The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science

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DDA 3005 — Numerical Methods

Solutions — Midterm Exam

Please make sure to present your solutions and answers in a comprehensible way and give explanations of your steps and results!

- *The exam time is 90 minutes.*
- *The total number of achievable points is* 100 *points.*
- *Please abide by the honor codes of CUHK-SZ.*

Good Luck!

Exercise 1 (LU Factorization): (*22* points)

Consider the matrix $\boldsymbol{A} \in \mathbb{R}^{4\times 4}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 & 8 \\ 2 & 0 & 2 & 4 \\ 0 & 1 & 0 & 3 \\ 1 & -2 & 1 & 0 \end{bmatrix}.$$

a) Compute an LU factorization with pivoting of the matrix $\boldsymbol{A}$. State the final $\boldsymbol{L}$ and $\boldsymbol{U}$ matrices and the obtained permutation matrix $\boldsymbol{P}$.

b) Solve the linear system $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{b} = [3,\, 2,\, 2,\, 1]^\top$.

Solution :

a) We can choose 2 as first pivot and swap rows 1 and 2 of $\boldsymbol{A}$. This yields

$$\boldsymbol{M}_1 = \begin{bmatrix} 1 & & & \\ -\frac{1}{2} & 1 & & \\ 0 & 0 & 1 & \\ -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{P}_1\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 1 & 0 & 2 & 8 \\ 0 & 1 & 0 & 3 \\ 1 & -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & 1 & & \\ 0 & 0 & 1 & \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 & 4 \\ & 0 & 1 & 6 \\ & 1 & 0 & 3 \\ & -2 & 0 & -2 \end{bmatrix} = \boldsymbol{L}_1\boldsymbol{U}_1.$$

The next pivot is -2, i.e., we first swap rows 2 and 4 of $\boldsymbol{P}_1\boldsymbol{A}$, $\boldsymbol{L}_1$, and $\boldsymbol{U}_1$:

$$\boldsymbol{P}_2\boldsymbol{P}_1\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & 1 & & \\ 0 & 0 & 1 & \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 & 4 \\ & -2 & 0 & -2 \\ & 1 & 0 & 3 \\ & 0 & 1 & 6 \end{bmatrix} = \boldsymbol{L}_1\boldsymbol{U}_1.$$

This yields

$$\boldsymbol{M}_2 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & \frac{1}{2} & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{P}_2\boldsymbol{P}_1\boldsymbol{A} = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & 1 & & \\ 0 & -\frac{1}{2} & 1 & \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 & 4 \\ & -2 & 0 & -2 \\ & & 0 & 2 \\ & & 1 & 6 \end{bmatrix} = \boldsymbol{L}_2\boldsymbol{U}_2.$$

We now swap the last rows of $\boldsymbol{P}_2\boldsymbol{P}_1\boldsymbol{A}$, $\boldsymbol{L}_2$, and $\boldsymbol{U}_2$ to obtain:

$$\boldsymbol{P}_3\boldsymbol{P}_2\boldsymbol{P}_1\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & 2 & 8 \\ 0 & 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & 1 & & \\ \frac{1}{2} & 0 & 1 & \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 & 4 \\ & -2 & 0 & -2 \\ & & 1 & 6 \\ & & & 2 \end{bmatrix} = \boldsymbol{LU}.$$

In summary, we have:

$$\boldsymbol{P} = \begin{bmatrix} & 1 & & \\ & & & 1 \\ 1 & & & \\ & & 1 & \end{bmatrix}, \quad \boldsymbol{L} = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & 1 & & \\ \frac{1}{2} & 0 & 1 & \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} 2 & 0 & 2 & 4 \\ & -2 & 0 & -2 \\ & & 1 & 6 \\ & & & 2 \end{bmatrix}.$$

b) Using $\boldsymbol{PA} = \boldsymbol{LU}$, it follows $\boldsymbol{Ax} = \boldsymbol{b} \iff \boldsymbol{PAx} = \boldsymbol{Pb} \iff \boldsymbol{LUx} = \boldsymbol{Pb}$. We have $\boldsymbol{Pb} = [2,\, 1,\, 3,\, 2]^\top$. We first solve $\boldsymbol{Ly} = \boldsymbol{Pb}$ via forward-substitution:

$$\boldsymbol{Ly} = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & 1 & & \\ \frac{1}{2} & 0 & 1 & \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}.$$

