UNIT-II
Integral Calculus

UNIT - II: Integral Calculus **(12 periods)**

Integration as inverse operation of differentiation. Simple integration by substitution, by parts and by partial fractions (for linear factors only). Introduction to definite integration. Use of formulae $\int_0^{\frac{\pi}{2}} \sin^n x dx$, $\int_0^{\frac{\pi}{2}} \cos^n x dx$, $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$ for solving problems ,where m and n are positive integers. Applications of integration for (i). Simple problems on evaluation of area bounded by a curve and axes. (ii). calculation of volume of a solid formed by revolution of an area about axes. (Simple problems).

Q.70:- परवलय $y^2 = 4ax$ तथा उसके नाभिलम्ब द्वारा कटी लम्बाई ज्ञात करें।

Find the length cut off by the parabola $y^2 = 4ax$ and its latus rectum.

$y^2 = 4ax$

d. w. r. to y

$2y = 4a \cdot \frac{dx}{dy}$

$\frac{dx}{dy} = \frac{2y}{4a}$

$\frac{dx}{dy} = \frac{y}{2a}$ —— (1)

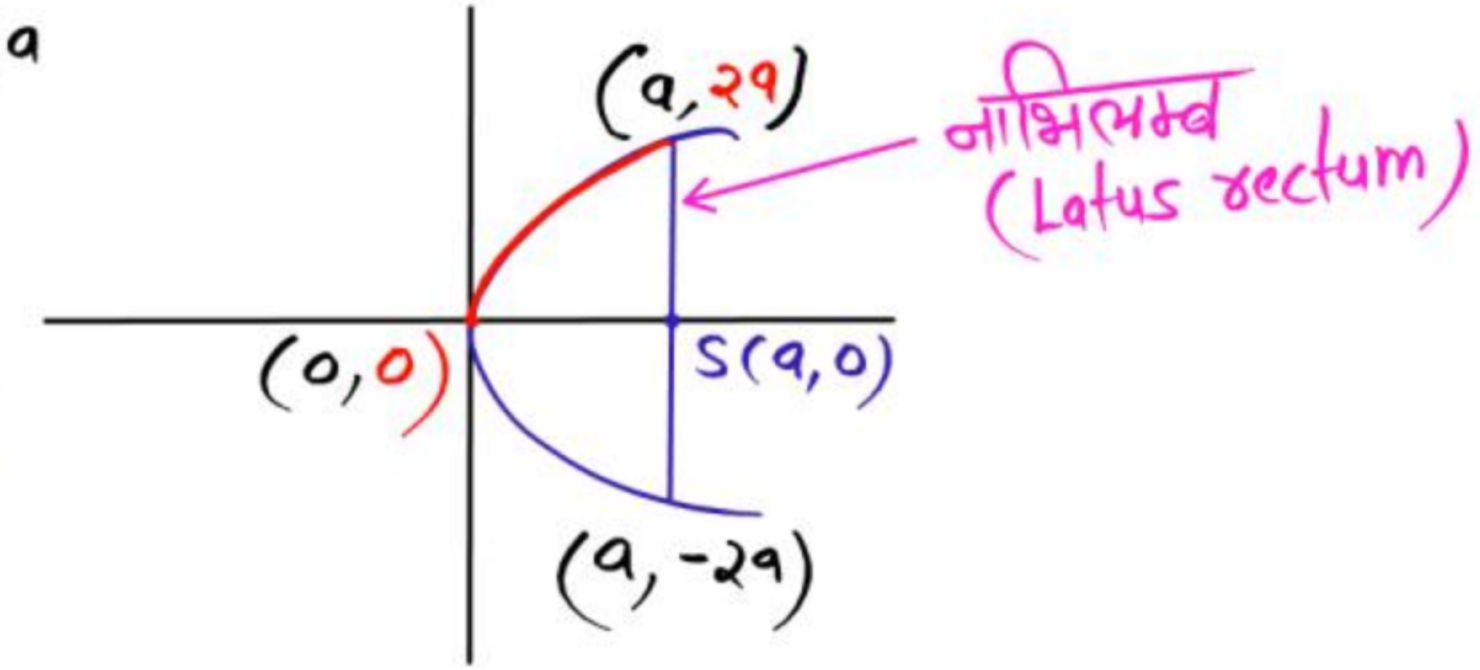
Limits $y = 0$ तथा $y = 2a$

Length

$= 2 \times \int_0^{2a} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy$

$S = 2 \times \int_0^{2a} \sqrt{1 + \left(\frac{y}{2a}\right)^2} \cdot dy$

$= 2 \times \int_0^{2a} \sqrt{\frac{4a^2 + y^2}{4a^2}} \cdot dy$



$$= 2 \times \int_0^{2a} \frac{\sqrt{4a^2+y^2}}{2a} \cdot dy$$

$$= \frac{\cancel{2}}{\cancel{2}a} \int_0^{2a} \sqrt{(2a)^2+y^2} \cdot dy \qquad \int \sqrt{a^2+x^2} \cdot dx = \frac{1}{2}\left[x \cdot \sqrt{a^2+x^2} + a^2 \cdot \log\left(x+\sqrt{a^2+x^2}\right)\right]$$

