Midterm
OCT 20th

1. Let $Z_0, Z_1, Z_2, \ldots$ be independent, identically distributed random variables with common p.m.f.:

k	0	1	2	3
p_k	0.1	0.3	0.2	0.4

For $n \geq 0$, define $X_n = \min(Z_0, Z_1, \ldots, Z_n)$.

(a) Give the transition matrix P of the Markov chain (X_n). **[5 marks]**

(b) Calculate $\mathbb{P}(X_3 = 2, X_2 = 1, X_1 = 2 | X_0 = 3)$ **[3 marks]**

(c) Calculate $\mathbb{P}(X_3 = 0, X_2 = 1, X_1 = 2 | X_0 = 3)$. **[3 marks]**

(d) Find the communication classes of the chain. **[3 marks]**

(e) Find the recurrent and transient states of the chain. **[4 marks]**

(f) What are the periods of the states? **[4 marks]**

(g) Does the chain have a stationary (invariant) distribution? **[3 marks]**

(h) Find the limiting probabilities of the chain (if any). **[5 marks]**

Solution.

(a) We have $X_n = \min(Z_0, Z_1, \ldots, Z_n), n \geq 1$
The transition matrix is shown as follows:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.9 & 0 & 0 \\ 0.1 & 0.3 & 0.6 & 0 \\ 0.1 & 0.3 & 0.2 & 0.4 \end{bmatrix}$$

(b) $\mathbb{P}(X_3 = 2, X_2 = 1, X_1 = 2 | X_0 = 3) = P_{32}P_{21}P_{12} = 0$

(c) $\mathbb{P}(X_3 = 0, X_2 = 1, X_1 = 2 | X_0 = 3) = P_{32}P_{21}P_{10} = 0.2 * 0.3 * 0.1 = 0.006$

(d) Since $X_m \leq X_{m-1} (\forall m \geq 1)$,
$i \to j$ is impossible if $i < j$.
Therefore, all classes singleton and we have 4 classes: $0, 1, 2, 3$

(e) For $i = 1, 2, 3$, $P_{i,i-1} > 0$ and once at $i - 1$, the MC cannot return to i.
Therefore $f_i < 1$ and i is transient.
For $i = 0$ is absorbing, thus recurrent.

(f) For all i, $p_{ii} > 0$, so "i" is aperiodic.

(g) For $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$ Solving $\pi P = \pi$, i.e.

$$\begin{cases} \pi_0 = \pi_0 + 0.1\pi_1 + 0.1\pi_2 + 0.1\pi_3 \\ \pi_1 = 0.9\pi_1 + 0.3\pi_2 + 0.3\pi_3 \\ \pi_2 = 0.6\pi_2 + 0.2\pi_3 \\ \pi_3 = 0.4\pi_3 \end{cases}$$

From the equations: $\pi_3 = 0$, then $\pi_2 = 0, \pi_1 = 0, \pi_0 = 1$
There is one unique stationary distribution$\pi = (1, 0, 0, 0)$

(h) Since $j = 1, 2, 3$ are transient,
$P_{ij}^n \to 0$ as $n \to \infty$ for all $0 \le i \le 3$ and $1 \le j \le 3$
Moreover,

$$1 = P_{i0}^n + P_{i1}^n + P_{i2}^n + P_{i3}^n$$

implies $P_{i0}^n \to 1 (\forall 0 \le i \le 3)$
Therefore, as $n \to \infty$

$$P^n \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which are the limiting probability of the MC.

2. The No Claims Discount scale operated by a motor insurer has 3 levels of discount, namely 0%, 30%, and 50%. The rules for moving between discount levels are:
- if a driver makes no claims in a year he or she moves in the following year to the next higher rate of discount (or stays at 50% if already at that level).
- if a driver makes 1 claim in a year he or she moves in the following year to the next lower rate of discount (or stays at 0% if already at that level).
- if a driver makes 2 or more claims in a year he or she moves in the following year to the 0% discount level (or stays at 0% if already at that level).

The probability of one claim in a year for a certain risk group is 0.14, and the probability of 2 or more claims is 0.01.

