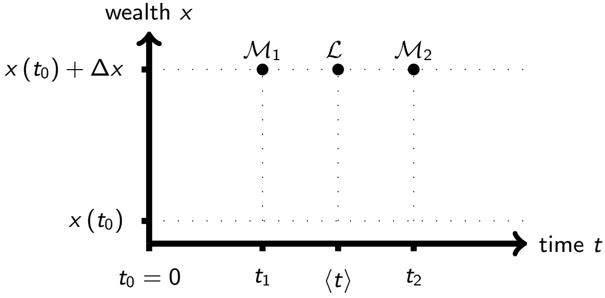
Mark Kirstein



# Commercial Break : Risk Preferences in Time Lotteries

Full paper at: bit.ly/TimeLotteries

- Known payment amount: $\Delta x$
- Two possible payment times
  earlier $t_1$
  later $t_2$ $(> t_1)$
- Probability $0 \leq p \leq 1$ to receive $\Delta x$ earlier at $t_1$
  $(1 - p$ later at $t_2)$
- Every time lottery $L$ defines a unique timed payment $\mathcal{L}$: $\Delta x$ is received with certainty at the expected payment time
  $\langle t \rangle = pt_1 + (1-p)\, t_2$




$\hookrightarrow$ Growth-optimal preferences vs EDUT preferences

# What are we weighting for?

## A mechanistic model for probability weighting

Ole Peters Alexander Adamou Yonatan Berman Mark Kirstein

✉ m.kirstein@lml.org.uk

@nonergodicMark

Full paper at bit.ly/lml-pw-r1

Live paper at bit.ly/lml-pw

Erich-Schneider-Seminar Economics Department, CAU Kiel

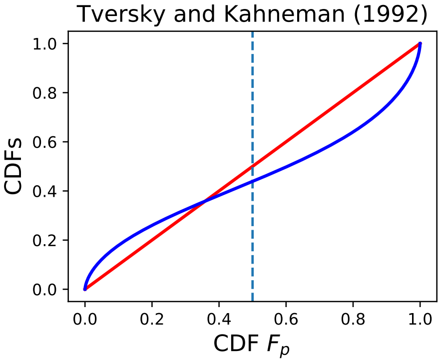
02 November 2020

Max Planck Institute for Mathematics in the Sciences

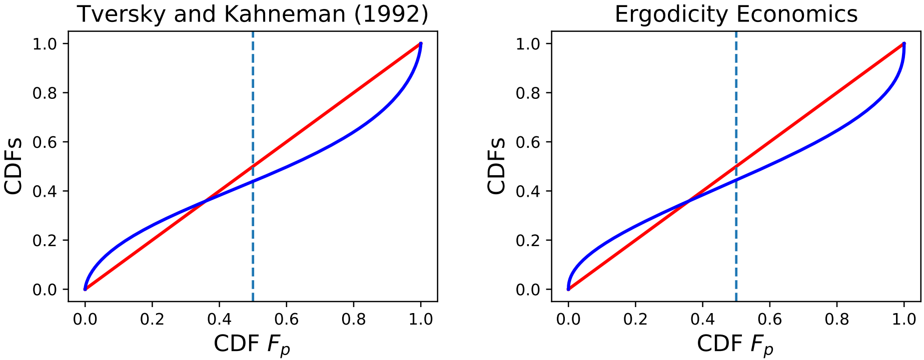
UNIVERSITÄT LEIPZIG Mathematisches Institut

LONDON MATHEMATICAL LABORATORY

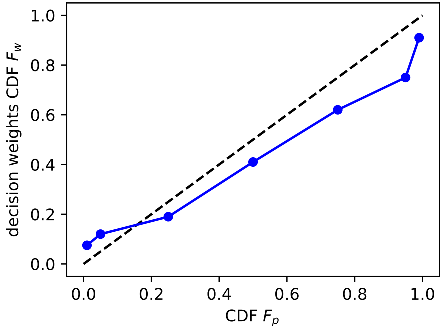
# Main result



# Main result


Tversky and Kahneman (1992)
Ergodicity Economics
CDFs
CDF $F_p$
0.0
0.2
0.4
0.6
0.8
1.0

# Definition of Probability Weighting (PW)




(Preston and Baratta 1948, p. 188, Fig. 1, relabelled axes)

"The existence [...] a scale of *psychological probability* and its functional relationship to the scale of *mathematical probability*" (p. 186)

- empirical pattern: inverse-S shape
- important component in behavioural economics (Cumulative Prospect Theory)

Classical interpretation of PW:

- maladaptive irrational cognitive bias

$\hookrightarrow$ How does this pattern emerge?

