Introduction to Linear Algebra

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1 Introduction

This is an introductory guide for beginners in linear algebra. As a non-professional, there will inevitably be inaccuracies. Readers are welcome to point out my mistakes.

We will go through the fundamental element of linear algebra, including but not limited to: **systems of linear equations, linear space, determinant, matrix...**

At last, we will go back to the DE to see how linear algebra can be used in DE.

2 Systems of linear equations and vector linear space

2.1 Systems of linear equations

We have learnt how to solve systems of equations like this in junior high school(or earlier):

$$\begin{cases} x + 2y = 4 \cdots\cdots (1) \\ 2x - y = 3 \cdots\cdots (2) \end{cases}$$

At that time, we add equation(2) twice to equation(1), we get $5x = 10 \Rightarrow x = 2$, bring the result back to the equation(1), we get $y = 1$, so that:

$$\begin{cases} x = 2 \\ y = 1 \end{cases}$$

It seems that we have solved all the **Systems of linear algebra** through this **Elimination method**. However, this method is not **enough trivial**, in another world, we can conclude this method and find a more trivial expression to get the result, so that we can save our hands and only need to tell the computer how to get the answer.

We get start from finding a form notation for elimination: Notice that the exact notation of each variable is never mind when doing the elimination. Thus, we can use the column indicator of each variable to completely represent itself. What is more, the '$+, -, =$' are never important if we make a convention that the last column represents the constant term in the equation.

Then, equation(1) and equation(2) can be written in this way:

$$\begin{cases} x+2y=4\cdots\cdots(1) \\ 2x-y=3\cdots\cdots(2) \end{cases} \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \end{pmatrix}$$

It is called **MATRIX**. It's appearance is so mundane that you may be disappointed with it, but don't worry, it's just an introduction to matrices, let's finish solving the linear system of equations first.

Figuratively speaking, we can always obtain a matrix of this form by eliminating (that is, adding and subtracting between the rows of the matrix):

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1r} & c_{1r+1} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2r} & c_{2r+1} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & c_{rr} & c_{rr+1} & \cdots & c_{m} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

It's definition is: ***The column index of the first non-zero element of each row increases as the row index increases***, we call it row echelon form figuratively.

Now we want to find a general way to prove that we can always get the row echelon form.

First, we specify that the element in the row i and the column j is a_{ij}. And we already know that **swapping the order of two rows, adding k times of one row to the other, multiplying all elements of one row by a non-zero number** are permitted (since it is a system of linear equations).

Then, you can add $(-\frac{a_{j1}}{a_{11}})$ times of row 1 to row j($j \neq 1$), so that the first row only have one non-zero number, a_{11}. Do the same to column 2, but this time, add$(-\frac{a_{j2}}{a_{22}})$times of row 2 to row j($j \neq 1,2$). And the same to column 3 ......(some columns may not need to operate if the elements are all zeros, except for those rows that were previously added to other rows as a base)

Ok! When you do this to all the columns (expect the last column, because it represent for the constant term), you will get echelon matrix. But it is not enough to get the answer. We have to go further.

Remember that **multiplying all elements of one row by a non-zero number is permitted**. We assume the first non-zero number of row i is b_i, if it exists (which means some rows may be all zero). We multiply row i by $\frac{1}{b_i}$, and the first non-zero number of each row will become zero(expect for the last column).

Then, we start from the last non-zero row i (expect for the last column), assume that the first non-zero number is in column j, add k times of it to other rows to make other elements in column j become zero.

Now, we get a matrix like this:(it is only an example)

$$\begin{pmatrix} 1 & 0 & 1/2 & 0 & 2 \\ 0 & 1 & -1/3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

It is called **row canonical form**.

In conclusion, a matrix is in row canonical form only if it satisfies the following conditions:

•It is in row echelon form.

•The leading entry in each nonzero row is a 1 (called a leading 1).

•Each column containing a leading 1 has zeros in all its other entries.

Now, you can find the answer of the system of linear equations have already written in the row canonical form if only you remember the convention we have made to the variables.

Let i be the numbers of leading 1, j be the numbers of variables. Then we can divide the answer into three cases:

•if other rows(rows that do not have leading 1) are all zero

$$\begin{cases} i = j \Rightarrow unique \quad solution \\ i < j \Rightarrow infinite \quad solution \end{cases}$$

•if other rows(rows that do not have leading 1) are not all zero

$$no\ solution$$

Exercise

1.Write down the row canonical form of following system and find the answer of the system

$$\begin{cases} 2x + y - z = 8 \\ -3x - y + 2z = -11 \\ -2x + y + 2z = -3 \end{cases}$$

2.*Converting a system of linear equations to the corresponding homogeneous system of linear equations means replacing all constant terms with zeros. Assume that a system has infinite solutions, prove that all the solutions can be expressed by a solution of the corresponding homogeneous system of linear equations add a solution of itself.

$$x_i = x_i(general \quad of \quad homogeneous) + x_i(particular)$$

2.2 Vector Linear Space

Linear Space is a general concept, in this guide I will only talk about vector linear space and use vector to represent linear equation system, for it is simple and useful.

