15.1 Notation

Logic

Table 15.1.1: Logic.

Symbol	Meaning
$\land$	Conjunction, logical AND
$\lor$	Disjunction, logical OR
$\neg$	Negation, logical NOT
$\oplus$	Exclusive OR
$\rightarrow$	Conditional
$\leftrightarrow$	Biconditional
$\equiv$	Logical equivalence
$\forall$	Universal quantifier, "for all"
$\exists$	Existential quantifier, "there exists"
$\therefore$	Therefore

Proofs

Table 15.1.2: Proofs.

Symbol	Meaning

■	End of proof

Sets

Table 15.1.3: Sets.

Symbol	Meaning
$A = \{\ldots, \ldots\}$	Roster notation of set A
$\emptyset, \{\}$	Empty set
$\|A\|$	Cardinality of set A
$\in$	Element of a set
$\notin$	Not an element of a set
$\mathbb{N}$	The set of natural numbers $\{0, 1, 2, \ldots\}$
$\mathbb{Z}$	The set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Z}^+$	The set of positive integers $\{1, 2, \ldots\}$
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of positive real numbers
$A = \{x \in S : P(x)\}$	Set builder notation of A of all x in S such that $P(x)$ is true
U	Universal set
$\subseteq$	Subset
$\subset$	Proper subset
$P(A)$	Power set of set A

$\cap$	Intersection
$\cup$	Union
$\overline{}$	Complement
$\bigcap_{i=1}^{n} A_i$	Intersection of sets $A_1, A_2, \ldots$, $A_1 \cap A_2 \cap \ldots$
$\bigcup_{i=1}^{n} A_i$	Union of sets $A_1, A_2, \ldots$, $A_1 \cup A_2 \cup \ldots$
$-$	Set difference
$\oplus$	Symmetric difference
$\times$	Cartesian product
λ	Empty string

Functions

Table 15.1.4: Functions.

Symbol	Meaning
$f : x \to Y$	Function f that maps domain X to target Y
$\lfloor x \rfloor$	Floor of x, largest integer less than or equal to x
$\lceil x \rceil$	Ceiling of x, smallest integer greater than or equal to x
f^{-1}	Inverse function of function f
$f \circ g$	Composition of function f with function g
I_A	Identity function on A

Boolean algebra

Table 15.1.5: Boolean algebra.

Symbol	Meaning
$\cdot$	Boolean multiplication
$+$	Boolean addition
$-$	Boolean complement
$\equiv$	Boolean equivalence
$\uparrow$	Boolean NAND
$\downarrow$	Boolean NOR

Relations and digraphs

Table 15.1.6: Relations and digraphs.

Symbol	Meaning
aRb	Binary relation R between $a \in A$ and $b \in B$
$\langle v_0, v_1, \ldots, v_l \rangle$	Walk from vertex v_0 to vertex v_l
$S \circ R$	Composition of relation R on relation S
G^k	kth graph power of G
G^+	Transitive closure of graph G
$a \preceq b$	aRb such that R is a partial order
$a \prec b$	aRb such that R is a strict order
$a \sim b$	aRb such that R is an equivalence relation
$[a]$	Equivalence class of a

Computation

Table 15.1.7: Algorithms and analysis.

Symbol	Meaning
$:=$	Assignment
$f = O(g)$	Function f is Oh of function g
$f = \Omega(g)$	Function f is Omega of function g
$f = \Theta(g)$	Function f is Theta of function g

Table 15.1.8: Finite state machines.

Symbol	Meaning
Q	Finite set of states
$q_0 \in Q$	q_0 is the start state.
I	Finite set of input actions
O	Finite set of output responses
$\delta : Q \times I \to Q \times O$	Transition function

Table 15.1.9: Turing machines.

Symbol	Meaning
Q	Finite set of states
Γ	Finite set of tape symbols
$\Sigma \subset \Gamma$	A subset of the tape symbols are input symbols
$q_0 \in Q$	q_0 is the start state
$q_{\text{acc}} \in Q$	q_{acc} is the accept state
$q_{\text{rej}} \in Q$	q_{rej} is the reject state
$\delta : (Q - \{q_{\text{acc}}, q_{\text{rej}}\}) \times \Gamma \to Q \times \Gamma \times \{L, R\}$	Transition function

Induction and recursion

Table 15.1.10: Notation for sequences and summations.