$\hookrightarrow$ Can we derive a functional form (rather than fit a function)?

# Related Literature

- no motivation of the functional form of weighting curve other than fit
- origin of PW preferences? no stable mappings (Stewart et al. 2015)
- experimental design is a key confounder $\rightarrow$ description-experience gap, *i.e.* less (even under-) overweighting in decisions-from-experience (Hertwig et al. 2004; Hertwig and Erev 2009)
- meta-analyses find all possible weighting curves (Ungemach et al. 2009; Wulff et al. 2018, Tab. 9)

Statistical explanations

- PW due to biased estimation (Fox and Hadar 2006)
- PW is optimal in sequential learning problems (Seo et al. 2019)
- PW heuristic as an approximate Bayesian solution for the inference problem (Martins 2006)

$\hookrightarrow$ reproducibility, context dependence, (mal)adaptive

# Setup

No *truth*, only difference in models of uncertainty

**Task:** model payout, $x$, of a gamble as a random variable.

**Disinterested Observer (DO)**

DO assigns PDF $p(x)$
$\hookrightarrow$ CDF $F_p(x)$

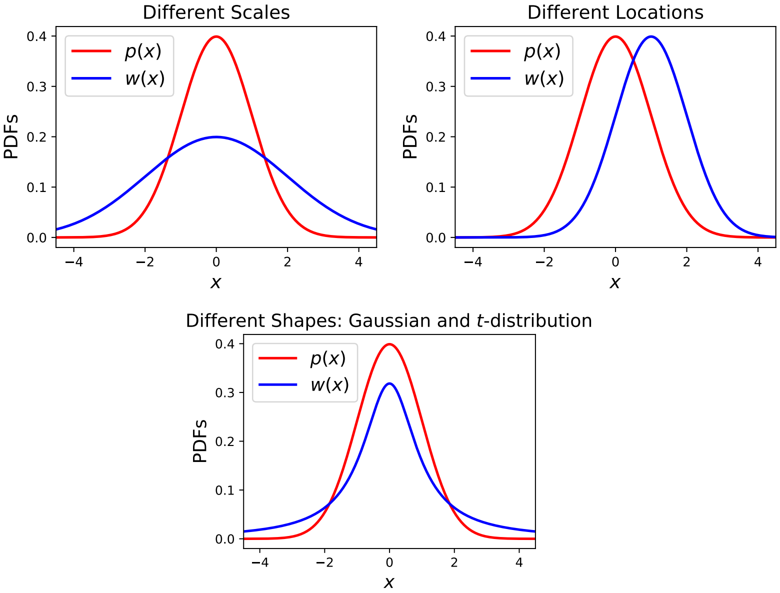
$$F_p(x) = \int_{-\infty}^{x} p(s)\,\mathrm{d}s$$

