The Book of Special Relativity

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Special relativity is a theory proposed by Albert Einstein that describes the propagation of matter and light at high speeds. It was invented to explain the observed behavior of electric and magnetic fields, which it beautifully reconciles into a single so-called electromagnetic field, and also to resolve a number of paradoxes that arise when considering travel at large speeds. Special relativity also explains the behavior of fast-traveling particle, including the fact that fast-traveling unstable particles appear to decay slower than identical particles traveling slower. Special relativity is an indispensable tool of modern physics, and its predictions have been experimentally tested time and time again without any discrepancies turning up. Special relativity reduces to Newtonian mechanics in the limit of small speeds.

According to special relativity, no wave or particle may travel at a speed greater than the speed of light c. Therefore, the usual rules from Newtonian mechanics do not apply when adding velocities that are large enough. For example, if a particle travels at a speed v with respect to a stationary observer, and another particle travels at a speed v' with respect to the first particle, the speed u of particle two seen by the observer is not as would be the case in Newtonian mechanics, but rather

$$u = \frac{v + v'}{1 + \frac{vv'}{c^2}}$$

1 Invariance of Lengths Measured Perpendicular to Relative Motion

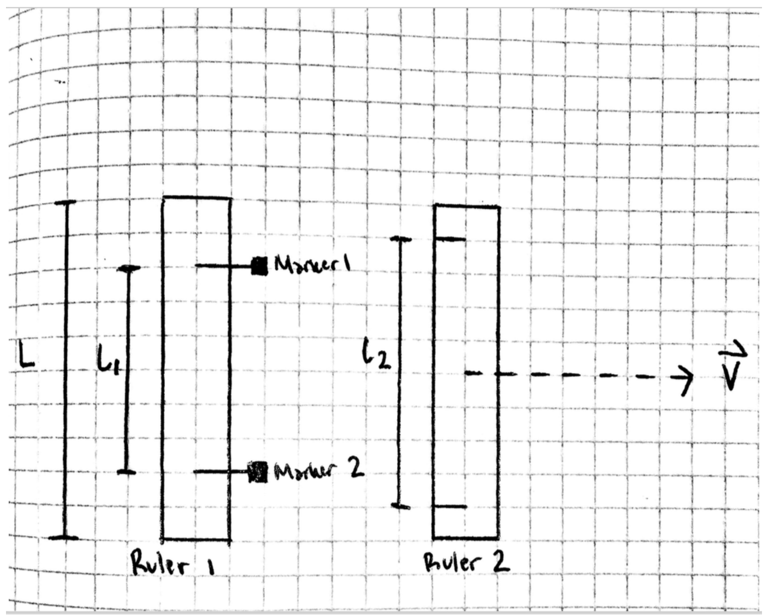

Figure 1: Two rulers perpendicularly flying past eachother

In the figure above, we are presented with a thought experiment:

Suppose we are given two rulers, Ruler 1 and Ruler 2 of length L. Ruler 1 has two markers attached on it with a separation of L_1. Our hypothesis is that if Ruler 2 is flying perpendicularly by Ruler 1 with a speed $\vec{v}$, then Ruler 2 will become shorter than Ruler 1.

If indeed Ruler 2 does become shorter, then as it travels by Ruler 1, the tick marks drawn by the marker on Ruler 1 will not have a separation of L_1 but rather of L_2 where $L_2 > L_1$. However, according to the principle of relativity it is equally valid to think of Ruler 2 as stationary and Ruler 1 as moving during the flyby. From this perspective, the moving Ruler (now Ruler 1) is *longer* than the stationary stick, as that is the only way for $L_2 > L_1$ to be true. Thus, our hypothesis that our moving Ruler will become shorter faces a contradition, and is **rejected**.

