

Midterm Solution

Nov 11th

1. We throw a fair die, then we throw a coin as many times as the die indicates points. Let X be the number of Tails obtained. Determine $E(X)$ and $\text{var}(X)$. [10 markers]

Solution: Consider that the die shows N points. Then $X = Y_1 + \dots + Y_N$ is a random sum where the Y_i 's are independent Bernoulli ($\frac{1}{2}$) variables. Note that $E(N) = \frac{7}{2}$ and $\text{var}(N) = \frac{35}{6}$. Therefore,

- $E(X) = \mu E(N) = \frac{1}{2} \cdot \frac{7}{2} = \frac{7}{4}$.

$$-\text{var}(X) = \sigma^2 E(N) + \mu^2 \text{var}(N) = \frac{1}{4} \cdot \frac{7}{2} + \left(\frac{1}{2}\right)^2 \cdot \frac{35}{6} = \frac{77}{48}.$$

2. Consider n tosses of a fair die and let X_1 and X_2 be the number of tosses the die shows the numbers 1 and 2, respectively.

(a) Find $E(X_1^2)$.

(b) Find $E(X_1^2 | X_2 = 0)$.

(c) Find $E(X_1^2 | X_2 > 0)$.

Solution:

(a) X_1 has binomial $(n, 1/6)$ distribution, so $E(X_1^2) = (n/6)^2 + n \frac{1}{6} \frac{5}{6} = n(n+5)/6^2$.

(b) Conditionally to $X_2 = 0$, X_1 has binomial $(n, 1/5)$ distribution, so $E(X_1^2 | X_2 = 0) = (n/5)^2 + n \frac{1}{5} \frac{4}{5} = n(n+4)/5^2$.

(c) We have

$$\mathbb{E}(X_1^2) = \mathbb{E}(X_1^2 | X_2 = 0) \mathbb{P}(X_2 = 0) + \mathbb{E}(X_1^2 | X_2 > 0) \mathbb{P}(X_2 > 0)$$

This leads to

$$\begin{aligned} \mathbb{E}(X_1^2 | X_2 > 0) &= \frac{\mathbb{E}(X_1^2) - \mathbb{E}(X_1^2 | X_2 = 0) \mathbb{P}(X_2 = 0)}{\mathbb{P}(X_2 > 0)} \\ &= \frac{n(n+5)/6^2 - n(n+4)/5^2(5/6)^n}{1 - (5/6)^n} \end{aligned}$$

3. A Markov chain $\{X_n, n \geq 0\}$ with states 0, 1, 2 has the transition probability matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

(a) Find $P(X_1 = 1, X_2 = 2 | X_0 = 0)$ and $P(X_2 = 1 | X_0 = 0)$.

(b) If $P(X_0 = 0) = P(X_0 = 1) = \frac{1}{4}$, find $E[X_2]$.

(c) Is this Markov chain irreducible? Why?

(d) In the long run, what proportion of time is the process in each of the three states?

Solution:

$$(a) P(X_1 = 1, X_2 = 2 | X_0 = 0) = P(X_1 = 1 | X_0 = 0) P(X_2 = 2 | X_1 = 1, X_0 = 0) = p_{01}p_{12} = \frac{1}{3}\frac{2}{3} = \frac{2}{9}$$

The P^2 of the probability matrix is

$$\begin{pmatrix} \frac{1}{3} & \frac{5}{18} & \frac{7}{18} \\ \frac{1}{3} & \frac{1}{6} & \frac{5}{9} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

$$P(X_2 = 1 | X_0) = p_{01}^2 = \frac{5}{18}$$

$$(b) P(X_0 = 2) = \frac{1}{2}. \text{ Hence}$$

$$E[X_2] = P(X_2 = 1) + 2P(X_2 = 2) = \frac{1}{4}P_{01}^2 + \frac{1}{4}P_{11}^2 + \frac{1}{2}P_{21}^2 + 2\left(\frac{1}{4}P_{02}^2 + \frac{1}{4}P_{12}^2 + \frac{1}{2}P_{22}^2\right) = \frac{71}{72}$$

(c) Yes, since the three states communicate with each other.

(d) Let $\pi_i, i = 0, 1, 2$ be the limiting probabilities. Then $\pi_0 + \pi_1 + \pi_2 = 1$, and

$$\begin{aligned} \pi_0 &= \frac{1}{2}\pi_0 + \frac{1}{2}\pi_2 \\ \pi_1 &= \frac{1}{3}\pi_0 + \frac{1}{3}\pi_1 \end{aligned}$$

$$\text{The solution is } \pi_0 = \frac{2}{5}, \pi_1 = \frac{1}{5}, \pi_2 = \frac{2}{5}.$$

4. Consider a reservoir with a total capacity of h units of water. Let X_n represent the amount of water in the reservoir at the end of the n -th day. You are given the following information.

