一. 解: 2014年

Smith标准形:
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda(\lambda-1) & 0 & 0 & 0 \\ 0 & 0 & \lambda^2(\lambda-1)^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 5}$$

各阶行列式因子:
$D_1(\lambda)=1$
$D_2(\lambda)=\lambda(\lambda-1)$
$D_3(\lambda)=\lambda^3(\lambda-1)^4$

二. 解:
$$A=\begin{bmatrix} 1 & 3i & 0 \\ -3i & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad |\lambda E-A|=\begin{vmatrix} \lambda-1 & -3i & 0 \\ 3i & \lambda-1 & 0 \\ 0 & 0 & \lambda-4 \end{vmatrix}=(\lambda-4)^2(\lambda+2)$$
$\lambda_1=\lambda_2=4 \quad \lambda_2=-2$

当 $\lambda=4$ 时
$$4E-A=\begin{bmatrix} 3 & -3i & 0 \\ 3i & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\simeq\begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$\alpha_1=(i,1,0)^T$, $\alpha_2=(0,0,1)^T \Rightarrow V_1=(\frac{i}{\sqrt2},\frac{1}{\sqrt2},0)^T$, $V_2=(0,0,1)^T$

当 $\lambda=-2$ 时
$$-2E-A=\begin{bmatrix} -3 & -3i & 0 \\ 3i & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix}\simeq\begin{bmatrix} i & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$\alpha_3=(1,i,0)^T \Rightarrow V_3=(\frac{1}{\sqrt2},\frac{i}{\sqrt2},0)^T$

$$\therefore U=\begin{bmatrix} \frac{i}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & 0 & \frac{i}{\sqrt2} \\ 0 & 1 & 0 \end{bmatrix} \quad U^HAU=\begin{bmatrix} 4 & & \\ & 4 & \\ & & -2 \end{bmatrix}$$

令 $X=Uy$, 其中 $y=(y_1,y_2,y_3)$. 代入二次型得:
$f(x_1,x_2,x_3)=X^HAX=y^H(U^HAU)y=4\bar{y}_1y_1+4\bar{y}_2y_2-2\bar{y}_3y_3$.

三. 解: $|\lambda I-AB|=0$ $\because$ A正定H阵 $\therefore\exists$可逆矩阵Q: $A=Q^HQ$
$\therefore 0=|\lambda I-AB|=|\lambda I-Q^HQB|=|\lambda Q^H(Q^H)^{-1}-Q^HQBQ^H(Q^H)^{-1}|$
$=|Q^H||\lambda I-QBQ^H||(Q^H)^{-1}|=|\lambda I-QBQ^H|$
$\therefore\lambda$也是矩阵 QBQ^H 的特征值.
~~同理~~ 又$\because QBQ^H$ 为H阵 $\therefore\lambda$为实数 同理证BA的特征值为实数.

四. 解:
$$A=\begin{bmatrix} 2 & -i \\ 0 & 0 \\ -2 & i \end{bmatrix} \quad A^H=\begin{bmatrix} 2 & 0 & -2 \\ i & 0 & -i \end{bmatrix} \quad AA^H=\begin{bmatrix} 5 & 0 & -5 \\ 0 & 0 & 0 \\ -5 & 0 & 5 \end{bmatrix} \quad |\lambda E-AA^H|=\lambda^2(\lambda-10)$$
$\lambda_1=10$. $\lambda_2=\lambda_3=0$ $\therefore$ 正奇异值: $\sigma=\sqrt{10}$: $\Delta=\sqrt{10}$

当 $\lambda_1=10$ 时:
$$10E-AA^H=\begin{bmatrix} 5 & 0 & 5 \\ 0 & 0 & 0 \\ 5 & 0 & 5 \end{bmatrix}\simeq\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$\alpha_1=(-1,0,1)^T \to \mu_1=(\frac{-1}{\sqrt2},0,\frac{1}{\sqrt2})^T \quad U_1=\mu_1$

$$V_1=A^HU_1\Delta^{-1}=\begin{bmatrix} 2 & 0 & -2 \\ i & 0 & -i \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt2} \\ 0 \\ \frac{1}{\sqrt2} \end{bmatrix}\frac{1}{\sqrt{10}}=\begin{bmatrix} \frac{-2}{\sqrt5} \\ \frac{-i}{\sqrt5} \end{bmatrix}$$

~~当 $\lambda_2=0$ 时~~

~~$-AA^H\begin{bmatrix} -5 & 0 & 5 \\ 0 & 0 & 0 \\ 5 & 0 & -5 \end{bmatrix}\simeq\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\alpha_2=(0,1,0)^T$~~

$\alpha_1\bar{\alpha}_2=\bar{\alpha}_2\alpha_1=0$

$V_2=(\frac{-i}{\sqrt5},\frac{2}{\sqrt5})^T$

$$\therefore V=\begin{bmatrix} -\frac{2}{\sqrt5} & \frac{-i}{\sqrt5} \\ -\frac{i}{\sqrt5} & \frac{2}{\sqrt5} \end{bmatrix}$$