$$= \frac{1}{a} \times \frac{1}{2}\left[y \cdot \sqrt{4a^2+y^2} + (2a)^2 \cdot \log\left(y+\sqrt{4a^2+y^2}\right)\right]_0^{2a}$$

$$= \frac{1}{2a}\left[2a \cdot \sqrt{4a^2+(2a)^2} + (2a)^2 \cdot \log\left(2a+\sqrt{4a^2+(2a)^2}\right) - \left(0+4a^2 \cdot \log\left(0+2a\right)\right)\right]$$

$$= \frac{1}{2a}\left[2a \cdot \sqrt{8a^2} + 4a^2 \cdot \log\left(2a+\sqrt{8a^2}\right) - 4a^2 \cdot \log 2a\right]$$

$$= 2\sqrt{2}a + 2a \cdot \log\left(2a+2\sqrt{2}a\right) - 2a \cdot \log 2a$$

$$= 2a\left[\sqrt{2} + \log(2a + 2\sqrt{2}a) - \log 2a\right]$$

$$= 2a\left[\sqrt{2} + \log\left(\frac{2a + 2\sqrt{2}a}{2a}\right)\right]$$

$$= 2a\left[\sqrt{2} + \log\left(\frac{\cancel{2a}(1+\sqrt{2})}{\cancel{2a}}\right)\right]$$

$$= 2a\left(\sqrt{2} + \log(1+\sqrt{2})\right) \quad \underline{\underline{\text{Ans}}}$$

Q.70:- परवलय $y^2 = 4ax$ तथा उसके नाभिलम्ब द्वारा कटी लम्बाई ज्ञात करें।

Find the length cut off by the parabola $y^2 = 4ax$ and its latus rectum.

$y^2 = 4ax$

d.w.r. to x

$2y \cdot \frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{4a}{2y}$

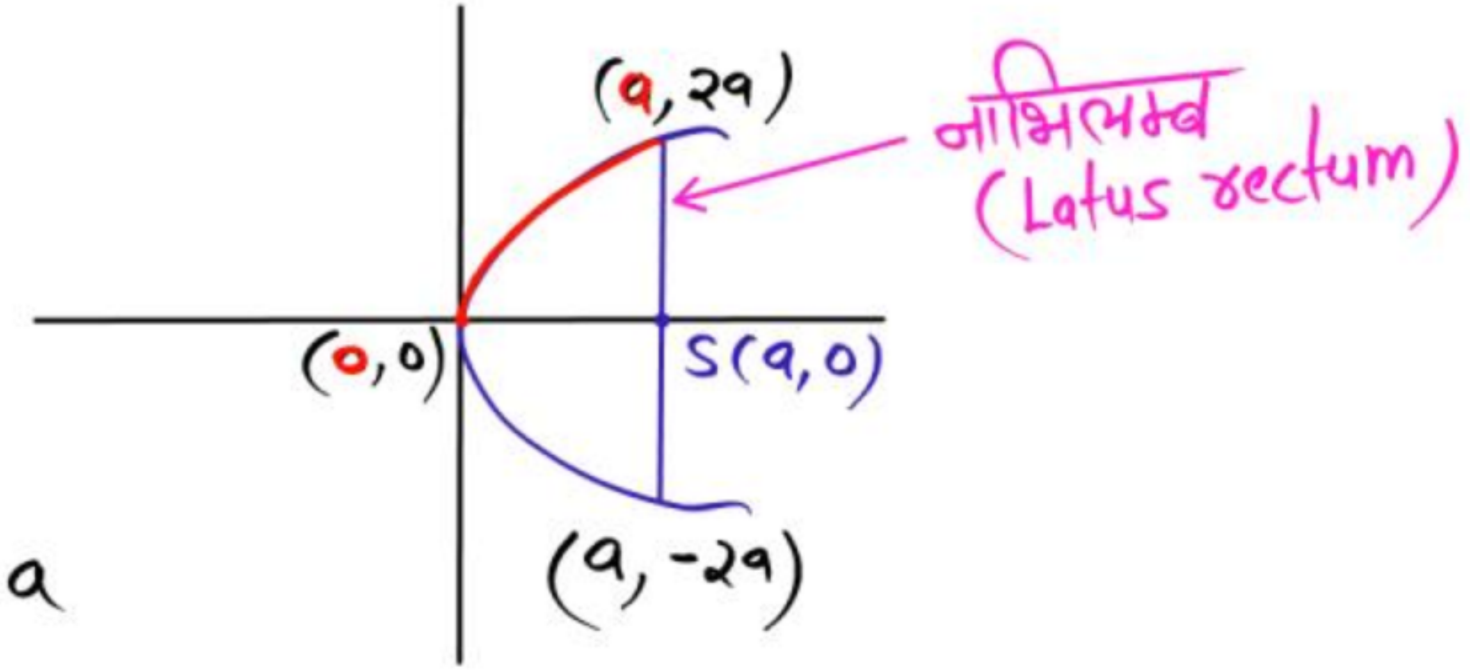
$\frac{dy}{dx} = \frac{2a}{\sqrt{4ax}}$

$\frac{dy}{dx} = \frac{\cancel{2a}}{\cancel{2} \cdot \cancel{\sqrt{a}} \cdot \sqrt{x}}$

