

Rational Number Teaching and Learning - Literature Review

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Section 1: Introduction

This literature review on rational numbers was part of the work undertaken in the 2017-2018 year of the fractions research project. Having focused primarily on fractions, in the continuous work that started in 2011 through the KNAER project (Phase 1), and led to the development of the Fractions Learning Pathways, the research team along with the Ministry were interested in better understanding what the research had to say about rational numbers in general, beyond fractions. What does the research suggest about effective ways of connecting fractions with other rational number systems to improve student understanding, for example? What are the differences and similarities between the findings related to fractions teaching and learning, and the research on students' learning of ratios and decimals?

In our review of the literature, we found that studies tend to focus one type of rational number (i.e., a study is likely to place focus on fractions *or* decimals, but less likely to present research on both). Key concepts and foci of each article were identified as a way to organize the literature found on rational numbers into major themes. Upon an extensive search using key terms (fraction, decimal, ratio, percent, rational number), over 10,000 peer-reviewed texts were found. For example, <rational number> in Proquest houses 4,792 peer reviewed texts. Combining <decimal> and <fraction> results in 555 peer-reviewed texts. 115 sources were used as the base of this literature review due to their quality, reliability, range and specificity related to rational number research that are in the arena of education and educational implications. The 81 texts referred to directly in the reference list provide a sound survey of the current literature.

Articles were grouped by their focus on fractions, decimals, mixed (fractions and decimals), etc., so the separation in the literature became more obvious. Despite a title including the term “rational numbers”, studies often tended to discuss only fractions or only decimals. In cases where a study looked at more than one rational number system, it was most common to look at fractions and decimals (to examine, for example, student understanding across these types, as opposed to, say, fractions and ratios). The research all tended to focus on positive rational numbers (it was noted that there was limited research available on student learning of negative rational numbers). The following table provides an overview of the literature on rational numbers presented in this review, to give a sense of the separation within the literature, and the large emphasis on fractions over other types of rational numbers.

Table 1. Overview of presently reviewed literature on rational numbers

Fractions	Decimals	Ratio	Mixed	Other*
61	7	3	21	23

*From the literature search on rational numbers, these articles focused on general math instruction, teaching challenges, etc.

Section 2: Defining rational numbers

A rational number is defined as any number that is or can be expressed as the quotient

of two integers, $\frac{a}{b}$, where $b \neq 0$. The relationship between the digits is a key consideration for interpreting a rational number. “As a logical mathematical construction, a rational number can be thought of as a number that derives its meaning from the pair of numbers used in its denotation” (Behr, Khoury, Harel, Post, & Lesh, 1997, p. 49). The relationship between a and b may also be described as an ordered pair

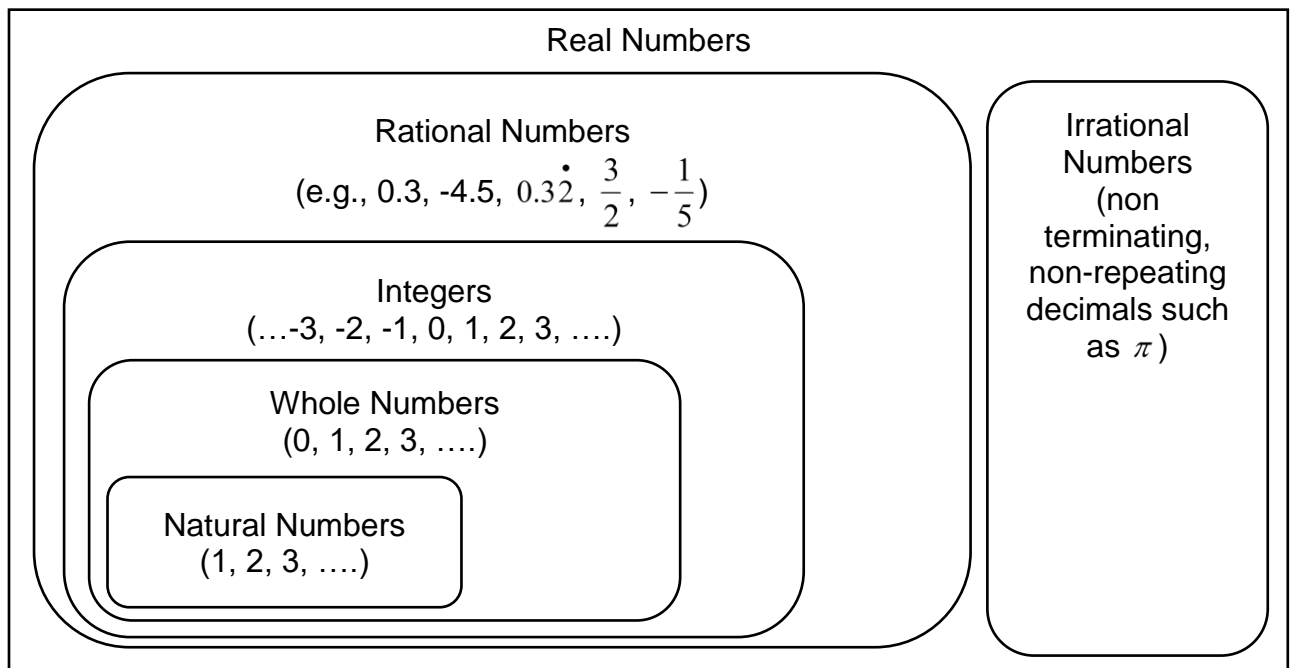


Figure 1. Number systems

(a , b) (Ni, 2000), or as a ratio, $a:b$ (Rosen (2005) as cited in Vamvakoussi & Vosniadou, 2010). Rational numbers include fractions and decimals as well as all integers, whole numbers and natural numbers (see Figure 1)

DeWolf, Bassok and Holyoak (2015) discuss the relational structure of fractions and

decimals. The notation $\frac{a}{b} = c$ “connects rational numbers to their magnitudes. Note that

fractions and decimals play distinct roles within this structure. Specifically, the $\frac{a}{b}$

component, which expresses a ratio between integers, has the form of a fraction; in contrast, the output of the implied division, c , expresses the magnitude of that relation, Importantly, though a fraction and its corresponding decimal convey (to some arbitrarily close approximation) the same magnitude, their distinct roles in the relational model give rise to meaningful conceptual distinctions” (p. 129).

Smith III (1995) states “due to the intimate relation between fractions and rational numbers, these terms are often used interchangeably, causing substantial confusion” (p. 6). Olive (1999) explained this in part by stating, “the term *rational numbers* has been used to refer to the formal mathematical definition (as elements of a quotient field) as well as the fractions of elementary school arithmetic” (p. 279). *For purposes of clarity, within this literature review, the term rational number refers to the number system rather than solely elementary school fractions.*

Ontario students see rational numbers across their academic trajectory, as depicted in Figure 2.

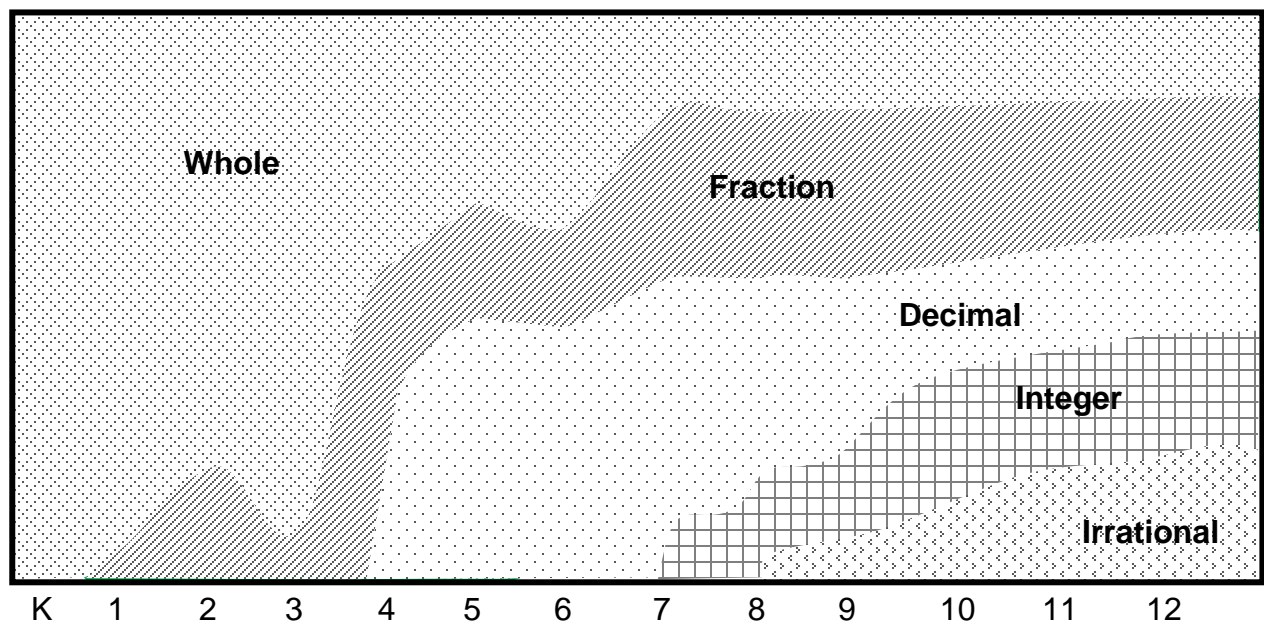


Figure 2. Number systems within Ontario mathematics curriculum

Rational numbers are the first number system that students encounter which requires an understanding of density of numbers (Brousseau, Brousseau, & Warfield, 2007, Vamvakoussi & Vosniadou, 2010). Within the integer/whole/natural numbers, there are a finite number of numbers between any two numbers. For example, between 4 and 6 there is only one number, 5. Moving to rational numbers, between 4 and 6 there are an infinite number of numbers. Vamvakoussi and Vosniadou (2010) describe this as “rational...numbers are densely ordered...[and] natural numbers are discrete” (p. 181).

Additionally, rational number notation differs from whole number notation. “Fraction notation for rational numbers refers to two integers, a and b , written with a bar between them $\frac{a}{b}$... Like fractions, decimals represent the relationship between integers;

however, decimal notation is based upon powers of 10 and is an extension of base-10 notation for integers” (Shaughnessy, 2011, p. 8).

Finally, the rules of operations with whole numbers do not hold true for rational numbers. For example, combining two quantities does not necessarily result in a larger quantity (such as $2.5 + (-3.4) = -0.9$). The commonly taught generalization of whole number multiplication and division that multiplication results in a larger quantity and division results in a smaller do not hold true (Vamvakoussi & Vosniadou, 2010).

Consider $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Pitkethly and Hunting (1996) discussed four mechanisms for rational number knowledge-building: 1. Whole number schemes; 2. Partitioning schemes, 3. Measuring schemes; and, 4. Equivalencing schemes and state, “growth in each of the mechanisms moves from context related action to more formal fraction knowing” (p. 8). For example, with fractions, students in kindergarten explore fair share contexts such as ‘three cakes shared fairly amongst twelve people’. In junior grades, these students learn that three divided by 12 can be written as $\frac{3}{12}$ and that each person would receive that much cake.

Brousseau, Brousseau, and Warfield (2007) stated that “rational numbers are usually presented in the form of fractions, but they are conceived in several different mathematical concepts: 1. Fractions as measurement... 2. Fractions as *linear mappings*... 3. Fractions as *ratios*” (p. 283). As with the constructs for fractions identified within *Foundations to Learning and Teaching Fractions: Addition and Subtraction*, Behr et al (1997) state, “the concept of rational number consists of a number of possible subconstructs – part whole, quotient, ratio, number, operator, and measure” (p. 48). Through a consideration of pre-service elementary teachers’ rational number problems solving, Behr et al. (1997) deduced that without a grasp of each construct, complete rational number understanding is not possible.

Section 3. How are rational number concepts developed?

3.1 The important role of magnitude and quantity in developing understanding of rational numbers

Understanding numbers, including rational numbers, as magnitudes is an important part of numerical development across number systems: “numerical development involves coming to understand that all real numbers have magnitudes that can be ordered and assigned specific locations on number lines”, including a mental number line (Siegler, Thompson & Schneider, 2011, p. 274). Put simply, “Precise representations of numerical magnitudes are foundational for learning mathematics” and are linked to “whole number and fraction arithmetic skill, memory for numbers, and other aspects of mathematical knowledge” (Fazio, Bailey, Thompson & Siegler, 2014, p. 54).

Fazio, Bailey, Thompson and Siegler (2014) examined the relationship between symbolic and non-symbolic magnitude representations and math achievement among 53 5th graders in the United States, and conducted a meta-analysis of 19 research studies. They examined both symbolic and non-symbolic magnitude understanding in whole numbers and fractions, using tasks on magnitude comparison and number line estimation. In the study, numerical magnitude understanding is defined as “the ability to comprehend, estimate, and compare the sizes of numbers (both symbolic and non-symbolic whole numbers and fractions)” (p. 54). The study found that “both symbolic and non-symbolic numerical magnitude understandings were uniquely related to mathematics achievement, but the relation was much stronger for symbolic numbers” (p. 53). In other words, children’s precision in representing symbolic whole and rational numbers was predictive of their overall math achievement. Having a sense of magnitude – the relative quantity of numbers – is a crucial foundation to all mathematics; “Findings suggest that interventions designed to improve symbolic magnitude representations might be useful for children” (p. 70).

In their study on problem solving involving rational numbers, Kent, Empson and Nielsen (2015) researched 5th grade students’ thinking about multiplication and division with fractions. They considered student strategies related to fractional quantities in one classroom. Students were asked the following: “You have $4\frac{2}{3}$ cups of hot chocolate powder. Each serving requires $\frac{2}{3}$ cup hot chocolate powder. How many servings can you make?” (p. 85). The researchers refer to this as a “multiple group problem” and discuss how these types of problems “can be used to promote the development of children’s understanding of fractional quantities and their relationships before the introduction of generalized procedures for multiplying and dividing fractions” (p. 85). The class of 5th grade students made use of different strategies, including: (1) direct modeling (for example, where $4\frac{2}{3}$ was represented in a drawing of circles and the 7 servings were counted directly on each of the circles); (2) using fraction relationships, such as repeated addition (where $\frac{2}{3}$ was added repeatedly to make $4\frac{2}{3}$); and, (3) by doing multiplication instead of division (e.g., by considering how many groups of $\frac{2}{3}$ are needed to make $4\frac{2}{3}$ cups). The researchers demonstrate how these types of problems

allow students to make meaning of complex operations such as multiplication and division without the need of formal algorithms, and, importantly, help to build children's understanding of fractions as quantities. They emphasize that, when "students do not see fractions as quantities, they have difficulty making sense of operations on quantities, such as adding or multiplying" (p. 89).

In a quantitative study of operations with fractions in grade 6 and 8, Siegler and Pyke (2012) again identified magnitude as a foundation for conceptual understanding of fractions. The researchers found that the gap between high and low-achieving students in 6th grade had grown substantially by 8th grade (their overall achievement as well as their knowledge of arithmetic with fractions). The researchers found that these differences were predicted by the students' knowledge of fraction magnitude and whole number division in Grade 6. These findings led them to conclude that "Placing greater instructional emphasis on the need to view each fraction as an integrated magnitude that expresses the relation between its numerator and its denominator might avoid subsequent difficulties not only in fraction arithmetic but in learning of mathematics more generally" (p. 2003).

Most recently, Hansen, Jordan, Fernandez, Siegler, Fuchs, Gersten, and Micklos (2015) studied the predictors of 6th grade fraction concepts and procedures ($n = 334$). They found that fractions concepts were strongly connected to "whole number line estimation, non-symbolic proportional reasoning, long division, working memory, and attentive behavior" and that fractions procedures were strongly connected to "whole number line estimation, multiplication fact fluency, division, and attention" (p. 34). Importantly, for both procedures and concepts, researchers found that the ability to locate whole numbers on the number line was the strongest contributing factor of success, and emphasized "the particular importance of numerical magnitude knowledge for acquiring this fraction knowledge" (Hansen et al., 2015, p. 47).

Ratio processing, which generally involves studying the brain's speed at processing ratios, is another area of recent research in the domain of the psychology of mathematics. Matthews, Lewis and Hubbard (2016) have studied systems that help to develop understanding of sophisticated mathematics concepts, such as fractions (Matthews, Lewis, & Hubbard, 2016). They describe what is known as a "ratio-processing system" (RPS). Ratio processing involves the comparison of ratios in visual forms, including line lengths comparisons and dot quantities comparisons where the dots vary in colour, size and quantity. Essentially participants performed tasks such as comparing a dot ratio of blue to yellow dots for example, to another set of dots of two colours – a second dot ratio of blue and yellow dots. A group of 183 undergraduate students participated in the study and completed a series of these non-symbolic dot ratio comparison tasks as well as symbolic fraction comparisons, number line estimations and algebra tasks. It was found that "non-symbolic-ratio magnitudes predict[ed] symbolic math competence. Participants with higher RPS acuity performed better on three measures of symbolic fraction knowledge and on a measure of algebra knowledge" (p. 198). In other words, magnitude perception was found to predict overall success in fractions as well as other areas of mathematics.

i. the integrated theory of numerical development

The integrated theory of numerical development grounded in the work of Bob Siegler and his colleagues proposes “a key developmental continuity across all types of real numbers” (Siegler et al, 2011, p. 274) and is often associated with rational number magnitude understanding in the literature. The theory posits that magnitude is the cornerstone of children’s numeric development, impacting both conceptual and procedural understanding. As Bailey, Zhou, Zhang, Cui, Fuchs, Jordan, and Siegler (2015) explain, “children who do not understand fraction magnitudes are viewed as being at a significant disadvantage for learning fraction procedures because they cannot estimate solutions to arithmetic problems and, therefore, are unlikely to detect solutions that are implausible or to reject flawed procedures that produced such answers” (p.69).