First, we write a vector **x** in this way:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Defining the addition and subtraction of vectors:

$$z = x + y \Rightarrow z_i = x_i + y_i$$

Defining the multiply(by number) of vectors: (k is number, x is vector)

$$(k \cdot x)_i = k \cdot x_i$$

Defining 0 vector:

$$0_i = 0$$

Now the vector space is basically complete. We are going to use vector to express the solutions of systems of linear equations. But before that, we have to define **rank** and **dimension**.

For a group of vector $v_1, v_2 ... v_k$, they are said to be linearly dependent, if there exist scalars $a_1, a_2 ... a_k$ not all zero, such that

$$a_1 v_1 + a_2 v_2 + ... + a_k v_k = 0$$

They are said to be linearly independent, if there do not exist scalars $a_1, a_2 ... a_k$ not all zero, such that

$$a_1 v_1 + a_2 v_2 + ... + a_k v_k = 0$$

And for a sequence of vectors, it has many subsequence, it is easy to prove that ,if you chose a vector and add other vectors into it to form a subsequence and request it to be linear independent, the maximum number of vectors in the subsequence is a constant. We define the maximum number as the rank of the vector sequence. And the largest sub sequence is called **maximal linearly independent group** of the vector sequence. (***the proof is not required in this guide***)

All the vectors can be expressed linearly in terms of maximal linearly independent groups, otherwise there exist a vector linearly independent with maximal linearly independent groups and the maximal linearly independent groups is not maximal.

Now, we consider a group of vector:(v_i are vectors and a_i are numbers)

$$\mathscr{A} = \{x | x = a_1 v_1 + a_2 v_2 + ... + a_k v_k\}$$

If $v_1, v_2 ... v_k$ are linearly independent, we say that the group of vector consist of the space spanned by the vector group $v_1, v_2 ... v_k$. And can be written in this way:

$$\mathscr{A} = < v_1, v_2, ..., v_k >$$

It is not difficult to find that the rank of $\mathscr{A}$ is k, and we use dimension to represent the rank of a vector space. So formally we say dim($\mathscr{A}$)=k rather than rank($\mathscr{A}$)=k.

Okay! Now we can look back to systems of linear equations. We are going to talk about homogeneous system of linear equations. And the solutions of non homogeneous system of linear equations consist of a manifold* according to the exercise 2 in last section (***it is not required in this guide***)

The solutions of homogeneous system of linear equations consist of a vector space because if $v_1, v_2 ... v_k$ are linearly independent solutions of the system, then $a_1v_1 + a_2v_2 + ... + a_kv_k$ is a solution of the system. So the solutions can be expressed by finite vectors! (***It is easy to find such group of vector, but it is not required in this guide.***)

Exercise

1.Prove following vectors are linearly independent

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad x_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

2.* Google the exact definition of linear space and prove that vector space is linear space

3 Determinant and Matrix

3.1 Permutation

We first introduce the definition of Determinant, and if you go further in math, you will find it is associated with a lot of concepts, including but not limited to the volume factor of linear maps and the inverse of a matrix.

First, we have to introduce the *permutation.* A permutation of the set { 1 , 2 , ⋯ , n } is a function σ that reorders this set of integers. And define the value of σ:

•if the order is 1,2,3,...,n , the value is 1

•when you swap the order of two of the numbers, multiply the value by -1

Example

1.Given a permutation {1,2,4,3}, find the value of σ.

Notice that, σ{1,2,3,4}=1. And we can swap the last two numbers to get {1,2,4,3}, so that σ{1,2,4,3}$=-1 \cdot 1 = -1$

3.2 The Definition of Determinant

OK. Now we continue to define the determinant. Assume that there are permutations of column index. These permutations consists of a set, we use S_n to represent it. And we use σ_i to represent the i-th item of the permutation σ.

Definition of Determinant:

$$\det(A) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{n} a_{i,\sigma_i} \right)$$

$\operatorname{sgn}(\sigma)$ refers to the value of σ, and a_{i,σ_i} refers to the element at row i, column σi

To put it figuratively, it is to take an element in each row and column and multiply them together, then multiply it by the value of the permutation, and finally sum it up.

Example

1.Calculate the determinant of this 2×2 matrix :

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

First, the column index have two permutations: {1,2} and {2,1}. And the value is 1 and -1.

