Useful formulas for "Stochastic Processes"

Chapter 1

- $\mathbb{P}\left(\bigcup_1^n A_i\right) = \sum_{k=1}^{n}(-1)^{k+1} \sum_{\{i_1,\ldots,i_k\}} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$
- The coupon collection problem: $\mathbb{E}(N) = \sum_{k=1}^n (-1)^{k+1} \sum_{\{i_1,\ldots,i_k\}} \frac{1}{p_{i_1}+\cdots+p_{i_k}}$
- $X \geq 0$: $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > u)du$, $\mathbb{E}[X] = \sum_{k=0}^\infty \mathbb{P}(X > k)$.
- Order statistics $(X_{(1)}, \ldots, X_{(n)})$ has joint density $n! f(x_1)\cdots f(x_n) 1_{\{x_1<\cdots<x_n\}}$
- ith order statistic $X_{(i)}$ has density $i\binom{n}{i} f(x)F^{i-1}(x)\{1-F(x)\}^{n-i}$.
- $\mathbb{E}(X) = \mathbb{E}\{\mathbb{E}(X|Y)\}$.
- $\mathrm{var}(X) = \mathbb{E}\{\mathrm{var}(X|Y)\} + \mathrm{var}\{\mathbb{E}(X|Y)\}$.
- Random sum $S_N = X_1 + \cdots + X_N$: $\mathbb{E}(S_N) = \mu\,\mathbb{E}(N)$, $\mathrm{var}(S_N) = \sigma^2 E(N) + \mu^2 \mathrm{var}(N)$.

Chapter 2

- Hitting time of i: $\{\tau_i = n\} = \{X_n = i, X_{n-1} \neq i, \ldots, X_1 \neq i\}$.
- $f_{ii}^{(n)} = \mathbb{P}(\tau_i = n \mid X_0 = i)$, $f_i = \sum_{n=1}^\infty f_{ii}^{(n)} = \mathbb{P}(\tau_i < \infty \mid X_0 = i)$.
- $P_{ii}^n = \sum_{k=1}^n f_{ii}^{(k)} P_{ii}^{n-k}, \quad n \geq 1.$
- $f_{jk}^{(n)} = \mathbb{P}\{X_n = k, X_{n-1} \neq k, \ldots, X_1 \neq k \mid X_0 = j\}$.
- $P_{jk}^n = \sum_{\ell=1}^n f_{jk}^{(\ell)} P_{kk}^{n-\ell}, \quad n \geq 1.$
- $U_{jk}(s) = \sum_{n=0}^\infty P_{jk}^n s^n, \quad F_{jk}(s) = \sum_{n=1}^\infty f_{jk}^{(n)} s^n.$
- $U_{jk}(s) - 1_{\{j=k\}} = F_{jk}(s) U_{kk}(s)$.
- Gambler's probabilities of reaching a fortune N, $0 \leq i \leq N$ and $r = q/p$:
$$p_i = \begin{cases} (1-r^i)/(1-r^N), & \text{if } p \neq 1/2 \\ i/N, & \text{if } p = 1/2 \end{cases}$$
Ruin probabilities: $q_i = 1 - p_i$.
- $s_{ij} = \mathbb{E}\left[\sum_{n=0}^\infty 1_{\{X_n=j\}} \Big| X_0 = i\right] = \sum_{n=0}^\infty \mathbb{P}(X_n = j \mid X_0 = i) = \sum_{n=0}^\infty P_{ij}^n.$
- $s_{ij} = 1_{\{i=j\}} + \sum_{k=1}^M P_{ik} s_{kj}$, or in matrix form, $S = I + P_T S$. Equivalently, $S = (I - P_T)^{-1}$.
- $f_{ij} = \sum_{n=1}^\infty f_{ij}^{(n)} = \frac{s_{ij} - 1_{\{i=j\}}}{s_{jj}}, \quad i, j \in T.$

