What are we weighting for?

A mechanistic model for probability weighting

Ole Peters Alexander Adamou Yonatan Berman Mark Kirstein

✉ m.kirstein@lml.org.uk

@nonergodicMark

Full paper at bit.ly/lml-pw-r1

Live paper at bit.ly/lml-pw

MPI MiS, Group Seminar, 03 November 2020

Max Planck Institute for Mathematics in the Sciences

UNIVERSITÄT LEIPZIG Mathematisches Institut

LONDON MATHEMATICAL LABORATORY

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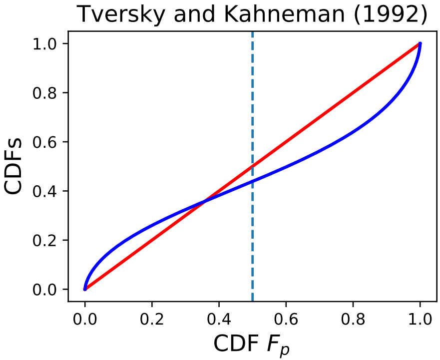
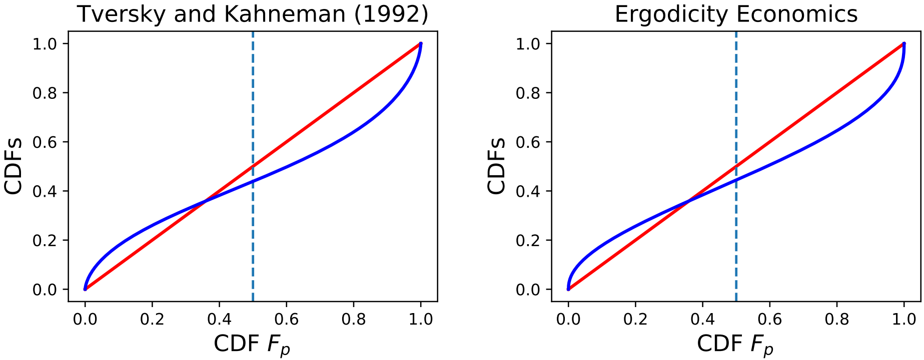
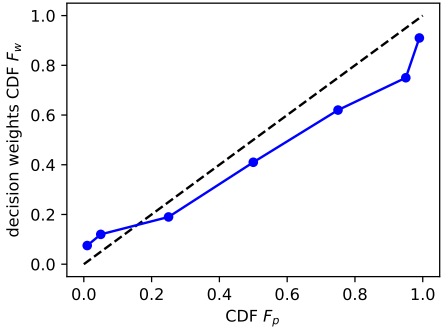
Main result



Main result


Tversky and Kahneman (1992)
Ergodicity Economics
CDFs
CDF F_p
0.0
0.2
0.4
0.6
0.8
1.0

Definition of Probability Weighting (PW)



(Preston and Baratta 1948, p. 188, Fig. 1, relabelled axes)

"The existence [...] a scale of *psychological probability* and its functional relationship to the scale of *mathematical probability*" (p. 186)

- empirical pattern: inverse-S shape
- important component in behavioural economics (Cumulative Prospect Theory)

Classical interpretation of PW:

- as a cognitive process
- maladaptive irrational cognitive bias

↪ How does this pattern emerge?

↪ Can we derive a functional form (rather than fit a function)?

Related Literature

- no motivation of the functional form of weighting curve other than fit
- origin of PW preferences? no stable mappings (Stewart et al. 2015)
- experimental design is a key confounder $\rightarrow$ description-experience gap, *i.e.* less (even under-) overweighting in decisions-from-experience (Hertwig et al. 2004; Hertwig and Erev 2009)
- meta-analyses find all possible weighting curves (Ungemach et al. 2009; Wulff et al. 2018, Tab. 9)
- non-human animals (Andrews et al. 2018; Constantinople et al. 2019; Trimmer et al. 2011)

Statistical explanations

- PW maladaptive, due to biased estimation (Fox and Hadar 2006)
- PW adaptive in sequential learning problems (Seo et al. 2019)
- PW adaptive heuristic as an approximate Bayesian solution for the inference problem (Martins 2006)

$\hookrightarrow$ reproducibility, context dependence, (mal)adaptive

Setup

No *truth*, only difference in models of uncertainty

Task: model payout, x, of a gamble as a random variable.

