Lista 7

1) a) $Cov(X,Y) = E(XY) - E(X)E(Y) = 1/2 - 0 = 1/2$

$\therefore$ X e Y são correlacionados

b) $$f_{(X,Y)}(x,y) = \frac{1}{2\pi\sqrt{3/4}} \exp\left\{-\frac{1}{2(3/4)}\left[x^2 - x(y-1) + (y-1)^2\right]\right\}$$

$$= \frac{1}{\pi\sqrt{3}} \exp\left\{-\frac{2}{3}\left[x^2 - x(y-1) + (y-1)^2\right]\right\}$$

c) $$f_{X|Y}(x|y) \sim N\left(u_x + \rho(y-u_y)\frac{\sigma_x}{\sigma_y} ; \sigma_x^2(1-\rho^2)\right)$$

$$f_{X|Y}(x|y) \sim N\left(1/2\,(y-1) ; 3/4\right)$$

Assim, teremos:

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x - y/2)^2}{3/2}\right\}$$

Analogamente, teremos

$$f_{Y|X}(y|x) \sim N\left(1 + 1/2\,x ; 3/4\right)$$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(y - (1 + 1/2\,x))^2}{3/2}\right\}$$

2)a) $f_X(x) = \int_0^x \frac{1}{x}\,dy = \frac{y}{x}\Big|_0^x = 1$

$$f_Y(y) = \int_y^1 \frac{1}{x}\,dx = \ln(x)\Big|_y^1 = -\ln y = \ln(1/y)$$

b) $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1/x}{1} = \frac{1}{x}$

c) $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1/x}{\ln(1/y)}$

d) $E(Y|X=2)^* =$

3)a) $f(x) = \int_0^\infty x e^{-x(y+1)}\,dy$

$$= e^{-x}\cancelto{1}{\int_0^\infty x e^{-xy}\,dy}$$

$$= e^{-x}$$

$f(y) = \int_0^\infty x e^{-x(y+1)}\,dx$

$$\overset{\text{Por partes}}{=} \left[x\,\frac{e^{-x(y+1)}}{y+1}\right]_0^\infty - \int_0^\infty -\frac{e^{-x(y+1)}}{(y+1)}\,dx$$

$$= 0 - \left[\frac{e^{-x(y+1)}}{(y+1)^2}\right]_0^\infty$$

$$= \frac{1}{(y+1)^2}$$

b) $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{x e^{-x(y+1)}}{e^{-x}} = x e^{-xy}$$

c) $$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{x e^{-x(y+1)}}{\frac{1}{(y+1)^2}} = x(y+1)^2 e^{-x(y+1)}$$

d) $$E(X|Y=1/3) = \int_0^\infty x \cdot x(1/3+1)^2 e^{-x(\frac{1}{3}+1)} \, dx$$

$$= \int_0^\infty x^2 \cdot \frac{16}{9} e^{-x\frac{4}{3}} \, dx$$

$c = \frac{4}{3}$

$$= \int_0^\infty x^2 \cdot c^2 e^{-xc}$$

$$= c^2 \int_0^\infty x^2 e^{-xc}$$

por partes

$$= c^2 \left(\overset{0}{\cancel{\left[\frac{-x^2 e^{-xc}}{c} \right]_0^\infty}} - \int_0^\infty \frac{-2x e^{-xc}}{c} \, dx \right)$$

por partes

$$= 2c^2 \left(\left[\frac{-x e^{-xc}}{c^2} \right]_0^\infty - \int_0^\infty \frac{-e^{-xc}}{c^2} \right) = 2c^2 \cdot \left[-\frac{e^{-xc}}{c^3} \right]_0^\infty$$

$$= \frac{2}{c} = \frac{2}{\frac{4}{3}} = \frac{3}{2}$$

4) a) $f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y}\,dy = \lambda \int_x^\infty \lambda e^{-\lambda y}\,dy = \lambda e^{-\lambda x}$