Symbol	Meaning
$\{a_k\} = a_m, a_{m+1}, \ldots, a_n$	Sequence with index k starting at initial index m and ending at final index n
$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$	Summation with index k starting at initial index m and ending at final index n
$!$	Factorial

Integer properties

Table 15.1.11: Notation for integer properties.

Symbol	Meaning
$x \mid y$	x divides y
$x \nmid y$	x does not divide y
$q = n \text{ div } d$ $r = n \text{ mod } d$	For $n = qd + r$, div returns the quotient q when n is divided by d, and mod returns the remainder r when n is divided by d
$\mathbb{Z}_m$	The ring of integers $\{0, 1, 2, \ldots, m-1\}$
$\equiv$	Modular congruence
$(n)_b$	Base b representation of n

Counting

Table 15.1.12: Counting.

Symbol	Meaning
$P(n, r)$	Number of r-permutations from a set with n elements
$\binom{n}{r}, C(n, r)$	Number of r-subsets from a set with n elements

Discrete probability

Table 15.1.13: Discrete probability.

Symbol	Meaning
$p(E)$	Probability of event E
$p(E\|F)$	Probability of event E given event F
$X(S)$	Random variable with range S
$E[X]$	Expected value of random variable X

Graphs and trees

Table 15.1.14: Graphs and trees.

Symbol	Meaning
V_G	Vertex set of graph G
E_G	Edge set of graph G
$\{\ldots, \ldots\}$	Undirected edge

$(\ldots,\ldots)$	Directed edge
K_n	Complete graph on n vertices
C_n	Cycle on n vertices
Q_n	n-dimensional hypercube
$K_{n,m}$	Bipartite graph from n vertices to m vertices
$\kappa(G)$	Vertex connectivity of graph G
$\lambda(G)$	Edge connectivity of graph G
$\delta(G)$	Minimum degree of graph G
$\Delta(G)$	Maximum degree of graph G
$X(G)$	Chromatic number of graph G
$\omega(G)$	Clique number of graph G

15.2 Set notation and quantified logical statements

Using set notation to specify the domain of variables in quantified statements.

A ***predicate*** is a logical statement with a truth value that depends on one or more variables. For example, let $P(x)$ denote the predicate that says "x is a prime number." The truth value of $P(x)$ depends on the value of the variable x. When a predicate is used in a logical expression, the domain of each variable must be specified. The ***domain*** of a variable is the set of possible values for the variable. A reasonable domain for the predicate "x is a prime number" is the set of all positive integers.

The domain of each variable used in a logical expression can be defined before the logical expression. Ex:

The domain for variable x is $\mathbb{Z}^+$.
$\exists x P(x)$

An alternative notation that is equivalent to the example above uses set notation inside the logical expression to denote the domain of a variable.

$\exists x \in \mathbb{Z}^+, P(x)$

This material refers to the notation in which the domain of a variable is defined inside the logical expression as defining the domain of a variable ***inline***. The set used in an inline definition of a domain must be explicitly defined beforehand or must use a standard symbol for a mathematical set, such as $\mathbb{Z}^+$ for the set of positive integers or $\mathbb{R}$ for the set of real numbers.

PARTICIPATION ACTIVITY 15.2.1: Matching equivalent statements.

How to use this tool

$\exists x \in \mathbb{Z}^+, (x^2 < x)$ $\forall x \in \mathbb{R}^+, (x^2 > x)$ $\forall x \in \mathbb{Z}, (x^2 = x)$

$\exists x \in \mathbb{R}, (x^2 < x)$

The square of every positive real number is greater than that number.

There exists a positive integer whose square is less than that number.

The domain of variable x is the set of real numbers.
$\exists x(x^2 < x)$

The domain of variable x is the set of integers.
$\forall x(x^2 = x)$

Reset

PARTICIPATION ACTIVITY 15.2.2: Select the equivalent statement.

Select the option that is equivalent to the given expression. The predicate $R(x)$ is defined to mean that person x was given a raise.

1) The domain for variable x is the set of employees at a company.
$\forall x R(x)$

- Let C be the set of employees at a company.
$\forall x R(x)$
- $\forall x \in C, R(x)$
- Let C be the set of employees at a company.
$\forall x \in C, R(x)$
- Let C be the set of employees at a company.
$x \in C, R(x)$

2) The domain for variable x is the set of employees at a company.
$\exists x \notin R(x)$

- Let C be the set of employees at a company.
$x \in C, \notin R(x)$
- $\exists x \in C, \notin R(x)$
- Let C be the set of employees at a company.
$\exists x \notin R(x)$
- Let C be the set of employees at a company.
$\exists x \in C, \notin R(x)$

Inline domain definitions with more than one variable

The domains of the variables in a logical expression can be defined inline, even when there is more than one variable and those variables have different domains. In the example below, variables x and y have different domains.

