Homework 2
Due Oct 8-th Midnight

1. A total of m white and m black balls are distributed among two urns, with each urn containing m balls. At each stage, a ball is randomly selected from each urn and the two selected balls are interchanged. Let X_n denote the number of black balls in urn 1 after the nth interchange.

   (a) Give the transition probabilities of the Markov chain $X_n, n \geq 0$.

   (b) Find the limiting probabilities and show that the stationary chain is time reversible.

   Solution.

   (a) If $X_n = i$, then the first urn contains i black balls and $m - i$ white balls, and the second urn contains i white balls and $m - i$ black balls. The transition $i \to i + 1$ occurs if the ball selected from the first urn is white and the ball selected from the second urn is black. Thus,

   $$p_{i,i+1} = \frac{m-i}{m} \cdot \frac{m-i}{m} = \frac{(m-i)^2}{m^2}$$

   Similarly, we have

   $$p_{i,i-1} = \frac{i^2}{m^2},\ p_{i,i} = 2\frac{i(m-i)}{m^2}$$

   (b) By the above Proposition, we would like to solve:

   $$\pi_i p_{i,i+1} = \pi_{i+1} p_{i+1,i}, \quad i = 0, 1, ..., m-1, \quad \sum_{i=0}^{m} \pi_i = 1$$

   We have

   $$\frac{\pi_{i+1}}{\pi_i} = \frac{p_{i,i+1}}{p_{i+1,i}} = \left(\frac{m-i}{i+1}\right)^2$$

   $$\frac{\pi_0}{\pi_1} = \left(\frac{m}{1}\right)^2 = \binom{m}{1}^2, \quad \frac{\pi_2}{\pi_0} = \left(\frac{m}{1} \cdot \frac{m-1}{2}\right)^2 = \binom{m}{2}^2, \quad \ldots$$

   $$\frac{\pi_i}{\pi_0} = \binom{m}{i}^2, \quad \text{(by induction)}$$

   Therefore

   $$\pi = \frac{\binom{m}{i}^2}{\binom{m}{0}^2 + \binom{m}{1}^2 + \ldots + \binom{m}{m}^2}$$

   In fact, we have

   $$\binom{m}{0}^2 + \binom{m}{1}^2 + \ldots + \binom{m}{m}^2 = \binom{2m}{m}$$

   Thus

   $$\pi_i = \frac{\binom{m}{i}^2}{\binom{2m}{m}}$$

   Since we have found a solution, the chain is time reversible and π_i's are its stationary probabilities.

2. Consider a Markov chain $\{X_n, n \geq 0\}$ on the state space $E = \{1,2,3\}$ with transition probability matrix given by

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Assume that the (initial) distribution of X_0 is the uniform distribution on E.

(a) Calculate $\mathbb{P}(X_3 = 2, X_1 = 3)$.

(b) Calculate $\mathbb{P}(X_4 = 1, X_3 \neq 1, X_2 \neq 1 | X_1 = 1)$.

(c) Determine whether a limiting distribution exists for this MC, justifying your answer. If so, find the limiting distribution.

Solution.

(a) $\mu_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \mathbb{P}(X_3 = 2 | X_1 = 3) * \mathbb{P}(X_1 = 3)$
$\mu_1 = \mu_0 P \Rightarrow P(X_1 = 3) = \frac{5}{18}$
$\mathbb{P}(X_3 = 2, X_1 = 3) = P^2_{32}$

$$P^2_{32} \quad = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{6} \\ \frac{1}{3} \end{pmatrix} = \mathbb{P}(X_3 = 2, X_1 = 3) = \frac{7}{18}$$

$\mathbb{P}(X_3 = 2, X_1 = 3) = \frac{5}{18} * \frac{7}{18} = \frac{35}{324}$

(b)

$$\begin{aligned} \mathbb{P}(X_4 = 1, X_3 \neq 1, X_2 \neq 1 | X_1 = 1) &= \sum_{a,b \in 2,3} \mathbb{P}(X_4 = 1, X_3 = b, X_2 = a | X_1 = 1) \\ &= \sum_{a,b \in 2,3} P_{1a} P_{ab} P_{b1} \\ &= P_{12}P_{22}P_{21} + P_{12}P_{23}P_{31} + P_{13}P_{32}P_{21} + P_{13}P_{33}P_{31} \\ &= \frac{53}{216} \end{aligned}$$

(c) All states communicate with each other $\Rightarrow$ one class 0,1,2, irreducible.
Also, since $P^1_{11} = \frac{1}{6} > 0$, d(1)=1 and so, 0,1,2 is aperiodic.
Since this is a finite-state MC, it is positive recurrent.
By basic limit theorem, a unique limiting distribution exists.
Solving $\pi P = \pi$, i.e.

