The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 3
Due: 11:59pm, Oct. 20 (Friday), 2023

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- Please submit your assignment on Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student must not copy homework solutions from another student or from any other source.

Problem 1 (20pts).
Consider the following linear program:

$$\begin{array}{lll} \text{maximize} & x_1 + 2x_2 + 3x_3 + 8x_4 & \\ \text{subject to} & x_1 - x_2 + x_3 & \leq 2 \\ & x_3 - x_4 & \leq 1 \\ & 2x_2 + 3x_3 + 4x_4 & \leq 8 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array}$$

Use simplex tableau to completely solve it. For each step, draw the simplex tableau. Clearly mark what is the current basis, the current basic solution, and the corresponding objective function value.

Solution.
We first express the problem in standard form:

$$\begin{array}{lll} \text{minimize} & -x_1 - 2x_2 - 3x_3 - 8x_4 & \\ \text{subject to} & x_1 - x_2 + x_3 + s_1 & = 2 \\ & x_3 - x_4 + s_2 & = 1 \\ & 2x_2 + 3x_3 + 4x_4 + s_3 & = 8 \\ & x_1, x_2, x_3, x_4, s_1, s_2, s_3 & \geq 0. \end{array}$$

Since the right-hand side is nonnegative, we can choose the slack variables as basic variables and set the original variables to zero, i.e., the point $(x; s) = (0, 0, 0, 0, 2, 1, 8)^\top$ is a valid initial BFS with basis $B = \{5, 6, 7\}$. Notice that the initial tableau is already in canonical form, i.e., we have:

B	-1	-2	-3	-8	0	0	0	0
5	1	-1	1	0	1	0	0	2
6	0	0	1	-1	0	1	0	1
7	0	2	3	4	0	0	1	8

and the current objective function value is 0. The pivot column is $\{1\}$; the pivot row is $\{5\}$; the pivot element is 1; after the row updates we obtain the new tableau:

B	0	-3	-2	-8	1	0	0	2
1	1	-1	1	0	1	0	0	2
6	0	0	1	-1	0	1	0	1
7	0	2	3	4	0	0	1	8

The basis updated to $\{1,6,7\}$; the current BFS is $(x;s) = (2,0,0,0,0,1,8)^\top$; the current objective function value is -2. The pivot column in the updated tableau is $\{2\}$; the pivot row is $\{7\}$; the pivot element is 2; after the row updates the new tableau is given by:

B	0	0	$\frac{5}{2}$	-2	1	0	$\frac{3}{2}$	14
1	1	0	$\frac{5}{2}$	2	1	0	$\frac{1}{2}$	6
6	0	0	1	-1	0	1	0	1
2	0	1	$\frac{3}{2}$	2	0	0	$\frac{1}{2}$	4

The basis is updated to $\{1,6,2\}$; the current BFS is $(x;s) = (6,4,0,0,0,1,0)^\top$; the current objective function value is -14. The pivot column in the new tableau is $\{4\}$; the pivot row is $\{2\}$; the pivot element is 2; after the row updates the new tableau is given by:

B	0	1	4	0	1	0	2	18
1	1	-1	1	0	1	0	0	2
6	0	$\frac{1}{2}$	$\frac{7}{4}$	0	0	1	$\frac{1}{4}$	3
4	0	$\frac{1}{2}$	$\frac{3}{4}$	1	0	0	$\frac{1}{4}$	2

Since all reduced costs are nonnegative, the method stops with the optimal solutions $x^* = (2,0,0,2)^\top$ and $s^* = (0,3,0)^\top$; the optimal basis is $B = \{1,4,6\}$ and the optimal value (of the original problem) is 18. ■

Problem 2 (30pts).

