2009年

一. $(\beta_1\beta_2\beta_3)=(\alpha_1\alpha_2\alpha_3)P$

得 $P=(\alpha_1\alpha_2\alpha_3)^{-1}(\beta_1\beta_2\beta_3)$

$$=\begin{bmatrix}0&0&1\\0&1&-1\\1&-1&0\end{bmatrix}\begin{bmatrix}3&0&0\\0&2&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&1\\0&2&-1\\3&-2&0\end{bmatrix}$$

简易算法: $(\alpha_1,\alpha_2,\alpha_3 \mid \beta_1,\beta_2,\beta_3) \xrightarrow{\text{行变换}} \left[\begin{array}{ccc|ccc}1&0&0&0&0&1\\0&1&0&0&2&-1\\0&0&1&3&-2&0\end{array}\right]$，右边即为 P

(1) $B=P^{-1}AP$

(2) 求解 $Ax=0$

$A\sim\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}$ $\quad \alpha=(1,-1,1)^T$

核为 $\alpha=\alpha_1-\alpha_2+\alpha_3=(1,0,1)^T$

f的值域

$R(f)=span\{f(\alpha_1),f(\alpha_2),f(\alpha_3)\}$

$=span\{\alpha_1+\alpha_2-\alpha_3,\alpha_2+2\alpha_3,-\alpha_1+3\alpha_3\}$

$=span\{\alpha_1+\alpha_2-\alpha_3,\alpha_2+2\alpha_3\}$

二. 解:

$$A=\begin{bmatrix}\frac{1}{2}&0&\frac{3}{2}i\\0&2&0\\-\frac{3}{2}i&0&\frac{1}{2}\end{bmatrix}$$

$|A-\lambda E|=(2-\lambda)(\lambda-2)(\lambda+1)$

$\lambda=2$时. $A-2E\sim\begin{bmatrix}1&0&-i\\0&0&0\\0&0&0\end{bmatrix}$ $\quad \alpha_1=(0,1,0)^T$, $\alpha_2=(i,0,1)^T \to u_2=(\frac{i}{\sqrt2},0,\frac{1}{\sqrt2})^T$

$\lambda=-1$时, $A+E\sim\begin{bmatrix}1&0&i\\0&1&0\\0&0&0\end{bmatrix}$ $\quad \alpha_3=(-i,0,1)^T \to u_3=(-\frac{i}{\sqrt2},0,\frac{1}{\sqrt2})^T$

$$U=(u_1\,u_2\,u_3)=\begin{bmatrix}0&\frac{i}{\sqrt2}&-\frac{i}{\sqrt2}\\1&0&0\\0&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}$$

令$X=UY$, $Y=(y_1,y_2,y_3)$ 则

$f(x)=X^HAX=Y^HU^HAUY=2\bar{y}_1y_1+2\bar{y}_2y_2-\bar{y}_3y_3$

三. 解:

共轭 $AA^H=\begin{bmatrix}2&0\\0&-i\\0&0\end{bmatrix}\begin{bmatrix}2&0&0\\0&+i&0\end{bmatrix}=\begin{bmatrix}4&0&0\\0&+1&0\\0&0&0\end{bmatrix}$

奇异值为$\sqrt{\lambda}=2,1$

当$\lambda=4$时, $AA^H-4E\sim\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$ 故 $u_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$

当$\lambda=1$时, $AA^H-E\sim\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$ 故 $u_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$

当$\lambda=0$时, $AA^H\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ 故 $u_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$

则 $U=(u_1,u_2,u_3)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $U_1=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$

令 $V_1=A^HU_1\Delta^{-1}=\begin{bmatrix}1&0\\0&-i\end{bmatrix}$

则 $V=\begin{bmatrix}1&0\\0&+i\end{bmatrix}$

此时, $A=U\begin{bmatrix}2&0\\0&1\\0&0\end{bmatrix}V^H$

四. ① $\|UA\|_2=\max\limits_j(\lambda_i[(UA)^HUA])^{\frac12}$
$=\max\limits_j(\lambda_i[A^HU^HUA])^{\frac12}$
$=\max\limits_j(\lambda_i[A^HA])^{\frac12}=\|A\|_2$

② $\|AV\|_2=\max\limits_j(\lambda_i[(AV)^HAV])^{\frac12}$
$=\max\limits_j(\lambda_i[V^HA^HAV])^{\frac12}$
$=\max\limits_j(\lambda_i[A^HA])^{\frac12}=\|A\|_2$

③ $(UAV)^H(UAV)=V^HA^HU^HUAV=V^HA^HAV$

$\therefore \|UAV\|_2=\max\limits_j(\lambda_i[V^HA^HAV])^{\frac12}$
$=\max\limits_j(\lambda_i[A^HA])^{\frac12}=\|A\|_2$

从而 $\|UA\|_2=\|AV\|_2=\|UAV\|_2=\|A\|_2$

五.

