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Final
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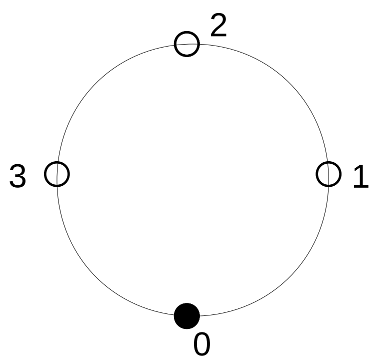
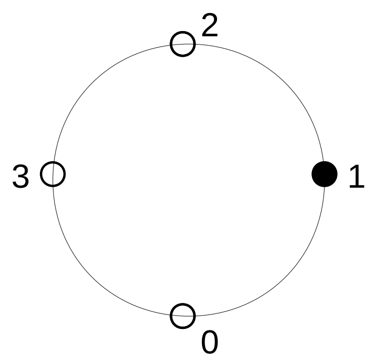
1. Consider four equidistant points marked on a circle with numbers 0, 1, 2 and 3. A *wolf* is laid at point 0 (figure (a)). A *sheep* is going to occupy one of the remaining points (white circles) and stay there forever. The wolf takes a random walk on the circle. This means that it goes in each step with probability $\frac{1}{2}$ to one of the two neighbouring points. If this is its first visit at a point where there is the sheep, the wolf eats the poor sheep.



(a) Starting position



(b) Possible position after one step

For $n \geq 0$, let X_n be the position of the wolf after n moves.

(a) Explain briefly that $\{X_n, n \geq 0\}$ is a discrete time Markov chain. **[5 marks]**

(b) Give the transition matrix of the chain. **[5 marks]**

(c) Calculate $\mathbb{P}(X_3 = 1, X_1 = 3)$. **[5 marks]**

(d) Calculate $\mathbb{P}(X_3 = 1, X_2 \neq 1, X_1 \neq 1)$. **[5 marks]**

(e) Find the best point for the sheep to occupy, that is the point which has the largest probability that the wolf visits it last. If the best point does not exist, explain the reason.
Hint. Let p_k be the probability that the wolf visits point k last ($k = 1, 2, 3$). Find a system of equations for the p_ks. **[10 marks]**

[30 marks]

Solution.

(a) The wolf is making a random walk in the circle: at each time n, its position X_n is a function of X_{n-1}, but not of the previous ones $X_{n-2}, X_{n-3}, \ldots$. So (X_n) is a Markov Chain.

(b) States $T = \{0, 1, 2, 3\}$

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

(c) $X_0 = 0$. Note that odd number states $\{1, 3\}$ can only be reached at odd times $2k+1, (k \geq 0)$, and even number states $\{0, 2\}$ can only be reached at even times $2k, (k \geq 0)$
The event in $(X_3 = 1, X_1 = 3)$ with positive probability are $(X_3 = 1, X_2 = 0, X_1 = 3)$ and $(X_3 = 1, X_2 = 2, X_1 = 3)$, each with probability $\frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8}$.
So $\mathbb{P}(X_3 = 1, X_1 = 3) = \frac{1}{4}$

(d) "$X_2 \neq 1$" is always true, "$X_1 \neq 1$" means "$X_1 = 3$".
So $\mathbb{P}(X_3 = 1, X_2 \neq 1, X_1 \neq 1) = \mathbb{P}(X_3 = 1, X_1 = 3) = \frac{1}{4}$

(e) - By definition: $p_1 + p_2 + p_3 = 1$
- By symmetry of the problem: $p_1 = p_3$
- Consider p_1, by examine the first two moves X_1 and X_2,
we have $p_1 = \frac{1}{2} * 0 + \frac{1}{2} * (\frac{1}{2} + \frac{1}{2}p_1)$, so $p_1 = \frac{1}{3}$
Therefore, $p_1 = p_2 = p_3 = \frac{1}{3}$. No best point to pick for the poor sheep.

2. An insurance company has two insurance portfolios, namely A and B. Claims in portfolio A occur in accordance with a Poisson process with mean 3 per year. Claims in portfolio B occur in accordance with a Poisson process with mean 5 per year. The two processes are independent.