According to the definition,

$$\det(A) = 1 \times a_{11} \times a_{22} + (-1) \times a_{12} \times a_{21} = ad - bc$$

3.3 Determinant Expansion

In theory, we already know the calculation method of any determinant, but in fact, even the calculation of the third-order determinant is not easy, we need some methods to help us calculate, this part I will introduce the most common methods: Expand by row or column.

The minor $M_{i,j}$ is defined to be the determinant of the $(n-1) \times (n-1)$-matrix that results from A by removing the row i and the column j. The expression $(-1)^{i+j} M_{i,j}$ is known as a cofactor. Then we have:(***the proof is easy, but again, is not required in this guide***)

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

In the same way, youcan do this expansion to the column j by changing the sum notation to sum the i.

Example

1.Calculate the determinant of this matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 899 & 2 & 1 \\ 65534 & 1 & 2 \end{pmatrix}$$

Expand by the first row:

$$\det(A) = 1 \times 1 \times M_{11} + (-1) \times 0 \times M_{12} + 1 \times 0 \times M_{13} = M_{11}$$

And,

$$M_{11} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 \times 2 - 1 \times 1 = 3$$

Exercise

1.Calculate the determinant of this matrix:(The element in row i, column i of this matrix is λ_i, and the other elements are 0)

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

3.4 Definition of Matrix

We have seen the concept of matrix many times before, and now we formally introduce the matrix.

A $m \times n$ matrix can be expressed in this form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3j} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

Now we are going to define the operations on matrices.

Defining *addition*:**only if A and B are both $m \times n$ matrix**

$$C = A + B \Rightarrow c_{ij} = a_{ij} + b_{ij}$$

Defining *multiply(by number)*:**k is number and A is matrix**

$$(k \cdot A)_{ij} = k \cdot (a_{ij})$$

Defining *multiply(between matrices)*:**only if the number of columns of A equals to the number of rows of B**

$$C = AB \Rightarrow c_{ij} = \sum_k a_{ik} \cdot b_{kj}$$

Pay attention:the multiply between matrices is generally not allowed to swap the order, even if they are both $n \times n$ matrices. But when multiplying multiple matrices, any adjacent two of them can be calculated first, for example:$ABCD = (AB)CD = A(BC)D = AB(CD)$. What is more, matrix multiplication satisfies the distributive law of multiplication. That means: $AB + AC = A(B + C)$, if B+C exists. Further more, $ABD + ACD = A(B + C)D$, if B+C exists

Example

1.Calculate this multiply:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} =?$$

Calculate each element: $c_{11} = a_{11} \times b_{11} + a_{12} \times b_{21} = 1 \times 1 + 1 \times 4 = 5$......

$$Answer : \mathbf{C} = \begin{bmatrix} 5 & 7 & 9 \\ 10 & 14 & 18 \\ 15 & 21 & 27 \end{bmatrix}$$

3.5 Special Operations on Matrix

3.5.1 Transpose

Definition:

$$(A^{\mathrm{T}})_{ij} = a_{ji}$$

Example

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 10 & 14 & 18 \\ 15 & 21 & 27 \end{bmatrix} \quad \Rightarrow \quad A^{\mathrm{T}} = \begin{bmatrix} 5 & 10 & 15 \\ 7 & 14 & 21 \\ 9 & 18 & 27 \end{bmatrix}$$

3.5.2 Inverse

Definition:**only if A is a square matrix**

$$\mathbf{A}\mathbf{A}^{-1} = I \quad ; \quad I_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

A^{-1} is called the inverse of A.

You have to know following truth:(***again, easy to prove but not required in this guide***)

$\bullet\mathbf{A}^{-1}$ exists only if $\det(A) \neq 0$
$\bullet\mathbf{A}^{-1}$ once exists, it is unique

Now we are going to find inverse:
We can write A($n \times n$) and I($n \times n$) into a $n \times 2n$ matrix:

$$\begin{bmatrix} \mathbf{A} & I \end{bmatrix}$$

It is easy to find that doing operations to the rows like what we have done to the system of linear equations equals to multiply a matrix on the left(If you think it's not easy, just admit it, it doesn't matter). So if we do a series of row operations and turn the A matrix to I(that is to say, change A to its row canonical form, because it is full rank[1]), then the I matrix will turn to A^{-1}. That is because:

$$\mathbf{A}^{-1}\begin{bmatrix} \mathbf{A} & I \end{bmatrix} = \begin{bmatrix} I & \mathbf{A}^{-1} \end{bmatrix}$$

Then, we find the inverse of A

Pay attention: only row operations are required. And do not worry the order of the operations, because the inverse of A is unique.

