



MAT3007 · Homework 2

Due: 11:59pm, September 28 (Thursday), 2023

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- Please submit your assignment on Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student must not copy homework solutions from another student or from any other source.
- For those questions that ask you to write MATLAB/Python codes to solve the problem. Please attach the code to the homework. You also need to clearly state (write or type) the optimal solution and the optimal value you obtained. However, you do not need to attach the outputs in the command window of MATLAB/Python.

Problem 1 (15pts). True or False

Consider an LP in its standard form and the corresponding constraint set $P = \{x | Ax = b, x \geq 0\}$. Suppose that the matrix A has dimensions $m \times n$ and that its rows are linearly independent. For each of the following statements, state whether it is true or false. Please explain your answers (if not true, please show a counterexample).

- (a) The set of all optimal solutions (assuming existence) must be bounded;
- (b) At every optimal solution, no more than m variables can be positive;
- (c) If there is more than one optimal solution, then there are uncountably many optimal solutions.

Solution.

- (a) False. Consider the following counterexample:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{s.t.} & x \geq 0 \end{array}$$

then the optimal solution set is unbounded.

- (b) False. Consider the following counterexample: *minimize* 0. Then any feasible x is optimal no matter how many positive components it has.

- (c) True. If x_1 and x_2 are optimal solution, then, any convex combination of x_1, x_2 is also optimal. Specifically, $\bar{x} = \gamma x_1 + (1 - \gamma)x_2$ is optimal for any $\gamma \in [0, 1]$.

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Problem 2 (20pts). Graphical Method

Solve the following 2-dimensional linear optimization problem using the graphical method.

$$\begin{aligned} &\text{maximize} && x_1 + x_2 \\ &\text{s.t.} && -x_1 + x_2 \leq 2.5 \\ &&& x_1 + 2x_2 \leq 9 \\ &&& 0 \leq x_1 \leq 4 \\ &&& 0 \leq x_2 \leq 3 \end{aligned}$$

Which constraints are active at optimal solution? Also list all the vertices of the feasible region.

Solution.

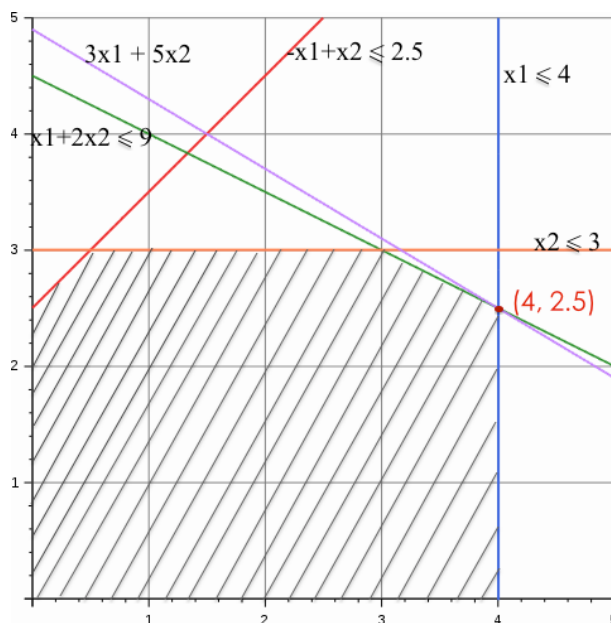


Figure 1: Graphical Illustration of Problem 2

From Figure 1, we can find all the basic feasible solutions of the feasible region:

$$(0, 0), (0, 2.5), (0.5, 3), (3, 3), (4, 2.5), (4, 0).$$

We can find that the optimal solution is $(4, 2.5)$, and the optimal value is 24.5. Since the optimal solution is at $(4, 2.5)$, the following two lines that intersect at that point are the active constraints.

$$\begin{aligned}x_1 + 2x_2 &\leq 9 \\x_1 &\leq 4\end{aligned}$$

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Problem 3 (20pts). Basic solutions and basic feasible solutions

Consider the following linear optimization problem:

$$\begin{aligned}\text{maximize} \quad & x_1 + 2x_2 + 4x_3 \\ \text{s.t.} \quad & x_1 + x_3 \leq 8 \\ & x_2 + 2x_3 \leq 15 \\ & x_1, x_2, x_3 \geq 0\end{aligned}$$

- Transform it into standard form;
- Argue without solving this LP that there must exist an optimal solution with no more than 2 positive variables;
- List all the basic solutions and basic feasible solutions (of the standard form);
- Find the optimal solution by using the results in step (c).