$f_Y(y) = \int_0^y \lambda^2 e^{-\lambda y}\,dx = \lambda^2 y e^{-\lambda y}$

b) $f_{Y|X}(y|x) = \dfrac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} = \lambda e^{-\lambda(y-x)}$

c) $f_{X|Y}(x|y) = \dfrac{\lambda^2 e^{-\lambda y}}{\lambda^2 y e^{-\lambda y}} = \dfrac{1}{y}$

d) $E(Y|X=1) = \int_1^\infty y \lambda e^{-\lambda(y-1)}\,dy$

$$= e^{\lambda} \int_1^\infty y \lambda e^{-\lambda y}\,dy$$

$$= e^{\lambda} \cdot \lambda \cdot \left[-\frac{y e^{-\lambda y}}{\lambda} \Bigg|_1^\infty - \int_1^\infty -\frac{e^{-\lambda y}}{\lambda} \right]$$

$$= e^{\lambda} \cdot \lambda \left[\frac{e^{-\lambda}}{\lambda} + \frac{e^{-\lambda}}{\lambda^2} \right]$$

$$= e^{\lambda} \cdot \frac{(\lambda+1) e^{-\lambda}}{\lambda} = \frac{(\lambda+1)}{\lambda}$$

e) $E(X|Y=5) = \int_0^5 \frac{x \cdot 1}{5} dx = \frac{1}{5} \cdot \left[\frac{x^2}{2}\right]_0^5 = \frac{5}{2}$ //

5) Seja $Z = X_1 + X_2$, então teremos

$$Var(Z \cdot X_3) = E(Z^2 \cdot X_3^2) - (E(ZX_3))^2$$

$$\overset{indep}{=} E(Z^2) \cdot E(X_3^2) - (E(Z) \cdot E(X_3))^2$$

$$= (Var(Z) + E(Z)^2)(Var(X_3) + E(X_3)^2) - (E(Z) \cdot E(X_3))^2$$

Como $Z \sim gamma(2,2)$ e $X_3 \sim EXP(1)$, teremos

$$= \left(\frac{2}{4} + \left(\frac{2}{2}\right)^2\right)\left(\frac{1}{4} + (1)^2\right) - \left(\frac{2}{2} \cdot 1\right)^2$$

$$= \left(\frac{3}{2} \cdot \frac{5}{4}\right) - 1 = \frac{15}{8} - \frac{8}{8} = \frac{7}{8}$$ //

6) X: valor do dado
Y: "valor" da moeda

$X \sim$ Uniforme discreta $\{0,6\}$

$Y = \begin{cases} 1/2 & P(1/2) = 1/2 \\ 2 & P(2) = 1/2 \end{cases}$

$$E(E(X|Y)) = E\left(Y \cdot \frac{6+0}{2}\right) = E(Y \cdot 3) = 3E(Y) = 3$$ //

7) $E[(X-Y)^2] = E(X^2 - 2XY + Y^2)$

$$= E(X^2) + E(Y^2) - 2E(XY)$$

$$\overset{indep}{=} (Var(X) + E(X)^2) + (Var(Y) + E(Y)^2) - 2E(X) \cdot E(Y)$$

$$= 2\sigma^2 + 2\mu^2 - 2\mu^2 = 2\sigma^2$$

8) $E((2+X)^2) = E[4+4X+X^2]$

$$= E(4) + 4E(X) + E(X^2)$$

$$= 4 + 4 + (5+1^2) = 14$$

$Var(4+3X) = 9Var(X) = 45$

9) Seja T_1 o lançamento 1 e T_2 o lançamento 2 e que T_1 e T_2 sejam independentes e identicamente distribuídos. Assim teremos:

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= E[(T_1+T_2)(T_1-T_2)] - E(T_1+T_2).E(T_1-T_2)$$

$$\overset{indep}{=} E(T_1^2 - T_2^2) - (E(T_1)+E(T_2))(E(T_1)-E(T_2))$$

$$\overset{indep}{=} E(T_1^2) - E(T_2^2) - (E(T_1)^2 - E(T_2)^2)$$

$$= Var(T_1) - Var(T_2) = 0$$

10) a) Pela desigualdade de markov teremos:

$$P(X \geq 85) \leq \frac{75}{85} = \frac{15}{17}$$

b) Usando a desigualdade de chebyshev, teremos:

$$P(|X-75| \leq 10) \geq 0,25$$

11) Pela desigualdade de markov teremos que

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Como $g(x) \geq b > 0$ quando $X \geq a$, então teremos que

$$P(X \geq a) \leq \frac{E(X)}{a} = P(g(X) \geq b) \leq \frac{E(g(X))}{b}$$

$$P(X \geq a) \leq \frac{E(g(X))}{b}$$

12) Calculando $E(X)$, teremos:

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\, dx$$

$$\overset{*}{\geq} \int_{-\infty}^{+\infty} a\, f(x)\, dx$$

$* \; P(a \leq x \leq b) = 1$

$$= a \int_{-\infty}^{+\infty} f(x)\, dx = a$$

Assim teremos que $E(X) \geq a$. Analogamente teremos:

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\, dx$$

$$\overset{*}{\leq} \int_{-\infty}^{+\infty} b\, f(x)\, dx$$

$(*) \; P(a < X < b) = 1$

$$= b \int_{-\infty}^{+\infty} f(x)\, dx = b \quad \Rightarrow \quad E(X) \leq b$$

$* \quad a \leq E(X) \leq b$

13) $f(x) = \int_0^x \frac{1}{2x} dy = \frac{1}{2}$

$f(y) = \int_y^2 \frac{1}{2x} dx = \frac{\ln(2) - \ln(y)}{2}$ //

$E(X) = \int_0^2 x \cdot \frac{1}{2} dx = x^2 \Big|_0^2 = 4$

$E(Y) = \int_0^2 \frac{\ln(2)}{2} - \frac{\ln(y)}{2} dy = \frac{y\ln(2)}{2}\Big|_0^2 - y(\ln(y)-1)\Big|_0^2$

$= \ln(2) - 2\ln(2) + 2$

$= 2 - \ln(2)$

$E(XY) = \int_0^2 \int_y^2 \frac{xy}{2x} dx\, dy$

$= \int_0^2 \frac{y(2-y)}{2} dy = \int_0^2 y - \frac{y^2}{2} dy$

$= \frac{y^2}{2}\Big|_0^2 - \frac{y^3}{6}\Big|_0^2$

$= 2 - \frac{8}{6} = \frac{2}{3}$ //

$\mathrm{COV}(X,Y) = E(XY) - E(X)E(Y)$

$= \frac{2}{3} - 4(2 - \ln(2))$

$= \frac{2 - 24 + 12\ln(2)}{3} = \frac{12\ln(2) - 22}{3}$

(4) $COV(X_i-\bar{X};\bar{X}) = COV(X_i;\bar{X}) - COV(\bar{X};\bar{X})$

$$= COV\left(X_i; \sum_{l=1}^{n} \frac{X_l}{n}\right) - Var(\bar{X})$$

$$= \frac{1}{n} \sum_{l=1}^{n} COV(X_i; X_l) - \frac{\sigma^2}{n}$$

Note que:

$$COV(X_i;X_l) = \begin{cases} 0 & i \neq l \text{ pela independencia} \\ \sigma^2 & i = l \end{cases}$$

Então teremos:

$$COV(X_i-\bar{X};\bar{X}) = \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0$$

(5) $X \sim$ geometrica$(1/6)$
$Y \sim$ geometrica$(1/6)$

a) $E(X) = \frac{1}{p} = 6$

b) $E(X|Y=1) = 1+6 = 7$

c) Construindo a probabilidade condicional, teremos:

$$P_{X|Y}(X|Y=5) = \begin{cases} \left(\frac{4}{5}\right)^{x-1} \cdot \frac{1}{5} & x<5 \\ 0 & x=5 \\ \frac{4}{5}^{4} \cdot \left(\frac{5}{6}\right)^{x-6} \cdot \left(\frac{1}{6}\right) & x>5 \end{cases}$$

Assim, calculando a esperança, teremos

$$E(X|Y=5) = \sum_{x=1}^{4} x\left(\frac{4}{5}\right)^{x-1} \cdot \frac{1}{5} + \sum_{x=6}^{\infty} x \frac{4}{5}^{4} \left(\frac{5}{6}\right)^{x-6} \left(\frac{1}{6}\right)$$

$$= \frac{1}{5} \sum_{x=1}^{4} x\left(\frac{4}{5}\right)^{x-1} + \frac{1}{6} \cdot \frac{4}{5}^{4} \underbrace{\sum_{x=6}^{\infty} x\left(\frac{5}{6}\right)^{x-6}}_{(*)}$$

