The Chinese University of Hong Kong, Shenzhen
School of Data Science

<div align="center">

MAT3007 · Homework 1

Due: 11:59pm, September 22 (Friday), 2023

</div>

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.

- Please submit your assignment on Blackboard.

- The homework must be written in English.

- Late submission will not be graded.

- Each student must not copy homework solutions from another student or from any other source.

- For those questions that ask you to write MATLAB/Python codes to solve the problem. Please attach the code to the homework. You also need to clearly state (write or type) the optimal solution and the optimal value you obtained. However, you do not need to attach the outputs in the command window of MATLAB/Python.

Problem 1 (25pts). Modeling

A company produces two kinds of products. A product of the first type requires 1/8 hours of assembly labor, 1/2 hours of testing, and \$1.2 worth of raw materials. A product of the second type requires 1/4 hours of assembly, 1/6 hours of testing, and \$0.9 worth of raw materials. Given the current personnel of the company, there can be at most 90 hours of assembly labor and 80 hours of testing each day. Products of the first and second type have a market value of \$9 and \$8 respectively.

(a) Formulate a linear optimization that maximizes the daily profit of the company.

(b) Write the standard form of the LP you formulated in part (a).

(c) Consider the following modification to the original problem: Suppose that up to 40 hours of overtime assembly labor can be scheduled, at a cost of \$8 per hour. Can it be easily incorporated into the linear optimization formulation and how?

(d) Solve the LP using software (for the optimization problem formulated in part (a)).

Solution.

(a) Let x_1 be the number of type 1 product, and x_2 be the number of type 2 product.

$$
\begin{array}{ll}
\text{maximize} & (9 - 1.2)x_1 + (8 - 0.9)x_2 \\
\text{s.t.} & \frac{1}{8}x_1 + \frac{1}{2}x_2 \leq 90 \\
& \frac{1}{4}x_1 + \frac{1}{6}x_2 \leq 80 \\
& x_1, x_2 \geq 0
\end{array}
$$

<div align="center">

1

</div>

(b) The standard form is as follows.

$$\begin{aligned}
\text{minimize} \quad & -(9 - 1.2)x_1 - (8 - 0.9)x_2 \\
\text{s.t.} \quad & \tfrac{1}{8}x_1 + \tfrac{1}{2}x_2 + s_1 = 90 \\
& \tfrac{1}{4}x_1 + \tfrac{1}{6}x_2 + s_2 = 80 \\
& x_1, x_2, s_1, s_2 \geq 0
\end{aligned}$$

(c) Let x_3 be the number of overtime hours. We can form the following LP.

$$\begin{aligned}
\text{maximize} \quad & (9 - 1.2)x_1 + (8 - 0.9)x_2 - 8x_3 \\
\text{s.t.} \quad & \tfrac{1}{8}x_1 + \tfrac{1}{2}x_2 \leq 90 + x_3 \\
& \tfrac{1}{4}x_1 + \tfrac{1}{6}x_2 \leq 80 \\
& x_3 \leq 40 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

(d) The optimal objective value is about 2724 . The optimal solutions are $x_1 = 240, x_2 = 120$.

```
cvx_begin
variables x1 x2
minimize -(9-1.2)*x1 - (8-0.9)*x2
subject to
1/8*x1 + 1/2*x2 <= 90;
1/4*x1 + 1/6*x2 <= 80;
x1 >= 0;
x2 >= 0;
cvx_end
```

∎

Problem 2 (25pts). Reformulate NLP as LP
Reformulate the following problems as linear programming:

$$\begin{aligned}
\text{minimize} \quad & 2x_2 + |x_1 - x_3| \\
\text{subject to} \quad & |x_1 + 2| + |x_2| \leq 5 \\
& x_3^2 \leq 1
\end{aligned}$$

Please also write down its standard form.

Solution.

Let $y_1 = |x_1 - x_3|$, $y_2 = |x_1 + 2|$, $y_3 = |x_2|$.

