The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 5

Due: 11:59am, Nov. 2nd (Thursday), 2023

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- Please submit your assignment on Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student must not copy homework solutions from another student or from any other source.

Problem 1 (50pts). Consider the following linear program:

$$\begin{array}{lll} \text{maximize} & 3x_1 + 4x_2 + 3x_3 + 6x_4 & \\ \text{subject to} & 2x_1 + x_2 - x_3 + x_4 & \geq 12 \\ & x_1 + x_2 + x_3 + x_4 & = 8 \\ & -x_2 + 2x_3 + x_4 & \leq 10 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array} \tag{1}$$

After transforming the problem into standard form and apply Simplex method, we obtain the final tableau as follow:

B	0	2	9	0	3	0	36
1	1	0	-2	0	-1	0	4
4	0	1	3	1	1	0	4
6	0	-2	-1	0	-1	1	6

a) Derive the dual problem of the linear program (1) and calculate a dual solution based on complementarity conditions. Given that the optimal solution to the primal solution is unique, investigate whether the dual solution is unique.

b) Do the optimal solution and the objective function value change if we

- decrease the objective function coefficient for x_3 to 0?
- increase the objective function coefficient for x_3 to 9?
- decrease the objective function coefficient for x_4 to 5?
- increase the objective function coefficient for x_1 to 7?

c) Find the possible range for adjusting the coefficient 8 of the second constraint such that the current basis is kept optimal.

Solution 1.

a) The dual of problem (1) is given by

$$\begin{array}{lll} \text{minimize} & 12y_1 + 8y_2 + 12y_3 & \\ \text{subject to} & y_1 \le 0, y_2 \text{ free}, y_3 \ge 0 & \\ & 2y_1 + y_2 & \ge 3 \\ & y_1 + y_2 - y_3 & \ge 4 \\ & -y_1 + y_2 + 2y_3 & \ge 3 \\ & y_1 + y_2 + y_3 & \ge 6. \end{array}$$

Using the complementarity conditions, we can infer $2y_1 + y_2 - 3 = 0$, $y_1 + y_2 + y_3 - 6 = 0$, and $y_3(-x_2^* + 2x_3^* + x_4^* - 10) = -6y_3 = 0$. (Since the optimal slack variables are not all zero, one of the primal constraints needs to be inactive). This yields $y_3 = 0$ and:

$$2y_1 + y_2 = 3, \quad y_1 + y_2 = 6 \quad \implies \quad y_1 = -3, \quad y_2 = 9.$$

Since the primal solution is unique and the complementarity conditions fully characterize the dual solution $y^* = (-3, 9, 0)^\top$, the dual problem has a unique solution as well.

b) Using the final simplex tableau, we obtain:

$$A_B^{-1}A_N = \begin{pmatrix} 0 & -2 & -1 \\ 1 & 3 & 1 \\ -2 & -1 & -1 \end{pmatrix} \quad \text{and we have} \quad A_B^{-1} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

We now discuss the different questions step by step:

- Decreasing $c_3 = 3$ to 0 means that the new costs (in standard form) are given by: $\tilde{c} = (-3, -4, 0, -6, 0, 0)^\top$. Due to $\{3\} \notin B$, we can simply check:

  $$r_N^\top + \begin{pmatrix} 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 12 & 3 \end{pmatrix} \ge 0.$$

  In this case, both the optimal value as well as the optimal solution do not change.

- Increasing $c_3 = 3$ to 9 means that the new costs (in standard form) are given by: $\tilde{c} = (-3, -4, -9, -6, 0, 0)^\top$. Due to $\{3\} \notin B$, we can simply check:

  $$r_N^\top + \begin{pmatrix} 0 & -6 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 3 \end{pmatrix} \ge 0.$$

  In this case, both the optimal value as well as the optimal solution do not change.

- Decreasing $c_4 = 6$ to 5 means that the new costs (in standard form) are given by: $\tilde{c} = (-3, -4, -3, -5, 0, 0)^\top$. We then have:

  $$\tilde{c}_N^\top - \tilde{c}_B^\top A_B^{-1}A_N = \begin{pmatrix} -4 & -3 & 0 \end{pmatrix} - \begin{pmatrix} -3 & -5 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 & -1 \\ 1 & 3 & 1 \\ -2 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 2 \end{pmatrix}.$$

  Thus, $x^* = (4, 0, 0, 4, 0, 6)^\top$ will remain optimal solution with new optimal value 32.