(a) Calculate the probability that the 3rd claim in portfolio A occurs before the 3rd claim in portfolio B in a certain year. **[5 marks]**

(b) Given that exactly eight claims had occurred in the first year, find the expected arrival time of the first claim. **[10 marks]**

[15 marks]

Solution.

(a) Let $S_{X,n}$ be the arrival time of the nth claim in portfolio X. Then,

$$\begin{aligned}\mathbb{P}(S_{A,3} < S_{B,3}) &= \sum_{k=3}^{3+3-1} \binom{3+3-1}{k} (\frac{3}{3+3})^k (\frac{5}{3+5})^{3+3-1-k} \\ &= \sum_{k=3}^{5} \binom{5}{k} (\frac{3}{8})^k (\frac{5}{8})^5 - k \\ &= 0.2752\end{aligned}$$

(b) The supposed claim process $\{N(t)\}_{t \geq 0}$ is a Poisson process with rate $\lambda = 8$ per year.
Given that N(1)=8, the arrival time are distributed as the order statistics of $U_1, \ldots, U_8 \overset{iid}{\sim} U(0, 1)$.
In particular, $U_{(1)} = \min U_1, \ldots, U_n$

$$\begin{aligned}\mathbb{P}(U_{(1)} > t) &= \mathbb{P}(U_1 > t, \ldots, U_8 > t) \\ &= \prod_{i=1}^{8} \mathbb{P}(U_i > 8) \\ &= (1-t)^8, t > 0\end{aligned}$$

Therefore, the conditional expected arrival time of S_1 is

$$\mathbb{E}(S_1 | N(1) = 8) = \mathbb{E}(U_{(1)}) = \int_0^1 (1-t)^8 dt = \frac{1}{9}$$

3. (a) Let $(B_t)_{t \geq 0}$ be a standard Brownian motion. Let $0 \leq s < t < r$.

(i) Calculate $\mathbb{E}(B_s B_t^2)$. **[2 marks]**

(ii) Calculate $\mathbb{E}(B_t | B_s)$. **[2 marks]**

(iii) Calculate $\mathbb{E}(B_s|B_t)$. **[3 marks]**

(iv) Determine the distribution of $B_s + B_t$. **[3 marks]**

(b) Suppose $(\widetilde{B}_t)_{t\geq 0}$ is another standard Brownian motion independent of $(B_t)_{t\geq 0}$. Let $\rho \in [0,1]$ be a constant and define for all $t \geq 0$,

$$X_t = \rho B_t + \sqrt{1-\rho^2}\widetilde{B}_t.$$

Is $(X_t)_{t\geq 0}$ a Brownian motion? **[10 marks]**

[20 marks]

Solution.

(a) (i) We have $B_t = B_s + \Delta_{st}$ where $\Delta_{st} = B_t - B_s \sim \mathbb{N}(0, s-t)$ is independent of B_s. So,

$$\mathbb{E}(B_S B_t^2) = \mathbb{E}\left\{B_S(B_s + \Delta_{st})^2\right\} = \mathbb{E}\{B_S^3 + 2B_s^2\Delta_{st} + B_s\Delta_{st}^2\} = 0 + 0 + 0 = 0.$$

[L,E marks]

(ii) $\mathbb{E}(B_t \mid B_s) = \mathbb{E}(B_s + \Delta_{st} \mid B_s) = B_s + \mathbb{E}(\Delta_{st}) = B_s$. **[L,E marks]**

(iii) We have $(B_s, B_t) \sim \mathbb{N}_2(0, \Sigma)$ with $\Sigma = \begin{pmatrix} s & s \\ s & t \end{pmatrix}$. The conditional distribution of B_s given $B_t = y$ is a normal with mean

$$\mathbb{E}(B_s) + \frac{\mathrm{Cov}(B_s, B_t)}{\mathrm{Var}(B_t)}\{y - \mathbb{E}(B_t)\} = 0 + \frac{s}{t}(y - 0) = \frac{s}{t}y.$$

Therefore, $\mathbb{E}(B_t \mid B_s) = \frac{s}{t}B_t$. **[L,E marks]**

(iv) By the joint distribution of (B_s, B_t) in (iii), $B_s + B_t$ is normal distributed with mean $\mathbb{E}(B_s + B_t) = 0 + 0 = 0$, and variance

$$\mathrm{Var}(B_s + B_t) = \mathrm{Var}(B_s) + \mathrm{Var}(B_t) + 2\mathrm{Cov}(B_s, B_t) = 3s + t.$$

[L,E marks]

(b) We verify the 3 defining properties of a BM.