$\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}}$ —— (1)

Limits $x = 0$ तथा $x = a$

$$\text{Length} = 2 \times \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$



$$S = 2x\int_0^a \sqrt{1+\left(\frac{\sqrt{a}}{\sqrt{x}}\right)^2}\cdot dx$$

$$S = 2x\int_0^a \sqrt{1+\frac{a}{x}}\cdot dx$$

$$S = 2x\int_0^a \sqrt{\frac{x+a}{x}}\, dx$$

Hints $\div$ $(x = a\tan^2\theta)$

Q.71:- समाकलन विधि से वृत्त $x^2 + y^2 = a^2$ की परिधि ज्ञात करें।

Find the circumference of the circle $x^2 + y^2 = a^2$ by integration method.

$$x^2+y^2=a^2$$

d. w. r. to x

$$2x + 2y\cdot\frac{dy}{dx} = 0$$

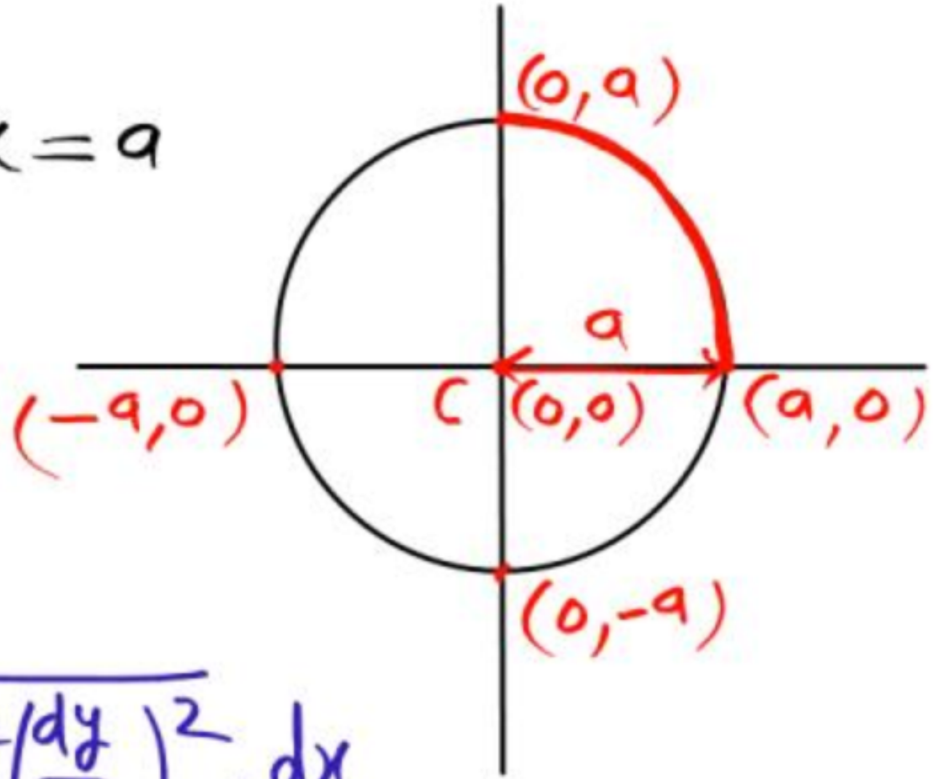
$$2y\cdot\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{\cancel{2}x}{\cancel{2}y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Limits

$x = 0$ तथा $x = a$



Length

$$S = 4 \times \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$$

$$S = 4\int_0^a \sqrt{1+\left(\frac{-x}{y}\right)^2}\cdot dx$$

$$= 4\int_0^a \sqrt{1+\frac{x^2}{y^2}}\cdot dx \qquad (y^2 = a^2 - x^2)$$

$$= 4\int_0^a \sqrt{1+\frac{x^2}{a^2-x^2}}\, dx$$

$$= 4\int_0^a \sqrt{\frac{a^2-\cancel{x^2}+\cancel{x^2}}{a^2-x^2}}\cdot dx$$

$$= 4\int_0^a \frac{a}{\sqrt{a^2-x^2}}\, dx$$

$$S = 4a\int_0^a \frac{1}{\sqrt{a^2-x^2}}\cdot dx$$

$$S = 4a\cdot\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4a\left[\sin^{-1}\left(\frac{a}{a}\right) - \sin^{-1}(0)\right]$$

