1. Continuous Wavelet Transform

1.2 Wavelet Properties

i.

$$g(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t)^2}$$

$$m(t) = -\frac{d^2\, g(t)}{dt^2}$$

$$\frac{d\, g(t)}{dt} = \frac{1}{\sqrt{2\pi}} \times \frac{-2t}{2} \times e^{-\frac{1}{2}(t)^2} = -\frac{t}{\sqrt{2\pi}} e^{-\frac{1}{2}(t)^2}$$

$$\frac{d^2\, g(t)}{dt^2} = \frac{-1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t)^2} + -\frac{t}{\sqrt{2\pi}} \times \frac{-2t}{2} \times e^{-\frac{1}{2}(t)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t)^2} (t^2 - 1)$$

$$\therefore \boldsymbol{m(t)} = \frac{\mathbf{1}}{\sqrt{\boldsymbol{2\pi}}} \boldsymbol{e^{-\frac{1}{2}(t)^2}(1 - t^2)}$$

ii. Let the normalizing factor be k.

$$k^2 \int_{-\infty}^{\infty} m^2(t)\, dt = 1$$

$$k^2 \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-t^2} (1 - t^2)^2\, dt = 1$$

$$k^2 \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-t^2} (t^4 - 2t^2 + 1)\, dt = 1$$

Consider,

$$I = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-t^2} (t^4 - 2t^2 + 1)\, dt$$

$$I = \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} e^{-t^2} t^4\, dt - 2 \int_{-\infty}^{\infty} e^{-t^2} t^2\, dt + \int_{-\infty}^{\infty} e^{-t^2}\, dt \right]$$

We know that,

e^{-t^2} is an even function. Also, t^2 and t^4 are even functions.

$\therefore e^{-t^2} t^2$ and $e^{-t^2} t^4$ both are even functions.

So, we can rewrite I as,

$$I = \frac{1}{2\pi}\left[2\int_0^\infty e^{-t^2} t^4\ dt\ -\ 2\times 2\ \int_0^\infty e^{-t^2} t^2\, dt\ +\ 2\int_0^\infty e^{-t^2}\ dt\right]$$

$$I = \frac{1}{\pi}\left[\int_0^\infty e^{-t^2} t^4\ dt\ -\ 2\ \int_0^\infty e^{-t^2} t^2\, dt\ +\ \int_0^\infty e^{-t^2}\ dt\right]$$

Substitute $t^2 = x$,

$$2t\, dt = dx \quad \Rightarrow \quad dt = {dx}/{2\sqrt{x}}$$

$$I = \frac{1}{\pi}\left[\int_0^\infty \frac{e^{-x}x^2}{2\sqrt{x}}\ dx\ -\ 2\ \int_0^\infty \frac{e^{-x}x}{2\sqrt{x}}\, dx\ +\ \int_0^\infty \frac{e^{-x}}{2\sqrt{x}}\, dx\right]$$

$$I = \frac{1}{2\pi}\left[\int_0^\infty e^{-x}x^{\frac{3}{2}}\ dx\ -\ 2\ \int_0^\infty e^{-x}x^{\frac{1}{2}}\, dx\ +\ \int_0^\infty e^{-x}x^{-\frac{1}{2}}\, dx\right]$$

$$I = \frac{1}{2\pi}\left[\int_0^\infty e^{-x}x^{\frac{5}{2}-1}\ dx\ -\ 2\ \int_0^\infty e^{-x}x^{\frac{3}{2}-1}\, dx\ +\ \int_0^\infty e^{-x}x^{\frac{1}{2}-1}\, dx\right]$$

$$I = \frac{1}{2\pi}\left[\Gamma\left(\frac{5}{2}\right)\ -\ 2\,\Gamma\left(\frac{3}{2}\right) +\ \Gamma\left(\frac{1}{2}\right)\right]$$

We know that,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\ \text{ and }\ \Gamma\left(\frac{n}{2}\right) = \sqrt{\pi}\ \frac{(n-2)!!}{2^{\frac{n-1}{2}}}\ \text{ where n} \in \mathrm{Z}^+$$

$$\therefore\ \Gamma\left(\frac{5}{2}\right) = \sqrt{\pi}\ \frac{3!!}{2^{\frac{4}{2}}} = \sqrt{\pi}\times\frac{3}{4}$$

$$\Gamma\left(\frac{3}{2}\right) = \sqrt{\pi}\ \frac{1!!}{2^{\frac{2}{2}}} = \sqrt{\pi}\times\frac{1}{2}$$

Therefore,

$$\boldsymbol{I = \frac{1}{2\pi}\left[\frac{3}{4}\sqrt{\pi} - 2\times\frac{1}{2}\sqrt{\pi} + \sqrt{\pi}\right] = \frac{3}{8\sqrt{\pi}}}$$

$$\Rightarrow\ k^2\ \times\ \frac{3}{8\sqrt{\pi}} = 1$$

$$\therefore \quad \boldsymbol{k = \sqrt{\frac{8\sqrt{\pi}}{3}}}$$

iii. $$\varphi(t) = \sqrt{\frac{8\sqrt{\pi}}{3}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t)^2}(1 - t^2)$$

$$\boldsymbol{\varphi(t) = \frac{2}{\sqrt{3\sqrt{\pi}}}\ e^{-\frac{1}{2}(t)^2}(1 - t^2)}$$

The generic wavelet function with the scaling factor,

$$\boldsymbol{\varphi(t; s) = \frac{2}{\sqrt{3s\sqrt{\pi}}}\ e^{-\frac{1}{2}\left(\frac{t}{s}\right)^2}\left(1 - \left(\frac{t}{s}\right)^2\right)}$$

iv.

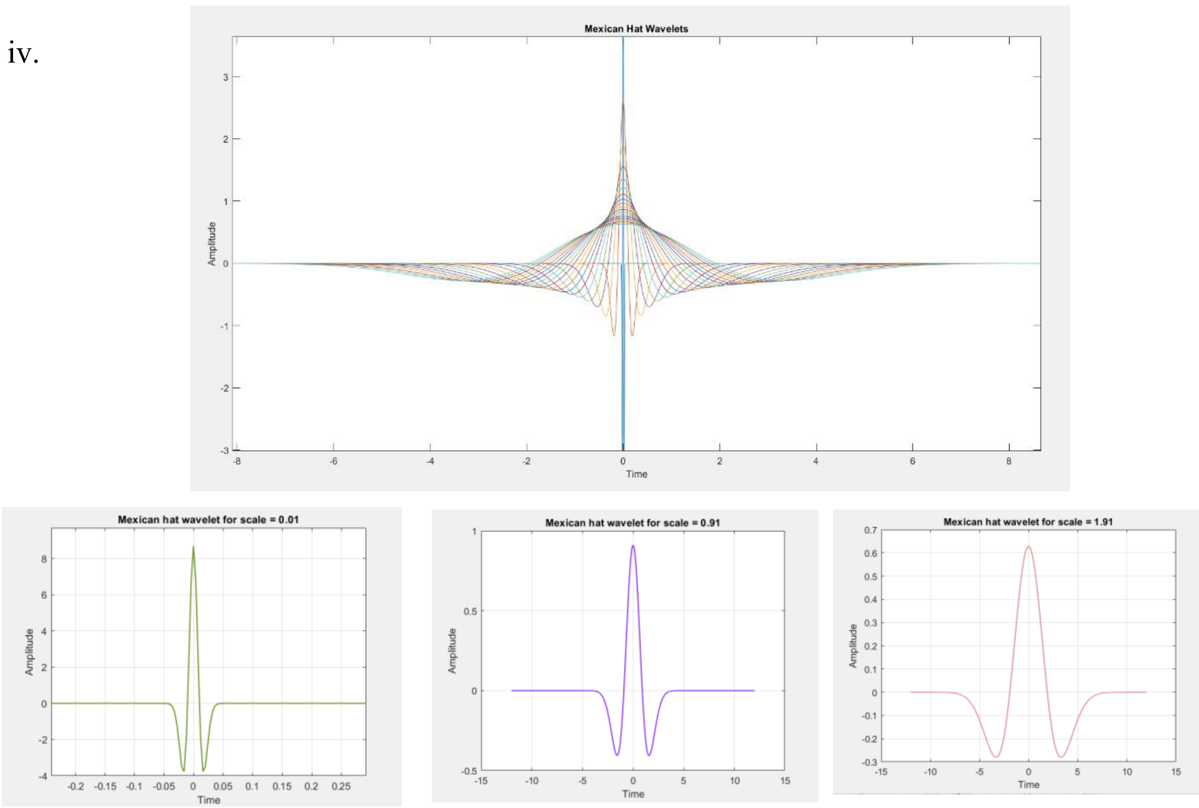


Figure 1: Time domain representation of the normalized Mexican hat wavelet function at different scales

- Increasing the scale of a wavelet spreads it more in the time domain, while decreasing the scale localizes it.
- Higher scales result in lower amplitude of the wavelet signal, whereas lower scales yield higher amplitude.