$$\begin{cases} \pi_0 &= \frac{1}{3}\pi_0 + \frac{1}{6}\pi_1 = \frac{2}{3}\pi_2 \\ \pi_1 &= \frac{1}{2}\pi_0 + \frac{1}{6}\pi_1 + \frac{1}{3}\pi_2 \\ \pi_2 &= \frac{1}{6}\pi_0 + \frac{2}{3}\pi_1 \\ 1 &= \pi_0 + \pi_1 + \pi_2 \end{cases}$$

It yields,

$$\begin{cases} \pi_0 &= \frac{22}{59} \approx 0.37288 \\ \pi_1 &= \frac{20}{59} \approx 0.33898 \\ \pi_2 &= \frac{17}{59} \approx 0.2881 \end{cases} \tag{1}$$

3. Four people sit at a round table for dinner. As appetizer, they play a game where a ball is passed from one to another as follows. At each round of the game, the person holding the ball will choose at random one of his two neighbors and then pass him the ball. The four people are numbered 1, 2, 3 and 4 in a clockwise order.

(a) Explain briefly why the successive positions of the ball in the game form a Markov chain.

(b) Give the transition matrix of the Markov chain.

(c) Find the communication classes of the chain.

(d) Find the periods of the states of the chain.

(e) Does the chain have limiting probabilities?

(f) Mr. Kwok, the person with the number 4, is now holding the ball. Find the expected number of rounds to be played until the ball goes back to him.

Solution.

(a) The position of the ball after round n depends only on its position when the round is played, that is, the position after round $n-1$. This property defines a Markov chain.

(b) The state space of the chain is $\{1, 2, 3, 4\}$, the four people numbered in a clockwise order. The transition matrix is

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

(c) Clearly this chain has only one class (with all the 4 states): the chain is irreducible.

(d) As the chain is irreducible, all states have a same period. Consider state "1". Clearly, $p_{11}^{2k+1} = 0$, so that return to state "1" is impossible with an odd number of moves (rounds).

Also $p_{11}^{2} = \frac{1}{2} > 0$, and this implies that $p_{11}^{2k} > 0$ for all $k \geq 1$. Therefore $p_{11}^{n} > 0$ iff n is an even integer: state "1" has period 2; all other states have period 2 as well.

(e) Since the chain is periodic, P^n will not converge to a row-constant matrix when $n \to \infty$: no limiting probabilities exist.

(f) Let E_i denote the expected rounds that the ball goes to player i. Then we have:

$$\begin{aligned} E_4 &= 0.5E_1 + 0.5E_3 + 1, \\ E_3 &= 0.5E_2 + 1, \\ E_2 &= 0.5E_1 + 0.5E_3 + 1, \\ E_1 &= 0.5E_2 + 1. \end{aligned}$$

Therefore,

$$E_4 = 4.$$

Thus, the expected round to be played until the ball goes back to Mr. Kwok is 4.

4. A coin is successively flipped and its probability of showing up heads is $p \in (0, 1)$. Let $Y_1, Y_2, \ldots$ be the sequence of outcomes. For $n \geq 0$, let

$$X_n = (Y_{n+1}, Y_{n+2}, Y_{n+3}),$$

that is the nth triple of consecutive outcomes from the flips. The sequence X_n is a Markov chain on the state space

$$E = \{\text{TTT,TTH,THT,HTT,THH,HTH,HHT,HHH}\} := \{1, 2, 3, 4, 5, 6, 7, 8\},$$

where T and H denotes tail and head shown in the flips while the 8 possible states are also numbered from 1 to 8 in the order given above.

In all the calculations, we denote $q = 1 - p$ to simplify the formulas.

(a) Suppose we just see THH as the last three outcomes. Determine the probability to observe, after two more flips, the triples THH and HTT.

(b) Determine the transition matrix of the chain.

(c) Determine the communication classes of the chain.

(d) Determine the periods of the states.

(e) Show that the long run proportions of the eight states in E exist for the chain and determine these proportions.

(f) The triples appearing in the chain are overlapping for near times: for example by definition the last two flip outcomes $\{Y_{n+2}, Y_{n+3}\}$ in X_n are identical to the first two ones in X_{n+1}. We now consider the *non-overlapping* triples in the sequence as follows:

$$Y_1Y_2Y_3,\ Y_4Y_5Y_6,\ Y_7Y_8Y_9,\ \ldots.$$

Clearly, the outcomes from these non-overlapping triples are still the eight ones in E.

Show that the long run proportions of the eight states in E from these non-overlapping triples still exist and they coincide with those found in (e).

Solution.

(a) After two more flips, the new last three outcomes would start with "H".
So $\mathbb{P}(THH) = 0$, and $\mathbb{P}(HTT) = q^2$

(b) The transition matrix:

$$P = (p_{ij}) = \begin{pmatrix} q & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & p & 0 & 0 \\ q & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & p \\ 0 & 0 & q & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & p \end{pmatrix}$$

(c) There exists a path connecting all 8 states:

$$(HTT) \to TTT \to THH \to HHH \to HHT \to HTH \to THT \to HTT \to (TTT)$$

All states communicate with each other.

