Math Teaching for Learning: Meanings of Multiplication and Division of Fractions

It is helpful to relate our knowledge of whole number operations to fraction operations, and, therefore, to consider meanings that can be used for both division of whole numbers and division of fractions. However, in North America, multiplication is often strongly connected to repeated-addition, which is more difficult to conceive of with fractions. For example, while it is relatively straightforward to visualize, or model, adding 3 to itself 4 times (as with 4×3), it is more difficult to visualize adding $\frac{1}{4}$ to itself $\frac{1}{3}$ times (as with $\frac{1}{3} \times \frac{1}{4}$).

As with whole numbers, the following properties are true for fractions:

a. The Commutative Property: $a \times b = b \times a$

e.g., $2 \times 3 = 3 \times 2$ $\qquad$ $\frac{1}{2} \times \frac{3}{5} = \frac{3}{5} \times \frac{1}{2}$

b. The Associative Property: $(a \times b) \times c = a \times (b \times c)$

e.g., $(2\times3) \times 4 = 2 \times (3\times4)$ $\qquad$ $(\frac{1}{2} \times \frac{3}{5}) \times \frac{4}{9} = \frac{1}{2} \times (\frac{3}{5} \times \frac{4}{9})$

c. The Identity Property: $a \times 1 = a$; $a \div 1 = a$ [Since $\frac{b}{b} = 1$, this could be written as $a \times \frac{b}{b} = a$; $a \div \frac{b}{b} = a$.]

e.g., $2 \times 1 = 2$ $\qquad$ $\frac{1}{2} \times 1 = \frac{1}{2}$ [and, since $\frac{3}{3} = 1$, $\frac{1}{2} \times \frac{3}{3} = \frac{1}{2}$; $\frac{1}{2} \div \frac{3}{3} = \frac{1}{2}$]

d. The Distributive Property $a \times (b + c) = a \times b + a \times c$

e.g., $2 \times (3+4) = 2\times3 + 2\times4$ $\qquad$ $\frac{1}{2} \times \left(\frac{3}{5} + \frac{4}{9}\right) = \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{4}{9}$

e. The Zero Property: $0 \div a = 0$; $0 \times a = 0$; $a \div 0$ is undefined ($a \neq 0$)

e.g., $0 \div 2 = 0$; $0 \times 2 = 0$; $2 \div 0$ is undefined $\qquad$ $0 \div \frac{1}{2} = 0$; $0 \times \frac{1}{2} = 0$; $\frac{1}{2} \div 0$ and $\frac{1}{0}$ are undefined

Paper folding, arrays and number lines are particularly powerful representations for multiplication and division of fractions. Students have early experiences with multiplication of fractions, such as when they consider what one third of one half is. The following images show how this can be determined using paper folding.


Strip folded in half.


One half folded in thirds.


One third of one half is one sixth.

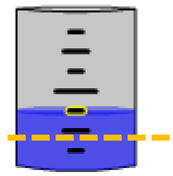
Within North America, research clearly indicates that instruction tends to focus on procedures for multiplying and dividing fractions, with less attention to developing the concepts underlying these procedures (see Procedural vs. Conceptual, Chapter 2 of *Fractions Operations: Multiplication and Division Literature Review)*. This means that the emphasis is on how to follow an algorithm (such as invert and multiply for division) rather than on how the procedures act on the fractional quantities (e.g., what does it mean to divide fractions) and the reasoning behind the operations and/or algorithms themselves (e.g., why do we invert and multiply?). Furthermore, in North America, instruction focuses on conceptual understanding separately from procedural fluency. Notably, in Korea the two are developed simultaneously. Additionally, unlike North American instruction which focuses on narrow interpretations of multiplication and division, the Korean curricula uncovers and addresses multiple meanings of each operation (Son and Senk, 2010). These differences are important as Korea has higher mean achievement than the US on multiple measures. See *Fractions Operations: Multiplication and Division Literature Review* for further information.

Multiplication Meanings

The following multiplication strategies apply across number systems, including whole numbers:

- Measurement multiplication (repeated addition)
  This is based upon the understanding of multiplication as relating to equal groups. The known values are the number of groups and the size of groups, which are multiplied to obtain the product or total quantity.

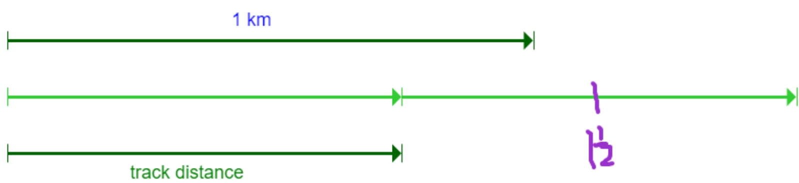
  E.g., A recipe calls for $\frac{3}{8}$ cup of water. Determine how much water is required to make $\frac{1}{2}$ of the recipe.



$$\frac{1}{2} \times \frac{3}{8} = ?$$

- Partial groups multiplication
  In partial groups, two fractions are multiplied together.

