The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 6

Due: 11:59pm, Nov 17

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 Optimality Conditions for Unconstrained Problem — I (20 pts).
Consider the function

$$f(x) = x_1^3 - x_2^3 + 3x_1^2 + 3x_2^2 - 9x_1$$

Use the first-order necessary condtion (FONC), second order necessary condition (SONC) and second order sufficient condition (SOSC) to find (i) saddle points, (ii) strict local minimizers and(iii) strict local maximizers.

Solution 1.

$$\nabla f = \begin{pmatrix} 3x_1^2 + 6x_1 - 9 \\ -3x_2^2 + 6x_2 \end{pmatrix}, \nabla^2 f = \begin{pmatrix} 6x_1 + 6 & 0 \\ 0 & -6x_2 + 6 \end{pmatrix}$$

Step 1: Calculate all stationary poinrs of f by solving $\nabla f = 0$:
Thus, the stationary points are: $x_1^* = (1; 0), x_2^* = (-3; 0), x_3^* = (1; 2), x_4^* = (-3; 2)$.

Step 2: Determine the definitenesss of the Hessian $\nabla^2 f(x^*)$ to decide whether the stationary points x^* are local minima, maxima or saddle points:

$$\nabla^2 f(x_1^*) = \begin{pmatrix} 12 & 0 \\ 0 & 6 \end{pmatrix}, \nabla^2 f(x_2^*) = \begin{pmatrix} -12 & 0 \\ 0 & 6 \end{pmatrix}, \nabla^2 f(x_3^*) = \begin{pmatrix} 12 & 0 \\ 0 & -6 \end{pmatrix}, \nabla^2 f(x_4^*) = \begin{pmatrix} -12 & 0 \\ 0 & -6 \end{pmatrix}$$

x_1^* is a strict local minimizer, x_4^* is a strict local maximizer and x_2^*, x_3^* are saddle points.

Problem 2 Optimality Conditions for Unconstrained Problem — II (20 pts).
Consider the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^2} \; f(x) = x_1^3 - x_1\left(1 + x_2^2\right) + x_2^4.$$

(a) Compute the gradient and Hessian of f and calculate all stationary points.

(b) For each stationary point, investigate whether it is a local maximizer, local minimizer, or saddle point and explain your answer.

Note: For a 2×2 Hessian, we can check the trace and determinant to verify their definiteness, as $\mathrm{tr}(Q) = \lambda_1 + \lambda_2$ and $\mathrm{let}(Q) = \lambda_1 \lambda_2$ for any matrix Q, where λ_1 and λ_2 are the two eigenvalues of Q.

Solution 2.

(a) The gradient and Hessian of f are given by

$$\nabla f(x) = \begin{pmatrix} 3x_1^2 - x_2^2 - 1 \\ -2x_1x_2 + 4x_2^3 \end{pmatrix} \quad \text{and} \quad \nabla^2 f(x) = \begin{pmatrix} 6x_1 & -2x_2 \\ -2x_2 & -2x_1 + 12x_2^2 \end{pmatrix}.$$

In total, f has the following four stationary points:

$$\bar{x}_1 = \left(\frac{\sqrt{3}}{3}, 0\right), \quad \bar{x}_2 = \left(\frac{-\sqrt{3}}{3}, 0\right), \quad \bar{x}_3 = \left(\frac{2}{3}, \frac{-\sqrt{3}}{3}\right), \quad \bar{x}_4 = \left(\frac{2}{3}, \frac{\sqrt{3}}{3}\right).$$

(b) We have

$$\nabla^2 f(\bar{x}_1) = \begin{pmatrix} 2\sqrt{3} & 0 \\ 0 & -\frac{2\sqrt{3}}{3} \end{pmatrix} \quad \text{and} \quad \nabla^2 f(\bar{x}_2) = \begin{pmatrix} -2\sqrt{3} & 0 \\ 0 & \frac{2\sqrt{3}}{3} \end{pmatrix}.$$

These two Hessians are diagonal matrices with eigenvalues $2\sqrt{3}, -\frac{2\sqrt{3}}{3}$ and $-2\sqrt{3}, \frac{2\sqrt{3}}{3}$ respectively and, hence $\nabla^2 f(\bar{x}_1)$ and $\nabla^2 f(\bar{x}_2)$ are indefinite and the stationary points $\bar{x}_1$ and $\bar{x}_2$ are saddle points. Furthermore, it holds that

$$\nabla^2 f(\bar{x}_3) = \begin{pmatrix} 4 & \frac{2\sqrt{3}}{3} \\ \frac{2\sqrt{3}}{3} & \frac{8}{3} \end{pmatrix} \quad \text{and} \quad \nabla^2 f(\bar{x}_4) = \begin{pmatrix} 4 & -\frac{2\sqrt{3}}{3} \\ -\frac{2\sqrt{3}}{3} & \frac{8}{3} \end{pmatrix}$$