This implies $y_1 = 2$, $y_2 = 0$, $y_3 = 2$, and $y_4 = 2$. We now solve $\boldsymbol{Ux} = \boldsymbol{y}$ via back-substitution:

$$\boldsymbol{Ux} = \begin{bmatrix} 2 & 0 & 2 & 4 \\ & -2 & 0 & -2 \\ & & 1 & 6 \\ & & & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

which yields $x_4 = 1$, $x_3 = -4$, $x_2 = -1$, and $x_1 = 1 - x_3 - 2x_4 = 3$.

Exercise 2 (Linear Data Fitting): (*27* points)

Let $\{(t_i, b_i)\}_{i=1}^4$ be a set of given data points. We want to find a linear mapping $f(t, \boldsymbol{x}) = x_1 + x_2 t$ that fits the data as much as possible, i.e., $f(t_i, \boldsymbol{x}) \approx b_i$ for all i.

The data points are given explicitly by:

$$t_1 = -1, \quad b_1 = -1 \quad \Big| \quad t_2 = 0, \quad b_2 = 1 \quad \Big| \quad t_3 = 1, \quad b_3 = 0 \quad \Big| \quad t_4 = 2, \quad b_4 = 1.$$

This problem can be expressed as a linear least squares problem of the form

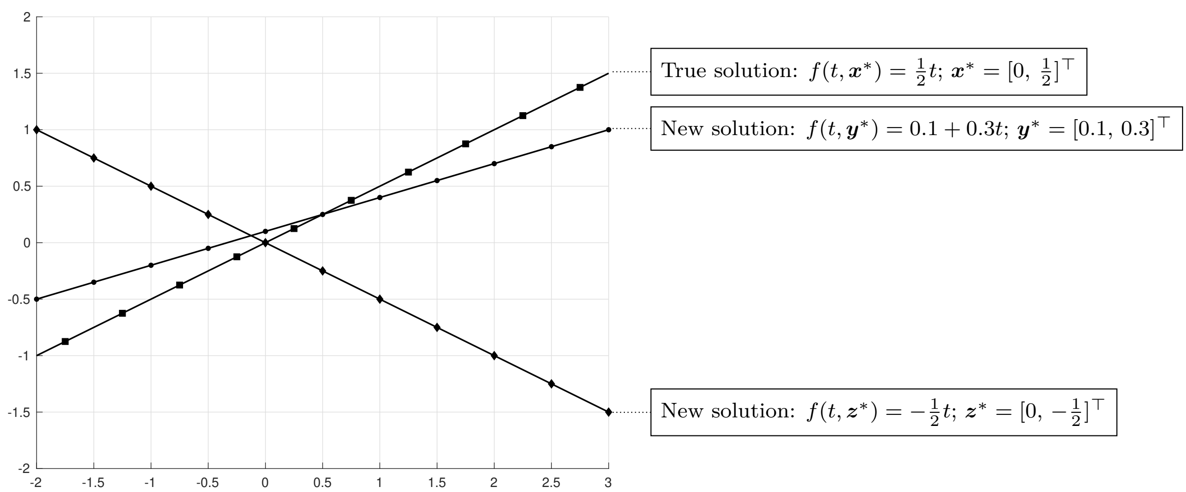
$$\min_{\boldsymbol{x} \in \mathbb{R}^2} \|\boldsymbol{Ax} - \boldsymbol{b}\|_2^2 \quad \text{where} \quad \boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \tag{1}$$

a) Compute a QR factorization of $\boldsymbol{A}$ using Householder transformations. State the obtained upper triangular matrix $\boldsymbol{R}$ of the factorization.