We can thus conclude: *A stick moving perpendicular to its length has the same length as an identical stick that is stationary.*

2 Time Dilation

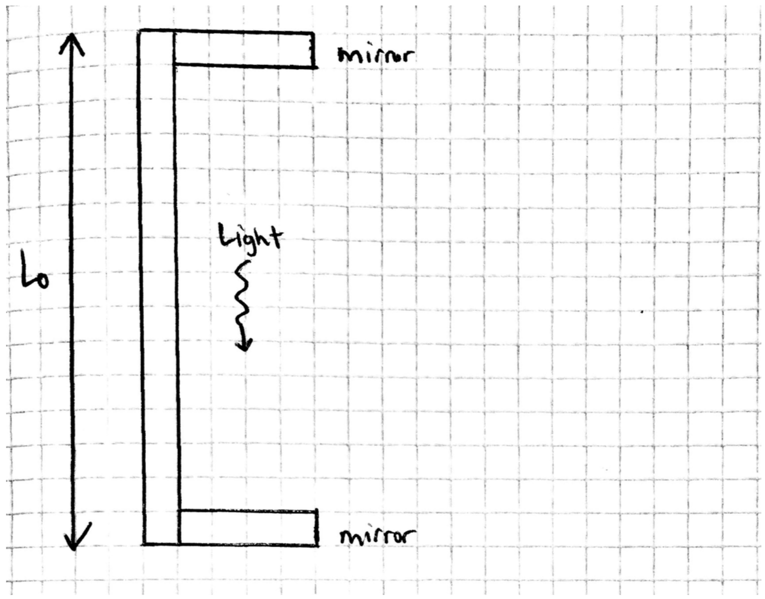

Figure 2: Light Clock

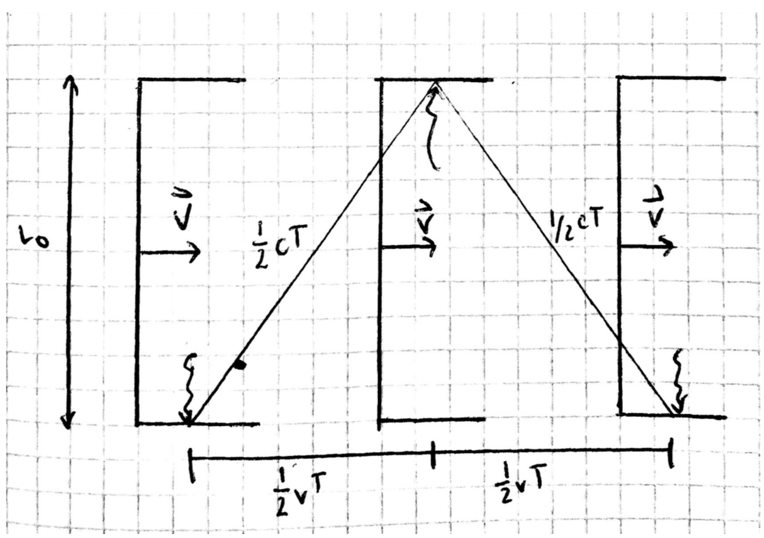

Figure 3: A Light Clock Moving with Velocity $\vec{v}$

Suppose we construct a Light Clock, which is defined as the structure in Figure 1. The time between ticks T_0 on a light clock is the time between when the light hits, say the lower mirror. Since the light travels a length L_0, we can relate the time between ticks by $2L_0 = cT_0$.

Consider when the light clock is moving at a velocity $\vec{(v)}$. In this frame of reference, the clock moves a distance vT_0 in between ticks, as demonstrated in Figure 2. Thus, the distance that the light is travelling by the Pythagorean theorem is now expressed as

$$cT_0 = 2\sqrt{L_0^2 + (\frac{1}{2}vT_0)^2}$$

But since we know that the speed of light is the same in all inertial frames, we can substitute in $L_0 = \frac{1}{2}cT_0$ and you get

$$cT = 2\sqrt{(\frac{1}{2}cT_0)^2 + (\frac{1}{2}vT)^2}$$

Solving for T yields

$$T = \frac{T_0}{\sqrt{1 - (\frac{\vec{v}^2}{c^2})}}$$

Since we know that $\vec{v} < c$, we can deduce from the given equation that the time T between ticks in a reference frame moving with velocity $\vec{(v)}$ is greater than time T_0 between ticks in the proper reference frame of the clocks.

