Homework 3

1. (a) Let X be exponential random variable with parameter λ, i.e. its density is

$$f(x) = \lambda e^{-\lambda x}, \qquad 0 < x < \infty.$$

Calculate the value of $\mathrm{E}(X|X > 1)$ and $\mathrm{E}(X|1 < X < 2)$.

Solution: From definition of conditional expectation,

$$E(X \mid X > 1) = \frac{1}{P(X > 1)} \cdot \int_1^\infty x f(x) dx = \frac{1}{e^{-\lambda}} \cdot \int_1^\infty \lambda x e^{-\lambda x} dx = \frac{1+\lambda}{\lambda}$$

$$\begin{aligned} E(X \mid 1 < X < 2) &= \int_1^2 x \lambda e^{-\lambda x} dx \cdot \frac{1}{P(1 < X < 2)} \\ &= \frac{\left(-x e^{-\lambda x} - e^{-\lambda x}/\lambda\right)\big|_1^2}{-\, e^{-\lambda x}\big|_1^2} \\ &= \frac{e^{-\lambda}(1 + 1/\lambda) - e^{-2\lambda}(2 + 1/\lambda)}{e^{-\lambda} - e^{-2\lambda}} \\ &= 1 + \frac{1}{\lambda} - \frac{1}{e^{\lambda} - 1} \end{aligned}$$

(b) Suppose X, Y and Z are independent exponential random variables with parameters λ, μ and v respectively, i.e. the pdf of X is $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$. Find $E[\max\{X, Y, Z\}]$.

Solution: Its probability distribution function is given by

$$\begin{aligned} P(\max\{X, Y, Z\} < t) &= P(X < t, Y < t, Z < t) && \text{(Property of max function)} \\ &= P(X < t)P(Y < t)P(Z < t) && \text{(Independence between } X, Y, Z) \\ &= \left(1 - e^{-\lambda t}\right)\left(1 - e^{-\mu t}\right)\left(1 - e^{-v t}\right) \end{aligned}$$

Note that for non-negative random variable, we have that $E(X) = \int_0^\infty P(X > t)dt$.

$$\begin{aligned} E(\max\{X, Y, Z\}) &= \int_0^\infty P(\max\{X, Y, Z\} > t) dt \\ &= \frac{1}{\lambda} + \frac{1}{\mu} + \frac{1}{v} - \frac{1}{\lambda + v} - \frac{1}{\lambda + \mu} - \frac{1}{v + \mu} + \frac{1}{\lambda + \mu + v} \end{aligned}$$

(c) Let X and Y be independent random variables distributed as exponential with parameters λ and μ, respectively (i.e. a probability density function (pdf) of X is $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$, and a pdf of Y is $f_Y(y) = \mu e^{-\mu y}$ for $y > 0$). Let I, independent of X and Y, be a Bernoulli random variable with success probability $P(I = 1) = \dfrac{\mu}{\lambda + \mu}$. Define

$$W = X - Y \quad \text{and} \quad Z = \begin{cases} X & \text{if } I = 1, \\ -Y & \text{if } I = 0. \end{cases}$$

Show, by using moment generating functions (e.g. $M_X(t) = E[e^{tX}] = \frac{\lambda}{\lambda - t}$ for $t < \lambda$), that W and Z have the same distribution.

Solution:

$$M_W(t) = E\left(e^{t(X-Y)}\right) = E\left[e^{tX}\right] \cdot E\left(e^{-tY}\right) = \frac{\lambda\mu}{(\lambda - t)(\mu + t)}$$

$$M_Z(t) = E\left(\frac{\mu}{\lambda + \mu} e^{tX} + \frac{\lambda}{\lambda + \mu} e^{-tY}\right) = \frac{\lambda\mu}{(\lambda - t)(\mu + t)}$$

We conclude that $M_W(t) = M_Z(t)$ and therefore W and Z have the same distribution.

(d) Let X and Y be independent exponential random variables with respective rates λ and μ, where $\lambda > \mu$. Let $c > 0$ and $\delta = \lambda - \mu > 0$.

(i) Show that the conditional density function of X, given that $X + Y = c$, is

$$f_{X|X+Y}(x|c) = \frac{\delta e^{-\delta x}}{1 - e^{-\delta c}}, \quad 0 < x < c.$$

(ii) Use part (i) to find $\mathbf{E}[X|X+Y]$.

(iii) Find $\mathbf{E}[Y|X+Y]$.