(*You don't need to form or calculate the orthogonal matrix $\boldsymbol{Q}$ explicitly*).

b) Show that $\boldsymbol{x}^* = [0, \frac{1}{2}]^\top$ is a solution of the least squares problem (1). Is $\boldsymbol{x}^*$ a unique solution to the problem (1)?

c) One of your friends, Kafka, wants to repeat the data fitting problem for two more choices of the input vector $\boldsymbol{b}$. He obtains the following plot of his results:



Unfortunately, Kafka forgot to record the different choices of $\boldsymbol{b}$ that he tested. He only remembers that one of the choices – say $\tilde{\boldsymbol{b}}$ – satisfies the condition $\|\Delta \boldsymbol{b}\| = \|\boldsymbol{b} - \tilde{\boldsymbol{b}}\| \le 1$.

Can you help Kafka to figure out which of the computed solutions $\boldsymbol{y}^*$ or $\boldsymbol{z}^*$ corresponds to $\tilde{\boldsymbol{b}}$? More specifically, choose one of the following statements as your response to Kafka:

1.) $\boldsymbol{y}^*$ *seems to be a potential solution to the updated least-squares problem using* $\tilde{\boldsymbol{b}}$*!*

2.) $\boldsymbol{z}^*$ *should be the correct solution to the updated least-squares problem!*

3.) *It seems that both of the points* $\boldsymbol{y}^*$ *and* $\boldsymbol{z}^*$ *are potential solutions to the updated least-squares problem.*

4.) *Strange! I am convinced that none of the points can be a solution to the new least-squares problem when using* $\tilde{\boldsymbol{b}}$ *as input.*

The following additional information can be used to make your decision:

$$\operatorname{cond}(\boldsymbol{A}) = \tfrac{\sqrt{5}+1}{2} \approx 1.62, \quad \|\boldsymbol{A}\boldsymbol{x}^*\| = \tfrac{\sqrt{6}}{2} \approx 1.23, \quad \|\boldsymbol{b}\| = \sqrt{3} \approx 1.74, \quad \|\boldsymbol{x}^*\| = \tfrac{1}{2}.$$

Provide detailed explanations for your choice and decision!

Solution :

a) Noticing $\|\boldsymbol{a}_1\| = 2$, we can choose $\alpha_1 = -2$ and $\boldsymbol{v}_1 = \boldsymbol{a}_1 + 2\boldsymbol{e}_1 = [3,\, 1,\, 1,\, 1]^\top$ It then follows $\|\boldsymbol{v}_1\|^2 = 9 + 3 = 12$ and

$$\boldsymbol{Q}_1 = \boldsymbol{H}_{v_1} = \boldsymbol{I} - \frac{1}{6}\boldsymbol{v}_1\boldsymbol{v}_1^\top, \quad \boldsymbol{Q}_1\boldsymbol{a}_2 = \boldsymbol{a}_2 - 0 \cdot \boldsymbol{v}_1, \quad \boldsymbol{A} \to \boldsymbol{Q}_1\boldsymbol{A} = \begin{bmatrix} -2 & -1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}.$$

Next, due to $\|[0,\, 1,\, 2]^\top\| = \sqrt{5}$, we can choose $\alpha_2 = \pm\sqrt{5}$ and $\boldsymbol{v}_2 = [0,\, 1,\, 2]^\top \mp [\sqrt{5},\, 0,\, 0]^\top = [\mp\sqrt{5},\, 1,\, 2]^\top$. We obtain $\|\boldsymbol{v}_2\|^2 = 5 + 1 + 4 = 10$ and

$$\boldsymbol{H}_{v_2} = \boldsymbol{I}_2 - \frac{1}{5}\boldsymbol{v}_2\boldsymbol{v}_2^\top, \quad \boldsymbol{A} \to \boldsymbol{Q}_2\boldsymbol{Q}_1\boldsymbol{A} = \begin{bmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}_{v_2} \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & \pm\sqrt{5} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(This step actually does not require any calculations). Thus, we have

