Final
Dec 21st

1. Let $U_1, U_2, \ldots,$ be a sequence of independent uniform (0, 1) random variables, and let $S_n = U_1 + \cdots + U_n$ for $n \geq 1$. Define
$$N = \min\{n \geq 2 \ : \ U_n > U_{n-1}\},$$
and for $0 < x \leq 1$
$$M(x) = \min\{n \geq 1 \ : S_n > x\}.$$
That is, N is the index of the first uniform random variable that is larger than its immediate predecessor, and $M(x)$ is the number of uniform random variables we need sum to exceed x.

   (a) Show that for all $n \geq 0$, $\quad \mathbb{P}(N > n) = \dfrac{1}{n!}$. **[4 marks]**

   (b) Note that
$$\{M(x) > n\} = \{S_1 \leq x, \ldots, S_n \leq x\}.$$
Show that for $n \geq 1$, **[6 marks]**
$$\mathbb{P}[M(x) > n+1 \mid U_1] = \mathbb{P}[M(x - U_1) > n] \cdot 1_{\{U_1 \leq x\}}.$$

   (c) Show, by induction on $n \geq 1$, that for all $n \geq 1$, we have **[6 marks]**
$$\text{for all } 0 < x \leq 1, \quad \mathbb{P}[M(x) > n] = \frac{x^n}{n!}.$$

   (d) Compare the distributions of N and $M(1)$, and find their means. **[4 marks]**

[Total: 20 marks]

Solution:

(a) We have, $\quad \mathbb{P}(N > n) = \mathbb{P}(U_1 \geq U_2 \geq \cdots \geq U_{n-1} \geq U_n) = \dfrac{1}{n!}$, because this is one of the $n!$ orderings which all have the same probability $1/n!$.

(b) We need to show that
$$\mathbb{P}(M(x) > n+1 \mid U_1 = y] = \mathbb{P}[M(x-y) > n] \cdot 1_{\{y \leq x\}}.$$
Clearly, if $y > x$, the probability is zero.
For $U_1 = y \leq x$, we have
$$\begin{aligned}
& \mathbb{P}[M(x) > n+1 \mid U_1 = y] \\
= \ & \mathbb{P}(S_1 \leq x, S_2 \leq x, \ldots, S_{n+1} \leq x \mid U_1 = y) \\
= \ & \mathbb{P}(y \leq x, y + U_2 \leq x, \ldots, y + U_2 + \cdots + U_{n+1} \leq x \mid U_1 = y) \\
= \ & \mathbb{P}(U_2 \leq x - y, \ldots, U_2 + \cdots + U_{n+1} \leq x - y \mid U_1 = y) \\
= \ & \mathbb{P}(U_2 \leq x - y, \ldots, U_2 + \cdots + U_{n+1} \leq x - y) \\
= \ & \mathbb{P}[M(x-y) > n],
\end{aligned}$$
The last equality results from the clear fact being obtained clearly the n-uple $(U_2, \ldots, U_{n+1})$ has the same distribution as $(U_1, \ldots, U_n)$. The claim is proved.

(c) Proof by induction.

- $n = 1$: $\mathbb{P}[M(1) > x] = \mathbb{P}(S_1 \leq x) = \mathbb{P}(U_1 \leq x) = x$, so the property is true for $n = 1$.
- Suppose the property is true for n. For $\mathbb{P}[M(x) > n + 1]$, from the previous question, we have

$$\begin{aligned}\mathbb{P}[M(x) > n+1] &= \mathbb{E}\left\{\mathbb{P}[M(x-U_1) > n] \cdot 1_{\{U_1 \leq x\}}\right\} \\ &= \int_0^x \mathbb{P}[M(x-y) > n] dy = \int_0^x \frac{(x-y)^n}{n!} dy = \frac{x^{n+1}}{(n+1)!},\end{aligned}$$

where in the last integral, we have used the induction hypothesis.

The property is thus proved for all $n \geq 1$.