Example

1.Calculate inverse matrix of this matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

First, write down a 2×4 matrix:

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}$$

[1]full rank $\Leftrightarrow$ det(A)=0 $\Leftrightarrow$ row canonical form expressed as I (not required in this guide again)

Then, do row operations:

$$\left[\begin{array}{cccc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{row(2)+(-\frac{c}{a})\times row(1)} \left[\begin{array}{cccc} a & b & 1 & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \xrightarrow[row(2)\times\frac{a}{ad-bc}]{row(1)\times\frac{1}{a}} \left[\begin{array}{cccc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{row(1)+(-\frac{b}{a})\times row(2)} \left[\begin{array}{cccc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Thus,

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right]$$

Exercise

1.Use matrix multiply to express this system of linear equations:

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \quad \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

Hint:the solution can be expressed by a vector(a vector is also a matrix):

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

2.Prove following truth of Transpose and Inverse:

$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$\left(\mathbf{A}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{\mathrm{T}}$$

Hint:Use component expressions of Transpose and the definition of Inverse.

4 Similar Matrix and Eigenvalues

We will have a short look at the similar Matrix and eigenvalues, for they will be used in DE solutions.

Definition: for $\alpha \neq 0, \lambda \neq 0$, ($\alpha$ is vector and λ is number) if:

$$\mathbf{A}\alpha = \lambda\alpha, (\alpha \neq 0; \lambda \neq 0)$$

λ is called a eigenvalue of $\mathbf{A}$, and α is called a eigenvector of λ.

Ok, now we are going to first get eigenvalues and then trough eigenvalues get eigenvectors.

Remember the definition of $\mathbf{I}$, and we have:[2]

$$\mathbf{A}\alpha = \lambda\mathbf{I}\alpha$$

Remember the matrix multiply, we have:

$$(\lambda\mathbf{I} - \mathbf{A})\,\alpha = 0$$

Which means if we want to get non-zero α, this is required :[3]

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0$$

This is a polynomial equation is called characteristic equation and we can always find n($\mathbf{A}$ is $n \times n$ matrix) solutions.[4] (maybe imaginary numbers or multiple roots)

Then, take λ into the origin equation to solve the α.

Example

1.Find the eigenvalues and corresponding eigenvectors of this matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

First, write down the characteristic equation:

$$(\lambda - 4)(\lambda - 2) = 0$$

So the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2, \lambda_2 = 4$.

- $\lambda_1 = 2$:

$$\begin{bmatrix} 3-2 & -1 \\ -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution of this system of linear equations are: $x_1 = x_2$, so the corresponding eigenvector can be this one:

$$p_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- $\lambda_2 = 4$:

$$\begin{bmatrix} 3-4 & -1 \\ -1 & 3-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution of this system of linear equations are: $x_1 = -x_2$, so the corresponding eigenvector can be this one:

$$p_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

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[2]It is easy to prove I$\alpha = \alpha$ through the definition of I

[3]According to several truth from systems of linear equations, not required in this guide

[4]That is ***Fundamental Theorem of Algebra***

5 Orthogonal Matrix and Group *

Do not worry, this chapter is for entertainment only.

What is group?

Let G be the group, it just need to satisfy following rules:

- it has defined multiply and if $a, b \in G$, $ab \in G$
- there exists a unit e, $ea = ae = a$
- it associative law of multiplication: $ab + ac = a(b + c)$;$abd + acd = a(b + c)d$
- for every elements exists an unique inverse element: $a^{-1}a = e$

And what is orthogonal matrix?

Let P be a orthogonal matrix, it just need to satisfy:

$$\mathbf{P}\mathbf{P}^{\mathrm{T}} = \mathbf{I}$$

It is easy to prove that orthogonal matrix consist of a group. What is more, we want to look at the group of 3×3 orthogonal matrix.

Imagine that you are living in a 3-dimension space, and we use the Space Cartesian Coordinate System, and you are now at somewhere. Your place can be expressed by a vector:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Now, the Space Cartesian Coordinate System occurs a rotation around z axis by θ

Your place was turned to:

$$r' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} xcos\theta - ysin\theta \\ ysin\theta + xcos\theta \\ z \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{P}r$$

You can choose the appropriate axis to illustrate that all the rotation are associated with a orthogonal matrix. So we can learn the rotation of the rigid body by the rotation group!

6 The End

It is a bit off-topic. This chapter only briefly introduces the application of matrices and the more essential algebraic structure behind matrices. If you are interested in these content (even physics content), I can continue to discuss this content.

That's all for this week!

Thanks for your reading. Any questions can be asked through email: 2100017810@stu.pku.edu.cn or Wechat.