$$\boldsymbol{R} = \begin{bmatrix} -2 & -1 \\ 0 & \pm\sqrt{5} \end{bmatrix}.$$

b) We only need to verify that $\boldsymbol{x}^*$ satisfies the normal equation $\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{A}^\top\boldsymbol{b}$:

$$\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{A}^\top \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and

$$\boldsymbol{A}^\top\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Hence, $\boldsymbol{x}^*$ is an optimal solution to the least squares problem (1). As $\boldsymbol{A}$ has full column rank, $\boldsymbol{x}^*$ is the only, unique solution to (1).

c) As discussed in the lectures, we have

$$\frac{\|\tilde{\boldsymbol{x}} - \boldsymbol{x}^*\|}{\|\boldsymbol{x}^*\|} \leq \frac{\operatorname{cond}(\boldsymbol{A})}{\cos(\theta)} \frac{\|\Delta \boldsymbol{b}\|}{\|\boldsymbol{b}\|} = \frac{\operatorname{cond}(\boldsymbol{A})\|\boldsymbol{b}\|}{\|\boldsymbol{A}\boldsymbol{x}^*\|} \frac{\|\Delta \boldsymbol{b}\|}{\|\boldsymbol{b}\|} = \frac{\operatorname{cond}(\boldsymbol{A})}{\|\boldsymbol{A}\boldsymbol{x}^*\|}\|\Delta \boldsymbol{b}\|$$

where $\cos(\theta) = \frac{\|\boldsymbol{A}\boldsymbol{x}^*\|}{\|\boldsymbol{b}\|}$. Hence, we can infer

$$\|\tilde{\boldsymbol{x}} - \boldsymbol{x}^*\| \leq \frac{\operatorname{cond}(\boldsymbol{A})\|\boldsymbol{x}^*\|}{\|\boldsymbol{A}\boldsymbol{x}^*\|}\|\Delta \boldsymbol{b}\| \leq \frac{\sqrt{5}+1}{2\sqrt{6}} \leq \frac{1}{2}\Big(1 + \frac{1}{\sqrt{6}}\Big) \leq \frac{1}{2}\Big(1 + \frac{1}{2}\Big) = \frac{3}{4}.$$

Due to $\|\boldsymbol{z}^* - \boldsymbol{x}^*\| = 1 > \frac{3}{4}$, $\boldsymbol{z}^*$ cannot be a solution. Due to $\|\boldsymbol{y}^* - \boldsymbol{x}^*\| = \sqrt{0.01 + 0.04} = \frac{\sqrt{5}}{10} < 0.3 < 0.75$, $\boldsymbol{y}^*$ can be a potential solution to the problem. Hence, option 1.) is the only meaningful response.

Exercise 3 (Error Analysis of the Absolute Value): (*18* points)
We consider the problem of evaluating the absolute value function

$$f(x) := \sqrt{x^2}. \tag{2}$$

For all floating-point systems appearing in this question, we assume:

- The IEEE-Standard 754 holds, i.e., for any machine numbers x, y, we have $x \circledast y = \mathrm{fl}(x * y) = (1+\varepsilon)(x \circledast y)$ and $\mathrm{sqrt}(x) = \mathrm{fl}(\sqrt{x}) = (1+\varepsilon)\sqrt{x}$ for some ε with $|\varepsilon| \leq \varepsilon_{\text{mach}}$. Here, $*$ can represent any arithmetic operation $\{+, -, \cdot, /\}$.

a) Is the problem (2) well- or ill-conditioned for $x \neq 0$? Provide detailed explanations!

b) Consider the following algorithm:

$$\hat{f}(x) = \mathrm{sqrt}(\mathrm{fl}(x) \odot \mathrm{fl}(x)),$$

where $\mathrm{sqrt}(\cdot)$ and $\odot$ correspond to the arithmetic operations in the floating-point system.

Show that $\hat{f}$ is a stable algorithm to compute $f(x) = \sqrt{x^2} = |x|$.

