# 一、填空题

1. $\underline{n}$ . $\underline{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}}$ . $\underline{\mathbb{R}[x]_3 \text{ 或 } \mathrm{span}\{1, x, x^2\}}$

~~两种写法都对.~~

2. 不变因子为 $\underline{d_1(\lambda)=1,\ d_2(\lambda)=1,}$ $\underline{d_3(\lambda)=(\lambda-4)(\lambda^2-9)}$

不写 $d_1(\lambda)$, $d_2(\lambda)$ $d_3(\lambda)$ 也可以

初等因子为 $\underline{\lambda-4,\ \lambda+3,\ \lambda-3}$

3. $\underline{5}$ , $\underline{4+\sqrt{2}}$ $\underline{e^{24}}$

4. $\underline{\begin{bmatrix} 2 & e^{-t} \\ -\sin t & 0 \end{bmatrix}}$ $\underline{\begin{bmatrix} x^2 & e^{-x} \\ \sin x & x \end{bmatrix}}$

二.

$$A=\begin{bmatrix} 2 & 2 & 1 \\ -2 & 6 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

A 的 Jordan 标准形为 $\underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \text{或} \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{\text{二者均可以}}$

因此 A 的最小多项式为 $(\lambda-4)^2$

$$A=PJ_AP^{-1}$$

$$f(A)=Pf(J)P^{-1}=\begin{bmatrix} f(4)-2f'(4) & 2f'(4) & f'(4) \\ -2f'(4) & f(4)+2f'(4) & f'(4) \\ 0 & 0 & f(4) \end{bmatrix}$$

不管用何种方法 只要算出 $f(A)$ 具有这种形式即可.

$$\cos\frac{\pi}{2}A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=I_{3\times3}=I_3.$$

$$e^{tA}=e^{4t}\begin{bmatrix} 1-2t & 2t & t \\ -2t & 1+2t & t \\ 0 & 0 & 1 \end{bmatrix}$$

三.

$$AA^H = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ $AA^H$ 特征值为

$\lambda_1 = \lambda_2 = 9$. $\lambda_3 = 0$

A 的奇异值为 $\delta_1 = \delta_2 = 3$. $\Delta = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

$(9I - AA^H)X = 0 \rightsquigarrow \eta_1 = [1.\ 0.\ 0]^T$

$\eta_2 = [0.\ 1.\ 0]^T$

$(0I - AA^H)X = 0 \rightsquigarrow \eta_3 = [0.\ 0.\ 1]^T$

$U = [\eta_1 . \eta_2 . \eta_3]$ . $\underset{\overset{\|}{U^{-1}AU}}{U^H A U} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$

$U_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $V_1 = A^H U_1 \Delta^{-H}$

$V_1 = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} = V$ $D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$

$A = UDV^H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

四.

A为正规矩阵.

$|\lambda I-A|=0$. $(\lambda^2+4)\lambda=0$ $\lambda_1=0$. $\lambda_2=2i$

$\lambda_3=-2i$.

$\lambda_1=0 \rightsquigarrow (0I-A)X=0 \rightsquigarrow \eta_1=[0,0,1]^T$

$\lambda_2=2i \rightsquigarrow (2iI-A)X=0 \rightsquigarrow \eta_2=[\frac{1}{\sqrt{2}},\frac{i}{\sqrt{2}},0]^T$

$\lambda_3=-2i \rightsquigarrow (-2iI-A)X=0 \rightsquigarrow \eta_3=[1/\sqrt{2},-i/\sqrt{2},0]^T$

($\eta_1$. $\eta_2$. $\eta_3$ 为单位特征向量)

只要三者都是单位特征向量即可. $\eta_1$. $\eta_2$. $\eta_3$

表达方式不唯一.

$$G_1=\eta_1\eta_1^H=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$G_2=\eta_2\eta_2^H=\begin{bmatrix}1/2&-i/2&0\\i/2&1/2&0\\0&0&0\end{bmatrix}$$

$A=0\cdot G_1+2iG_2+(-2i)G_3$

这里带级含有三部分.

缺一不可.

$$G_3=\eta_3\eta_3^H=\begin{bmatrix}1/2&i/2&0\\-i/2&1/2&0\\0&0&0\end{bmatrix}$$

五.

$A^H = A$ 为 Hermite 矩阵.

$|\lambda I - A| = 0$ . $\lambda_1 = 1$ . $\lambda_2 = 2$ $\lambda_3 = 5$

$\lambda_1 = 1$ . $(I-A)X = \theta \rightsquigarrow \eta_1 = [1 . 0 . 0]^T$.

