1 二元关系和偏序集

A binary relation $\gtrsim$ on X is a subset of $X \times X$, i.e., $\gtrsim \subset X \times X$. We write $x \gtrsim y$ instead of $(x, y) \in \gtrsim$.

eg, let $X = \{a, b, c\}$. A binary relation $\gtrsim$ on X can be defined as:

$$\gtrsim = \{(a, a), (a, b), (b, a)\} \subseteq X \times X$$

We write $a \not\gtrsim b$ instead of $(a, b) \in \gtrsim$, and $a \gtrsim b$ instead of $(a, b) \in \gtrsim$.

A binary relation $\gtrsim$ on X is a partial order if it satisfies the following conditions for all $x, y, z \in X$:

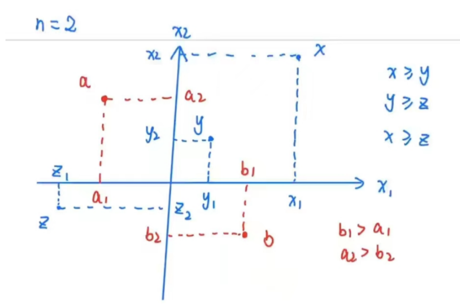
(1) Transitivity: $x \gtrsim y$ and $y \gtrsim z$ imply $x \gtrsim z$.

(2) Reflexivity: $x \gtrsim x$

(3) Antisymmetry: $x \gtrsim y$ and $y \gtrsim x$ imply $x = y$

A partially ordered set is a set X with a partial order $\gtrsim$ on X, denoted as $(X, \gtrsim)$.

eg, the set $\mathbb{R}^n$ with the partial order $\gtrsim$ defined as $x \gtrsim y$ if and only if $x_i \gtrsim y_i$ for $i = 1, 2, \cdots, n$ is a partially ordered set: $(\mathbb{R}^n, \gtrsim)$.



(2) $(2^x, \supset)$, where 2^x is the set of all subset of x, $\supset$ means "contain", is a partially ordered set.

$$x = \{a, b, c\}$$
$$2^x = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \emptyset, x\}$$
$$\{a\} \nsupseteq \{b\}, \{b\} \nsupseteq \{a\}.$$

A partial order Z on x is a total order if it is complete, ie., for any $x, y \in X, x \gtrsim y$ or $y \gtrsim x$.

eg. $(R, \gtrsim)$ is totally ordered set.

A subset of partially order set that is totally ordered is called a chain.

Suppose $X = \{a, b, c\}$. Then $(2^x, \supset)$, is a partially ordered set.

Consider $S \subset 2^x, S = \{\emptyset, \{a\}, \{a, b\}, x\}$. Then S is a chain of $(2^x, \supset)$.

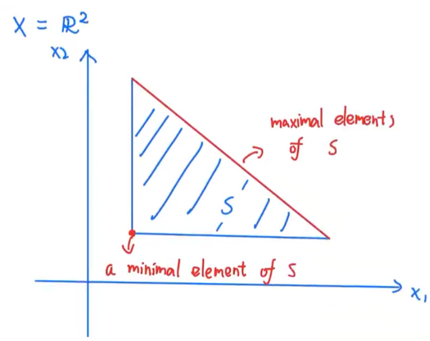
2 85

Suppose (x, z) is a partially ordered set.Consider $S \subset X$.

A maximal element of S is an element $\bar{x} \in S$ with no $x \in S$ such $x \geq \bar{x}$.

A minimal element of S is an element $\underline{x} \in S$ with no $x \in S$ such $\underline{x} \gtrsim x$.

eg $x \in X$ is an upper bound of S if for all $y \in S, x \geqslant y$

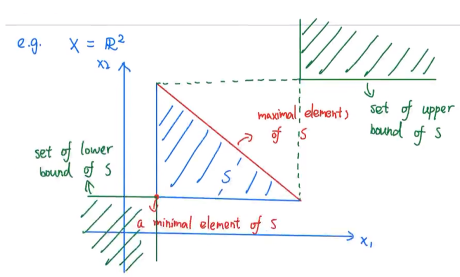


$x \in X$ is a lower bound of S if for all $y \in S, y \geq x$.