Disinterested Observer (DO)

DO assigns PDF $p(x)$
$\hookrightarrow$ CDF $F_p(x)$

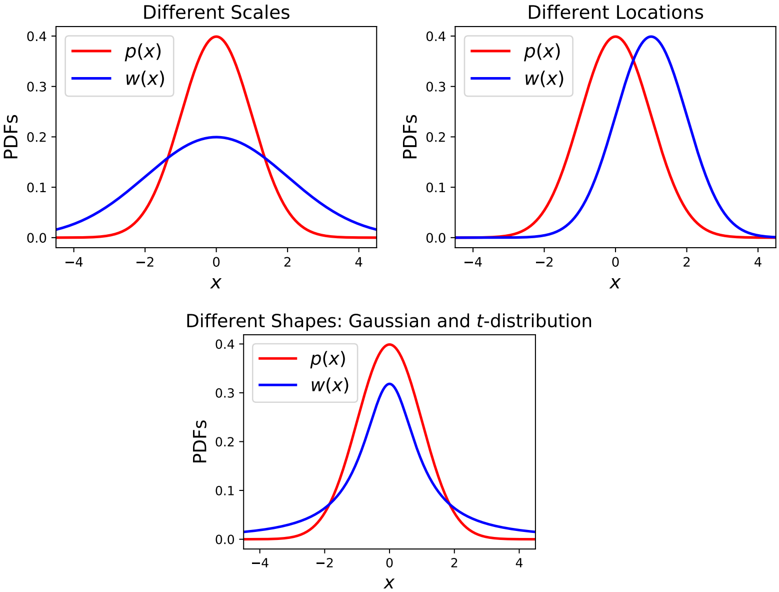
$$F_p(x) = \int_{-\infty}^{x} p(s)\,\mathrm{d}s$$

Decision Maker (DM)

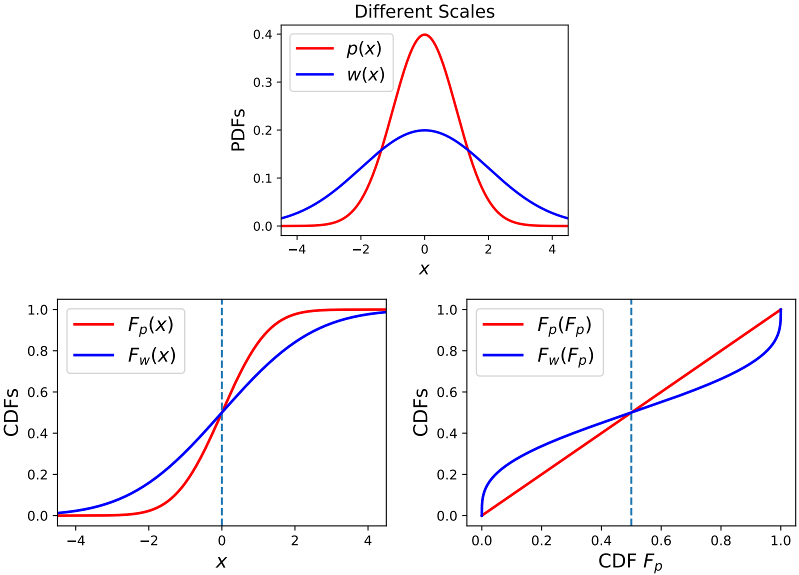
DM assigns different PDF $w(x)$
$\hookrightarrow$ CDF $F_w(x)$