Solution.

- Standard form:

$$\begin{aligned}\text{minimize} \quad & -x_1 - 2x_2 - 4x_3 \\ \text{s.t.} \quad & x_1 + x_3 + x_4 = 8 \\ & x_2 + 2x_3 + x_5 = 15 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0\end{aligned}$$

- First, we will argue that there exist an optimal solution for this problem. From the constraints we know the feasible set is bounded. The objective function is to maximize the sum of bounded variables, so an optimal solution must exist. Moreover, since all the rows of the constraint matrix are linearly independent with $m = 2$ and $n = 5$, there must exist an optimal solution with no more than 2 positive variables (from the lecture slides).

- See the following table for the basic solutions and basic feasible solutions:

B	$\{1, 2\}$	$\{1, 3\}$	$\{1, 5\}$	$\{2, 3\}$	$\{2, 4\}$	$\{3, 4\}$	$\{3, 5\}$	$\{4, 5\}$
x_B	(8, 15)	(0.5, 7.5)	(8, 15)	(-1, 8)	(15, 8)	(7.5, 0.5)	(8, -1)	(8, 15)
Obj. Val.	-38	-17.5	-8	-30	-30	-30	-32	0
BFS	Y	Y	Y	N	Y	Y	N	Y

(d) The optimal solution is $(8, 15, 0, 0, 0)$.

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Problem 4 (20pts). Vertex Covering Problem

Write a MATLAB/Python code to solve the vertex covering problem discussed in our Lecture (the graph is in Figure 2). When you solve it, use constraints $0 \leq x_i \leq 1$ rather than $x_i \in \{0, 1\}$. What are the optimal solution and optimal value returned by CVX (suppose we label the variables as x_a, \dots, x_j)? What is the optimal value of the true problem? So whether one can remove the integer constraint when solving this problem?

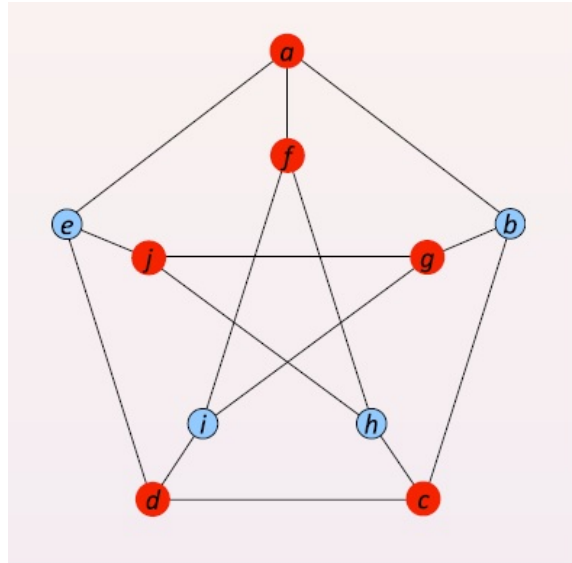


Figure 2: Graph for vertex covering

Solution.

The optimal solution to the vertex cover problem without the integrality constraint is $x_a = x_b = x_c = x_d = x_e = x_f = x_g = x_h = x_i = x_j = 0.5$ and the optimal value is 5.