.

v.

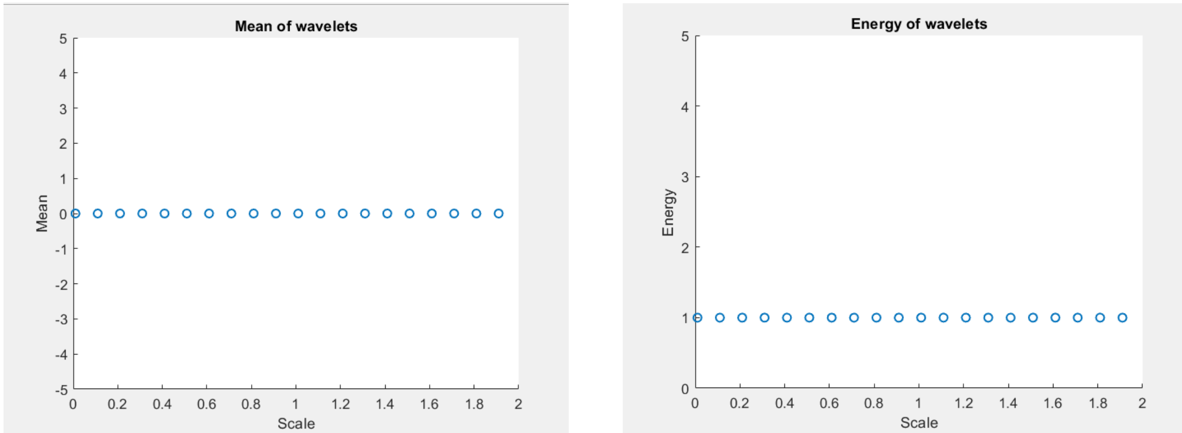


Figure 2: Mean and energy variation among daughter wavelets

- According to the above plots, the mean of all the daughter wavelets is zero and energy equals to unity.

Compact Support visualization

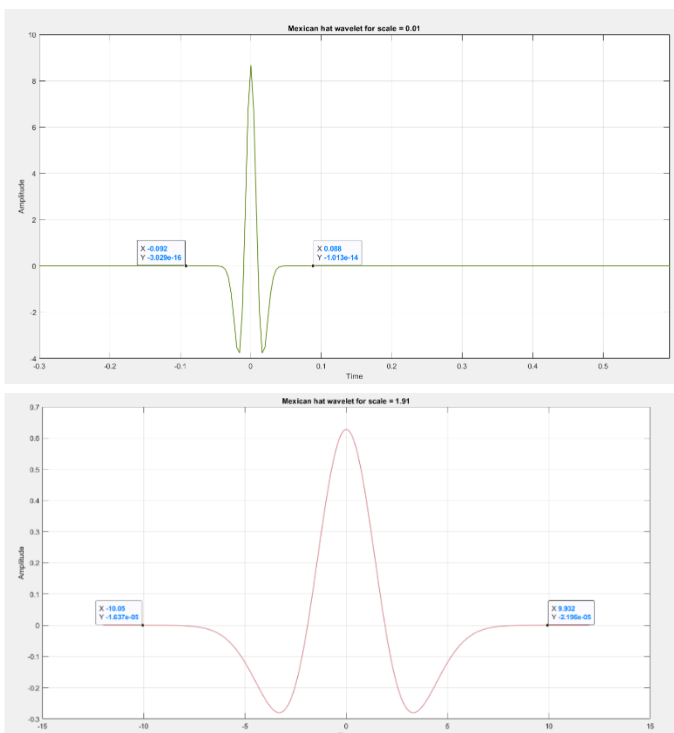


Figure 3: Visualization of compactness of the lowest and highest daughter wavelets

- Even at the highest and lowest scales, all the nonzero wavelet values are confined to a compact, finite range in the time domain, and this applies to all intermediate wavelets as well.

vi.

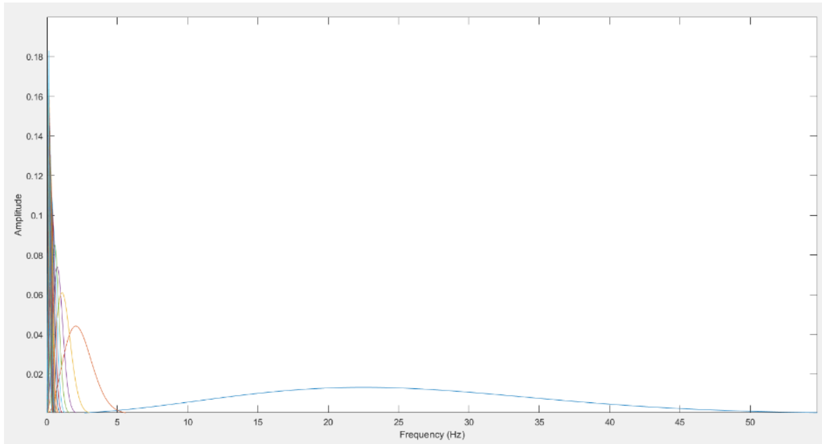


Figure 4: Spectra of daughter wavelets

- The spectrum of higher-scale wavelets is concentrated in the low-frequency range, with magnitude increasing as the scale increases.
- Conversely, when the scale decreases, the spectrum spreads across the frequency domain, resulting in a decrease in magnitude.

1.3 Continuous Wavelet Decomposition

i.

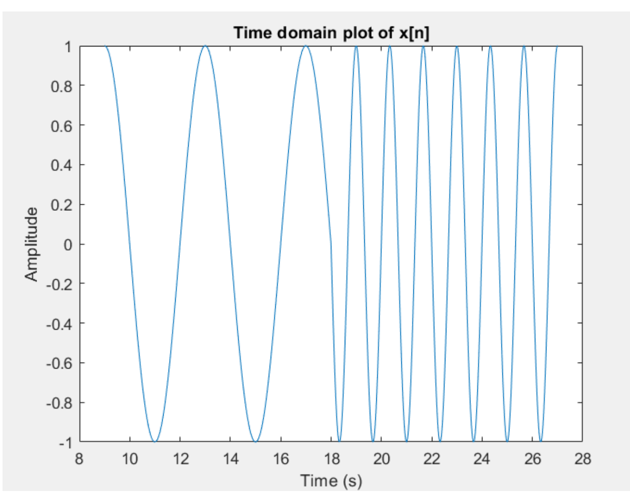


Figure 5: Constructed signal

ii.

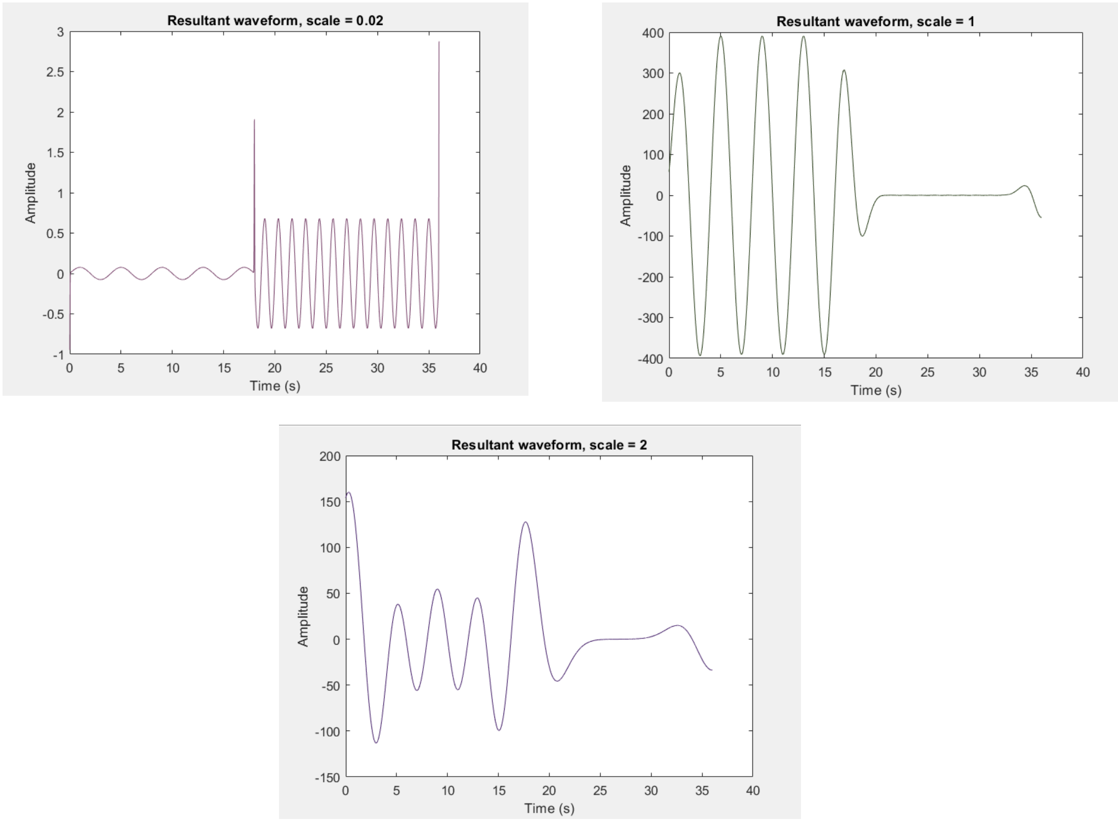


Figure 6: Resultant waveforms after applying scales Mexican hat wavelet on x[n]. (Plots are for three selected scales)

iii.