~~$A=UDV^H=$~~

当$\lambda_2=\lambda_3=0$时

$$-AA^H=\begin{bmatrix}-5&0&5\\0&0&0\\5&0&-5\end{bmatrix}\simeq\begin{bmatrix}-1&0&1\\0&0&0\\0&0&0\end{bmatrix}\quad \alpha_2=(1,0,1)^T,\ \alpha_3=(0,1,0)^T \Rightarrow \mu_2=(\tfrac{1}{\sqrt2},0,\tfrac{1}{\sqrt2})^T,\ \mu_3=(0,1,0)^T.$$

$$A=UDV^H=\begin{bmatrix}-\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&0\\0&0&1\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&0\end{bmatrix}\begin{bmatrix}\sqrt{10}&0\\0&0\\0&0\end{bmatrix}\begin{bmatrix}-\frac{2}{\sqrt5}&\frac{1}{\sqrt5}\\-\frac{1}{\sqrt5}&\frac{2}{\sqrt5}\end{bmatrix}.$$

$$A=\begin{bmatrix}1&-\frac12\\0&0\\0&0\end{bmatrix}\quad B=\begin{bmatrix}2\\0\\-2\end{bmatrix}\quad C=\begin{bmatrix}1&-\frac12\end{bmatrix}$$

$A=BC$

$$A^+=C^H(CC^H)^{-1}(B^HB)^{-1}B^H=\begin{bmatrix}-\frac{1}{\sqrt{10}}&0&\frac{1}{\sqrt{10}}\\\frac{1}{\sqrt{10}}&0&-\frac{1}{\sqrt{10}}\end{bmatrix}$$

五、证明:

非负性: 当$\|A\|\neq0$时, $|a_{ij}|>0$ $\therefore \|A\|>0$

当$\|A\|=0$时, $|a_{ij}|=0$ $\therefore \|A\|=0$

齐次性. $\|kA\|=\sqrt{mn}\max\limits_{i,j}|ka_{ij}|=|k|\sqrt{mn}\max\limits_{i,j}|a_{ij}|=|k|\|A\|$

三角不等式: 设$A\in C^{m\times n}$, $B\in C^{m\times n}$

$\|A+B\|=\sqrt{mn}\max\limits_{i,j}|a_{ij}+b_{ij}|\le\sqrt{mn}\max\limits_{i,j}|a_{ij}|+\sqrt{mn}\max\limits_{i,j}|b_{ij}|=\|A\|+\|B\|$

矩阵乘法相容性: 设$A\in C^{m\times n}$, $B\in C^{n\times l}$

$\|AB\|=\sqrt{ml}\max\limits_{i,j}\left|\sum\limits_{k=1}^{n}a_{ik}b_{kj}\right|\le\sqrt{ml}\max\limits_{i,j}\sum\limits_{k=1}^{n}|a_{ik}||b_{kj}|\le\sqrt{ml}\cdot n\max\limits_{i,k}|a_{ik}|\max\limits_{k,j}|b_{kj}|$

$=\sqrt{mn}\cdot\max\limits_{i,k}|a_{ik}|\cdot\sqrt{nl}\max\limits_{k,j}|b_{kj}|=\|A\|\|B\|$

六、解: $|\lambda E-A|=\begin{vmatrix}\lambda&-1&-i\\1&\lambda&0\\-i&0&\lambda\end{vmatrix}=\lambda(\lambda^2+1)\quad \lambda_1=0\quad\lambda_2=i\quad\lambda_3=-i$

① $\lambda_1=0$时

$-A=\begin{bmatrix}0&-1&-i\\1&0&0\\-i&0&0\end{bmatrix}\simeq\begin{bmatrix}0&1&i\\1&0&0\\0&0&0\end{bmatrix}$ ~~$\alpha_1=(0,-i,1)^T$~~ $\alpha_1=(0,\frac{-i}{\sqrt2},\frac{1}{\sqrt2})^T$

② $\lambda_2=i$时?

$iE-A=\begin{bmatrix}i&-1&-i\\1&i&0\\-i&0&i\end{bmatrix}\simeq\begin{bmatrix}1&1&0\\i&-1&-1\\-i&0&i\end{bmatrix}\simeq\begin{bmatrix}1&i&0\\0&-1&0\\-i&0&i\end{bmatrix}\simeq\begin{bmatrix}1&i&0\\0&1&2i\\0&&\end{bmatrix}\simeq\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$\alpha_2=($

③ $\lambda_3=-i$时

$-iE-A=\begin{bmatrix}-i&-1&-i\\1&-i&0\\-i&0&-i\end{bmatrix}\simeq\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix}$ ~~$\alpha_3=(-1,0,1)^T$~~ $\alpha_3=(\frac{-1}{\sqrt2},0,\frac{1}{\sqrt2})^T$

$\therefore$

∴解: $|\lambda E-A| = \begin{vmatrix} \lambda & -1 & -i \\ 1 & \lambda & 0 \\ -i & 0 & \lambda \end{vmatrix} = \cancel{\lambda^3+\lambda} = \lambda(\lambda^2+2)$