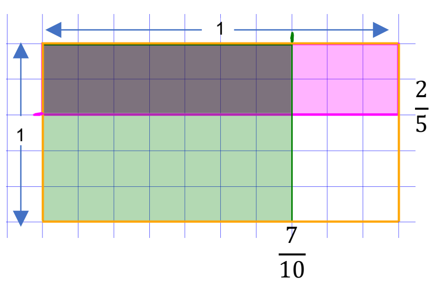
  E.g., The distance around a running track is $\frac{3}{4}$ km. Determine how far a person would run if they travel $1\frac{1}{2}$ of the running track.



$$1\frac{1}{2} \times \frac{3}{4} = ?$$

- Cartesian product
  This interprets multiplication as the shared space of two quantities. An array is used to represent this (see *Math Teaching for Learning: Representing Multiplication and Division of Fractions with Arrays and Number Lines* for more information about arrays).

  E.g., A rectangle has dimensions of $\frac{2}{5}$m and $\frac{7}{10}$m. What is the area of the rectangle?

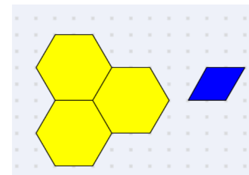


$$\frac{2}{5} \times \frac{7}{10} = ?$$

Division Meanings

The following division strategies apply across number systems, including whole numbers:

- Measurement division (Quotative; repeated subtraction)
  This is based upon the understanding of division as relating to equal groups. The known values are the total quantity and the size of groups, which are divided to obtain the quotient or number of groups (total quantity ÷ size of the groups = number of groups).
  E.g., Three yellow hexagons represent one whole. What fraction is represented by one blue rhombus?



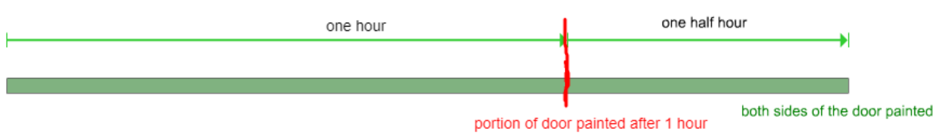
- Partitive Division (Fair Share)
  When we share things fairly we are using partitive division. The known values are the total quantity and the number of groups, which are divided to obtain the quotient or size of groups (total quantity ÷ number of groups = size of the groups).
  E.g., There is $\frac{1}{3}$ of a piece of paper which three friends are to share fairly. What fraction of the whole sheet does each person receive?

- Cartesian product (Product and Factors Division)
  This connects to the interpretation of multiplication as the shared space of two quantities. As the inverse of multiplication, division connects to determining an unknown dimension if the total area and one dimension are known. An array is used to represent this (see One pager 1 for more information about arrays).
  E.g., A rectangle has an area of $\frac{7}{25}$m^2 and one dimension of $\frac{7}{10}$m. What is the length of the other side ?

The following division strategies relate to fractions specifically:

- Unit Rate
  This is based upon the understanding of division as relating to equal groups but is focused on determining the size of one group.

  E.g., Jose can paint both sides of a door in $1\frac{1}{2}$ hours. How much of the door can Jose paint in 1 hour?



$$1 \div 1\frac{1}{2} = ?$$

- Inverse of Multiplication

  This model is the one that is most often emphasized in North American instruction and is recognized as the algorithm for division involving fractions. Since division is the inverse of multiplication, by inverting a fraction and multiplying, the inverse is applied.

  E.g., $\frac{7}{25} \div \frac{7}{10} = \frac{7}{25} \times \frac{10}{7}$

Myths about Multiplication and Division

Students often over generalize patterns in whole number multiplication and division, resulting in confusion when working with fractions and other rational numbers. In addition to using representations, such as number lines and arrays, real life contexts, such as fair share contexts, help students to realize that these are myths. Additionally, using number patterns supports students in sense making about operations with fractions (e.g., 3 × 5, 2 × 5, 1 × 5, 0 ×5, $\frac{1}{2}$× 5, $\frac{1}{4}$ × 5, etc.).

Myth 1: Multiplication always results in a larger answer.

Although this is always true with positive whole numbers, it is not true with fractions.

For example, $\frac{1}{2} \times \frac{5}{3} = \frac{5}{6}$ $\quad \frac{1}{2} < \frac{5}{6} < \frac{5}{3}$

This is also not true with integers (-2 x 3 = -6) and decimals (0.5 x 1.5 = 0.75).

Myth 2: Division always results in a smaller answer.

Again, although this is true with positive whole numbers, it is not true with fractions.

For example, $\frac{5}{6} \div \frac{1}{2} = \frac{5}{3}$ $\quad \frac{1}{2} < \frac{5}{6} < \frac{5}{3}$

This is also not true with integers (-6 ÷ -2 = 3) and decimals (0.75 ÷ 1.5 = 0.5).

Myth 3: The larger number is always divided by the smaller number.

This is a pattern with whole number division questions presented to learners in an effort to avoid answers less than one. However, the order of the division statement is related to the meaning of the quantities rather than their values.

Consider the previous example: There is $\frac{1}{3}$ of a piece of paper which three friends are to share fairly. What fraction of the whole sheet does each person receive?

The number sentence would be $\frac{1}{3} \div 3$.

Reference:

Son, J-W., & Senk, S. L. (2010). How reform curricula in the USA and Korea present multiplication and division of fractions. *Educational Studies in Mathematics, 74*: 117-142.