Solution:

(i)

$$f_{X|X+Y}(x \mid c) = \frac{f(x, x+y=c)}{f(x+y=c)} = \frac{(\lambda-\mu)e^{-\lambda-\mu}x}{1-e^{-\lambda-\mu_c}}\frac{\delta e^{-\delta x}}{1-e^{-\delta c}}$$

(ii)

$$E(X \mid X+Y) = \int_0^c x f_{x|x+y} dx = \int_0^c \frac{\delta x e^{-\delta x}}{1-e^{-\delta c}} dx = \frac{1}{\delta\left(1-e^{-\delta c}\right)}\left[1-(\delta c+1)e^{-\delta c}\right]$$

(iii)

$$E(Y \mid X+Y) = c - E(X \mid X+Y) = c - \left(\frac{1}{\delta\left(1-e^{-\delta c}\right)}\left[1-(\delta c+1)e^{-\delta c}\right]\right)$$

2. Let T_1 and T_2 be exponential random variables with parameter λ, and let S be an exponential random variable with parameter μ. We assume that all three of these random variables are independent. Derive an expression for the expected value of $\min\{T_1 + T_2, S\}$

Solution:

We view the random variables T_1 and T_2 as interarrival times in two independent Poisson processes both with rate λS as the interarrival time in a third Poisson process (independent from the first two) with rate μ. We are interested in the expected value of the time Z until either the first process has had two arrivals or the second process has had an arrival. Given that the first arrival was from the second process, the expected wait time for that arrival would be $\frac{1}{\mu+\lambda}$. The probability of an arrival from the second process is $\frac{\mu}{\mu+\lambda}$. Given that the first arrival time was from the first process, the expected wait time would be that for first arrival, $\frac{1}{\mu+\lambda}$, plus the expected wait time for another arrival from the merged process. Similarily, the probability of an arrival from the first process is $\frac{\lambda}{\mu+\lambda}$. Thus,

$$\begin{aligned}\mathbf{E}[Z] = &\ \mathbf{P}(\text{Arrival from second process})\mathbf{E}[\text{wait time} \mid \text{Arrival from second process}] + \\ &\ \mathbf{P}(\text{Arrival from first process})\mathbf{E}[\text{wait time} \mid \text{Arrival from first process}] \\ = &\ \frac{\mu}{\mu+\lambda}\cdot\frac{1}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}\cdot\left(\frac{1}{\mu+\lambda}+\frac{1}{\mu+\lambda}\right)\end{aligned}$$

After some simplifications, we see that

$$\mathbf{E}[Z] = \frac{1}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}\cdot\frac{1}{\mu+\lambda}$$

3. Consider a Poisson process of rate λ. Let random variable N be the number of arrivals in $(0, t]$ and M be the number of arrivals in $(0, t+s]$, where $t, s \geq 0$.

(a) Find the conditional PMF of M given $N, p_{M|N}(m \mid n)$, for $m \geq n$.

(b) Find the joint PMF of N and $M, p_{N,M}(n, m)$.

(c) Find the conditional PMF of N given $M, p_{N|M}(n \mid m)$, for $n \leq m$, using your answer to part (b).

(d) Rederive your answer to part (c) without using part (b). As a hint, consider what kind of distribution the k^{th} arrival time has if we are given the event $\{M = m\}$, where $k \leq m$.

(e) Find $E[NM]$.

Solution:

(a) We know there are n arrivals in t amount of time, so we are looking for how many extra arrivals there are in s amount of time.

$$p_{M|N}(m \mid n) = \frac{(\lambda s)^{m-n} e^{-\lambda s}}{(m-n)!} \quad \text{for } m \geq n \geq 0$$

(b) By definition:

$$\begin{aligned} p_{N,M}(n, m) &= p_{M|N}(m \mid n) p_N(n) \\ &= \frac{\lambda^m s^{m-n} t^n e^{-\lambda(s+t)}}{(m-n)! n!} \quad \text{for } m \geq n \geq 0 \end{aligned}$$

(c) By definition:

$$\begin{aligned} p_{N|M}(n \mid m) &= \frac{p_{M,N}(m, n)}{p_M(m)} \\ &= \binom{m}{n} \frac{s^{m-n} t^n}{(s+t)^m} \quad \text{for } m \geq n \geq 0 \end{aligned}$$

(d) We want to find: $\mathbf{P}(N = n \mid M = m)$. Given $M = m$, we know that the m arrivals are uniformly distributed between 0 and $t+s$. Consider each arrival a success if it occurs before time t , and a failure otherwise. Therefore given $M = m, N$ is a binomial random variable with m trials and probability of success $\frac{t}{t+s}$. We have the desired probability:

$$\mathbf{P}(N = n \mid M = m) = \binom{m}{n} \left(\frac{t}{t+s}\right)^n \left(\frac{s}{t+s}\right)^{m-n} \quad \text{for } m \geq n \geq 0$$