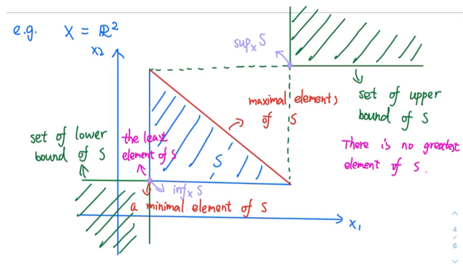
If $x \in S$ is an upper (lower) bound of 5 , then x is the greatest (least) element of S.

If the set of upper bound of s has a least element, then it is called a supremum of S, denoted as $\sup_x S$.

(1) For any $y \in S, \sup_x S \gtrsim y$.



(2) For any $y' \in x$ such that $y' \gtrsim y$ for all $y \in S$, then $y' \gtrsim \sup_x S$. If the set of lower bound of s has a greatest element. then it is called a



infimum of s, denoted as inf_x.

(1) For any $y \in S, y \gtrsim \inf x$.

(2) For any $y' \in X$ such that $y \gtrsim y'$ for all $y \in S$, then $inf_x S \gtrsim y'$.

3 格与子格

Suppose $(x, \leq)$ is a partially ordered set.Consider $S \subset X$.

The supremum of S, denoted as $\sup_x S$ satisfies:

(1) For any $y \in S, y \leq \sup_x S$.

(2) For any $y' \in x$ such that $y \leq y'$ for all $y \in S$, then $\sup_x S \leq y'$.

The infimum of S, denoted as $\inf_x S$ satisfies:

(1) For any $y \in S, \inf_x S \leq y$.

(2) For any $y' \in x$ such that $y' \leq y$ for all $y \in S$, then $y' \leq \inf_x S$.

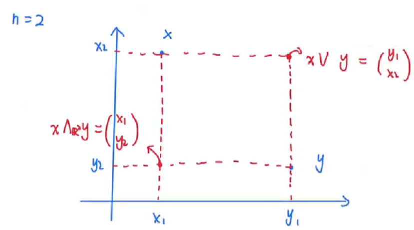
Suppose $(x, \leq)$ is a partially ordered set. For $x, y \in X$, we write

$$x \vee_x y = \sup_x \{x, y\} \quad \text{join}$$
$$x \wedge_x y = \inf_x \{x, y\} \quad \text{meet}$$

A partially ordered set $(x, \leq)$ is a lattice if for all $x, y \in X$, $x \vee_x y$ and $x \wedge_x y$ exist.

e.g. (1) $(\mathbb{R}^n, \leq)$ is a lattice. For any $x, y \in \mathbb{R}^n$,

$$x \vee_x y = \sup_x \{x, y\} = \begin{pmatrix} \max\{x_1, y_1\} \\ \max\{x_2, y_2\} \\ \vdots \\ \max\{x_n, y_n\} \end{pmatrix}, \quad x \wedge_x y = \inf_x \{x, y\} = \begin{pmatrix} \min\{x_1, y_1\} \\ \min\{x_2, y_2\} \\ \vdots \\ \min\{x_n, y_n\} \end{pmatrix}$$

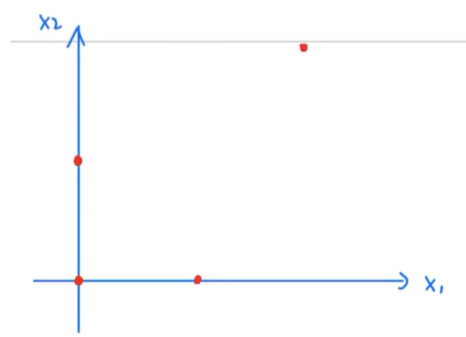


(2) $(2^x, \subseteq)$ is a lattice. For $S, T \in 2^x$,

$$S \wedge T = S \cap T$$
$$S \vee T = S \cup T$$

(3) $x = \{(0,0),(0,1),(1,0)\} \subset \mathbb{R}^2$, $(x, \subseteq)$ is not a lattice. Because $(0,1) \vee_x (1,0) = \sup_x\{(0,1),(1,0)\}$

(4) $x = \{(0,0),(0,1),(1,0),(2,2)\} \subset \mathbb{R}^2$ $(x, \geq)$ is a lattice.