and $\mathrm{tr}(\nabla^2 f(\bar{x}_3)) = \mathrm{tr}(\nabla^2 f(\bar{x}_4)) = \frac{20}{3} > 0$ and $\det(\nabla^2 f(\bar{x}_3)) = \det(\nabla^2 f(\bar{x}_4)) = \frac{32}{3} - \frac{4}{3} > 0$. This shows that $\nabla^2 f(\bar{x}_3)$ and $\nabla^2 f(\bar{x}_4)$ are positive definite. Thus, by the second order sufficient conditions, $\bar{x}_3$ and $\bar{x}_4$ are strict local minimizer.

Problem 3 KKT Conditions for Constrained Problem — I (20 pts).
Consider the following problem:

$$\begin{aligned} \underset{x_1, x_2 \in \mathbb{R}}{\text{minimize}} \quad & (x_1 - 4)^2 + \left(x_2 - \frac{7}{2}\right)^2, \\ \text{s.t.} \quad & x_2 - x_1^2 \geqslant 0, \\ & x_1 + x_2 \leqslant 6, \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

(a) Write down the KKT optimality conditions.

(b) Find a KKT pair (x^*, λ^*) where $x^* = (x_1^*, x_2^*)$ and λ^* is the corresponding multiplier vector.

Solution 3.

(a) The KKT conditions are:

$$2\left(x_1 - 4\right) + 2\lambda_1 x_1 + \lambda_2 - \lambda_3 = 0, \quad 2\left(x_2 - \frac{7}{2}\right) - \lambda_1 + \lambda_2 - \lambda_4 = 0,$$
$$\lambda_1\left(x_1^2 - x_2\right) = 0, \quad \lambda_2\left(x_1 + x_2 - 6\right) = 0, \quad \lambda_3 x_1 = 0, \quad \lambda_4 x_2 = 0, \quad \lambda_i \geqslant 0.$$

(Remark: it is possible to not use λ_3 and λ_4.)

(b) **Case1:** When $x_1 = 0$,
suppose $x_2 = 0$, then

$$\begin{aligned} \lambda_2 - \lambda_3 &= 8 \\ -\lambda_1 + \lambda_2 - \lambda_4 &= 7 \\ -6\lambda_2 &= 0 \end{aligned}$$

Therefore, $\lambda_1 + \lambda_4 = -7 \leq 0$, contradicting that $\lambda_i \geq 0$. So, $x_2 \neq 0$, we have

$$\begin{aligned} \lambda_1 = \lambda_4 &= 0 \\ -8 + \lambda_2 - \lambda_3 &= 0 \\ -7 + 2x_2 + \lambda_2 &= 0 \\ \lambda_2(x_2 - 6) &= 0 \end{aligned}$$

Then we have $-\frac{1}{2}\lambda_2(\lambda_2 + 5) = 0$, since $\lambda_2 \geq 0$, take $\lambda_2 = 0$.
Then $\lambda_3 = -8$, impossible.
Hence, "$x_1 = 0$" is impossible.
Case2: $x_1 \neq 0$,
Then $\lambda_3 = 0$, $x_2 \geq x_1^2 > 0$, so $\lambda_4 = 0$.
Now we have:

$$\begin{aligned} 2x_1 - 8 + 2\lambda_1 x_1 + \lambda_2 &= 0 \\ 2x_2 - 7 - \lambda_1 + \lambda_2 &= 0 \\ \lambda_1(x_1^2 - x_2) &= 0 \\ \lambda_2(x_1 + x_2 - 6) &= 0 \end{aligned}$$

If $\lambda_1 = 0$, then $x_1 - x_2 = \frac{1}{2}$, $x_2 - x_1^2 = -x_2^2 - \frac{1}{4} < 0$, impossible.
Therefore $\lambda_1 \neq 0, x_1^2 = x_2$.
Now, suppose $\lambda_2 \neq 0, x_1 + x_2 = 6$, then $x_1 = 2, x_2 = 4$ since $x_1^2 - x_2 = 0$, we have

$$\begin{aligned} -4 + 4\lambda_1 + \lambda_2 &= 0 \\ 1 - \lambda_1 + \lambda_2 &= 0 \end{aligned}$$

Solve λ_1, λ_2 from above we get $\lambda_1 = 1, \lambda_2 = 0$, λ_2 should be 0.
Therefore we have $x_1^* = 2, x_2^* = 4, \lambda_1^* = 1, \lambda_2^* = 0, \lambda_3^* = 0, \lambda_4^* = 0$.