$\lambda_1=0$ $\lambda_2=\sqrt{2}i$ $\lambda_3=-\sqrt{2}i$ $\alpha_1^H=(0,\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}})$

① $\lambda_1=0$时 $\alpha_1=(0,\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}})^T$

② $\lambda_2=\sqrt{2}i$时. $\alpha_2=(\frac{-i}{\sqrt{2}},\frac{1}{2},-\frac{i}{2})^T$ $\alpha_2^H=(\frac{i}{\sqrt{2}},\frac{1}{2},\frac{i}{2})$

③ $\lambda_3=-\sqrt{2}i$时, $\alpha_3=(\frac{i}{\sqrt{2}},\frac{1}{2},\frac{-i}{2})^T$ $\alpha_3^H=(\frac{-i}{\sqrt{2}},\frac{1}{2},\frac{i}{2})$

$G_1=\alpha_1\alpha_1^H=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ $G_2=\alpha_2\alpha_2^H=\begin{bmatrix} \frac{1}{2} & \frac{-i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & \frac{-i}{4} & \frac{1}{4} \end{bmatrix}$ $G_3=\alpha_3\alpha_3^H=\begin{bmatrix} \frac{1}{2} & \frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{-i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & \frac{-i}{4} & \frac{1}{4} \end{bmatrix}$

$A=\sqrt{2}iG_2-\sqrt{2}iG_3$

七. 解: (1). $|\lambda E-A|=(\lambda-3)^4$ 最小多项式: $(\lambda-3)^3$ $m=3$ $J=\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ & & & 3 \end{bmatrix}$.

(2) $p(x)=a_0+a_1x+a_2x^2$ $p'(x)=a_1+2a_2x$ $p''(x)=2a_2$

$$\begin{cases} f(3)=p(3)=a_0+3a_1+9a_2 \\ f'(3)=p'(3)=a_1+6a_2 \\ f''(3)=p''(3)=2a_2. \end{cases} \Rightarrow \begin{cases} a_0=f(3)-3f'(3)+\frac{9}{2}f''(3) \\ a_1=f'(3)-3f''(3) \\ a_2=\frac{1}{2}f''(3) \end{cases} \Rightarrow f(A)=a_0E+a_1A+a_2A^2$$

① 当 $f(A)=e^{tA}$ $f'(A)=te^{tA}$ $f''(A)=t^2e^{tA}$

$f(3)=e^{3t}$ $f'(3)=te^{3t}$ $f''(A)=t^2e^{3t}$

$$f(A)=\begin{bmatrix} f(3) & & & \\ f(3)-\frac{5}{2}f'(3) & f(3) & & \\ & f(3)-\frac{3}{2}f'(3) & f(3) & \\ & & & f(3) \end{bmatrix} = \begin{bmatrix} e^{3t} & & & \\ (t-\frac{5}{2}t)e^{3t} & e^{3t} & & \\ & (t-\frac{3}{2}t)e^{3t} & e^{3t} & \\ & & & e^{3t} \end{bmatrix}$$

② 当 $f(A)=\cos\frac{\pi}{2}A$ $f'(A)=-\frac{\pi}{2}\sin\frac{\pi}{2}A$ $f''(A)=-\frac{\pi^2}{4}\cos\frac{\pi}{2}A$

$f(3)=\cancel{\cos\frac{3\pi}{2}}0$ $f'(3)=\frac{\pi}{2}$ $f''(3)=0$

$$\cancel{f(\cos\frac{\pi}{2}A)}=\cos\frac{\pi}{2}A=\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\pi}{2} & 0 & 0 & 0 \\ 0 & \frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

八 解: 证明:

$\cos(2\pi E+A)=\cos 2\pi E\cdot\cos A+\sin 2\pi E\cdot\sin A=E\cdot\cos A=\cos A$

$\because E=\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{n\times n}$ $2\pi E=\begin{bmatrix} 2\pi & & \\ & \ddots & \\ & & 2\pi \end{bmatrix}$ $\cos 2\pi E=\begin{bmatrix} \cos 2\pi & & \\ & \ddots & \\ & & \cos 2\pi \end{bmatrix}=\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

同理: $\sin 2\pi E=\begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}$

1.题解:

$$\frac{dA(t)}{dt}=\begin{bmatrix} 2e^{2t} & 0 & 0 \\ 0 & 0 & 2t \\ 0 & 2t & 0 \end{bmatrix}. \qquad \frac{d^2A(t)}{dt^2}=\begin{bmatrix} 4e^{2t} & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\int_0^{x^2} A(t)\,dt=\begin{bmatrix} \frac{1}{2}e^{2x^2}-\frac{1}{2} & 0 & 0 \\ 0 & 3x^2 & \frac{1}{3}x^6 \\ 0 & \frac{1}{3}x^6 & 3x^2 \end{bmatrix}.$$

$$\therefore \frac{d}{dx}\left(\int_0^{x^2} A(t)\,dt\right)=\begin{bmatrix} \frac{e^{2x^2}}{2x} & 0 & 0 \\ 0 & 6x & 2x^5 \\ 0 & 2x^5 & 6x \end{bmatrix}$$