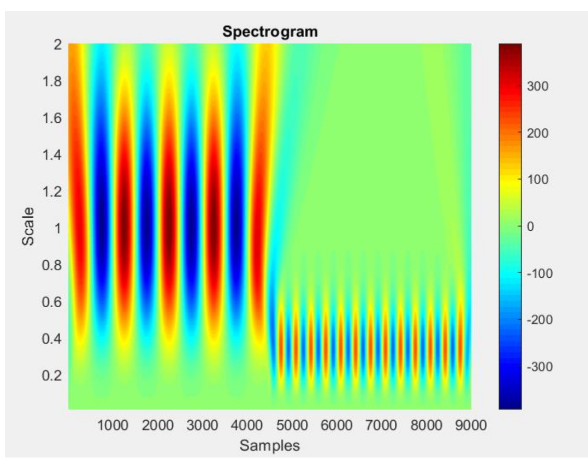


Figure 7: Spectrogram of the derived coefficients

- The image shows a noticeable change in the frequency pattern around the midpoint. The initial segment has a broader wave pattern, suggesting a lower frequency, while the second segment has more closely spaced waves, indicating a higher frequency.

- The presence of these two visually distinct patterns suggests that the signal contains two different frequency components over the given range of samples.

2. Discrete Wavelet Transform

$$\psi_{m,n}(t) = \frac{1}{\sqrt{s_0^m}}\psi\left(\frac{t - n\tau_0 s_0^m}{s_0^m}\right)$$

$s_0 = scaling\ step\ size, \qquad \tau_0 = translation\ step\ size$

Typically, $s_o = 2$ and $\tau_0 = 1$

2.2 Applying DWT with the Wavelet Toolbox in MATLAB

i.

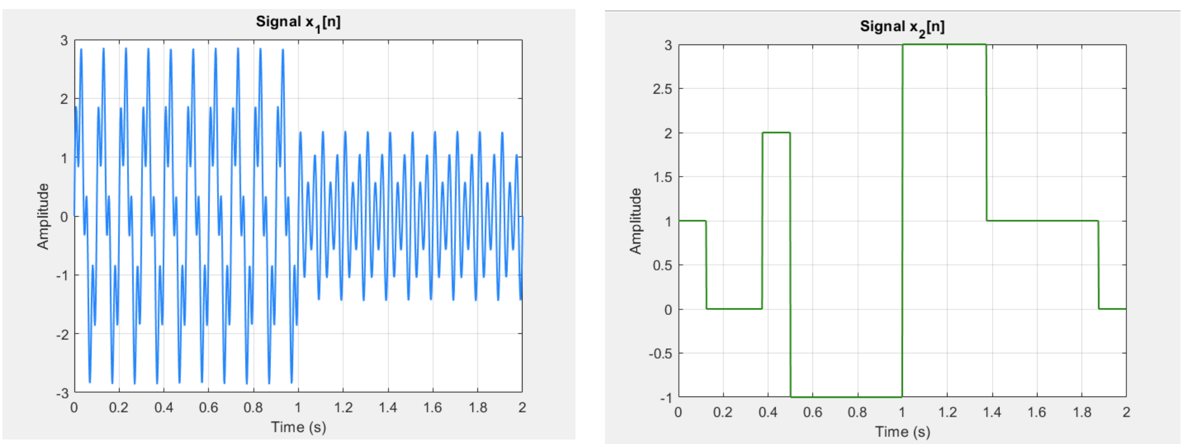


Figure 8: Clean x_1[n] and x_2[n] signals

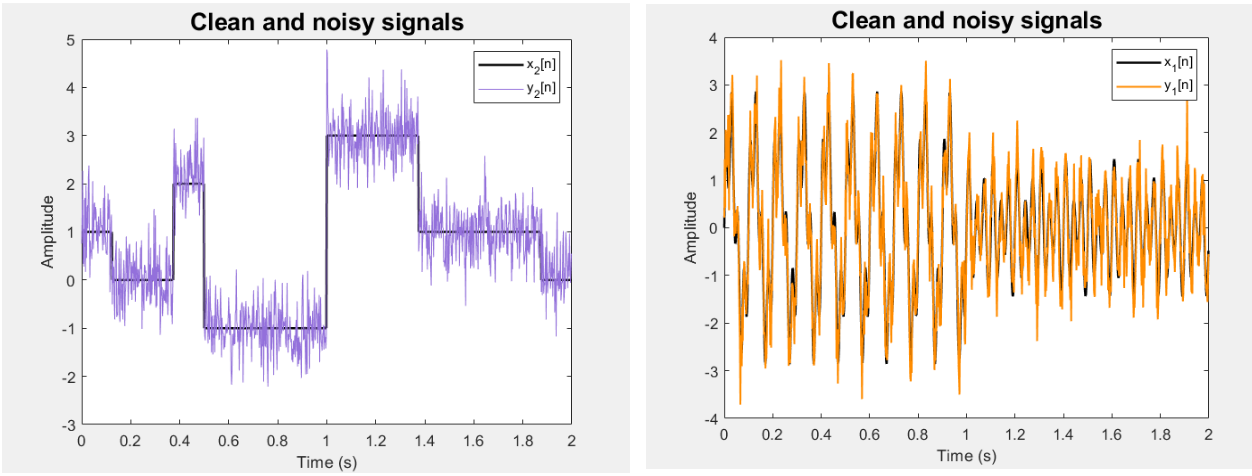


Figure 9: Corrupted $x_1[n]$ and $x_2[n]$ signals with 10dB noise

ii.

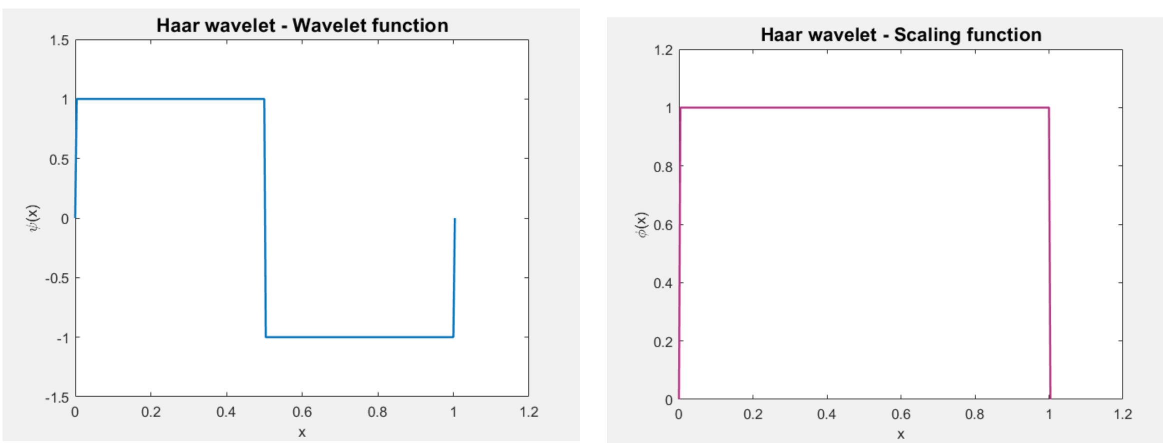


Figure 10 : Wavelet function and scaling function of Haar wavelet using wavefun()