Problem 4 Failure of KKT Conditions for Constrained Problem — II (20 pts).
(**Note:** This example shows that a global minimizer may not satisfy the KKT conditions, though there are KKT points.)
Consider the optimization problem:

$$\begin{array}{ll} \text{minimize} & -x^2 + x^3 \\ \text{subject to} & x^3(x+1)^3 \leq 0. \end{array}$$

(a) Write down the KKT conditions for this problem.

(b) Find out the KKT pionts (the primal and dual variable pairs satisfying KKT conditions).

(c) Show that $x = -1$ is a global minimizer.

Solution 4.

(a) First we associate a Lagrangian multiplier λ for the constraint and write down the Lagrangian function:

$$L(x, \lambda) = -x^2 + x^3 + \lambda x^3(x+1)^3. \tag{1}$$

We have the following KKT conditions:

- Main conditions

$$-2x + 3x^2 + \lambda\left(3x^2(x+1)^3 + 3x^3(x+1)^2\right) = 0 \tag{2}$$

- Dual feasibility condition: $\lambda \geq 0$;
- Complementarity conditions:

$$\lambda x^3(x+1)^3 = 0; \tag{3}$$

- Primal feasibility conditions:

$$x^3(x+1)^3 \leq 0; \tag{4}$$

(b) We consider λ:

- If $\lambda = 0$, then we have $-2x + 3x^2 = 0$, thus $x = 0$ or $x = \frac{2}{3}$(Omitted because it can't satisfy primal feasibility).
- If $\lambda \neq 0$, then we have $x^3(x+1)^3 = 0$, thus $x = 0$ or $x = -1$(Omitted because it can't satisfy main conditions).

Therefore, KKT points are $(x = 0, \lambda \geq 0)$.

(c) Since $x^3(x+1)^3 \leq 0$, we have $x \in [-1, 0]$. Let $f(x) = -x^2 + x^3$, then $f'(x) = -2x + 3x^2 \geq 0$, which means $f(x)$ is monotonically increasing in $[-1, 0]$. Therefore, $x = -1$ is a global minimizer.

Problem 5 KKT Conditions for Constrained Problem — II (20 pts).
(**Note:** This problem is actually convex and any KKT points must be globally optimal, and we will study convex optimization soon.)
Consider the optimization problem:

$$\begin{array}{rl} \text{minimize} & 2x_1 + x_2 + x_3 \\ \text{subject to} & \frac{2}{x_1} + \frac{9}{x_2} + \frac{4}{x_3} \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(a) Write down the KKT conditions for this problem.

(b) Find the KKT points (the primal and dual variable pairs satisfying KKT conditions).

Solution 5.

(a) First we associate a Lagrangian multiplier λ for the constraint and write down the Lagrangian function:

$$L(x_1, x_2, x_3, \lambda) = 2x_1 + x_2 + x_3 + \lambda_0 \cdot (2/x_1 + 9/x_2 + 4/x_3 - 1) - \lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3.$$

We have the following KKT conditions:

- Main conditions

$$2 - \frac{2\lambda_0}{x_1^2} = 0; \quad 1 - \frac{9\lambda_0}{x_2^2} = 0; \quad 1 - \frac{4\lambda_0}{x_3^2} = 0;$$

- Dual feasibility condition: $\lambda_i \geq 0$;
- Complementarity conditions:

$$\lambda_0 \cdot (\frac{2}{x_1} + \frac{9}{x_2} + \frac{4}{x_3} - 1) = 0; \quad \lambda_1 x_1 = 0; \quad \lambda_2 x_2 = 0; \quad \lambda_3 x_3 = 0;$$

- Primal feasibility conditions:

$$\frac{2}{x_1} + \frac{9}{x_2} + \frac{4}{x_3} \leq 1; \quad x_1, x_2, x_3 \geq 0$$

(b) From the primal feasibility condition, we can tell that $x_1 \neq 0$, $x_2 \neq 0$, $x_3 \neq 0$. Therefore, from the complementarity conditions, we must have

$$x_1^2 = \lambda_0; \quad x_2^2 = 9\lambda_0; \quad x_3^2 = 4\lambda_0.$$

Thus $x_1 = \sqrt{\lambda_0}$, $x_2 = 3\sqrt{\lambda_0}$, $x_3 = 2\sqrt{\lambda_0}$. Then because $x_1 \neq 0$, we know that $\lambda_0 \neq 0$. Thus, from the complementarity conditions, we must have

$$\frac{2}{x_1} + \frac{9}{x_2} + \frac{4}{x_3} = 1.$$

Plugging $x_1 = \sqrt{\lambda_0}$, $x_2 = 3\sqrt{\lambda_0}$, $x_3 = 2\sqrt{\lambda_0}$ into the above equation, we have $\lambda = 49$.

Therefore, the only solution to the KKT condition is $x_1 = 7$, $x_2 = 21$, $x_3 = 14$.