(d) For $x = 1$, we have $\mathbb{P}[M(1) > n] = \frac{1}{n!}$: therefore the two variables have a same distribution. We have

$$\mathbb{E}[M(1)] = \mathbb{E}(N) = \sum_0^\infty \mathbb{P}(N > n) = \sum_0^\infty \frac{1}{n!} = e.$$

2. (a) A transition probability matrix P is said to be doubly stochastic if the sum over each column equals one; that is,

$$\sum_i P_{ij} = 1, \quad \text{for all } j.$$

If such a chain is irreducible and aperiodic and consists of $M + 1$ states $0, 1, \ldots, M$, show that the long-run proportions are given by

$$\pi_j = \frac{1}{M+1}, \quad j = 0, 1, \ldots, M.$$

[5 marks]

(b) Let Y_n be the sum of n independent rolls of a fair die. Find

$$\lim_{n\to\infty} P(Y_n \text{ is a multiple of } 5).$$

Hint: Define an appropriate Markov chain on the set $\{0, 1, 2, 3, 4\}$ of remainders from integer divisions by 5 and apply the results of Part (2a)

[10 marks]

[Total: 15 marks]

Solution:

(a) If $\sum_{i=0}^m P_{ij} = 1$ for all j; then $r_j = \frac{1}{M+1}$ satisfies

$$r_j = \sum_{i=0}^m r_i P_{ij}, \quad \sum_{j=0}^m r_j = 1.$$

Under the assumption, the chain is positive recurrent, so by uniqueness, these are the limiting probabilities.

(b) Let X_n denote the value of Y_n modulo 5. That is, X_n is the remainder when Y_n is divided by 5. Now X_n is a Markov chain with states $0, 1, \ldots, 4$.

The transition matrix is

$$P = \frac{1}{6}\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It is easy to verify that all the column sums equal 1. Hence, from Part (2a), the limiting probabilities are $r_i = \frac{1}{5}$, and $r_0 = \frac{1}{5}$ is the required probability.

3. (a) Let $\{M(t), t \geq 0\}$ be a counting process with inter-arrival times of events independent and identically distributed as uniform from 0 to 1. Does the process $\{M(t), t \geq 0\}$ have independent and stationary increment property? Give your reasons. **[10 marks]**

(b) Let $\{N_t\}_{t\geq 0}$ be a Poisson process with rate 3.

i. Calculate $P(N_4 = 3)$. **[2 marks]**

ii. Calculate $P(N_4 = 3, N_6 = 6, N_{10} = 12)$. **[3 marks]**

iii. Calculate $P(N_6 = 6|N_4 = 3)$. **[3 marks]**

iv. Calculate $P(N_6 = 6|N_{10} = 12)$. **[3 marks]**

v. Determine $E(N_{10}|N_6)$ and $E(N_{10}|N_4, N_6)$. **[4 marks]**

[Total: 25 marks]

Solution:

(a) This process does not have an independent increment property since the interarrival time is not memoryless. Let T_1 be the arrival time of the first event. Then $P(N(0.5) = 0) = P(T_1 > 0.5) = 0.5$. Using same argument as in part (a), $P(N(1) - N(0.5) = 0|T_1 > 0.5) = 0$. Now if the first event occurs in $(0, 0.5)$, then there would be a positive probability that at least one events occur in $(0.5, 1)$, i.e. $P(N(1) - N(0.5) = 0|T_1 < 0.5) < 1$. As a result,

$$P(N(1) - N(0.5) = 0) = P(N(1) - N(0.5) = 0|T_1 < 0.5)\, P(T_1 < 0.5)$$

$$< P(T_1 < 0.5) = 0.5 = P(N(0.5) = 0).$$

So the numbers of events occurring in $(0, 0.5)$ and $(0.5, 1)$ do not have the same distribution. Hence the process does not have stationary increments.