Siegler, Thompson and Schneider (2011) conducted a study involving 11- and 13- year olds, measuring fraction understanding and accuracy on a variety of tasks, including a number line estimation task, magnitude comparison, fraction arithmetic and general math achievement. Findings indicated that strategies useful for whole number magnitude proficiency (e.g., using the mental number line) were also useful for developing proficiency in fractions. The authors considered the “emphasis on acquisition of knowledge about numerical magnitudes... [to be] a basic process uniting the development of understanding of all real numbers” (p. 285). The similarities between strategies for whole number and fraction magnitudes provided support for an integrated theory of number.

Siegler and Lortie-Forgues (2014) subsequently found that “A basic tenet of the integrated theory is that numerical development is largely a process of broadening the range and types of numbers... Thus, accurate representations of numerical magnitude can be seen as the common core of numerical development” (p. 148). The authors identified four major areas in which magnitude understanding progresses along a developmental continuum: (1) through non-symbolic magnitude representation and understanding (which begin as early as 6 months); (2) through symbolic magnitude representation and understanding; (3) through whole number magnitude understanding; and, (4) through rational number magnitude understanding. Rational number magnitude understanding follows comprehension of magnitude for other number systems; all numbers can be seen as integrated at this stage.

To confirm that this integrated development of number understanding was not simply a North American phenomenon, Torbeyns, Schneider, Xin and Siegler (2014) compared 6th and 8th grade ($n = 187$) mathematics achievement in three countries with distinctly different educational practices (the United States, China and Belgium). Again this study found that, for students, “fractions magnitude understanding was positively related to their general mathematical achievement in all countries...and suggest that instructional interventions should target learners’ interpretation of fractions as magnitudes, e.g., by practicing translating fractions into positions on number lines” (p. 1).

3.2 Cognitive and neuroscience research on rational numbers

Another area of research on rational numbers provides a glimpse at a rare intersection of the educational sciences, cognitive science, neuroscience and psychology – the area of study that examines how, and where, numeric quantities (including rational number quantities) are processed or perceived in the brain. One aspect of this work includes the investigation of whether humans (and other primates) show an innate ability to process quantities (Matthews & Chesney, 2015, p. 30). In a study by Vallentin and Nieder (2008) for example, both humans and non-human primates processed absolute quantity (i.e., a specific count of objects or the length of a line) in the brain's fronto-parietal cortical networks (Vallentin & Nieder, 2008). Although not specifically considering fractions or decimals, Vallentin and Nieder (2008) found rhesus monkeys had the ability to reason proportionally, and could differentiate between different ratios, such as 1:4, 2:4, 3:4 and 4:4, suggesting that primates indeed have innate numerical processing abilities. Much of this research is focused on the “approximate number system” (ANS), which refers to the theory that we have an “evolutionary ancient cognitive system that allows us to approximate ratios” (Gabriel, Szucs & Content, 2013, p.1). As Gabriel, Szucs and Content (2013) explain, there is evidence in the research for the ANS – a system “designed for proportional understanding” – in monkeys, infants and young children (p.1). For instance, an intuitive sense of half and an intuitive ability to reason proportionately were both found in a study of Amazonian children: “Infants, children, adults, and many non-human animals are able to represent large numbers of objects, sounds and events in an imprecise fashion, using the Approximate Number System (ANS)... the ANS supports these computations without the use of symbols... its signature is the imprecise representations of number, in which the discrimination of two quantities is determined by their ratio and not absolute difference” (McCrink, Spelke, Dehaene, & Pica, 2012, p. 451). The ANS is also used in non-symbolic arithmetic calculations, including addition and subtraction. In studying fifteen 7-11 year old children growing up in the Amazon, the authors found that despite not having number words or formal education, the children were able to conceptualize non-symbolic magnitudes of half, and as such, concluded that the ANS supports intuitive abilities to reason about halving. An innate rational number processing ability is thus further established. This seemingly innate ability (prior to formal instruction) to reason with fractions is also discussed by Siegler and Lortie-Forgues (2014), who considered an innate ability to process magnitude, and more specifically focused on pre-verbal magnitude abilities. It appeared that even without the ability to use number words, young children have the capability of representing numerical magnitudes. Mix, Levine and Huttenlocher (1999) also argued that, prior to formal instruction, children as young as four have a sense of fractions, and “could calculate with fractional amounts less than or equal to one” (p. 172). Furthermore, in a consideration of proportional matching in 3-4 year olds, the ability to reason proportionately was present when children were shown whole pizzas and half pizzas, or whole boxes of chocolate and half boxes of chocolates. These young children demonstrated a developing sense of proportional equivalence, prior to formal instruction (Singer-Freeman & Goswami, 2001).

The intuitive sense of rational numbers was further considered by researchers who used equal sharing tasks to assess young children's understanding of fractional

quantities (Sophian, Garyantes, & Chang, 1997). Experiments with 5-7 year old children (sample sizes of roughly 20) required the children to consider the division of a pizza into different numbers of plates and asked whether or not one set would give more or less pizza per plate than another (i.e., if the number of plates reduces or increases the amount of pizza per person). A variety of results were found, including incorrectly associating the number of plates with the amount of pizza per plate (i.e., not seeing the inverse relation between the two; that the more plates you have, the less pizza on each plate). The incorrect reasoning was associated with equal partitioning for students of all ages in the study. When subtraction was used instead of equal partitioning, 5 years olds were successful. And when simpler equal partitioning occurred, 7 year olds outperformed their younger peers. "The evidence that children are able to make sense of fractional relations by at least 5 years of age, raise[s] the possibility that the origins of fraction knowledge may not derive entirely from counting. Children's understanding of partitioning appears to be another potentially important source of early fraction knowledge (p. 743). All in all, despite some apparent age-related developmental changes, the authors maintained the belief that reasoning about fractions does have an innate foundation.

Infant number sense and ability to work with approximate numerical magnitudes were explored in another study (McCrink & Wynn, 2007). A total of 44 6-month-old infants were involved in the study. Infants were shown both close ratios (e.g., 2:1, 3:1, 4:1) and distant ratios (e.g., 14:7; 38:19; 28:7). The ability of the infants to discriminate proportions was impressive and led the authors to make the following statement about number sense in infancy: "the present results, in tandem with other findings on numerical abilities in infancy, support the claim that the number sense is a core capacity that is likely to be similar from organism to organism despite variations in culture or directed experience" (p. 744). Combined, the studies presented in this subsection provide intriguing evidence for the innate or intuitive abilities in rational number processing.

Gabriel, Szucs and Content (2013) point out a puzzling aspect of these findings; if children are able to perceive and process ratios from a very early age, it seems "striking that fractions are one of the most difficult topics of early mathematics education to grasp" (p. 1), which is particularly "surprising as several theorists assume that the understanding of magnitude and ratios is evolutionarily hardwired" (p. 1).

Scientists are also interested in *where* this hardwiring might be occurring in the brain, and this might lead in the direction of answering the question, why are fractions so difficult if ratios are hardwired in the brain? Recent studies in the field of neuroscience have provided evidence that the intraparietal sulcus (located on the brain's parietal lobe) as the main area of the brain in which magnitude comparisons occur for whole numbers), however less is known about the location of magnitude representation for rational numbers. In a study of 16 participants from UCLA (mean age of 21 years), participants were presented with pairs of numbers (fractions, decimals or whole numbers) and were asked to compare their magnitudes (DeWolf, Chiang, Bassok, Holyoak, & Monti, 2016). Magnetic Resonance Imaging (MRI) was used to assess

neural activity while participants completed the task. Results indicated that, although the intraparietal sulcus was activated across all number types (i.e., rational and natural), brain activity differed depending on the numerical representation (i.e., base-10 versus fractions). In the proceedings of the 2016 meeting of the International Mind, Brain and Education Society, Gomez, Hubbard and Fazio (2016) referred to studies which argue that, despite the approximate number system and natural number processing mapping to the parietal cortex region of the brain, there is no such neural representation for fractions learning. This theory could help provide an explanation for the difficult nature of fractions; it could be that no neural pre-disposition exists. The recent work of Matthews and colleagues however, provides evidence that supports a neural pre-disposition for processing fractions. These conflicting results in the literature, as identified by Gomez, Hubbard and Fazio further point to the need for continued research in the area.

While fascinating, this field of research at this time may perhaps be of limited practical application for educators (or at least, requires more work to bridge the distance between this body of research and practical applications in the classroom). The new field of Mind, Brain and Education – which has the primary goal of bringing together research in education and the cognitive sciences to impact practice at the classroom level – may provide a fertile ground for future studies in this area (Matthews & Chesney, 2015; Matthews, Lewis & Hubbard, 2016).

3.3 Working memory

The role of working memory on rational number learning is also important to the learning of rational numbers. In research by Hansen et al. (2015) with 6th grade students, working memory was identified as one of the key predictors of fraction knowledge. Working memory “involves the ability to store and manipulate information in short-term memory” (p. 37). The study considered the role working memory plays in fraction knowledge by assessing students using a counting recall and working memory test, that had students recall the number of items on a card. Results indicated that “working memory explained unique variance in fractions concepts, but not fraction procedures” (p. 46). The authors recommend more work in the area to further assess the relationship between working memory and complex fractions understanding. The findings indicate, however, “that a constellation of cognitive processes independently support the development of fractions understanding” (p. 46), including working memory.

Fuchs (2013) also considered the role of working memory in fraction interventions. In the study, the author assessed “whether fluency practice differentially benefits students with weaker working memory” (p. 499-500). Working with 4th grade students, Fuchs studied the outcomes of two 12-week teaching interventions. One intervention focused on building fluency in fraction knowledge (n=94), while the other focused more on building conceptual understanding (n=91). There was also a control condition (n=92). Results related to working memory (WM) indicated that there was “an aptitude-treatment interaction, in which WM moderated the effects of the fluency versus conceptual condition on the number line measure. Supplementary conceptual work was superior to the fluency condition for students with very weak WM, but supplementary

fluency work promoted better learning for students with more adequate WM” (p. 510). In other words, differences in rational number fluency were found to be related to working memory.

Section 4. Challenges in the Teaching and Learning of Rational Numbers

Three major challenges emerge in the research as being particularly strong barriers that impede learning in rational number instruction. The first is a challenge related to the structures of rational number systems themselves. The second relates to challenges with understanding magnitude, which we have seen is an important foundation to understanding rational numbers. The third relates to the “whole number bias” that children bring into their learning of rational numbers. Finally, we find challenges that result from the historic and ongoing instructional tendency to emphasize rules and procedures over conceptual understanding in the teaching of rational numbers.

4.1 Challenges presented by the structures of rational number systems

The systems that form the foundation of fractions and decimals can cause confusion for students. This confusion stems particularly from fundamental structural differences; for example, with decimals, the number of partitions is restricted by place value (factors of 10), whereas for fractions the number of partitions is essentially unlimited. Because they are different systems, including distinct conventions for notation, converting fractions to decimals and vice versa involves understanding these structures. For example, Moskal and Magone (2000) discuss an early misconception sometimes seen in student thinking – the misconception that the fractions bar and the decimal point mean the same thing (e.g., that $\frac{3}{4}$ could be expressed as 3.4 and or that .5 could be expressed as $\frac{1}{5}$). This example highlights the complication of translating symbolic notation from one system to another (Moskal & Magone 2000).

DeWolf, Bassok and Holyoak (2015) explored how perception of numbers is impacted by the fundamental differences between fractions and decimals, including what they describe as the “2D”, “bipartite” (i.e., made up of two separate numbers) symbolic notation of fractions vs. the “1D” nature of a decimal (notation where the digits are side by side on a common plane). These perceptual differences may result in fractions or decimals being easier to process depending on the numerical context. In this study, researchers report that, for participants, fractions tended to implicitly lend themselves well to representing the “relations between two distinct [countable] sets” (p. 129) – or discrete quantities, whereas decimals more easily conveyed a “portion of a continuous unit.” – continuous quantities. In addition, the researchers argue that fractions “allow more accurate reasoning about bipartite relational structures, such as ratios” (p. 130). This notion of discrete versus continuous quantity is well illustrated in Figure 3.

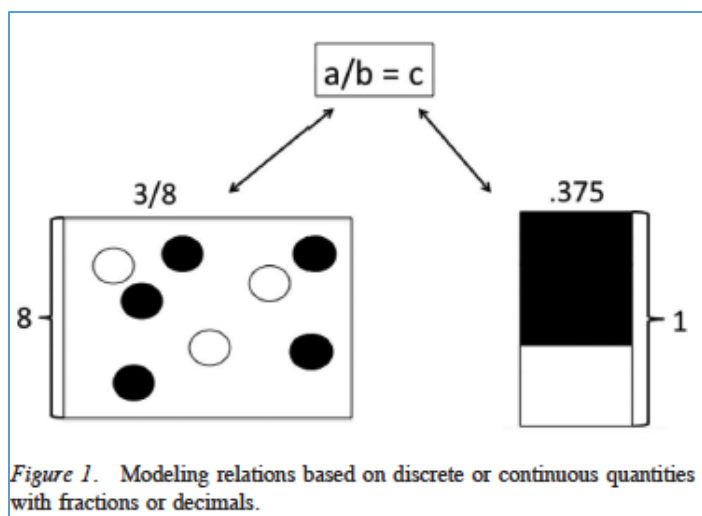


Figure 3. Modeling relations based on discrete or continuous quantities with fractions or decimals (DeWolf, Bassok & Holyoak, 2015, p. 129)

The importance of context and attending to discrete or continuous representations was born out in a series of experiments (with varying sample sizes, ranging from 48 to 75 participants) that focused on undergraduate student understanding of fractions and decimals. The researchers considered the contexts in which participants preferred (and chose to use) one type of rational number over another. Participants were shown a series of pictures, such as discrete set models (i.e., images of countable, discrete objects) or continuous area models (such as a rectangular bar divided into two colours, either partitioned with lines or left un-partitioned, as in the figure above), and were asked to select which notation (either a fraction or decimal) best represented the relation depicted in the picture. Participants selected fractions to represent the set models and decimals to represent the area models. The structural differences between the two types of rational numbers were found to be ultimately “linked to differences in the mathematical procedures they afford” (DeWolf, Bassok & Holyoak, 2015, p. 130). Overall, the study found that fractions were preferred to decimals when quantities consisted of discrete, countable sets, whereas, decimals more easily represented continuous masses that depicted relative magnitudes and proportions. (Similar conclusions were also found in Rapp, Bassok, DeWolf and Holyoak, 2015.)

Research has suggested that fractions may indeed be more difficult than decimals for students to process. DeWolf and colleagues (DeWolf, Grounds, Bassok, & Holyoak, 2014) also considered American undergraduate student understanding of rational numbers over three experiments ($n = 26$ to 95 students, depending on the experiment). Participants were asked to compare magnitudes of groups of numbers (e.g., $22/37$, 0.595 and 595) while their response times were recorded. In general, it was found that subjects generally thought about decimals in the same way that they thought about integers, while fractions were processed in a unique way. *More specifically, processing was slower for fractions than for decimals or integers.* The researchers linked this difference in processing speed to differences in the formal notational structures of fractions (i.e., the bipartite structure, with a numerator and denominator), which takes longer to process, and decimals (i.e., the base-10 structure), which are similar to

integers and whole numbers therefore allowing for quicker processing due to greater familiarity. The authors attributed the more challenging nature of fractions (as compared to decimals) to these fundamental differences, and stated that despite the earlier focus on (and therefore longer exposure to) fractions in school contexts, there is a “greater ease of comparing decimals than fractions” (p. 73) that persists into adulthood.

4.2 Challenges related to rational number quantity and magnitude

In a presentation to the International Mind, Brain and Education Society (IMBES) conference Gomez, Hubbard and Fazio (2016) stated that “a problem often observed in research and practice is the lack of understanding that fractions have an associated magnitude that depends not on the absolute magnitudes of their components (numerator and denominator), but on their relative magnitudes” and this is challenging for many students, who “lack basic intuitions about fraction magnitude”(p. 1). The relationship between the numerator and the denominator define the quantity of the number (the fraction), and this quantity can only be understood as the relationship.