Matlab

```
cvx.begin
variables xa xb xc xd xe xf xg xh xi xj
minimize xa + xb + xc + xd + xe + xf + xg + xh + xi + xj
subject to
xa + xb >= 1
xa + xe >= 1
xa + xf >= 1
xb + xc >= 1
xb + xg >= 1
xc + xd >= 1
xc + xh >= 1
xd + xe >= 1
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xd + xi >= 1
xe + xj >= 1
xf + xh >= 1
xf + xi >= 1
xg + xi >= 1
xg + xj >= 1
xh + xj >= 1
0 <= xa, xb, xc, xd, xe, xf, xg, xh, xi, xj <= 1
cvx_end

```

However, for this problem, it is easy to observe that one cannot remove the integer constraint, as the returned optimal solution is not integer at all. One can also solve the integer version of the linear optimization problem using Gurobi or Mosek and obtain that the optimal value is 6. ■

Problem 3 (25pts). A Robust LP Formulation

In this exercise, we consider the following optimization problem:

$$\min_{x \in \mathbb{R}^n} c^\top x \quad \text{subject to} \quad \|Ax - b\|_\infty \leq \delta, \quad x \geq 0 \quad (1)$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $\delta \geq 0$ are given and $\|y\|_\infty = \max_{1 \leq i \leq p} |y_i|$ denotes the maximum norm of a vector $y \in \mathbb{R}^p$. In the case $\delta = 0$, problem (1) coincides with the standard form for linear programs. The choice $\delta > 0$ can be useful to model situations where A and/or b are not fully or exactly known, e.g., when A and/or b contains certain uncertainty (can be caused by noise). In this case, problem (1) belongs to the so-called robust optimization.

- Rewrite the optimization problem (1) as a linear problem.
- We now consider a specific application of problem (1).

The fruit store in Kuai Le Shi Jian is producing two different fruit salads A and B . The smaller fruit salad A consists of “1/4 mango, 1/8 pineapple, 3 strawberries”; the larger fruit salad B consists of “1/2 mango, 1/4 pineapple, 1 strawberry”. The profits per fruit salad and the total number of fruits in stock are summarized in the following table:

	Mango	Pineapple	Strawberry	Net profit
Fruit salad A	1/4	1/8	3	10 RMB
Fruit salad B	1/2	1/4	1	20 RMB
Stock / Resources	25	10	120	

Suppose all fruits need to be processed and *completely used* to make the fruit salads A and B . Given these constraints, formulate a linear program to maximize the total profits of the fruit store. Show that this program can be expressed in standard form

$$\min_{x \in \mathbb{R}^n} c^\top x \quad \text{subject to} \quad Ax = b, \quad x \geq 0,$$

with $n = 2$ and $m = 3$. In addition, is this linear programming solvable?

Note: Since we want to produce “complete fruit salads”, the variables x_1 and x_2 should actually be modeled as integer variables: $x_1, x_2 \in \mathbb{Z}$. However, since we do not know how to

deal with these integer constraints in general at this moment, you may just ignore them for now.

- (c) One of the employee found some additional fruits in a storage crate and the manager of the fruit shop decides to determine the production plan by using the robust formulation (1). Consider the robust variant of the problem in part (b) with $\delta = 5$.

- Sketch the feasible set of this problem.
- Solve the problem graphically, i.e., Calculate the optimal value and the optimal solution set.
- Which constraints are active in the solution?
- Find one integer solution of this problem.

Solution.

- (a) By the definition of infinity norm, we have

$$\|Ax - b\|_\infty \leq \delta \iff |a_i^\top x - b_i| \leq \delta \quad \forall i \iff -\delta \leq a_i^\top x - b_i \leq \delta \quad \forall i.$$

Using this connection, we propose the following reformulation of the robust problem (1)

$$\min_x c^\top x \quad \text{s.t.} \quad \begin{cases} a_i^\top x - b_i + \delta \geq 0 & \forall i = 1, \dots, m, \\ \delta - a_i^\top x + b_i \geq 0 & \forall i = 1, \dots, m, \\ x \geq 0. \end{cases} \quad (2)$$

- (b) Let x_1 and x_2 denote the number fruit salads A and B , respectively. Since all fruits need to be processed completely, we need to satisfy the equality constraints

$$\frac{1}{4}x_1 + \frac{1}{2}x_2 = 25, \quad \frac{1}{8}x_1 + \frac{1}{4}x_2 = 10, \quad 3x_1 + x_2 = 120$$

and the total profit is given by $10x_1 + 20x_2$. This yields the following linear program

$$\min_x -10x_1 - 20x_2 \quad \text{s.t.} \quad \begin{cases} x \geq 0, \\ x_1 + 2x_2 = 100, \\ x_1 + 2x_2 = 80, \\ 3x_1 + x_2 = 120. \end{cases}$$

This linear programming is not solvable, as the first two constraints are not consistent.