2

$$\text{minimize} \quad 2x_2 + y_1$$
$$\text{s.t.} \quad y_1 \geq x_1 - x_3$$
$$y_1 \geq -x_1 + x_3$$
$$y_2 + y_3 \leq 5$$
$$y_2 \geq x_1 + 2$$
$$y_2 \geq -x_1 - 2$$
$$y_3 \geq x_2$$
$$y_3 \geq -x_2$$
$$x_3 \leq 1$$
$$x_3 \geq -1$$

The standard form:

$$\text{minimize} \quad 2x_2 + y_1$$
$$\text{s.t.} \quad y_1 - \left(x_1^+ - x_1^-\right) + \left(x_3^+ - x_3^-\right) - s_1 = 0$$
$$y_1 + \left(x_1^+ - x_1^-\right) - \left(x_3^+ - x_3^-\right) - s_2 = 0$$
$$y_2 + y_3 - 5 + s_3 = 0$$
$$y_2 - \left(x_1^+ - x_1^-\right) - 2 - s_4 = 0$$
$$y_2 + \left(x_1^+ - x_1^-\right) + 2 - s_5 = 0$$
$$y_3 - \left(x_2^+ - x_2^-\right) - s_6 = 0$$
$$y_3 + \left(x_2^+ - x_2^-\right) - s_7 = 0$$
$$\left(x_3^+ - x_3^-\right) + 1 - s_8 = 0$$
$$\left(x_3^+ - x_3^-\right) - 1 + s_9 = 0$$
$$y_1, y_2, y_3, x_1^+, x_1^-, x_2^+, x_2^-, x_3^+, x_3^- \geq 0$$
$$s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9 \geq 0$$

∎

Problem 3 (25pts). The China Railroad Ministry is in the process of planning relocations of freight cars among 5 regions of the country to get ready for the fall harvest. Table1 shows the cost of moving a car between each pair of regions. Table2 shows the current number of cars in each region and the number needed for harvest shipping.
Write down a linear optimization to compute the least costly way to move the cars such that the need is met.

(a) Formulate an optimization problem to achieve this task.

(b) Solve the formulated optimization problem using software. If you have integer constraints in your formulation, you can first ignore the integer constraints and solve the relaxed problem. What are the optimal solution of the relaxed problem? What is the optimal value of the true problem and why?

Solution.

From/To	1	2	3	4	5
1	–	20	13	11	28
2	20	–	18	8	46
3	13	18	–	9	27
4	11	8	9	–	20
5	28	46	27	20	–

Table 1: Cost of moving a car

	1	2	3	4	5
Present	110	335	400	420	610
Need	150	200	600	200	390

Table 2: Number of current and needed cars

Let x_{ij} denote the number of cars moving from place i to j.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{5} \sum_{j=1}^{5} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i \neq 1} x_{i1} - \sum_{j \neq 1} x_{1j} \geq 150 - 110 \\
& \sum_{i \neq 2} x_{i2} - \sum_{j \neq 2} x_{2j} \geq 200 - 335 \\
& \sum_{i \neq 3} x_{i3} - \sum_{j \neq 3} x_{3j} \geq 600 - 400 \\
& \sum_{i \neq 4} x_{i4} - \sum_{j \neq 4} x_{4j} \geq 200 - 420 \\
& \sum_{i \neq 5} x_{i5} - \sum_{j \neq 5} x_{5j} \geq 390 - 610 \\
& x_{ij} \geq 0 \quad \forall i, j \\
& x_{ij} \in \mathbb{Z} \quad \forall i, j.
\end{aligned}
$$

```
c = [0 20 13 11 28;
20 0 18 8 46;
13 18 0 9 27;
11 8 9 0 20;
28 46 27 20 0];

cvx_begin
variables x(5,5)

minimize sum(sum(c.*x))
subject to

x(2,1)+x(3,1)+x(4,1)+x(5,1)-x(1,2)-x(1,3)-x(1,4)-x(1,5)>=150-110;
x(1,2)+x(3,2)+x(4,2)+x(5,2)-x(2,1)-x(2,3)-x(2,4)-x(2,5)>=200-335;
x(1,3)+x(2,3)+x(4,3)+x(5,3)-x(3,1)-x(3,2)-x(3,4)-x(3,5)>=600-400;
x(1,4)+x(2,4)+x(3,4)+x(5,4)-x(4,1)-x(4,2)-x(4,3)-x(4,5)>=200-420;
x(1,5)+x(2,5)+x(3,5)+x(4,5)-x(5,1)-x(5,2)-x(5,3)-x(5,4)>=390-610;
x >= zeros(5,5);
cvx_end
```