$\lambda_2 = 2$ . $(2I-A)X = \theta \rightsquigarrow \eta_2 = [0, \frac{i-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]^T$

$\lambda_3 = 5$ $(5I-A)X = \theta \rightsquigarrow \eta_3 = [0 . \frac{1}{\sqrt{3}}, \frac{1+i}{\sqrt{3}}]^T$

只要让 $\eta_1$. $\eta_2$. $\eta_3$ 长度为1彼此正交即可.

$$U = [\eta_1 . \eta_2 . \eta_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{i-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \end{bmatrix}$$

$$U^H A U = U^{-1} A U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$ $X = UY$ 代入

$f(x) = X^H A X$. 可得 $f(x) = \bar{y}_1 y_1 + 2\bar{y}_2 y_2 + 5\bar{y}_3 y_3$

所有特征值都大于零. 所以 $f(x)$ 为正定二次型.

六.

$$\|A\|_2^2 = \max_{X\neq\theta} \frac{\|AX\|_2^2}{\|X\|_2^2} = \max_{X\neq\theta} \frac{X^H A^H A X}{X^H X} = \rho(A^H A)$$

$$= \rho^2(A)$$ . $\rho(A)$为A的谱半径.

$$\|A\|_2 = \rho(A)$$

或另一种证法:

A为正规矩阵. 存在酉矩阵U使得

$$U^H A U = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$A = U\Lambda U^H \qquad A^H = U\Lambda^H U^H$$

$$AA^H = U\Lambda\Lambda^H U^H = U \begin{bmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_n|^2 \end{bmatrix} U^H$$

$$\|A\|_2 = \max_{j=1}^{n} (\lambda_j(AA^H))^{1/2} = \underbrace{|\lambda_j|}_{\text{最大者}} = \rho(A).$$

$$\|A\|_F^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2 = Tr(AA^H)$$

$$= \sum_{i=1}^{m} \lambda_i(AA^H)$$

$\lambda_i(AA^H)$ 表示 $AA^H$ 的任意一个特征值

$\rho(AA^H)$ 为所有特征值模长最大者.

并且 $AA^H$ 所有特征值均大于或等于零.

故 $\rho(AA^H) \le Tr(AA^H) = \|A\|_F^2$.

t.

$$\sum_{k=1}^{\infty} \frac{k}{10^k} x^k$$ 收敛半径为 10.

$|\lambda I - A| = 0$. $\lambda_1 = 4$. $\lambda_2 = -2$

$\rho(A) = 4 < 10$. $\sum \frac{k}{10^k} A^k$ 收敛(且绝对收敛)

$$\sum \frac{k}{10^k} A^k = \sum_{k=1}^{\infty} k B^k, \quad B = A/10$$

$$\sum_{k=1}^{\infty} k x^k = x(1-x)^{-2}$$.

$$\sum_{k=1}^{\infty} kB^k = B(I-B)^{-2} \qquad B = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 1/10 \end{bmatrix}$$

$$I-B = \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}.$$

$$\sum_{k=1}^{\infty} kB^k = B(I-B)^{-2} = \begin{bmatrix} 35/72 & 5/8 \\ 5/8 & 35/72 \end{bmatrix}.$$

11. A为复数域上矩阵. 存在可逆矩阵P.使得

$$P^{-1}AP = J_A$$

$$J_A = \begin{bmatrix} J_{k_1}(\lambda_1) & & & \\ & J_{k_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{k_s}(\lambda_s) \end{bmatrix}$$

$$J_{k_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \ddots & 1 \\ & & & & \lambda_i \end{bmatrix}_{k_i \times k_i}$$

$$J_{k_i}(\lambda_i) = \lambda_i I_{k_i} + \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots & 1 \\ & & & & 0 \end{bmatrix}_{k_i \times k_i}$$

$$= \lambda_i I_{k_i} + N_{k_i}$$

$$J_A = \begin{bmatrix} \lambda_1 I_{k_1} & & & \\ & \lambda_2 I_{k_2} & & \\ & & \ddots & \\ & & & \lambda_s I_{k_s} \end{bmatrix} + \begin{bmatrix} N_{k_1} & & & \\ & N_{k_2} & & \\ & & \ddots & \\ & & & N_{k_s} \end{bmatrix}$$

$$A = P J_A P^{-1}$$

$$= P \begin{bmatrix} \lambda_1 I_{k_1} & & & \\ & \lambda_2 I_{k_2} & & \\ & & \ddots & \\ & & & \lambda_s I_{k_s} \end{bmatrix} P^{-1} + P \begin{bmatrix} N_{k_1} & & & \\ & N_{k_2} & & \\ & & \ddots & \\ & & & N_{k_s} \end{bmatrix} P^{-1}$$

$$= B + C$$

显然 $B$ 为可对角化矩阵. $C$ 为幂零矩阵. 且

$$BC = CB.$$