(i) Initial value: $X_0 = \rho B_0 + \sqrt{1-\rho^2}\tilde{B}_0 = 0 + 0 = 0$.

(ii) Independence of increments: let $0 \leq s < t < s' < t'$. We have

$$X_t - X_s = \rho(B_t - B_s) + \sqrt{1-\rho^2}(\tilde{B}_t - \tilde{B}_s),$$
$$X_{t'} - X_{s'} = \rho(B_{t'} - B_{s'}) + \sqrt{1-\rho^2}(\tilde{B}_{t'} - \tilde{B}_{s'}).$$

The four increments $B_t - B_s$, $\tilde{B}_t - \tilde{B}_s$, $B_{t'} - B_{s'}$ and $\tilde{B}_{t'} - \tilde{B}_{s'}$ are all independent. Therefore $X_t - X_s$ and $X_{t'} - X_{s'}$ are independent.

(iii) Stationary Gaussian increments: let $0 \leq s < t$. Then from (ii) $X_t - X_s$ is a sum of two independent normals: so it is also normal with mean

$$\rho\,\mathbb{E}(B_t - B_s) + \sqrt{1-\rho^2}\,\mathbb{E}(\tilde{B}_t - \tilde{B}_s) = 0 + 0 = 0,$$

and variance

$$\mathrm{Var}\left\{\rho(B_t - B_s) + \sqrt{1-\rho^2}(\tilde{B}_t - \tilde{B}_s)\right\} = \rho^2(t-s) + (1-\rho^2)(t-s) = t - s.$$

Conclusion: $(X_t)_{t\geq 0}$ is a standard BM. **[L marks]**.

4. Let $\{X_t\}$ be a CTMC with state space $\{0, 1, 2\}$ and transition matrix

$$P(t) = \begin{pmatrix} \frac{3}{16} + \frac{3}{16}e^{-8t} + Ke^{-4t} & \frac{3}{8} - \frac{3}{8}e^{-8t} & \frac{7}{16} + \frac{3}{16}e^{-8t} - \frac{5}{8}e^{-4t} \\ \frac{3}{16} - \frac{5}{16}e^{-8t} + \frac{1}{8}e^{-4t} & h(t) & \frac{7}{16} - \frac{5}{16}e^{-8t} + Le^{-4t} \\ \frac{3}{16} + \frac{3}{16}e^{-8t} - \frac{3}{8}e^{-4t} & \frac{3}{8} - \frac{3}{8}e^{-8t} & \frac{7}{16} + \frac{3}{16}e^{-8t} + \frac{3}{8}e^{-4t} \end{pmatrix}$$

where K and L are constants and $h(t)$ is an unknown function.

(a) Determine the values of K and L. **[5 marks]**

(b) Determine the functional form of $h(t)$. **[5 marks]**

(c) Given $P(X_{1.5} = 1) = P(X_{1.5} = 2) = 1/2$, determine the probability that $X_{2.5} = 0$. **[5 marks]**

(d) Find the stationary distribution of this CTMC. **[5 marks]**

(e) Suppose $X_0 = 0$ and $X_2 = 2$. Given this, determine the probability that $X_1 = 1$. **[5 marks]**

(f) Assuming that the current state is $i \in \{0, 1, 2\}$, how long does it take on average for the system to transition to other states? In other words, if we denote $T_i = \min\{t : X_t \neq i | X_0 = i\}$, what's the expectation of $\mathbb{E}[T_i]$? **[10 marks]**

[35 marks]

Solution.

(a) Since $P_{00}(0) = 1$, we have $K = 5/8$. Since $P_{12}(0) = 0$, we have $L = -1/8$.

(b) Since the rows of $P(t)$ add to 1, $h(t) = 1 - P_{10}(t) - P_{12}(t) = \frac{3}{8} + \frac{5}{8}e^{-8t}$.

(c) $P(X_{2.5} = 0) = P_{10}(1)/2 + P_{20}(1)/2 = 3/16 - e^{-8}/16 - e^{-4}/8$.

(d) Letting $t \to \infty$, $(\pi_0, \pi_1, \pi_2) = (3/16, 3/8, 7/16)$.

