8200 HW 1 Xiaofei Xue

2. $E(w_t) = 0$ $Var(w_t) = \sigma_w^2$

$x_t = \beta_0 + \beta_{1t} + w_t$ $\beta_0$ $\beta_1$ constant

a) $E(x_t) = E(\beta_0 + \beta_{1t} + w_t)$

$= E(\beta_0) + E(\beta_1 t) + E(w_t)$

$= \beta_0 + E(\beta_1 t) + 0$

If $x_t$ is stationary, then $E(x_t)$ is constant.

Since $\beta_0 + E(\beta_1 t)$ is not constant,

$x_t$ is non-stationary.

2.b Show $\Delta x_t = X_t - X_{t-1}$ is stationary

① $\mu_t$ is independent of $t$

② for each $h$, $\gamma_x(t+h, t)$ is independent of $t$

$$E(\Delta x_t) = E(x_t - x_{t-1})$$

$$= E(\beta_0 + \beta_1 t + w_t - (\beta_0 + \beta_1 t{-}1 + w_{t-1}))$$

$$= E(\cancel{\beta_0} + \beta_1 t + w_t - \cancel{\beta_0} - \beta_1 t{-}1 - w_{t-1})$$

$$= E((\beta_1 (t - (t-1)) + w_t - w_{t-1})$$

$$= E(\beta_1 \cdot 1) + E(w_t) - E(w_{t-1})$$

$$= \beta_1.$$

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Autocovariance function of a stationary time series:

$$\gamma(t,s) = \underset{h=t-s}{cov}(X_t, X_s) = E((X_t - \mu) \cdot (X_s - \mu))$$

$X_t = \beta_0 + \beta_1 \cdot t + W_t \qquad X_{t-1} = \beta_0 + \beta_1 (t-1) + W_{t-1}$

$\Delta x_t = X_t - X_{t-1} = \beta_1 + W_t - W_{t-1}$

$Var(\Delta X_t) = Var(\beta_1 + W_t - W_{t-1})$

$= Var(\beta_1) + Var(W_t) + (-1)^2\, Var(W_{t-1})$

$= 0 + \sigma_w^2 + \sigma_w^2 = 2\sigma_w^2$

Let $g = \Delta X_t = X_t - X_{t-1} = \beta_1 + W_t - W_{t-1}$

time $n$, and time $m$ $\qquad h = n - m$

$\gamma(n, m) = cov(g_n, g_m) = E((g_n - \mu) \cdot (g_m - \mu))$

$= E((\cancel{\beta_1} + W_n - W_{n-1} - \cancel{\beta_1}) \cdot (\cancel{\beta_1} + W_m - W_{m-1} - \cancel{\beta_1}))$

$= E((W_n - W_{n-1}) \cdot (W_m - W_{m-1}))$

$= E(W_n \cdot W_m - W_{n-1} \cdot W_m - W_n \cdot W_{m-1} + W_{n-1} \cdot W_{m-1})$

$= E(W_n \cdot W_m) - E(W_{n-1} \cdot W_m) - E(W_n \cdot W_{m-1}) + E(W_{n-1} \cdot W_{m-1})$

$= E(W_n)E(W_m) - E(W_{n-1})E(W_m) - E(W_n)E(W_{m-1}) + E(W_{n-1})E(W_{m-1})$

$= 0 - 0 - 0 + 0$

$= 0$

Autocovariance is constant independent of time.

2.C $Y_t$ is a general stationary process

$E(Y_t) = 0$ autovariance $\gamma(h)$ is independent of time

$$X_t = \beta_0 + \beta_1 \cdot t + Y_t \qquad X_{t-1} = \beta_0 + \beta_1(t-1) + Y_{t-1}$$

$$\Delta X_t = X_t - X_{t-1} = \cancel{\beta_0} + \beta_1 t + Y_t - \cancel{\beta_0} - \beta_1(t-1) - Y_{t-1}$$

$$= \beta_1 \cdot (t - (t-1)) + Y_t - Y_{t-1} = \beta_1 + Y_t - Y_{t-1}$$

$$E(\Delta X_t) = E(\beta_1 + Y_t - Y_{t-1}) = E(\beta_1) + E(Y_t) - E(Y_{t-1})$$

$$= \beta_1 + 0 - 0 = \beta_1$$

Again let $g(t) = \Delta X_t = \beta_1 + Y_t - Y_{t-1}$

Let $t = n$, $t = m$ $\quad h = n - m$

$$\gamma(n, m) = Cov(g_n, g_m) = E\big((g_n - \mu_n)(g_m - \mu_m)\big)$$

$$= E\big((\cancel{\beta_1} + Y_n - Y_{n-1} - \cancel{\beta_1})(\cancel{\beta_1} + Y_m - Y_{m-1} - \cancel{\beta_1})\big)$$

$$= E\left[(Y_n - Y_{n-1}) \cdot (Y_m - Y_{m-1})\right]$$

$$= E(Y_n \cdot Y_m - Y_{n-1} \cdot Y_m - Y_n \cdot Y_{m-1} + Y_{n-1} \cdot Y_{m-1})$$

$$= E(Y_n \cdot Y_m) - E(Y_{n-1} \cdot Y_m) - E(Y_n \cdot Y_{m-1}) + E(Y_{n-1} \cdot Y_{m-1})$$