The optimal solutions are $x_{41} = 40, x_{43} = 200, x_{24} = 20$, with the rest $x_{ij} = 0$. Since the solution satisfy the integer constraint, then it must be optimal to the original problem since we enlarge the feasible region by relaxing the integer constraint. The optimal objective value is 3195.

Problem 4 (25pts). Write a software code to use linear optimization to solve the shortest path problem. Suppose the input of the problem will be an $n \times n$ matrix of W, where w_{ij} is the length of the path from i to j. In our implementation, we always use 1 to denote the source node (the S node in the lecture slides), and n to denote the terminal node (the T node in the lecture slides). In addition, we assume for any i and j, there is a path, i.e., the set of E is all pairs of nodes. This is without loss of generality since one can set w_{ij} and w_{ji} to be an extremely large number if there is no edge between i and j, effectively eliminating it from consideration.
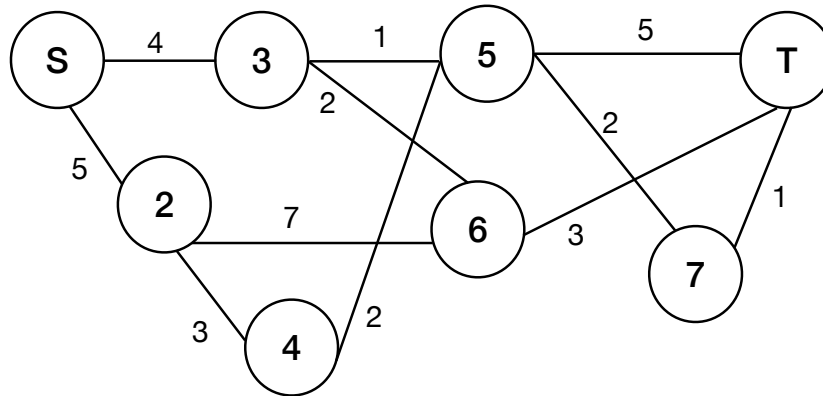

Figure 1: The graph of the shortest path problem

You are asked to solve the optimization problem formulated in the lecture slides using software, with the given labeling shown in Figure 1. Basically, you need to input the W matrix for this case, and then solve it. To solve this problem, you can relax the binary integer constraint $x_{ij} \in \{0, 1\}$ to $0 \leq x_{ij} \leq 1$. What is the optimal solution, i.e., the optimal path, for the relaxed problem? Is it optimal to the original problem? If yes, justify it.

Solution.
The optimal solutions are $x(1, 3) = x(3, 5) = x(5, 7) = x(7, 8) = 1$, with the rest $x(i, j) = 0$. The optimal objective value is 8. Since the solutions are either 0 or 1, satisfying the binary integer constraints, it must be optimal to the original binary integer program since we have enlarged the feasible region.

```
M = 100;
W = [M, 5, 4, M, M, M, M, M;
5, M, M, 3, M, 7, M, M;
4, M, M, M, 1, 2, M, M;
M, 3, M, M, 2, M, M, M;
M, M, 1, 2, M, M, 2, 5;
M, 7, 2, M, M, M, M, 3;
M, M, M, M, 2, M, M, 1;
```

```
M, M, M, M, 5, 3 ,1, M];
[n, m] = size(W);
cvx_begin
variables x(n,n)
minimize sum(sum(W .* x))
subject to
sum(x(1,:)) - sum(x(:,1))== 1;
sum(x(:,n)) - sum(x(n,:)) == 1;
sum(x(2:n-1,:), 2) - sum(x(:,2:n-1))' == 0;
0 <= x <= 1
cvx_end
```

∎