$$= 4a\left[\sin^{-1}(1) - 0\right]$$

$$= 4a\left[\frac{\pi}{2}\right]$$

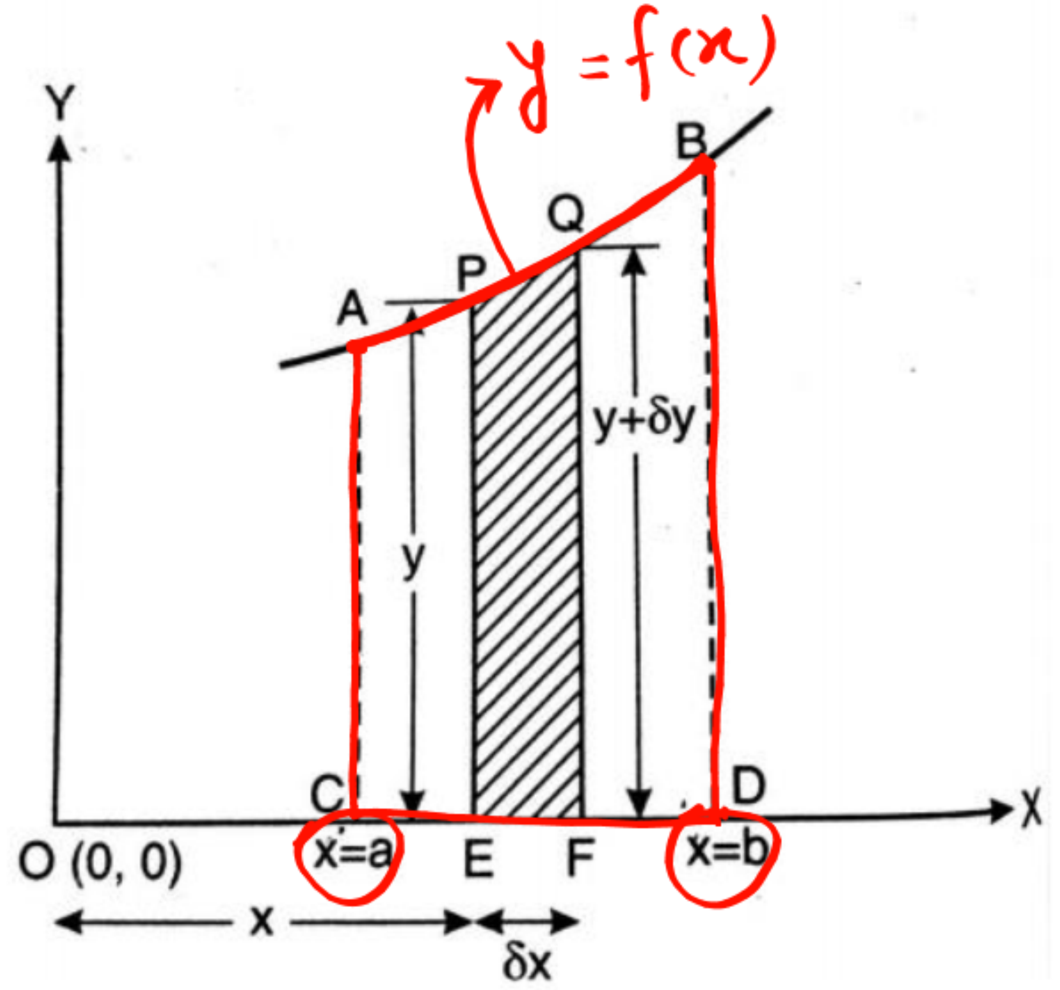
$$= \underline{\underline{2a\pi}} \text{ } \underline{\underline{\text{Ans}}}$$

समाकलन द्वारा क्षेत्रफल ज्ञात करना (To find the Area by Integration)

- किसी वक्र $y = f(x)$ के लिये, सीमाओं $x = a, x = b$ तथा X-अक्ष द्वारा घिरे क्षेत्रफल को ज्ञात करना
  For a curve $y = f(x)$, find the area bounded by the limits $x = a$, $x = b$ and the X-axis

$$\boxed{\text{क्षेत्रफल } ACDBA = A = \int_{x=a}^{x=b} f(x)\,dx = \int_{x=a}^{x=b} y.dx}$$