The importance of understanding quantity of rational numbers was highlighted in research conducted by Behr and colleagues, who found that students had difficulty with fraction comparisons (determining relative quantities to determine greater than/less than comparisons) (Behr, Wachsmuth, Post, & Lesh, 1984). This seminal study consisted of an 18-week teaching intervention, where 12 fourth grade students in Illinois were provided instruction that focused on fraction naming, equivalence, comparison, addition and multiplication. Following the intervention, students were interviewed. The researchers found that rational number quantity understanding was underdeveloped: “The results suggest that... by the middle of fourth grade, or during initial instruction in fractions, children have not developed a quantitative notion of fractions that is strong enough to deal with questions of their order—even in the case of unit fractions with small denominators” (p. 333). The study, therefore, calls for development of quantity understanding in rational numbers, a notably challenging area for students.

Indeed, challenges regarding magnitude may be a factor that increases student difficulty with fractions over decimals. Hurst and Cordes (2016) investigated whether rational number magnitudes (fractions, decimals, and whole numbers) lie “along the same ordered mental continuum” (p. 281). After 62 college students in Boston compared number magnitudes under two conditions (within notations, e.g., fractions to fractions, or between notations, e.g., fractions to decimals), it was found that “fraction magnitudes are much more difficult than either decimal and whole-number magnitudes” (p. 291). Interestingly, the study also included eye-tracking data, which examined the length of time that participants spent looking at the numerators and denominators of fractions, and found that participants spent significantly longer amounts of time looking at the numerators compared to the denominators. Researchers found a relationship between participants who focused longer on the numerator, and lower scores on their assessment on fractions procedures. The researchers noted that “Given the inverse relationship between the value of the denominator and the value of the fraction, optimal responding should have involved either longer or at least comparable looking at the denominator relative to the numerator” and concluded that this was further confirming

evidence that “both adults and children often treat fraction components like separate whole numbers” rather than a quantity determined by the relationship between the numerator and the denominator (p. 291). Related findings from a 2010 study with 10- and 12-year-olds examining their ability to compare magnitudes of fractions with common components (same numerator or same denominator) found that “response times were slower for fractions with common numerators than for fractions with common denominators, indicating an interference of the magnitude of the denominators with the selection of the larger fraction”; in other words, students are cognitively challenged by the relational nature of the numerator and denominator in establishing fractional magnitudes (p. 244).

4.3 Challenges resulting from emphasis on procedures over conceptual understanding

In the research examining the roles of procedural and conceptual understanding in developing student understanding of rational number (particularly in fractions, where much of the research is concentrated), there is consensus that: 1) there has been an instructional bias in North America towards procedures over concepts; and, 2) this overemphasis on procedures and algorithms may actually impede the development of student understanding of foundational rational number concepts, in both the short and long term.

In general, procedural knowledge is described as knowledge that: focuses on algorithms (Bailey, et al., 2015); involves the ability to apply a sequence of steps to a problem (Gabriel, et al., 2013); and does not necessarily require “understanding of what the elements implemented in the procedure mean” (Hallett, Nunes, & Bryant, 2010, p. 396). Conceptual knowledge, on the other hand, is described as knowledge that: focuses on the properties of numbers (as opposed to just algorithms) (Bailey, et al., 2015); involves generalized understanding of the principles behind rational numbers (Gabriel, et al., 2013); and, according to Byrnes as cited in Hallett, Nunes, and Bryant (2010), involves an understanding of “‘relational representations,’ which consist of two or more represented entities that are mentally linked through a relation of some sort” (p. 396). In terms of fractions specifically, Bailey, et al. (2015) explain that “procedural knowledge of fractions consists of fluency with the four fraction arithmetic operations: addition, subtraction, multiplication, and division. Conceptual knowledge in this area involves understanding the properties of fractions, including their magnitudes (e.g., $\frac{4}{5}$ is greater than $\frac{1}{2}$), principles relevant to fractions (e.g., an infinite number of fractions can be placed between any two other fractions) and notations for expressing fractions (e.g., $\frac{3}{4} = \frac{6}{8} = .75$)” (p. 69). These distinctions can be generalized and applied to all rational numbers.

Research has shown that a solid understanding of rational numbers requires both procedural and conceptual understanding (Siegler & Pyke, 2013); however, much of the research has indicated a common instructional bias towards “execution of mathematical procedures... [over an] understanding of quantitative relations” (DeWolf, Bassok, & Holyoak, 2015, p. 127) and a tendency to emphasize the memorization of algorithms

without working to develop a solid understanding of how or why the procedure works (Yetim & Alkan, 2013).

A substantial body of research over the past 35 years points to the problems inherent in the overemphasis on procedures and algorithms compared to conceptual understanding, which leads to student use (or mis-use) of algorithms absent of meaning. In their study of a classroom-based intervention focused on using number lines to teach fractions and integers, Saxe, Diakow and Gearhart (2013) observed that “fractions require students to conceptualize multiplicative relations between quantities, but students often treat the numerals in representations of fractions as whole numbers, and memorize algorithms for fractions operations without deep understanding of numerator/denominator relationships” (Saxe, Diakow, & Gearhart, 2013, p. 344).

Brown and Quinn (2006) conducted an error analysis on assessments of 143 students in high school elementary algebra classes, in which they set out to better understand the content challenges that algebra students have with fractions, in order to “span the gap between arithmetic and algebra” (p. 28). In their words, the “building materials [for solid algebra understanding] are conceptual understanding and the ability to perform arithmetic manipulation on whole numbers, decimal fractions, and common fractions”; in other words, fractions understanding is foundational to algebra. (p. 28) Brown and Quinn point to seminal research by Kieren (1980) and colleagues who explain the confounding effect of procedural emphasis: “Too often simply an algorithm has been taught, abandoning the student deep in the rational number construct. This provides no connection for understanding, and leaves the student clinging to a prescribed step-by-step set of instructions” (p. 28). The researchers in this study analyzed errors on a 25-item assessment, designed to focus on specific fractions and algebra-related skills and understanding, such as the use of algorithms when computing with fractional quantities, applications of basic fractions concepts in word problems, and computational fluency (i.e., considering whether or not students “have control over fraction concepts and algorithms that would allow them to demonstrate fluent computation in unfamiliar contexts”) (p. 36). The analysis revealed large error percentages in all categories of questions, and noted the tendency for students to apply algorithms through strict memorization and often illogically, sometimes in cases when no procedure was necessary to solve the problem.

Further, when algorithms were used, students were unsure of the correct methods and often made mistakes. The researchers were direct in their identification of the problem, worth citing here at length:

Regrettably, many students are taught algorithms before they have had the time to develop the fundamental concept. Their only alternative when confronted with fraction operations is to match what is being presented with one of the disconnected, previously-memorised algorithms from earlier mathematics experiences. If the situation being presented is novel or is not in a recognisable form, then a student's best effort is no more than a good guess. The errors that were made repeatedly demonstrate that a good guess is not sufficient. The results of the error analysis reveal an overall lack of experience with basic

fraction concepts — experience that should have been gained through an informal treatment of fractions providing an abundance of concrete referents. Only a few students used pictorial representations to help them answer some of the questions. A few more students were able to apply the concept directly and provide the correct answer without resorting to an algorithm. For most of the students, however, the strategy of choice was to select an algorithm and then use it. This approach yielded a host of illogical answers, which more often than not went undetected by the student. Such inconsistent results demonstrate a lack of fluency with fraction computation, the fluency that becomes a necessity when students begin to work with algebraic fractions. (p. 38-39)

The results of Hasemann's 1981 study in this area remain relevant (Hasemann, 1981). The study involved 12-15-year-old students who struggled with fractions. Participants were presented with a test that included word problems, diagrams and computational questions. Overwhelmingly, the students relied on memorized rules when solving the problems, without fully understanding the rules or why the rules were applicable to certain questions. As Hasemann pointed out, these instances can cause the false impression that students understand the procedure and the concept behind it, because they may still arrive at the correct answer (especially in computational questions) in spite of lacking or limited understanding. These findings lead Hasemann to reflect that "the concept of 'understanding' must [therefore]...be considered in more detail" (Hasemann, 1981, p. 81). Students using the algorithm for multiplication of two fractions (involving multiplying the numerator by the numerator, and the denominator by the denominator) is a good example of this phenomenon: "here the pupils do not need to know why the rule is so and not otherwise, nor do they need to have understood the concept 'fraction' property. Therefore one can say that most pupils have understood the rule for multiplication instrumentally" (p. 81).

A more recent study affirms the importance of providing students with opportunities to build conceptual knowledge (Gabriel, et al., 2013). The authors proposed that a limited conceptual knowledge contributes to children's struggles with rational numbers. They tested 439 grade 4-6 students in Belgium on a number of conceptual fractions concepts (i.e., understanding of part-whole partitioning and proportion) and procedural fractions concepts (i.e., operations like addition, subtraction and multiplication, and simplification). Results indicated that, "teaching practice seems to focus more on procedures than on conceptual understanding of fractions. But our results showed that procedures are not sufficient to carry out operations with fractions for instance. Even if pupils are intensively trained with finding the least common denominators procedure, the percentage of correct responses for addition and subtraction with different denominators remained low. Conceptual understanding is essential to ensure a deep understanding of fractions." (p. 9)

Studies have indicated that when presented with rational number problems, students who are capable of thinking beyond basic algorithms, rules and general strategies outperform their peers who do not (both in terms of speed and accuracy). Over twenty years ago, Smith III (1995) conducted in-depth task-based interviews with students

across school levels (10 from each of elementary, middle and secondary school), administering a series of tasks related to fractional order, comparison, equivalence, and density. In his study, Smith III was interested in competence with rational numbers, which he defined as follows: "Competence with rational numbers is indicated by capable performance across a range of task settings that involve their core mathematical properties. More specifically, it requires the ability to solve both novel and familiar problems and to handle all the fractions and decimals that might appear in those contexts. It also demands substantial insight into the underlying mathematical structure of the domain" (p. 4). It was of interest to Smith III whether or not students resorted to traditional, procedural methods (such as finding the common denominator) or whether students applied more general strategies that involved less computation and more reasoning, including invented strategies. It was his goal to move beyond performance evaluation to examine the strategies of the most competent students. Smith III developed a framework for analyzing the knowledge used by the students who performed most successfully on the assessment tasks, intended to provide "an analytical basis for characterizing the knowledge that underlies competent reasoning with fractions" (p. 16). Four main strategies were identified: (1) The "Part Perspective" involves equi-partitioning or using a "mental model of divided quantities" (p. 16), and was used in comparison questions and "supported more efficient, analytic strategies" (p. 17); (2) The "Components Perspective" considered the relationships between the numerator and the denominator or the parts and the whole; (3) The "Reference Point Perspective" involved the location of fractions in relation to benchmark (well-known) fractions on a mental number line for comparison and addend tasks; (4) Finally, the "Transform Perspective" involved the use of conversion of quantities to friendlier numbers (e.g., transformations could involve using equivalent fractions, such as fractions with common numerators or common denominators, or turning fractions into decimals), and again was used in comparison and addend problems. Smith III also conducted textbook analyses to determine whether these strategies were taught and found no evidence for support in the textbooks, suggesting that students had somehow borrowed these strategies from other number knowledge they had acquired through prior experiences, in order to cope with the complexities of the mathematics they were being asked to do. The study indicated that, "competent reasoning depends instead on a much more diverse and complicated knowledge base than textbook content suggests. ... Competent reasoning with rational numbers requires both general strategies and strategies that apply only in numerically specific situations" (p. 38). In other words, students with the ability to think flexibly and reason with a range of strategies could think conceptually about rational numbers with a high degree of competency. The researcher's view is that we have underestimated the "larger role for student-generated knowledge" (p. 39) and points out that this is one "of the problems in teaching rational numbers using the traditional textbook-based curriculum that so strongly emphasizes numerical transformations" (p. 41). He writes that the danger in this traditional approach lies in its

"failure to recognize explicitly students' specific and constructed strategies as legitimate mathematical knowledge. Insightful student strategies are often criticized and rejected only because they fail to match the textbook methods. Over time, students learn that mathematics is about remembering general

numerical procedures and has little to do with their own creative thinking and understanding.” (p. 41)

The study highlights the crucial importance of providing students with opportunities to construct conceptual meaning:

“If more students are to discover and understand the nature of rational numbers and the power of their own sense-making abilities, classroom instruction must do much more to identify, discuss, evaluate, and, where necessary, validate or refine student-generated strategies. Most centrally, the criteria for what counts as knowledge must change, from a literal match to textbook procedures to consistency with the mathematical properties of rational numbers.” (p. 41)

However, Smith III notes the importance of increased teacher content knowledge of rational numbers to enact such a pedagogy effectively.

4.4 Natural number bias

“The natural number bias is described as the (inappropriate) application of natural number features in rational number tasks” (Van Hoof, Verschaffel, & Van Dooren, 2015, p. 40). The natural number bias, also referred to as the whole number bias or the natural number interference, plays into difficulties in understanding rational numbers (Behr, Wachsmuth, Post, & Lesh, 1984; DeWolf, Grounds, Bassok, & Holyoak, 2014; Iuculano & Butterworth, 2011; Kainulainen, McMullen, & Lehtinen, 2017; Moskal & Magone, 2000; Pitkethly & Hunting, 1996; Post, Wachsmuth, Lesh, & Behr, 1985; Seethaler, Fuchs, Star, & Bryant, 2011; Sophian, Garyantes, & Chang, 1997; Van Hoof, Verschaffel, & Van Dooren, 2016). When learning about rational numbers, whole number schemas often overpower developing rational number schemas leading to misconceptions and an overgeneralization of rules. For example, Ni and Zhou (2005) define the whole number bias as the “tendency in children to use the single-unit counting scheme applied to whole numbers to interpret instructional data on fractions” (p. 27). This bias leads to errors in rational number thinking “because of [the] characteristic differences [of rational numbers] from whole numbers, both conceptually and in format” (DeWolf, Grounds, Bassok, & Holyoak, 2014, p. 81).

What are the causes of the whole number bias? The issue is a complicated one. In their very thorough review of the literature (including developmental, neuropsychological and teaching experiment studies), Ni and Zhou (2005) explain that “issues of whole number bias reflect not merely a matter of interference between prior and new knowledge in children’s construction of fraction concepts, but a constellation of more general questions with regard to the origin and development of numerical cognition” (p. 27). The authors examine three competing hypotheses regarding the origins of the whole number bias: the innate constraint hypothesis, the undifferentiated amount hypothesis, and the learning account. The *innate constraint hypothesis* proposes an innate nature of the whole number bias: “the discrete nature of the mental magnitude representation ... is assumed to impede children’s acquisition of fractional numbers that are very different from whole numbers” (p. 31). In other words, rational numbers (which are continuous) do not easily fit into well-developed knowledge of whole numbers (which are discrete) and a preference for the latter can persist into adulthood. The *undifferentiated amount hypothesis* proposes “that there is no innate bias privileging discrete quantity and that

the [whole number] bias has resulted from a developmental process of differentiation between discrete and continuous quantification” (p. 30). Finally, the *learning account* presumes that whole number bias results from “instruction that makes inappropriate use of children’s prior whole number knowledge and fails to help children differentiate fraction and rational numbers from whole numbers” (p. 30). The authors determined that there is insufficient evidence to determine which of the hypotheses in fact accounts for the whole number bias, and concluded that there is evidence for effects from both nature (innate views of number) and nurture (based on experience and learning). They also point out – importantly – that the bias is both cognitively efficient (when applied in whole number situations) and inflexible (when overgeneralized to rational number situations), and that it is “a challenge to manage the trade-off in instruction. ... Children use their knowledge about whole numbers to make sense about fraction numbers; and instruction on fractions makes use of the prior knowledge in children. The practice follows a common wisdom about learning and instruction, that is, to use what is already known to figure out what to be learned. There is nothing wrong with the wisdom because this is how our constructive minds work. The problem lies in that the instruction does not take measures accordingly to reduce the possible risk of reinforcing the whole number bias by taking advantage of the prior knowledge” (p. 40).

In contrast, other researchers place the responsibility for whole number bias squarely on instructional emphases on whole numbers in education. As Varma and Karl (2013) investigated decimal proportions (numbers between 0 and 1) in undergraduate and graduate students ($n=54$) and confirmed the importance of the decimal system for “using a small set of objects to express a large set of numbers” (p. 299). Importantly, the authors noted the “interference” of whole number knowledge when processing decimals (e.g., when incorrectly applying knowledge of the whole number 29 to proportionally reason about 0.29). This emphasis on whole numbers also makes it probable that prior to learning about rational numbers, “students have formed a rather coherent domain-specific, naïve theory of number” (Vamvakoussi, Christou, Mertens, & Van Dooren, 2011, p. 677). In fact, Vamvakoussi and Vosniadou (2010) viewed the whole number bias as one of the biggest difficulties to overcome, arguing that development past the bias is difficult.