(a) Calculate the proportion of this risk group at each discount level once a stationary state is reached. **[10 marks]**

(b) Calculate the expected premium per customer in this risk group if the base (i.e. undiscounted) premium is \$1,000. **[10 marks]**

Solution.

(a) The probability matrix is given by

$$\mathbf{P} = \begin{pmatrix} 0.15 & 0.85 & 0 \\ 0.15 & 0 & 0.85 \\ 0.01 & 0.14 & 0.85 \end{pmatrix}$$

Let π be a stationary probability vector.
$\pi P = \pi$, $\pi(I - P) = 0$
$\Rightarrow \pi_1 = 0.0352, \pi_2 = 0.1447, \pi_3 = 0.8201$

(b)

$$\begin{aligned} \mathbb{E}(\text{premium}) &= 1000 * 0.0352 + 700 * 0.1447 + 500 * 0.8201 \\ &= 546.54 \end{aligned}$$

3. (a) Suppose in the gambler's ruin problem that the probability of winning a bet depends on the gambler's present fortune. Specifically, suppose that α_i is the probability that the gambler wins a bet when his or her fortune is i. Given that the gambler's initial fortune is i $(0 \le i \le N)$, let P_i denote the probability that the gambler's fortune reaches N before 0.

i. Derive a formula that relates P_i to P_{i-1} and P_{i+1}. **[5 marks]**

ii. Using the same approach as in the gambler's ruin problem, solve the equation of part (i) for P_i. **[5 marks]**

iii. Suppose that i balls are initially in urn 1 and $N-i$ are in urn 2, and suppose that at each stage one of the N balls is randomly chosen, taken from whichever urn it is in, and placed in the other urn. Find the probability that the first urn becomes empty before the second. **[10 marks]**

(b) Consider $n+1$ vertices on a circle numbered as $0, 1, \ldots, n$ clockwise. A particle moves among these vertices as follows. *A particle moves among $n+1$ vertices that are situated on a circle in the following manner.* At each step it moves one step either in the clockwise direction with probability $0 < p < 1$ or the counterclockwise direction with probability $q = 1-p$. Starting at a specified vertex, say vertex 0, let T be the time of the first return to vertex 0. Find the probability that all vertices have been visited by time T. (Hint: use a first-step analysis.) **[10 marks]**

Solution.

(a) The state space in $\{0, 1, 2, \ldots, N\}, (N \geq 1)$

i) $1 \leq i \leq N-1$: conditioning on the first move, it is easily seen that

$$P_i = \alpha_i P_{i+1} + (1-\alpha_i)P_{i-1}, 1 \leq i \leq N-1$$

with initial/terminal conditions: $P_0 = 0, P_N = 1$

ii) writing $P_i = \alpha_i P_i + (1-\alpha_i)P_i$,
we have $\alpha_i(P_{i+1} - P_i) = (1-\alpha_i)(P_i - P_{i-1})$
$\Rightarrow P_{i+1} - P_i = \beta_i(P_i - P_{i-1})$, where $\beta_i = \frac{1-\alpha_i}{\alpha_i}$, with $c_i = \beta_i \ldots \beta_1$
Therefore, $P_i - P_{i-1} = \beta_{i-1} \ldots \beta_1(P_1 - P_0)$ and $P_i = (1 + c_1 + \ldots + c_{i-1})P_1$
Finally, with $P_N = 1, P_i = \frac{c_0 + \ldots + c_{i-1}}{c_0 + \ldots + c_{N-1}}$, with $c_0 = 1$

iii) For $i \leq i \leq N-1$, urn with i balls has the probability $\alpha_i = \frac{N-i}{N}$ to win an additional ball.
The requested probability is P_{N-i} with $\alpha_i = \frac{N-i}{N}$

(b) Let A be the event that all states have been visited by time T. Then conditioning on the direction of the first step gives:

$$\mathbb{P}(A) = p\,\mathbb{P}(A|\text{clockwise}) + q\,\mathbb{P}(A|\text{counterclockwise})$$