This brings us to question: Does this fact only hold true for light clocks or all clocks in general? It turns out that the answer is yes. Consider the following thought experiment:

In the proper reference frame of the clocks, the time between ticks of both clocks is exactly one second. A light-sensitive film is placed on the face of the conventional clock, behind a rotating disk. Each time the light pulse reflects off the lower mirror, a narrow region of the light-sensitive paper directly behind the slot gets exposed. These exposed regions will be aligned with the tick marks, and all observers must agree with this permanent record.

In the reference frame where the clocks are both moving, the light pulse exposes the film behind the slot on the clock face each time the pulse reflects off the lower mirror. Because the light clock is moving, the time between these reflections is greater than 1 s, in accordance with equation for T. When an observer of original reference frame sees that the lines appearing where the film was exposed are aligned with the tick marks, they realize that in their reference frame the conventional clock runs slow in exactly the same manner that the light clock runs slow and that this has nothing to do with the mechanism of the conventional clock. Thus, we conclude that all moving clocks run slow in exactly the same manner that a moving light clock runs slow. Because this is the case, we conclude that it is time itself that runs slow. We call this phenomena **Time Dilation**.

3 Length Contraction

Earlier, we demonstrated that the length of a stick moving perpendicular to its length and the length of an identical stationary stick are equal. However, the key takeaway from this is that this only applies when the stick is perpendicular. Here, we analyze what happens when we analyze a situation where a stick is moving parallel to its length.

Suppose we are given a Light Clock in two scenarios as demonstrated by the figures below.

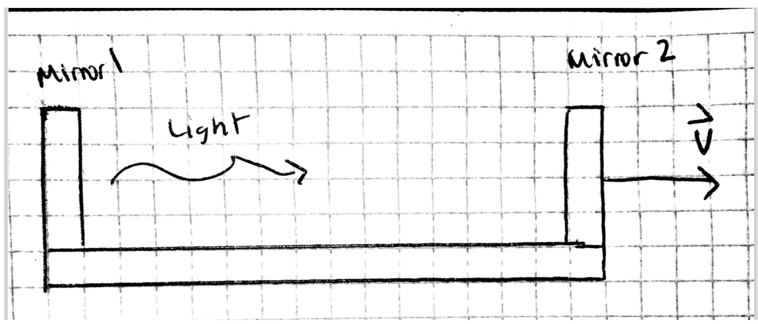

Figure 4: Light Clock travelling Parallel to Length

Suppose the Light Clock has length L_0. Thus, the time T_0 between ticks of the light clock when $\vec{(v)} = 0$ is $\frac{2L_0}{c}$

However, if $\vec{(v)} \neq 0$ then the light has to travel a greater distance to hit the right side and a shorter distance to come back and hit the left side. The distance travelled by the light in stationary reference frame is cT_0.

When the box is not stationary, let us define three events:
Event 0: Light pulse reflects off the mirror at the left end.
Event 1: Light pulse reflects off the mirror at the right end.
Event 2: Light pulse reflects off at the mirror at the left end.

The times of occurences of these events are labeled $t_0 t_1 t_2$. In the time between events 0 and 1, the clock moves a distance $\vec{v}(t_1 - t_0)$ and the light pulse travels a distance $c(t_1 - t_0)$. We can express this as

$$c(t_1 - t_0) = L + v(t_1 - t_0)$$

Similarly for the time between events 1 and 2, we can express it as

$$c(t_2 - t_1) = L + v(t_2 - t_1)$$

Using substitution to elimate t_1, we can now express this as

$$t_2 - t_0 = \frac{\frac{2L}{c}}{1 - \frac{\vec{v}^2}{c^2}}$$

Employing our Time Dilation equation to Events 0 and 2, we can state that

$$t_2 - t_0 = \frac{\frac{2L_0}{c}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}$$

By setting these equations equal to eachother and solving for L, we get that

$$L = L_0\sqrt{1 - \frac{\vec{v}^2}{c^2}}$$

Note that this formula does not involve any properties of the stick but rather properties of space and time.