Velocity-time and acceleration-time graphs (physics and sports exercise and health science)

TOK: Is it possible for an area of knowledge to describe the world without transforming it?

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SL 5.6

Content	Guidance, clarification and syllabus links
Derivative of x^n ($n \in \Theta$), $\sin x$, $\cos x$, e^x and $\ln x$. Differentiation of a sum and a multiple of these functions.	
The chain rule for composite functions. The product and quotient rules.	**Example:** $f(x) = e^{(x^2+2)}$, $\quad f(x) = \sin(3x-1)$ **Link to:** composite functions (SL2.5).

Connections

Links to other subjects: Uniform circular motion and induced emf (physics).

TOK: What is the role of convention in mathematics? Is this similar or different to the role of convention in other areas of knowledge?

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SL 5.7

Content	Guidance, clarification and syllabus links
The second derivative. Graphical behaviour of functions, including the relationship between the graphs of f, f' and f''.	Use of both forms of notation, $\frac{d^2y}{dx^2}$ and $f''(x)$. Technology can be used to explore graphs and calculate the derivatives of functions. **Link to:** function graphing skills (SL2.3).

Connections

Links to other subjects: Simple harmonic motion (physics).

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SL 5.8

Content	Guidance, clarification and syllabus links
Local maximum and minimum points. Testing for maximum and minimum.	Using change of sign of the first derivative or using sign of the second derivative where $f''(x) > 0$ implies a minimum and $f''(x) < 0$ implies a maximum.
Optimization.	Examples of optimization may include profit, area and volume.
Points of inflexion with zero and non-zero gradients.	At a point of inflexion, $f''(x) = 0$ **and** changes sign (concavity change), for example $f''(x) = 0$ is not a sufficient condition for a point of inflexion for $y = x^4$ at $(0, 0)$. Use of the terms "concave-up" for $f''(x) > 0$, and "concave-down" for $f''(x) < 0$.

Connections

Other contexts: Profit, area, volume.

Links to other subjects: Velocity-time graphs, simple harmonic motion graphs and kinematics (physics); allocative efficiency (economics).

TOK: When mathematicians and historians say that they have explained something, are they using the word "explain" in the same way?

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SL 5.9

Content	Guidance, clarification and syllabus links
Kinematic problems involving displacement s, velocity v, acceleration a and total distance travelled.	$v=\frac{\mathrm{d}s}{\mathrm{d}t};\quad a=\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}^2s}{\mathrm{d}t^2}$ Displacement from t_1 to t_2 is given by $\int_{t_1}^{t_2} v(t)\mathrm{d}t$. Distance between t_1 to t_2 is given by $\int_{t_1}^{t_2} \|v(t)\|\mathrm{d}t$. Speed is the magnitude of velocity.

Connections

Links to other subjects: Kinematics (physics).

International-mindedness: Does the inclusion of kinematics as core mathematics reflect a particular cultural heritage? Who decides what is mathematics?

TOK: Is mathematics independent of culture? To what extent are we people aware of the impact of culture on what we they believe or know?

Download connections template

SL 5.10

Content	**Guidance, clarification and syllabus links**
Indefinite integral of $x^n \left(n \in \Theta\right)$, $\sin x$, $\cos x$, $\frac{1}{x}$ and e^x.	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$
The composites of any of these with the linear function $ax + b$.	**Example:** $f'(x) = \cos(2x+3) \Rightarrow f(x) = \frac{1}{2}\sin(2x+3) + C$
Integration by inspection (reverse chain rule) or by substitution for expressions of the form: $\int kg'(x) f(g(x)) dx$.	**Examples:** $\int 2x(x^2+1)^4 dx$, $\int 4x \sin x^2 dx$, $\int \frac{\sin x}{\cos x} dx$.

Connections

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SL 5.11

Content	Guidance, clarification and syllabus links
Definite integrals, including analytical approach.	$\int_a^b g'(x)\mathrm{d}x = g(b) - g(a)$. The value of some definite integrals can only be found using technology. **Link to**: definite integrals using technology (SL5.5).
Areas of a region enclosed by a curve $y = f(x)$ and the x-axis, where $f(x)$ can be positive or negative, without the use of technology. Areas between curves.	Students are expected to first write a correct expression before calculating the area. Technology may be used to enhance understanding of the relationship between integrals and areas.

Connections

International-mindedness: Accurate calculation of the volume of a cylinder by Chinese mathematician Liu Hui; Ibn Al Haytham: first mathematician to calculate the integral of a function, in order to find the volume of a paraboloid.

TOK: Consider $f(x) = \frac{1}{x},\ 1 \le x$. An infinite area sweeps out a finite volume. Can this be reconciled with our intuition? Do emotion and intuition have a role in mathematics?

Enrichment: Exploring numerical integration techniques such as Simpson's rule or the trapezoidal rule.

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AHL content

Recommended teaching hours: 27

The aim of the AHL content in the calculus topic is to extend and build upon the aims, concepts and skills from the SL content. Further powerful techniques and useful applications of differential and integral calculus are introduced.