(5) $x = \{(0,0),(0,1),(1,0)\} \cup \{(x_1,x_2) \mid x_1 = x_2, x_1 > 2\}$

$(x, \geqslant)$ is not a lattice.

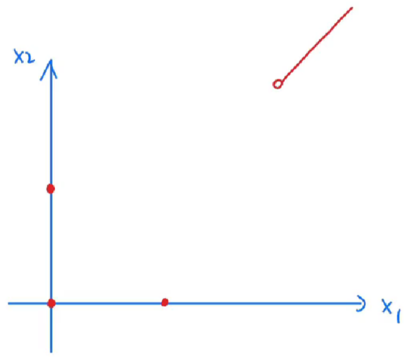
For a lattice (x, z), $K \subset x$ is a sublattice of (x, z) if for all $x, y \in K$, $x \vee y \in K$ and $x \wedge y \in K$.

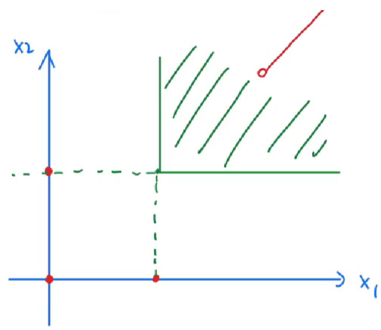
For example,

(1)$K = \{(0,0),(1,0),(0,1),(1,1)\} \subset \mathbb{R}^2$ is a sublattice of $(\mathbb{R}^2, \geqslant)$.

(2) $K = \{(0,0),(1,0),(0,1),(2,2)\} \subset \mathbb{R}^2$ is not a sublattice of $(\mathbb{R}^2, \geqslant)$.

This is because $(1,0) \vee_{\mathbb{R}^2} (0,1) = \sup_{\mathbb{R}^2}\{(1,0),(0,1)\} = (1,1) \notin K$. Note that $(1,1)$ is not an element of K, so K does not satisfy the closure property under the join operation in $(\mathbb{R}^2, \geqslant)$.





4 完备格

Suppose $(X, \succsim)$ is a partially ordered set. For $x, y \in X$, we write:

$$x \vee_x y = \sup_X \{x, y\} \quad \text{join}$$

$$x \wedge_x y = \inf_X \{x, y\} \quad \text{meet}$$

A partially ordered set $(X, \succsim)$ is a lattice if for all $x, y \in X$, both $x \vee_x y$ and $x \wedge_x y$ exist.

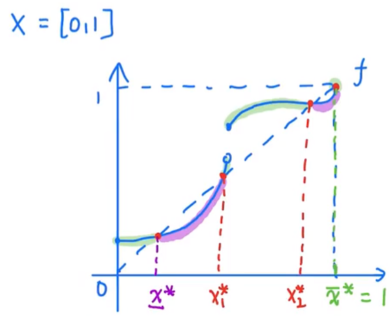
For a lattice $(X, \geqslant)$, a subset $K \subset X$ is a sublattice of $(X, \geqslant)$ if for all $x, y \in K$, both $x \vee_x y$ and $x \wedge_x y$ belong to K.

A lattice $(X, \geqslant)$ is complete if for all $S \subset X$, both $\sup_X S$ and $\inf_X S$ exist.

For example:

(1) $X = [0, 1] \subset \mathbb{R}$ with the partial order $\geqslant$ is a complete lattice.

(2) $X = [0, 1] \subset \mathbb{R}$ with the partial order $\geqslant$ is a lattice.



However, $(X, \geqslant)$ is not a complete lattice because $\sup_X [0,1] \notin X$.

5 Tarski 不动点定理

For a partially ordered set $(X, \succsim_X)$ and $(Y, \succsim_Y)$, a function $f : X \to Y$ is non-decreasing if for any $x, x' \in X$, $x \succsim_X x'$ implies $f(x) \succsim_Y f(x')$.

Tarski's Fixed Point Theorem

Suppose $(X, \geq)$ is a complete lattice, and $f : X \to X$ is a non-decreasing function.

