STA4001 Stochastic Processes Fall 2024
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Homework 5
Due by Dec 8, 2024

1. Let $B(t), t \geq 0$ be a standard Brownian motion.
   (a) Determine $P(B(2.8) > 1.3 \mid B(1.2) = 0.5)$
   (b) Determine $P(B(1.2) \leq 0.9 \mid B(3.1) = 1.6)$
   (c) Show that for any $0 < s < t, B(t)$ and $B(s) - \frac{s}{t}B(t)$ are independent.
   (d) Let $M(t) = \max_{0 \leq s \leq t} B(s)$. Determine $\mathbb{E}[M(3)]$.

Solution:

(a)

$$\begin{aligned}
P(B(2.8) > 1.3 \mid B(1.2) = 0.5) &= P(B(2.8) - B(1.2) > 0.8 \mid B(1.2) = 0.5) \\
&= P(B(2.8) - B(1.2) > 0.8) \\
&= P(B(1.6) > 0.8) \\
&= P(P(1.6)/\sqrt{1.6} > 0.8/\sqrt{1.6}) \\
&= 1 - \Phi(0.8/\sqrt{1.6})
\end{aligned}$$

(b) Since $P(B(1.2) = t \mid B(3.1) = 1.6) \sim \mathcal{N}(\frac{1.2}{3.1} \cdot 1.6, \frac{1.2}{3.1} \cdot 1.9)$, we have

$$P(B(1.2) \leq 0.9 \mid B(3.1) = 1.6) = \Phi\left(\frac{0.9 - \frac{1.2}{3.1} \cdot 1.6}{\sqrt{\frac{1.2}{3.1} \cdot 1.9}}\right)$$

(c) Consider the two normal random variables $B(s) - \frac{s}{t}B(t)$ and $B(t)$

$$\begin{aligned}
\operatorname{Cov}\left(B(s) - \frac{s}{t}B(t), B(t)\right) &= \operatorname{Cov}(B(s), B(t)) - \frac{s}{t}\operatorname{Cov}(B(t), B(t)) \\
&= s - \frac{s}{t} \cdot t = 0
\end{aligned}$$

Since uncorrelated normal variables are also independent, $B(s) - \frac{s}{t}B(t)$ and $B(t)$ are independent.

(d) Since $P(M(t) = x) = \frac{\sqrt{2}}{\sqrt{\pi t}}e^{-\frac{x^2}{2t}}$, we have

$$\mathbb{E}[M(3)] = \sqrt{\frac{2}{3\pi}} \int_0^\infty x e^{-\frac{x^2}{6}} dx = \sqrt{\frac{2}{3\pi}} \cdot 3 = \sqrt{\frac{6}{\pi}}$$

2. Let $B(t), t \geq 0$ be a standard Brownian motion.

   (a) What is the distribution of $B(s) + 2B(t), s \leq t$?

   (b) Compute the conditional distribution of $B(s)$ given that $B(t_1) = A$ and $B(t_2) = B$, where $0 < t_1 < s < t_2$

   (c) Calculate $\mathbb{E}(B(s)B(t)^2)$, where $0 < s < t$.

   (d) Calculate $\mathbb{E}(B(t_1)B(t_2)B(t_3))$, where $t_1 < t_2 < t_3$.

   Solution:

   (a) $B(s) + 2B(t) = B(s) + 2(B(s) + B(t) - B(s)) = 3B(s) + 2(B(t) - B(s))$. Since $3B(s) \sim \mathcal{N}(0, 9s)$ and $2(B(t) - B(s)) \sim \mathcal{N}(0, 4(t-s))$. As $B(s)$ and $B(t) - B(s)$ are independent, we have $B(s) + 2B(t) \sim \mathcal{N}(0, 4t + 5s)$

   (b) The conditional distribution $B(s) - A$ given that $B(t_1) = A$ and $B(t_2) = B$ is the same as the conditional distribution of $B(s - t_1)$ given that $B(0) = 0$ and $B(t_2 - t_1) = B - A$, which is normal with mean $\frac{s-t_1}{t_2-t_1}(B - A)$ and variance $\frac{(s-t_1)}{t_2-t_1}(t_2 - s)$. Hence the desired conditional distribution is normal with mean $A + \frac{(s-t_1)(B-A)}{t_2-t_1}$ and variance $\frac{(s-t_1)(t_2-s)}{t_2-t_1}$.