Abrindo $(*)$ teremos

$$6 + 7\left(\frac{5}{6}\right) + 8\left(\frac{5}{6}\right)^2 + \dots$$

$$6 + (6+1)\cdot\frac{5}{6} + (6+2)\cdot\left(\frac{5}{6}\right)^2 + \dots$$

$$\left(6 + 6\cdot\frac{5}{6} + 6\cdot\left(\frac{5}{6}\right)^2 + \dots\right) + \left(\frac{5}{6} + 2\cdot\left(\frac{5}{6}\right)^2 + 3\cdot\left(\frac{5}{6}\right)^3 + \dots\right)$$

$$\frac{6}{\left(1-\frac{5}{6}\right)} + \frac{\frac{5}{6}}{\left(1-\frac{5}{6}\right)^2} = \frac{6}{\frac{1}{6}} + \frac{\frac{5}{6}}{\left(\frac{1}{6}\right)^2} = 6\cdot(6+5)$$

Então teremos que:

$$E(X|Y=5) = \frac{1}{5}\sum_{x=1}^{4} x\left(\frac{4}{5}\right)^{x-1} + \frac{1}{6}\cdot\left(\frac{4}{5}\right)^{4}\cdot 6\cdot(5+6)$$

$$= \frac{1}{5}\left(1 + 2\left(\frac{4}{5}\right) + 3\left(\frac{4}{5}\right)^2 + 4\left(\frac{4}{5}\right)^3\right) + \frac{4}{5}^{4}\cdot 11$$

$$=$$

18) X: número de tempestade
Y = ano bom

Calculando a esperança, teremos:

$E(X) = E(X|Y=1)\,p(Y=1) + E(X|Y=0)\,p(Y=0)$

$3 . 0{,}4 + 5 . 0{,}6 = 1{,}2 + 3 = 4{,}2$

Assim, fazendo $E(X^2)$, teremos

$E(X^2) = E(X^2|Y=1) + p(Y=1) + E(X^2|Y=0) . p(Y=0)$

Usando $E(X^2|Y) = Var(X|Y) + [E(X|Y)]^2$, teremos

$E(X^2) = (3+3^2) . 0{,}4 + (5+5^2) . 0{,}6$
$= 22{,}8$

Por fim, calculando a variancia, teremos

$Var(X) = E(X^2) - [E(X)]^2 = 5{,}16$

20) X: numero de pessoas que fazem compra
Y: numero de pessoas que chegam

$$E(X) = E(E(X|Y)) = \sum_{y=0}^{\infty} E(X|Y=y) \cdot P(Y=y)$$

$$= \sum_{y=0}^{\infty} E(X|Y=y) \cdot \frac{20^y \cdot e^{-20}}{y!}$$

$$\overset{*}{=} \sum_{y=1}^{\infty} y \cdot \frac{3}{4} \cdot \frac{20^y \cdot e^{-20}}{y!}$$

binomial

$E(X|Y) = yp$

$$= \frac{3}{4} e^{-20} \sum_{y=1}^{\infty} \frac{20^y}{(y-1)!}$$

$$= \frac{60}{4} e^{-20} \sum_{y=1}^{\infty} \frac{20^{(y-1)}}{(y-1)!} = 15\, e^{-20} \cdot e^{20} = 15$$

$$E(Var(X|Y)) = E\left(Y \cdot \frac{3}{4} \cdot \frac{1}{4}\right) = \frac{3}{16} E(Y) = \frac{60}{16}$$

$$Var(E(X|Y)) = Var\left(Y \cdot \frac{3}{4}\right) = \frac{9}{16} Var(Y) = \frac{180}{16}$$

$$Var(X) = E(Var(X|Y)) + Var(E(X|Y))$$

$$= \frac{60}{16} + Var\left(Y \cdot \frac{3}{4}\right)$$

$$= \frac{60}{16} + \frac{9}{16} \cdot 20 = \frac{240}{16} = 15$$