Let $X^* = \{x \in X \mid x = f(x)\}$ be the set of fixed points of f.

(1) $X^* \neq \varnothing$.

(2) Let $\bar{x}^* = \sup\{x \in X \mid f(x) \geq x\}$ and $x^* = \inf\{x \in X \mid x \geq f(x)\}$. Then $\bar{x}^*, x^* \in X^*$ and for any $x^* \in X^*$, we have $\underline{x}^* \leq x^* \leq \bar{x}^*$.

(3) $(X^*, \geq)$ is a complete lattice.

eg Proof: Let $x' = \{x \in X \mid f(x) \geq x\}$.

Since $(X, \geq)$ is complete, $\inf x \in x$.

Also, $\inf x \in x'$ because $f(\inf x) \geq \inf x$.

Therefore, $x' \neq \emptyset$.

Let $\bar{x}^* = \sup_x x' \in X$.

We need to show that $\bar{x}^* = f(x^*)$.

Take any $x \in x'$, then $f(x) \geq x$ and $\bar{x}^* \geq x$.

Since f is non-decreasing, $f(\bar{x}^*) \geq f(x)$.

By transitivity, $f(\bar{x}^*) \geq x$.

Then $f(\bar{x}^*) \geq \bar{x}^*$.

Again, since f is non-decreasing, $f(f(\bar{x}^*)) \geq f(\bar{x}^*)$.

Thus, $f(\bar{x}^*) \in x'$.

Therefore, $\bar{x}^* \geq f(\bar{x}^*)$.

By antisymmetry, $\bar{x}^* = f(\bar{x}^*)$.

Hence, (1) is done.

Now, take any $x^* \in x'$, $x^* = f(x^*)$.

Then, $x^* \in x'$.

Therefore, $\bar{x}^* \geq x^*$.

By a symmetric argument, $x^* = \inf\{x \in X \mid x \geq f(x)\} \in x'$ and for any $x^* \in x', x^* \geq x^*$.

Hence, (2) is done.

Take any $s \subset x^*$, let $\bar{s} = \sup_x s$ and $z = \{x \in X \mid x \geq \bar{s}\}$.

We want to show that $(z, \geq)$ is complete.

Take any $T \subset z$. We need to show that $\inf T \geq \bar{s}$ and $\sup T \geq \bar{s}$.

For any $y \in T \subset z$, we have $\sup T \geq y \geq \bar{s}$.

Also, for any $y \in T \subset z$, we have $y \geq \bar{s}$.

Then $\inf T \geq \bar{s}$.

Now, take any $z \in Z$, i.e., $z \geq j$.

For any $x \in S \subset x^*$, we have $z \geq x$.

Since f is non-decreasing, $f(z) \geq f(x) = x$.

Then $f(z) \geq f(x)$.

Let $f|_z : z \to z$ be the restriction of f on z such that for any $z \in z$, $f|_z(z) = f(z)$.

Let z^* be the set of fixed points of $f|_z$. By (1), $z^* \neq \emptyset$.

By (2), there exists $\underline{z}^* \in z^*$ such that for all $z^* \in z^*$, $\underline{z}^* \geq z^*$.

Then $\underline{z}^* = \sup_{x^*} S$.

Similarly, $\inf_{n^*} S$ exists.

Therefore, $(x^*, \geq)$ is complete lattice.

(3) is done.

6 超模函数

For a lattice $(X, \vee)$, a function $f : X \to \mathbb{R}$ is said to be supermodular if for all $x, x' \in X$,

$$f(x \vee x') + f(x \wedge x') \geqslant f(x) + f(x')$$

For example, let $X = \mathbb{R}^2$, and let $f(x_{11}, x_2) = x_1 x_2$. This function is supermodular. Let's consider $x = (x_1, x_2)$ and $x' = (x'_1, x'_2)$.

If $x \geqslant x'$, that is, $x_1 \geqslant x'_1$ and $x_2 \geqslant x'_2$, then $x \vee x' = x$ and $x \wedge x' = x'$. In this case, the inequality holds trivially.