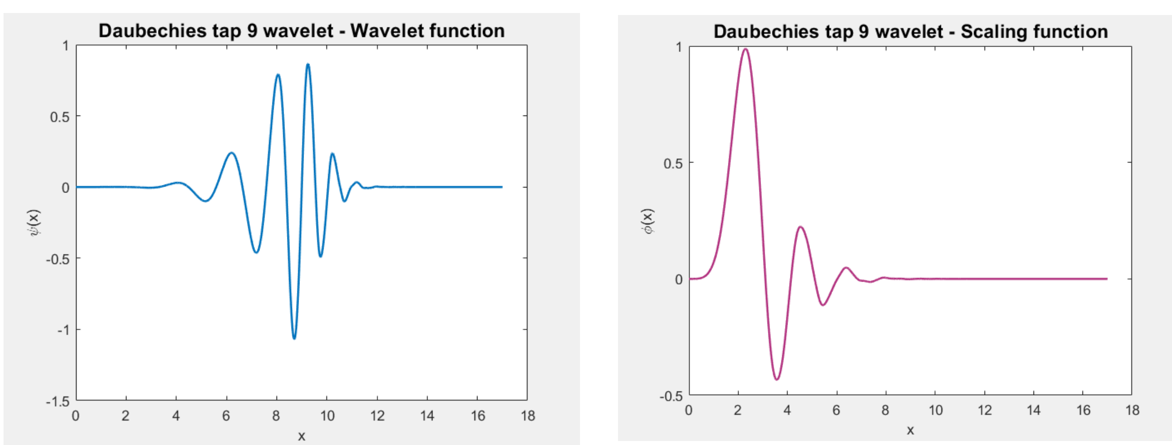


Figure 11: Wavelet function and scaling function of Daubechies tap 9 wavelet using wavefun()

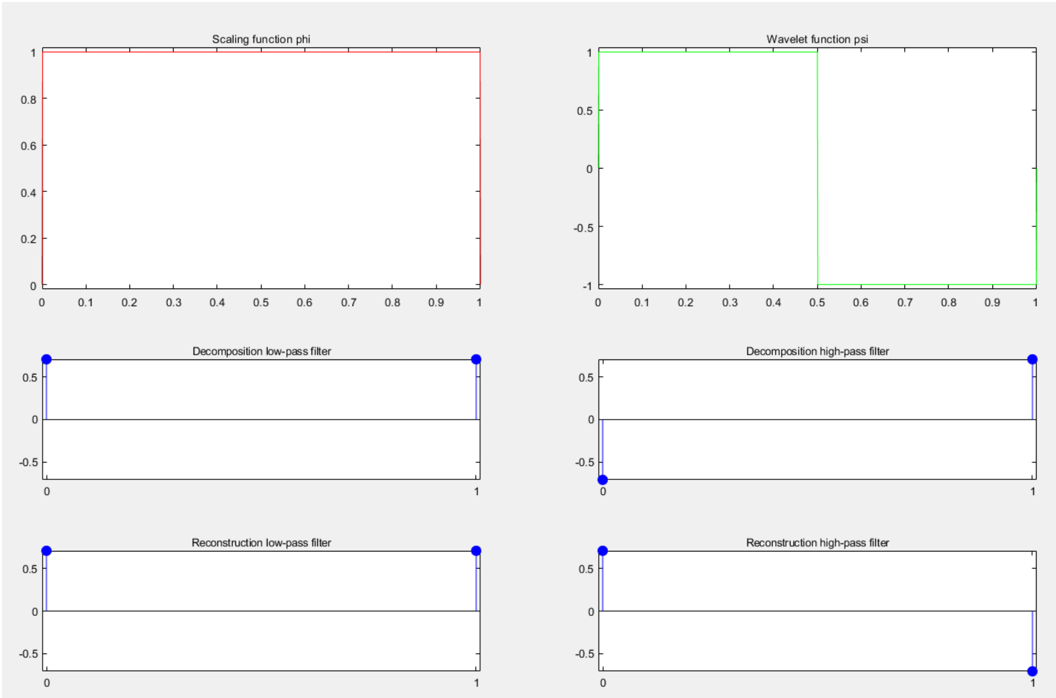


Figure 12: Wavelet function and scaling function of Haar wavelet using waveletAnalyzer GUI

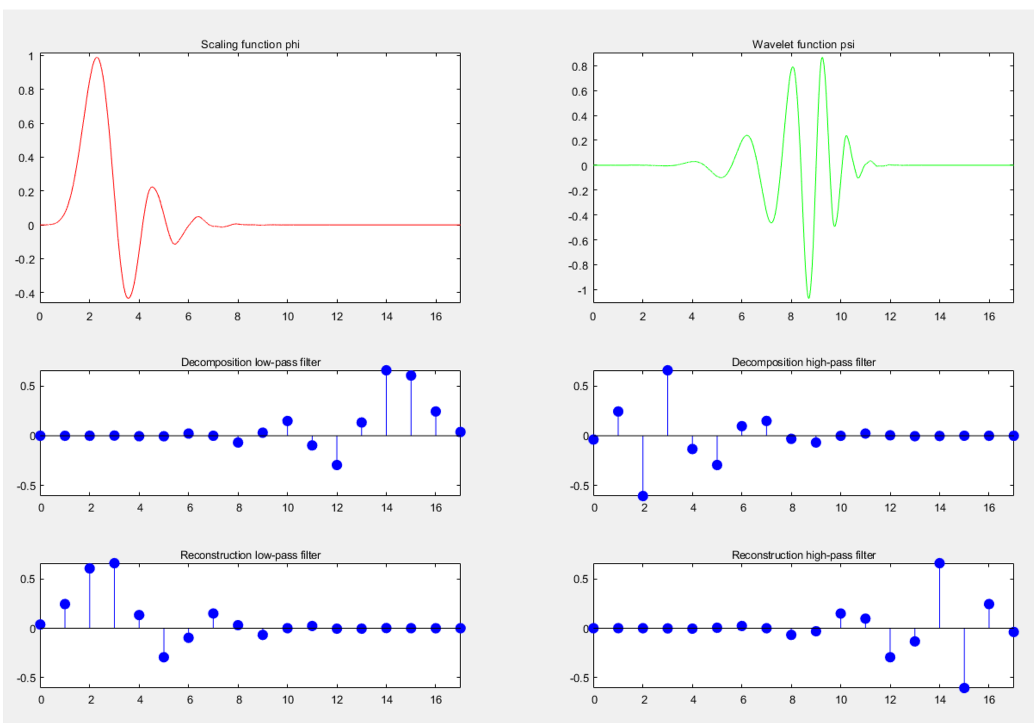


Figure 13: Wavelet function and scaling function of Daubechies tap 9 wavelet using waveletAnalyzer GUI

iii.