**Decision Maker (DM)**

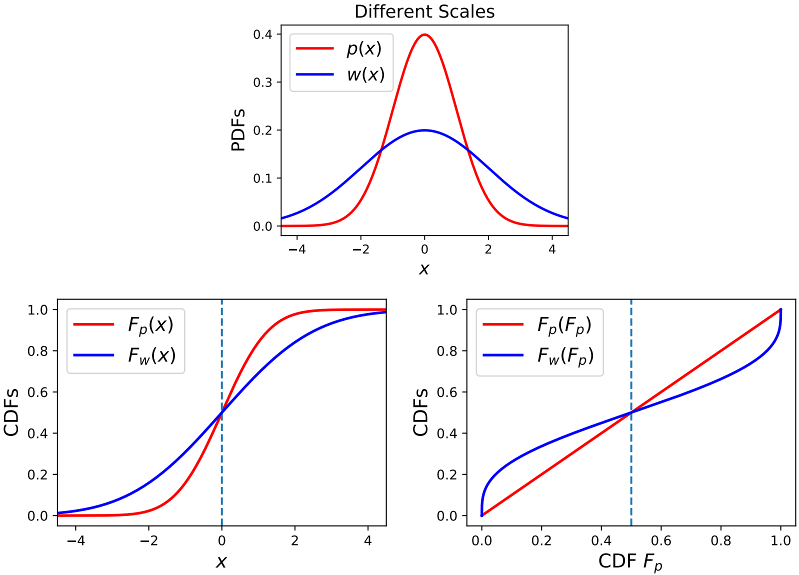
DM assigns different PDF $w(x)$
$\hookrightarrow$ CDF $F_w(x)$

$$F_w(x) = \int_{-\infty}^{x} w(s)\,\mathrm{d}s$$

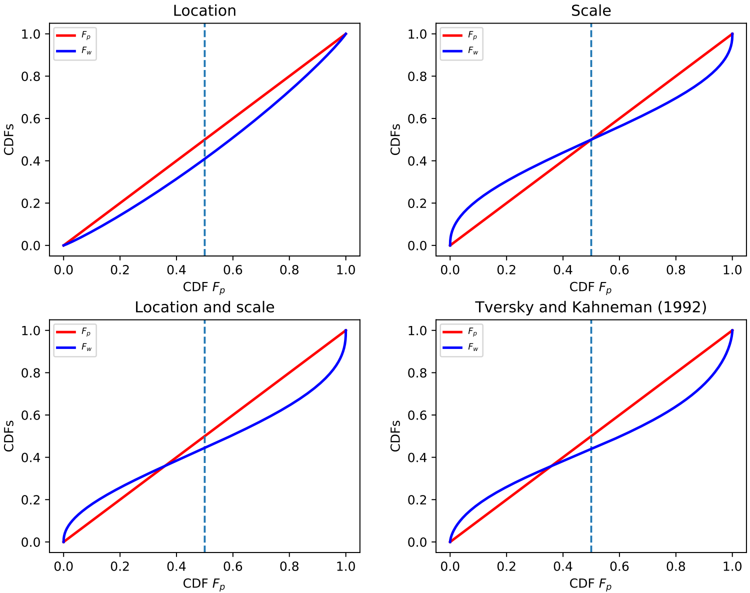
# Scales, Locations, Shapes



# Thought Experiment: DM assumes greater scale



# Asymmetric Inverse-S = diff. in uncertainty + diff. in locations



Numerically feasible for arbitrary distributions:

1. list values of DO's CDF, $F_p(x)$, at set $x_i$
2. list values of DM's CDF, $F_w(x)$, at same $x_i$
3. plot $F_w(x)$ *vs.* $F_p(x)$

# Functional form of the weighting function

Gaussian case with different scale:

$$w(p) = p^{\frac{1}{\alpha^2}} \underbrace{\frac{\left(2\pi\sigma^2\right)^{\frac{1-\alpha^2}{2\alpha^2}}}{\alpha}}_{\text{normalisation factor}} , \tag{1}$$

where

- DO's scale is $\sigma$, $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$
- DM's scale is $\alpha\sigma$, $X \sim \mathcal{N}\left(\mu, (\alpha\sigma)^2\right)$
- DO uses greater uncertainty $\alpha < 1 \rightarrow$ S shape
- DM uses greater uncertainty $\alpha > 1 \rightarrow$ inverse-S shape
- uncertainty measured by the standard deviation

# Functional Forms

Tversky and Kahneman (1992, $\gamma = 0.68$)

$$F_w^{TK}\left(F_p;\gamma\right) = \left(F_p\right)^{\gamma} \frac{1}{\left[\left(F_p\right)^{\gamma} + \left(1 - F_p\right)^{\gamma}\right]^{1/\gamma}} \tag{2}$$