Fundamental differences between rational numbers and whole numbers themselves present contributing factors. In their research on 5-to-7-year-old childrens’ understanding of fractional quantities, Sophian, Garyantes and Chang (1997) listed two main differences between fractions and whole numbers that cause difficulties: “First, fractions are not consecutively arranged, as are counting numbers. Between any two fractions there are always infinitely many others. ... And second, large numbers do not always imply large quantities in a fractional context as they do in counting” (p. 743). Additionally, differences between the structures of rational and whole numbers limit the accuracy of applying whole number constructs to rational numbers. For example, the notation of fractions is in the form of numerator and denominator, representing a single quantity, but students often consider them as two separate numbers (Gabriel, et al., 2013). Gabriel et al. based this observation on their work with 439 Grade 4-6 students and their conceptual and procedural understanding of fractions.

A specific example of how whole number bias can lead to faulty reasoning about rational numbers can be found in a study of Grade 6 students in Australia ($n=323$), who were asked to identify the larger fraction in pairs of numbers (Clarke & Roche, 2009). Students' fraction comparison strategies were analyzed and a couple of misapplied whole number related strategies were identified: gap thinking, "where the student is not considering the size of the denominator and therefore the size of the relevant parts (or the ratio of numerator to denominator), but merely the absolute difference between numerator and denominator" (p. 129) and the general assumption that, as in whole numbers, the larger the numbers, the larger the fraction.

Fractions are more complex numbers than whole numbers as they involve consideration of both the ratio and the division of two whole numbers (Torbeyns, et al., 2014). While whole numbers have one representation for every number, "rational numbers have different representations of the same number (e.g., 0.5 and $\frac{1}{2}$)" (Van Hoof, et al., 2013, p. 155). It is also possible to confuse rules for whole number symbols when comparing decimals, for example, it is common to think of 0.40 as ten times more than 0.4, to perceive 0.245 as having a bigger value than 0.45 (Moskal & Magone, 2000), or to consider the numerator and denominator of a fraction as two unrelated whole numbers (Vamvakoussi & Vosniadou, 2010) because of whole number bias.

The rules and patterns for operations also differ between rational numbers and whole numbers. The integrated theory of numerical development posits that there is continuity in development across types of number. Within the theory, it becomes apparent that the "invariant" rules learned for properties of whole numbers are dispelled in development of rational number understanding: "several invariant properties of whole numbers are not invariant properties of all numbers" (Siegler & Lortie-Forgues, 2014, pp. 147-148). For example, the whole number bias may lead to the assumption that multiplication results in larger products and division makes quantities smaller for rational numbers since it is a steadfast whole number rule. Of course this does not hold true for rational numbers; with rational numbers, multiplication tends to make quantities smaller and division tends to result in larger quantities (it is dependent upon the values involved, since multiplication by a number greater than 1 will increase the quantity but not multiplication by a number less than 1) (see Van Hoof, et al., 2013 and Siegler & Lortie-Forgues, 2014). Learning that the "salient and invariant properties of whole numbers are not true of all numbers" (Siegler, Duncan, Davis-Kean, Duckworth, Claessens, Engel, & Chen, 2012, p. 692), may occur for the first time when operations with rational numbers are introduced. To counteract the whole number bias, it is very important to be "aware that rational numbers behave differently than natural numbers (e.g., with respect to ordering and operations)" (Vamvakoussi, Christou, Mertens, & Van Dooren, 2011, p. 677).

In the discussion of the whole number bias, the concept of density of rational numbers provides an illustrative example of how the properties of rational numbers make them particularly challenging, because they disrupt student notions of number properties learned in the context of whole numbers. "Density" of rational numbers refers to the fact that an infinite number of rational numbers can be found between any two rational

numbers (Gabriel, et al., 2013). This is not the case with whole numbers, which are discrete (there is no other whole number between 2 and 3, for example). The continuous – and “densely ordered” – nature of rational numbers (Vamvakoussi & Vosniadou, 2010) is a novel and incredibly challenging number property for students to comprehend; research shows this understanding to be either fragile or non-existent. For example, in a study of 7th, 9th and 11th grade students in Greece, Vamvakoussi and Vosniadou (2010) asked 549 students how many, and what type, of numbers lie between two rational numbers. Students tended to view rational numbers as discrete (like whole numbers) and also tended to respond that, where there could be quantities between two rational numbers, these needed to be of a like type (i.e., a decimal couldn’t be placed between two fractions and vice versa). The researchers attributed the problems with density not only to misconceptions around the discreteness of rational numbers, (which they noted was in line with prior research), but also to problems with notation, and argued that both of these issues were “related to a more general problem of conceptual change in the transition from natural to rational numbers” (p. 3). The researchers argued that “these problems are interconnected and reveal deep conceptual difficulties with the rational number concept, rather than occasional intrusions of prior knowledge or mere confusion with symbols” (p. 30). They go on to write that their framework theory approach of conceptual change begins with the assumptions that prior to exposure to, and instruction in, rational numbers, students have “consolidated a complex system of interrelated beliefs and background assumptions about what counts as a number and how it behaves, namely an initial framework theory of number within which number is conceptualized as *counting number*” (p. 31). Therefore, integrating new understandings about properties of rational numbers such as density requires “fundamental ontological and epistemological changes to take place in students’ conceptual organization of number, i.e., requires conceptual change” (p. 31). Interestingly, while age and instruction did have an effect, with 7th graders showing the highest frequency of what the researchers called “naïve” responses, “no significant performance differences were found between ninth and eleventh graders, which suggests that further instruction about rational numbers did not address certain aspects of students’ rational numbers thinking, e.g., the idea of discreteness of numbers” (p.38). In other words, there appears to be evidence for a need for continued experience and instruction into the secondary school years to build full understanding of rational numbers. The major concern of the researchers was that, in spite of years of practice converting fractions and decimals, students still were lacking solid understanding of the interconnections between them.

Vamvakoussi, Christou, Mertens and Van Dooren (2011) then extended this research with the goal of further investigating the finding that “students mistakenly assign the property of discreteness to rational numbers” (p. 676). In their a replication study, Vamvakoussi et al. (2011) compared Greek (n=84) and Flemish (n=128) students’ performance on a test asking about numbers in an interval (i.e., how many numbers exist between a pair of numbers, including numbers, decimals and fractions?). They found that students from both countries were more successful on questions involving whole numbers than fractions or decimals” (p. 683).

Section 5. Evidence-based approaches for supporting student learning

5.1 Overview

There is a growing body of research on evidence-based effective approaches to support student learning of rational number systems, including decimals and fractions. Here in Ontario, both primary research (in the form of collaborative action research with teachers from Grades 3-10) and secondary research (in the form of literature reviews conducted as part of the Ontario fractions research) have been conducted to investigate effective approaches to instruction for fractions in Ontario, Canada. The findings, developed based on ongoing research conducted since 2011, are reflected in the Fractions Learning Pathways and accompanying resources (see www.edugains.ca). In brief, the research team (Bruce, Flynn, Yearley, Bennett, Condon, & Shaw) have found the following practices to be important for development of fractions concepts and skills: (1) grounding the understanding of rational numbers on the unit, in particular focusing on unit fractions; (2) the use of visual models that increase student understanding and have longevity over the grades (in particular rectilinear area and linear models); (3) providing students with experiences with physical equipartitioning of models; (4) developing greater teacher understanding of the different meanings of fractions important to increase precision with fractions teaching; (5) an emphasis on comparing, ordering and equivalent fractions, as well as (6) the continued return to fractions ideas (from unit fractions, to comparing and equivalence of fractions, to operations with fractions) throughout the school year. For extensive details on these ideas, please see the Fractions Learning Pathways as well as the previously submitted literature reviews (available at www.tmerc.ca).

In a recent synthesis of research, Rittle-Johnson and Jordan (2016) summarized the 28 studies funded by the Institute of Education Sciences in the US between 2002 and 2013. Some of these contributions specific to fractions and algebra in the middle grades are worthy of note here because of the high calibre of the original research as well as the synthesis. The contributions related directly to student learning about fractions include the following:

- 1) Research findings show that there are many factors and processes that affect fraction learning, including “numerical magnitude understanding, arithmetic fluency, attention, memory, and verbal skills” (p.15; citing work by Seethaler et al., 2011; Hansen, et al., 2013; Siegler & Lortie-Forgues, 2014; Siegler & Lortie-Thompson, 2014; Siegler, Thompson & Schneider, 2011);
- 2) The research underscores the importance of using number lines to represent fraction magnitudes for both students with and without mathematical difficulties (citing Siegler, Thompson & Schneider, 2011; Fuchs, 2013; Saxe, Diakow & Gearhart, 2013; Gearhardt & Saxe, 2014);
- 3) Results indicate that “Adolescents with mathematics difficulties benefit from fractions instruction that builds fractions skills and concepts alongside problems anchored in everyday contexts” (citing work by Bottge et al., 2010a; Bottge et al., 2014; Stephens, Bottge, & Rueda, 2009; Bottge et al., 2010b).

Further recommendations for the teaching of fractions, have been provided by Son (2012) who referred to the Korean strategy for teaching fraction multiplication; In Korea, only one textbook is used for Grades 1-6 and fraction multiplication and is introduced at the beginning of Grade 5. Specific recommendations for fraction multiplication provided in this study are beneficial to the teaching of rational numbers in general, and are in alignment with the recommendations by Rittle-Johnson and Jordan (2016). These are:

1. Focus on “multiple meanings” whereby students explore the varied meanings of an operation to develop conceptual understanding of procedures. For example, multiplication is emphasized as repeated addition, or a number of groups of a specific size, but not often as scaling;
2. Apply models showing multiple representations (i.e., combine the use of area models with set models and measurement/number line models), to develop conceptual understanding;
3. Consider “multiple computational strategies” for solving a problem, instead of just using the same procedure over and over (as is the case in U.S. textbooks); and,
4. Focus on developing both conceptual and procedural understanding simultaneously by “1. developing the meaning of the operation using a real-life context; 2. developing strategies for computing; and 3. using and applying strategies” (p. 391).

Together, the studies presented in this introductory section argue for the need to establish a firm foundation in rational number concepts, including fraction concepts. In this literature review, we were interested in expanding beyond fractions to report on the research for effective approaches to teaching rational numbers generally. Since much of the research does tend to focus on fractions, they are still heavily represented in this section, with some additional recommendations from research dealing with other rational number systems. The goal of developing both conceptual knowledge and fluency in procedures and the importance of using multiple representations are of particular focus.

5.2 Providing learning that focuses on the magnitude of rational numbers

Magnitude understanding, as we have seen in Section 4.1., is recognized as a vital foundation to number development, including the development of rational numbers. Recognizing that all numbers have magnitude makes comparing and relating numbers across number systems possible. The number sense research conducted by Van Hoof, Verschaffel and Van Dooren (2016) suggested that rational number learning “should focus first on a deep exploration and understanding of the magnitude of rational numbers... [and draw] explicit attention to the differences with natural number magnitude, before focusing on the other aspects of rational number understanding” (p. 12).

It cannot be overstated how important magnitude understanding is to the development of rational number concepts. A strong case for focusing instruction on the magnitude and quantify of rational numbers has already been made in Section 4, which examined research highlighting the importance of magnitude to understanding rational numbers, and which presented research that illuminated student challenges with understanding

rational numbers as quantities. As the work of Siegler, Thompson and Schneider (2011) shows, understanding magnitudes can help students make connections across all number systems. For example, it is possible to place, order and compare all numbers on a number line, since they can all be expressed with a magnitude (i.e., whole numbers and rational numbers can both appear on the same line). The integrated theory of numerical development, which proposes that number systems are interconnected, has clear implications for instruction: “if magnitudes are central to understanding fractions as well as whole numbers, then instruction that emphasizes magnitude understanding is more likely to succeed than instruction that does not emphasize magnitude understanding” (p. 293).

5.3 Emphasizing conceptual understanding

Section 4.3 focused on problems and issues that result from the tendency to emphasize procedural knowledge over conceptual understanding in instruction on rational numbers (especially when it comes to operations). In this subsection, the focus shifts to suggestions, based on research evidence, for instruction that emphasizes conceptual knowledge and the strong consensus of the need to provide ample opportunities for students to build this conceptual understanding in rational number instruction. Several key studies are summarized below.

In Arnon, Nesher and Nirenburg’s 2001 study with 11-12 year olds ($n=30$), the researchers reported on a teaching experiment that used a computer program which allowed students to make sense of complex rational number ideas, without using algorithms. Instead of simply applying a rule (which is often forgotten), students could represent rational numbers and learned “to add, subtract, compare fractions, and more, often beyond what they [tend to] learn in conventional classrooms and syllabi” (Arnon, Nesher, & Nirenburg, 2001, p. 209). The researchers concluded that “When we teach a concept in one of its narrower meanings, we often create a foundation for misconceptions” (p. 210).

In their landmark study, (outlined in detail in Section 5.4), Moss and Case (1999) noted that “The domain of rational numbers has traditionally been a difficult one for middle school students to master. Although most students eventually learn the specific algorithms that they are taught, their general conceptual knowledge often remains remarkably deficient” (Moss & Case, 1999, p. 122). Results indicated that the sequence of instruction is particularly important in developing conceptual fluency; the students who received the experimental and carefully sequenced curriculum outperformed, and had a deeper understanding of rational numbers than, their control group peers.

Applying real-world contexts, pictures and manipulatives as a way to promote “meaning for the operations on rational numbers rather than [just] the application of rote procedures” (p. 729) was recommended by Bezuk and Armstrong (1992). The authors framed their article around the challenging nature of fraction multiplication and argued that, while there may be a perception that it is more efficient to apply algorithms to multiplication (over using models and representations), the potential for developing conceptual understanding is greatly reduced. “The decision to forego understanding can

have disastrous effects... consigning students to many years of memorization without meaning” (p. 729). In a sequence of activities designed for grades 5-9 students, Bezuk and Armstrong offered strategies beyond procedures (i.e., drawing area models and building on knowledge of partitioning) for introducing fraction multiplication in a way that helps students build conceptual understanding. Further contributing to the argument that students need to have an understanding beyond just rules, Hasemann (1981) wrote: “pupils ought to have relational understanding to succeed constantly in applying the rule” (p. 82).

The research of Cramer, Post and delMas (2002) also provided evidence for the importance of developing conceptual understanding prior to focusing on procedural fluency. Implementing a 28-30 day teaching program with over 1600 grade 4 and 5 students, the mixed-methods study involved student performance on rational number tasks (examining fraction concepts, order, estimation, operations) as well as student interviews. Students were taught using either a commercial (traditional) curriculum or the Rational Number Project curriculum, which focused on using multiple representations (i.e., on combining the use of pictures, manipulatives, real-world contexts, and symbolic representations). The post assessment included 33 questions which assessed student knowledge of basic fractions concepts, equivalence, order, and addition and subtraction. Student responses were categorized as correct, incorrect or blank, with correct or incorrect responses further divided into conceptual, procedural, no explanation, used manipulatives or unclear. Students who were taught using the Rational Number Project curriculum, where conceptual understanding was the focus, far outperformed their peers who were taught in the traditional way, relying on rote, procedural understanding. Researchers concluded that “developing an understanding of the meaning of the symbols, examining relationships, and building initial concepts of order and equivalence should be the focus of instruction. Conceptual understanding should be developed before computational fluency” (p. 112).

5.4 The important interaction of procedural knowledge and conceptual understanding of rational numbers

Refuting the superficial and popular pitting of conceptual understanding against procedural knowledge, other studies have suggested that there are nuances in the interaction between conceptual and procedural knowledge that need to be further explored; these studies suggest a positive interaction between the two that may argue for an approach that integrates both. It may be that some children may learn concepts first, some may learn and retain procedures first, and others learn procedures and concepts simultaneously (Hallett, Nunes, & Bryant, 2010). This finding came from a study of Grade 4 and 5 students (n=318) who completed an assessment of fraction knowledge (evaluating both procedural and conceptual understanding). The authors were interested in determining how individuals differ in their application of conceptual and procedural knowledge of fraction. Results indicated that not all students apply concepts and procedures in the same order or fashion. For example, some students referred more to their knowledge of procedures, while others preferred to use their conceptual understanding to reason through fractions tasks. However, students who had high levels of both types of knowledge outperformed their peers who have had

a high level of knowledge in one or the other. Interestingly, individuals who relied more on conceptual understanding than procedural understanding, also did better than their peers who relied on procedural knowledge. A similarly structured study by Hallett and colleagues considered the role of school experience and student ability (grade 6 and 8; $n=233$) in conceptual and procedural understanding of fractions. Findings made “a strong argument that understanding individual differences in conceptual and procedural knowledge may be an important part of understanding the development of mathematical cognition” (Hallett, Nunes, Bryant, & Thorpe, 2012, p. 485).

Like Hallett and colleagues, in an article titled *Patterns Of Strengths and Weakness in Children’s Knowledge about Fractions*, Hecht and Vagi (2012) argued for the importance of exploring “individual learning profiles” in terms of conceptual and procedural understanding. Grade 4 and 5 students ($n=181$) were tested on their procedural and conceptual understanding, using fraction computation tasks, pictorial representations (i.e., identifying the shaded portion or fraction of a polygon), fraction size comparison, word problems, and estimation tasks. It became clear that conceptual knowledge was linked with higher performance, but that “children’s reliance on procedural and conceptual knowledge about fractions appears to be in large part a dynamic process over time” (p. 225).