- The daily inputs to the reservoir are independent and discrete random variables. On a given day,

$$\mathbb{P}(\text{input is } j \text{ units}) = \frac{1}{2^{j+1}}, j = 0, 1, 2, \dots$$

- Any overflow (when the total amount of water exceeds the capacity of h units) is regarded as a loss.
- Provided the reservoir is not empty, one unit is released at the end of the day.
- The value of X_n for day n is the content of the reservoir after the release at the end of the day.

Therefore, the stochastic process $\{X_n, n = 1, 2, \dots\}$ is a Markov chain with state space $\{0, 1, 2, \dots, h-1\}$.

- (a) Explain why the stochastic process is a Markov Chain. [5 marks]
 - (b) Show that it is irreducible, aperiodic and positive recurrent. [5 marks]
- For the remaining part of this question, assume $h = 3$.
- (c) Determine the one-step transition matrix. [5 marks]
 - (d) Find the limiting probability [5 marks]

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_n = 0)$$

(e) The reservoir is initially having 2 units of water. Find the expected number of days until the reservoir is empty. [10 marks]

Solution:

1. The stochastic process takes a discrete time index and discrete state space. Moreover, the Markov property is satisfied. Hence, it is a Markov Chain.
2. We show them one by one.

- Irreducible: there exists a path between any two states such that all states communicate with each other.
- Aperiodic: it is trivial that $d(i) = 1$ for all $i = 0, 1, 2, \dots, h - 1$.
- Positive recurrent: the Markov Chain is irreducible and has a finite state space; this forces the Markov chain to positive recurrent.

3. Note that

$$\begin{aligned}\mathbb{P}(\text{input is 0 units}) &= f(0) &= 0.5 \\ \mathbb{P}(\text{input is 1 units}) &= f(1) &= 0.25 \\ \mathbb{P}(\text{input is 2 units}) &= f(2) &= 0.125 \\ \mathbb{P}(\text{input is } \geq 3 \text{ units}) &= f(\geq 3) &= 0.125\end{aligned}$$

This gives the one step transition matrix as

$$P = \begin{pmatrix} f(0) + f(1) & f(2) & f(\geq 3) \\ f(0) & f(1) & f(\geq 2) \\ 0 & f(0) & f(\geq 1) \end{pmatrix} = \begin{pmatrix} 0.75 & 0.125 & 0.125 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

4. Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the limiting probability. We have

$$\begin{aligned}\pi P &= \pi \\ \pi_0 + \pi_1 + \pi_2 &= 1\end{aligned}$$

or equivalently,

$$\begin{cases} 0.75\pi_0 + 0.5\pi_1 = \pi_0 \\ 0.125\pi_0 + 0.25\pi_1 + 0.5\pi_2 = \pi_1 \\ 0.125\pi_0 + 0.25\pi_1 + 0.5\pi_2 = \pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

Solving gives,

$$\pi_0 = 0.5, \pi_1 = 0.25, \pi_2 = 0.25$$

such that

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_n = 0) = 0.5$$

(e) Let E_{j0} be the expected number of days to transit from j to 0. Conditioning gives

$$\begin{cases} E_{20} = 1 + 0.5E_{20} + 0.5E_{10} \\ E_{10} = 1 + 0.25E_{10} + 0.25E_{20} \end{cases}$$

Solving gives $E_{10} = 3$ and $E_{20} = 5$.

5. Suppose cars enter a one-way infinite-lane highway with Poisson rate $\lambda = 1$. The time spent by each car on the highway is independent of each other and is an exponential distribution with rate $\mu = 1$.

(a) Calculate the probability that the first car to enter the highway will leave the highway earlier than the second car to enter the highway. [5 marks]

- (b) Let car A be the i -th car entering the highway. Calculate the expectation of the time at which car A leaves the highway. [5 marks]
Now suppose there are two cars having entered the highway by time $t = 1$.
- (c) What's the expected number of cars that enter the highway before time $t = 1$ but on the highway at time $t = 2$. [5 marks]
- (d) What's the expected number of cars that enter the highway after time $t = 1$ but on the highway at time $t = 2$. [5 marks]
- (e) What's the expected number of cars on the highway at time $t = 2$. [5 marks]

Solution: Denote the counting process of the number of cars having entered the highway as $\{N(t), t \geq 0\}$. For the i -th car, we denote the enter time as S_i , the stay time as D_i , and the inter-enter time between the $i-1$ th car and the i th car as T_i , i.e.,

$$T_1 = S_1, \quad T_i = S_i - S_{i-1}, \forall i > 1.$$

Here $T_1, D_1, T_2, D_2, \dots \stackrel{i.i.d.}{\sim} Exp(1)$.