$$F_w(x) = \int_{-\infty}^{x} w(s)\,\mathrm{d}s$$

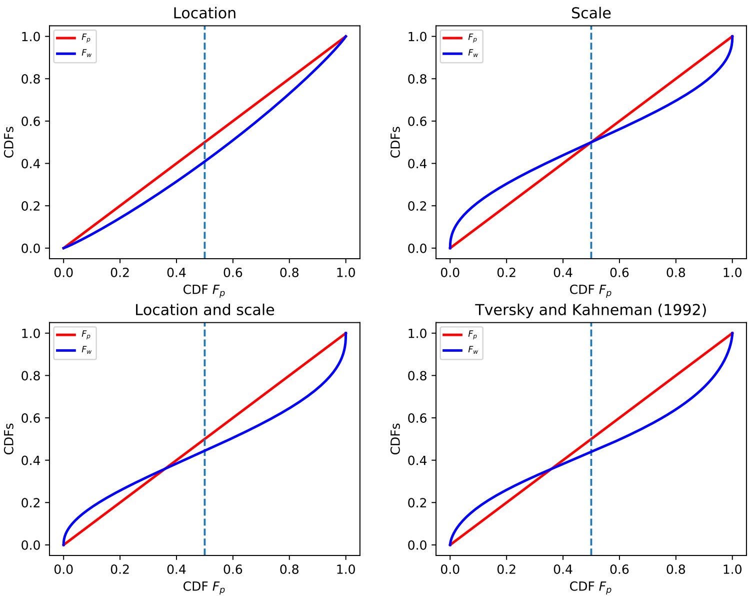
Scales, Locations, Shapes



Thought Experiment: DM assumes greater scale



Asymmetric Inverse-S = diff. in uncertainty + diff. in locations



Numerically feasible for arbitrary distributions:

1. list values of DO's CDF, $F_p(x)$, at set x_i
2. list values of DM's CDF, $F_w(x)$, at same x_i
3. plot $F_w(x)$ *vs.* $F_p(x)$

Functional form of the weighting function

Gaussian case with different scale:

$$w(p) = p^{\frac{1}{\alpha^2}} \underbrace{\frac{\left(2\pi\sigma^2\right)^{\frac{1-\alpha^2}{2\alpha^2}}}{\alpha}}_{\text{normalisation factor}},$$

where

- DO's scale is σ, $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$
- DM's scale is $\alpha\sigma$, $X \sim \mathcal{N}\left(\mu, (\alpha\sigma)^2\right)$
- DO uses greater uncertainty $\alpha < 1 \rightarrow$ regular-S shape
- DM uses greater uncertainty $\alpha > 1 \rightarrow$ inverse-S shape
- uncertainty measured by the standard deviation

Functional Forms

(EE mechanism) Gaussian case:

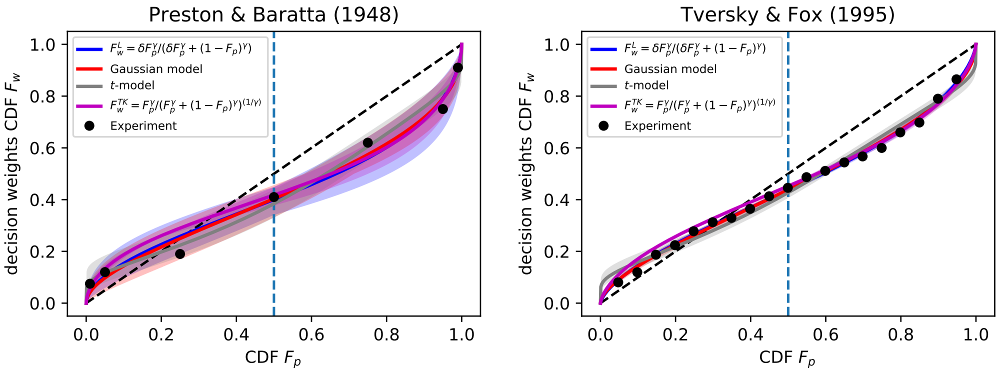
$$\boxed{w(p) = p^{\frac{1}{\alpha^2}} \frac{\left(2\pi\sigma^2\right)^{\frac{1-\alpha^2}{2\alpha^2}}}{\alpha}} \tag{1}$$

