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Homework 4

Due by Nov 25, 2024

1. Consider a CTMC $X = \{X(t), t \geq 0\}$ on $S = \{A, B, C\}$ with generator G given by

$$G = \begin{bmatrix} -8 & 3 & 5 \\ 2 & -6 & 4 \\ 2 & 0 & -2 \end{bmatrix}$$

   (a) Draw the rate diagram.

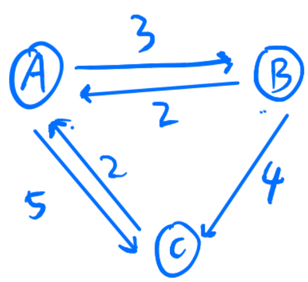
   (b) Calculate the jump matrix and holding time rates.

   (c) Starting from state B, use `Python` or another language to simulate a sample path of the CTMC until the 8th jump of the CTMC occurs. Plot the sample path (by hand is ok). Please specify as much detail as possible.

   (d) Use a computer software like `Python` or a good calculator to directly compute the transition probability matrix $P(t)$ at $t = 0.4$ and $t = 1$

   (e) Find $P\{X(1.2) = C|X(0) = A\}$ and $P\{X(3) = A|X(1) = B\}$.

Solution:



(a)

(b) Jump matrix: $J = \begin{bmatrix} 0 & \frac{3}{8} & \frac{5}{8} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 1 & 0 & 0 \end{bmatrix}$.

Holding time rates:

$$\lambda(A) = 8 \quad \lambda(B) = 6 \quad \lambda(C) = 2$$

(c) omit

(d) $P(0.4) = e^{0.4G} = \begin{bmatrix} 0.21465251 & 0.13436959 & 0.6509779 \\ 0.19633687 & 0.17078581 & 0.63287732 \\ 0.19633687 & 0.08006786 & 0.72359527 \end{bmatrix}$

$P(1) = e^{G} = \begin{bmatrix} 0.20003632 & 0.10121214 & 0.69875154 \\ 0.19999092 & 0.10186587 & 0.69814321 \\ 0.19999092 & 0.09938712 & 0.70062196 \end{bmatrix}$

(e) $P\{X(1.2) = C|X(0) = A\} = P_{AC}(1.2) = 0.69962548$, $P\{X(3) = A|X(1) = B\} = P_{BA}(2) = 0.2$

2. A small barber shop, operated by a single barber, has room for only two costumers. Potential costumers arrive at a Poisson rate of 3 per hour, and the successive serving times are independent exponential random variables of mean 1/4 hour.

(a) What is the average number of customers in the shop?

(b) What is the proportion of potential customers that enters the shop? (Hint: A customer gets rejected upon arrival if and only if he sees two customers in the system)

(c) If the barber could work 3 times as fast, how much more businesss would he do (in average)? (Hint: Compare potential customers that enters the shop.)

Solution:

(a) We define states $\{0, 1, 2\}$ as the number of customers in the barber. The generator matrix is calculated as:

$$G = \begin{pmatrix} -3 & 3 & 0 \\ 4 & -7 & 3 \\ 0 & 4 & -4 \end{pmatrix}$$

Solving the equation $\pi G = 0$ we get

$$\pi = \left[\frac{16}{37}, \frac{12}{37}, \frac{9}{37}\right].$$

Then the average number of customers is calculated as:

$$E[n] = \pi^T[0, 1, 2] = \frac{30}{37}$$

State	Input rate to	= Output rate from
0	$(1/2)\pi_d + (2/3)\pi_b$	$= 2\pi_0 + (3/2)\pi_0$
2	$(3/2)\pi_d + 2\pi_b$	$= (1/2)\pi_2 + (2/3)\pi_2$
d	$(2/3)\pi_2 + 2\pi_0$	$= (1/2)\pi_d + (3/2)\pi_d$
b	$(1/2)\pi_2 + (3/2)\pi_0$	$= (2/3)\pi_b + 2\pi_b$

Table 1: Problem 3

(b) The proportion is calculated as $\pi_0 + \pi_1 = \frac{28}{37}$.

(c) The new generator matrix is

$$G_{\text{new}} = \begin{pmatrix} -3 & 3 & 0 \\ 12 & -15 & 3 \\ 0 & 12 & -12 \end{pmatrix}$$

and the new stationary distribution is

$$\pi_{\text{new}} = \left[\frac{16}{21}, \frac{4}{21}, \frac{1}{21}\right]$$

and the new proportion is $\frac{20}{21}$. Therefore, business will be 25.9% more.