Despite the fact that conceptual knowledge is often assumed to be superior or indicative of more sophisticated knowledge, some literature highlights that it may not be an emphasis of conceptual knowledge but rather a focus on each type in tandem, that best supports learning. As Son (2012) states, procedural flexibility is linked to the ability to compare and consider a variety of solutions and mathematical concepts. The article by Son (2012) examined Korean and American instruction of fraction multiplication. The vast difference between the two methods of teaching was that there was emphasis on procedure and algorithms in the U.S., compared to a focus on developing both conceptual and procedural knowledge of fraction multiplication in Korea. After a detailed analysis of Korean textbooks, the recommendation for other jurisdictions to “develop conceptual understanding and procedural fluency simultaneously” (p. 392) was made.

The argument for a bi-directional process between procedural knowledge and conceptual knowledge (i.e., that there exists a positive interaction between the two) was also in focus in the study by Bailey, et al. (2015). In their analysis of grade 6 and 8 students’ development of fraction knowledge, the authors argued that it appears that procedural and conceptual knowledge facilitates “acquisition of the other” (p. 78) in a cyclical manner, with both types of knowledge contributing to the other. When students were able to draw on their knowledge of fraction magnitudes (a conceptual notion), they were more likely “to learn fraction arithmetic procedures more effectively, which in turn could facilitate acquisition of further conceptual knowledge of magnitudes and of the conceptual basis of fraction procedures” (p. 81). In their study of 251 3rd-6th-grade students, Kainulainen, McMullen, and Lehtinen (2017) also considered this bi-directional nature of procedural and conceptual learning. Since children differ in their use of procedural and conceptual knowledge in working with both fractions and decimals, it was decided that an interaction between the two knowledge types occurred. Continuing

in this vein is the argument that the two types of knowledge work in tandem – “neither would necessarily precede the other” (Gabriel, et al., 2013, p. 2); both types of knowledge seem important when learning fractions and may interact in a cyclical manner.

Furthermore, procedural learning on its own is certainly not enough to build a strong foundation for later mathematics. Brown and Quinn are unequivocal in their conclusions about the importance of a focus on conceptual learning. The results of their 2006 study of involving 143 grade 9 and 10 algebra students (reported in detail in 5iii) led the researchers to the following recommendations for children in the earlier primary years:

Informal practice with fraction concepts should be limited to experiences that arise naturally, like fair sharing or situations that involve money. Lamon (1999) claims that studies have shown that if children are given the time to develop their own reasoning for at least three years without being taught standard algorithms for operations with fractions and ratios, then a dramatic increase in their reasoning abilities occurred, including their proportional thinking (p. 5)” (p. 39).

For students in the later primary years, Brown and Quinn recommend informal exploration with fractions, including:

the manipulation of concrete objects and the use of pictorial representations, such as unit rectangles and number lines. Fraction notation must be developed, but formal fraction operations using teacher-taught algorithms should be postponed. Learning the subject of fractions will revolve around informal strategies for solving problems involving fractions. The objective at this level is to build a broad base of experience that will be the foundation for a progressively more formal approach to learning fractions” (p. 39).

For middle school children, “More time is needed to allow students to invent their own ways to operate on fractions rather than memorising a procedure (Huinker, 1998).

Progressively this development should lead to more formal definitions of fraction operations and algorithms that prepare students for the abstractions that arise later in the study of algebra” (p. 39). Their final conclusion is worth quoting, for it raises the alarm about even larger issues created by a lack of time and attention to conceptual development and an overemphasis on rote procedures – the impact on students’ experience and perception of mathematics:

A philosophy that seemingly ignores established guidelines regarding a child’s cognitive development (Wadsworth, 1996) and forces children into the belief that learning mathematics is memorising facts and algorithms is worse than a problem. It causes children to lose control over numbers and to perceive doing mathematics as a drudgery. (p. 40)

5.5 Making connections across rational number systems

Current research emphasizes the importance of connecting rational number systems, particularly fractions and decimals (though some research also includes integers in that list). Certainly the goal is for students to understand and connect meaning across all of these systems. It is important for students to understand, for example, that both decimals and fractions represent particular numeric magnitudes and, moreover, that a decimal or a fraction can be used to represent the *same* magnitude or quantity. As the

research by Vamvakoussi and Vosniadou (2010) highlighted, however, in spite of years converting fractions to decimals and vice versa (which may give us the impression that they understand that they are representing the same quantity in different ways), many students do *not* consolidate this learning. Something beyond the procedure for converting fractions and decimals is required to meaningfully connect the two number systems. The research literature on decimals and fractions generally focuses on the order in which to teach them or whether to teach them concurrently.

Moss and Case (1999) conducted an investigation of a 4th-grade experimental curriculum (Moss & Case, 1999) (n=29). The curriculum presented rational numbers in the following order: percentages, then decimals, and then fractions, using “a simple unidimensional representation of number...” and focusing “benchmark equivalencies among percents, decimals, and fractions throughout the curriculum” (p. 145). The order of instruction is certainly of interest, however, the researchers observed that more important than order of instruction is fidelity to the development of children’s thinking (the idea “that the general sequence of coordinations remains progressive and closely in tune with children’s original understanding”) and that the “teaching of one form of representation in some depth is preferable to the superficial teaching of several forms at once” (p. 125). The researchers reasoned that by the age of 10 or 11 (the age of the students participating in the study), students have a well-developed sense of quantity to 100, and began instruction in this landmark study with a volume model (a beaker of water) in which students were asked to assign a number (percent) out of 100 to describe its level of fullness. Decimal numbers were then introduced (specifically, 2-place decimals), followed by “exercises in which fractions, decimals and percents were to be used interchangeably” (p. 126). Researchers hypothesized that students would be able to apply a “ratio-measurement structure” (e.g., represented by the beaker of water) to fractions and decimals, allowing children to draw on experiences with percentages in everyday life and the observation that it is easier to convert tidily from percentages to a fraction or a decimal, while the reverse is not always true. After receiving the experimental curriculum, students demonstrated a more solid foundational understanding of rational numbers, including complex properties of rational numbers such as density, as compared to their control group peers who did not receive the modified curriculum. Students in the experimental group also placed less emphasis on whole number referents and had an enhanced ability to reason proportionately and in novel situations, again as compared to their control group peers.

In another more recent study examining undergraduate student understanding of decimals and fractions, DeWolf, Bassok and Holyoak (2014) recommended that students be introduced to meanings of fractions other than part-whole relationships, which is almost exclusively the focus on fractions instruction in North America: “teaching students about the part-to-part relation in addition to the part-to-whole relation might help to expand children’s understanding of fractions. This more general understanding might, in turn, aid students in eventually learning more abstract mathematics, such as algebra” (p. 143). As previously discussed, in their series of experiments with undergraduate students, they found that participants tended to represent sets of discrete objects with fractions, but represented continuous quantities with decimals.

They also found that “whereas previous research has established that decimals are more effective than fractions in supporting magnitude comparisons, the present study reveals that fractions are relatively advantageous in supporting relational reasoning with discrete (or discretized) concepts” (p. 127). Based on their findings from this study, the researchers conclude that “it might well be easier for children to learn about magnitudes less than one by being introduced to decimals prior to fractions. Fractions might be taught later than decimals, with an emphasis on their status as a *relationship* between two natural numbers” (DeWolf, Grounds, Bassok, & Holyoak, 2014, p. 143). They explain that decimal magnitudes may be easier to learn because “their implicit denominator is a constant (base 10), rather than a variable, so that a decimal inherently expresses the unidimensional concept of magnitude” (p. 143). These researchers confirmed these findings in another study (DeWolf, Bassok, & Holyoak, 2015), again with undergraduate students, which showed that response times for participants were identical whether they were comparing whole numbers or decimals, and slower for fractions, indicating again that decimals are perhaps the better representation for magnitude, and fractions the better representation for relations. This aligns with findings from an earlier study (Iuculano & Butterworth, 2011) which found that participants were able to place integers and decimals with relatively similar accuracy on a number line, but were less accurate in placing fractions.

One study recognized that applying prior knowledge of fractions to new learning about decimals could either help or hinder a student: “knowledge of fractions can assist and confuse students as they develop an understanding of the decimal system” (Moskal & Magone, 2000, p. 317). The authors considered the role that “fractional referents” (i.e., their prior knowledge of fractions) play in developing understanding of decimals in their examination of student responses to the Decimal Magnitude Task. The similarities between fractions and decimals helped students with prior knowledge of fractions to understand decimals, since both types of rational numbers can express values between two whole numbers (e.g., both 2.25 and $2\frac{1}{4}$ are between 2 and 3). Differences between fraction and decimals systems, on the other hand, present complications. The number of partitions possible for each type of rational number differs: for fractions, the number of partitions is defined by the denominator (the unit we are working in), whereas, for decimals, the base-10 place value system limits the number of partitions to factors of 10. As with whole number bias, overgeneralization of fractions knowledge to decimals can cause problems for students. Some students, for example, interpret the decimal point as having the same meaning/function as the fraction bar, leading students to see 3.4 and $\frac{3}{4}$ as equivalent quantities. Despite the possibility of overgeneralization of fractions rules to decimals, however, “it is through building relationships within and between referent systems, and recording and extending these ideas, that students eventually develop” understanding (p. 320).

5.6 Connecting instruction to prior knowledge of whole numbers

Although the sequencing of teaching rational number systems is still being studied, researchers have clearly recognized the “positive role of whole number magnitude knowledge in learning fractions” (Torbeys, et al., 2014, p. 6) and that whole number

magnitude knowledge is tied directly to understanding decimal numbers (Moskal & Magone, 2000). Brown and Quinn (2006) stated that “the rational number concept is pivotal to extending whole number concepts while building fraction concepts, which can then be extended to form algebraic concepts” (p. 40). Several key studies in this area are summarized below.

A longitudinal study following 6-year-olds for four years (to the age of 10) examined differences children’s understanding of fractions (Vukovic, Fuchs, Geary, Jordan, Gersten, & Siegler, 2014). Assessment changed over time and moved from general competencies at age 6 (i.e., nonverbal reasoning, attention, executive control language) to whole number computation at age 7, to fractions concepts at age 10. A main goal was to discern the “conceptual leap from whole-number understanding to fraction knowledge” and the associated predictors (p. 1471). Overall, both early general competencies and early math skills predicted later fraction understanding. Results also suggested “whole-number skills are developmental precursors of fraction learning... and that the whole-number bias may be alleviated in part by addressing gaps in children’s understanding of whole-number procedures and principles” (p. 1473).

Moss and Case (1999) based their theoretical framework on educational psychology research which hypothesises that whole number and rational number schemas develop in a similar way. For example, whole number knowledge begins with children’s ability to count verbally and compare quantities, and rational number knowledge begins with an ability to reason proportionately and halve and double numbers. Knowledge (schema) for these number systems develop separately, until they, after some time and development of knowledge, merge to form a “core understanding [that] is... extended to more complex numbers and forms of representation until the overall structure of the entire field is understood” (p. 125). The authors explain that “one of the most important roles that instruction can play is to refine and extend the naturally occurring process whereby new schemas are first constructed out of old ones, then gradually differentiated and integrated” (p. 125).

A longitudinal study that followed pairs of students from Grade 3 to 5 focused on the development of rational number arithmetic and *how* students reorganized their number schemas to accommodate new information (Olive, 1999). The research considered children’s prior whole number understanding as a foundation for fraction knowledge. A specific goal of the research project was to understand the “reorganizations children need to make in their abstract number sequences to operate meaningfully with fractions” (p. 280) with the view that prior knowledge of whole numbers can be beneficial to rational number learning. Olive (1999), like Moss and Case (1999), notes the importance of schemas and argued that number schemas develop through “the interiorization [i.e., construction of mental structures] of activities that children engage in through applications of their prior number sequence” (p. 281). Using computer programs as the method of teaching, students were exposed to a variety of rational number concepts, beginning with an exploration of a fractional unit and partitioning, and moving to determining fractions of fractions in fair sharing problems (by partitioning area models). In order to confirm that the students deeply understood fractions as quantities

and operations, the measurement model of fractions was used. Through close analysis of 50 teaching episodes with two of the children in particular, the study found that students' whole number understanding contributed to their solving rational number arithmetic problems. Instead of interfering with developing fractional schemes, it was found that whole number knowledge "contributed to the reorganization" (p. 309) of the schemes, and that this "reorganization involved an integration of their whole-number knowledge with their fractional schemes whereby whole-number division was regarded as the same as multiplication by the reciprocal fraction" (p.279).

The whole number knowledge that students bring to their experiences with rational numbers present a conundrum; on the one hand, this knowledge is a foundation that they can draw on to understand and reason about rational numbers and their quantities, on the other hand, as we have seen in the phenomenon of the natural number bias, whole number knowledge can be applied in unhelpful and inappropriate ways to rational numbers. In their study with 4th grade students, Kainulainen, McMullen and Lehtinen (2017) also suggest that it is beneficial to bring knowledge of natural numbers to the learning of rational numbers, and recognized that instructional strategies can be employed to reduce the whole number bias and promote development of fraction order and equivalence. The 4th grade participants commonly drew on whole number knowledge when ordering fractions, and a category of observed strategies was called "whole number dominance". The results showed that despite "children's schemas for ordering whole numbers [being] very strong and, at least during initial instruction in fractions... overgeneralized" (p. 333), after prolonged instruction, the majority of students were able to reason about fractions in an appropriate way (i.e., one that resulted in a correct answer and indicated fractional understanding).

Of particular practical interest is research on the use of mathematics thinking tools which span number systems and become central to developing the connections from whole numbers to fractions and decimals, for example (Siegler, Thompson and Schneider, 2011, discussed in greater detail in Section 4). Specifically, Siegler and his colleagues found that the mental number line proved effective in reasoning about both whole numbers and fraction magnitudes, and that magnitude understanding of both types of number was greatly influenced by general math achievement and proficiency in arithmetic. Specific strategies used by the students to reason about fractions magnitudes also involved reasoning about whole numbers. The researchers note that: one common, and effective, strategy was to translate the fraction being estimated into a percentage of the distance between the two endpoints and then to use the percentage as if it were a whole number on a 0–100 number line. For example a child might reason that $\frac{4}{5}$ was 80% of the distance between 0 and 1 and proceed as if locating 80 on a 0–100 number line. (p. 291)

The authors ultimately argued that "difficulty learning about fractions stems from drawing *inaccurate* analogies to whole numbers, rather than from drawing analogies between whole numbers and fractions" (p. 291).

Further, Mohring, Newcombe, Levine and Frick (2016) discussed the potential of other spatial reasoning strategies to help students overcome whole number bias. Their study

assessed 8-10 year olds' (n=52) proportional reasoning and fraction knowledge using tests mapped to *the Common Core State Standards for Mathematics*. In order to incorporate visual-spatial reasoning into ratio tasks, students were presented with rectangular images divided into blue and red portions, representing water and juice respectively. They were then asked to rate (on a scale), which mixture would have a strong juice flavour or weak juice flavour (both stacked and side-by-side rectangular diagrams were used). Spatial estimations of proportion were found to be linked to fraction knowledge. Drawing on intuition around proportional reasoning, enabled a focus on reasoning with ratios (see also Beswick (2011) who conducted similar experiments where students made paint charts of varying ratios of light paint to dark paint).

5.7 Precision with using representations, models and tools

i. importance of using a variety of models

The use of models is fundamental to developing rational number understanding. Unfortunately, models may often be left aside in favour of algorithms (Bezuk & Armstrong, 1992). In rational number instruction, the use of physical models (e.g., cuisinaire rods, fraction strips/circles) or visual models (e.g. area models, set models, number lines) is helpful as they help children “concretize” or make meaning of difficult and complex rational number concepts (Behr, Wachsmuth, Post, & Lesh, 1984; Cramer, Post, & delMas, 2002; Post, 1989; Steffe, 2004). It is important for students to have opportunities to create their own models as a way of reasoning through a problem (Kent, Empson, & Nielsen, 2015) and to make meaning of concepts and procedures. Researchers argue that it is only through taking the time to model relationships (i.e., through objects or diagrams) of fractions (as opposed to exclusively symbolic representations) that understanding is achieved for most students.

Yetim and Alkan (2013) discussed the importance of having exposure to many different representations to help clarify rational number misconceptions. Common rational number mistakes and misconceptions were examined amongst 73 grade 7 students. A diagnostic test was administered and results indicated that one of the areas in which students struggled was rational number representation. The authors argued that emphasis should be placed on encouraging the use of a variety of representations, so as to make rational numbers more concrete; “models and modeling is [a] very effective method in visualizing the abstract forms in the brain” (p. 219).

Working with Grade 4-6 classes in Belgium, Gabriel, Coche, Szucs, Carette, Rey and Content (2013) found that students “seem to have a stereotypic representation of fractions” (p. 8) and resorted to the same models, despite their challenge for a specific rational number (for example, using a circle area model instead of a set model to represent a difficult-to-partition fraction like $\frac{1}{7}$). Knowing how to represent rational numbers with a range of tools, therefore, helped students select the appropriate model to use in solving a problem.

