The Odd Questions of Chapter 3

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3.1: Find a closed form expression for the following expression

$$\sum_{i=0}^{n-1}(2i+1) \tag{1}$$

This equation represents the sum of the first n odd integers. Let's look at a couple answers

$$\sum_{i=0}^{1-1}(2i+1) = (2(0)+1) = 1$$
$$\sum_{i=0}^{2-1}(2i+1) = (2(0)+1)+(2(1)+1) = 1+3 = 4$$
$$\sum_{i=0}^{4-1}(2i+1) = (2(0)+1)+(2(1)+1)+(2(2)+1)+(2(3)+1) = 1+3+5+7 = 16$$

Each of these answers look like perfect squares. Let's conjecture that:

$$\sum_{i=0}^{n-1}(2i+1) = n^2 \tag{2}$$

Now let's prove induction. We know that the equations match when $n = 0$ as $0 - 1$ results in a value less than the begining value of i, which by convention is 0. Now $n+1$ must be proven

$$\begin{aligned}
\sum_{i=0}^{(n+1)-1}(2i+1) &= \sum_{i=0}^{n}(2i+1) \\
&= \sum_{i=0}^{n-1}(2i+1)+(2n+1) && \text{(breaking off last term)} \\
&= (n^2)+(2n+1) && \text{(by induction hypothesis)} \\
&= n^2+2n+1 \\
&= (n+1)^2 && \text{(factoring)}
\end{aligned}$$

Proven to simplify exactly to conjectured $(n+1)^2$

3.3: Prove the Extended Pigeonhole Principal by induction on $|Y|$

The Extended Pigeonhole Principal can be written as...

if $|X| > |Y| * k$ then there are $k+1$ members of X that share at least 1 member of Y, or that $k = \lceil \frac{|X|}{|Y|} \rceil - 1$

Base Case:

$$\begin{aligned}
|X| &> 1 * k \\
\frac{|X|}{1} &> k \\
\frac{|X|}{1} &> \lceil \frac{|X|}{1} \rceil - 1 && \text{(by induction hypothesis)}
\end{aligned}$$

This must be true as a ceiling function may never raise the value by 1 or more, so subtracting 1 will always result in a smaller number than the original fraction
Induction Case:

$$\begin{aligned}
|X| &> (|Y| + 1) * k \\
\frac{|X|}{|Y| + 1} &> k \\
\frac{|X|}{|Y| + 1} &> \lceil \frac{|X|}{|Y| + 1} \rceil - 1 && \text{(by induction hypothesis)}
\end{aligned}$$

This must be true for the same reason as the base case.

3.5: Prove the following by induction that for any $n \geq 0$, ...

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{3}$$

Base Case:

$$\sum_{i=0}^{0} i^2 = 0^2 = 0 \frac{0(0+1)(2(0)+1)}{6} = \frac{0(1)(1)}{6} = \frac{0}{6} = 0 \tag{4}$$

Both sides equal 0 in the base case
Induction Case:
First, let's rewrite the right half of the equation with (n+1)

$$\begin{aligned}
&\frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} = \\
&\qquad \frac{(n+1)(n+2)(2n+3)}{6} = \frac{2n^3 + 9n^2 + 13n + 6}{6}
\end{aligned}$$

This is determined by simplifying and multiplying the terms together Now, let's rewrite the left half of the equation with (n+1)

$$\begin{aligned}
&\sum_{i=0}^{n+1}(2i+1) = \\
&\sum_{i=0}^{n+1}(2i+1) = \sum_{i=0}^{n}(2i+1) + (n+1)^2 && \text{(breaking off last term)} \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 && \text{(by induction hypothesis)} \\
&= \frac{2n^3+3n^2+n}{6} + (n+1)^2 \\
&= \frac{2n^3+3n^2+n}{6} + (n^2+2n+1) \\
&= \frac{2n^3+3n^2+n}{6} + \frac{6n^2+12n+6)}{6} \\
&= \frac{2n^3+9n^2+13n+6}{6}
\end{aligned}$$

Now if we compare the final results for both halfs, they are equivalent

3.7: Prove the following by induction that for any $n \geq 0$, ...

$$\sum_{i=0}^{n} i^3 = (\sum_{i=0}^{n} i)^2 \tag{5}$$

Base Case:

$$\sum_{i=0}^{0} i^3 = 0^3 = 0(\sum_{i=0}^{0} i)^2 = (0)^2 = 0 \tag{6}$$

Both sides equal to 0 in the base case Induction Case:

First, let's rewrite $\sum_{i=0}^{n} i$ to equal $\frac{n(n+1)}{2}$ which can be proven by induction here

$$
\begin{aligned}
\sum_{i=0}^{n+1} i &= \\
\sum_{i=0}^{n+1} i &= \sum_{i=0}^{n} +(n+1) \\
&= \frac{n(n+1)}{2} + (n+1) && \text{(by induction hypothesis)} \\
&= \frac{n^2+n}{2} + (n+1) \\
&= \frac{n^2+n}{2} + \frac{2n+2}{2} \\
&= \frac{n^2+3n+2}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{(n+1)^2+(n+1)}{2} &= \frac{(n^2+2n+1)+(n+1}{2} \\
&= \frac{n^2+3n+2}{2}
\end{aligned}
$$

Now, let's rewrite $\sum_{i=0}^{n} i^3$ to equal $\frac{n^2(n^2+2n+1)}{4}$ which can be proven by induction here

$$\begin{aligned}
\sum_{i=0}^{n+1} i^3 &= \\
\sum_{i=0}^{n+1} i^3 &= \sum_{i=0}^{n} i^3 + (n+1)^3 \\
&= \frac{n^2(n^2+2n+1)}{4} + (n+1)^3 && \text{(by induction hypothesis)} \\
&= \frac{n^4+2n^3+n^2}{4} + (n+1)^3 \\
&= \frac{n^4+2n^3+n^2}{4} + (n^3+3n^2+3n+1) \\
&= \frac{n^4+2n^3+n^2}{4} + \frac{(4n^3+12n^2+12n+4)}{4} \\
&= \frac{n^4+6n^3+13n^2+12n+4}{4}
\end{aligned}$$

and

$$\begin{aligned}
\frac{(n+1)^2((n+1)^2+2(n+1)+1)}{4} &= \frac{(n+1)^2((n+1)^2+2n+3)}{4} \\
&= \frac{(n^2+2n+1)((n+1)^2+2n+3)}{4} \\
&= \frac{(n^2+2n+1)((n^2+2n+1)+2n+3)}{4} \\
&= \frac{(n^2+2n+1)(n^2+4n+4)}{4} \\
&= \frac{n^4+6n^3+13n^2+12n+4}{4}
\end{aligned}$$

So now it must be proven that $\frac{n^2(n^2+2n+1)}{4} = (\frac{n(n+1)}{2})^2$:

$$\begin{aligned}
\sum_{i=0}^{n} i^3 &= (\sum_{i=0}^{n} i)^2 \\
\frac{n^2(n^2+2n+1)}{4} &= (\frac{n(n+1)}{2})^2 && \text{(by proven equivalence)} \\
\frac{n^2(n^2+2n+1)}{4} &= (\frac{n^2(n+1)^2}{2^2}) \\
\frac{n^2(n^2+2n+1)}{4} &= (\frac{n^2(n^2+2n+1)}{4}
\end{aligned}$$

3.9: What is the flaw in the proof, "All Horses are the Same Color"

There is no mathematical or logical process which proves that the horses in a set truly are the same color, the proof relies on its own singular assumption to prove itself

3.11: Prove the following about the Thue Sequence

a) For every $n \geq 1$, T_{2n} is a palindrome

Base case $n = 1$, $2n = 2$, $T_2 = 0110$. $T_2 n$ reads the same both directions

Induction hypothesis Fix $n \geq 1$, and assume that $T_2 n$ is a palindrome

Induction step $T_{2(n+1)}$ becomes $T_2 n + 2$. Because T_{n+1} concatonates T_n and its compliment, the ending values take 2 iterations of n for the compliments to revert and create another palindrome. So, because $T_2 n$ is a palindrome, so will $T_2 n + 2$

b) For every n, if 0 is replaced by 01 and 1 is replaced by 10 simultaneously everywhere in T_n, the result is T_{n+1}

Base case $n = 1$, $T_1 = 01$ and $T_2 = 0110$. T_2 could be created by replace T_1's 0s with 01 and 1s with 10s

Induction hypothesis Assume that T_{n+1} can be created by replacing T_n's 0s and 1s with 01s and 10s respectively

Induction step T_{n+1} takes the complementary bits from T_n and T_{n-1} the same, all the way down to T_0. This self iterative process is no different from repeating the steps from T_0 to T_1 over and over. Replacing the 0s and 1s with 01s and 10s is just breaking the sequence into small T_0 chunks and adding their complements back.