Note that once arrived at "n" clock-wisely, the particle can move to "0", or go backward to delay the return to "0".
- Conditioning on "clockwise", the probability is to reach 0 clock-wisely (by state n) before reaching 0 (counterclockwisely). Visiting all states by the returning time to "0" means hitting "n" before "0": this is the winning probability in the gambler's ruin problem that a gambler that starts with 1\$ will reach n\$ before going broke: with $r = q/p$,
$\mathbb{P}(A|\text{clockwise}) = \frac{1-r}{1-r^n}$, if $r \neq 1$, or $1/n$ if $r = 1$.
- Similarly, once at n, Visiting all states by the returning time to "0" means hitting "1" before "0": this is the ruin probability in the gambler's ruin problem that a gambler that starts with $(n-1)$\$ will broke before reaching n\$:
$\mathbb{P}(A|\text{counterclockwise}) = 1 - \frac{1-r^{n-1}}{1-r^n} = \frac{r^{n-1}-r^n}{1-r^n}$ if $r \neq 1$, or $1 - \frac{n-1}{n} = 1/n$ if $r = 1$.
(Notice that this equals $\frac{1-r^{-1}}{1-(r^{-1})^n}$, ie the previous one by exchanging p and q).
- Finally

$$\mathbb{P}(A) = p\frac{1-r}{1-r^n} + q\frac{r^{n-1}-r^n}{1-r^n} = \frac{1-r}{1-r^n}(p + qr^{n-1}), \quad r \neq 1,$$

and $1/n$ if $r = 1$.

4. Each entering customer must be served first by server 1, then by server 2, and finally by server 3. The amount of time it takes to be served by server i is an exponential random variable with rate μ_i, $i = 1, 2, 3$. Suppose you enter the system when it contains a single customer who is being served by server 3.

(a) Find the probability that server 3 will still be busy when you move over to server 2. **[5 marks]**

(b) Find the probability that server 3 will still be busy when you move over to server 3. **[5 marks]**

(c) Find the expected amount of time that you spend in the system. (Whenever you encounter a busy server, you must wait for the service in progress to end before you can enter service.) **[10 marks]**

Solution.

(a) $\mathbb{P}(X_1 < X_3) = \frac{\mu_1}{\mu_1+\mu_3}$.

(b) Conditional on (a), the desired prob is $\mathbb{P}(X_2 < X_3) = \frac{\mu_2}{\mu_2+\mu_3}$, so the total prob is $\frac{\mu_1}{\mu_1+\mu_3} \cdot \frac{\mu_2}{\mu_2+\mu_3}$.

(c) The expected amount of time you spend in the first two servers is $\frac{1}{\mu_1} + \frac{1}{\mu_2}$, in the third server is $\frac{\mu_1}{\mu_1+\mu_3} \cdot \frac{\mu_2}{\mu_2+\mu_3} \cdot \frac{2}{\mu_3} + (1 - \frac{\mu_1}{\mu_1+\mu_3} \cdot \frac{\mu_2}{\mu_2+\mu_3})\frac{1}{\mu_3}$, so the total expected amount of time is $\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} + \frac{\mu_1}{\mu_1+\mu_3} \cdot \frac{\mu_2}{\mu_2+\mu_3} \cdot \frac{1}{\mu_3}$.

Useful formulas for "Stochastic Processes"

Chapter 1

- $\mathbb{P}\left(\bigcup_1^n A_i\right) = \sum_{k=1}^n (-1)^{k+1} \sum_{\{i_1,\dots,i_k\}} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$
- The coupon collection problem: $\mathbb{E}(N) = \sum_{k=1}^n (-1)^{k+1} \sum_{\{i_1,\dots,i_k\}} \frac{1}{p_{i_1}+\cdots+p_{i_k}}$
- $X \geq 0$: $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > u)du$, $\mathbb{E}[X] = \sum_{k=0}^\infty \mathbb{P}(X > k)$.
- Order statistics $(X_{(1)}, \dots, X_{(n)})$ has joint density $n! f(x_1) \cdots f(x_n) 1_{\{x_1 < \cdots < x_n\}}$
- ith order statistic $X_{(i)}$ has density $i\binom{n}{i} f(x) F^{i-1}(x)\{1 - F(x)\}^{n-i}$.
- $\mathbb{E}(X) = \mathbb{E}\{\mathbb{E}(X|Y)\}$.
- $\mathrm{var}(X) = \mathbb{E}\{\mathrm{var}(X|Y)\} + \mathrm{var}\{\mathbb{E}(X|Y)\}$.
- Random sum $S_N = X_1 + \cdots + X_N$:
  $\mathbb{E}(S_N) = \mu\,\mathbb{E}(N)$, $\mathrm{var}(S_N) = \sigma^2 E(N) + \mu^2 \mathrm{var}(N)$.