Moss and Case (1999) cite research by Sowder (1995) and Markovits and Sowder (1991), who suggest that “children need to learn how to move among the various

possible representations of rational number in a flexible manner” (including symbolic representations) (p. 124). However, citing work by Mack (1990), they also caution that “the teaching of one form of representation in some depth is preferable to the superficial teaching of several different forms at once” (p. 126). We believe based on this body of research as well as our own primary research that there is a middle ground, where instruction provides children with experiences with a variety of physical and visual models that are purposefully chosen and used - importantly these models become tools for thinking about rational numbers (see *Math Teaching for Learning: Purposeful Representations* for further information).

ii. number lines as a particularly powerful thinking tool

The number line serves as a particularly powerful representation and thinking tool for rational numbers (Hansen, et al., 2015; Gomez, Hubbard & Fazio, 2016; Ni, 2000; Rittle-Johnson & Jordan, 2016; Saxe, Diakow, & Gearhart, 2013; Torbeyns, Schneider, Xin & Siegler, 2015; Vukovic, et al., 2014). The number line is mathematically powerful because of its ability to enable comparisons of multiple number systems at once (i.e., both whole numbers and all types of rational numbers can appear on the same line; see Iuculano & Butterworth, 2011; Kallai & Tzelgov, 2009; Saxe, Diakow, & Gearhart, 2013; Shaughnessy, 2011). For example, putting both whole numbers and fractions on the same number line helped students with conceptualizing and generalizing fractions (Kallai & Tzelgov, 2009).

In a study of the number line, Saxe, Diakow, and Gearhart (2013) examined student interpretations of fractions and integers with 4th and 5th grade students. The *Learning Mathematics through Representations* curriculum used in the study includes number lines as the primary thinking tool for integers and fractions. A total of 571 students (315 fourth grade and 256 fifth grade students) and 19 teachers participated in the intervention which involved 9 lessons on integers and 10 lessons on fractions using the number line. Students in the treatment group outperformed a control group at post-test (75% greater gains than the control group), and these results were retained 5 months later. By exploring the number line first with integers, the students were better able to use it effectively with fractions. The number line, “as the principal representational context” (p. 360) was effective in supporting solutions for challenging problems and was effective across domains (i.e., for both integers and fractions).

As with all representations, students need time to work with the representation to fully understand it. A number line has a particular anatomy – a starting point (which may or may not be zero), ‘hash’ marks or partition lines that indicate specific locations on the line, spaces between the hash marks that demarcate units and show distance and on which additional number relationships may be mapped. Unfortunately, students may not have opportunities to learn about the physicality of the number line before being asked to work with it, and the number line representations are often static (unchanging); Elementary school textbooks, for example, frequently ask students to place, order and compare rational numbers (fractions and decimals) using a number line from 0 to 1, but there are many other ways that number lines can be used (e.g., stacked number lines) to locate numbers beyond 1, and to show relationships between numbers and number

systems. When interviewing 31 fifth grade students on their abilities to use a number line (by identifying points on the line using both fractions and decimals), Shaughnessy (2011) observed several challenges students experienced including, using unconventional notation, redefining the unit, or using a “two-count” strategy, as shown in the figure below:

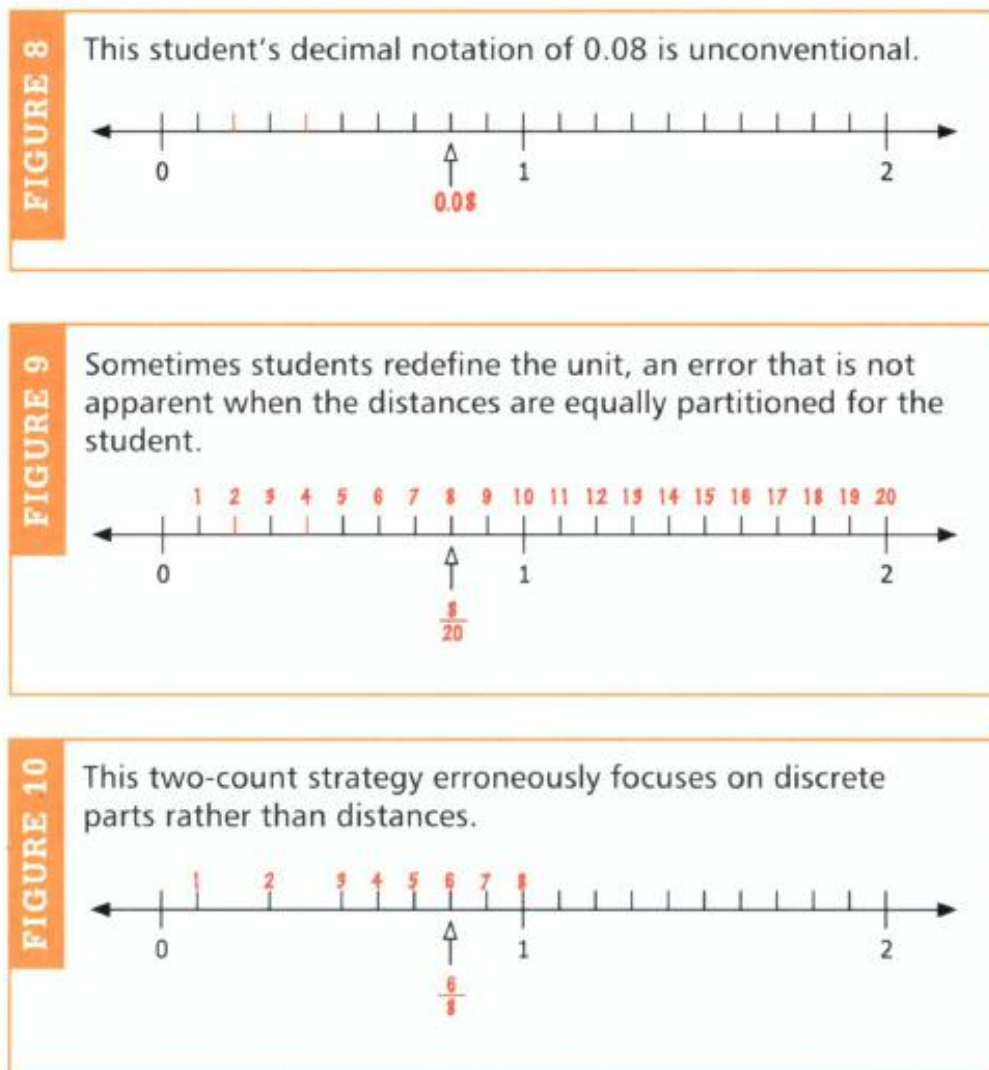


Figure 4. Challenges with representations on number lines (Shaughnessy, 2011, p. 432)

Based on the observed student errors and misconceptions, and her recognition of the importance of number line fluency for rational numbers, Shaughnessy (2011) suggested it would be beneficial to provide students with a range of different number line tasks (i.e., both partitioned and unpartitioned lines, where students have to either label an existing point or create the point to be labelled).

Some literature focuses on a more abstract notion of the number line; the mental number line. The mental number line, unlike a physical number line,

is a dynamic, continually changing structure rather than a fixed, static one. Initially useful for organizing knowledge of nonsymbolic numbers and then of small, positive, symbolic whole numbers, the mental number line is progressively extended rightward to represent larger symbolic whole numbers, leftward to represent negative numbers, and interstitially to represent symbolic fractions and decimals (Siegler & Lortie-Forgues, 2014, p. 145).

iii. [measurement as a powerful context for learning about rational numbers](#)

A discussion of measurement as a context or approach for instruction in rational numbers is appropriate here, because it often involves the use of number lines and puts rational numbers into a relevant context. Beyond the obvious linear relationship between a number line and a metre stick, for example, measurement readily allows for a bridging of whole number and rational number knowledge. Almost forty years ago, Kieren (1980) identified five key interpretations of rational number knowledge: whole-part relations, ratios, quotients, operators, and importantly, measures. In 1989, Kastner subsequently reported that using a measurement to approach rational number tasks allowed students to visually and concretely represent complex concepts with precision: “As students gain measurement experience, the need for better precision becomes apparent... The idea of subdividing a unit that is inadequate for a proposed measurement task into halves or thirds or tenths is easily accepted, or even suggested, by a student who sees a measurement as being, say between five and six units. Operations on rational numbers are also readily seen in concrete terms in the measurement setting” (p. 43). For example, adding one-fifth units of ribbon along the metre stick is a concrete way of adding fractions. Concepts important to number sense, in general, can be enhanced through the use of “measurement-based” curriculum and “measurement practices” (p. 46).

Vamvakoussi and Vosniadou’s 2010 study examined student reasoning about the ordering and density of rational numbers. The authors provide support for using measurement as an important context for rational number instruction. The number line can help “students to conceptualize natural and non-natural numbers as members of the same family, and that the different symbolic representations of a number actually refer to the same mathematical object” (p. 204). By placing rational numbers and whole numbers together on the same number line, concepts like order and density become more apparent.

In an article titled *Using paper folding, fraction walls, and number lines to develop understanding of fractions for students from Years 5-8*, Pearn (2007) discussed a specific application of the measurement model: connecting paper folding of fractional strips to number lines. For example, students used number lines marked 0 to 1, and were asked to estimate a one-third distance of the number line, and then check it with their folded strips of paper thirds. The combined use of the two linear strategies (paper strip folding and the use of number lines) lent itself well to comparing and ordering rational numbers with a visual exploration of number densities.

Furthermore, “with fractions as measurement units, division of fractions becomes meaningful. For example, the questions ‘How much of $\frac{3}{4}$ is $\frac{1}{8}$?’ or ‘How many 8ths in $\frac{3}{4}$?’ require finding the measure of $\frac{3}{4}$ in terms of $\frac{1}{8}$ as a measurement unit” (Olive, 1999, p. 305). Connecting rational numbers to measurement has further positive outcomes, such as aiding in the conceptualization of division of fractions, a complex phenomenon.

5.8 Other pedagogical strategies

i. real life contexts

A few other pedagogical strategies are of note in the research on rational number instruction, namely: grounding rational number instruction in real life contexts, the potential of solution comparisons to allow students further opportunities to conceptualize and make sense, and a strategy called interleaving.

Real life contexts provide opportunities for students to apply math concepts and prior knowledge in meaningful ways. Joram, Resnick and Gabriele (1995) consider the appearance of numbers in everyday texts (real world applications). By examining magazines, the authors found percent, fractions, and averages were all commonly referenced. “Numbers are pervasive in our daily lives” (p. 346), so drawing connections to how rational numbers appear in everyday contexts (such as data displays in magazines and newspapers) is important for drawing attention to their relevance to real life. Rapp, Bassok, DeWolf and Holyoak (2015) recommend that instruction work “to find the best real-life examples that correspond to the target mathematical concepts” (p. 54) to improve student performance and draw on prior knowledge.

With a focus on decimals, Irwin (2001) discussed the importance of providing contextualized problems and making everyday contexts accessible to students. Eight pairs of students, aged 11-12, were presented with problems that addressed common misconceptions about decimals (for example, that a zero at the end of a decimal increases it by ten times or that $\frac{1}{4}$ can be expressed as either 0.4 or 0.25). The study aimed to observe whether the application of a relevant context around decimal problems would help students develop understanding; i.e., what difference does it make if, instead of simply asking students to multiply two decimals, a contextualized condition is applied for the same multiplication question to help students insert themselves into a real-life context. Pairs of students were given either appropriately contextualized or noncontextualized problems. Overwhelmingly, students who had worked through contextualized problems improved their decimal understanding more than their peers who worked with noncontextualized problems.

ii. comparing solutions

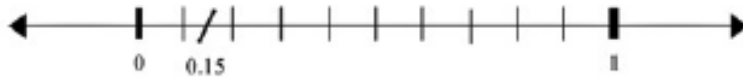
Rittle-Johnson and colleagues (in a number of articles) commented on the importance of comparing different solution strategies: “Comparison seems to be a fundamental learning process... In learning to solve equations, it pays to compare [solution methods]” (Rittle-Johnson & Star, 2007, p. 573). The work by Rittle-Johnson and Star (2009) involved 162 Grade 7 and 8 students who solved and compared problems with

same and different solution methods. Although focusing more on algebraic equations, the link to rational number is apparent, both because of the influence rational numbers have on later algebraic understanding and because algebraic expressions often involve the use of fractions and decimals. Solution comparisons fit into one of three categories: 1) solving the same type of problem using the same solution method; 2) solving different types of problem using the same solution method; and 3) solving the same type of problem using different solution methods. Results indicated that there was value in comparing methods, as it increased both procedural and conceptual knowledge. Again investigating grade 7 and 8 students' approaches to solving algebraic problems Rittle-Johnson, Star, and Durkin (2009) found that comparing multiple examples and solution strategies was helpful for student understanding (i.e., it was important to focus not just on the same solution methods or the same problem types); solution "comparison seems to be a fundamental learning process" (Rittle-Johnson & Star, 2007, p. 573).

Subsequently, in a consideration of learning about decimal magnitude, a solution comparison strategy was again found beneficial to student learning (Durkin & Rittle-Johnson, 2012). In particular, the study found that opportunities to compare incorrect solutions provided particularly rich learning. This study followed 4th and 5th graders (n=74) as they were learning about decimals. Students were placed in either an "incorrect condition" where they focused on comparing correct solutions with incorrect solutions, or in a "correct condition" where they compared only correct solutions. Students were shown number lines with decimals placed on them and an explanation for why the number was placed where it was (either correct or incorrect). The following image show examples of incorrect and correct conditions from the study:

Incorrect Condition

Correct



Matt said, "The line is divided into 10 pieces. I put 0.15 between 1 tenth and 2 tenths because 0.15 is 1 tenth plus some."

Incorrect

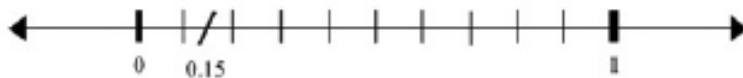


Justin said, "0.15 has two numbers after the decimal point, which makes it a medium-sized number. So I put 0.15 near the middle."

1. How is Matt's way different from Justin's way?
2. Why can't you solve the problem Justin's way?

Correct Condition

Correct



Matt said, "The line is divided into 10 pieces. I put 0.15 between 1 tenth and 2 tenths because 0.15 is 1 tenth plus some."

Correct



Laura said, "I know 15 hundredths is small and close to 0. So I marked 0.15 a little after 0."

1. How is Matt's way different from Laura's way?
2. Explain why both ways of thinking are correct.

Fig. 1. Sample intervention packet page for each condition.

Figure 5. Sample of correct and incorrect conditions (Durkin & Rittle-Johnson, 2012, p. 208)

The comparison of correct and incorrect answers, as in the first example above, was most beneficial as it made valuable learning from incorrect examples possible. In fact, examining incorrect examples was most beneficial and led to improved performance, as

“the *incorrect* condition supported greater learning of correct procedures and retention of correct concepts, in part because of a reduction of whole number misconceptions” (Durkin & Rittle-Johnson, 2012, p. 212). Focusing on the incorrect solution dispelled misconceptions, helped identify clarify the correct answer, and built student conceptual and procedural knowledge.

iii. interleaving

Interleaving is a strategy that ensures that different types of practice problems are intermixed, rather than presented in chunks of the same problem type one after the other (Rittle-Johnson & Jordan, 2016). For example, questions on three different concepts (a, b, c) are presented as abc abc abc instead of aaa, bbb, ccc. The use of interleaving, in general, helps students make sense of related concepts, instead of confusing them (Rohrer, 2012). Focusing on interleaving helped grade 7 students (n=126) make sense of similar concepts and encouraged student reasoning and selection of appropriate solution methods for a problem, since students could not just routinely apply the same method question after question (Rohrer, Dedrick, & Stershic, 2015). In other words, this method prevents students from being able to apply a recently learned strategy to a set of similar questions, without a solid understanding of why the specific strategy works for a specific problem. For example, in their study, Rohrer, Dedrick and Stershic (2015) gave students assignments where they were asked to answer questions on graphs, slope and then a series of seemingly unrelated concepts, like fractions, percentages and proportions. It was found that “the mere rearrangement of practice problems improved mathematics learning in the classroom... that the... benefit of interleaving does not diminish over time... [and] interleaved practice provided near immunity against forgetting” (p. 905-906).

Section 6: Analysis of rational number treatment in Ontario text resources

Given the findings of this literature review, we wanted to also survey the ways that rational numbers are represented in resources that are widely available for teacher use in Ontario. Looking at five current textbooks used in Ontario (two Grade 6 mathematics textbooks, and three Grade 8 mathematics textbooks), a brief analysis was conducted to look for general trends both in terms of the frequency of lessons on rational numbers as well as the representations used to introduce and reinforce concepts related to rational numbers. We acknowledge that there are, of course, variations in instruction from teacher to teacher, and a textbook analysis certainly does not capture all of these variations in terms of strategies or pedagogy in general, but it is nonetheless an interesting exercise for considering how rational numbers are generally represented in Ontario textbooks.