Tversky and Kahneman (1992, $\gamma = 0.68$):

$$F_w^{TK}\left(F_p; \gamma\right) = \left(F_p\right)^{\gamma} \frac{1}{\left[\left(F_p\right)^{\gamma} + \left(1 - F_p\right)^{\gamma}\right]^{1/\gamma}} \tag{2}$$

Lattimore et al. (1992):

$$F_w^{L}\left(F_p; \delta, \gamma\right) = \frac{\delta F_p^{\gamma}}{\delta F_p^{\gamma} + \left(1 - F_p\right)^{\gamma}} \tag{3}$$


Preston & Baratta (1948)
Tversky & Fox (1995)
$F_w^L = \delta F_p^\gamma/(\delta F_p^\gamma + (1-F_p)^\gamma)$
Gaussian model
t-model
$F_w^{TK} = F_p^\gamma/(F_p^\gamma + (1-F_p)^\gamma)^{(1/\gamma)}$
Experiment
decision weights CDF F_w
CDF F_p
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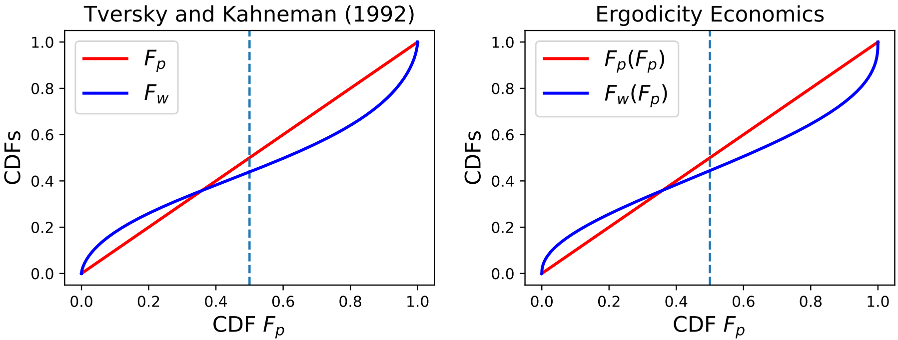
Interim conclusion



- DM's greater scale gives inverse-S shape (unimodal distributions)
- difference in locations gives asymmetry
- reproduces observations of probability weighting

Job done. Thank you for your attention ;)

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The Ergodicity Question

Typical DO concern

What happens on average to the ensemble of subjects?

$\neq$

Typical DM concern

What happens to me on average over time?

The Ergodicity Question

Typical DO concern $\neq$ **Typical DM concern**

What happens on average to the ensemble of subjects?

What happens to me on average over time?

↪ Broken ergodicity is more fundamental than being just about *correct* behavioural models

- broken ergodicity leads to a collision of ensemble and time perspective in mathematical statements like $F_w(F_p)$

Reasons for different uncertainty models

- DM has no control over experiment
- experiment may be unclear to DM
- DM may not trust DO
- ...

Experiencing probabilities

- "probability" is polysemous
- natural frequencies: "10 out of 100" vs 10% (Gigerenzer 1991, 2018; Hertwig and Gigerenzer 1999)
- probabilities are not observable
- probabilities encountered as
  - known frequencies in ensemble of experiments (DO)
  - frequencies estimated over time (DM)

$\hookrightarrow$ **estimates have uncertainties**

$\hookrightarrow$ **cautious (rational?) DM accounts for these uncertainties**

$\hookrightarrow$ **small probabilities come with large relative uncertainties**

probabilities are not observable, but counts are

Rare Event

- $p(x) = 0.001$
- $T = 100$ observations
- $\sim 99.5\%$ get 0 or 1 events
- $\hat{p}(x) = 0$ or $\hat{p}(x) = 0.01$

