Lista 5

1) a) Para f seja fdp teremos que encontrar c tal que $\int_{-\infty}^{\infty} f(x)dx = 1$, então teremos que:

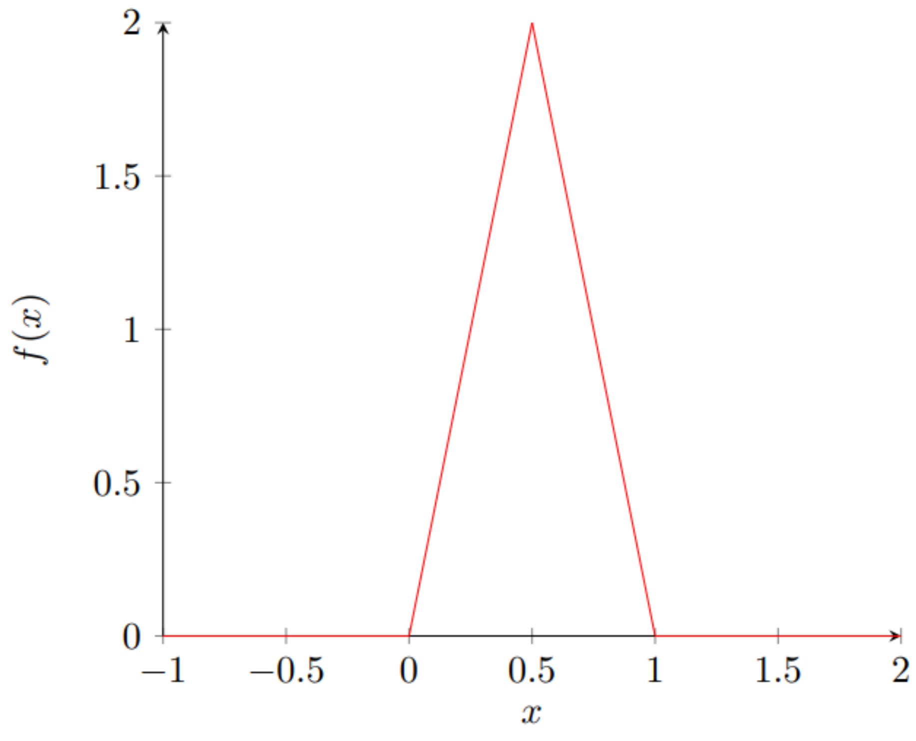
$$\int_{-\infty}^{\infty} f(x)dx = \int_0^{1/2} cx\,dx + \int_{1/2}^{1} c(1-x)\,dx = 1$$

$$= c \cdot \left(\left[\frac{x^2}{2}\right]_0^{1/2} + \left[x - \frac{x^2}{2}\right]_{1/2}^{1} \right) = 1$$

$$= c\left(\frac{1}{8} + \frac{1}{2} - \frac{1}{2} + \frac{1}{8}\right) = 1$$

$$= \frac{c}{4} = 1 \Rightarrow c = 4$$

b)



g) $P(X \leq 1/2) = \int_{-\infty}^{1/2} f(x)dx = \int_{0}^{1/2} 4x\,dx = 2x^2\Big|_0^{1/2} = \frac{1}{2}$

$P(X > 1/2) = 1 - P(X \leq 1/2) = 1 - \frac{1}{2} = \frac{1}{2}$

$P(1/4 < X < 3/4) = \int_{1/4}^{3/4} f(x)dx$

$= \int_{1/4}^{1/2} 4x\,dx + \int_{1/2}^{3/4} 4(1-x)dx$

$= 4\left(\frac{x^2}{2}\Big|_{1/4}^{1/2} + x - \frac{x^2}{2}\Big|_{1/2}^{3/4} \right)$

$= 4\left(\frac{1}{8} - \frac{1}{32} + \frac{3}{4} - \frac{9}{32} - \frac{1}{2} + \frac{1}{8} \right)$

$= 4\left(\frac{4}{32} - \frac{1}{32} + \frac{24}{32} - \frac{9}{32} - \frac{16}{32} + \frac{4}{32} \right)$

$= \frac{4 \cdot 6}{32} = \frac{6}{8}$

2) Pela definição de esperança teremos

$$E(X) = \int_{-\infty}^{+\infty} f(x)\,dx = 0 + \int_0^{\pi/2} x\,\text{sen}(x)\,dx$$

