समाकलन के अनुप्रयोग
(Applications of Integration)

- निश्चित समाकलन का प्रयोग, निम्नलिखित क्षेत्रों में किया जा सकता है-
  Definite integral can be used in the following areas-

1. वक्रों के चापों की लम्बाई निकालने में (In determining the length of arcs of curves)
2. समाकलन द्वारा क्षेत्रफल निकालने में (In determining areas by integration)
3. ठोसों के पृष्ठ तथा आयतन निकालने में (In determining surfaces and volumes of solid), इत्यादि।

वक्रों के चापों की लम्बाईयाँ (Lengths of the Arcs of Curves)

- प्राय: इस लम्बाई को 'S' से प्रदर्शित किया जाता है।)
  Usually this length is represented by 'S'.
- किसी वक्र y=f(x) के लिये समाकलन द्वारा वक्र के कुछ भाग या पूरे वक्र की लम्बाई निकालने। विधि/सूत्र) इस प्रकार है-
  For any curve y=f(x), find the length of some part of the curve or the whole curve by integration. The method/formula is as follows-

1. वक्र y = f (x) के बिन्दुओं x = a तथा x = b के बीच चाप की लम्बाई

The length of the arc between the points x = a and x = b of the curve y = f (x)

$$S = \int_{x=a}^{x=b} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.dx$$

2. वक्र x = f (y) के बिन्दुओं y = a तथा y = b के बीच चाप की लम्बाई,

2. The length of the arc between the points y = a and y = b of the curve x = f (y),

$$S = \int_{y=a}^{y=b} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,.dy$$

Note:-

उपरोक्त सभी में ध्यान देने वाली बात यह है कि जिसके सापेक्ष समाकलन होगा सीमायें भी उसी के लिये रखी जायेगी।

The point to be noted in all the above is that the limits will be set for the person with respect to whom the integration will be done.

Q.67:- वक्र $y^2 = x^3$ के $x = 0$ से $x = \frac{5}{9}$ तक के चाप की लम्बाई ज्ञात कीजिये।

Find the length of the arc.

$$y^2 = x^3 \Rightarrow y = x^{3/2}$$

d.w.r. to x

$$\Rightarrow 2y \cdot \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3 \cdot x^2}{2 \cdot x^{3/2}}$$

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2} \quad \text{——— (1)}$$

Limits $x = 0$ and $x = \frac{5}{9}$

Length of Arc, $$S = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$S = \int_0^{5/9} \sqrt{1 + \left(\frac{3\sqrt{x}}{2}\right)^2} \cdot dx$$

$$S = \int_0^{5/9} \sqrt{1+\frac{9x}{4}}\cdot dx$$

$$S = \int_0^{5/9} \sqrt{\frac{4+9x}{4}}\cdot dx$$

$$S = \frac{1}{2}\int_0^{5/9} (4+9x)^{1/2}\cdot dx$$

$$= \frac{1}{2}\left[\frac{(4+9x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\times 9}\right]_0^{5/9}$$

$$= \frac{1}{2}\left[\frac{(4+9x)^{3/2}}{\frac{27}{2}}\right]_0^{5/9}$$

$$= \frac{1}{2}\times\frac{2}{27}\left[(4+9x)^{3/2}\right]_0^{5/9}$$

$$= \frac{1}{27}\left[\left(4+\cancel{9}\times\frac{5}{\cancel{9}}\right)^{3/2} - \left(4+9\times 0\right)^{3/2}\right]$$

$$= \frac{1}{27}\left[(9)^{3/2} - (4)^{3/2}\right]$$

$$= \frac{1}{27}\left[\left(3^{\cancel{2}}\right)^{3/\cancel{2}} - \left(2^{\cancel{2}}\right)^{3/\cancel{2}}\right]$$

$$= \frac{1}{27}\left[27-8\right]$$

$$= \frac{19}{27}\text{ unit}$$ Ans

Q.68:- वक्र $y = x^{3/2}$ की $x = 0$ से $x = 5$ के बीच चाप की लम्बाई ज्ञात करों।

Find the length of the arc.

$$y = x^{3/2}$$

d. w. r. to x

$$\frac{dy}{dx} = \frac{3}{2} \cdot x^{\frac{3}{2}-1}$$

$$\frac{dy}{dx} = \frac{3}{2} \cdot x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{3\sqrt{x}}{2} \quad \text{——— (1)}$$