Now let's consider the case where $x_1 \geqslant x'_1$ and $x_2 < x'_2$. We have $x \vee x' = (x_1, x'_2)$ and $x \wedge x' = (x'_1, x_2)$.

$$\begin{aligned} f(x \vee x') + f(x \wedge x') &= x_1(x'_2) + x'_1(x_2) \\ &= x_1 x'_2 + x'_1 x_2 \\ &= f(x) + f(x') \end{aligned}$$

So the inequality holds in this case as well.

A function f is said to be strictly supermodular if for all $x, x' \in X$ such that neither $x \geq x'$ nor $x' \geq x$, we have

$$f(x \vee x') + f(x \wedge x') > f(x) + f(x')$$

A function f is said to be (strictly) submodular if $-f$ is (strictly) supermodular.

For partially ordered sets $(X, \gtrsim_X)$ and $(Y, \gtrsim_Y)$, a function $f : X \times Y \to \mathbb{R}$ satisfies increasing differences if for all $x'', x' \in X$ and $y'', y' \in Y$ such that $x'' \gtrsim_X x'$ and $y'' \gtrsim_Y y'$, we have

$$f(x'', y'') - f(x', y'') \geqslant f(x'', y') - f(x', y')$$

eg

7 超模函数的最优化

(1) If $f : X \to \mathbb{R}$ is supermodular, then $\operatorname{argmax} f(x)$ is a sublattice of X.

computer

$u(m, c)$ complemetarity

monitor

$u(m+\theta, c+\delta) - u(m, c+\delta) \geq u(m+\theta, c) - u(m, c)$

(2) If $f : X \to \mathbb{R}$ is strictly supermodular, then $\operatorname{argmax} f(x)$ is a chain, i.e., for any $x, x' \in \operatorname{argmax} f(x)$, either $x \geq x'$ or $x' \geq x$.

Proof:

(1) For any $x, x' \in \operatorname{argmax} f(x)$,

$$f(x) \geq f(x \vee_x x')$$
$$f(x') \geq f(x \wedge_x x')$$

Since f is supermodular,

$$f(x \vee_x x') + f(x \wedge_x x') \geq f(x) + f(x')$$
$$0 \geq f(x \wedge_x x') - f(x') \geq f(x) - f(x \vee_x x') \geq 0$$

Therefore,

$$f(x) = f(x') = f(x \vee_x x') = f(x \wedge_x x') = \max_{x \in X} f(x)$$

Thus, $x \vee_x x', x \wedge_x x' \in \operatorname{argmax} f(x)$.

(2) By contradiction.

Suppose for $x, x' \in \operatorname{argmax} f(x)$, neither $x \geq x'$ nor $x' \geq x$.

By strict supermodularity of f,

$$f(x \vee_x x') + f(x \wedge_x x') > f(x) + f(x')$$

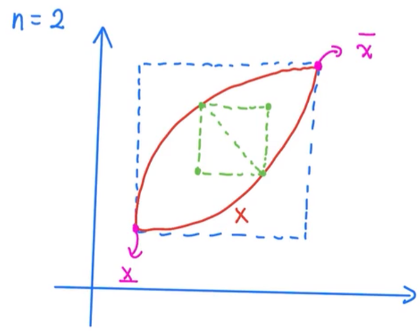
However, $f(x) \geq f(x \vee_x x')$ and $f(x') \geq f(x \wedge_x x')$, which leads to a contradiction.

Let $(X, \vee_X, \wedge_X)$ and $(Y, \vee_Y, \wedge_Y)$ be lattices.

Suppose $f : X \times Y \to \mathbb{R}$ is supermodular.

Assume $v(y) = \sup_{x \in X} f(x, y)$ is finite for all $y \in Y$. Then v is supermodular.

Proof: Take any $y, y' \in Y$.



