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MAT3007 · Homework 4

Due: 11:59pm, Oct. 27 (Friday), 2023

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Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.

- Please submit your assignment on Blackboard.

- The homework must be written in English.

- Late submission will not be graded.

- Each student must not copy homework solutions from another student or from any other source.

Problem 1 (10+10+10=30pts).
Consider the following linear program:

$$
\begin{aligned}
\text{maximize} \quad & 3x_1 + x_2 + 4x_3 \\
\text{subject to} \quad & x_1 + 3x_2 + x_3 \leq 5 \\
& x_1 + 2x_2 + 2x_3 \leq 7 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

(a). What is the corresponding dual problem?

(b). Solve the dual problem graphically.

(c). Use complementarity conditions for the primal-dual pair to solve the primal problem.

Solution.

(a). The dual problem is

$$
\begin{aligned}
\text{minimize} \quad & 5y_1 + 7y_2 \\
\text{s.t.} \quad & y_1 + y_2 \geq 3 \\
& 3y_1 + 2y_2 \geq 1 \\
& y_1 + 2y_2 \geq 4 \\
& y_1, y_2 \geq 0
\end{aligned}
$$

(b). The optimal solution is $(2, 1)$ and the optimal value is 17.

(c). We first notice that the second constraint in the dual problem is not tight: $3y_1^* + 2y_2^* - 1 \neq 0$, which implies $x_2^* = 0$.

Then, we observe that $y_1^* \neq 0, y_2^* \neq 0$, we have that

$$x_1^* + x_3^* = 5$$
$$x_1^* + 2x_3^* = 7$$

As a result, we get that $x_1^* = 3, x_3^* = 2$, which is primal feasible. Thus, the optimal solution of the primal problem is $(3, 0, 2)$.

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Problem 2 (5 + 10 + 10 + 10 = 35pts).
Consider the following table of food and corresponding nutritional values:

	Protein, g	Carbohydrates, g	Calories	Cost
Bread	4	7	130	3
Milk	6	10	120	4
Fish	20	0	150	9
Potato	1	30	70	1

The ideal intake for an adult is at least 30 grams of protein, 40 grams of carbohydrates, and 400 calories per day. The problem is to find the **least** costly way to achieve those amounts of nutrition by using the four types of food shown in the table.

(a). Formulate this problem as a linear optimization problem (specify the meaning of each decision variable and constraint).

(b). Solve it using MATLAB, find an optimal solution and the optimal value.

(c). Formulate the dual problem. Interpret the dual problem. (Hint: Suppose a pharmaceutical company produces each of the nutrients in pill form and sells them each for a certain price.)

(d). Use MATLAB to solve the dual problem. Find an optimal solution and the optimal value.

Solution.

(a). Let x_i denote the amount of food $i = 1, \cdots, 4$. The LP formulation is as follows:

$$
\begin{aligned}
\text{minimize} \quad & 3x_1 + 4x_2 + 9x_3 + x_4 \\
\text{s.t.} \quad & 4x_1 + 6x_2 + 20x_3 + x_4 \geq 30 \\
& 7x_1 + 10x_2 + 30x_4 \geq 40 \\
& 130x_1 + 120x_2 + 150x_3 + 70x_4 \geq 400 \\
& x_i \geq 0, \quad \forall i = 1, \ldots, 4
\end{aligned}
$$

The first three constraints are the minimum daily value requirements for each nutrient.

(b). The optimal solution is (0, 0, 1.36, 2.8), and the objective value is 15.04 at optimal.

(c). Dual problem:

$$\begin{aligned}
\text{maximize} \quad & 30y_1 + 40y_2 + 400y_3 \\
\text{s.t.} \quad & 4y_1 + 7y_2 + 130y_3 \leq 3 \\
& 6y_1 + 10y_2 + 120y_3 \leq 4 \\
& 20y_1 + 150y_3 \leq 9 \\
& y_1 + 30y_2 + 70y_3 \leq 1 \\
& y_i \geq 0, \quad \forall i = 1, \ldots, 3
\end{aligned}$$

Dual interpretation: A pharmaceutical company makes 3 kinds of pills. Each kind of pill contains 1 gram of a type of nutrient. The dual variable y_i represent the price for each pill of nutrient i. The company wants to sell the pills and maximize its profit. The objective function is the revenue of the company when consumers are buying the pills to satisfy their daily nutrient requirements. The constraints are restrictions on the prices of the pills to make it attractive to the customers. The total cost that can purchase nutrients equivalent to the food should not exceed the price of buying the food directly.

(d). The optimal solution is (0.384, 0, 0.0088), and the objective value is 15.04 at optimal.

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Problem 3 (15pts) Farkas's Lemma.

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following two condition holds:

(1). $\exists x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.

(2). $\exists y \in \mathbb{R}^m$ such that $A^\top y \geq 0$ and $y^\top b < 0$.

Solution.
Consider the primal and dual problem pair $\min_y b^\top y, s.t. A^\top y \geq 0$ and $\max_x 0, s.t. Ax = b, x \geq 0$. If there exists y such that $A^\top y \geq 0$ and $y^\top b < 0$, then the primal is unbounded, which implies that the dual problem is infeasible. That is, there exists no x such that $Ax = b$ and $x \geq 0$. If there exists no y such that $A^\top y \geq 0$ and $y^\top b < 0$, then for all y's satisfy $A^\top y \geq 0$, we have $y^\top b \geq 0$. Therefore, the primal problem has finite optimum $y = 0$ and the optimal value is 0. From strong duality theorem, the dual problem must be feasible, which implies the existence of x such that $Ax = b$ and $x \geq 0$.

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Problem 4 (10 + 10 = 20pts). Special Dual problem.

Suppose M is a square matrix such that $M = -M^T$, for example,

$$M = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -4 \\ -2 & 4 & 0 \end{pmatrix}$$

Consider the following optimization problem:

$$\begin{aligned}
\text{minimize}_x \quad & c^T x \\
\text{subject to} \quad & Mx \geq -c \\
& x \geq 0
\end{aligned}$$

(a). Show that the dual problem of it is equivalent to the primal problem.

(b). Argue that the problem has optimal solution if and only if there is a feasible solution.

Solution.

(a). The dual is as follows:
$$\max -c^T y$$
$$\text{s.t. } M^T y \leq c$$
$$y \geq \mathbf{0}$$

Since $M = -M^T$, $M^T y \leq c$ is equivalent to $My \geq -c$.
Therefore, we can reformulate the dual problem as a minimization problem as follows:

$$\min c^T y$$
$$\text{s.t. } My \geq -c$$
$$y \geq \mathbf{0}$$

Therefore, the dual problem is equivalent to the primal problem.

(b). First, it is obvious that if the problem has optimal solution, then it must have a feasible solution.
Now we prove the other direction.
If the problem has a feasible solution x, then $y = x$ is also feasible to the dual problem.
Therefore, both the primal and dual problems are feasible.
According to the weak duality theorem, they both have finite optimal solution.

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