Hint: The following facts can be useful. Let ε_i, $i = 1, \ldots, m$ be a collection of numbers satisfying $|\varepsilon_i| \leq \varepsilon_{\text{mach}}$. Then, (for all i) it holds that $\sqrt{1+\varepsilon_i} = 1 + \mathcal{O}(\varepsilon_{\text{mach}})$ and $\prod_{i=1}^{m}(1 + \varepsilon_i) = 1 + \mathcal{O}(\varepsilon_{\text{mach}})$. *(You don't need to verify these facts).*

Solution :

a) Notice that, for $x \neq 0$, we have

$$\operatorname{cond}_f(x) = \left|\frac{xf'(x)}{f(x)}\right| = \left|\frac{x/(2\sqrt{x^2}) \cdot 2x}{\sqrt{x^2}}\right| = \left|\frac{x^2}{x^2}\right| = 1.$$

Thus, evaluating f is well-conditioned for all $x \neq 0$.

b) As the IEEE-standard 754 holds in this system, there exists ε_i, $i = 1, 2, 3$ with $|\varepsilon_i| \le \varepsilon_{\text{mach}}$ such that

$$\text{fl}(x) = (1 + \varepsilon_1)x, \quad \text{fl}(x) \odot \text{fl}(x) = (1 + \varepsilon_2)(1 + \varepsilon_1)^2 x^2, \quad \text{and}$$

$\hat{f}(x) = \text{sqrt}((1 + \varepsilon_2)(1 + \varepsilon_1)^2 x^2) = (1 + \varepsilon_3)\sqrt{1 + \varepsilon_2}(1 + \varepsilon_1) \cdot \sqrt{x^2}$. Consequently, defining $\hat{x} := (1 + \varepsilon_3)\sqrt{1 + \varepsilon_2}(1 + \varepsilon_1)x$, we have

$$\hat{f}(x) = f(\hat{x}).$$

In addition, using the hint, it follows $(1+\varepsilon_3)\sqrt{1 + \varepsilon_2}(1+\varepsilon_1) = (1+\mathcal{O}(\varepsilon_{\text{mach}}))(1+\mathcal{O}(\varepsilon_{\text{mach}})) = 1 + \mathcal{O}(\varepsilon_{\text{mach}})$ and $\hat{x} = (1 + \mathcal{O}(\varepsilon_{\text{mach}}))x$ Thus, we can infer

$$\frac{|\hat{x} - x|}{|x|} = \frac{|(1 + \mathcal{O}(\varepsilon_{\text{mach}}))x - x|}{|x|} = \mathcal{O}(\varepsilon_{\text{mach}}).$$

This shows that $\hat{f}$ is a stable algorithm.

Exercise 4 (Miscellaneous): (*12* points)

State whether each of the following statements is *True* or *False*. For each statement provide a short explanation, counterexample, or proof. Only answers with full explanations will be graded. (Short answers of the form "true" or "false" will not be accepted).

a) Let $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. Then, it holds that $\|\boldsymbol{Q}\|_F \le 1$.

b) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a singular matrix. Then, $\boldsymbol{A}$ cannot have an LU factorization of the form $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ where $\boldsymbol{L} \in \mathbb{R}^{n\times n}$ is a lower unit triangular matrix and $\boldsymbol{U}$ is an upper triangular matrix.

c) Mesmer wants to numerically solve a linear system of equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \quad \text{with} \quad \boldsymbol{A} \in \mathbb{R}^{3005\times 3005} \text{ and } \boldsymbol{b} \in \mathbb{R}^{3005}, \tag{3}$$

where $\boldsymbol{A}$ is a symmetric matrix. He wants to solve the system (3) as quickly as possible (with the smallest computational costs). Accuracy is not Mesmer's primary concern, but the method should not break or return a potential error. Mesmer has the following options:

1.) Use a Householder QR factorization to factorize $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$ and compute $\boldsymbol{x}_{\text{sol}} = \boldsymbol{R}^{-1}\boldsymbol{Q}^\top\boldsymbol{b}$.