$\hookrightarrow$ $\hat{p}(x)$ off by 1000%

Common Event

- $p(x) = 0.5$
- $T = 100$ observations
- $\sim 99.5\%$ get between 35 and 65 events,
- $0.35 < \hat{p}(x) < 0.65$

$\hookrightarrow$ $\hat{p}(x)$ off by 30%

$\boxed{\hookrightarrow \text{ small } p(x)\text{, small count} \rightarrow \text{big uncertainty}}$

Scaling of uncertainty: small count $\rightarrow$ big uncertainty

1. scaling of counts $n(x)$:

$$n(x) \sim p(x)\delta x T$$

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2. uncertainty in Poisson-distributed counts $\sim \sqrt{n(x)}$

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3. estimate of the asymptotic probability density

$$p(x) \approx \underbrace{\frac{n(x)}{T\delta x}}_{\text{estimate}} \pm \underbrace{\frac{\sqrt{n(x)}}{T\delta x}}_{\text{relative uncertainty}} \tag{4}$$

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4. express the uncertainty in terms of the estimate itself

$$\varepsilon\left[\hat{p}(x)\right] \equiv \frac{\sqrt{n(x)}}{T\delta x} = \sqrt{\frac{\hat{p}(x)}{T\delta x}} \tag{5}$$

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6. relative error in the estimate $\lim\limits_{p(x)\to 0} 1/\sqrt{\hat{p}(x)T\delta x}$ grows,
7. **Prediction:** $\lim_{T\to\infty} \hat{p} \to p$

Cautionary principle : DMs don't like surprises

To avoid surprises, DMs add estimation uncertainty $\varepsilon\left[\hat{p}(x)\right]$ to every estimated probability, then normalize, s.t.

$$w(x) = \frac{\hat{p}(x) + \varepsilon\left[\hat{p}(x)\right]}{\int \left(\hat{p}(s) + \varepsilon\left[\hat{p}(s)\right]\right) ds} \tag{6}$$

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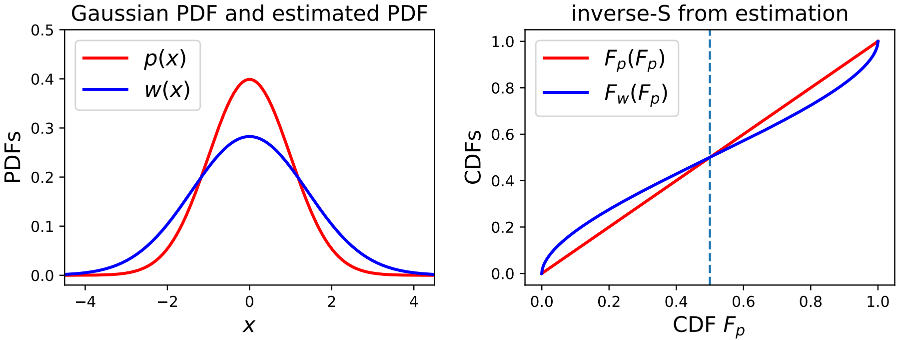
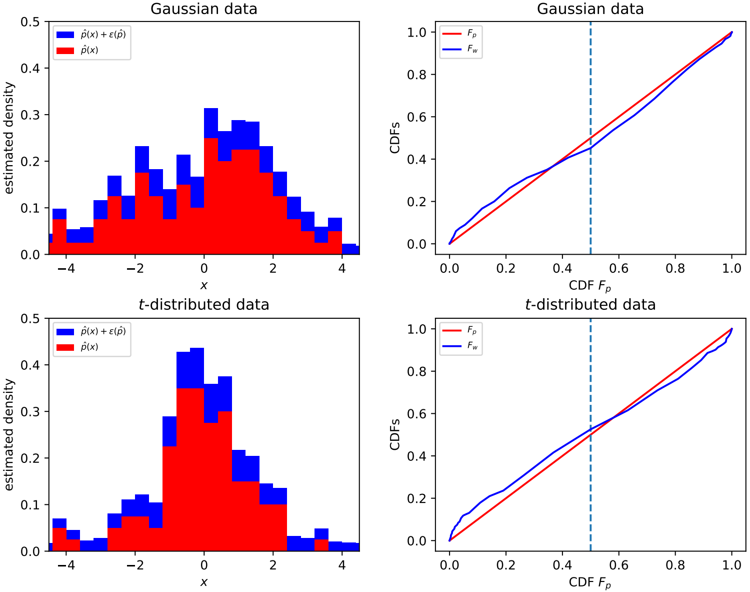


