The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science

Andre Milzarek
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DDA 3005 — Numerical Methods

Solutions — Sample: Midterm Exam

Please make sure to present your solutions and answers in a comprehensible way and give explanations of your steps and results!

- *The exam time is 90 minutes.*
- *There are five exercises on four sheets (including this sheet).*
- *The total number of achievable points is* 100 *points.*
- *Please abide by the honor codes of CUHK-SZ.*

Good Luck!

Exercise 1 (LU Factorization): (*22* points)

Consider the matrix $\boldsymbol{A} \in \mathbb{R}^{4\times 4}$ where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 4 & 0 & 2 \\ 2 & 2 & 2 & 4 \\ 1 & 3 & 0 & 0 \end{bmatrix}.$$

a) Compute an LU factorization with pivoting of the matrix $\boldsymbol{A}$. State the final $\boldsymbol{L}$ and $\boldsymbol{U}$ matrices and the obtained permutation matrix $\boldsymbol{P}$.

b) Solve the linear system $\boldsymbol{A}^\top \boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = [0,\ 4,\ 2,\ 8]^\top$.

Solution :

a) We can choose 2 as first pivot and swap row 1 and 3 of $\boldsymbol{A}$. This yields

$$\boldsymbol{M}_1 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ -\frac{1}{2} & 0 & 1 & \\ -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{P}_1\boldsymbol{A} = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 0 & 4 & 0 & 2 \\ 1 & 1 & 1 & 4 \\ 1 & 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ \frac{1}{2} & 0 & 1 & \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 & 4 \\ & 4 & 0 & 2 \\ & 0 & 0 & 2 \\ & 2 & -1 & -2 \end{bmatrix} = \boldsymbol{L}_1\boldsymbol{U}_1.$$

The next pivot is 4 (no permutations are required); we have:

$$\boldsymbol{M}_2 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix}, \quad \boldsymbol{P}_1\boldsymbol{A} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ \frac{1}{2} & 0 & 1 & \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 & 4 \\ & 4 & 0 & 2 \\ & 0 & 0 & 2 \\ & & -1 & -3 \end{bmatrix} = \boldsymbol{L}_2\boldsymbol{U}_2.$$

We now swap the last rows of $\boldsymbol{P}_1\boldsymbol{A}$, $\boldsymbol{L}_2$, and $\boldsymbol{U}_2$ to obtain:

$$\boldsymbol{P}_2\boldsymbol{A} = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 0 & 4 & 0 & 2 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ \frac{1}{2} & \frac{1}{2} & 1 & \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 & 4 \\ & 4 & 0 & 2 \\ & & -1 & -3 \\ & & & 2 \end{bmatrix} = \boldsymbol{L}\boldsymbol{U}.$$

In summary, we have:

$$\boldsymbol{P} = \begin{bmatrix} & & 1 & \\ & 1 & & \\ & & & 1 \\ 1 & & & \end{bmatrix}, \quad \boldsymbol{L} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ \frac{1}{2} & \frac{1}{2} & 1 & \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} 2 & 2 & 2 & 4 \\ & 4 & 0 & 2 \\ & & -1 & -3 \\ & & & 2 \end{bmatrix}.$$

b) Using $\boldsymbol{P}\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, it follows $\boldsymbol{A}^\top\boldsymbol{x} = \boldsymbol{b} \iff \boldsymbol{A}^\top\boldsymbol{P}^\top\boldsymbol{P}\boldsymbol{x} = \boldsymbol{b} \iff \boldsymbol{U}^\top\boldsymbol{L}^\top\boldsymbol{P}\boldsymbol{x} = \boldsymbol{b}$. We first solve $\boldsymbol{U}^\top\boldsymbol{v} = \boldsymbol{b}$ via forward-substitution:

$$\boldsymbol{U}^\top\boldsymbol{v} = \begin{bmatrix} 2 & & & \\ 2 & 4 & & \\ 2 & 0 & -1 & \\ 4 & 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \\ 8 \end{bmatrix}.$$

This implies $v_1 = 0$, $v_2 = 1$, $v_3 = -2$, and $v_4 = 0$. We now solve $\boldsymbol{L}^\top\boldsymbol{y} = \boldsymbol{v}$ via back-substitution:

$$\boldsymbol{L}^\top\boldsymbol{y} = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ & 1 & \frac{1}{2} & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

which yields $y_4 = 0$, $y_3 = -2$, $y_2 = 2$, and $y_1 = 1$. Finally, the solution of the linear system of equations $\boldsymbol{A}^\top \boldsymbol{x} = \boldsymbol{b}$ is given by

$$\boldsymbol{x} = \boldsymbol{P}^\top \boldsymbol{y} = \begin{bmatrix} & & & 1 \\ & 1 & & \\ 1 & & & \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -2 \end{bmatrix}.$$