Chapter 2

- Hitting time of i:
  $\{\tau_i = n\} = \{X_n = i, X_{n-1} \neq i, \dots, X_1 \neq i\}$.
- $f_{ii}^{(n)} = \mathbb{P}(\tau_i = n \mid X_0 = i)$,
  $$f_i = \sum_{n=1}^\infty f_{ii}^{(n)} = \mathbb{P}(\tau_i < \infty \mid X_0 = i) .$$
- $P_{ii}^n = \sum_{k=1}^n f_{ii}^{(k)} P_{ii}^{n-k}, \quad n \geq 1.$
- $f_{jk}^{(n)} = \mathbb{P}\{X_n = k, X_{n-1} \neq k, \dots, X_1 \neq k \mid X_0 = j\}$.
- $P_{jk}^n = \sum_{\ell=1}^n f_{jk}^{(\ell)} P_{kk}^{n-\ell}, \quad n \geq 1.$
- $U_{jk}(s) = \sum_{n=0}^\infty P_{jk}^n s^n, \quad F_{jk}(s) = \sum_{n=1}^\infty f_{jk}^{(n)} s^n.$
- $U_{jk}(s) - 1_{\{j=k\}} = F_{jk}(s) U_{kk}(s)$.
- Gambler's probabilities of reaching a fortune N, $0 \leq i \leq N$ and $r = q/p$:
  $$p_i = \begin{cases} (1 - r^i)/(1 - r^N), & \text{if } p \neq 1/2 \\ i/N, & \text{if } p = 1/2 \end{cases}$$
  Ruin probabilities: $q_i = 1 - p_i$.
- $$s_{ij} = \mathbb{E}\left[\sum_{n=0}^\infty 1_{\{X_n = j\}} \Big| X_0 = i\right] = \sum_{n=0}^\infty \mathbb{P}(X_n = j \mid X_0 = i) = \sum_{n=0}^\infty P_{ij}^n.$$
- $s_{ij} = 1_{\{i=j\}} + \sum_{k=1}^M P_{ik} s_{kj}$, or in matrix form,
  $S = I + P_T S$. Equivalently, $S = (I - P_T)^{-1}$.
- $f_{ij} = \sum_{n=1}^\infty f_{ij}^{(n)} = \frac{s_{ij} - 1_{\{i=j\}}}{s_{jj}}, \quad i, j \in T.$

Chapter 3

- $\min\{X_1, \dots, X_n\} \sim \mathcal{E}(\lambda_1 + \cdots + \lambda_n)$
- $\mathbb{P}\{X_1 < X_2\} = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$
- $\mathbb{E}\left[X_1 1_{\{X_1 < X_2\}}\right] = \dfrac{\lambda_1}{(\lambda_1 + \lambda_2)^2}$
- $\mathbb{P}\{X_1 < X_2 < X_3\} = \dfrac{\lambda_1 \lambda_2}{(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)}$
- $\mathbb{P}(R_n = i) = \dfrac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$

Some common distributions

	pdf/pmf	mean	variance
Geometric	$(1-p)^{n-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	$e^{-\lambda}\frac{\lambda^k}{k!}$	λ	λ
Exponential	$\theta e^{-\theta x}$	$\frac{1}{\theta}$	$\frac{1}{\theta^2}$
Gamma	$\frac{\theta^k x^{k-1}}{\Gamma(k)} e^{-\theta x}$	$\frac{k}{\theta}$	$\frac{k}{\theta^2}$