(b) i. $N_4 \sim \mathcal{P}(12)$ so that $P(N_4 = 3) = e^{-12}\dfrac{12^3}{3!} \approx 0.00177$.

ii.

$$\begin{aligned}
&P(N_4 = 3, N_6 = 6, N_{10} = 12) = P(N_4 = 3, N_6 - N_4 = 3, N_{10} - N_6 = 6)\\
=&P(N_4 = 3)P(N_6 - N_4 = 3)P(N_{10} - N_6 = 6)\\
=&e^{-12}\frac{12^3}{3!} \times e^{-6}\frac{6^3}{3!} \times e^{-12}\frac{12^6}{6!}\\
=&e^{-30}\frac{12^8}{10} \approx 4 \cdot 10^{-6}.
\end{aligned}$$

iii. We have $N_6 = N_4 + (N_6 - N_4)$ where the two summands are independent.

$$\begin{aligned}
P(N_6 = 6|N_4 = 3) &= \frac{P(N_4 = 3, N_6 - N_4 = 3)}{P(N_4 = 3)} = \frac{P(N_4 = 3)P(N_6 - N_4 = 3)}{P(N_4 = 3)}\\
&= P(N_6 - N_4 = 3) = e^{-6}\frac{6^3}{3!} = e^{-6}6^2 \approx 0.0892
\end{aligned}$$

iv. As in (c),

$$\begin{aligned}
P(N_6 = 6|N_{10} = 12) &= \frac{P(N_{10} = 12|N_{N_6} = 6)\ P(N_6 = 6)}{P(N_{10} = 12)}\\
&= \frac{P(N_{10} - N_{N_6} = 6)\ P(N_6 = 6)}{P(N_{10} = 12)}\\
&= \binom{12}{6}\left(\frac{3}{5}\right)^6\left(\frac{2}{5}\right)^6 \approx 0.1766.
\end{aligned}$$

v. We have $N_{10} = N_6 + (N_{10} - N_6)$ where the two summands are independent. Thus $E[N_{10}|N_6] = E[N_6 + (N_{10} - N_6)|N_6] = N_6 + E[(N_{10} - N_6)|N_6] = N_6 + E[N_{10} - N_6] = N_6 + 12.$ Similarly, observing that $(N_{10} - N_6)$ is independent of (N_4, N_6),

$$
\begin{aligned}
E[N_{10}|N_4, N_6] &= E[N_6 + (N_{10} - N_6)|N_4, N_6] = N_6 + E[(N_{10} - N_6)|N_4, N_6] \\
&= N_6 + E[N_{10} - N_6] = N_6 + 12.
\end{aligned}
$$

4. Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion.

   (a) Let $\{W_t\}_{t\geq 0}$ be another standard Brownian motion independent of $\{B_t\}_{t\geq 0}$. For each $t \geq 0$, define

   $$Z_t = \alpha B_t + \beta W_t,$$

   where $\alpha > 0$ and $\beta > 0$ are two given positive constants.
   Find the condition on (α, β) such that $(Z_t)_{t\geq 0}$ is again a standard Brownian motion. **[8 marks]**

   (b) Let $a > 0$ be a fixed constant. Show that the process $\{C_t\}_{t\geq 0}$ with $C_t = aB_{t/a^2}$ is a standard Brownian motion. **[8 marks]**

   (c) Let $a > 0$ be a fixed constant. Recall the hitting time of level a is

   $$T_a = \inf\{t \geq 0 : B_t = a\}.$$

   By using the result of (b), show that T_a has the same distribution as $a^2 T_1$. **[4 marks]**

[Total: 20 marks]

Solution:

(a) It is easy to see that the process $Z_t = \alpha B_t + \beta W_t$ has
   - null initial value;
   - independent increments ;
   - stationary increments.

   It remains to check that for all $t \geq 0$, $Z_t \sim N(0, t)$. As B_t, W_t are i.i.d. $N(0, t)$-distributed, Z_t is normal with mean zero and variance $(\alpha^2 + \beta^2)t$. So the process $\{Z_t\}_{t\geq 0}$ is standard Brownian motion *if and only if* the condition $\boxed{\alpha^2 + \beta^2 = 1}$ is fulfilled.