Exercise 2 (Weighted Linear Regression): (*22* points)

Let $\{(\boldsymbol{y}_i, z_i)\}_{i=1}^5$ be a set of data points, where $\boldsymbol{y}_i \in \mathbb{R}^3$ denotes the input data and $z_i \in \mathbb{R}$ is the corresponding output data. We want to solve the following *weighted least squares problem*:

$$\min_{\boldsymbol{x} \in \mathbb{R}^3} \sum_{i=1}^{5} w_i \cdot (\boldsymbol{x}^\top \boldsymbol{y}_i - z_i)^2, \tag{1}$$

where $w_i \geq 0$, $i = 1, \ldots, 5$. The data points and weights are given explicitly by:

$\boldsymbol{y}_1 = [0,\ \frac{1}{2},\ 1]^\top$,	$z_1 = 1$,	$w_1 = 4$	$\boldsymbol{y}_2 = [0,\ 0,\ -1]^\top$,	$z_2 = 0$,	$w_2 = 1$
$\boldsymbol{y}_3 = [3,\ 1,\ -1]^\top$,	$z_3 = 7$,	$w_3 = 0$	$\boldsymbol{y}_4 = [0,\ 0,\ \frac{\sqrt{2}}{2}]^\top$,	$z_4 = -\sqrt{2}$,	$w_4 = 2$
$\boldsymbol{y}_5 = [-1,\ -2,\ 4]^\top$,	$z_5 = 1$,	$w_5 = 1$			

a) Find a matrix $\boldsymbol{A}$ and a vector $\boldsymbol{b}$ to express problem (1) in the following format:

$$\min_{\boldsymbol{x} \in \mathbb{R}^3} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2. \tag{2}$$

b) Compute a QR factorization of $\boldsymbol{A}$. State the obtained upper triangular matrix $\boldsymbol{R}$ of the factorization. (*You don't need to form or calculate the orthogonal matrix $\boldsymbol{Q}$ explicitly*).

c) Show that $\boldsymbol{x}^* = [-13,\ 4,\ -1]^\top$ is a solution of the weighted least squares problem (1).

Solution :

a) Noticing $w_i \cdot (\boldsymbol{x}^\top \boldsymbol{y}_i - z_i)^2 = (\boldsymbol{x}^\top (\sqrt{w_i}\boldsymbol{y}_i) - \sqrt{w_i} z_i)^2$, we can set

$$\boldsymbol{A} = \begin{bmatrix} \sqrt{w_1}\boldsymbol{y}_1^\top \\ \sqrt{w_2}\boldsymbol{y}_2^\top \\ \sqrt{w_3}\boldsymbol{y}_3^\top \\ \sqrt{w_4}\boldsymbol{y}_4^\top \\ \sqrt{w_5}\boldsymbol{y}_5^\top \end{bmatrix} = \begin{bmatrix} 2\boldsymbol{y}_1^\top \\ \boldsymbol{y}_2^\top \\ \boldsymbol{0} \\ \sqrt{2}\boldsymbol{y}_4^\top \\ \boldsymbol{y}_5^\top \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & 4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} \sqrt{w_1}z_1 \\ \sqrt{w_2}z_2 \\ \sqrt{w_3}z_3 \\ \sqrt{w_4}z_4 \\ \sqrt{w_5}z_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

b) The matrix $\boldsymbol{A}$ is already close to triangular form. Since row permutations are orthogonal operations, we can first exchange rows of $\boldsymbol{A}$:

$$\boldsymbol{A} \to \boldsymbol{P}\boldsymbol{A} = \begin{bmatrix} -1 & -2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, $\boldsymbol{P}$ swaps rows $1 \leftrightarrow 5$; $2 \leftrightarrow 5$; $3 \leftrightarrow 5$. We can now use a Givens rotation to eliminate the entry "1" in the last column: $c = -\frac{1}{\sqrt{2}}$, $s = \frac{1}{\sqrt{2}}$, $\alpha = \sqrt{2}$,

$$\boldsymbol{Q} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ & & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \\ & & & & 1 \end{bmatrix}, \quad \boldsymbol{A} \to \boldsymbol{QPA} = \begin{bmatrix} -1 & -2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, we have

$$\boldsymbol{R} = \begin{bmatrix} -1 & -2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

c) We only need to verify that $\boldsymbol{x}^*$ satisfies the normal equation $\boldsymbol{A}^\top \boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{A}^\top \boldsymbol{b}$:

$$\boldsymbol{A}^\top \boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{A}^\top \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & 4 \end{bmatrix} \begin{bmatrix} -13 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -2 \\ 2 & -1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}$$

and

$$\boldsymbol{A}^\top \boldsymbol{b} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -2 \\ 2 & -1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}.$$