2.) Use an LU factorization $\boldsymbol{P}\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ to solve (3) and recover $\boldsymbol{x}_{\text{sol}} = \boldsymbol{U}^{-1}(\boldsymbol{L}^{-1}\boldsymbol{P}\boldsymbol{b})$.

3.) Apply a Cholesky factorization to obtain $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^\top$ and set $\boldsymbol{x}_{\text{sol}} = \boldsymbol{L}^{-\top}(\boldsymbol{L}^{-1}\boldsymbol{b})$.

4.) Use an LU factorization without pivoting $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ to obtain $\boldsymbol{x}_{\text{sol}} = \boldsymbol{U}^{-1}(\boldsymbol{L}^{-1}\boldsymbol{b})$.

Then, option 2.) is the best choice for Mesmer.

Solution :

a) *False.* For $\boldsymbol{Q} = \boldsymbol{I}$, we obtain $\|\boldsymbol{Q}\|_F = \sqrt{n}$. This is a counter-example if $n \ge 2$.

b) *False.* Counter-example: $\boldsymbol{A} = 0$; we can write it as $\boldsymbol{A} = 1 \cdot 0$ which satisfies all requirements. We can also consider the $n \times n$ case: $\boldsymbol{0} = \boldsymbol{A} = \boldsymbol{I} \cdot \boldsymbol{0}$.

c) *True.* The approaches 3.) and 4.) might break down (as $\boldsymbol{A}$ is not assumed to be positive definite). The Householder QR factorization requires more flops than the LU factorization.

Exercise 5 (Updating the Cholesky Factorization): (*21* points)
Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a symmetric, positive definite matrix and let $\boldsymbol{v}, \boldsymbol{b} \in \mathbb{R}^n$ be given. Suppose that we have access to a Cholesky factorization of the matrix $\boldsymbol{A}$, i.e., $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^\top$, where $\boldsymbol{L} \in \mathbb{R}^{n\times n}$ is a lower triangular matrix. Our goal is to develop a suitable factorization for the updated matrix

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{v}\boldsymbol{v}^\top.$$

Let us further set $\boldsymbol{w} := \boldsymbol{L}^{-1}\boldsymbol{v}$, $\gamma := \sqrt{1+\|\boldsymbol{w}\|^2} - 1$, and $\tilde{\boldsymbol{L}} := \boldsymbol{L}(\boldsymbol{I} + \gamma\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2})$.

a) Show that $\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{L}}\tilde{\boldsymbol{L}}^\top$.

b) Verify that the inverse of $\boldsymbol{I} + \gamma\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2}$ is given by $\boldsymbol{I} - \frac{\gamma}{\gamma+1}\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2}$.

c) Based on the given Cholesky factorization of $\boldsymbol{A}$, develop an algorithm to compute the solution $\boldsymbol{x}^*$ of the updated system $\tilde{\boldsymbol{A}}\boldsymbol{x}^* = \boldsymbol{b}$ that only requires $\mathcal{O}(n^2)$ operations. Explain why your algorithm achieves such a complexity.

   Remark: You don't need to write `MATLAB` or `Python` code here. Try to be brief and use concise steps or compact pseudocode.

Solution :

a) We directly compute

$$\begin{aligned}\tilde{\boldsymbol{L}}\tilde{\boldsymbol{L}}^\top &= \boldsymbol{L}\Big(\boldsymbol{I} + \gamma\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2}\Big)\Big(\boldsymbol{I} + \gamma\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2}\Big)\boldsymbol{L}^\top = \boldsymbol{L}\Big(\boldsymbol{I} + 2\gamma\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2} + \gamma^2\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2}\Big)\boldsymbol{L}^\top \\ &= \boldsymbol{L}\boldsymbol{L}^\top + \frac{2\gamma+\gamma^2}{\|\boldsymbol{w}\|^2}\boldsymbol{L}\boldsymbol{w}(\boldsymbol{L}\boldsymbol{w})^\top.\end{aligned}$$