(d) Since $P_{11} = q > 0$, state "TTT" is aperiodic. By class property, all states are aperiodic as well as the Markov Chain.

(e) The Markov Chain is irreducible and aperiodic on a finite-state space. The MC is hence positive recurrent. The long run proportions exist and all equal to the unique invariant probability distribution. $(\pi_1, \ldots, \pi_8)$.
Solving

$$(\pi_1, \ldots, \pi_8)P = (\pi_1, \ldots, \pi_8)$$
$$(\pi_1, \ldots, \pi_8) = (q^3, pq^2, pq^2, pq^2, p^2q, p^2q, p^2q, p^3)$$

(f) Let $Z_0 = Y_1Y_2Y_3, Z_1 = Y_4Y_5Y_6, \ldots, Z_n = Y_{3n+1}Y_{3n+2}Y_{3n+3}, n \geq 0$
Clearly, $(Z_0, Z_1, Z_2, \ldots) = (X_0, X_3, X_6, \ldots)$ is a subsequence of (X_n)
$Ł(Z_n|Z_{n-1}, \ldots, Z_0) = Ł(Z_n|Z_{n-1}) = Ł(X_{3n}|X_{3(n-1)})$
So Z_n is a Markov Chain with transition matrix $Q = P^3$.
The convergence of P_{ij}^n implies that of Q_{ij}^n: long-run proportion of (Z_n) exists.
$\pi P = \pi \Rightarrow \pi Q = \pi$
The proportions are identical of that of (X_n).

5. I have 4 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.

   (a) If the probability of rain is p, what is the probability that I get wet?

   (b) Current estimates show that $p = 0.6$ in Edinburgh. How many umbrellas should I have so that, if I follow the strategy above, the probability I get wet is less than 0.1?

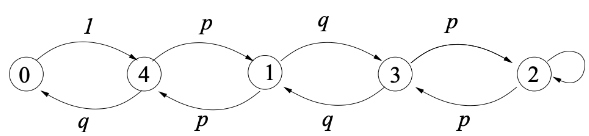
   Solution. To solve the problem, consider a Markov chain taking values in the set $S = \{i : i = 0, 1, 2, 3, 4\}$, where i represents the number of umbrellas in the place where I am currently at (home or office). If $i = 1$ and it rains then I take the umbrella, move to the other place, where there are already 3 umbrellas, and, including the one I bring, I have next 4 umbrellas. Thus,

$$p_{1,4} = p$$

   because p is the probability of rain. If $i = 1$ but does not rain then I do not take the umbrella, I go to the other place and find 3 umbrellas. Thus,

$$p_{1,3} = 1 - p \equiv q$$

   Continuing in the same manner, We can form a Markov chain with the following diagram:



   Let us find the stationary distribution. By equating fluxes, we have:

$$\pi(2) = \pi(3) = \pi(1) = \pi(4)$$

$$\pi(0) = \pi(4)q$$

   Also,

$$\sum_{i=0}^{4} \pi(i) = 1$$

   Expressing all probabilities in terms of $\pi(4)$ and inserting in this last equation, we find

$$\pi(4)q + 4\pi(4) = 1$$

   or

$$\pi(4) = \frac{1}{q+4} = \pi(1) = \pi(2) = \pi(3), \quad \pi(0) = \frac{q}{q+4}$$

   I get wet every time I happen to be in state 0 and it rains. The chance I am in state 0 is $\pi(0)$. The chance it rains is p. Hence

$$P(WET) = \pi(0) \cdot p = \frac{qp}{q+4}$$

   With $p = 0.6$, i.e., $q = 0.4$, we have

$$P(WET) \approx 0.0545$$

   If I want the chance to be less than 1% then, clearly, I need more umbrellas. So, suppose I need N umbrellas. Set up the Markov chain as above. It is clear that

$$\pi(N) = \pi(N-1) = ... = \pi(1)$$

$$\pi(0) = \pi(N)q$$

Inserting in $\sum_{i=0}^{N} \pi(i)$, we find

$$\pi(N) = \frac{1}{q+N} = \pi(N-1) = ... = \pi(1), \quad \pi(0) = \frac{q}{q+N}$$

and so,

$$P(WET) = \frac{pq}{q+N}$$

We want $P(WET) = 1/100$, or $q + N > 100pq$, or

$$N > 100pq - q = 100 \times 0.4 \times 0.6 - 0.4 = 23.6$$

So to reduce the chance of getting wet from around 6% to less than 1% I need 24 umbrellas instead of 4. That's too much. I'd rather get wet.