- (a) The right interpretation of this question is "the probability of the leave time of first car less than the leave time of second car", so the answer is

$$\begin{aligned} P(S_1 + D_1 < S_2 + D_2) &= P(D_1 < T_2 + D_2) = 1 - P(D_1 \geq T_2 + D_2) \\ &= 1 - P(D_1 \geq T_2 + D_2 | D_1 \geq T_2)P(D_1 \geq T_2) \\ &= 1 - P(D_1 \geq D_2) \cdot P(D_1 \geq T_2) \quad [\text{memoryless property}] \\ &= 1 - \frac{1}{1+1} \cdot \frac{1}{1+1} = \frac{3}{4} \end{aligned}$$

Most of the students think the question is "the probability of the leave time of first car less than the enter time of second car", in this case, the answer will be as follows:

$$\begin{aligned} P(S_1 + D_1 < S_2) &= P(D_1 < T_2) = 1 - P(D_1 \geq T_2) \\ &= 1 - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

Since there are many misleadings among students, if your answer is $1/2$, you still have full marks. If your answer is $3/4$, you would get one more mark

(b)

$$E[S_i + D_i] = E\left[\sum_{j=1}^i T_j + D_i\right] = i + 1$$

For the part (c) and (d), we will use the following result:

Given there are n arrivals before time t , the joint distribution of the n arrival (waiting) times S_1, S_2, \dots, S_n is the same as that of the order statistics from a sample of n i.i.d. random variables uniformly distributed over the interval $(0, t)$.

- (c) Condition on there are two cars between $(0, 1)$, the joint distribution of S_1, S_2 is the same as the distribution of the order statistics from $U_1, U_2 \stackrel{i.i.d.}{\sim} Unif(0, 1)$.

Denote $Z_i = \mathbb{1}_{\text{ith car enter before 1 and stay at time 2}} = \mathbb{1}_{S_i < 1, S_i + D_i > 2}$, where for any event A , $\mathbb{1}_A \sim \text{Bern}(P(A))$. Then we have

$$\begin{aligned}
& E[\# \text{ cars enter before 1 and stay at time 2} | \text{exactly two cars enter before 1}] = E[Z_1 + Z_2 | N_1 = 2] \\
& = 1 \cdot \sum_{i=1}^2 P(S_i < 1, S_i + D_i > 2 | N_1 = 2) \quad [\text{linearity of expectation}] \\
& = \sum_{i=1}^2 P(U_i + D_i > 2) \quad [\text{conditional distribution of arrival time}] \\
& = \sum_{i=1}^2 \int_0^1 \int_{2-u}^{\infty} e^{-x} dx \cdot 1 du \\
& = 2 \cdot \int_0^1 e^{u-2} du = 2 \cdot (e^{-1} - e^{-2})
\end{aligned}$$

- (d) Here we consider the i th car entering within time $(1, 2)$ as the i th car. Cars who enter in $(0, 1)$ don't matter anymore.

Denote the number of arrivals within $(1, 2)$ as $N \sim \text{Poisson}(1)$. Condition on $N = n$, the joint distribution of S_1, \dots, S_n is the same as the distribution of the order statistics from $U_1, \dots, U_n \stackrel{i.i.d.}{\sim} \text{Unif}(1, 2)$.

Denote $X_i = \mathbb{1}_{\text{ith car enter after time 1 and stay at time 2}} = \mathbb{1}_{1 < S_i < 2, S_i + D_i > 2}$. Then we have

$$\begin{aligned}
& E[\# \text{ cars enter after time 1 and stay at time 2} | \text{exactly two cars enter before 1}] \\
& = E[\# \text{ cars enter after time 1 and stay at time 2}] = E\left[\sum_{i=1}^N X_i\right] \\
& = E[E\left[\sum_{i=1}^N X_i | N\right]] \quad [\text{Tower property}] \\
& = E\left[\sum_{i=1}^N P(1 < S_i < 2, S_i + D_i > 2 | N)\right] \\
& = E\left[\sum_{i=1}^N P(U_i + D_i > 2)\right] \\
& = E\left[\sum_{i=1}^N \int_1^2 \int_{2-u}^{\infty} e^{-x} dx \cdot 1 du\right] \\
& = E[N \cdot \int_1^2 e^{u-2} du] = E[N \cdot (1 - e^{-1})] = 1 - e^{-1}.
\end{aligned}$$

(e)

$$\begin{aligned}
& E[\# \text{ cars stay at time 2} | \text{exactly two cars enter before 1}] \\
& = E[\# \text{ cars enter before 1 and stay at time 2} | \text{exactly two cars enter before 1}] \quad (\text{c}) \\
& \quad + E[\# \text{ cars enter before 1 and stay at time 2} | \text{exactly two cars enter before 1}] \quad (\text{d}) \\
& = 2 \cdot (e^{-1} - e^{-2}) + 1 - e^{-1} = 1 + e^{-1} - 2e^{-2}
\end{aligned}$$