AHL 5.12

Content	Guidance, clarification and syllabus links
Informal understanding of continuity and differentiability of a function at a point.	In examinations, students will not be asked to test for continuity and differentiability.
Understanding of limits (convergence and divergence). Definition of derivative from first principles $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.	**Link to:** infinite geometric sequences (SL1.8). Use of this definition for polynomials only.
Higher derivatives.	Familiarity with the notations $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$, $f^{(n)}(x)$. **Link to**: proof by mathematical induction (AHL 1.15).

Connections

Links to other subjects: Theory of the firm (economics).

International-mindedness: How the Greeks' distrust of zero meant that Archimedes' work did not lead to the Calculus; investigate attempts by Indian mathematicians (500-1000AD) to explain division by zero.

TOK: Does the fact that Leibniz and Newton came across the Calculus at similar times support the argument of Platonists over Constructivists?

Enrichment: Fundamental theorem of calculus.

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AHL 5.13

Content	Guidance, clarification and syllabus links
The evaluation of limits of the form $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ and $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}$ using l'Hôpital's rule or the Maclaurin series.	The indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. For example: $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$. **Link to:** horizontal asymptotes (SL2.8) .
Repeated use of l'Hôpital's rule.	

Connections

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AHL 5.14

Content	Guidance, clarification and syllabus links
Implicit differentiation. Related rates of change. Optimisation problems.	Appropriate use of the chain rule or implicit differentiation, including cases where the optimum solution is at the end point.

Connections

Other contexts: Links between mathematical and physical models.

TOK: Euler was able to make important advances in mathematical analysis before calculus had been put on a solid theoretical foundation by Cauchy and others. However, some work was not possible until after Cauchy's work. What does this suggest about the nature of progress and development in mathematics? How might this be similar/different to the nature of progress and development in other areas of knowledge?

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AHL 5.15

Content	Guidance, clarification and syllabus links
Derivatives of $\tan x,\ \sec x,\ \operatorname{cosec} x,\ \cot x,\ a^x,$ $\log_a x,\ \arcsin x,\ \arccos x,\ \arctan x.$	
Indefinite integrals of the derivatives of any of the above functions. The composites of any of these with a linear function.	Indefinite integral interpreted as a family of curves. **Examples:** $\int \frac{1}{x^2+2x+5}\mathrm{d}x = \frac{1}{2}\arctan\frac{(x+1)}{2} + C$ $\int \sec^2(2x+5)\mathrm{d}x = \frac{1}{2}\tan(2x+5) + C$
Use of partial fractions to rearrange the integrand.	$\int \frac{1}{x^2+3x+2}\mathrm{d}x = \ln\left\lvert\frac{x+1}{x+2}\right\rvert + C$ **Link to:** partial fractions (AHL1.11)

Connections

TOK: Can a mathematical statement be true before it has been proven?

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AHL 5.16

Content	Guidance, clarification and syllabus links
Integration by substitution.	On examination papers, substitutions will be provided if the integral is not of the form $\int kg'(x)f(g(x))\mathrm{d}x$. **Link to:** integration by substitution (SL5.10).
Integration by parts.	**Examples:** $\int x\sin x\mathrm{d}x$, $\int \ln x\mathrm{d}x$, $\int \arcsin x\mathrm{d}x$
Repeated integration by parts.	**Examples:** $\int x^2\mathrm{e}^x\mathrm{d}x$ and $\int \mathrm{e}^x\sin x\mathrm{d}x$.

Connections

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AHL 5.17

Content	Guidance, clarification and syllabus links
Area of the region enclosed by a curve and the y-axis in a given interval. Volumes of revolution about the x-axis or y-axis.	

Connections

Other contexts: Industrial design.

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AHL 5.18

Content	Guidance, clarification and syllabus links
First order differential equations. Numerical solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ using Euler's method.	$x_{n+1} = x_n + h$, where h is a constant.
Variables separable.	**Example:** the logistic equation $\frac{\mathrm{d}n}{\mathrm{d}t} = kn(a-n), \quad a, \quad k \in P$ **Link to**: partial fractions (AHL1.11) and use of partial fractions to rearrange the integrand (AHL5.15).
Homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$using the substitution $y = vx$.	
Solution of $y + P(x)y = Q(x)$, using the integrating factor.	

Connections

Other contexts: Newton's law of cooling, population growth, carbon dating.

Links to other subjects: Decay curves (physics); first order reactions (chemistry)

TOK: Does personal experience play a role in the formation of knowledge claims in mathematics? Does it play a different role in mathematics compared to other areas of knowledge?

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AHL 5.19

Content	Guidance, clarification and syllabus links
Maclaurin series to obtain expansions for e^x, $\sin x$, $\cos x$, $\arctan x$, $\ln(1+x)$, $(1+x)^p$, $p \in \Theta$.	
Use of simple substitution, products, integration and differentiation to obtain other series.	**Example**: for substitution: replace x with x^2 to define the Maclaurin series for e^{x^2}. **Example** *:* the expansion of $e^x \sin x$.
Maclaurin series developed from differential equations.	

Connections

International-mindedness: Comparison of the Bourbaki to the Kerala School.

TOK: Is there always a trade-off between accuracy and simplicity?

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