$$\boxed{\text{Area } (A) = \int_{x=a}^{b} y.\,dx}$$

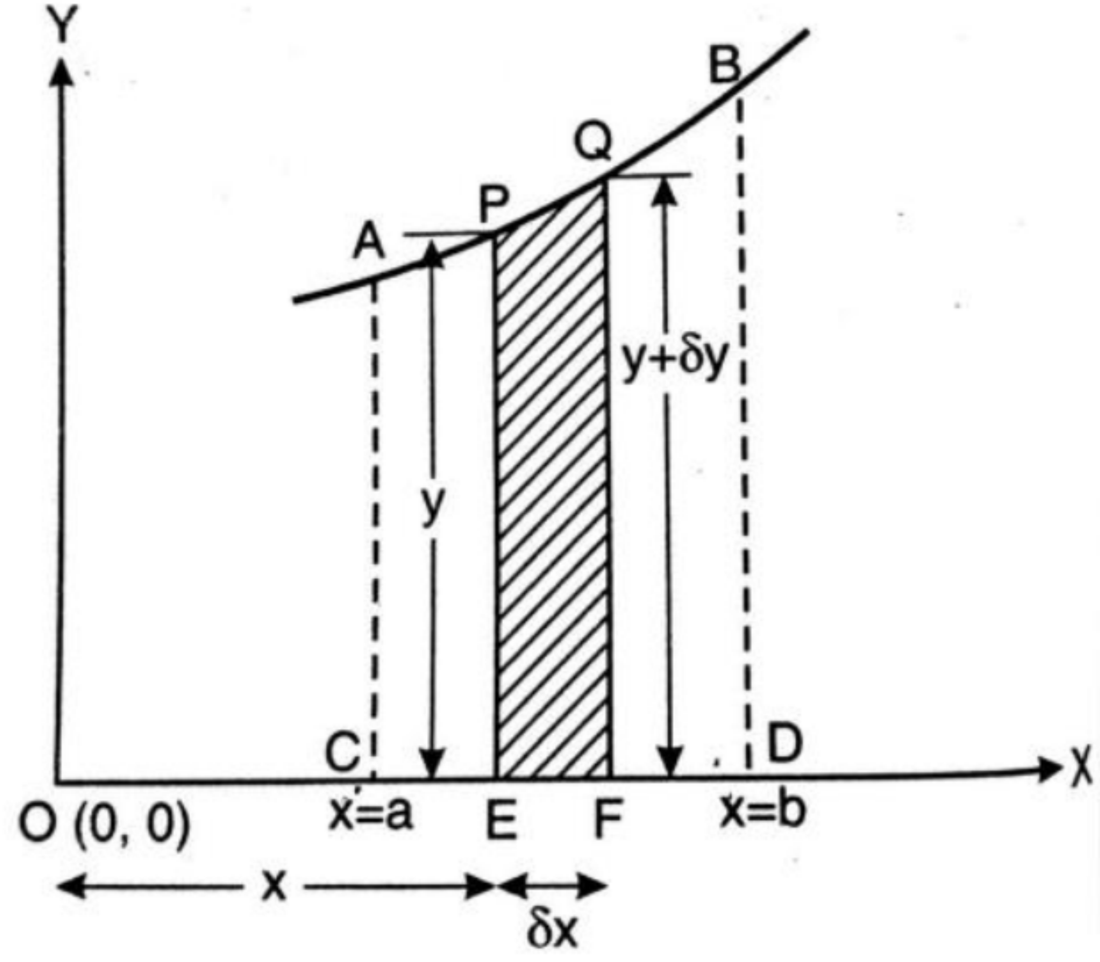


- इसी प्रकार, वक्र x = f (y), (y), सीमाओं y = c, y = d तथा अक्ष द्वारा घिरे क्षेत्रफल को ज्ञात किया जाता है।

Similarly, the area bounded by the curve x = f (y), (y), the boundaries y = c, y = d and the axis is found.

$$\text{क्षेत्रफल } 'A' = \int_{y=c}^{y=d} f(y)\,dy = \int_{y=c}^{y=d} x.dy$$

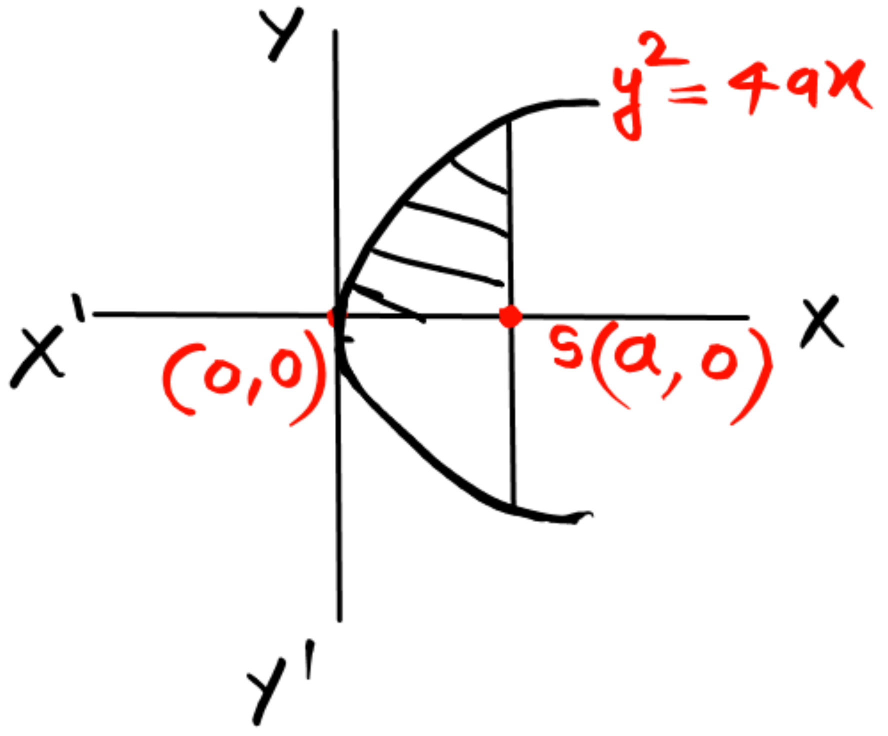
$$\text{Area}(A) = \int_{y=c}^{d} x.dy$$



$\frac{1}{2}+\frac{3}{2}=\frac{4}{2}=2$

Q.72:- परवलय $y^2 = 4ax$ तथा इसके नाभिलम्ब के बीच घिरे क्षेत्र का क्षेत्रफल ज्ञात करें।

Find the area of the region enclosed between the parabola $y^2 = 4ax$ and its foci.