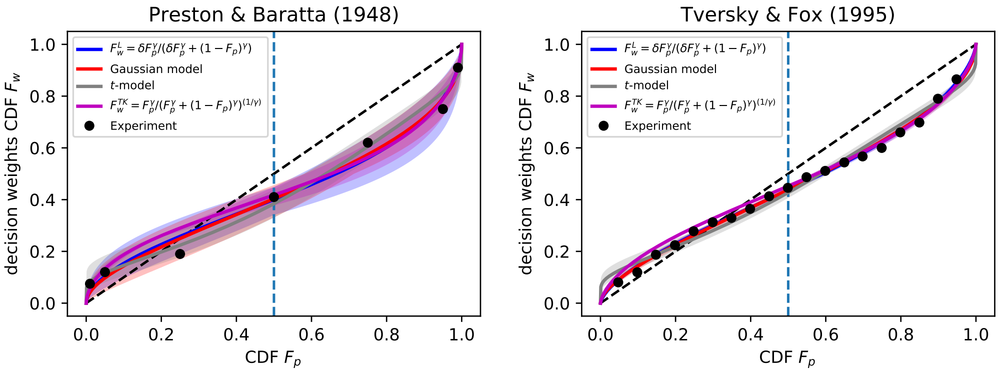
Lattimore et al. (1992)

$$F_w^{L}\left(F_p;\delta,\gamma\right) = \frac{\delta F_p^{\gamma}}{\delta F_p^{\gamma} + \left(1 - F_p\right)^{\gamma}} \tag{3}$$

Gaussian case with greater DM scale $\alpha\sigma$

$$w(p) = p^{\frac{1}{\alpha^2}} \frac{\left(2\pi\sigma^2\right)^{\frac{1-\alpha^2}{2\alpha^2}}}{\alpha}, \tag{4}$$

which is a power law in $p$ with a pre-factor to ensure normalisation


Preston & Baratta (1948)
Tversky & Fox (1995)
$F_w^L = \delta F_p^\gamma / (\delta F_p^\gamma + (1-F_p)^\gamma)$
Gaussian model
$t$-model
$F_w^{TK} = F_p^\gamma / (F_p^\gamma + (1-F_p)^\gamma)^{(1/\gamma)}$
Experiment
decision weights CDF $F_w$
CDF $F_p$
0.0
0.2
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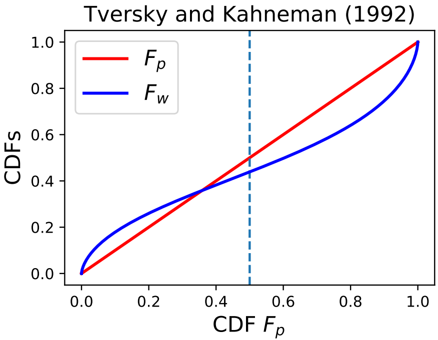
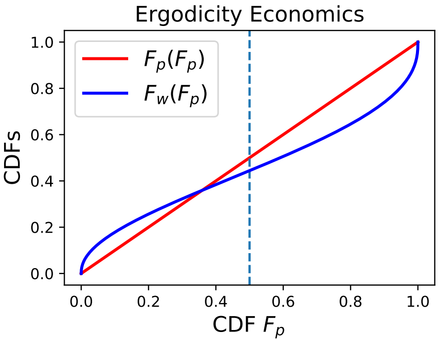
# Interim conclusion




- DM's greater scale gives inverse-S shape (unimodal distributions)
- difference in locations gives asymmetry
- reproduces observations of probability weighting

*Job done. Thank you for your attention ;)*

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# The Ergodicity Question

**Typical DO concern**

What happens on average to the ensemble of subjects?

$\neq$

**Typical DM concern**

What happens to me on average over time?

# Why DM's greater scale?

- DM has no control over experiment
- experiment may be unclear to DM
- DM may not trust DO
- ...