   Using $2\gamma+\gamma^2 = (1+\gamma)^2-1 = 1+\|\boldsymbol{w}\|^2-1 = \|\boldsymbol{w}\|^2$ and $\boldsymbol{L}\boldsymbol{w} = \boldsymbol{v}$, this implies $\tilde{\boldsymbol{L}}\tilde{\boldsymbol{L}}^\top = \boldsymbol{A}+\boldsymbol{v}\boldsymbol{v}^\top$.

b) We have

$$\Big(\boldsymbol{I} + \gamma\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2}\Big)\Big(\boldsymbol{I} - \frac{\gamma}{\gamma+1}\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2}\Big) = \boldsymbol{I} + \gamma\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2} - \frac{\gamma}{\gamma+1}\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2} - \frac{\gamma^2}{\gamma+1}\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2}$$

   Due to $(\gamma+1)\gamma - \gamma - \gamma^2 = 0$, this proves that $\boldsymbol{I} - \frac{\gamma}{\gamma+1}\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2}$ is the inverse of $\boldsymbol{I} + \gamma\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2}$.

c) Since $\boldsymbol{L}$ is lower triangular, we can compute the vector $\boldsymbol{w} = \boldsymbol{L}^{-1}\boldsymbol{v}$ using one forward substitution which requires $\mathcal{O}(n^2)$ flops. Computation of γ requires $2n+2 = \mathcal{O}(n)$ (or $2n+1$) flops. To obtain $\boldsymbol{x}^*$, we can perform the steps

$$\tilde{\boldsymbol{A}}\boldsymbol{x}^* = \boldsymbol{b} \quad\iff\quad \tilde{\boldsymbol{L}}\tilde{\boldsymbol{L}}^\top\boldsymbol{x}^* = \boldsymbol{b} \quad\iff\quad \boldsymbol{y}^* = \tilde{\boldsymbol{L}}^{-1}\boldsymbol{b} \text{ and } \boldsymbol{x}^* = \tilde{\boldsymbol{L}}^{-\top}\boldsymbol{y}^*.$$

   Based on part b) and setting $\boldsymbol{y} = \boldsymbol{L}^{-1}\boldsymbol{b}$, we have

$$\boldsymbol{y}^* = \tilde{\boldsymbol{L}}^{-1}\boldsymbol{b} = \Big(\boldsymbol{I} - \frac{\gamma}{\gamma+1}\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2}\Big)\boldsymbol{L}^{-1}\boldsymbol{b} = \boldsymbol{y} - \frac{\gamma}{\gamma+1}\cdot\boldsymbol{w}^\top\boldsymbol{y}\cdot\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|^2}.$$

   The computation of $\boldsymbol{y}$ involves another forward substitution (costing $\mathcal{O}(n^2)$). If $\|\boldsymbol{w}\|^2$ is stored, then computation of $\boldsymbol{y}^*$ requires additional $n + (2n-1) + n + 4 = 4n+3 = \mathcal{O}(n)$ flops. The final costs to compute $\boldsymbol{x}^*$ are identical to this step:

$$\boldsymbol{x}^* = \tilde{\boldsymbol{L}}^{-\top}\boldsymbol{y}^* = (\boldsymbol{L}^\top)^{-1}\Big(\boldsymbol{I} - \frac{\gamma}{\gamma+1}\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2}\Big)\boldsymbol{y}^*.$$

   The computation of $(\boldsymbol{I} - \frac{\gamma}{\gamma+1}\frac{\boldsymbol{w}\boldsymbol{w}^\top}{\|\boldsymbol{w}\|^2})\boldsymbol{y}^*$ can be done within $\mathcal{O}(n)$ steps. Since $\boldsymbol{L}^\top$ is upper triangular, $\boldsymbol{x}^*$ can then be obtained via a final back-substitution ($\mathcal{O}(n^2)$). In total the overall complexity is given by $\mathcal{O}(n^2)$.