Limits $x=0$ and $x=5$

Length of Arc

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_0^5 \sqrt{1+\left(\frac{3\sqrt{x}}{2}\right)^2}\, dx$$

$$= \int_0^5 \sqrt{1+\frac{9 \cdot x}{4}}\, dx$$

$$= \int_0^5 \sqrt{\frac{4+9x}{4}}\, dx$$

$$= \frac{1}{2}\int_0^5 (4+9x)^{1/2} \cdot dx$$

$$= \frac{1}{2}\left[\frac{(4+9x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\times 9}\right]_0^5$$

$$= \frac{1}{2}\left[\frac{(4+9x)^{3/2}}{\frac{3}{2}\times 9}\right]_0^5$$

$$= \frac{1}{2}\times\frac{2}{27}\left[(4+9x)^{3/2}\right]_0^5$$

$$= \frac{1}{27}\left[(4+9\times 5)^{3/2} - (4+9\times 0)^{3/2}\right]$$

$$= \frac{1}{27}\left[(49)^{3/2} - (4)^{3/2}\right]$$

$$= \frac{1}{27}\left[(7^{\cancel{2}})^{3/\cancel{2}} - (2^{\cancel{2}})^{3/\cancel{2}}\right]$$

$$= \frac{1}{27}(343 - 8)$$

$$= \frac{335}{27} \text{ unit} \qquad \text{Ans}$$

Q.69:- वक्र $y^2 = x^3$ की बिन्दुओं $\overset{x_1, y_1}{(0,0)}$ तथा $\overset{x_2, y_2}{(1, 2)}$ के बीच चाप की लम्बाई ज्ञात करों।

Find the length of the arc.

$$S = \frac{13\sqrt{13} - 8}{27} \quad \underline{\underline{\text{Ans}}}$$

Limits $x = 0$ तथा $x = 1$

$$y^2 = x^3$$

d. w.r.to x

$$2y \cdot \frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3 \cdot x^2}{2y}$$

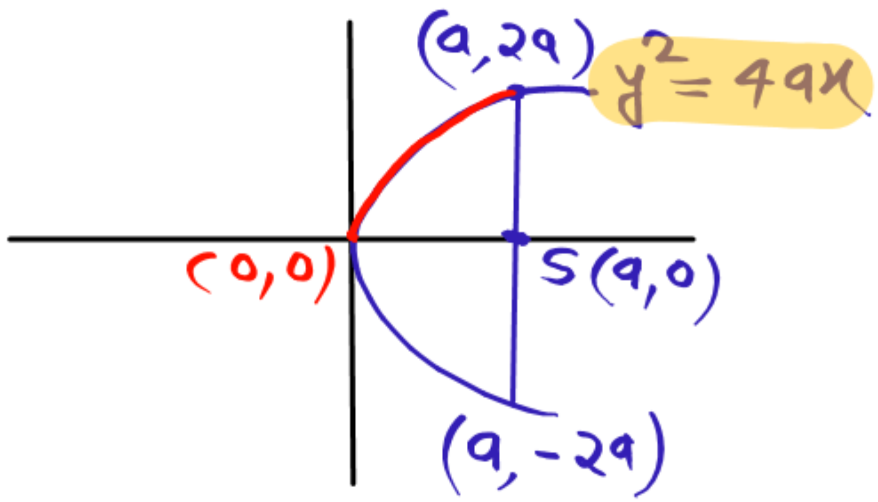
$$\frac{dy}{dx} = \frac{3 \cdot x^2}{2 \cdot x^{3/2}}$$