Chapter 3

- $\min\{X_1, \ldots, X_n\} \sim \mathcal{E}(\lambda_1 + \cdots + \lambda_n)$
- $\mathbb{P}\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$
- $\mathbb{E}\left[X_1 1_{\{X_1<X_2\}}\right] = \frac{\lambda_1}{(\lambda_1+\lambda_2)^2}$
- $\mathbb{P}\{X_1 < X_2 < X_3\} = \frac{\lambda_1\lambda_2}{(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)}$
- $\mathbb{P}(R_n = i) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$
- $\mathbb{P}(S_n^1 < S_m^2) = \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} p^k (1-p)^{n+m-1-k}$

Chapter 4

- *Kolmogorov backward equation*: $P'(s) = GP(s)$, where $G_{ij} > 0$ for $i \neq j$, and $G_{ii} = -\sum_{j\neq i} G_{ij}$.
- $\lambda_i = -G_{ii}$ and $J_{ij} = G_{ij}/\lambda_i$ for $i \neq j$, and $J_{ii} = 0$.
- $P(t) = e^{tG}$, for $t \geq 0$.
- *M/M/1* queue stationary distribution $\pi(i) = (1-\theta)\theta^i$, where $\theta = \lambda/\mu < 1$.
- Ergodic Theorem: with $\{\pi_i\}$ the stationary distribution, $\mathbb{P}\Big\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t))dt = \sum_{i\in E} \pi_i f(i)\Big\} = 1.$

Chapter 5

- $f(\mathbf{x}) = (2\pi)^{-\frac{p}{2}} |\mathbf{\Sigma}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$
- For $\mathbf{X} = \begin{pmatrix}\mathbf{X}_1 \\ \mathbf{X}_2\end{pmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{\Sigma})$
$$\boldsymbol{\mu} = \begin{pmatrix}\boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2\end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix}\mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22}\end{pmatrix},$$
$\mathbf{X}_1|\mathbf{X}_2 = \mathbf{x}_2 \sim \mathcal{N}(\boldsymbol{\mu}_{1|2}, \mathbf{\Sigma}_{11|2})$, where
$\boldsymbol{\mu}_{1|2} = \mathrm{E}(\mathbf{X}_1|\mathbf{X}_2 = \mathbf{x}_2) = \boldsymbol{\mu}_1 + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$,
$\mathbf{\Sigma}_{11|2} = \mathrm{Var}(\mathbf{X}_1|\mathbf{X}_2 = \mathbf{x}_2) = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}$.
- For $0 = t_0 < t_1 < \cdots < t_n$, $(B(t_1), \ldots, B(t_n)) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$ where $\mathbf{\Sigma}_{ij} = t_i \wedge t_j$.
- Let $0 < t_1 \leq t_2$.
Forcasting: $B(t_2) \mid B(t_1) = x_1 \sim \mathcal{N}(x_1, t_2 - t_1)$.
Backcasting: $B(t_1) \mid B(t_2) = x_2 \sim \mathcal{N}\left(\frac{t_1}{t_2}x_2, \frac{t_1}{t_2}(t_2 - t_1)\right)$.
- $T_a = \inf\{t : B(t) = a\}$
- Reflection principle: $B_t^* = \begin{cases} B_t, & \text{if } t \leq T_a, \\ 2a - B_t, & \text{if } t > T_a. \end{cases}$
- Running maximum: $M_t = \max_{0\leq s\leq t} B(s), \quad t \geq 0.$
- For $0 < b \leq a$, $\mathbb{P}(M_t \geq a,\ B_t \leq b) = \mathbb{P}(B_t \geq 2a - b)$.
- M_t has the same distribution as $|B_t|$.
- T_a has the same distribution as $\frac{a^2}{B_1^2}$

Some common distributions

	pdf/pmf	mean	variance
Geometric	$(1-p)^{n-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	$e^{-\lambda}\frac{\lambda^k}{k!}$	λ	λ
Exponential	$\theta e^{-\theta x}$	$\frac{1}{\theta}$	$\frac{1}{\theta^2}$
Gamma	$\frac{\theta^k x^{k-1}}{\Gamma(k)} e^{-\theta x}$	$\frac{k}{\theta}$	$\frac{k}{\theta^2}$