   (c)
$$\begin{aligned}
\mathbb{E}\left(B(s)B(t)^2\right) &= \mathbb{E}\left(B(s)\left(B(t) - B(s) + B(s)\right)^2\right) \\
&= \mathbb{E}\left(B(s)\left(B(t) - B(s)\right)^2 + 2B(s)^2\left(B(t) - B(s)\right) + B(s)^3\right) \\
&= 0
\end{aligned}$$

   (d)
$$\begin{aligned}
E\left[B\left(t_1\right) B\left(t_2\right) B\left(t_3\right)\right] &= E\left[E\left[B\left(t_1\right) B\left(t_2\right) B\left(t_3\right) \mid B\left(t_1\right), B\left(t_2\right)\right]\right] \\
&= E\left[B\left(t_1\right) B\left(t_2\right) E\left[B\left(t_3\right) \mid B\left(t_1\right), B\left(t_2\right)\right]\right] \\
&= E\left[B\left(t_1\right) B\left(t_2\right) B\left(t_2\right)\right] \\
&= E\left[E\left[B\left(t_1\right) E\left[B^2\left(t_2\right) \mid B\left(t_1\right)\right]\right]\right] \\
&= E\left[B\left(t_1\right) E\left[B^2\left(t_2\right) \mid B\left(t_1\right)\right]\right] \\
&= E\left[B\left(t_1\right)\left\{\left(t_2 - t_1\right) + B^2\left(t_1\right)\right\}\right] \\
&= E\left[B^3\left(t_1\right)\right] + \left(t_2 - t_1\right) E\left[B\left(t_1\right)\right] \\
&= 0
\end{aligned}$$

3. Suppose $\{B_t : \ t \geq 0\}$ is a standard Brownian motion.

   (a) Find the density of $T_{1.1}$=min$\{t : B_t = 1.1\}$.

(b) Suppose that you purchased a stock at price 20 one year ago, and the current price is 0.4 due to the global financial crisis. Suppose the price of the stock changes according to a standard Brownian motion process. What is the probability that you recover your purchase price sometime within an additional time $t = 100$?

Solution:

(a) The distribution function of $T_{1.1}$ is

$$\Pr(T_{1.1} \le t) = 2\left[1 - \Phi\left(\frac{1.1}{\sqrt{t}}\right)\right]$$

and hence the density is

$$\begin{aligned} f_{T_{1.1}}(t) &= \frac{d}{dt} 2\left[1 - \Phi\left(\frac{1.1}{\sqrt{t}}\right)\right] \\ &= -2\phi\left(\frac{1.1}{\sqrt{t}}\right)\frac{-1.1}{2t^{3/2}} \\ &= 1.1t^{-3/2}\phi\left(\frac{1.1}{\sqrt{t}}\right) \\ &= \frac{1.1}{\sqrt{2\pi t^3}}\exp\left\{-\frac{1.21}{2t}\right\}, \quad t > 0 \end{aligned}$$

(b)

$$\begin{aligned} P(T_{20-0.4} \le 100) &= 2\left[1 - \Phi\left(\frac{20 - 0.4}{\sqrt{100}}\right)\right] \\ &= 2[1 - \Phi(1.96)] \\ &= 0.05. \end{aligned}$$

4. Suppose $\{B_t : t \ge 0\}$ is a Brownian motion process with drift coefficient 0 and variance parameter σ^2. The process starts from state 0. Find the mean and covariance functions of the following processes:

(a) $X_t = B_t^2, \ t \ge 0$;

(b) $Y_t = B_t - tB_1, \ 0 \le t \le 1$.