# Experiencing probabilities

- "probability" is polysemous
- natural frequencies "10 out of 100" vs 10% (Gigerenzer 1991, 2018; Hertwig and Gigerenzer 1999)
- probabilities are not observable
- probabilities encountered as
  - known frequencies in ensemble of experiments (DO)
  - frequencies estimated over time (DM)

$\hookrightarrow$ **estimates have uncertainties – cautious DM accounts for these**

# Estimating probabilities

**Rare Event**

- $p(x) = 0.001$
- $T = 100$ observations
- $\sim 99.5\%$ get 0 or 1 events
- $\hat{p}(x) = 0$ or $\hat{p}(x) = 0.01$

$\hookrightarrow$ $\hat{p}(x)$ off by 1000%

**Common Event**

- $p(x) = 0.5$
- $T = 100$ observations
- $\sim 99.5\%$ get between 35 and 65 events,
- $0.35 < \hat{p}(x) < 0.65$

$\hookrightarrow$ $\hat{p}(x)$ off by 30%

$\hookrightarrow$ small $p(x)$, small count $\rightarrow$ big uncertainty

# Scaling of uncertainty: small count $\rightarrow$ big uncertainty

1. scaling of counts $n(x)$:

$$n(x) \sim p(x)\delta x T \quad \iff \quad p(x) \sim {}^{n(x)}/_{T\delta x}$$

2. uncertainty in Poisson-distributed counts $\sim \sqrt{n(x)}$
3. estimate of the asymptotic probability density

$$p(x) \approx \underbrace{\frac{n(x)}{T\delta x}}_{\text{estimate}} \pm \underbrace{\frac{\sqrt{n(x)}}{T\delta x}}_{\text{relative uncertainty}} \tag{5}$$

4. express the uncertainty in terms of the estimate itself

$$\varepsilon\left[\hat{p}(x)\right] \equiv \frac{\sqrt{n(x)}}{T\delta x} = \sqrt{\frac{\hat{p}(x)}{T\delta x}} \tag{6}$$

5. standard error $\lim_{p(x)\rightarrow 0} \sqrt{\hat{p}(x)/T\delta x}$ in $\hat{p}$ shrinks
6. relative error in the estimate $\lim_{p(x)\rightarrow 0} 1/\sqrt{\hat{p}(x)T\delta x}$ grows,
7. $\lim_{T\rightarrow\infty} \hat{p} \rightarrow p$

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# Cautionary principle : DMs don't like surprises

To avoid surprises, DMs add estimation uncertainty $\varepsilon\left[\hat{p}(x)\right]$ to every estimated probability, then normalize, s.t.

$$w(x)=\frac{\hat{p}(x)+\varepsilon\left[\hat{p}(x)\right]}{\int\left(\hat{p}(s)+\varepsilon\left[\hat{p}(s)\right]\right)ds} \tag{7}$$

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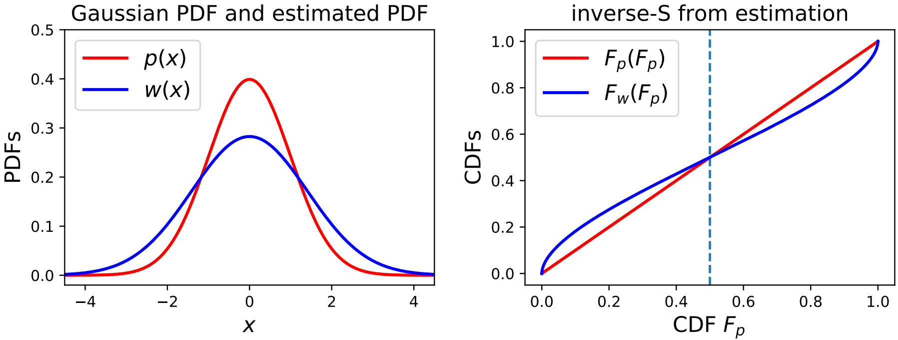
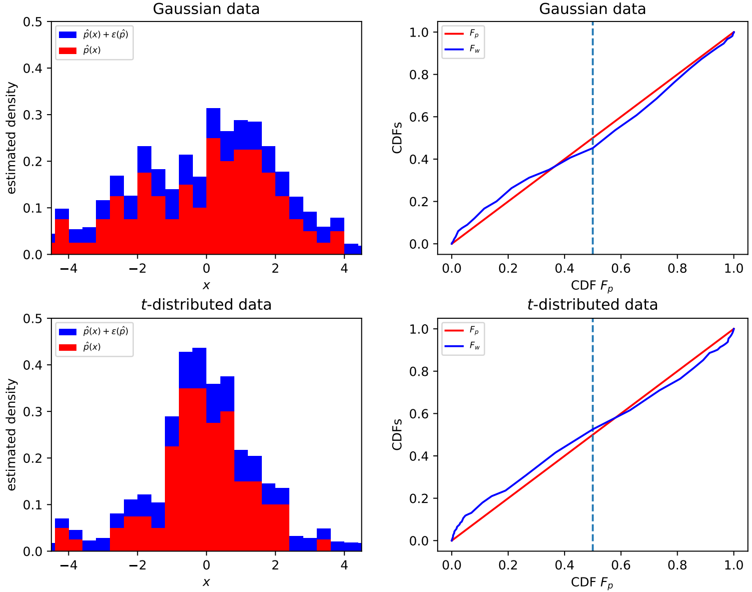