- Increasing $c_1 = 3$ to 7 means that the new costs (in standard form) are given by: $\tilde{c} = (-7, -4, -3, -6, 0, 0)^\top$. We then have:

$$\tilde{c}_N^\top - \tilde{c}_B^\top A_B^{-1} A_N = \begin{pmatrix} -4 & -3 & 0 \end{pmatrix} - \begin{pmatrix} -7 & -6 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 & -1 \\ 1 & 3 & 1 \\ -2 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \end{pmatrix}.$$

  The optimal solution will change in this case. (Calling CVX, we can find $x^* = (8, 0, 0, 0, 4, 10)^\top$ with optimal value 56).

c) We need to find the range of λ such that $x_B^* + \lambda A_B^{-1} e_2$ is nonnegative, i.e.,

$$\begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 - \lambda \\ 4 + 2\lambda \\ 6 - 2\lambda \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This yields $-2 \leq \lambda \leq 3$ and thus, the coefficient 8 can be chosen from the interval $[6, 11]$ without changing the current optimal basis.

Problem 2 (50pts). Consider the following linear program:

$$\begin{array}{lll} \text{maximize} & x_1 + cx_2 + 3x_3 + 8x_4 & \\ \text{subject to} & x_1 - x_2 + x_3 & \leq 2 \\ & x_3 - x_4 & \leq 1 \\ & 2x_2 + 3x_3 + 4x_4 & \leq b \\ & x_1, x_2, x_3, x_4 & \geq 0, \end{array} \tag{2}$$

where $c = 2, b = 8$. Denote $x = (x_1; x_2; x_3; x_4; s_1; s_2; s_3)$ as the decision variable to the standard form of the above problem, where s_1, s_2, s_3 are the slack variables corresponding to the three constraints. The following table gives the final simplex tableau when solving the standard form of the above problem:

B	0	1	4	0	1	0	2	18
1	1	-1	1	0	1	0	0	2
6	0	$\frac{1}{2}$	$\frac{7}{4}$	0	0	1	$\frac{1}{4}$	3
4	0	$\frac{1}{2}$	$\frac{3}{4}$	1	0	0	$\frac{1}{4}$	2

a) What is the current dual optimal solution? In what range can we change the coefficient c so that the current optimal basis still remains optimal?

b) If we change $c = 2$ to $c = -100$, what will be the new optimal solution to problem (2) and the optimal value of problem (2)?

c) In what range can we change the coefficient of the third constraint $b = 8$ so that the current optimal basis still remains optimal?

d) What will be the new optimal solution to problem (2) when we change $b = 8$ to $b = 4$ and what will be the optimal value?

Solution.

a) By the complementarity conditions, $-1 - y_1 = 0$, $y_3 = 0$ and $-y_2 + 4y_3 + 8 = 0$, from which $y_1 = -1$, $y_2 = 0$, $y_3 = -2$. Since $j \in N$, the condition is $r_N - \lambda e_j \geq 0$, where $e_j = e_1$ and we can read that $r_N = [1, 4, 1, 2]^\top$, the condition on is $1 - \lambda \geq 0$ which gives $\lambda \leq 1$. Thus, we can choose $c \leq 3$.

b) If $c_2 = -c = 100$, the optimal basis will remain the same and changing c does not affect primal optimal solution. Thus, we have the new primal optimal solution and the optimal value keep unchanged. From the simplex tableau, we have $[2, 0, 0, 2]^\top$ is the optimal solution and 18 is the optimal value of the maximization problem.

c) The condition is

$$x_B^* + \lambda A_B^{-1} e_3 \geq 0$$

from the tableau, we read that $A_B^{-1} = [1\ 0\ 0; 0\ 1\ 1/4; 0\ 0\ 1/4]$. Then the condition becomes

$$[2, 3, 2] + \lambda[0\ 1/4\ 1/4] \geq 0,$$

which gives $\lambda \geq -8$. Overall, we can choose any $b = b_3 \geq 0$.

d) $\tilde{x}_B = x_B^* + A_B^{-1}\Delta b = [2; 3; 2] + A_B^{-1}[0; 0; -4] = [2; 2; 1]$. Thus the new optimal primal solution to the standard form is $[2, 0, 0, 1, 0, 2]^2$. Thus the optimal solution to the original problem is $[2, 0, 0, 1]^\top$. Since the optimal dual solution is unchanged, we have $y^* = [-1, 0, -2]^\top$. The optimal value becomes 10.

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