Por partes
$u = x \quad dv = \text{sen}(x)$

$$\overset{*}{=} \Big[-x\cos(x)\Big]_0^{\pi/2} - \int_0^{\pi/2} -\cos(x)$$

$$= 0 - \Big[-\text{sen}(x)\Big]_0^{\pi/2} = 1$$

Calculando a Var(X), teremos

$*$ Por partes $u = x^2 \quad dv = \text{sen}(x)$

$$E(X^2) = \int_0^{\pi/2} x^2\,\text{sen}(x)\,dx \overset{*}{=} \Big[-x^2\cos(x)\Big]_0^{\pi/2} - \int_0^{\pi/2} -2x\cos(x)\,dx$$

$*$ Por partes $u = x \quad dv = \cos(x)$

$$= \int_0^{\pi/2} 2x\cos(x)\,dx \overset{*}{=} \Big[2x\,\text{sen}(x)\Big]_0^{\pi/2} - \int_0^{\pi/2} 2\,\text{sen}(x)$$

$$= \pi - \Big[2\cos(x)\Big]_0^{\pi/2} = \pi - 2$$

Então teremos:

$$\text{Var}(X) = E(X^2) - E(X) \Rightarrow \pi - 2 - 1^2 = \pi - 3$$

3) a) $P(X=1) = 1/4 \qquad P(X=2) = 1/6 \qquad P(X=3)\ 1/12$

b) $P(1/2 < X < 3/2) = F(3/2) - F(1/2)$

$= \frac{1}{2} + \frac{1}{8} - 1/8 = 1/2$

4) $X \sim N(10, 4)$

a) $P(8 \leq X \leq 10) = P(-1 \leq Z \leq 0) = 0{,}34$

b) $P(9 \leq X \leq 12) = P(-1/2 \leq Z \leq 1) = 0{,}19 + 0{,}34 = 0{,}53$

c) $P(X > 10) = P(Z > 0) = 0{,}5$

d) $P(X < 8 \vee X > 11) = 1 - P(8 \leq X \leq 11)$

$= 1 - P(-1 \leq Z \leq 1/2)$

$= 1 - P(-1/2 \leq Z \leq 1) = 1 - 0{,}53 = 0{,}47$

5) $\frac{X - 12}{2} \sim N(0,1)$

$P(X > c) = P\left(\frac{X-12}{2} > \frac{c-12}{2}\right) = 0{,}10$

$\frac{c-12}{2} \overset{\text{tabela}}{=} 1{,}28 \Rightarrow c = 14{,}56$

6) "a)" $Y = X^2$

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) \Rightarrow P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$F_X(x) = \int_{-\infty}^{+\infty} f(t)\,dt = \int_{-1}^{x} \frac{1}{2}\,dt = \frac{t}{2}\Big|_{-1}^{x} = \frac{x+1}{2}\, I_{[-1,1]}(x)$$

$$F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{\sqrt{y}+1}{2} - \frac{(-\sqrt{y}+1)}{2} = \sqrt{y}\, I_{[0,1]}(y)$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d\sqrt{y}}{dy} = \frac{1}{2\sqrt{y}}\, I_{[0,1]}(y)$$

Para verificar que é fdp, faremos

$$\int_{-\infty}^{+\infty} f(y)\,dy = \int_0^1 \frac{1}{2\sqrt{y}}\,dy = \sqrt{y}\Big|_0^1 = 1$$

Logo $f_Y(y)$ é f.d.p, então $Y \sim f_Y(y) = \frac{1}{2\sqrt{y}} I_{[0,1]}(w)$

"b)" $W = |X|$

$$F_W(w) = P(W < w) = P(|X| < w) = P(-w < X < w)$$

$$= F_X(w) - F_X(-w)$$

$$= \frac{w+1}{2} - \left(\frac{-w+1}{2}\right) = w\, I_{[0,1]}(w)$$

Então calculando a f.d.p. teremos:

$$f_W(w) = \frac{dF_W(w)}{dw} = \frac{dw}{dw} I(w)_{[0,1]} = 1\, I(w)_{[0,1]}$$

$$W = |X| \sim U(0,1)$$

7) Temos que:

$$\begin{cases} \int_0^1 f(x)\,dx = 1 \Rightarrow \int_0^1 (a + bx^2)\,dx = ax + \frac{bx^3}{3}\Big|_0^1 = 1 \\ \int_0^1 x f(x)\,dx = \frac{3}{5} \Rightarrow \int_0^1 (ax + bx^3)\,dx = \frac{ax}{2} + \frac{bx^4}{4}\Big|_0^1 = 3/5 \end{cases}$$

$$\begin{cases} a + \frac{b}{3} = 1 \Rightarrow a = 1 - b/3 \Rightarrow a = 3/5 \,// \\ \frac{a}{2} + \frac{b}{4} = 3/5 \Rightarrow \frac{1}{2} - \frac{b}{6} + \frac{b}{4} = \frac{3}{5} \Rightarrow \frac{1}{10} = \frac{b}{12} \Rightarrow b = \frac{6}{5} \,// \end{cases}$$

8) $Y = cX$

$$P(Y \leq y) = P(cX \leq y) = P\left(X < \frac{y}{c}\right) = F_X\left(\frac{y}{c}\right)$$

Então, como $X \sim Exp(\lambda)$, teremos:

$$F_Y(y) = F_X\left(\frac{y}{c}\right) = \int_0^{y/c} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_0^{y/c} = 1 - e^{-\lambda y/c}$$

$$F_Y(y) = 1 - e^{-\lambda y/c} \Rightarrow f_Y(y) = \frac{d(1 - e^{-\lambda \frac{y}{c}})}{dy} = \frac{\lambda}{c} e^{-\frac{\lambda}{c} y}$$

$$\therefore Y \sim Exp\left(\frac{\lambda}{c}\right)$$

9) No final

10) a) $P(Y=3) = P(X \leq 1) = \frac{1}{5}$

$$P(Y=8) = P(X \leq 4) - P(X \leq 1) = \frac{4}{5} - \frac{1}{5} = \frac{3}{5}$$

$$P(Y=10) = P(X > 4) = 1/5$$

b) $E(Y) = 3.P(Y=3) + 8.P(Y=8) + 10.P(Y=10)$

$$3.\frac{1}{5} + 8.\frac{3}{5} + 10.\frac{1}{5}$$

$$= \frac{3 + 24 + 10}{5} = 37/5 = 7{,}4$$

c)

$$P(Y=8 \mid X > 5/2) = \frac{P(Y=8 \cap X > 5/2)}{P(X > 5/2)}$$

$$= \frac{P(1 < X < 4 \cap X > 5/2)}{P(X > 5/2)}$$

$$= \frac{P(5/2 < X < 4)\,*}{P(X > 5/2)\,*}$$

$*$ $P(5/2 < X < 4) = P(X < 4) - P(X < 5/2)$
$\frac{4}{5} - \frac{5/2}{5} = 0{,}3$

$*$ $P(X > \frac{5}{2}) = 1 - P(X < \frac{5}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$

Então teremos:

$$P(Y=8 \mid X > 5/2) = \frac{P(5/2 < X < 4)}{P(X > 5/2)} = \frac{\frac{3}{10}}{\frac{1}{2}} = \frac{3}{5}$$

(ii) a) Para f.d.p ser bem definida, temos que obter

$$\int_0^{\infty} f(x)dx = 1$$

Então teremos:

$$\int_0^{\infty} f(x)dx = \int_0^{\ln(7)/3} c \, dx + \int_{\ln(7)/3}^{+\infty} 3e^{-3x} dx = 1$$

$$= \left[Cx\right]_0^{\ln(7)/3} + \left[-e^{-3x}\right]_{\ln(7)/3}^{+\infty} = 1$$

$$= c\frac{\ln(7)}{3} + e^{-3 \cdot \ln(7)/3} = 1$$

$$= c\frac{\ln(7)}{3} + \frac{1}{7} = 1 \Rightarrow C = \frac{18}{7\ln(7)}$$

b)

Da primitiva calculada em a), temos

$$F_x(x) = \begin{cases} \frac{18}{7\ln(7)} x & \text{se} \quad 0 < x \leq \ln(7)/3 \\ \frac{6}{7} + e^{-3 \cdot \ln(7)/3} - e^{-3x} & \text{se } x > \ln(7)/3 \end{cases}$$

$$F_x(x) = \begin{cases} \frac{18}{7\ln(7)} x & \text{se} \quad 0 < x \leq \ln(7)/3 \\ 1 - e^{-3x} & \text{se } x > \ln(7)/3 \end{cases}$$

