MSA 8150 homework 1 Xiaofei Xue

Q1 (a) show if $a_1b_2 - a_2b_1 \neq 0$, then $x = \dfrac{c_1b_2 - b_1c_2}{a_1b_2 - a_2b_1}$

$y = \dfrac{a_1c_2 - c_1a_2}{a_1b_2 - a_2b_1}$

$$\begin{cases} a_1x + b_1y = c_1 & \times a_2 \\ a_2x + b_2y = c_2 & \times a_1 \end{cases}$$

$$\begin{cases} a_1a_2x + a_2b_1y = a_2c_1 \\ a_1a_2x + a_1b_2y = a_1c_2 \end{cases} \times -1 \quad \begin{cases} a_1a_2x + a_2b_1y = a_2c_1 \\ -a_1a_2x - a_1b_2y = -a_1c_2 \end{cases}$$

$$a_2b_1y - a_1b_2y = a_2c_1 - a_1c_2$$

$$y(a_2b_1 - a_1b_2) = a_2c_1 - a_1c_2$$

$$\boxed{y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2}}$$

$$\begin{cases} a_1x + b_1y = c_1 & \times b_2 \\ a_2x + b_2y = c_2 & \times b_1 \end{cases} \quad \begin{cases} a_1b_2x + b_1b_2y = b_2c_1 \\ a_2b_1x + b_1b_2y = b_1c_2 \end{cases} \times -1$$

$$\begin{cases} a_1b_2x + b_1b_2y = b_2c_1 \\ -a_2b_1x - b_1b_2y = -b_1c_2 \end{cases}$$

$$a_1b_2x - a_2b_1x = b_2c_1 - b_1c_2$$

$$\boxed{x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}}$$

b) $y = \beta_1 g(x) + \beta_2 f(x)$ $\quad (x_1\ y_1)\ (x_2\ y_2) \ldots (x_n, y_n)$

$\hat{y}_1 = \beta_1 g(x_1) + \beta_2 f(x_1)$

$\hat{y}_2 = \beta_1 g(x_2) + \beta_2 f(x_2)$

$\vdots$

$\hat{y}_i = \beta_1 g(x_i) + \beta_2 f(x_i)$

$RSS = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$

$= \sum_{i=1}^{n} (y_i - \beta_1 g(x_i) - \beta_2 f(x_2))^2$

$\frac{\partial R}{\partial \beta_1} = 2 \cdot \sum_{i=1}^{n} (y_i - \beta_1 g(x_i) - \beta_2 f(x_i)) \cdot -g(x_i) = 0$

$\frac{\partial R}{\partial \beta_2} = 2 \cdot \sum_{i=1}^{n} (y_i - \beta_1 g(x_i) - \beta_2 f(x_i)) \cdot -f(x_i) = 0$

$\cancel{2} g_i \cdot \sum_{i=1}^{n} (y_i - \beta_1 g_i - \beta_2 f_i) = 0 \qquad \sum_{i=1}^{n} (y_i g_i - \beta_1 g_i^2 - \beta_2 g_i f_i) = 0$

$\sum_{i=1}^{n} y_i g_i - \sum_{i=1}^{n} \beta_1 g_i^2 - \sum_{i=1}^{n} \beta_2 g_i f_i = 0 \quad \Big| \quad \sum_{i=1}^{n} y_i g_i = \sum_{i=1}^{n} \beta_1 g_i^2 + \sum_{i=1}^{n} \beta_2 g_i f_i$

$\cancel{2} \cdot f_i \sum_{i=1}^{n} (y_i - \beta_1 g_i - \beta_2 f_i) = 0 \qquad \sum_{i=1}^{n} (y_i f_i - \beta_1 g_i f_i - \beta_2 f_i^2) = 0$

$\sum_{i=1}^{n} y_i f_i - \sum_{i=1}^{n} \beta_1 g_i f_i - \sum_{i=1}^{n} \beta_2 f_i^2 = 0 \quad \Big| \quad \sum_{i=1}^{n} y_i f_i = \sum_{i=1}^{n} \beta_1 g_i f_i + \sum_{i=1}^{n} \beta_2 f_i^2$

$$\begin{cases} \sum_{i=1}^{n} \beta_1 g_i^2 + \sum_{i=1}^{n} \beta_2 g_i f_i = \sum_{i=1}^{n} y_i g_i \\ \sum_{i=1}^{n} \beta_1 g_i f_i + \sum_{i=1}^{n} \beta_2 f_i^2 = \sum_{i=1}^{n} y_i f_i \end{cases} \qquad \begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \quad \Big| \quad \begin{matrix} x = \beta_1 \\ y = \beta_2 \end{matrix}$$

