矩阵函数的幂级数表示

定理： 设 $A\in C^{n\times n}$，一元函数 $f(x)$ 能够展开成关于 x 的幂级数 $f(x)=\sum_{k=0}^{\infty}c_k x^k$，收敛半径为 R. 当矩阵 A 的谱半径 $\rho(A)<R$ 时，矩阵幂级数 $\sum_{k=0}^{\infty}c_k A^k$ 绝对收敛，并且

$$f(A)=\sum_{k=0}^{\infty}c_k A^k.$$

证明：

$$\sum_{k=0}^{\infty} c_k A^k = \sum_{k=0}^{\infty} c_k (PJ^k P^{-1}) = P(\sum_{k=0}^{\infty} c_k J^k) P^{-1}$$

$$= P\mathrm{diag}\left[\sum_{k=0}^{\infty} c_k J_1^k(\lambda_1), \sum_{k=0}^{\infty} c_k J_2^k(\lambda_2), \cdots, \sum_{k=0}^{\infty} c_k J_r^k(\lambda_r)\right] P^{-1}$$

其中

$$\sum_{k=0}^{\infty} c_k J_i^k(\lambda_i) = \begin{bmatrix} \sum_{k=0}^{\infty} c_k \lambda_i^k & \sum_{k=0}^{\infty} c_k C_k^1 \lambda_i^{k-1} & \cdots & \sum_{k=0}^{\infty} c_k C_k^{d_i-1} \lambda_i^{k-d_i+1} \\ & \sum_{k=0}^{\infty} c_k \lambda_i^k & \ddots & \vdots \\ & & \ddots & \sum_{k=0}^{\infty} c_k C_k^1 \lambda_i^{k-1} \\ & & & \sum_{k=0}^{\infty} c_k \lambda_i^k \end{bmatrix}_{d_i \times d_i}$$

其中 $$C_k^l = \frac{k(k-1)\cdots(k-l+1)}{l!}, \quad (l \le k)$$

$$C_k^l = 0, \quad (l > k)$$

因为 $\rho(A) < R$, 所以 $f(\lambda_i) = \sum_{k=0}^{\infty} c_k \lambda_i^k, \quad f'(\lambda_i) = \sum_{k=0}^{\infty} c_k C_k^1 \lambda_i^{k-1},$

$$\frac{f''(\lambda_i)}{2!} = \sum_{k=0}^{\infty} c_k C_k^2 \lambda_i^{k-2}, \dots, \frac{f^{(d_i-1)}(\lambda_i)}{(d_i-1)!} = \sum_{k=0}^{\infty} c_k C_k^{d_i-1} \lambda_i^{k-d_i+1}.$$

$$\sum_{k=0}^{\infty} c_k J_i^k(\lambda_i) = \begin{bmatrix} f(\lambda_i) & f^{'}(\lambda_i) & \frac{1}{2!} f^{''}(\lambda_i) & \cdots & \cdots & \frac{1}{(d_i-1)!} f^{(d_i-1)}(\lambda_i) \\ & f(\lambda_i) & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & \frac{1}{2!} f^{''}(\lambda_i) \\ & & & & & f^{'}(\lambda_i) \\ & & & & & f(\lambda_i) \end{bmatrix}_{d_i \times d_i}$$

$$= f(J_i),$$

$$\sum_{k=0}^{\infty} c_k A^k = P\mathrm{diag}\left[\sum_{K=0}^{\infty} c_k J_1^k(\lambda_1), \sum_{K=0}^{\infty} c_k J_2^k(\lambda_2), \cdots, \sum_{K=0}^{\infty} c_k J_r^k(\lambda_r)\right] P^{-1}$$

$$= P\mathrm{diag}[f(J_1), f(J_2), \cdots, f(J_r)] P^{-1}$$

$$= f(A)$$

-----矩阵函数的幂级数表示

当 $|x| < +\infty$ 时,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \cdots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + (-1)^n \frac{1}{(2n)!}x^{2n} + \cdots$$

所以对于任意的矩阵 $A\in C^{n\times n}$，我们有

$$e^{A}=I+A+\frac{1}{2!}A^{2}+\cdots+\frac{1}{n!}A^{n}+\cdots$$

$$\sin A=A-\frac{1}{3!}A^{3}+\frac{1}{5!}A^{5}-\cdots+(-1)^{n}\frac{1}{(2n+1)!}A^{2n+1}+\cdots$$

$$\cos A=I-\frac{1}{2!}A^{2}+\frac{1}{4!}A^{4}-\cdots+(-1)^{n}\frac{1}{(2n)!}A^{2n}+\cdots$$

例： 已知 $$A=\begin{bmatrix}2 & 0 & 0\\ 1 & 1 & 1\\ 1 & -1 & 3\end{bmatrix},$$

求矩阵幂级数 $\sum\limits_{k=0}^{\infty}\dfrac{k+1}{10^{k+1}}A^k$ 的和.

解： A 的Jordan标准形为 $$J=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix},$$

所以 $\rho(A)=2.$

$$f(x)=\sum_{k=0}^{\infty}\frac{k+1}{10^{k+1}}x^{k}=\sum_{k=0}^{\infty}\left[\left(\frac{x}{10}\right)^{k+1}\right]'$$

$$=\left[\sum_{k=0}^{\infty}\left(\frac{x}{10}\right)^{k+1}\right]'=\left[-1+(1-\frac{x}{10})^{-1}\right]'$$

$$=\frac{1}{10}(1-\frac{x}{10})^{-2},\quad \left(|x|<R=10\right)$$

$$\therefore \quad \sum_{k=0}^{\infty} \frac{k+1}{10^{k+1}} A^k = f(A)$$

$$= \frac{1}{10}\left(I - \frac{A}{10}\right)^{-2} = \frac{5}{128}\begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 1 \\ 1 & -1 & 5 \end{bmatrix}.$$