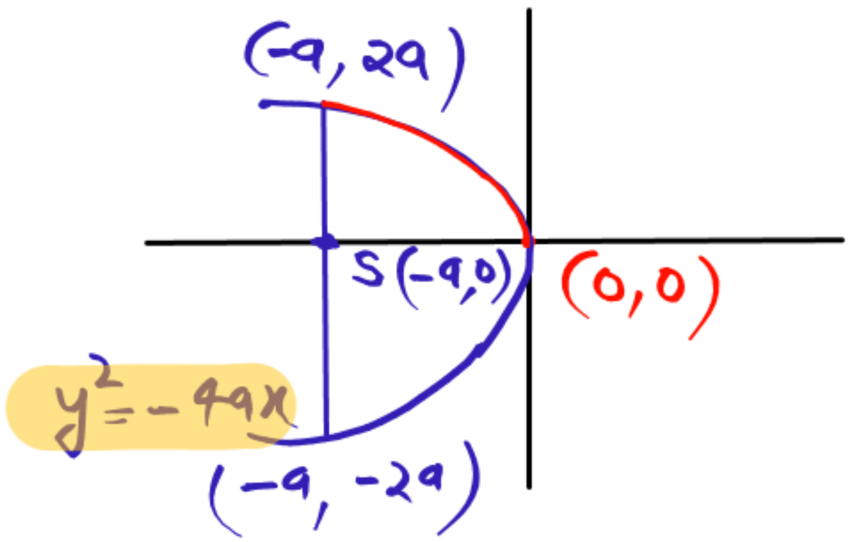
$$\frac{dy}{dx} = \frac{3x^{1/2}}{2}$$

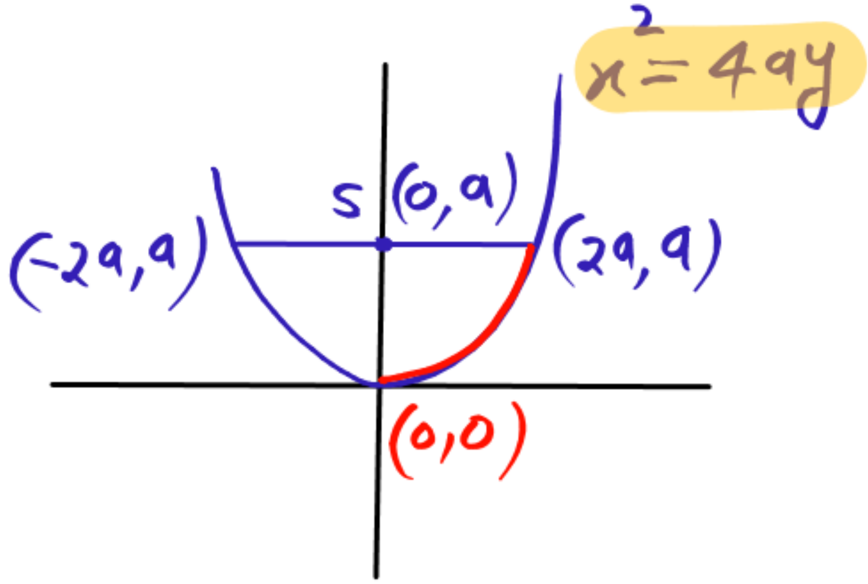
$$\frac{dy}{dx} = \frac{3\sqrt{x}}{2} \quad \text{—— (1)}$$

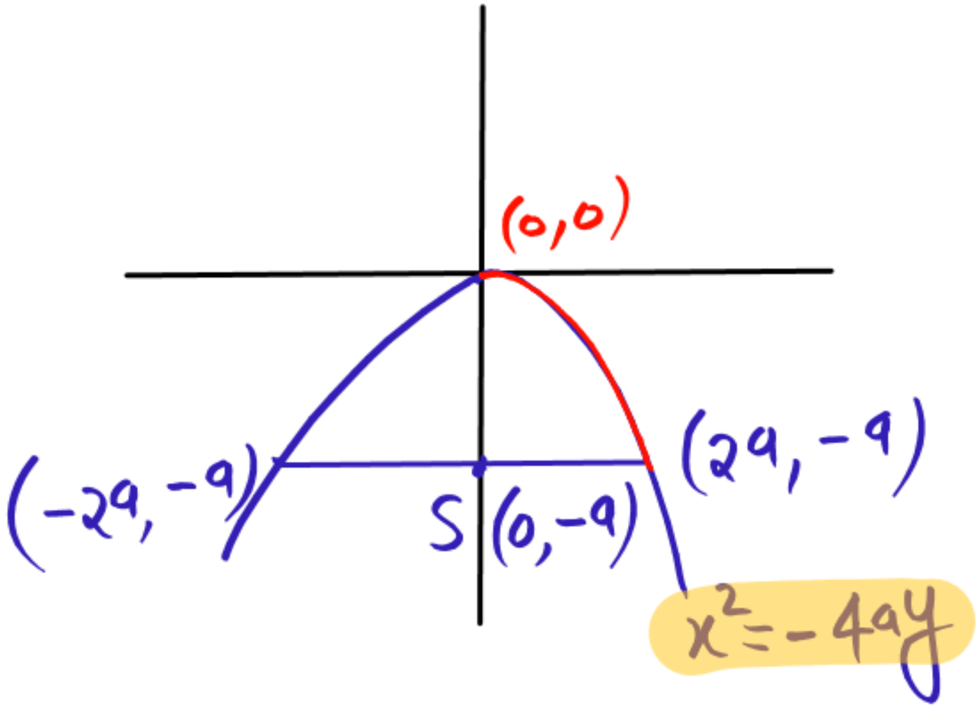
Length of Arc. $S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$ (H.W.)