Apply the two-phase simplex method (implemented by simplex tableau) to solve the following linear program. For each step, draw the simplex tableau. Clearly mark what is the current basis, the current basic solution, and the corresponding objective function value.

$$\begin{aligned} \text{minimize} \quad & x_1 - x_2 + 2x_3 \\ \text{subject to} \quad & 2x_1 - x_2 + 2x_3 \le -1 \\ & x_1 - x_2 - x_3 \le 4 \\ & x_2 - x_4 = 0 \\ & x_1, x_2, x_3, x_4 \ge 0. \end{aligned}$$

Solution.
We follow the procedure presented in the lecture and first express the problem in standard form:

$$\begin{array}{lll} \text{minimize} & x_1 - x_2 + 2x_3 & \\ \text{subject to} & 2x_1 - x_2 + 2x_3 + s_1 & = -1 \\ & x_1 - x_2 - x_3 + s_2 & = 4 \\ & x_2 - x_4 & = 0 \\ & x_1, x_2, x_3, x_4, s_1, s_2 & \geq 0. \end{array}$$

Notice that for this special problem we can also directly find an initial BFS by setting $(x; s) = (0, 1, 0, 1, 0, 5)^\top$ with $B = \{2, 4, 6\}$.
Phase I. If we apply the two-phase method, we first generate the auxiliary problem

$$\begin{array}{lll} \text{minimize} & y_1 + y_2 + y_3 & \\ \text{subject to} & -2x_1 + x_2 - 2x_3 - s_1 + y_1 & = 1 \\ & x_1 - x_2 - x_3 + s_2 + y_2 & = 4 \\ & x_2 - x_4 + y_3 & = 0 \\ & x_1, x_2, x_3, x_4, s_1, s_2, y_1, y_2, y_3 & \geq 0, \end{array}$$

where we rescaled the first equality constraint to obtain a nonnegative right-hand side. An initial BFS of the auxiliary problem is given by $(x; s; y) = (0, 0, 0, 0, 0, 0, 1, 4, 0)^\top$ with $B = \{7, 8, 9\}$. The reduced costs for the non-basic variables can be calculated by just summing the entries in the corresponding column of A and by multiplying the result with -1. (Notice that the problem is not in canonical form). In particular, we have

$$\big(0,0,0,0,0,0\big) - \big(1,1,1\big) \begin{pmatrix} -2 & 1 & -2 & 0 & -1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix} = \big(1,-1,3,1,1,-1\big)$$

and the simplex tableau is given by:

B	1	−1	3	1	1	−1	0	0	0	−5
7	−2	1	−2	0	−1	0	1	0	0	1
8	1	−1	−1	0	0	1	0	1	0	4
9	0	1	0	−1	0	0	0	0	1	0

The pivot column is $\{2\}$; the outgoing column is $\{9\}$; the pivot element is 1; the current BFS is degenerate; after the row updates we obtain the new tableau:

B	1	0	2	0	1	−1	0	0	1	−5
7	−2	0	−2	1	−1	0	1	0	−1	1
8	1	0	−1	−1	0	1	0	1	1	4
2	0	1	0	−1	0	0	0	0	1	0

The pivot column is $\{6\}$; the outgoing column is $\{8\}$; the pivot element is 1; after the row updates we obtain the new tableau:

B	2	0	1	-1	1	0	0	1	2	-1
7	-2	0	-2	1	-1	0	1	0	-1	1
6	1	0	-1	-1	0	1	0	1	1	4
2	0	1	0	-1	0	0	0	0	1	0

The pivot column is $\{4\}$; the outgoing column is $\{7\}$; the pivot element is 1; after the row updates the new tableau is given by:

B	0	0	0	0	0	0	1	1	1	0
4	-2	0	-2	1	-1	0	1	0	-1	1
6	-1	0	-3	0	-1	1	1	1	0	5
2	-2	1	-2	0	-1	0	1	0	0	1

Since the reduced costs are nonnegative we have reached an optimal point. Furthermore, the optimal value is zero, and hence $(x; s) = (0, 1, 0, 1, 0, 5)^\top$ is a BFS for the original problem with basis $B = \{2, 4, 6\}$.