(e)

$$\begin{aligned} &P(X_1 = 1 | X_0 = 0, X_2 = 2) \\ =&\frac{P(X_0 = 0, X_1 = 1, X_2 = 2)}{P(X_0 = 0, X_2 = 2)} \\ =&\frac{P_{01}(1)P_{12}(1)}{P_{02}(2)} \\ =&\frac{3}{8}\frac{(1 - e^{-8})(7 - 5e^{-8} - 2e^{-4})}{7 + 3e^{-16} - 10e^{-8}} \end{aligned}$$

(f) $G_{22} = P'_{22}(0) = -3$. Thus, $\mathbb{E}[T_3] = 1/3$. Similarly, $\mathbb{E}[T_1] = 1/4$ and $\mathbb{E}[T_2] = 1/5$.

Useful formulas for "Stochastic Processes"

Chapter 1

- $\mathbb{P}\left(\bigcup_1^n A_i\right) = \sum_{k=1}^n (-1)^{k+1} \sum_{\{i_1,\dots,i_k\}} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$
- The coupon collection problem:
  $\mathbb{E}(N) = \sum_{k=1}^n (-1)^{k+1} \sum_{\{i_1,\dots,i_k\}} \frac{1}{p_{i_1}+\cdots+p_{i_k}}$
- $X \geq 0$: $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > u)du$,
  $\mathbb{E}[X] = \sum_{k=0}^\infty \mathbb{P}(X > k)$.
- Order statistics $(X_{(1)}, \dots, X_{(n)})$ has joint density
  $n! f(x_1) \cdots f(x_n) 1_{\{x_1 < \cdots < x_n\}}$
- ith order statistic $X_{(i)}$ has density
  $i\binom{n}{i} f(x) F^{i-1}(x)\{1 - F(x)\}^{n-i}$.
- $\mathbb{E}(X) = \mathbb{E}\{\mathbb{E}(X|Y)\}$.
- $\mathrm{var}(X) = \mathbb{E}\{\mathrm{var}(X|Y)\} + \mathrm{var}\{\mathbb{E}(X|Y)\}$.
- Random sum $S_N = X_1 + \cdots + X_N$:
  $\mathbb{E}(S_N) = \mu\,\mathbb{E}(N)$, $\mathrm{var}(S_N) = \sigma^2 E(N) + \mu^2 \mathrm{var}(N)$.

Chapter 2

- Hitting time of i:
  $\{\tau_i = n\} = \{X_n = i, X_{n-1} \neq i, \dots, X_1 \neq i\}$.
- $f_{ii}^{(n)} = \mathbb{P}(\tau_i = n \mid X_0 = i)$,
  $f_i = \sum_{n=1}^\infty f_{ii}^{(n)} = \mathbb{P}(\tau_i < \infty \mid X_0 = i)$.
- $P_{ii}^n = \sum_{k=1}^n f_{ii}^{(k)} P_{ii}^{n-k}, \quad n \geq 1.$
- $f_{jk}^{(n)} = \mathbb{P}\{X_n = k, X_{n-1} \neq k, \dots, X_1 \neq k \mid X_0 = j\}$.
- $P_{jk}^n = \sum_{\ell=1}^n f_{jk}^{(\ell)} P_{kk}^{n-\ell}, \quad n \geq 1.$
- $U_{jk}(s) = \sum_{n=0}^\infty P_{jk}^n s^n, \quad F_{jk}(s) = \sum_{n=1}^\infty f_{jk}^{(n)} s^n.$
- $U_{jk}(s) - 1_{\{j=k\}} = F_{jk}(s) U_{kk}(s)$.
- Gambler's probabilities of reaching a fortune N, $0 \leq i \leq N$ and $r = q/p$:
  $$p_i = \begin{cases} (1-r^i)/(1-r^N), & \text{if } p \neq 1/2 \\ i/N, & \text{if } p = 1/2 \end{cases}$$
  Ruin probabilities: $q_i = 1 - p_i$.
- $$s_{ij} = \mathbb{E}\Big[\sum_{n=0}^\infty 1_{\{X_n=j\}} \Big| X_0 = i\Big] = \sum_{n=0}^\infty \mathbb{P}(X_n = j \mid X_0 = i) = \sum_{n=0}^\infty P_{ij}^n.$$
- $s_{ij} = 1_{\{i=j\}} + \sum_{k=1}^M P_{ik} s_{kj}$, or in matrix form,
  $S = I + P_T S$. Equivalently, $S = (I - P_T)^{-1}$.
- $f_{ij} = \sum_{n=1}^\infty f_{ij}^{(n)} = \dfrac{s_{ij} - 1_{\{i=j\}}}{s_{jj}}, \quad i, j \in T.$