(b) Checking the three defining conditions:
   - initial value: $C_0 = aB_0 = 0$;
   - independent increments: if (s_j, t_j)'s are k non overlapping intervals $(1 \leq j \leq k)$, the increments

     $$C_{t_j} - C_{s_j} = aB_{t_j/a^2} - aB_{s_j/a^2} = a(B_{t_j/a^2} - B_{s_j/a^2}), \quad 1 \leq j \leq k,$$

     are clearly independent as multiples of non-overlapping increments of $\{B_t\}$.
   - stationary increments: for $0 \leq s < t$, $C_t - C_s = a(B_{t/a^2} - B_{s/a^2})$ has normal distribution with mean 0 and variance $a^2(t/a^2 - s/a^2) = t - s$. These increments are stationary and coincide with those of a standard Brownian motion.

(c) Define the hitting time $\tilde{T}_a$ be the analogue of T_a but for the process $\{C_t\}$, that is,

   $$\tilde{T}_a = \inf\{t \geq 0 : C_t = a\}$$

   By (b), $\tilde{T}_a$ has the same distribution as T_a. Moreover,

   $$
   \begin{aligned}
   \tilde{T}_a &= \inf\{t \geq 0 : C_t = a\} \\
   &= \inf\{t \geq 0 : aB_{t/a^2} = a\} \\
   &= \inf\{t \geq 0 : B_{t/a^2} = 1\} \\
   &= a^2 \cdot \inf\{u \geq 0 : B_u = 1\} \qquad \text{(set } t = a^2 u\text{)} \\
   &= a^2 T_1.
   \end{aligned}
   $$

   Therefore, T_a has the same distribution as $\tilde{T}_a = a^2 T_1$.

5. A small barbershop, operated by a single barber, has room for at most two waiting customers, excluding the one being served. That is, the maximum number of customers in the shop is 3. Potential customers arrive according to a Poisson process with rate of 9 per hour, and the service times are exponentially distributed with mean 10 minutes. When a customer arrives and observes x customer in the shop, he/she will enter with probability $(2/3)^x$, x=0, 1, and 2. When a customer arrives and observes the barbershop full (with three customers inside), he/she leaves.

   (a) Let $X(t)$ be the number of customers at the barbershop at time t. Write down the state space and calculate the jump matrix for this CTMC. **[10 marks]**

   (b) Find out the long run proportion of time for all the states. Compute the average number of customers in the shop and the average waiting time for each admitted customer. **[5 marks]**

   (c) Compute the proportion of potential customers that enter the shop. **[5 marks]**

[Total: 20 marks]

Solution:

(a) The state space is $\{0, 1, 2, 3\}$. For $X(t) = 1$, the arrival rate of the customers who want to enter is $9 \cdot \frac{2}{3} = 6$. For $X(t) = 2$, the arrival rate of the customers who want to enter is $9 \cdot (\frac{2}{3})^2 = 4$. The jump matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) Since the holding time for four states are $\lambda_1 = 15, \lambda_2 = 12, \lambda_3 = 10, \lambda_4 = 6$, the generator matrix G is

$$\begin{pmatrix} -9 & 9 & 0 & 0 \\ 6 & -12 & 6 & 0 \\ 0 & 6 & -10 & 4 \\ 0 & 0 & 6 & -6 \end{pmatrix}$$

Solving $\pi G = 0$ and $\sum_i \pi = 1$, we have $\pi = \left(\frac{2}{10}, \frac{3}{10}, \frac{3}{10}, \frac{2}{10}\right)$. Then the average number of customers in the shop is $\sum_{i=1}^{3} \pi_i \cdot i = \frac{3}{2}$, the average waiting time is $\sum_{i=1}^{2} \pi_i \cdot \frac{1}{6} * i = \frac{3}{20}$

(c) The proportion is $\sum_{i=0}^{2} \pi_i \cdot (\frac{2}{3})^i = \frac{8}{15}$

Useful formulas for "Stochastic Processes"