(e) We can rewrite the expectation as:

$$\begin{aligned} \mathbf{E}[NM] &= \mathbf{E}\left[N(M-N) + N^2\right] \\ &= \mathbf{E}[N]\mathbf{E}[M-N] + \mathbf{E}\left[N^2\right] \\ &= (\lambda t)(\lambda s) + \left(\operatorname{var}(N) + (\mathbf{E}[N])^2\right) \\ &= (\lambda t)(\lambda s) + \lambda t + (\lambda t)^2 \end{aligned}$$

where the second equality is obtained via the independent increment property of the poisson process.

4. A delivery company makes deliveries 6 days a week. Accidents involving the vehicles of this company occur according to a Poisson process with a rate of 3 per day. In each accident, damage to the contents of the vehicles is independently distributed as follows:

Amount of damage	\$0	\$2,000	\$8,000
Probability	0.25	0.5	0.25

(a) Determine the mean and variance of the company's weekly aggregate damages.

(b) Determine the probability that the company's weekly aggregate damages will not exceed \$63,000.

Solution:

(a) Let $0 \leq X \leq 6$ be the number of accident each week and Y_i be the amount of damage in each accident $i = 1, \cdots, X$.

$$E\left(\sum_{i=1}^{X} Y_i\right) = 3 \times 6 \times (2000 \times 0.5 + 8000 \times 0.25) = 5.4 \times 10^4$$

$$\operatorname{Var}\left(\sum_{i=1}^{X} Y_i\right) = 18 \times E\left(Y_i^2\right) = 3.24 \times 10^8$$

(b) Let $Z \sum_{i=1}^{X} Y_i$. Applying central limit theorem, we approximate the probability by

$$P(Z \leq 63000) = P\left(\frac{Z - E(Z)}{\mathrm{Var}(Z)} \leq \frac{63000 - 54000}{3.24 \times 10^8}\right) = \Phi\left(\frac{1}{2}\right) = 0.6915$$

where $\Phi(\cdot)$ is the pdf of standard normal distribution.

5. A small supermarket has only one checkout counter. The inter-arrival times between the customers arriving at the counter are independent and identically distributed as exponential with mean $1/\lambda$. Upon arrival, a customer will be served immediately if there is no one being served; otherwise he/she is made to wait in queue until it is his/her turn to be served. The queue is reduced in a first-come, first-served manner. The service times are assumed to be independent and identically distributed as exponential with mean $1/\mu$. Once served, the customers will leave the supermarket.

   (a) Let $N_1(t)$ denote the number of customers having arrived at the counter by time t. Is $\{N_1(t), t \geq 0\}$ a Poisson process? Explain briefly.

   (b) Let $N_2(t)$ denote the number of customers having departed from the counter by time t. Is $\{N_2(t), t \geq 0\}$ a Poisson process? Explain briefly.

   (c) Define a transition to be the event that there is a change in the number of customers in the queue, including the one being served. Let X_n denote the number of customers in the queue after the n-th transition. Show that the sequence $\{X_n, n = 0, 1, 2, \ldots\}$ is a Markov chain with transition probabilities

$$P_{01} = 1, P_{i,i+1} = \frac{\lambda}{\lambda + \mu}, P_{i,i-1} = \frac{\mu}{\lambda + \mu}, \text{for } i = 1, 2, \ldots$$

Solution:
(a) $N_1(t)$ is a Poisson process because the inter-arrival times follows iid exponential distribution.
(b) $N_2(t)$ is not a Poisson process because the elapse times among departures are not identically distributed. Specifically, when the queue is non-empty, waiting time until the next departure is distributed as $\mathcal{E}(\mu)$. When the queue is empty, waiting time until the next departure is the sum of $\mathcal{E}(\lambda)$ and $\mathcal{E}(\mu)$.
(c) Let state space $E = \{0, 1, 2, \cdots\}$ be the number of customers in the queue.

- For state 0 , since no customer can be served and the only event for transition is the arrival of the new customer, thus $P_{01} = 1$.
- For state $1, 2\cdots$, two possible events may occur: (i) existing customer is served (its time is denoted as X); (ii) new customer arrives (its time denoted as Y). Two events follows exponential distribution with parameter μ and λ. The state $i > 0$ transfers to $i + 1$ if Y happens earlier, i.e., $P(X > Y) = \frac{\lambda}{\lambda+\mu}$. Otherwise it transfers to $i - 1$ with probability $\frac{\mu}{\lambda+\mu}$.