For any $x, x' \in X$,

$$\begin{aligned} & v(y \vee_Y y') + v(y \wedge_Y y') \\ \geq f(x \vee_X x', y \vee_Y y') + f(x \wedge_X x', y \wedge_Y y') & \\ \geq f(x, y) + f(x', y') & \end{aligned}$$

Since this holds for any $x, x' \in X$, then

$$\begin{aligned} & v(y \vee_Y y') + v(y \wedge_Y y') \\ \geq \sup_{x, x' \in X} (f(x, y) + f(x', y')) & \\ = v(y) + v(y') & \end{aligned}$$

8 欧氏空间中的紧格

If $X \subset \mathbb{R}^n$ is nonempty and compact, and $(X_1, \geq)$ is a lattice, then there exist $\bar{x}, \underline{x} \in X$ such that for any $x \in X$, $\bar{x} \geq x \geq \underline{x}$.

eg

Proof: Since X is compact, for each $i \in \{1, 2, \ldots, n\}$, there exists $z^i \in X$ such that $z^i \in \operatorname{argmax}_{x \in X} x_i$. Then, let $\bar{x} = (((z^1 V_x z^2)\, V_x z^3)\, V_x \cdots)\, V_x z^n$, where $\vee$ denotes the component-wise maximum. Since $z^i \in \operatorname{argmax}_{x \in X} x_i$, we have $\bar{x} \geq z^i$ for all i. Therefore, $\bar{x} \geq x$ for all $x \in X$.

Similarly, let $\underline{x} = (z^1 \wedge z^2 \wedge \cdots \wedge z^n)$, where $\wedge$ denotes the component-wise minimum. Then, we have $z^i \leq \underline{x}$ for all i, which implies $x \geq \underline{x}$ for all $x \in X$. Therefore, $\bar{x}$ and $\underline{x}$ satisfy the desired properties.

Note that in your original notes, you used x to denote both the set and an element of the set, which can be confusing. Also, the notation $z^i V_x z^j$ is not standard in mathematics. It is more common to use $\vee$ and $\wedge$ to denote the maximum and minimum operations on lattices, respectively.

9 单调比较静态

For a lattice $(X, \vee, \wedge)$, a function $f : X \to \mathbb{R}$ is said to be supermodular if for all $x, x' \in X$,

$$f(x \vee x') + f(x \wedge x') \geqslant f(x) + f(x')$$

If $f : X \to \mathbb{R}$ is supermodular, then $\operatorname{argmax}_{x \in X} f(x)$ is a sublattice of X.

For partially ordered sets $(X_1, \geq)$ and $(Y, \leq)$, a function $f : X \times Y \to \mathbb{R}$ satisfies increasing differences if for

$$\begin{gathered} x'', x' \in X, y'', y' \in Y, x'' \geq x', y'' \geq y', \\ f(x'', y'') - f(x', y'') \geqslant f(x'', y') - f(x', y') \end{gathered}$$

Monotone Comparative Statics

Suppose $X \subset \mathbb{R}^n$ is compact, $(X_1, \geq)$ is a lattice, $(Y, \leq)$ is a partially ordered set, and $f : X \times Y \to \mathbb{R}$ satisfies:

(1) For each $y \in Y$, $f(\cdot, y)$ is continuous.

(2) For each $y \in Y$, $f(\cdot, y)$ is supermodular.

(3) f satisfies increasing differences.

Then for each $y \in Y$, $\operatorname{argmax}_{x \in X} f(x, y)$ has a greatest and a least elements, denoted as $\bar{x}(y)$ and $\underline{x}(y)$, respectively. For $y, y' \in Y_1$ if $y \geq y'$, then $\bar{x}(y) \geq \bar{x}(y')$ and $\underline{x}(y) \geq \underline{x}(y')$.

Proof:

For each $y \in Y$, since X is compact and $f(\cdot, y)$ is continuous, by Weierstrass's Theorem, $\operatorname{argmax}_{x \in X} f(x, y) \neq \emptyset$ and $\operatorname{argmax}_{x \in X} f(x, y)$ is compact. We know that $\operatorname{argmax}_{x \in X} f(x, y)$ is a sublattice of X. Then, $(\operatorname{argmax}_{x \in X} f(x, y), \geq)$ is a lattice.

Together with $\text{argmax}_{x \in X} f(x, y)$ being compact, there exist $\bar{x}(y), \underline{x}(y) \in \text{argmax}_{x \in X} f(x, y)$ such that for any $x \in \text{argmax}_{x \in X} f(x, y)$, $\bar{x}(y) \geq x \geq \underline{x}(y)$. Take $y, y' \in Y$, $y \geq y'$.