Interactive code at https://bit.ly/lml-pw-count-b



$T=100$, $\delta x=0.4$, estimates of $\hat{p}(x)$ in red, estimates with one standard error $\hat{p}(x)+\varepsilon\left[\hat{p}(x)\right]$ in blue.

Switch to sketch of experimental design

Conclusion

Classical interpretation of PW

- overestimation of low probability events
- underestimation of high probability events

$\hookrightarrow$ maladaptive irrational cognitive bias

Ergodicity Economics and PW

- inverse-S shape: neutral indicator of different models of uncertainty
- reported observations consistent with DM's extra uncertainty
- may arise from DM estimating probabilities over time

$\hookrightarrow$ Probability weighting is rational cautious behaviour under uncertainty over time

- testable prediction(s) $\rightarrow$ Let's run an experiment!
- Manuscript at `bit.ly/lml-pw-r1`
- Interactive code at `bit.ly/lml-pw-count-b`

Thank you for your attention!

I'm looking forward to the discussion
Comments & questions are very welcome, here or to

✉ m.kirstein@lml.org.uk
@nonergodicMark

Ergodicity Economics Online Conference
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BACK UP

Probability Weighting as an Estimation Issue

"It is important to distinguish overweighting, which refers to a property of decision weights, from the overestimation that is commonly found in the assessment of the probability of rare events. [...] In many real-life situations, overestimation and overweighting may both operate to increase the impact of rare events." (Kahneman and Tversky 1979, p. 281)

↪ distinguish between
- uncertainty estimation and
- "weighting"

we analyse the former and find very good agreement with the empirical inverse-S pattern

↪ How big is the residual "probability weighting" after accounting for uncertainty estimation?

Main Results

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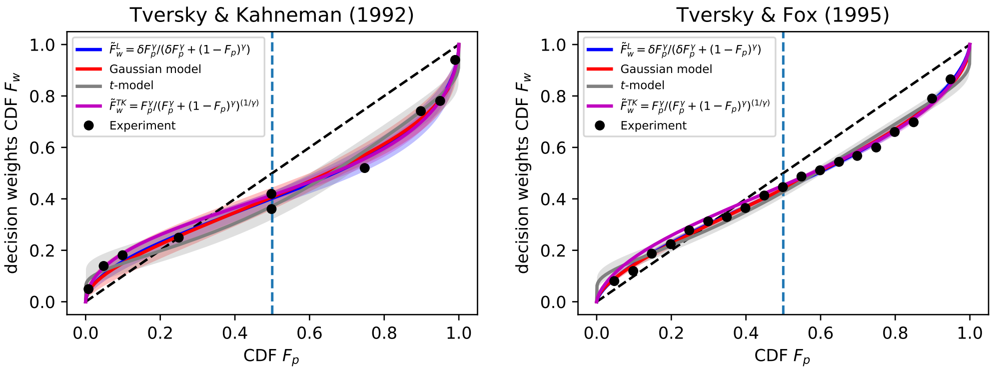
Estimation Error Explains 99% of Probability Weighting



- similar fits of Gaussian & t-distributed model

$\hookrightarrow$ How big is the residual "probability weighting" after accounting for estimation errors?

◂ Main Results

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