The domain for variable x is $\mathbb{Z}^+$.
The domain for variable y is $\mathbb{R}^+$.

$\forall x \exists y \left(y = \frac{1}{x}\right)$

The logical expression below is equivalent to the example above.

$\forall x \in \mathbb{Z}^+, \exists y \in \mathbb{R}^+, \left(y = \frac{1}{x}\right)$

The domains for two variables can also be the same set as in the example below.

$\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+, \left(y = \frac{1}{x}\right)$

PARTICIPATION ACTIVITY 15.2.3: Matching equivalent statements.

How to use this tool

$\forall x \in \mathbb{Z}, \exists y \in \mathbb{R}, (y^2 + x = 2)$

$\forall y \in \mathbb{Z}, \exists x \in \mathbb{R}, (y^2 + x = 2)$

$\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, (y^2 + x = 2)$

$\forall y \in \mathbb{R}, \exists x \in \mathbb{Z}, (y^2 + x = 2)$

The domain for variable x is $\mathbb{Z}$.
The domain for variable y is $\mathbb{R}$.
$\forall x \exists y (y^2 + x = 2)$

The domain for variable x is $\mathbb{R}$.
The domain for variable y is $\mathbb{Z}$.
$\forall x \exists y (y^2 + x = 2)$

The domain for variable y is $\mathbb{Z}$.
The domain for variable x is $\mathbb{R}$.

$\forall y \exists x(y^2 + x = 2)$

The domain for variable y is $\mathbb{R}$.
The domain for variable x is $\mathbb{Z}$.
$\forall y \exists x(y^2 + x = 2)$

Reset

PARTICIPATION ACTIVITY | 15.2.4: Inline domain definitions with multiple variables.

A denotes the set of people who work for Company 1. B denotes the set of people who work for Company 2. The predicate $K(x, y)$ means that person x knows person y. Select the logical expression that is equivalent to the given statement.

1) Everyone at Company 1 knows someone at Company 2.

- ○ $\forall x \in A, \exists y \in A, K(x, y)$
- ○ $\forall x \in A, \exists y \in B, K(x, y)$
- ○ $\forall x \in B, \exists y \in A, K(x, y)$

2) Two people at Company 1 do not know each other.

- ○ $\exists x \in A, \exists y \in A, (\notin K(x, y) \wedge \notin K(y, x))$
- ○ $\exists x \in A, \exists y \in B, (x \neq y \wedge \notin K(x, y) \wedge \notin K(y, x))$
- ○ $\exists x \in A, \exists y \in A, (x \neq y \wedge \notin K(x, y) \wedge \notin K(y, x))$

Additional exercises

EXERCISE | 15.2.1: Quantified expressions with inline domain definitions: mathematical examples.

Indicate whether each statement is true or false. Justify the answer.

(a) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, (x^2 = -y)$

(b) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}^+, (x^2 = -y)$

(c) $\forall x \in \mathbb{Z}^+, \exists y \in \mathbb{R}, (x^2 = y)$

(d) $\forall x \in \mathbb{Z}^+, \exists y \in \mathbb{Z}, \left(\frac{1}{x}\right) = y$

(e) $\exists x \in \mathbb{Z}^+, \forall y \in \mathbb{R}, (y^2 \geq x)$

(f) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, ((x \neq y) \to x^2 + y^2 > 0)$

(g) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{R}, ((y \geq x) \to y^2 \geq 2y)$

EXERCISE 15.2.2: Quantified expressions with inline domain definitions: English to logic.

$T =$ set of students in the third grade at a school.
$F =$ set of students in the fourth grade at a school.

Define the following predicates.

$O(x, y)$: x is older than y.
$S(x, y)$: x is a sibling of y.

Translate each English sentence into a quantified logical expression. Define the domain of each variable inline.

(a) Sam is older than every student in the third grade.

(b) Every fourth grader is older than every third grader.

(c) There is a third grader who is older than at least one fourth grader.

(d) There is a fourth grader who is older than every third grader.

(e) Every fourth grader is older than at least one third grader.

(f) There is a third grader who is a sibling of another third grader.

(g) There is a third grader who is a sibling of a fourth grader.

(h) There is a third grader who does not have a sibling in the fourth grade.