Textbooks that are used in Ontario classrooms for mathematics must be approved by the Minister of Education and identified on the Trillium List. Consequently, this analysis is limited to the five textbooks for the selected grades which appear on the Trillium List. It is important to note that textbooks approved on the Trillium List for Ontario are often published for other Canadian school systems as well and so do not necessarily align entirely to the Ontario curriculum (i.e., there may be additional content which extends beyond the Ontario curriculum).

For the purposes of this analysis, textbooks from Grades 6 and 8 were considered. In the Ontario curriculum, by the end of Grade 6, students should have acquired considerable understanding of rational numbers, including reading, representing, comparing and ordering decimals and fractions as well as operations with decimals. Thus, an examination of Grade 6 textbooks would provide insight into the methods and models emphasized at this critical transition point. It is within the Grade 7 and 8 curricula that students learn about the operations with fractions, so examination of the Grade 8 textbooks allows for a similar insight as students transition to secondary schools and streamed mathematics courses. It is important to note that the Curriculum was last revised in 2005 and the textbooks were published in 2005 and 2006.

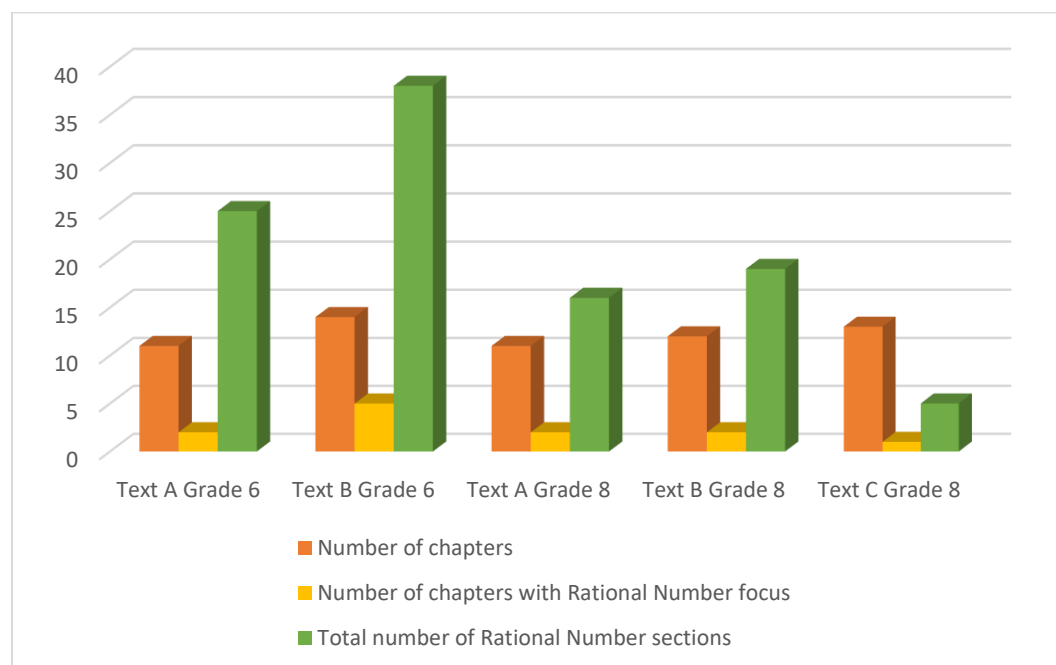
Textbook and teacher resource availability varies greatly between boards and schools across the province. Similarly, there is a wide range of supplemental resources used. (A supplementary resource is defined by the Ontario Ministry of Education as “a resource that supports only a limited number of curriculum expectations, or the curriculum expectations in a single strand, outlined in the curriculum policy document for a specific subject or course, or a limited number of expectations for a Kindergarten learning area. Such a resource may be intended for use by an entire class or group of students. Examples are readers, novels, spelling programs, dictionaries, atlases, and computer software and instructional guides.”) (Ministry of Education, 2008, p. 4) However, since textbooks and their accompanying teacher resources do act as a driver in program planning in many schools across Ontario, a cursory overview of the structure of the texts provides great insight into the range of time allocation and usage of visual representations or models.

The first chart reveals the range in the total number of chapters across the five textbooks (11 – 14 chapters) in orange. Of these 11-14 chapters, between 1 and 5 chapters focus on rational numbers (see yellow bars). Within these textbooks, there is a wide variance in the number of instructional sections on Rational Numbers (green bars) that appear within each chapter (5 to 38 subsections). Note that in order to assess instructional components, pre-chapter reviews, post-chapter practices and other special interest sections were not included in the counts. We conclude from this brief analysis that there is little consistency across textbooks in terms of the weight given to rational numbers compared to mathematics in the whole text, and there is a lack of sub-topic treatments.

Table 1. Comparison of rational number foci across Ontario Trillium-listed textbooks for Grades 6 and 8

	Text A Grade 6	Text B Grade 6	Text A Grade 8	Text B Grade 8
Number of chapters	11	14	11	12
Number of chapters with Rational Number focus	2	5	2	2
Total number of Rational Number sections	25	38	16	19

Table 2. Graphical comparison of rational number foci across Ontario Trillium-listed textbooks for Grades 6 and 8

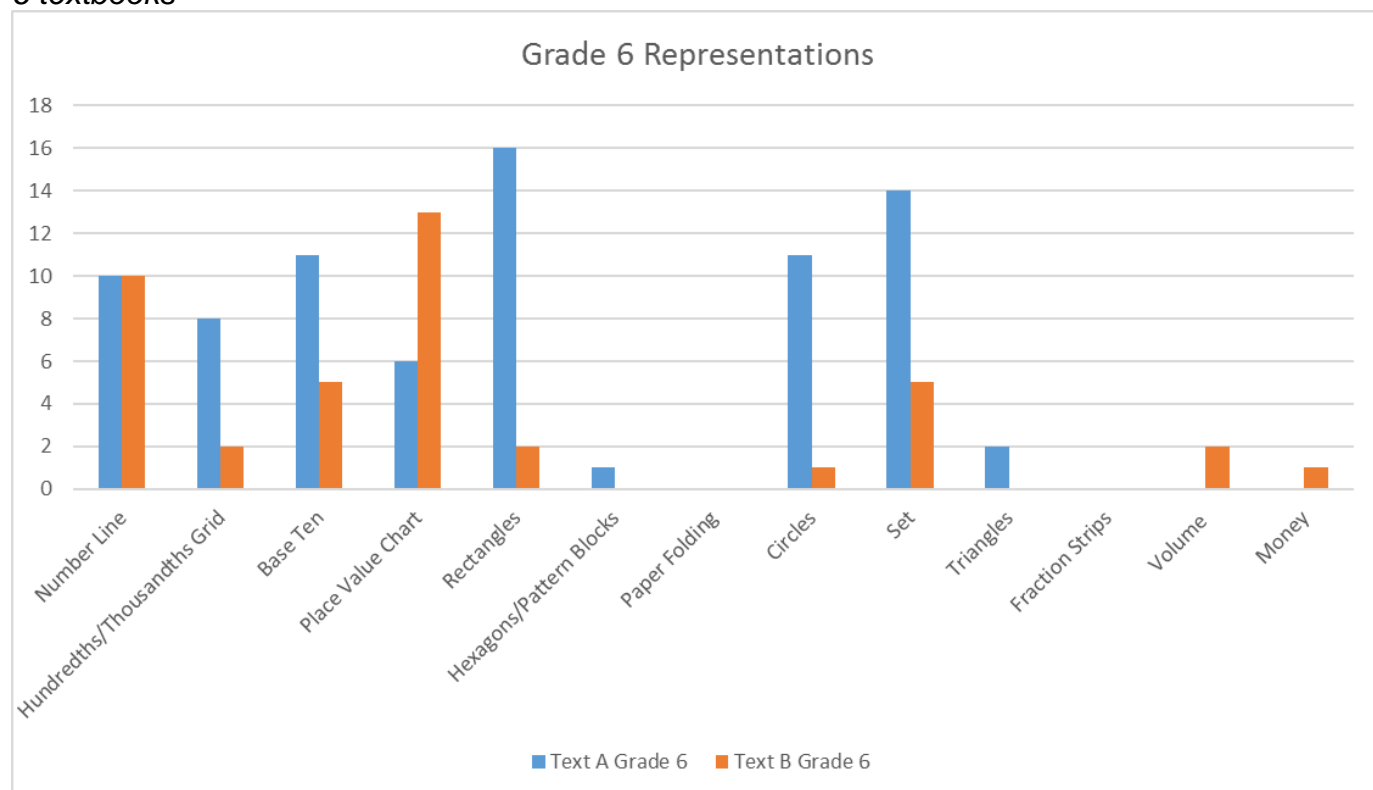


A more detailed examination of the representations used in comparing amongst the two grade 6 and amongst the three grade 8 textbooks reveals, again, a wide range and lack of consistency both in the number and type of representations used.

Table 3. Representations of rational numbers appearing in Grade 6 textbooks

	Text A Grade 6	Text B Grade 6
Number Line	10	10
Hundredths/Thousandths Grid	8	2
Base Ten	11	5
Place Value Chart	6	13
Rectangles	16	2
Hexagons/Pattern Blocks	1	
Paper Folding		
Circles	11	1
Set	14	5
Triangles	2	
Fraction Strips		
Volume		2
Money		1

Table 4. Graphical depiction of representations of rational numbers appearing in Grade 6 textbooks



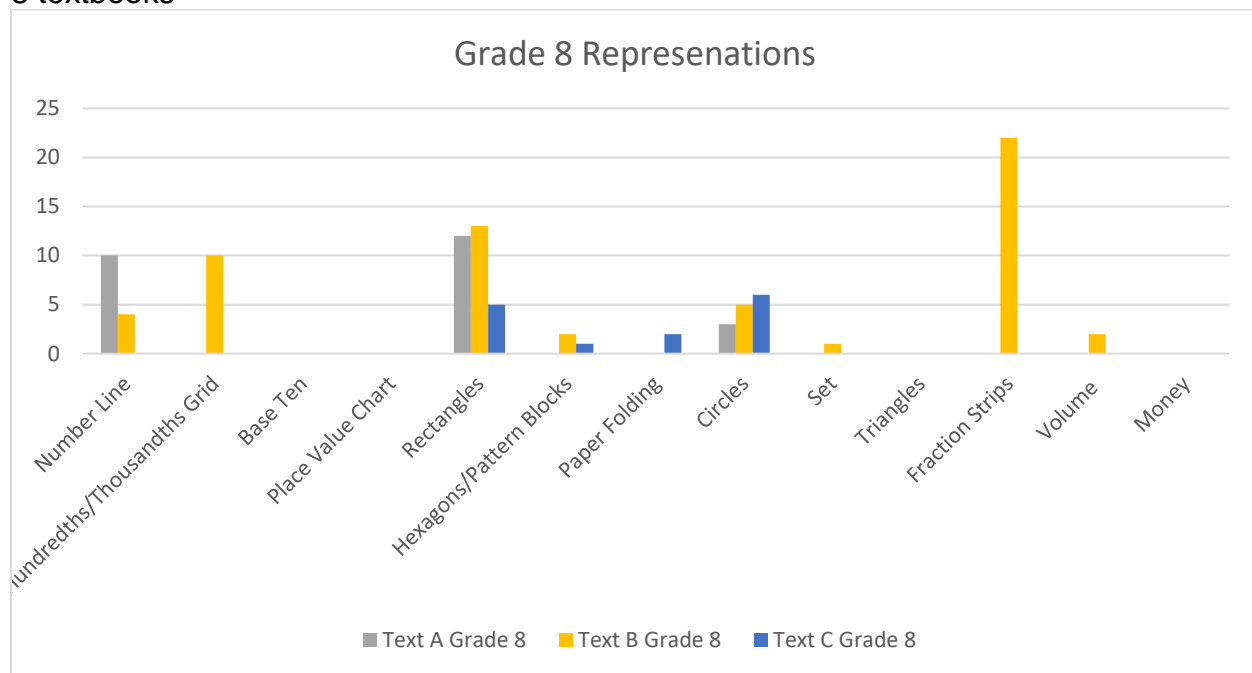
From the data, we can also see that Text A has more representations available to students compared to Text B, but both texts use a range of 9 representations. Neither text illustrated paper folding or fraction strips, although both used number lines somewhat frequently.

Table 5. Representations of rational numbers appearing in Grade 8 textbooks

	Text A Grade 8	Text B Grade 8	Text C Grade 8
Number Line	10	4	
Hundredths/Thousandths Grid		10	
Base Ten			
Place Value Chart			
Rectangles	12	13	5
Hexagons/Pattern Blocks		2	1
Paper Folding			2
Circles	3	5	6
Set		1	
Triangles			
Fraction Strips		22	
Volume		2	

Money			
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Table 6. Graphical depiction of representations of rational numbers appearing in Grade 8 textbooks



In the grade 8 textbooks, two of the texts used three different representations while the third (Text B) used 8 different representations. Text C was the only text to represent rational numbers using fraction strips (highest frequency), volume measurement, a set model and hundredths-thousandths grid.

While there is a wide range of representations (both frequency and type) used across the five textbooks, it is also our finding that, in general, models are not used consistently across the chapters within a given textbook. For example, in Textbook B Grade 6, base ten materials and place value charts are used within the multiplying decimals chapter while number lines and place value charts are used within the dividing decimals chapter. Similarly, Textbook A Grade 8 uses one circle representation in the chapter on ratio, rates and percents and then uses number lines, rectangles and circles when converting from fractions to decimals and operations with fractions. Context, of course, plays a role in which representations are used, however overall the representations were not linked together or used consistently to build on previous models and the concepts they represented.

Privileging of algorithms and procedural knowledge over conceptual understanding was evident to varying degrees across the textbooks. In one grade 8 textbook section on addition and subtraction of fractions, a sample solution using paper folding is followed by a solution using arrays and then another using the algorithm. The summary box for the section, however, includes only the algorithm. In a grade 6 textbook section on dividing decimals, a number line is drawn to show the mathematics but the division

algorithm is used to determine the answer. In grade 6 textbooks, the procedure for determining equivalent fractions is presented as multiplying both the numerator and denominator by the same number rather than through connections to the splitting or merging of fractional regions.

There is also a wide range of incorporation of rational numbers across other topics. One Grade 8 textbook uses financial contexts throughout, allowing students to apply and refine their understanding of decimals across multiple strands. Another grade 8 textbook includes decimals in the data management and measurement sections. The other textbooks appear to have limited inclusion of rational numbers across these topics.

These trends reinforce the importance of supporting educators in planning mathematics instruction, including comparing the textbook content and methodology with the Ontario mathematics curriculum and more recent research-informed practices such as those discussed in the “Paying Attention to ...” series of Ministry-produced supplementary resources. Indeed, these textbooks appear to align with the long-standing observation by Cramer, Post and Behr (1989) that

[t]extbook-dominated school programs are static. The children have little opportunity to manipulate materials and to vary other aspects of the problem situation. Problems generally are presented in a ready-to-solve form, and the children learn to take what is given, manipulate it according to predetermined rules, and proceed directly to a solution (Cramer, Post, & Behr, 1989) p. 103).

However, since textbooks and teacher resources play an important role in the planning and delivery of mathematics programs, it is important to note that often small changes can yield significant results. For example, Rittle-Johnson and Jordan (2016) found that “[s]imple changes to how teachers and textbooks sequence practice problems can substantially improve learning and knowledge retention” (Rittle-Johnson & Jordan, 2013, p. 13).

Section 7: Summary and Discussion

The literature review presented here on the teaching and learning of rational numbers aims to serve three purposes. First, it provides a sense of the overall educational landscape, with particular attention to the Ontario context. Essentially, there is much room for improvement as students continue to struggle with the meaning of rational numbers and how to work with them.

Second, the review offers the reader some considerations for the development of procedural knowledge and conceptual understanding, and information on the current challenges students face. The intention here is to provide some background for the education community so that we might further enhance student understanding of rational numbers in ways that are coherent, precise, and based on evidence from current research.

An example of increased coherence would involve students using the number line representation as a thinking tool from unitizing to equi-partitioning, to comparing quantities to performing operations with rational numbers (such as fractions, decimal numbers and percentages). A second example of increased coherence would involve returning continually to the concept of magnitude as a foundation for all rational numbers. An example of enhanced pedagogy based on research evidence would involve taking up wrong answers regularly - and in a positive learning environment – to help dispel persistent misconceptions. An example of greater precision would involve identifying these misconceptions in the classroom, and addressing them with precisely aligned and powerful models (such as working with linear measurement contexts) that further understanding.

The third purpose of this literature review, related to the second, is to point to some reasonably manageable, immediately implement-able, promising and researched-effect strategies for supporting students in their work with rational numbers. Examples of best practices highlighted in this literature review include interleaving and regular use of number lines, fractions strips and paper folding models, for supporting the development of student understanding of rational numbers.