Hence, $\boldsymbol{x}^*$ is an optimal solution to the least squares problem (1).

Exercise 3 (Error Analysis): (*21* points)

We consider the problem of evaluating the function

$$f(x) := \sqrt{1+x} - 1, \quad x \geq -\frac{1}{2}. \tag{3}$$

For all floating-point systems appearing in this question, we assume:

- The IEEE-Standard 754 holds, i.e., for any machine numbers x, y, we have $x \circledast y = \mathrm{fl}(x * y) = (1+\varepsilon)(x \circledast y)$ and $\mathrm{sqrt}(x) = \mathrm{fl}(\sqrt{x}) = (1+\varepsilon)\sqrt{x}$ for some ε with $|\varepsilon| \leq \varepsilon_{\mathrm{mach}}$. Here, $*$ can represent any arithmetic operation $\{+, -, \cdot, /\}$.

a) Is the problem (3) well- or ill-conditioned? Provide detailed explanations!

b) Let $x \geq -\frac{1}{2}$ be a machine number. Consider the following algorithm:

$$\hat{f}(x) = \mathrm{sqrt}(1 \oplus x) \ominus 1,$$

where $\mathrm{sqrt}(\cdot)$, $\oplus$, and $\ominus$ correspond to the arithmetic operations in the floating-point system.

- Consider a normalized floating-point system with $\beta = 2$, $L < 0$, $U > 0$, $p > 0$, and rounding by chopping. What is the machine precision $\varepsilon_{\mathrm{mach}}$ in this system?
- Show that $\mathrm{sqrt}(1 \oplus \varepsilon_{\mathrm{mach}}) = 1$ and use this result to justify that $\hat{f}$ is not accurate.

Solution :

a) Notice that

$$\mathrm{cond}_f(x) = \left|\frac{xf'(x)}{f(x)}\right| = \left|\frac{x/(2\sqrt{1+x})}{\sqrt{1+x}-1}\right| = \left|\frac{x}{2\sqrt{1+x}} / \frac{x}{\sqrt{1+x}+1}\right|$$
$$= \frac{\sqrt{1+x}+1}{2\sqrt{1+x}} = \frac{1}{2} + \frac{1}{2\sqrt{1+x}}.$$

Then for any $x \geq -\frac{1}{2}$, we have $\mathrm{cond}_f(x) \leq \frac{1+\sqrt{2}}{2}$. Thus, f is well conditioned for all $x \geq -\frac{1}{2}$.

b) In this system, we have $\varepsilon_{\mathrm{mach}} = 2^{1-p}$. Notice that $1 + 2^{1-p}$ is always a machine number in this system, i.e., $1 \oplus \varepsilon_{\mathrm{mach}} = 1 + \varepsilon_{\mathrm{mach}}$. Since $1 + \varepsilon_{\mathrm{mach}} < (1 + \frac{\varepsilon_{\mathrm{mach}}}{2})^2$, we know

$$1 \leq \sqrt{1 \oplus \varepsilon_{\mathrm{mach}}} = \sqrt{1 + \varepsilon_{\mathrm{mach}}} < 1 + \frac{\varepsilon_{\mathrm{mach}}}{2}.$$