Limits $x=0$ तथा $x=a$

$\because y^2 = 4ax$

या $y = \sqrt{4ax}$

$$\text{Area} = 2\times\int_0^a y\cdot dx$$

$$= 2\int_0^a \sqrt{4ax}\cdot dx$$

$$A = 2\times 2\sqrt{a}\int_0^a x^{1/2}\cdot dx$$

$$= 4\cdot\sqrt{a}\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^a$$

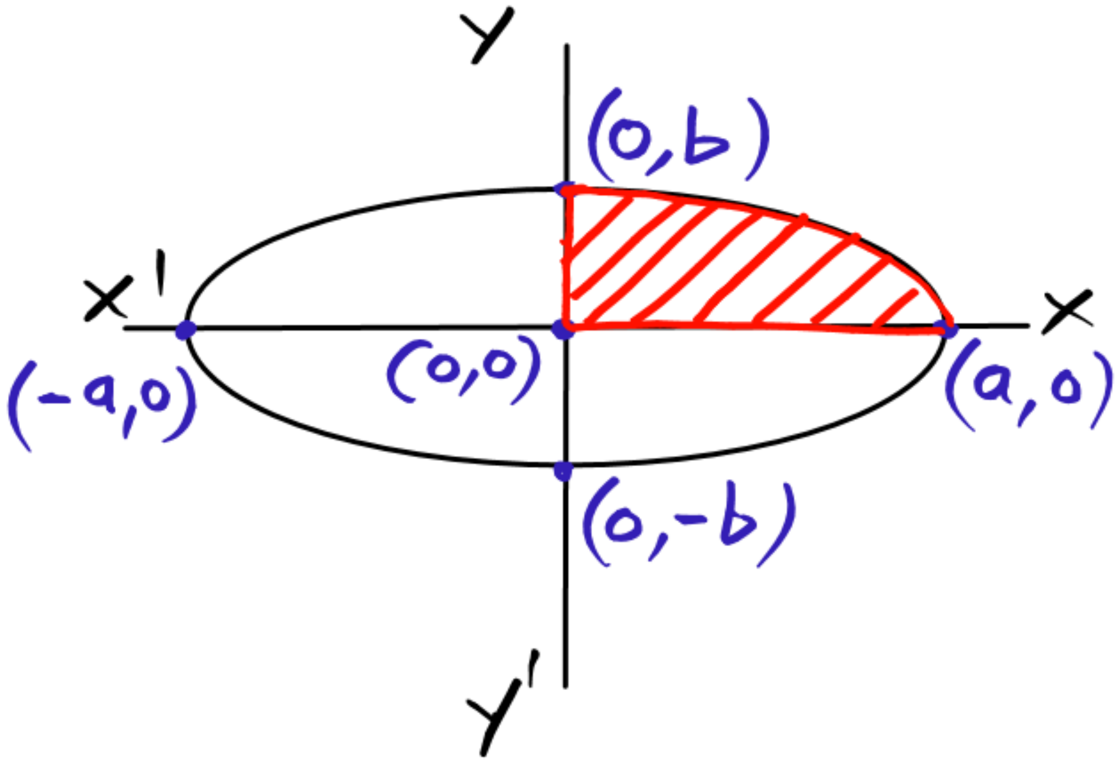
$$= 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$$

$$= 4\cdot(a)^{1/2}\times\frac{2}{3}\left[a^{3/2}-0\right]$$

$$= \frac{8a^2}{3}\ \text{unit}^2\ \text{Ans}$$

Q.73:- दीर्घवृत्त $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ का क्षेत्रफल निकालें।

Find the area of the ellipse.



Limits, $x=0$ तथा $x=a$

$\because \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$\frac{y^2}{b^2}=\frac{a^2-x^2}{a^2}$$

$$y^2=\frac{b^2(a^2-x^2)}{a^2}$$

$$y=\frac{b\sqrt{a^2-x^2}}{a}$$

$$\text{Area of ellipse} = 4 \times \int_0^a y \cdot dx$$

$$= 4 \times \int_0^a \frac{b\sqrt{a^2-x^2}}{a} \cdot dx$$

$$= \frac{4b}{a} \int_0^a \sqrt{a^2-x^2} \cdot dx$$

$$\int \sqrt{a^2-x^2}\, dx = \frac{1}{2}\left[x \cdot \sqrt{a^2-x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{\overset{2}{\cancel{4}}b}{a}\left[\frac{1}{\cancel{2}}\left\{x\sqrt{a^2-x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right\}\right]_0^a$$

$$= \frac{2b}{a}\left[a\sqrt{a^2-a^2} + a^2 \sin^{-1}\left(\frac{a}{a}\right) - (0+0)\right]$$

$$= \frac{\cancel{2}b}{\cancel{a}}\left[0 + a^{\cancel{2}}\frac{\pi}{\cancel{2}}\right] = \underline{\underline{\pi ab}} \text{ वर्ग इकाई} \quad \underline{\underline{\text{Ans}}}$$

Q.74:- परवलय $y = 6 - x - x^2$ तथा x-अक्ष के बीच घिरे स्थान का क्षेत्रफल ज्ञात कीजिये।

Find the area of the space enclosed between the parabola $y = 6 - x - x^2$ and the x-axis.