Chapter 1

- $\mathbb{P}\left(\bigcup_{1}^{n} A_i\right) = \sum_{k=1}^{n}(-1)^{k+1} \sum_{\{i_1,\ldots,i_k\}} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$
- The coupon collection problem:
  $\mathbb{E}(N) = \sum_{k=1}^{n}(-1)^{k+1} \sum_{\{i_1,\ldots,i_k\}} \frac{1}{p_{i_1}+\cdots+p_{i_k}}$
- $X \geq 0$: $\quad \mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > u)du$,
  $\mathbb{E}[X] = \sum_{k=0}^{\infty} \mathbb{P}(X > k)$.
- Order statistics $(X_{(1)}, \ldots, X_{(n)})$ has joint density
  $n! f(x_1)\cdots f(x_n) 1_{\{x_1 < \cdots < x_n\}}$
- ith order statistic $X_{(i)}$ has density
  $i\binom{n}{i} f(x) F^{i-1}(x)\{1 - F(x)\}^{n-i}$.
- $\mathbb{E}(X) = \mathbb{E}\{\mathbb{E}(X|Y)\}$.
- $\mathrm{var}(X) = \mathbb{E}\{\mathrm{var}(X|Y)\} + \mathrm{var}\{\mathbb{E}(X|Y)\}$.
- Random sum $S_N = X_1 + \cdots + X_N$:
  $\mathbb{E}(S_N) = \mu\,\mathbb{E}(N)$, $\mathrm{var}(S_N) = \sigma^2 E(N) + \mu^2 \mathrm{var}(N)$.

Chapter 2

- Hitting time of i:
  $\{\tau_i = n\} = \{X_n = i, X_{n-1} \neq i, \ldots, X_1 \neq i\}$.
- $f_{ii}^{(n)} = \mathbb{P}(\tau_i = n \mid X_0 = i)$,
  $f_i = \sum_{n=1}^{\infty} f_{ii}^{(n)} = \mathbb{P}(\tau_i < \infty \mid X_0 = i)$.
- $P_{ii}^{n} = \sum_{k=1}^{n} f_{ii}^{(k)} P_{ii}^{n-k}, \quad n \geq 1$.
- $f_{jk}^{(n)} = \mathbb{P}\{X_n = k, X_{n-1} \neq k, \ldots, X_1 \neq k \mid X_0 = j\}$.
- $P_{jk}^{n} = \sum_{\ell=1}^{n} f_{jk}^{(\ell)} P_{kk}^{n-\ell}, \quad n \geq 1$.
- $U_{jk}(s) = \sum_{n=0}^{\infty} P_{jk}^{n} s^n, \quad F_{jk}(s) = \sum_{n=1}^{\infty} f_{jk}^{(n)} s^n$.
- $U_{jk}(s) - 1_{\{j=k\}} = F_{jk}(s) U_{kk}(s)$.
- Gambler's probabilities of reaching a fortune N, $0 \leq i \leq N$ and $r = q/p$:
  $$p_i = \begin{cases} (1 - r^i)/(1 - r^N), & \text{if } p \neq 1/2 \\ i/N, & \text{if } p = 1/2 \end{cases}$$
  Ruin probabilities: $q_i = 1 - p_i$.
- $$s_{ij} = \mathbb{E}\Big[\sum_{n=0}^{\infty} 1_{\{X_n = j\}} \Big| X_0 = i\Big] = \sum_{n=0}^{\infty} \mathbb{P}(X_n = j \mid X_0 = i) = \sum_{n=0}^{\infty} P_{ij}^{n}.$$
- $s_{ij} = 1_{\{i=j\}} + \sum_{k=1}^{M} P_{ik} s_{kj}$, $\quad$ or in matrix form,
  $S = I + P_T S$. $\quad$ Equivalently, $S = (I - P_T)^{-1}$.
- $f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} = \dfrac{s_{ij} - 1_{\{i=j\}}}{s_{jj}}, \quad i, j \in T$.