Thus, we have $1 \leq \mathrm{sqrt}(1 \oplus \varepsilon_{\mathrm{mach}}) = \mathrm{fl}(\sqrt{1 \oplus \varepsilon_{\mathrm{mach}}}) \leq \mathrm{fl}(1 + \frac{\varepsilon_{\mathrm{mach}}}{2}) = 1$, i.e., $\mathrm{sqrt}(1 \oplus \varepsilon_{\mathrm{mach}}) = 1$. Taking $x = \varepsilon_{\mathrm{mach}}$, it follows $\hat{f}(x) = 0$ and

$$\frac{|\hat{f}(x) - f(x)|}{|f(x)|} = \frac{|f(x)|}{|f(x)|} = 1 \tag{4}$$

This implies

$$\max_{x \geq -\frac{1}{2}} \frac{|\hat{f}(x) - f(x)|}{|f(x)|} \geq 1$$

If $\hat{f}$ is accurate, then there is $C > 0$ such that $\max_{x \geq -\frac{1}{2}} \frac{|\hat{f}(x) - f(x)|}{|f(x)|} \leqslant C\varepsilon_{\mathrm{mach}}$. Taking $\varepsilon_{\mathrm{mach}} \to 0$ (i.e., $p \to \infty$), this yields a contradiction to (4). Hence, $\hat{f}$ cannot be accurate.

Exercise 4 (Miscellaneous): (*16* points)

State whether each of the following statements is *True* or *False*. For each statement provide a short explanation, counterexample, or proof. Only answers with full explanations will be graded. (Short answers of the form “true” or “false” will not be accepted).

a) Let $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ be an orthogonal matrix. Then, for $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, it holds that $\|\boldsymbol{QA}\|_2 = \|\boldsymbol{A}\|_2$.

b) Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a symmetric and nonsingular matrix. Then, $\boldsymbol{A}$ has an LU factorization of the form $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^\top$ where $\boldsymbol{L} \in \mathbb{R}^{n \times n}$ is a lower triangular matrix.

c) Denji, Aki, and Makima want to numerically solve a linear system of equations

$$\boldsymbol{Ax} = \boldsymbol{b} \quad \text{with} \quad \boldsymbol{A} \in \mathbb{R}^{1000 \times 1000} \text{ and } \boldsymbol{b} \in \mathbb{R}^{1000} \tag{5}$$

- Denji uses a Householder QR factorization to factorize $\boldsymbol{A} = \boldsymbol{QR}$. His solution $\boldsymbol{x}_{\mathrm{den}} = \boldsymbol{R}^{-1}\boldsymbol{Q}^\top\boldsymbol{b}$ has the residual norm $\|\boldsymbol{r}_{\mathrm{den}}\| = \|\boldsymbol{A}\boldsymbol{x}_{\mathrm{den}} - \boldsymbol{b}\|_2 = 4.545 \cdot 10^{-13}$.
- Aki uses an LU factorization $\boldsymbol{PA} = \boldsymbol{LU}$ to solve (5) and to recover $\boldsymbol{x}_{\mathrm{aki}} = \boldsymbol{U}^{-1}(\boldsymbol{L}^{-1}\boldsymbol{Pb})$. His calculated residual is $\|\boldsymbol{r}_{\mathrm{aki}}\| = \|\boldsymbol{A}\boldsymbol{x}_{\mathrm{aki}} - \boldsymbol{b}\|_2 = 9.181 \cdot 10^{1}$.
- Makima applies the classical Gram-Schmidt method to compute a QR factorization of $\boldsymbol{A}$. She obtains $\|\boldsymbol{r}_{\mathrm{mak}}\| = \|\boldsymbol{A}\boldsymbol{x}_{\mathrm{mak}} - \boldsymbol{b}\|_2 = 1.539 \cdot 10^{-11}$.

These results imply that the matrix $\boldsymbol{A}$ is ill-conditioned.

d) Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ be given and suppose $\boldsymbol{b} \in \mathrm{range}(\boldsymbol{A})$. Then, the linear least squares problem $\min_{\boldsymbol{x}} \|\boldsymbol{Ax} - \boldsymbol{b}\|_2^2$ has a unique solution.

Solution :

a) *True.* We have $\|\boldsymbol{Q}\boldsymbol{A}\|_2^2 = \lambda_{\max}(\boldsymbol{A}^\top\boldsymbol{Q}^\top\boldsymbol{Q}\boldsymbol{A}) = \lambda_{\max}(\boldsymbol{A}^\top\boldsymbol{A}) = \|\boldsymbol{A}\|_2^2$.

b) *False.* We can consider $\boldsymbol{A} = -1 \in \mathbb{R}^{1\times 1}$. Obviously there is no $\boldsymbol{L}$ such that $-1 = \boldsymbol{L}^2$.

c) *False.* Denji's result indicates that the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ can be solved fairly accurately with a very small error, i.e., the condition number of $\boldsymbol{A}$ can be moderate. Aki's surprising result can be caused by the potential inaccuracy of the LU factorization, i.e., $\boldsymbol{L}$ or $\boldsymbol{U}$ might be ill-conditioned (while $\boldsymbol{A}$ is not).

d) *False.* Take $\boldsymbol{A} = \boldsymbol{0}$ and $\boldsymbol{b} = \boldsymbol{0}$.