Solution:

(a) The mean is

$$\mathbb{E}[X(t)] = \mathbb{E}[B^2(t)] = \mathrm{Var}[B(t)] + \{\mathbb{E}[B(t)]\}^2 = \sigma^2 t.$$

and the covariance is

$$\begin{aligned}
\text{Cov}[X(s)X(t)] &= \mathbb{E}[B^2(s)B^2(t)] - \mathbb{E}[X(s)]\mathbb{E}[X(t)] \\
&= \mathbb{E}\{B^2(s)[B(t) - B(s) + B(s)]^2\} - \sigma^4 st \\
&= \mathbb{E}\{B^(s)\{[B(t) - B(s)]^2 + 2B(s)[B(t) - B(s)] + B^2(s)\}\} - \sigma^4 st \\
&= \mathbb{E}[B^2(s)]\mathbb{E}\{[B(t) - B(s)]^2\} + 2\mathbb{E}[B^3(s)]\mathbb{E}[B(t) - B(s)] + \mathbb{E}[B^4(s)] - \sigma^4 st \\
&= \sigma^2 s \cdot \left\{\text{Var}[B(t) - B(s)] + \{\mathbb{E}[B(t) - B(s)]\}^2\right\} + 3\sigma^4 s^2 - \sigma^4 st \\
&= \sigma^4 s(t-s) + 3\sigma^4 s^2 - \sigma^4 st \\
&= 2\sigma^4 s^2
\end{aligned}$$

for $0 \le s \le t$.

(b) The mean is

$$\mathbb{E}[Y(t)] = \mathbb{E}[B(t)] - t\mathbb{E}[B(1)] = 0 - 0 = 0$$

and the covariance is

$$\begin{aligned}
\mathbb{E}[Y(s)Y(t)] &= \mathbb{E}\{[B(s) - sB(1)][B(t) - tB(1)]\} - \mathbb{E}[Y(s)]\mathbb{E}[Y(t)] \\
&= \mathbb{E}[B(s)B(t) - sB(t)B(1) - tB(s)B(1) + stB^2(1)] \\
&= \sigma^2 s - s\sigma^2 t - t\sigma^2 s + st\sigma^2 \\
&= \sigma^2 s(1-t)
\end{aligned}$$

for $0 \le s \le t \le 1$.

5. Consider a process whose value changes every h time units; its new value being its old value multiplied either by the factor $e^{\sigma\sqrt{h}}$ with probability $p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h}\right)$, or by the factor $e^{-\sigma\sqrt{h}}$ with probability $1-p$. As h goes to zero, show that this process converges to geometric Brownian motion with drift coefficient μ and variance parameter σ^2

Solution: Let $X(t)$ denote the value of the process at time $t = nh$. Let $X_i = 1$ if the i th change results in the state value becoming larger, and let $X_i = 0$ otherwise. Then, with $u = e^{\sigma\sqrt{h}}, d = e^{-\sigma\sqrt{h}}$

$$\begin{aligned}
X(t) &= X(0)u^{\sum_{i=1}^n X_i} d^{n - \sum_{i=1}^n X_i} \\
&= X(0)d^n \left(\frac{u}{d}\right)^{\sum_{i=1}^n X_i}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\log\left(\frac{X(t)}{X(0)}\right) &= n\log(d) + \sum_{i=1}^{n} X_i \log(u/d) \\
&= -\frac{t}{h}\sigma\sqrt{h} + 2\sigma\sqrt{h}\sum_{i=1}^{t/h} X_i
\end{aligned}
$$

By the central limit theorem, the preceding becomes a normal random variable as $h \to 0$. Moreover, because the X_i are independent, it is easy to see that the process has independent increments. Also,

$$
\begin{aligned}
E\left[\log\left(\frac{X(t)}{X(0)}\right)\right] &= -\frac{t}{h}\sigma\sqrt{h} + 2\sigma\sqrt{h}\frac{t}{h}\frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h}\right) \\
&= \mu t
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}\left[\log\left(\frac{X(t)}{X(0)}\right)\right] &= 4\sigma^2 h \frac{t}{h} p(1-p) \\
&\to \sigma^2 t
\end{aligned}
$$

where the preceding used that $p \to 1/2$ as $h \to 0$.