Phase II. In order to start phase II, we need to construct the initial tableau (which can often be done by utilizing the final tableau from Phase I). We first compute the associated reduced costs:

$$\begin{aligned}
\bar{c}^\top &= c^\top - c_B^\top A_B^{-1} A \\
&= c^\top - \begin{pmatrix} -1, 0, 0 \end{pmatrix} \begin{pmatrix} -2 & 1 & -2 & 0 & -1 & 0 \\ -2 & 0 & -2 & 1 & -1 & 0 \\ -1 & 0 & -3 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1, 0, 0, 0, -1, 0 \end{pmatrix}.
\end{aligned}$$

Note that $A_B^{-1} A$ is directly read from the final tableau of Phase I.
Furthermore, we have $-c_B^\top (x; s)_B = 1$. The initial simplex tableau then is given by:

B	-1	0	0	0	-1	0	1
2	-2	1	-2	0	-1	0	1
4	-2	0	-2	1	-1	0	1
6	-1	0	-3	0	-1	1	5

Since all entries in the pivot column $\{1\}$ are negative, the problem is unbounded. ■

Problem 3 (30pts).
Use the two-phase simplex method (implemented by simplex tableau) to completely solve the linear optimization problem. For each step, draw the simplex tableau. Clearly mark what is the current basis, the current basic solution, and the corresponding objective function value.

$$\begin{aligned}
\text{minimize} \quad & x_1 + 3x_2 + x_4 - 2x_5 \\
\text{subject to} \quad & x_1 + 2x_2 + 4x_4 + x_5 && = 2 \\
& x_1 + 2x_2 - 2x_4 + x_5 && = 2 \\
& -x_1 - 4x_2 + 3x_3 && = 1 \\
& x_1, x_2, x_3, x_4, x_5 && \geq 0.
\end{aligned}$$

Solution.
The problem is already in standard form; we can immediately start with phase I.
Phase I. We first generate the auxiliary problem

$$\begin{array}{rll} \text{minimize} & y_1 + y_2 + y_3 & \\ \text{subject to} & x_1 + 2x_2 + 4x_4 + x_5 + y_1 & = 2 \\ & x_1 + 2x_2 - 2x_4 + x_5 + y_2 & = 2 \\ & -x_1 - 4x_2 + 3x_3 + y_3 & = 1 \\ & x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3 & \geq 0, \end{array}$$

An initial BFS of the auxiliary problem is given by $(x; y) = (0, 0, 0, 0, 0, 2, 2, 1)^\top$ with $B = \{6, 7, 8\}$. The reduced costs for the non-basic variables can again be calculated by $c_N - c_B^\top A_B^{-1} A_N = c_N - c_B^\top A_N$ since $A_B = I_m$. Hence, the non-basic reduced costs are given by:

$$\big(0, 0, 0, 0, 0\big) - \big(1, 1, 1\big) \begin{pmatrix} 1 & 2 & 0 & 4 & 1 \\ 1 & 2 & 0 & -2 & 1 \\ -1 & -4 & 3 & 0 & 0 \end{pmatrix} = \big(-1, 0, -3, -2, -2\big)$$

Therefore, we can construct the initial simplex tableau as:

B	-1	0	-3	-2	-2	0	0	0	-5
6	1	2	0	4	1	1	0	0	2
7	1	2	0	-2	1	0	1	0	2
8	-1	-4	3	0	0	0	0	1	1

The pivot column is $\{1\}$; the outgoing column is $\{6\}$; the pivot element is 1; after the row updates we obtain the new tableau:

B	0	2	-3	2	-1	1	0	0	-3
1	1	2	0	4	1	1	0	0	2
7	0	0	0	-6	0	-1	1	0	0
8	0	-2	3	4	1	1	0	1	3

The pivot column is $\{3\}$; the outgoing column is $\{8\}$; the pivot element is 3; after the row updates we obtain the new tableau:

B	0	0	0	6	0	2	0	1	0
1	1	2	0	4	1	1	0	0	2
7	0	0	0	-6	0	-1	1	0	0
3	0	$-\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	1

Since the reduced costs are nonnegative and the costs are zero, phase I of the two-phase simplex method stops with the BFS $(x; y) = (2, 0, 1, 0, 0, 0, 0)^\top$ and the basis $B = \{1, 3, 7\}$.