3. Johnson Medical Associates has two physicians on call, Drs. Dawson and Baick. Dr. Dawson is available to answer patients' calls for time periods that are exponentially distributed with mean 2 hours. Between those periods, he takes breaks, each of which being an exponential amount of time with mean 30 minutes. Dr. Baick works independently from Dr. Dawson, but with similar work patterns. The time periods she is available to take patients" calls and the times she is on break are exponential random variables with means 90 and 40 minutes, respectively.

   (a) In the long run, what is the proportion of time in which neither of the two doctors is available to take patients'calls?

   Solution: Let $X(t) = 0$ if neither Dr. Dawson nor Dr. Baick is available to answer patients' calls. Let $X(t) = 2$ if both of them are available to take the calls; $X(t) = d$ if Dr. Dawson is available to take the calls and Dr. Baick is not; $X(t) = b$ if Dr. Baick is available to take the calls but Dr. Dawson is not. Clearly, $\{X(t) : t \geq 0\}$ is a continuous-time Markov chain with state space $\{0, 2, d, b\}$. Let π_0, π_2, π_d, and π_b be the long-run proportions of time the process is in the stats $0, 2, d$, and b, respectively. The balance equations for $\{X(t) : t \geq 0\}$ are Solving any three of these equations along with $\pi_0 + \pi_2 + \pi_d + \pi_b = 1$, we obtain $\pi_0 = 4/65$, $\pi_2 = 36/65, \pi_d = 16/65$, and

$\pi_b = 9/65$. Therefore, the proportion of time neither of the two doctors is available to take patients' calls is $\pi_0 = 4/65 \approx 0.06$.

4. Each time a machine is repaired it remains up for an exponentially distributed time with rate λ. It then fails, and its failure is either of two types. If it is a type 1 failure, then the time to repair the machine is exponential with rate μ_1; if it is a type 2 failure, then the repair time is exponential with rate μ_2. Each failure is, independently of the time it took the machine to fail, a type 1 failure with probability p and a type 2 failure with probability $1 - p$.

   (a) What proportion of time is the machine down due to a type 1 failure?

   (b) What proportion of time is it down due to a type 2 failure?

   (c) What proportion of time is it up?

   Solution: Say the state is 0 if the machine is up, say it is i when it is down due to a type i failure, $i = 1, 2$. The balance equations for the limiting probabilities are as follows:

$$\begin{aligned} \lambda P_0 &= \mu_1 P_1 + \mu_2 P_2 \\ \mu_1 P_1 &= \lambda p P_0 \\ \mu_2 P_2 &= \lambda(1-p) P_0 \\ P_0 + P_1 + P_2 &= 1 \end{aligned}$$

   Solve these equations, we have

   (a) $P_1 = \lambda p \mu_1^{-1} \cdot (1 + \lambda p/\mu_1 + \lambda(1-p)/\mu_2)^{-1}$

   (b) $P_2 = \lambda(1-p)\mu_2^{-1} \cdot (1 + \lambda p/\mu_1 + \lambda(1-p)/\mu_2)^{-1}$

   (c) $P_0 = (1 + \lambda p/\mu_1 + \lambda(1-p)/\mu_2)^{-1}$

5. In a factory, there are m operating machines and s machines used as spares and ready to operate. The factory has k repair persons, and each repair person repairs one machine at a time. Suppose that,

   (i) Each machine works, independent of other machines, for a time period which is exponentially distributed with mean $\frac{1}{\mu}$, then it breaks down

   (ii) The time that it takes to repair an out-of-order machine is exponential with mean $\frac{1}{\lambda}$, independent of repair times for other machines;

   (iii) At times when all repair persons are busy repairing machines, the newly broken down machines will wait for repair

   (iv) When a machine breaks down, one of the spares will be used unless there is no spare machine available.

Let $X(t)$ be the number of machines operating or ready to operate at time t. Show that $X(t) : t \geq 0$ is a birth and death process and find the birth and death rates.

Solution: When $X(t) = n$, where $n \leq m + s$, the time until the next arrival and departure are exponentially distributed, and both processes are independent, then this process is a birth and death process. Denote the birth rate and death rate as λ_n and μ_n, which can be described as follows.

- $\lambda_n = \min\{k, m + s - n\} \cdot \lambda$
- $\mu_n = \min\{n, m\} \cdot \mu$