$\gamma(h)$ is an autocovariance function

therefore $\gamma(h) = \mathrm{cov}(Y_{t+h}, Y_t) = E\left[(Y_{t+h}-\mu)(Y_t-\mu)\right]$

$$= E\left(Y_{t+h}\cdot Y_t - \mu\cdot Y_t - Y_{t+h}\cdot\mu + \mu^2\right)$$

$$= E\left(Y_{t+h}\cdot Y_t - \gamma\cdot Y_t - Y_{t+h}\cdot\gamma + \gamma^2\right)$$

$$= E(Y_{t+h}\cdot Y_t) - E(\gamma\cdot Y_t) - E(\gamma\cdot Y_{t+h}) + E(\gamma^2)$$

$$= E(Y_{t+h}\cdot Y_t) - \gamma^2 - \gamma^2 + \gamma^2 = E(Y_{t+h}\cdot Y_t) - \gamma^2$$

$$E(Y_t)\cdot E(Y_{t+h}) = \gamma(h) + \gamma^2$$

$h = n-m \qquad E(Y_m)\cdot E(Y_n) = \gamma(h) + \gamma^2$

Let $t = m$

$t+h = m+h = n$

Then $E(Y_n\cdot Y_m) = \gamma(h) + \gamma^2$

$E(Y_{n-1}\cdot Y_m) = \gamma(h-1) + \gamma^2$

$E(Y_n\cdot Y_{m-1}) = \gamma(h+1) + \gamma^2$

$E(Y_{n-1}\cdot Y_{m-1}) = \gamma(h) + \gamma^2$

$$\boxed{\begin{aligned} n-m &= h \\ n-1-m &= h-1 \\ n-(m-1) &= h+1 \\ n-1-(m-1) &= h \end{aligned}}$$

$$\Gamma(n,m) = \gamma(h) + \cancel{\gamma^2} - \gamma(h-1) - \cancel{\gamma^2} - \gamma(h+1) - \cancel{\gamma^2} + \gamma(h) + \cancel{\gamma^2} = 2\gamma(h) - \gamma(h-1) - \gamma(h+1)$$

4. $X_t = W_{t-1} + 2W_t + W_{t+1}$ $\quad E(W_t)=0 \quad Var(W_t) = \sigma_w^2$

$Cov(W_{t+h}, W_t) = E[(W_{t+h}-\mu)\cdot(W_t-\mu)] = E(W_{t+h}\cdot W_t) =$

$E(W_{t+h})\cdot E(W_t) = 0$

$E(X_t) = E(W_{t-1} + 2W_t + W_{t+1}) = E(W_{t-1}) + 2\cdot E(W_t) + E(W_{t+1}) = 0$

lag: $h = s-t$

$\gamma(h) = \gamma(s-t) = Cov(X_s, X_t) =$

$E[(X_s-\mu)\cdot(X_t-\mu)] = E[(W_{s-1}+2W_s+W_{s+1}-0)(W_{t-1}+2W_t+W_{t+1}-0)]$

if $s=t$, $h=0$, then

$\gamma(s-t) = \gamma(0) = E(X_t\cdot X_t) = E(X_t^2) = E((W_{t-1}+2W_t+W_{t+1})^2)$

$= 6\sigma_w^2$

if $s=t+1$, $h=1$

$\gamma(s-t) = \gamma(1) = E(2W_{s-1}^2) + E(2W_s^2) = 4\sigma_w^2$

if $s=t+2$, $h=2$

$\gamma(s-t) = \gamma(2) = E(W_{s-1}^2) = \sigma_w^2$

$$\gamma(h) = \begin{cases} 6\sigma_w^2 & h=0 \\ 4\sigma_w^2 & h=1 \\ \sigma_w^2 & h=2 \\ 0 & h=3 \end{cases} \qquad \rho(h) = \begin{cases} 1 & h=0 \\ 2/3 & h=1 \\ 1/6 & h=2 \\ 0 & h=3 \end{cases}$$