Parabola (परवलय)









Q.70:- परवलय $y^2 = 4ax$ तथा उसके नाभिलम्ब द्वारा कटी लम्बाई ज्ञात करें।

Find the length cut off by the parabola $y^2 = 4ax$ and its latus rectum.

$$y^2 = 4ax$$

d.w.r.to y

$$2y = 4a \cdot \frac{dx}{dy}$$

$$\frac{dx}{dy} = \frac{2y}{4a}$$

$$\frac{dx}{dy} = \frac{y}{2a} \quad \text{—— (1)}$$

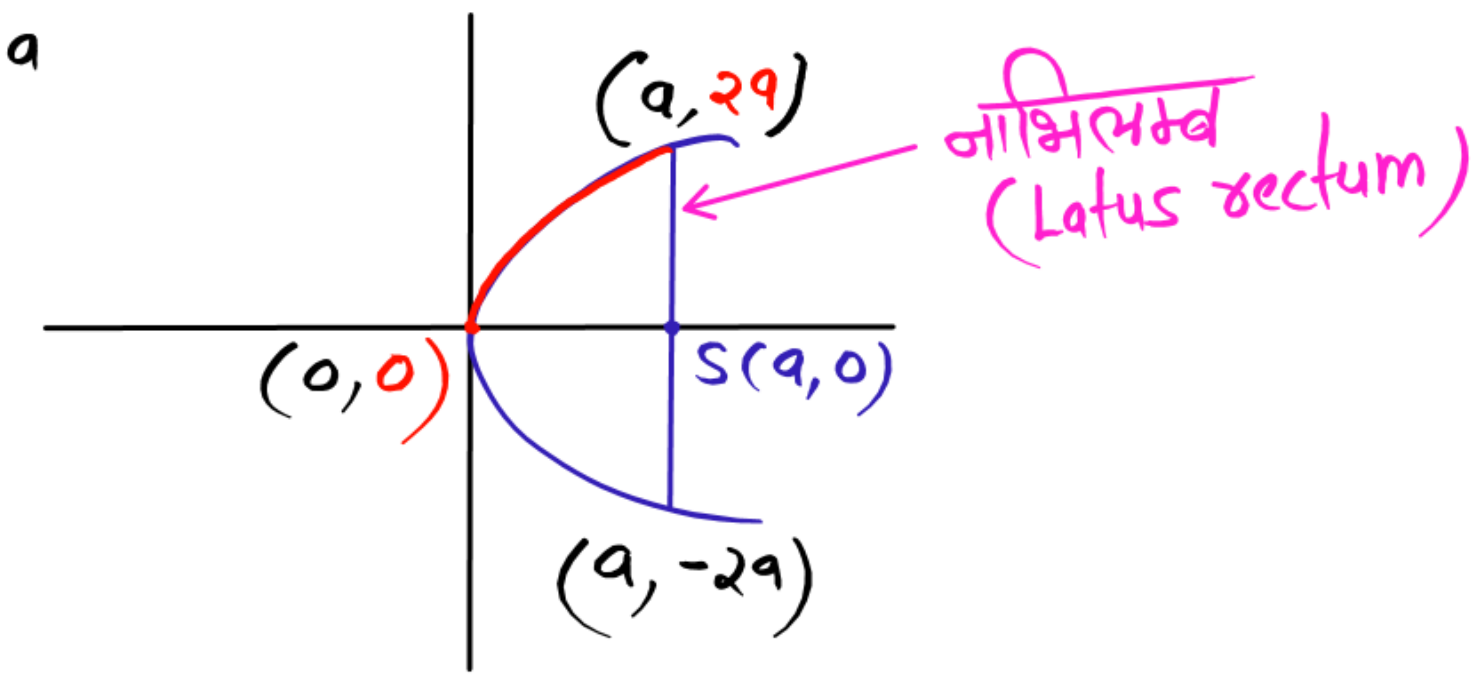
Limits $y = 0$ तथा $y = 2a$

Length

$$= 2 \times \int_0^{2a} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy$$

$$S = 2 \times \int_0^{2a} \sqrt{1 + \left(\frac{y}{2a}\right)^2} \cdot dy$$

$$= 2 \times \int_0^{2a} \sqrt{\frac{4a^2 + y^2}{4a^2}} \cdot dy$$



$$= 2 \times \int_0^{2a} \frac{\sqrt{4a^2+y^2}}{2a} \cdot dy$$

$$= \frac{\cancel{2}}{\cancel{2}a} \int_0^{2a} \sqrt{(2a)^2+y^2} \cdot dy \qquad \int \sqrt{a^2+x^2}\cdot dx = \frac{1}{2}\left[x\cdot\sqrt{a^2+x^2} + a^2\cdot\log\left(x+\sqrt{a^2+x^2}\right)\right]$$

$$= \frac{1}{a} \times \frac{1}{2} \left[y\cdot\sqrt{4a^2+y^2} + (2a)^2\cdot \log\left(y+\sqrt{4a^2+y^2}\right)\right]_0^{2a}$$

$$= \frac{1}{2a}\left[2a\cdot\sqrt{4a^2+(2a)^2} + (2a)^2\cdot\log\left(2a+\sqrt{4a^2+(2a)^2}\right) - \left(0+4a^2\cdot\log\left(0+2a\right)\right)\right]$$

$$= \frac{1}{2a}\left[2a\cdot\sqrt{8a^2}+4a^2\cdot\log\left(2a+\sqrt{8a^2}\right) - 4a^2\cdot\log 2a\right]$$

$$= 2\sqrt{2}a + 2a\cdot\log\left(2a+2\sqrt{2}a\right) - 2a\cdot\log 2a \,]$$

Q.70:- परवलय $y^2 = 4ax$ तथा उसके नाभिलम्ब द्वारा कटी लम्बाई ज्ञात करें।

Find the length cut off by the parabola $y^2 = 4ax$ and its latus rectum.