$a_1 = \sum_{i=1}^{n} g_i^2 \qquad b_1 = \sum_{i=1}^{n} g_i f_i$

$a_2 = \sum_{i=1}^{n} g_i f_i \qquad b_2 = \sum_{i=1}^{n} f_i^2$

$$\hat{\beta}_1 = \frac{c_1 b_2 - b_1 c_2}{a_1 b_2 - a_2 b_1} = \frac{(\sum_{i=1}^{n} y_i g_i)(\sum_{i=1}^{n} f_i^2) - (\sum_{i=1}^{n} g_i f_i)(\sum_{i=1}^{n} y_i f_i)}{(\sum_{i=1}^{n} g_i^2)(\sum_{i=1}^{n} f_i^2) - (\sum_{i=1}^{n} g_i f_i)(\sum_{i=1}^{n} g_i f_i)}$$

$$\hat{\beta}_2 = \frac{a_1 c_2 - c_1 a_2}{a_1 b_2 - a_2 b_1} = \frac{(\sum_{i=1}^{n} g_i^2)(\sum_{i=1}^{n} y_i f_i) - (\sum_{i=1}^{n} y_i g_i)(\sum_{i=1}^{n} g_i f_i)}{(\sum_{i=1}^{n} g_i^2)(\sum_{i=1}^{n} f_i^2) - (\sum_{i=1}^{n} g_i f_i)(\sum_{i=1}^{n} g_i f_i)}$$

c) Let $g(x)=0$ $\quad y=\beta_1 \cdot g(x)+\beta_2 \cdot f(x)=\beta_2 \cdot f(x)=\beta \cdot f(x)$

(x_1, y_1) $\quad \hat{y}_1=\beta \cdot f(x_1)$

(x_2, y_2) $\quad \hat{y}_2=\beta \cdot f(x_2)$

$\vdots \qquad \vdots$

(x_n, y_n) $\quad \hat{y}_n=\beta \cdot f(x_n)$

$$RSS=\sum_{i=1}^{n}(y_i-\hat{y}_i)^2$$
$$=\sum_{i=1}^{n}[y_i-\beta \cdot f(x_i)]^2$$

$$\frac{\partial R}{\partial \beta}=2 \cdot \sum_{i=1}^{n}(y_i-\beta f(x_i)) \cdot f(x_i)$$

use f_i for $f(x_i)$

$$\sum_{i=1}^{n}(y_i-\beta \cdot f_i) \cdot f_i=0 \qquad \sum_{i=1}^{n} y_i f_i-\sum_{i=1}^{n} \beta \cdot f_i^2=0$$

$$\sum_{i=1}^{n} y_i f_i=\sum_{i=1}^{n} \beta \cdot f_i^2 \qquad \hat{\beta}=\frac{\sum_{i=1}^{n} y_i f_i}{\sum_{i=1}^{n} f_i^2}$$

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d) $$\beta_2=\frac{\left(\sum_{i=1}^{n} y_i f_i\right)\left(\sum_{i=1}^{n} g_i^2\right)-\left(\sum_{i=1}^{n} y_i g_i\right)\left(\sum_{i=1}^{n} g_i f_i\right)}{\left(\sum_{i=1}^{n} g_i^2\right)\left(\sum_{i=1}^{n} f_i^2\right)-\left(\sum_{i=1}^{n} f_i g_i\right)^2}$$

$g(x_i)=g_i=0$

No, we can't get the answer for part c) by plugging in $g(x)=0$.

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e) $$E(\hat{\beta})=E\left(\frac{\sum_{i=1}^{n} y_i f_i}{\sum_{i=1}^{n} f_i^2}\right)=E\left(\frac{\sum_{i=1}^{n}(\beta \cdot \cancel{f_i}) \cancel{f_i}}{\sum_{i=1}^{n} \cancel{f_i^2}}\right)=\beta$$

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f) $g(x)=0$ $\quad y_i=\beta x_i^2+\varepsilon_i$ $\quad E(\varepsilon_i)=0$ $\quad Var(\varepsilon_i)=\sigma^2$

when $y=\beta \cdot f(x)+\varepsilon$, $f(x)=x^2$

$$\hat{\beta}=\frac{\sum_{i=1}^{n} y_i \cdot f_i}{\sum_{i=1}^{n} f_i^2}=\frac{\sum_{i=1}^{n} y_i x_i^2}{\sum_{i=1}^{n} x_i^4}=\beta+\frac{\varepsilon_i}{\sum_{i=1}^{n} x_i^2}$$

$$=\frac{\sum_{i=1}^{n}(\beta x_i^2+\varepsilon_i) x_i^2}{\sum_{i=1}^{n} x_i^4}=\frac{\sum_{i=1}^{n}(\beta x_i^4+\varepsilon_i x_i^2)}{\sum_{i=1}^{n} x_i^4}=\beta+\frac{\varepsilon_i \sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^4}$$

$$\hat{\beta} = \beta + \frac{\varepsilon_i}{\sum_{i=1}^{4} X_i^2} \qquad Var(\hat{\beta}) = Var\left(\beta + \frac{\varepsilon_i}{\sum_{i=1}^{4} X_i^2}\right) = Var(\beta) + Var\left(\frac{\varepsilon_i}{\sum_{i=1}^{4} X_i^2}\right)$$

$$= 0 + \frac{1}{\sum_{i=1}^{4} X_i^4} \cdot \sigma^2 = \frac{\sigma^2}{\sum_{i=1}^{4} X_i^4}$$

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g) Yes, I think based on $\frac{\sigma^2}{\sum_{i=1}^{4} X_i^4}$, with larger X_i, the variance will decrease, so does the uncertainty.

Q2

Q3 see R code