This basis still contains the auxiliary variable y_2. We substitute it with x_4. Notice that the fourth column is the only possible choice since columns $\{2\}$ and $\{5\}$ in A (of the original problem) do not form a linearly independent set of vectors with columns $\{1\}$ and $\{3\}$. In order to maintain the canonical form, we perform one additional row update to change the fourth column to a unit vector (help us to update the basis from $\{7\}$ to $\{4\}$). Then, we obtain ($A_B^{-1}A$ is read from the updated final tableau):

$$A_B^{-1}A = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & -\frac{2}{3} & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad c^\top - c_B^\top A_B^{-1}A = \big(0, 1, 0, 0, -3\big).$$

Phase II. Setting $B = \{1, 3, 4\}$, the initial costs are $-c_B^\top x_B = -2$. Based on the updated final tableau of Phase I, we can construct the initial simplex tableau for the original problem as:

B	0	1	0	0	-3	-2
1	1	2	0	0	1	2
4	0	0	0	1	0	0
3	0	$-\frac{2}{3}$	1	0	$\frac{1}{3}$	1

The pivot column is $\{5\}$; the outgoing column is $\{1\}$; the pivot element is 1; after the row updates we obtain the new tableau:

B	3	7	0	0	0	4
5	1	2	0	0	1	2
4	0	0	0	1	0	0
3	$-\frac{1}{3}$	$-\frac{4}{3}$	1	0	0	$\frac{1}{3}$

The method stops with the optimal solution $x = (0, 0, \frac{1}{3}, 0, 2)^\top$ and the optimal value -4. ■

Problem 4 (20pts).
Consider a linear optimization problem in the standard form, described in terms of the following initial tableau (Table 2):

B	0	0	0	δ	3	γ	ξ	0
2	0	1	0	α	1	0	3	β
3	0	0	1	-2	2	η	-1	2
1	1	0	0	0	-1	2	1	3

Table 1

The entries α, β, γ, δ, η and ξ in the tableau are unknown parameters, and $B = \{2, 3, 1\}$. For each of the following statements, find (sufficient) conditions of the parameter values that will make the statement true.

1. This is an acceptable initial tableau (i.e., the basic variables are feasible for the problem).

2. The first row (in the constraint) indicates that the problem is infeasible.

3. The basic solution is feasible but we have not reached an optimal basic set B.

4. The basic solution is feasible and the first simplex iteration indicates that the problem is unbounded.

5. The basic solution is feasible, x_6 is a candidate for entering B, and when we choose x_6 as the entering basis, x_3 leaves B.

Solution.

1. $\beta \geq 0$.
2. $\alpha \geq 0, \beta < 0$. The sum of all the positive variables has a negative value, which indicates infeasibility.
3. $\beta > 0$, at least one of $\delta, \gamma, \xi < 0$. The reduced cost of one of the non-basic variables is negative. Note that $\beta > 0$ is required, as in the *degenerate* case, we might have reached optimal but with a negative reduced cost. In the next iterate, the simplex method updates the basis with $y = x$, and y might be an optimal solution. In the case, the original x is already optimal.
4. $\beta \geq 0$, $\alpha \leq 0$, $\delta < 0$. The fourth column has all entries negative or zero.
5. $\beta \geq 0$, $\gamma < 0$, $\frac{2}{\eta} < \frac{3}{2}$ and $\eta > 0$ gives $\eta > \frac{4}{3}$. By the minimum ratio test,we want η to be the pivot element.

■