$y^2 = 4ax$

d. w. r. to x

$2y \cdot \frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{4a}{2y}$

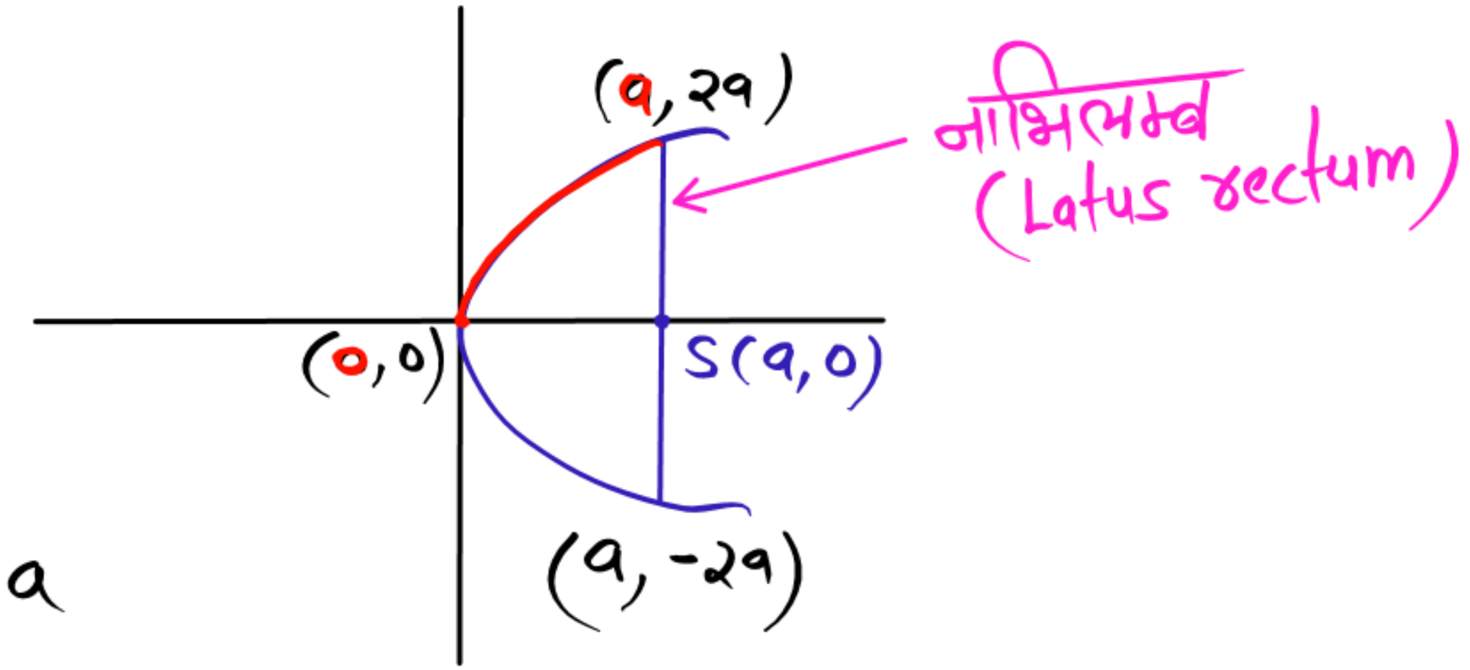
$\frac{dy}{dx} = \frac{2a}{\sqrt{4ax}}$

$\frac{dy}{dx} = \frac{\cancel{2a}}{\cancel{2}\cdot\cancel{\sqrt{a}}\cdot\sqrt{x}}$

$\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}}$ —— (1)

Limits $x = 0$ तथा $x = a$

$$\text{Length} = 2 \times \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$$



$$S = 2\pi \int_0^a \sqrt{1+\left(\frac{\sqrt{a}}{\sqrt{x}}\right)^2} \cdot dx$$

$$S = 2\pi \int_0^a \sqrt{1+\frac{a}{x}} \cdot dx$$

