A Brief Desription of the Pigeonhole Principal

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The Basic Pigeonhole Principal

The Pigeonhole Principle is one of many proven principles in the study of Discrete Mathematics. It is commonly used to introduce the basics of the field to beginners in a digestable and practical way, with a pigeon and hole analogy, hence the name. It states:
"If there are more pigeons than holes, then multiple pigeons must share the same hole"
When written in proper notion, the actual principle looks like this:

$$iff : X \mapsto Y, |X| > |Y| \Rightarrow \exists\ x_1, x_2 \in X : x_1 \neq x_2 \wedge f(x_1) = f(x_2)$$

To the uninformed, this may look and sound like complete nonsense, but with a bit of guidance the meaning becomes clear.

First, looking at the left side: If $f : X \mapsto Y, |X| > |Y|$
Functions - f : - similar to the functions you may find in algebra, these take in an input and return an output
Sets - X, Y - an orderless list of discrete elements. Represented here with the capital variables
Cardinality - $||$ - denotes the amount of elements in a set (its length)
Mapping - $\mapsto$ - used to compare sets, attempts to match elements of one set to another set

Now the equation is starting to make sense, it says "if there is a function" (If f :) "where set X is mapped to set Y" ($X \mapsto Y$) "and set X is larger than set Y" ($|X| > |Y|$) "THEN..." ($\Rightarrow$)

Now, the right side: $\exists\ x_1, x_2 \in X : x_1 \neq x_2 \wedge f(x_1) = f(x_2)$
Exists - $\exists$ - states that what follows will exist/is true
Elements - x_1, x_2 - variables to represent elements of a set. Numbered to denote difference, not order
Element of, "In" - $\in$ - states the variables are members of the following set
And - $\wedge$ - states what came before and what follows are both expressed at the same moment. Links the inequalities

Breaking it down the right side says "there exists" ($\exists$) "elements in set X" ($x_1, x_2 \in X$:) "that are inequal, but their outputs of the function are equal" ($x_1 \neq x_2 \wedge f(x_1) = f(x_2)$)

So putting it all together it creates:
"If there is a function where set X is mapped to set Y and set X is larger than set Y then... there exists elements in set X that are inequal, but their outputs of the function are equal"
OR
$iff : X \mapsto Y, |X| > |Y| \Rightarrow \exists\ x_1, x_2 \in X : x_1 \neq x_2 \wedge f(x_1) = f(x_2)$

Now think back to the original analogy, you can begin to understand how it relates to the principal at hand. Imagine that set X is the pigeons and set Y is the holes: if the amount of pigeons is larger than the amount of holes $\rightarrow$ 2 different pigeons will share the same hole.

The Extended Pigeonhole Principal

Now it is known there are pigeons sharing holes, but how many?
The Extended Pigeonhole explains exactly this:

For any finite sets X and Y and any positive integer k such that $|X| > k \times |Y|$, $iff : X \mapsto Y$, *then there are at least* $k + 1$ *distinct members* $x_1, ..., x_{k+1} \in X$ *such that* $f(x_1) = ... = f(x_{k+1})$

Thinking logically, if $|X| > k \times |Y|$ is true, then k can be found by $k = \lfloor \frac{|X|}{|Y|} \rfloor$ ($\lfloor n \rfloor$ being the floor or nearest integer that's $\leq$ than n)

If $k = 4$, then the members $x_1, ..., x_{k+1}$, or $x_1, ..., x_5$ all have the same output of the function.
"if there are 13 pigeons and 3 holes, k = 4, so there will be 5 pigeons that share a hole"

Alternate Extended Pigeonhole Principal

An alternate way to write the extended principal is as:
"For any finite sets X *and* Y, $iff : X \mapsto Y$ *then there is some* $y \in Y$ *such that* $f(x) = y$ *for atleast* $\lceil \frac{|X|}{|Y|} \rceil$ *values of* x*"*
This is very similar to the previous version, the main difference is the use of $\lceil n \rceil$ or ceiling rather than floor. Ceiling instead being the nearest integer that's $\geq$ than n. This removes the need for adding 1 to a k value