Chapter 3

- $\min\{X_1, \dots, X_n\} \sim \mathcal{E}(\lambda_1 + \cdots + \lambda_n)$
- $\mathbb{P}\{X_1 < X_2\} = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$
- $\mathbb{E}\left[X_1 1_{\{X_1 < X_2\}}\right] = \dfrac{\lambda_1}{(\lambda_1 + \lambda_2)^2}$
- $\mathbb{P}\{X_1 < X_2 < X_3\} = \dfrac{\lambda_1 \lambda_2}{(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)}$
- $\mathbb{P}(R_n = i) = \dfrac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$
- $\mathbb{P}(S_n^1 < S_m^2) = \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} p^k (1-p)^{n+m-1-k}$

Chapter 4

- *Kolmogorov backward equation*: $P'(s) = GP(s)$, where G is the generator matrix with $G_{ij} = \lambda_{ij}$ for $i \neq j$, and $G_{ii} = -\sum_{j \neq i} G_{ij}$.
- *M/M/1* queue stationary distribution $\pi(i) = (1 - \lambda/\mu)(\lambda/\mu)^i$.
- Ergodic Theorem: $\mathbb{P}\Big\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t))dt = \sum_{i \in E} \pi_i f(i)\Big\} = 1$, where $\{\pi_i\}$ is the stationary distribution.
- Little's law: $L = \lambda W$

Chapter 5

- $f(\mathbf{x}) = (2\pi)^{-\frac{p}{2}} |\mathbf{\Sigma}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \mathbf{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right\}$
- For $\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{\Sigma})$
  $$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix},$$
  $\mathbf{X}_1 | \mathbf{X}_2 = \mathbf{x}_2 \sim \mathcal{N}(\boldsymbol{\mu}_{1|2}, \mathbf{\Sigma}_{11|2})$, where
  $$\boldsymbol{\mu}_{1|2} = \mathrm{E}(\mathbf{X}_1 | \mathbf{X}_2 = \mathbf{x}_2) = \boldsymbol{\mu}_1 + \mathbf{\Sigma}_{12} \mathbf{\Sigma}_{22}^{-1} (\mathbf{x}_2 - \boldsymbol{\mu}_2),$$
  $$\mathbf{\Sigma}_{11|2} = \mathrm{Var}(\mathbf{X}_1 | \mathbf{X}_2 = \mathbf{x}_2) = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12} \mathbf{\Sigma}_{22}^{-1} \mathbf{\Sigma}_{21}.$$
- For $0 = t_0 < t_1 < \cdots < t_n$,
  $$\mathbf{W} = \begin{pmatrix} B(t_1) \\ B(t_2) \\ \vdots \\ B(t_n) \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \cdots & t_n \end{pmatrix}\right]$$
- Let $0 < t_1 \leq t_2$.
  Forcasting: $B(t_2) \mid B(t_1) = x_1 \sim \mathcal{N}(x_1, t_2 - t_1)$.
  Backcasting: $B(t_1) \mid B(t_2) = x_2 \sim \mathcal{N}\left(\frac{t_1}{t_2} x_2, \frac{t_1}{t_2}(t_2 - t_1)\right)$.
- $T_a = \inf\{t : B(t) = a\}$
- Reflection principle: $B_t^* = \begin{cases} B_t, & \text{if } t \leq T_a, \\ 2a - B_t, & \text{if } t > T_a. \end{cases}$
- Running maximum: $M_t = \max_{0 \leq s \leq t} B(s), \quad t \geq 0.$
- For $0 < b \leq a$, $\mathbb{P}(M_t \geq a,\ B_t \leq b) = \mathbb{P}(B_t \geq 2a - b)$.
- M_t has the same distribution as $|B_t|$:
  $$\mathbb{P}(M_t \leq x) = 2\Phi\left(\frac{x}{\sqrt{t}}\right) - 1, \quad x > 0.$$