For any $x \in \text{argmax}_{x \in X} f(x, y)$ and any $x' \in \text{argmax}_{x \in X} f(x, y')$, we have

$$\begin{aligned} f(x, y) &\geq f(x \vee x', y) \\ f(x', y') &\geq f(x \wedge x', y) \end{aligned}$$

Since $f(\cdot, y)$ is supermodular,

$$f(x \vee x', y) + f(x \wedge x', y) \geq f(x, y) + f(x', y)$$

Since f satisfies increasing differences,

$$f(x', y) - f(x \wedge x', y) \geq f(x', y') - f(x \wedge x', y')$$

Then,

$$\begin{aligned} 0 &\geq f(x \vee x', y) - f(x, y) \\ &\geq f(x', y) - f(x \wedge x', y) \\ &\geq f(x', y') - f(x \wedge x', y') \\ &\geq 0 \end{aligned}$$

Therefore,

$$\begin{aligned} x \vee x' &\in \text{argmax}_{x \in X} f(x, y) \\ x \wedge x' &\in \text{argmax}_{x \in X} f(x, y') \end{aligned}$$

Then,

$$\begin{aligned} \bar{x}(y) \vee \bar{x}(y') &\in \text{argmax}_{x \in X} f(x, y) \\ \underline{x}(y) \wedge \underline{x}(y') &\in \text{argmax}_{x \in X} f(x, y') \end{aligned}$$

Since $\bar{x}(y)$ is the greatest element of $\text{argmax}_{x \in X} f(x, y)$,

$$\bar{x}(y) \geq \bar{x}(y) \vee \bar{x}(y') \geq \bar{x}(y')$$

Since $\underline{x}(y)$ is the least element of $\text{argmax}_{x \in X} f(x, y')$,

$$\underline{x}(y) \leq \underline{x}(y) \wedge \underline{x}(y') \leq \underline{x}(y')$$

10 超模博弈

$I = \{1, \cdots, |I|\}$: set of players.

$S_i \subset \mathbb{R}^{n_i}$: a compact set of strategies of player $i \in I$, partially ordered by $\geqslant$.

$u_i : \prod_{j \in I} S_i \to \mathbb{R}$: payoff function of player $i \in I$.

$u_i(s_i, s_{-i})$: continuous in s_i for each s_{-i} and continuous in s_{-i} for each s_i.

Denote this normal form game by G.

G is a supermodular game if for each $i \in I$: (1) $(S_i, \geqslant)$ is a complete lattice; (2) $u_i(\cdot, s_{-i})$ is supermodular; (3) $u_i(s_i, s_{-i})$ satisfies increasing differences in (s_i, s_{-i}).

Best response correspondence of player i:

$$b_i(s_{-i}) = \{s_i \mid u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i\}$$

A normal form game has monotone best responses if for all $i \in I$: (1) for all $s_{-i} \in S_{-i}$, $b_i(S_{-i})$ has a greatest element $\bar{b}_i(s_{-i})$ and a least element $\underline{b}_i(s_{-i})$; (2) $\bar{b}_i(s_{-i})$ and $\underline{b}_i(s_{-i})$ are non-decreasing in s_i.

A supermodular game G has monotone best responses.

Proof: By Monotone Comparative Statics.

If a game G has monotone best responses, then G has a pure-strategy Nash equilibrium.

Proof: Define $\bar{b} : \prod_{j \in I} S_i \to \prod_{j \in I} S_i$ such that for any $s \in \prod_{j \in I} S_i$,

$$\bar{b}(s) = \left(\bar{b}_1(s_{-1}), \bar{b}_2(s_{-2}), \cdots, \bar{b}_n(s_{-n})\right)$$

Where $\bar{b}_i(s_{-i})$ is the greatest element of $b_i(s_{-i})$.

Then, $\bar{b}$ is a non-decreasing function from a complete lattice to itself.

By Tarski's fixed point theorem, $\exists s^* \in \prod_{j \in I} S_j$, $s^* = \bar{b}(s^*)$, which is a pure-strategy Nash equilibrium.