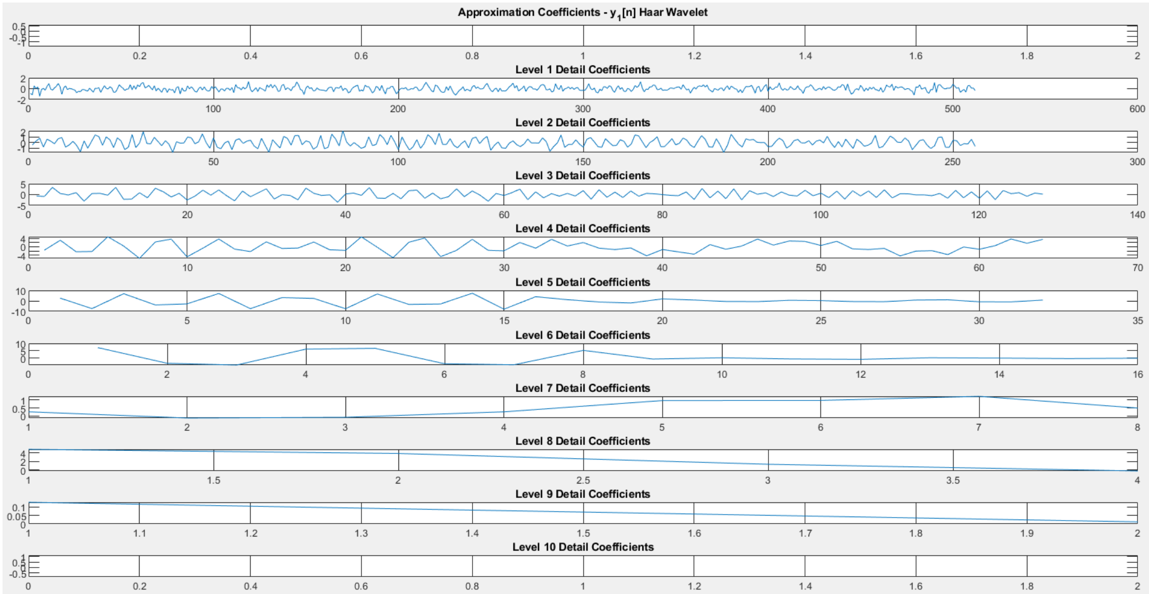


Figure 14: 10-Level decomposition of signal $y_1[n]$ using Haar wavelet

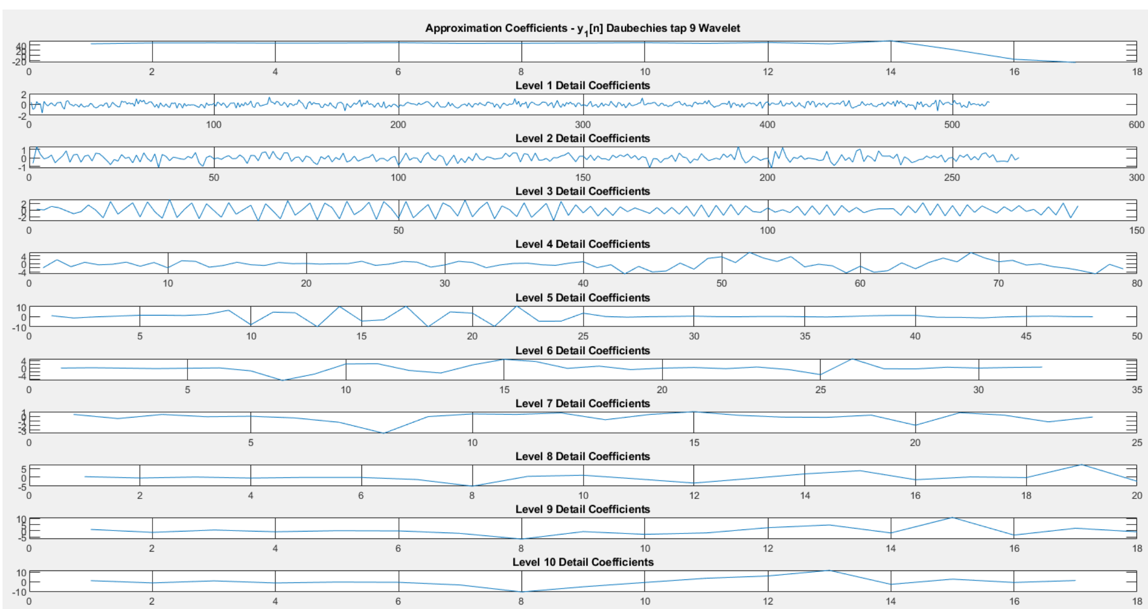


Figure 15: 10-Level decomposition of signal $y_1[n]$ using Daubechies tap 9 wavelet

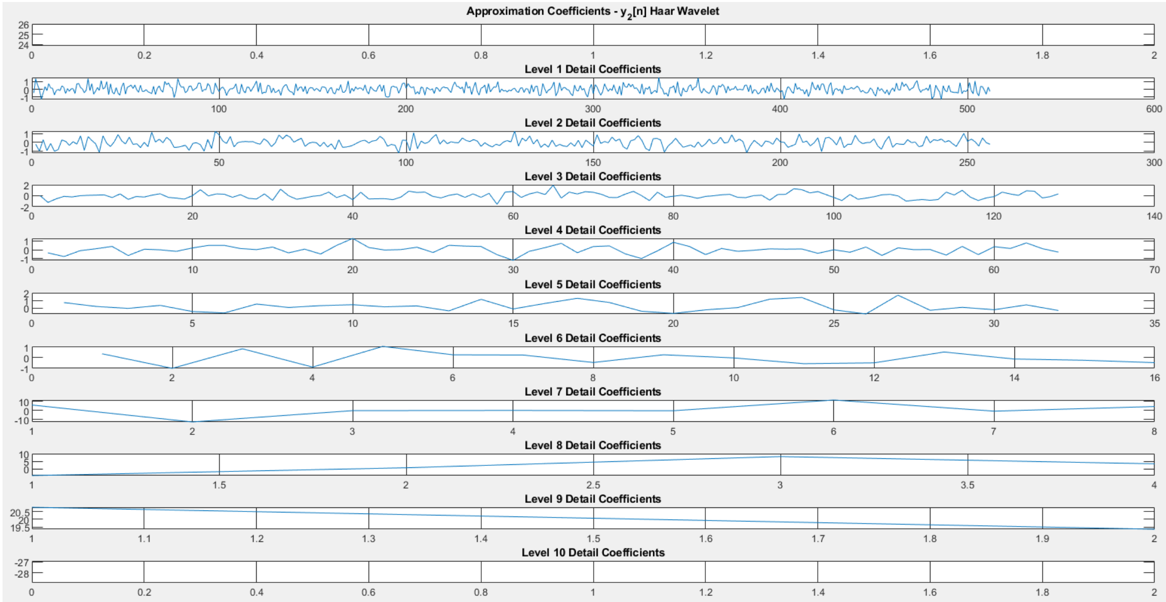


Figure 16: 10-Level decomposition of signal $y_2[n]$ using Haar wavelet

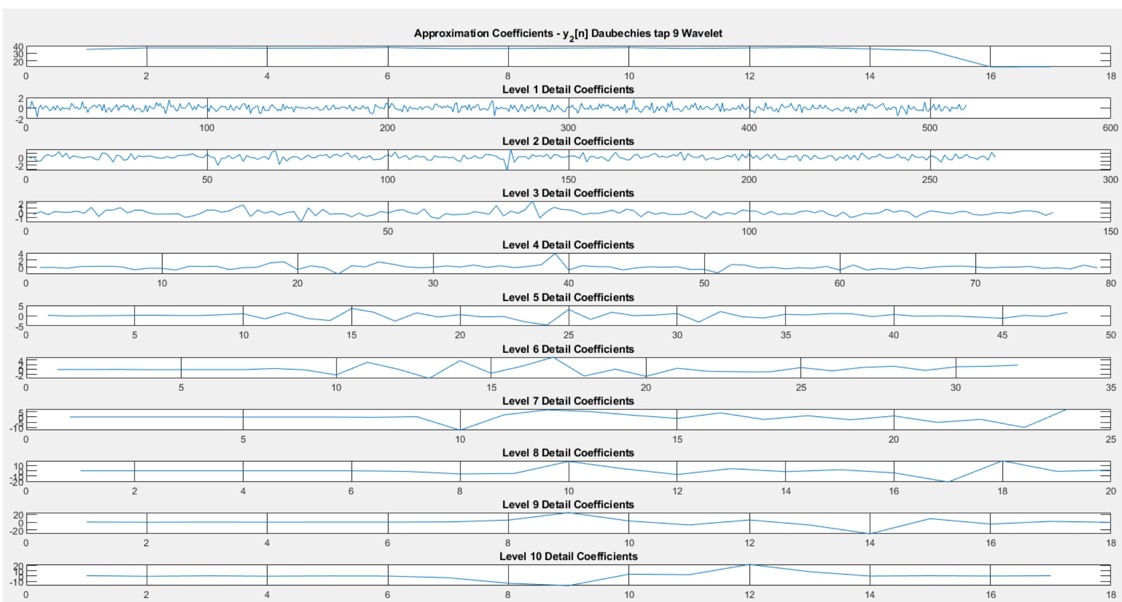


Figure 17: 10-Level decomposition of signal $y_2[n]$ using Daubechies tap 9 wavelet

iv. <u>Signal reconstruction using inverse DWT</u>

- The approximate coefficients are calculated using the detailed coefficients and length output from the wavelet decomposition step. We need to specify the wavelet function as well. These are the approximate coefficients at the lowest level of decomposition.

- Starting from the lowest level, iteratively reconstruct the signal by using the approximate coefficients and detailed coefficients at each level with the help of the inverse discrete wavelet transform (IDWT).

- The output of each IDWT application provides the approximate coefficients for the next higher level.

- At the highest level, the output of the IDWT is the fully reconstructed signal.