Limits $x = -3$ तथा $x = 2$

$y = 6 - x - x^2$ —— (1)

x-अक्ष का समी० $y = 0$ समी० (1) में रखने पर

$0 = 6 - x - x^2$

$0 = -(x^2 + x - 6)$

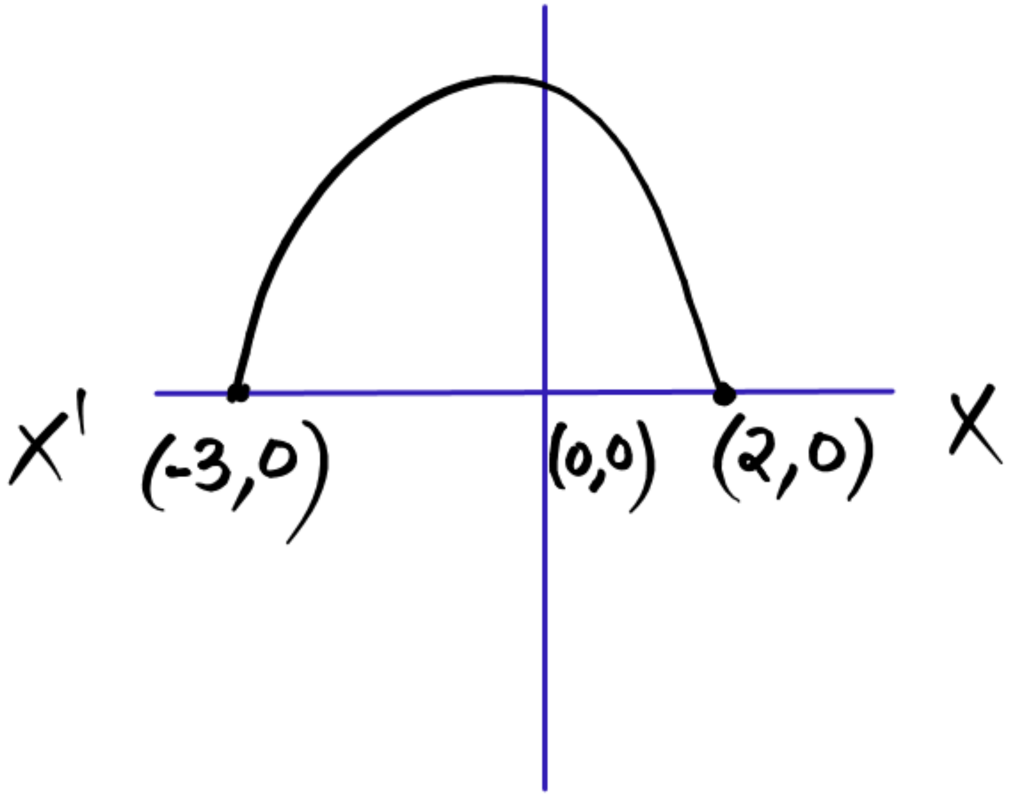
$x^2 + x - 6 = 0$

$x^2 + 3x - 2x - 6 = 0$

$x(x+3) - 2(x+3) = 0$

$(x+3)(x-2) = 0$

$x = 2, -3$



$$\text{Area} = \int_a^b y \cdot dx$$

$$= \int_{-3}^{2} (6 - x - x^2)\, dx$$

$$= \left[6.x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-3}^{2}$$

$$= \left[\left(6\times 2 - \frac{2^2}{2} - \frac{2^3}{3} \right) - \left(6\times(-3) - \frac{(-3)^2}{2} - \frac{(-3)^3}{3} \right) \right]$$

$$= \left[\left(12 - 2 - \frac{8}{3} \right) - \left(-18 - \frac{9}{2} + 9 \right) \right]$$

$$A = 12 - 2 - \frac{8}{3} + 18 + \frac{9}{2} - 9$$

$$= 30 - 11 - \frac{8}{3} + \frac{9}{2}$$

$$= \frac{19}{1} - \frac{8}{3} + \frac{9}{2}$$

$$= \frac{114 - 16 + 27}{6} = \frac{141 - 16}{6}$$

$$= \frac{125}{6} \text{ वर्गइकाई}$$

Ans

Q.75:- वक्र $\sqrt{x} + \sqrt{y} = \sqrt{a}$ और अक्षों से घिरा क्षेत्रफल ज्ञात करो।

Find the area bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the axes.