$$S = 2\pi \int_0^a \sqrt{\frac{x+a}{x}}\, dx$$ (Hold)

Q.71:- समाकलन विधि से वृत्त $x^2 + y^2 = a^2$ की परिधि ज्ञात करें।

Find the circumference of the circle $x^2 + y^2 = a^2$ by integration method.

$$x^2+y^2=a^2$$

d.w.r.to x

$$2x+2y\cdot\frac{dy}{dx}=0$$

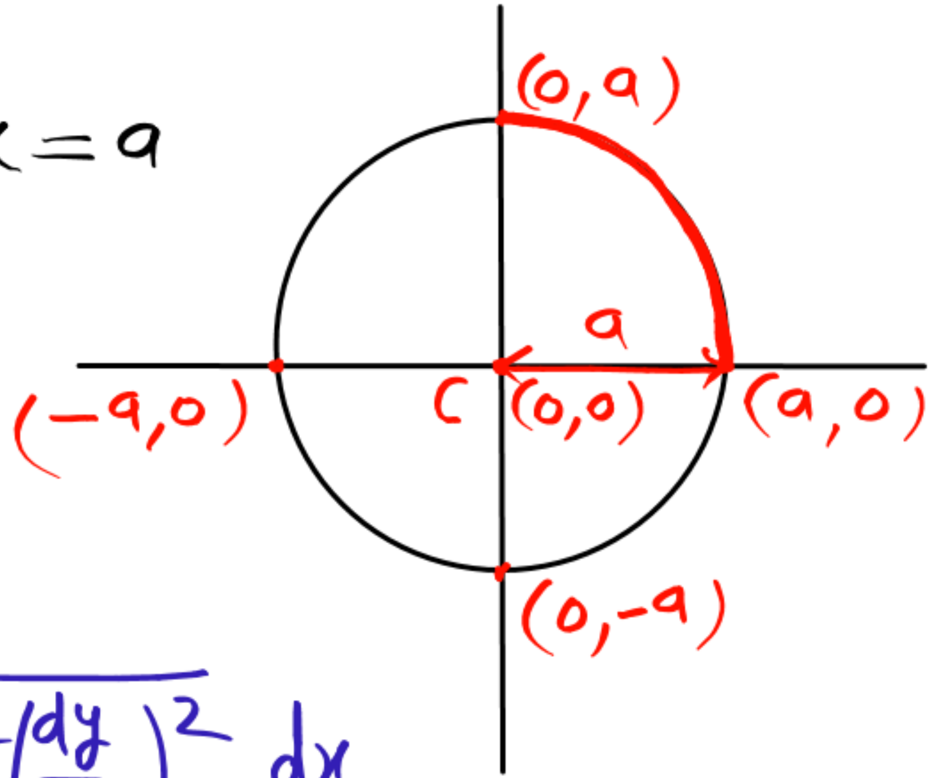
$$2y\cdot\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=-\frac{\cancel{2}x}{\cancel{2}y}$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

Limits

$x=0$ तथा $x=a$



Length

$$S=4\times\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$$