It would be remiss in this summary to not take note of the discussion on procedural knowledge and conceptual understanding. Research shows that students will not experience success if they are relying on procedures that are memorized without understanding how or why these algorithms ‘work’. The literature review also presents research that supports the learning of both rational number procedures and concepts simultaneously, ensuring that students do have a solid understanding of magnitude, number relationships, and the multiple underlying constructs of rational numbers. They also need to understand what they are doing when working with rational numbers as well as why, and how it makes sense.

Section 8: Recommendations

In light of this literature review on the teaching and learning of rational numbers, there are several recommendations that could serve well to enhance student understanding of these number systems.

There has been a reasonably rich tradition of research on teaching and learning fractions and some other rational number systems, such as the use of decimals, however there is minimal research on teaching rational number systems in an integrated fashion. Further, although there is sound research on the importance of linear measure and number line models for building deep understanding of rational numbers, there is limited evidence of their use in school mathematics or in school texts such as those surveyed in this literature review. It appears that these linear models are used sporadically at best in North America, including Ontario Canada.

Recommendation 1: Provide teachers with professional learning opportunities and resources which support the use of models which have been proven effective in building knowledge of rational numbers. This professional learning must be of sufficient intensity, including opportunities to experiment with these models in the classroom, that it has an impact on teaching practices. The effort required for shifts in the teaching practices related to rational numbers must not be underestimated – when educators have not experienced deep learning of rational numbers themselves, it is extremely challenging to adopt new strategies and models which are foreign to the educator.

Research is resoundingly clear that understanding magnitude (relative quantities), number relationships and their notation is fundamental to understanding the meaning of fractions and other rational numbers. Nonetheless, direct and explicit teaching of magnitude concepts have been incidental at best.

Recommendation 2: The level of attention to magnitude, quantity relationships and their notation is a second area for tremendous growth in education. Understanding proportions, relative quantities, and number density are foundational to rational number systems, and this fostering of these understandings begins in the early years, but must not be neglected in the later years of learning. A Ministry sponsored *Paying Attention to Magnitude and Relative Quantities* resource might be one way to heighten attention to these fundamental areas of understanding rational numbers. Highly illustrative examples across the grades in such a resource document, as well as compelling arguments for the benefits of focusing on magnitude concepts would be necessary to increase uptake in Ontario classrooms.

Finally, the research clearly points to a need for a purposeful balance between and sequence of conceptual and procedural knowledge. Students have increased success in mathematics, and in particular with rational numbers, when such instruction is in place. Without this integrated and bi-directional approach, students struggle to connect knowledge across number systems. This is particularly profound in the learning of operations in different number systems, where students are increasingly being exposed

to a range of strategies for whole number operations yet presented with algorithms for operations in other number systems, including operations with decimals and fractions.

Recommendation 3: Deepening educators' understanding of operations across the number systems (and into algebra) is another area which has great potential for growth. Flexible understanding of the operations, coupled with a solid understanding of magnitude, through the use of purposeful and powerful models, will allow all students to experience increased success in mathematics. As with magnitude, a Paying Attention to ... Operations across the Number Systems would be an example of a resource for supporting Ontario educators in building the understanding necessary for effective instruction. An alternate dynamic format such as a web-based resource could also be of interest. Building on resources already available, such as the Guides to Effective Instruction and the Fractions research materials, this document could bring current research to the forefront in a practical and concise resource.

Bibliography

- Arnon, I., Nesher, P., & Nirenburg, R. (2001). Where do fractions encounter their equivalents? Can this encounter take place in elementary-school? *International Journal of Computers for Mathematical Learning*, 6, 167-214.
- Bailey, D. H., Zhou, X., Zhang, Y., Cui, J., Fuchs, L. S., Jordan, N. C., . . . Siegler, R. S. (2015). Development of fraction concepts and procedures in U.S. and Chinese children. *Journal of Experimental Child Psychology*, 129, 68-83.
- Behr, M. J., Khoury, H. A., Harel, G., Post, T., & Lesh, R. (1997). Conceptual units analysis of preservice elementary school teachers' strategies on a rational-number-as-operator task. *Journal for Research in Mathematics Education*, 28(1), 48-69.
- Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education*, 15(5), 323-341.
- Beswick, K. (2011). Make your own paint chart. *Australian Mathematics Teacher*, 67(1), 6-11.
- Bezuk, N. S., & Armstrong, B. E. (1992). Understanding fraction multiplication. *The Mathematics Teacher*, 85(9), 729-732.
- Brousseau, G., Brousseau, N., & Warfield, V. (2007). Rationals and decimals as required in the school curriculum. Part 2: From rationals to decimals. *Journal of Mathematical Behavior*, 26, 281-300.
- Brown, G., & Quinn, R. J. (2006). Algebra students' difficulty with fractions: An error analysis. *Australian Mathematics Teacher*, 62(4), 28-40.
- Clarke, D. M., & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics*, 72, 127-138.
- Cramer, K. A., Post, T. R., & Behr, M. J. (1989). Cognitive restructuring ability, teacher guidance, and perceptual distracter tasks: An aptitude-treatment interaction study. *Journal for Research in Mathematics Education*, 20(1), 103-110.
- Cramer, K. A., Post, T. R., & delMas, R. C. (2002). Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the rational number project curriculum. *Journal for Research in Mathematics Education*, 33(2), 111-114.
- DeWolf, M., Bassok, M., & Holyoak, K. J. (2015). Conceptual structure and the procedural affordances of rational numbers: Relational reasoning with fractions and decimals. *Journal of Experimental Psychology: General*, 144(1), 127-150.
- DeWolf, M., Chiang, J. N., Bassok, M., Holyoak, K. J., & Monti, M. M. (2016). Neural representations of magnitude for natural and rational numbers. *NeuroImage*, 141, 304-312.

- DeWolf, M., Grounds, M. A., Bassok, M., & Holyoak, K. J. (2014). Magnitude comparison with different types of rational numbers. *Journal of Experimental Psychology: Human Perception and Performance*, 40(1), 71-82.
- Durkin, K., & Rittle-Johnson, B. (2012). The effectiveness of using incorrect examples to support learning about decimal magnitude. *Learning and Instruction*, 22, 206-214.
- Fazio, L. K., Bailey, D. H., Thompson, C. A., & Siegler, R. S. (2014). Relations of different types of numerical magnitude representations to each other and to mathematics achievement. *Journal of Experimental Child Psychology*, 123, 53-72.
- Fuchs, L. S. (2013). Improving At-Risk Learner's Understanding of Fractions. *Journal of Educational Psychology*, 105(3), 683-700.
- Gabriel, F., Coche, F., Szucs, D., Carette, V., Rey, B., & Content, A. (2013). A componential view of children's difficulties in learning fractions. *Frontiers in Psychology*, 4, 1-12.
- Gabriel, F.C., Szucs, D., & Content, A. (2013). Development of mental representations of fractions. *PLoS ONE*, 8(11), 1-15.
- Gomez, D. H., Hubbard, E. & Fazio, L. (2016). MBE Perspectives on the Learning of Fractions and their Magnitude. Toronto.
- Hallett, D., Nunes, T., & Bryant, P. (2010). Individual differences in conceptual and procedural knowledge when learning fractions. *Journal of Educational Psychology*, 102(2), 395-406.
- Hallett, D., Nunes, T., Bryant, P., & Thorpe, C. M. (2012). Individual differences in conceptual and procedural fraction understanding: The role of abilities and school experience. *Journal of Experimental Child Psychology*, 113, 469-486.
- Hansen, N., Jordan, N. C., Fernandez, E., Siegler, R. S., Fuchs, L., Gersten, R., & Micklos, D. (2015). General and math-specific predictors of sixth-graders' knowledge of fractions. *Cognitive Development*, 35, 34-49.
- Hasemann, K. (1981). On difficulties with fractions. *Educational Studies in Mathematics*, 12(1), 71-87.
- Hecht, S. A., & Vagi, K. J. (2012). Patterns of strengths and weaknesses in children's knowledge about fractions. *Journal of Experimental Child Psychology*, 111, 212-229.
- Hurst, M., & Cordes, S. (2016). Rational-number comparison across notation: Fractions, decimals, and whole numbers. *Journal of Experimental Psychology: Human Perception and Performance*, 42(2), 281-293.
- Irwin, K. C. (2001). Using everyday knowledge of decimals to enhance understanding. *Journal for Research in Mathematics Education*, 32(4), 399-420.
- Iuculano, T., & Butterworth, B. (2011). Rapid communication: Understanding the real value of fractions and decimals. *The Quarterly Journal of Experimental Psychology*, 64(11), 2088-2098.

- Joram, E., Resnick, L. B., & Gabriele, A. J. (1995). Numeracy as cultural practice: An examination of numbers in magazines for children, teenagers, and adults. *Journal for Research in Mathematics Education*, 26(4), 346-361.
- Kainulainen, M., McMullen, J., & Lehtinen, E. (2017). Early developmental trajectories toward concepts of rational numbers. *Cognition and Instruction*, 35(1), 4-19.
- Kallai, A. Y., & Tzelgov, J. (2009). A generalized fraction: An entity smaller than one on the mental number line. *Journal of Experimental Psychology: Human Perception and Performance*, 35(6), 1845-1864.
- Kastner, B. (1989). Number sense: The role of measurement applications. *The Arithmetic Teacher*, 36(6), 40-46.
- Kent, L. B., Empson, S. B., & Nielsen, L. (2015). The richness of children's fraction strategies. *Teaching Children Mathematics*, 22(2), 85-90.
- Kieren, T. E. (1980). The rational number construct – Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125-150). Columbus: ERIC/SMEAC.
- Matthews, P. G., & Chesney, D. L. (2015). Fractions as percepts? Exploring cross-format distance effects for fractional magnitudes. *Cognitive Psychology*, 78, 28-56.
- Matthews, P. G., Lewis, M. R., & Hubbard, E. M. (2016). Individual differences in nonsymbolic ratio processing predict symbolic math performance. *Psychological Science*, 27(2), 191-202.
- McCrink, K., & Wynn, K. (2007). Ratio abstraction by 6-month-old infants. *Psychological Science*, 18(8), 740-745.
- McCrink, K., Spelke, E. S., Dehaene, S., & Pica, P. (2012). Non-symbolic halving in an Amazonian indigene group. *Developmental Science*, 16(3), 451-462.
- Ministry of Education. (2005). *The Ontario Curriculum: Grades 1-8 Mathematics*. Toronto: Queen's Printer.
- Ministry of Education. (2008). *Guidelines for Approval of Textbooks*. Retrieved from Ministry of Education: http://www.trilliumlist.ca/files/Textbook_Guide_English_2008.pdf
- Mix, K. S., Levine, S. C., & Huttenlocher, J. (1999). Early fraction calculation ability. *Developmental Psychology*, 35(5), 164-174.
- Mohring, W., Newcombe, N. S., Levine, S. C., & Frick, A. (2016). Spatial proportional reasoning is associated with formal knowledge about fractions. *Journal of Cognition and Development*, 17(1), 67-84.
- Moskal, B. M., & Magone, M. E. (2000). Making sense of what students know: Examining the referents, relationships and modes students displayed in response to a decimal task. *Educational Studies in Mathematics*, 43, 313-335.

- Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*, 30(2), 122-147.
- Ni, Y. (2000). How valid is it to use number lines to measure children's conceptual knowledge about rational number? *Educational Psychology*, 20(2), 139-152.
- Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, 40(1), 27-52.
- Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning*, 1(4), 279-314.
- Pearn, C. A. (2007). Using paper folding, fraction walls, and number lines to develop understanding of fractions for students from Years 5-8. *Australian Mathematics Teacher*, 63(4), 31-36.
- Pitkethly, A., & Hunting, R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics*, 30(1), 5-38.
- Post, T. (1989). Fractions and other rational numbers. *The Arithmetic Teacher*, 37(1), 3-4.
- Post, T. R., Wachsmuth, I., Lesh, R., & Behr, M. J. (1985). Order and equivalence of rational numbers: A cognitive analysis. *Journal for Research in Mathematics Education*, 16(1), 18-36.
- Rapp, M., Bassok, M., DeWolf, M., & Holyoak, K. J. (2015). Modeling discrete and continuous entities with fractions and decimals. *Journal of Experimental Psychology: Applied*, 21(1), 47-56.
- Rittle-Johnson, B., and Jordan, N. C. (2016). Synthesis of IES-Funded Research on Mathematics: 2002–2013 (NCER 2016-2003) Washington, DC: National Center for Education Research, Institute of Education Sciences, U.S. Department of Education. This report is available on the Institute website at <http://ies.ed.gov/>.
- Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology*, 99(3), 561-574.
- Rittle-Johnson, B., & Star, J. R. (2009). Compared with what? The effects of different comparisons on conceptual knowledge and procedural flexibility for equation solving. *Journal of Educational Psychology*, 101(3), 529-544.
- Rittle-Johnson, B., Star, J. R., & Durkin, K. (2009). The importance of prior knowledge when comparing examples: Influences on conceptual and procedural knowledge of equation solving. *Journal of Educational Psychology*, 101(4), 836-852.
- Rohrer, D. (2012). Interleaving helps students distinguish among similar concepts. *Educational Psychology Review*, 24, 355-367.
- Rohrer, D., Dedrick, R. F., & Stershic, S. (2015). Interleaved practice improves mathematics learning. *Journal of Educational Psychology*, 107(3), 900-908.

- Saxe, G. B., Diakow, R., & Gearhart, M. (2013). Towards curricular coherence in integers and fractions: A study of the efficacy of a lesson sequence that uses the number line as the principal representational context. *ZDM Mathematics Education*, 45, 343-364.
- Schneider, M., & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance*, 36(5), 1227-1238.
- Seethaler, P. M., Fuchs, L. S., Star, J. R., & Bryant, J. (2011). The cognitive predictors of computational skills with whole versus rational numbers: An exploratory study. *Learning and Individual Differences*, 21, 536-542.
- Shaughnessy, M. M. (2011). Identify fractions and decimals on a number line. *Teaching Children Mathematics*, 17(7), 428-434.
- Siegler, R. S., & Lortie-Forgues, H. (2014). An integrative theory of numerical development. *Child Development Perspectives*, 8(3), 144-150.
- Siegler, R. S., & Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology*, 49(10), 1994-2004.
- Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., . . . Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, 23(7), 691-697.
- Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, 62, 273-296.
- Singer-Freeman, K. E., & Goswami, U. (2001). Does half a pizza equal half a box of chocolates? Proportional matching in an analogy task. *Cognitive Development*, 16, 811-829.
- Smith III, J. P. (1995). Competent reasoning with rational numbers. *Cognition and Instruction*, 13(1), 3-50.
- Son, J.-W. (2012). Fraction multiplication from a Korean perspective. *Mathematics Teaching in the Middle School*, 17(1), 388-393.
- Sophian, C., Garyantes, D., & Chang, C. (1997). When three is less than two: Early developments in children's understanding of fractional quantities. *Developmental Psychology*, 33(5), 731-744.
- Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning*, 6(2), 129-162.
- Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2014). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. *Learning and Instruction*, 37, 5-13.
- Vallentin, D., & Nieder, A. (2008). Behavioral and prefrontal representation of spatial proportions in the monkey. *Current Biology*, 18, 1420-1425.
- Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, 28(2), 181-209.

- Vamvakoussi, X., Christou, K. P., Mertens, L., & Van Dooren, W. (2011). What fills the gap between discrete and dense? Greek and Flemish students' understanding of density. *Learning and Instruction, 21*, 676-685.
- Van Hoof, J., Degrande, T., McMullen, J., Hannula-Sormunen, M., Lehtinen, E., Verschaffel, L., & Van Dooren, W. (2016). The relation between learners' spontaneous focusing on quantitative relations and their rational number knowledge. *Studia Psychologica, 58*(2), 156-170.
- Van Hoof, J., Lijnen, T., Verschaffel, L., & Van Dooren, W. (2013). Are secondary school students still hampered by the natural number bias? A reaction time study on fraction comparison tasks. *Research in Mathematics Education, 15*(2), 154-164.
- Van Hoof, J., Verschaffel, L., & Van Dooren, W. (2015). Inappropriately applying natural number properties in rational number tasks: Characterizing the development of the natural number bias through primary and secondary education. *Educational Studies in Mathematics, 90*, 39-56.
- Van Hoof, J., Verschaffel, L., & Van Dooren, W. (2016). Number sense in the transition from natural to rational numbers. *British Journal of Educational Psychology, 1-14*.
- Varma, S., & Karl, S. R. (2013). Understanding decimal proportions: Discrete representations, parallel access, and privileged processing of zero. *Cognitive Psychology, 66*, 283-301.
- Vukovic, R. K., Fuchs, L. S., Geary, D. C., Jordan, N. C., Gersten, R., & Siegler, R. S. (2014). Sources of individual differences in children's understanding of fractions. *Child Development, 85*(4), 1461-1476.
- Yetim, S., & Alkan, R. (2013). How middle school students deal with rational numbers? A mixed methods reserach study. *Eurasia Journal of Mathematics, Science & Technology Education, 9*(2), 213-221.
- Zhang, L., Fang, Q., Gabriel, F. C., & Szucs, D. (2016). Common magnitude representation of fractions and decimals is task dependent. *The Quarterly Journal of Experimental Psychology, 69*(4), 764-780.