Exercise 5 (Block Elimination): (*19* points)

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a nonsingular matrix and let $\boldsymbol{b}, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and $\alpha, b_{n+1} \in \mathbb{R}$ be given.

Suppose that we have access to an LU factorization of the matrix $\boldsymbol{A}$, i.e., $\boldsymbol{P}\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, where $\boldsymbol{P}$ is a permutation matrix, $\boldsymbol{L}$ is unit lower triangular, and $\boldsymbol{U}$ is an upper triangular matrix. We use this factorization to obtain the solution $\boldsymbol{x}^* = \boldsymbol{A}^{-1}\boldsymbol{b}$ of the linear system of equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Let us now consider the extended linear system of equations $\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}$, where

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{v} \\ \boldsymbol{u}^\top & \alpha \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}, \quad \tilde{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x} \\ x_{n+1} \end{bmatrix}, \quad \text{and} \quad \tilde{\boldsymbol{b}} = \begin{bmatrix} \boldsymbol{b} \\ b_{n+1} \end{bmatrix}.$$

Let us assume $\boldsymbol{u}^\top\boldsymbol{A}^{-1}\boldsymbol{v} \neq \alpha$. Based on $\boldsymbol{x}^*$ and the given LU factorization of $\boldsymbol{A}$, develop an algorithm to compute the solution $\tilde{\boldsymbol{x}}$ of the extended system $\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}$ that only requires $\mathcal{O}(n^2)$ operations. Explain why your algorithm achieves such a complexity.

Remark: You don't need to write `MATLAB` or `Python` code here. Try to be brief and use concise steps or compact pseudocode.

Solution : The linear system of equations $\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}$ is equivalent to

$$\boldsymbol{A}\boldsymbol{x} + \boldsymbol{v}\cdot x_{n+1} = \boldsymbol{b} \quad \text{and} \quad \boldsymbol{u}^\top\boldsymbol{x} + \alpha x_{n+1} = b_{n+1}$$

Taking $\boldsymbol{A}^{-1}$ in the first equation, we obtain $\boldsymbol{x} = \boldsymbol{A}^{-1}(\boldsymbol{b} - \boldsymbol{v}\cdot x_{n+1})$. Putting this in the second equation, it follows:

$$(\alpha - \boldsymbol{u}^\top\boldsymbol{A}^{-1}\boldsymbol{v})\cdot x_{n+1} = b_{n+1} - \boldsymbol{u}^\top\boldsymbol{x}^*$$

Thus, the assumption $\boldsymbol{u}^\top\boldsymbol{A}^{-1}\boldsymbol{v} \neq \alpha$ guarantees that $\tilde{\boldsymbol{A}}$ is invertible. To compute $\tilde{\boldsymbol{x}}$, we first calculate

$$\boldsymbol{y} = \boldsymbol{A}^{-1}\boldsymbol{v}$$

Since an LU factorization of $\boldsymbol{A}$ is given, this can be done, e.g., via $\boldsymbol{y} = \boldsymbol{U}^{-1}(\boldsymbol{L}^{-1}\boldsymbol{P}\boldsymbol{v})$. Since $\boldsymbol{L}$ and $\boldsymbol{U}$ are triangular matrices, such operation can be performed via one forward- and back-substitution which requires $2\times\mathcal{O}(n^2)$ flops. The matrix-vector product $\boldsymbol{P}\boldsymbol{v}$ simply permutes the elements in $\boldsymbol{v}$ which can be realized in $\mathcal{O}(n)$ operations. Hence, we can obtain x_{n+1} and $\boldsymbol{x}$ via

$$x_{n+1} = \frac{b_{n+1} - \boldsymbol{u}^\top\boldsymbol{x}^*}{\alpha - \boldsymbol{u}^\top\boldsymbol{y}} \quad \text{and} \quad \boldsymbol{x} = \boldsymbol{x}^* - \boldsymbol{y}\cdot x_{n+1}$$

This requires to additionally compute the inner products $\boldsymbol{u}^\top\boldsymbol{y}$ and $\boldsymbol{u}^\top\boldsymbol{x}^*$ ($2\times(2n-1)$ flops) and to perform $n+2$ additions and $n+1$ multiplications. Overall the complexity remains $\mathcal{O}(n^2)$.