- (c) The robust formulation is given by:

$$\max_x 10x_1 + 20x_2 \quad \text{s.t.} \quad \begin{cases} x \geq 0, \\ \left| \frac{x_1}{4} + \frac{x_2}{2} - 25 \right| \leq 5, \\ \left| \frac{x_1}{8} + \frac{x_2}{4} - 10 \right| \leq 5, \\ |5x_1 + x_2 - 120| \leq 5. \end{cases}$$

Expanding the abs value, we obtain

$$\max_x 10x_1 + 20x_2 \quad \text{s.t.} \quad \begin{cases} x \geq 0, \\ x_1 + 2x_2 - 100 \leq 20, \\ x_1 + 2x_2 - 100 \geq -20, \\ x_1 + 2x_2 - 80 \leq 40, \\ x_1 + 2x_2 - 80 \geq -40, \\ 3x_1 + x_2 - 120 \leq 5, \\ 3x_1 + x_2 - 120 \geq -5 \end{cases}$$

We can first draw the feasible region of this problem, which is shown in the Figure 3.

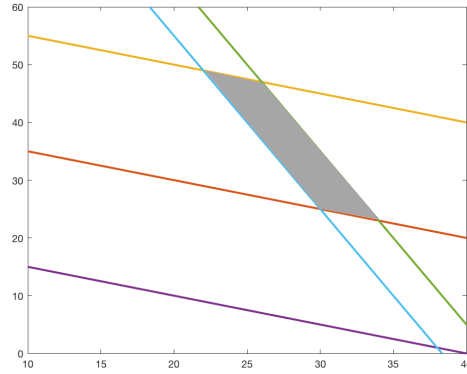


Figure 3: Feasible region

In order to get the optimal solution set and optimal value, we depict the contour lines $\{x : 10x_1 + 20x_2 = c\}$ for the several values of c . It can be seen when $c = 1200$, the contour line will touch the feasible region from above and achieves the maximum value, 1200; depicted as a dashed line in the Figure 4:

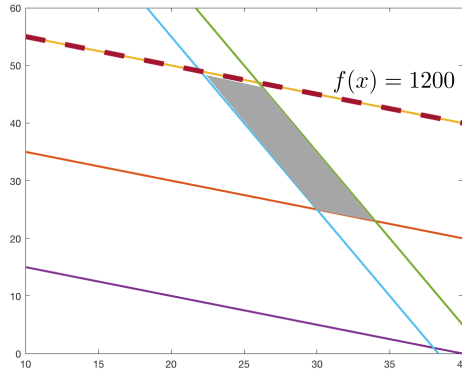


Figure 4: The solution set and optimal value

Furthermore, the optimal solution is not unique and they form a set:

$$X^* := \{x : x_1 + 2x_2 = 120, x_1 \in [22, 26]\}.$$

Thus, at an optimal solution $x \in X^*$ either the constraints $x_1 + 2x_2 = 120$ or $x_1 + 2x_2 = 120$ and $3x_1 + x_2 = 115$ or $x_1 + 2x_2 = 120$ and $3x_1 + x_2 = 125$ are active.

As for integer solution, we can let $x_1 \in \{22, 23, 24, 25, 26\}$, we will get $x_2 \in \{49, 48.5, 48, 47.5, 47\}$, respectively. Thus, we can select any of the following plans to achieve the maximum value 1200:

$$(x_1, x_2) = (22, 49), \text{ or, } (x_1, x_2) = (24, 48), \text{ or, } (x_1, x_2) = (26, 47).$$

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