Chapter 3

- $\min\{X_1, \ldots, X_n\} \sim \mathcal{E}(\lambda_1 + \cdots + \lambda_n)$
- $\mathbb{P}\{X_1 < X_2\} = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$
- $\mathbb{E}\left[X_1 1_{\{X_1 < X_2\}}\right] = \dfrac{\lambda_1}{(\lambda_1 + \lambda_2)^2}$
- $\mathbb{P}\{X_1 < X_2 < X_3\} = \dfrac{\lambda_1 \lambda_2}{(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)}$
- $\mathbb{P}(R_n = i) = \dfrac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$
- $\mathbb{P}(S_n^1 < S_m^2) = \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} p^k (1-p)^{n+m-1-k}$

Chapter 4

- *Kolmogorov backward equation*: $P'(s) = GP(s)$, where $G_{ij} > 0$ for $i \neq j$, and $G_{ii} = -\sum_{j \neq i} G_{ij}$.
- $\lambda_i = -G_{ii}$ and $J_{ij} = G_{ij}/\lambda_i$ for $i \neq j$, and $J_{ii} = 0$.
- $P(t) = e^{tG}$, for $t \geq 0$.
- *M/M/1* queue stationary distribution $\pi(i) = (1 - \theta)\theta^i$, where $\theta = \lambda/\mu < 1$.
- Ergodic Theorem: with $\{\pi_i\}$ the stationary distribution,
  $\mathbb{P}\Big\{\lim_{T \to \infty} \frac{1}{T} \int_0^T f(X(t))dt = \sum_{i \in E} \pi_i f(i)\Big\} = 1$.

Chapter 5

- $f(\mathbf{x}) = (2\pi)^{-\frac{p}{2}} |\boldsymbol{\Sigma}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right\}$
- For $\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
  $$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix},$$
  $\mathbf{X}_1 | \mathbf{X}_2 = \mathbf{x}_2 \sim \mathcal{N}(\boldsymbol{\mu}_{1|2}, \boldsymbol{\Sigma}_{11|2})$, where
  $\boldsymbol{\mu}_{1|2} = \mathrm{E}(\mathbf{X}_1 | \mathbf{X}_2 = \mathbf{x}_2) = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} (\mathbf{x}_2 - \boldsymbol{\mu}_2)$,
  $\boldsymbol{\Sigma}_{11|2} = \mathrm{Var}(\mathbf{X}_1 | \mathbf{X}_2 = \mathbf{x}_2) = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21}$.
- For $0 = t_0 < t_1 < \cdots < t_n$,
  $(B(t_1), \ldots, B(t_n)) \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ where $\boldsymbol{\Sigma}_{ij} = t_i \wedge t_j$.
- Let $0 < t_1 \leq t_2$.
  Forcasting: $B(t_2) \mid B(t_1) = x_1 \sim \mathcal{N}(x_1, t_2 - t_1)$.
  Backcasting: $B(t_1) \mid B(t_2) = x_2 \sim \mathcal{N}\left(\frac{t_1}{t_2} x_2, \frac{t_1}{t_2}(t_2 - t_1)\right)$.
- $T_a = \inf\{t : B(t) = a\}$
- Reflection principle: $B_t^* = \begin{cases} B_t, & \text{if } t \leq T_a, \\ 2a - B_t, & \text{if } t > T_a. \end{cases}$
- Running maximum: $M_t = \max_{0 \leq s \leq t} B(s), \quad t \geq 0$.
- For $0 < b \leq a$, $\mathbb{P}(M_t \geq a,\ B_t \leq b) = \mathbb{P}(B_t \geq 2a - b)$.
- M_t has the same distribution as $|B_t|$.
- T_a has the same distribution as $\dfrac{a^2}{B_1^2}$

Some common distributions

	pdf/pmf	mean	variance
Geometric	$(1-p)^{n-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	$e^{-\lambda}\frac{\lambda^k}{k!}$	λ	λ
Exponential	$\theta e^{-\theta x}$	$\frac{1}{\theta}$	$\frac{1}{\theta^2}$
Gamma	$\frac{\theta^k x^{k-1}}{\Gamma(k)} e^{-\theta x}$	$\frac{k}{\theta}$	$\frac{k}{\theta^2}$