$\sqrt{x} + \sqrt{y} = \sqrt{a}$ —— (1)

x-अक्ष का समी० $y = 0$ समी० (1) में रखने पर

$\sqrt{x} + 0 = \sqrt{a}$

Square on both side

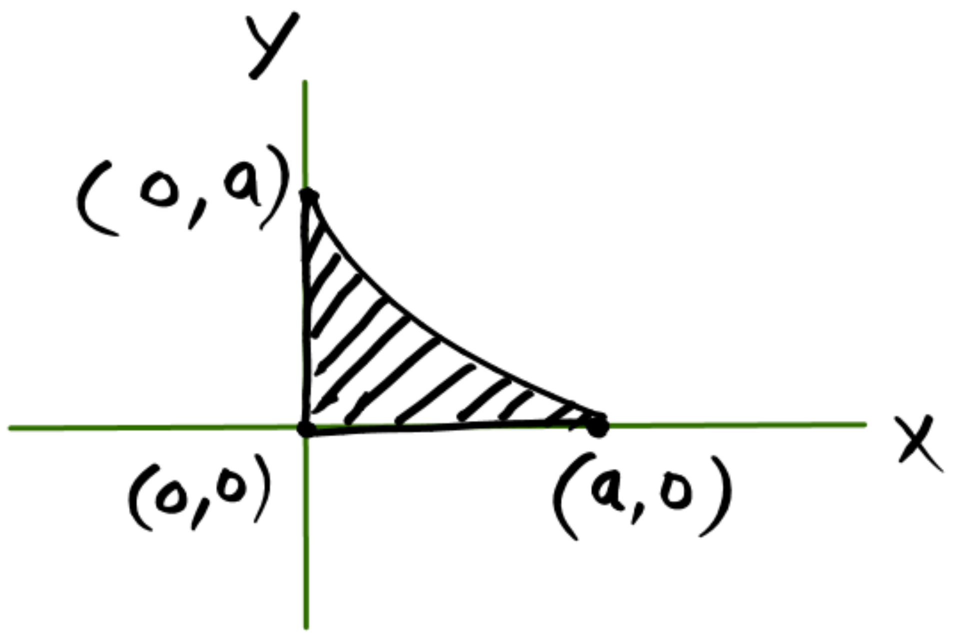
$\boxed{x = a}$

y-अक्ष का समी० $x = 0$ समी० (1) में रखने पर

$0 + \sqrt{y} = \sqrt{a}$

Square on both side

$\boxed{y = a}$



Limits $x = 0$ तथा $x = a$

समी० (1) से $\sqrt{y} = \sqrt{a} - \sqrt{x}$

square

$y = (\sqrt{a} - \sqrt{x})^2$ —— (2)

$$\text{Area } (A) = \int_a^b y \cdot dx$$

$$A = \int_0^a \left(\sqrt{a} - \sqrt{x}\right)^2 \cdot dx$$ (H.W.)

Q.76:- परवलय $ay = 3(a^2 - x^2)$ तथा x-अक्ष के बीच घिरे क्षेत्र का क्षेत्रफल ज्ञात कीजिये।

Find the area of the region enclosed between the parabola $ay = 3(a^2 - x^2)$ and the x-axis.

$ay = 3(a^2 - x^2)$ —— 1

x- अक्ष का समी० $y = 0$ समी० 1 में रखने पर

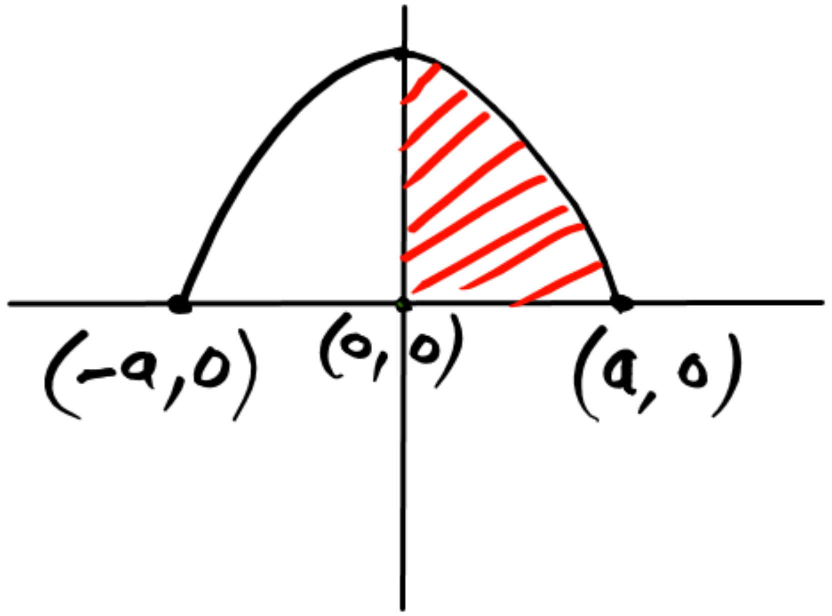
$0 = 3(a^2 - x^2)$

$0 = a^2 - x^2$

$x^2 = a^2$

squareroot

$\boxed{x = \pm a}$



Limits $x = 0$ तथा $x = a$

$y = \frac{3}{a}(a^2 - x^2)$ —— 2

$$\text{Area} = 2\int_0^a y \cdot dx$$

$$= 2\int_0^a \frac{3}{a}\left(a^2 - x^2\right) dx \quad \text{(H.W.)}$$