$\rho(A)=10-\sqrt{10}$

$\sum\limits_{k=0}^{\infty}\frac{k+1}{10^{k+1}}x^k=\left[\sum\limits_{k=0}^{\infty}\left(\frac{x}{10}\right)^{k+1}\right]'$

$=\left(\frac{\frac{x}{10}}{1-\frac{x}{10}}\right)'$, $\left|\frac{x}{10}\right|<1$

$=\frac{10}{(10-x)^2}$, $|x|<10$

(1) $\rho(A)<R$ 故 $\sum\limits_{k=0}^{\infty}\frac{k+1}{10^{k+1}}A^k$ 绝对收敛.

(2) $\sum\limits_{k=0}^{\infty}\frac{k+1}{10^{k+1}}A^k=10\cdot(10E-A)^{-2}$
$=10\cdot(\sqrt{10}E)^{-2}=E$

六. (1) $A^H=\begin{bmatrix}3&1&0&0\\0&3&1&0\\0&0&3&0\\0&0&0&3\end{bmatrix}$ $\quad A\sim A^H$ 故有相同Jordan形.

$\varphi_A(\lambda)=(\lambda-3)^3$

$p(x)=a_0+a_1x+a_2x^2$

$p(3)=a_0+3a_1+9a_2=f(3)$

$p'(3)=a_1+6a_2=f'(3)$

$p''(3)=2a_2=f''(3)$

$\therefore\begin{cases}a_0=f(3)-3f'(3)+\frac{9}{2}f''(3)\\a_1=f'(3)-3f''(3)\\a_2=\frac12f''(3)\end{cases}$

$\sin2\pi A=-6\pi E+2\pi A$

$\cos\frac{\pi}{2}A=$

(2) $|A-\lambda E| = (2-\lambda)(\lambda^2-1)$

$\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = 2$

$p(x) = a_0 + a_1 x + a_2 x^2$

$p(1) = f(1) = a_0 + a_1 + a_2$

$p(-1) = f(-1) = a_0 - a_1 + a_2$

$p(2) = f(2) = a_0 + 2a_1 + 4a_2$

$$\therefore \begin{cases} a_0 = f(1) + \frac{5}{6} f(-1) - \frac{1}{3} f(2) \\ a_1 = \frac{1}{2}[f(1) - f(-1)] \\ a_2 = -\frac{1}{2} f(1) - \frac{1}{3} f(-1) + \frac{1}{3} f(2) \end{cases}$$

代入

七.

证明：

设 $A = PJP^{-1} = P\,diag(J_1, J_2, \cdots, J_r)P^{-1}$

则 $e^{kA} = P\,diag(e^{kJ_1}, e^{kJ_2}, \cdots, e^{kJ_r})P^{-1}$

而 $$e^{kJ_i} = \begin{bmatrix} e^{k\lambda_i} & ke^{\lambda_i} & \cdots & \\ & e^{k\lambda_i} & \ddots & \\ & & \ddots & \\ & & & e^{k\lambda_i} \end{bmatrix}_{d_i \times d_i}$$

$$\begin{aligned} |e^{k\cdot A}| &= |P|\,|diag(e^{kJ_1}, e^{kJ_2}, \cdots e^{kJ_r})|\,|P^{-1}| \\ &= |diag(e^{kJ_1}, e^{kJ_2}, \cdots e^{kJ_r})| \\ &= e^{kd_1\lambda_1} e^{kd_2\lambda_2} \cdots e^{kd_r\lambda_r} \\ &= e^{k(d_1\lambda_1 + d_2\lambda_2 + \cdots + d_r\lambda_r)} \\ &= e^{k\,tr(A)} \end{aligned}$$

得证.

八.

$$\frac{d}{dx}\left(\int_0^{x^2} A(t)\,dt\right) = 2x \cdot A(x^2)$$

$$= 2x \cdot \begin{bmatrix} e^{2x^2} & x^2 e^{x^2} & x^4 \\ e^{-x^2} & 2e^{2x^2} & 0 \\ 3x^2 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2xe^{2x^2} & 2x^3 e^{x^2} & 2x^5 \\ 2xe^{-x^2} & 4xe^{2x^2} & 0 \\ 6x^3 & 0 & 0 \end{bmatrix}$$

九. $Ax = b$

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{C}$$

$\therefore A^+ = C^H(CC^H)^{-1}(B^HB)^{-1}B^H$

$x = A^+ b$.