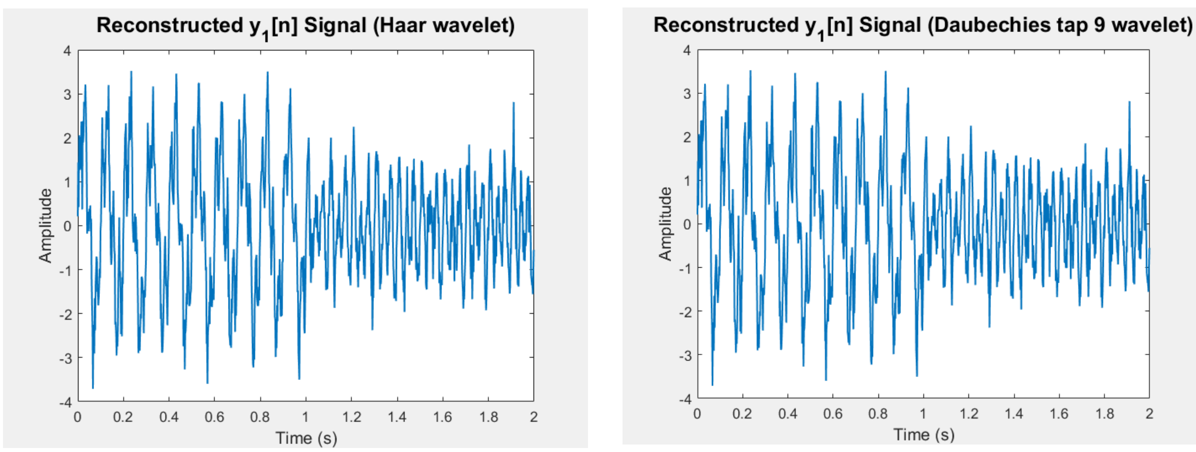


Figure 18: Reconstructed $y_1[n]$ using inverse DWT

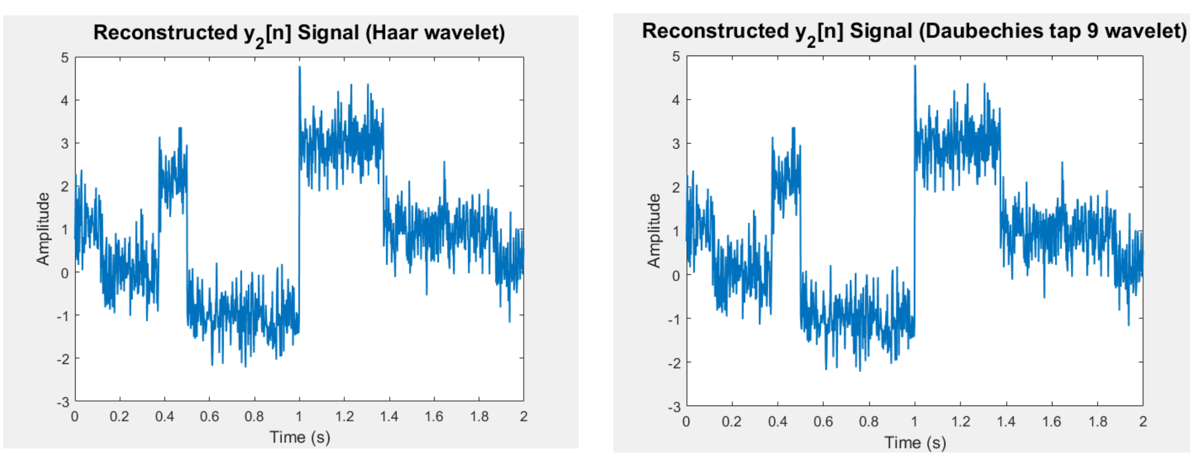


Figure 19: Reconstructed $y_2[n]$ using inverse DWT

- Given below is the energy difference between original signals and the reconstructed signals. Very low energy differences suggest that

$$y = \sum D^i + A$$

is accurate

```
Energy differences between original and reconstructed signals:

y1[n] reconstruction:
    Haar wavelet = 6.1216e-28
    Daubechies tap 9 wavelet = 1.5721e-17

y2[n] reconstruction:
    Haar wavelet = 5.6819e-27
    Daubechies tap 9 wavelet = 2.081e-17
```

Figure 20: Energy difference between signals output

2.3 Signal Denoising with DWT

i.

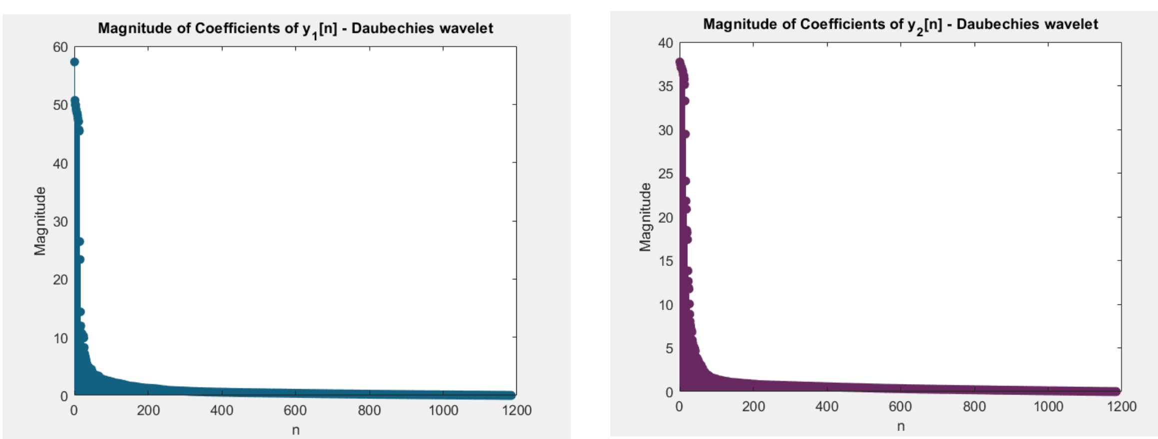
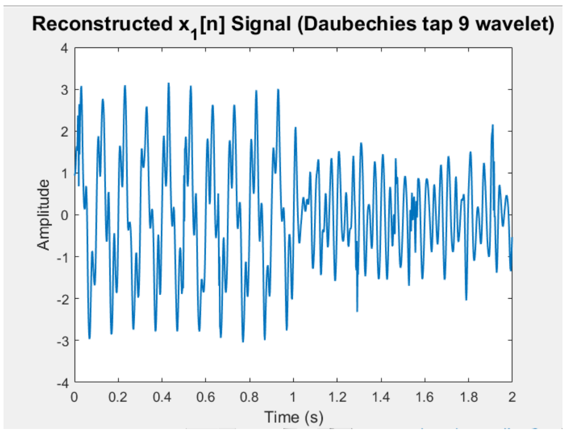


Figure 21: Magnitude plots of Daubechies tap 9 wavelet coefficients of $y_1[n]$ and $y_2[n]$

ii. The best threshold values were selected using a separate algorithm that tries to reduce the RMS error between the orginal signal and the reconstructed signal. According to that algorithm following thresholds were obtained.

Threshold values,

$y_1[n]$ – 1.12
$y_2[n]$ – 1.36



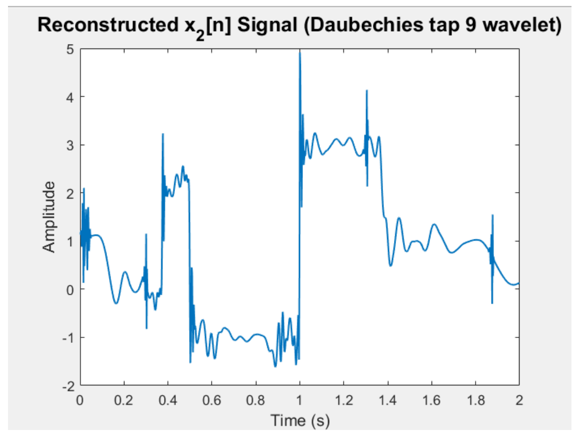


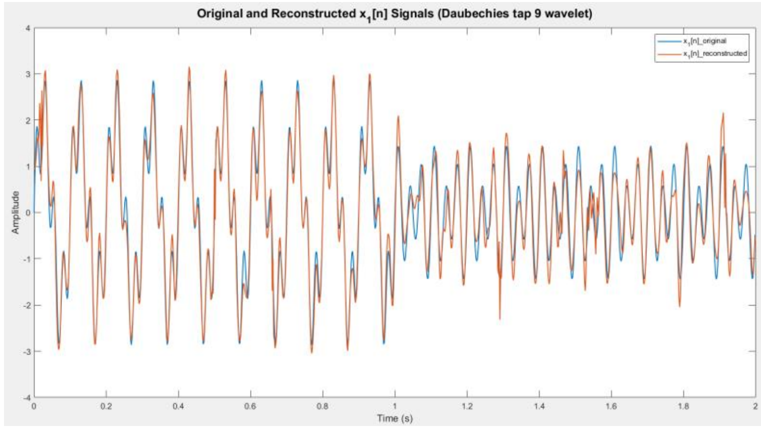
Figure 22: Reconstructed $x_1[n]$ and $x_2[n]$ using Daubechies tap 9 wavelet

- Given below is the command line output for best thresholds and related RMS error values.

```
best threshold for y1[n] : 1.12
    RMS error : 0.29044
best threshold for y2[n] : 1.36
    RMS error : 0.27887
```

Figure 23: Best thresholds and related RMS error values

iii.