Interactive code at `https://bit.ly/lml-pw-count-b`




$T = 100$, $\delta x = 0.4$, estimates of $\hat{p}(x)$ in red, estimates with one standard error $\hat{p}(x) + \varepsilon\left[\hat{p}(x)\right]$ in blue.

# Conclusion

Classical interpretation of PW

- overestimation of low probability events
- underestimation of high probability events

$\hookrightarrow$ maladaptive irrational cognitive bias

Ergodicity Economics and PW

- inverse-S shape: neutral indicator of different models of the world
- reported observations consistent with DM's extra uncertainty
- may arise from DM estimating probabilities over time

$\hookrightarrow$ Probability weighting is rational cautious behaviour under uncertainty over time

- testable prediction $\rightarrow$ Let's run an experiment!
- Manuscript at `bit.ly/lml-pw-r1`
- Interactive code at `bit.ly/lml-pw-count-b`

# Thank you for your attention!

I'm looking forward to the discussion
Comments & questions are very welcome, here or to

✉ m.kirstein@lml.org.uk
@nonergodicMark

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# BACK UP

# Probability Weighting as an Estimation Issue

"It is important to distinguish overweighting, which refers to a property of decision weights, from the overestimation that is commonly found in the assessment of the probability of rare events. [...] In many real-life situations, overestimation and overweighting may both operate to increase the impact of rare events." (Kahneman and Tversky 1979, p. 281)

↪ distinguish between

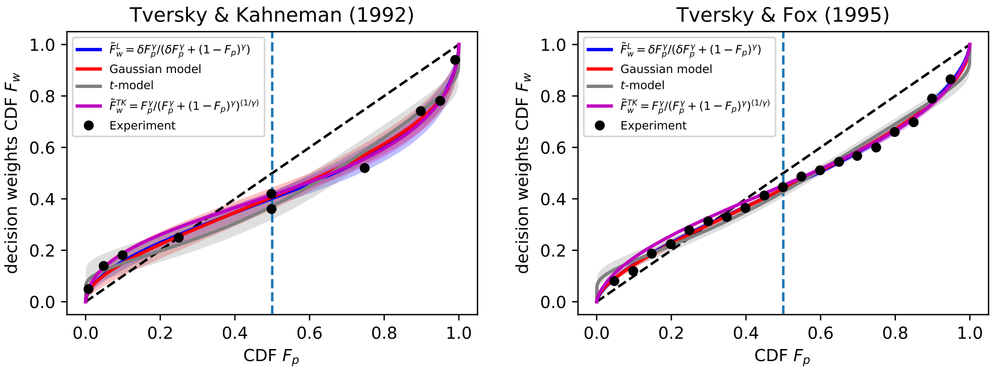
- uncertainty estimation and
- "weighting"

we analyse the former and find very good agreement with the empirical inverse-S pattern

↪ How big is the residual "probability weighting" after accounting for uncertainty estimation?

◂ Main Results

# Estimation Error Explains 99% of Probability Weighting




- similar fits of Gaussian & $t$-distributed model

$\hookrightarrow$ How big is the residual "probability weighting" after accounting for estimation errors?

◂ Main Results

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