(2) $p = 1/3000$ $n = 18000$ (>30) $np = 6$ (<10)

X: numero de celulas cancerígenas

$X \sim Poi(6)$

a) $P(X \geq 2) = 1 - P(X=0) - P(X=1)$

$$= 1 - \frac{e^{-6} 6^0}{0!} - \frac{e^{-6} \cdot 6^1}{1!} \qquad 1 - 7e^{-6}$$

b) Independencia entre as células serem cancerígenas ou não.

a) a) Como com $t > 0$, temos $e^{-t/2} > 0$ e $\frac{t}{4} > 0$, então $f(t) > 0, \forall t > 0$

Calculando a integral teremos

$$\int_{-\infty}^{\infty} f(t)dt = \int_0^{\infty} f(t)dt = \int_0^{\infty} \frac{1}{4} t e^{-t/2} = \frac{1}{4}\int_0^{\infty} t e^{-t/2} =$$

$$(*) = \frac{1}{4}\left[-2t \exp\left\{-\frac{1}{2}t\right\}\Big|_0^{\infty} - \int_0^{\infty} -2\exp\left\{-\frac{1}{2}t\right\}dt\right]$$

$$= \frac{1}{4}\left[0 + 2\left(-2\exp\left\{-\frac{1}{2}t\right\}\Big|_0^{\infty}\right)\right] = \frac{1}{2} \cdot (0 - (-2)) = 1$$

(*) integração por partes

b) $F_t(t) = \int_0^t f(x)dx = \int_0^t \frac{1}{4} x \exp\left\{\frac{-1x}{2}\right\} dx$

$$\overset{(*)}{=} \frac{1}{4}\left[-2x \exp\left\{-\frac{x}{2}\right\}\Big|_0^t - \int_0^t -2\exp\left\{-\frac{x}{2}\right\}dx\right]$$

$$= \frac{1}{4}\left[-2t\exp\left\{\frac{-t}{2}\right\} + 2\left[-2\exp\left\{-\frac{x}{2}\right\}\Big|_0^t\right]\right]$$

$$= \frac{1}{4}\left[-2t\exp\left\{-\frac{t}{2}\right\} + 2\left[-2\exp\{t/2\} + 2\right]\right]$$

$$= \frac{1}{4}\left[-2t\exp\left\{-\frac{t}{2}\right\} + \left(-4\exp\left\{-\frac{t}{2}\right\} + 4\right)\right]$$

$$= \frac{1}{4}\left[4 - 2(2+t)\exp\left\{\frac{-t}{2}\right\}\right] = 1 - \frac{(2+t)e^{-t/2}}{2} I(t \geq 0)$$

c) $E(T) = \int_{-\infty}^{+\infty} t f(t) dt = 0 + \int_0^{+\infty} t f(t) dt$

$$= \int_0^{+\infty} \frac{t^2}{4} e^{-\frac{t}{2}} dt$$

$$\overset{(*)}{=} \frac{1}{4}\left[-2t^2\exp\left\{-\frac{t}{2}\right\}\Big|_0^{\infty} - \int_0^{+\infty} -4t\exp\{-t/2\}\,dt\right.$$

$$= \frac{1}{4}\left[0 + 4\int_0^{\infty} t\exp^{-\frac{t}{2}}\,dt\right] = 4\cdot\int \frac{t}{4}\exp^{-t/2} = 4$$

$$E(T^2) = \int_0^{+\infty} t^2 f(t)\,dt = \int_0^{+\infty} t^2\,\frac{t}{4}\,e^{-t/2}\,dt = \int_0^{+\infty}\frac{t^3}{4}\,e^{-t/2}\,dt$$

$$\overset{(*)}{=} \frac{1}{4}\left(-2t^3\exp\left\{-\frac{t}{2}\right\}\Big|_0^{\infty} - \int_0^{\infty} -6t^2\exp\left\{-\frac{t}{2}\right\}dt\right)$$

$$\left(0 + 6\int_0^{+\infty}\frac{t^3}{4}\exp\{-t/2\}\,dt\right) = 6\cdot 4 = 24$$

Portanto teremos que

$$Var(T) = E(T^2) - E(T)^2 = 24 - 16 = 8$$

Então teremos:

$$\begin{cases} E(T) = 4 \text{ minutos} \\ DP(T) = 2\sqrt{2} \text{ minutos.} \end{cases}$$