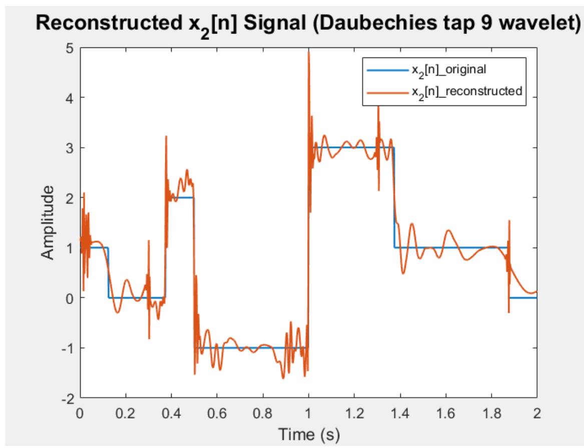


Figure 24: Reconstructed signal and original signal comparison – Daubechies tap 9 wavelet

- RMS error for $x_1[n]$ = 0.290440
- RMS error for $x_2[n]$ = 0.278873

- Reconstructed x1[n] signal is nearly equal to the actual signal. This indicates that Daubechies tap 9 wavelet is suitable for reconstructing smooth low and mid frequency signals,

- The reconstructed $x_2[n]$signals using Daubechies tap 9 wavelets still contain noticeable noise. Therefore, this wavelet seems to be less suitable for reconstruction of step signals like $x_2[n]$.

iv.

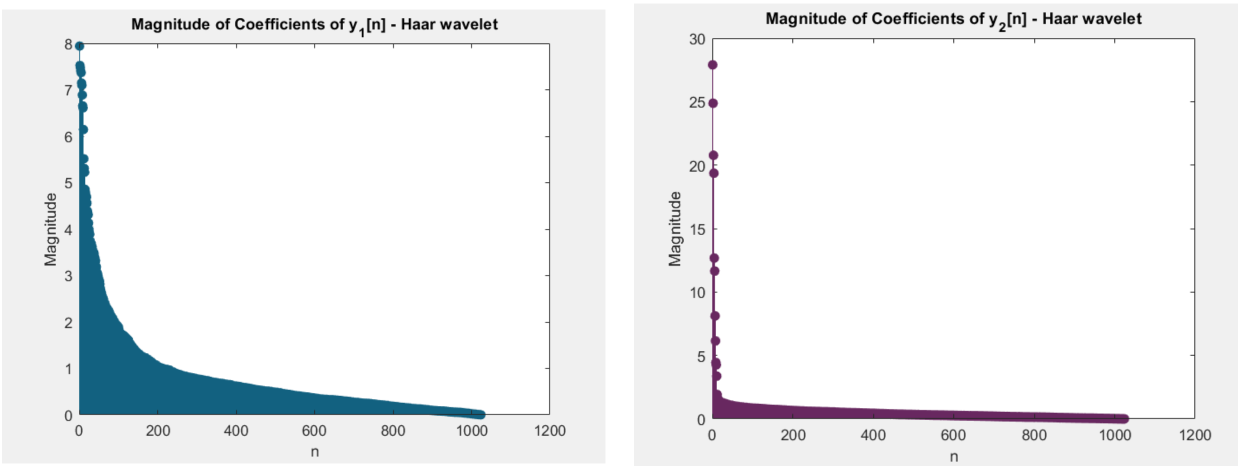


Figure 25: Magnitude plots of Haar wavelet coefficients of $y_1[n]$ and $y_2[n]$

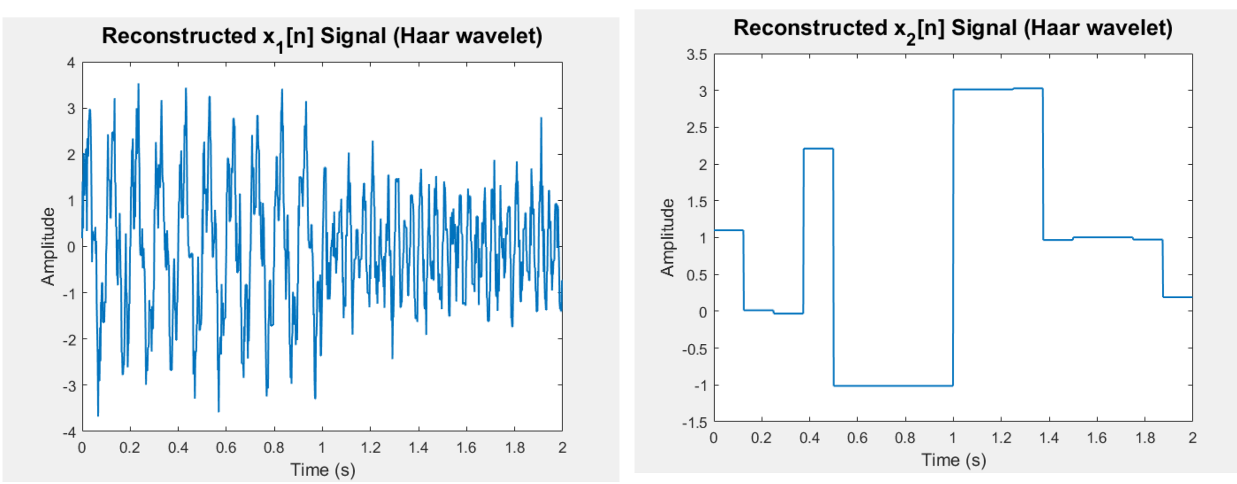


Figure 26: Reconstructed signal and original signal comparison - Haar wavelet

```
Haar wavelet:
best threshold for y1[n] : 0.37
    RMS error : 0.3893
best threshold for y2[n] : 1.921
    RMS error : 0.076963
```

Figure 27: Best thresholds and related RMS error values

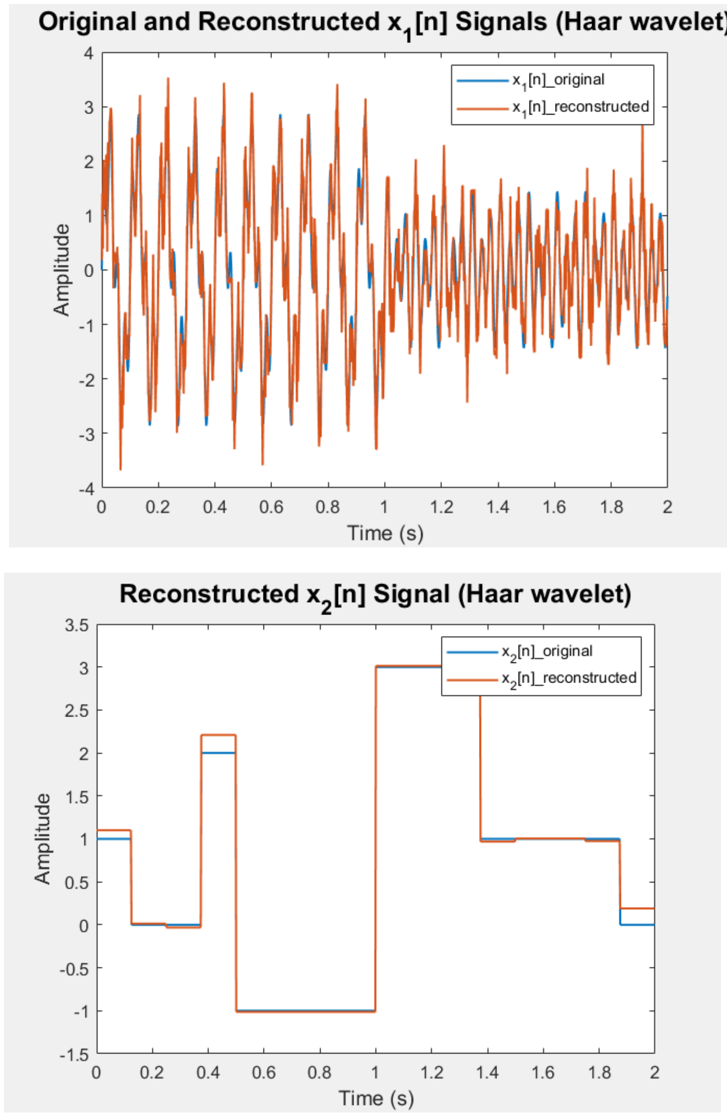


Figure 28: Reconstructed signal and original signal comparison – Haar wavelet

- The above plots suggest that Haar wavelet is suitable for reconstruction of both smooth signals and especially step signals.

v. Comparison of wavelets for signal reconstruction

Wavelet	$x_1[n]$	$x_2[n]$
Daubechies tap 9	0.290440	0.278873
Haar	0.3893	0.076963

- Haar wavelet: suitable for reconstructing step signals
- Daubechies tap 9 wavelets: suitable for reconstructing smooth signals

2.4 Signal Compression with DWT

i.

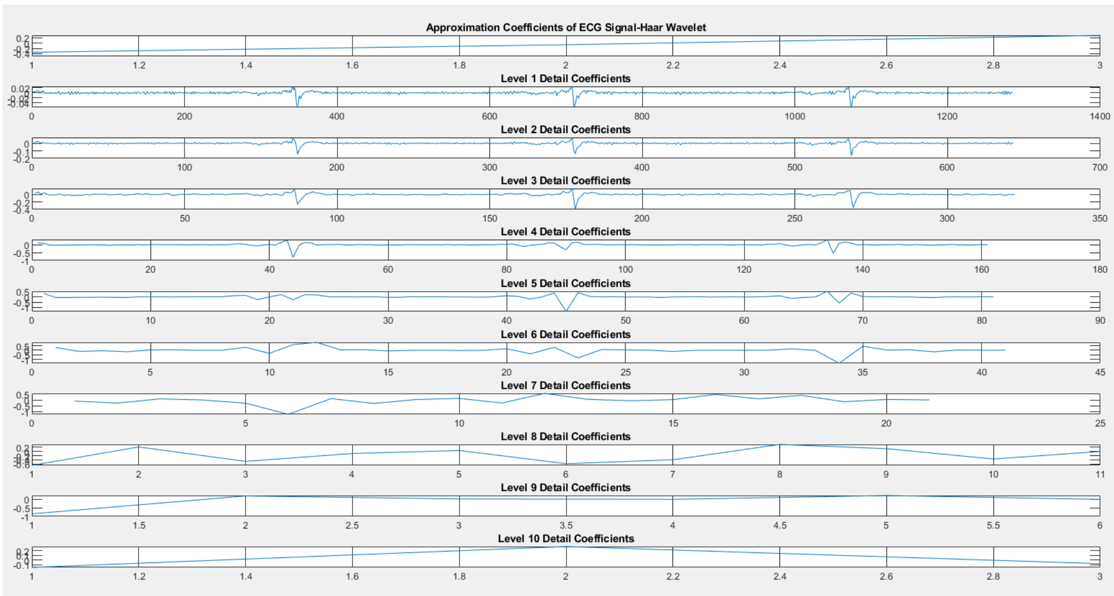


Figure 29: 10-Level decomposition of the ECG signal using Haar wavelet

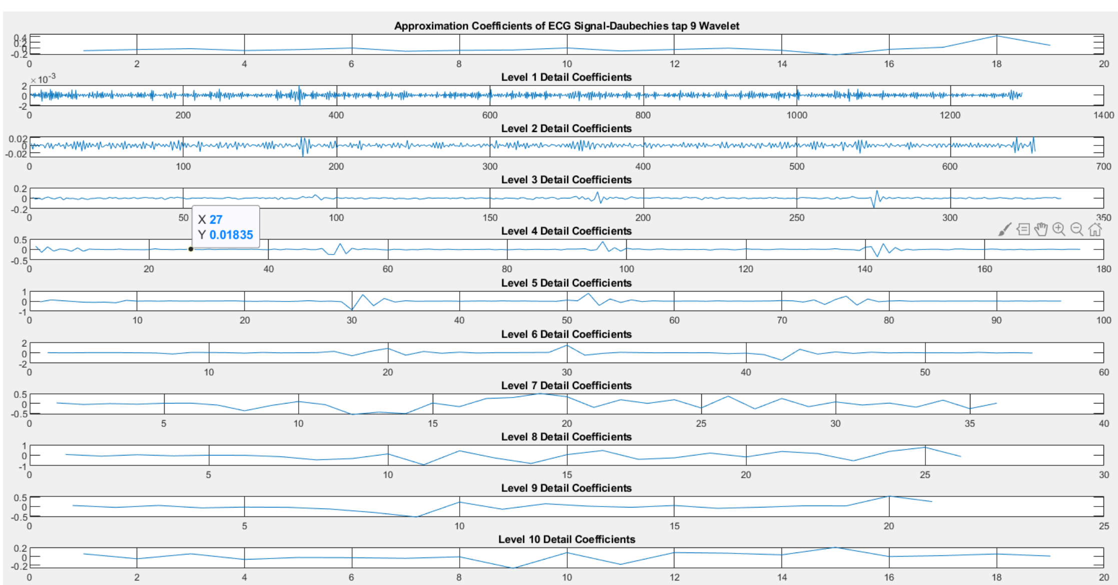
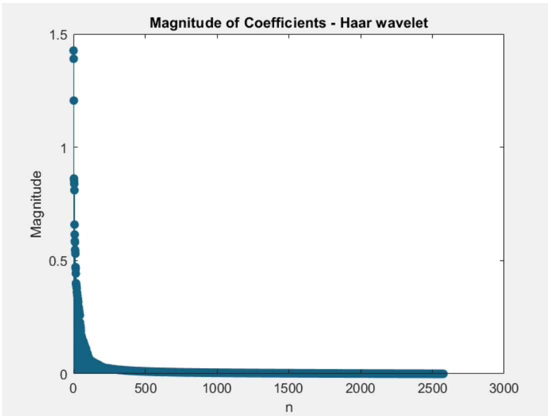


Figure 30: 10-Level decomposition of the ECG signal using Daubechies tap 9 wavelet

ii.



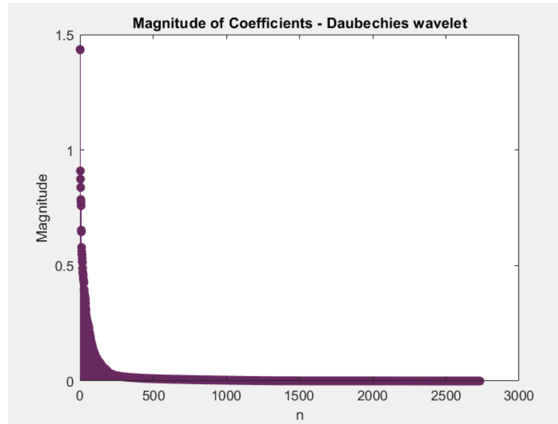


Figure 31: Magnitude plots of DWT coefficients of the ECG signal

```
number of coefficients (Haar wavelet) = 177
number of coefficients (Daubechies tap 9 wavelet) = 158
```

Figure 32: The number of coefficients required to preserve 99% energy of the original signal

iii.

```
Threshold for Haar wavelet = 3.865420e-02
Threshold for Daubechies tap 9 wavelet = 5.809124e-02
```

Figure 33: The magnitudes of threshold coefficients

```
Energy with respect to the original signal (Haar wavelet) = 0.990004
Compression ratio (Haar wavelet) = 14.559322

Energy with respect to the original signal (Daubechies tap 9 wavelet) = 0.990004
Compression ratio (Daubechies tap 9 wavelet) = 16.310127
```

Figure 34: Energy and compression ratios of compressed signals

- Given below is a visual comparison of the original signal and the compressed signals.

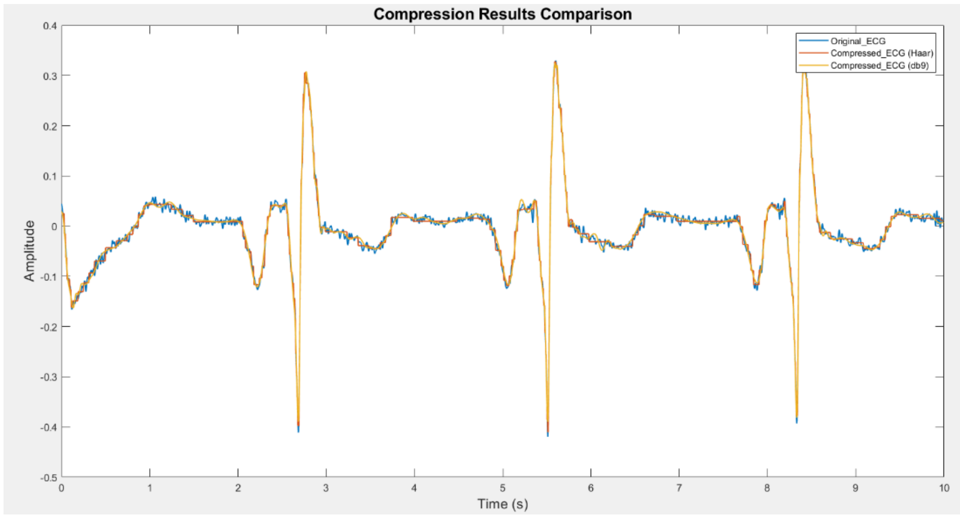


Figure 35: Compression of original signal and the compressed signals

- Due to compression, the high frequency fluctuations of the signal have been removed.