A Venn diagram will show why (1.2.7) holds, although a formal proof is not difficult (see Exercise 1.2). Using (1.2.7) and the fact that A and $B \cap A^c$ are disjoint (since A and A^c are), we have

$$P(A \cup B) = P(A) + P(B \cap A^c) = P(A) + P(B) - P(A \cap B) \tag{1.2.8}$$

from (a).

If $A \subset B$, then $A \cap B = A$. Therefore, using (a) we have

$$0 \le P(B \cap A^c) = P(B) - P(A),$$

establishing (c). □

Formula (b) of Theorem 1.2.9 gives a useful inequality for the probability of an intersection. Since $P(A \cup B) \le 1$, we have from (1.2.8), after some rearranging,

$$P(A \cap B) \ge P(A) + P(B) - 1. \tag{1.2.9}$$

This inequality is a special case of what is known as *Bonferroni's Inequality* (Miller 1981 is a good reference). Bonferroni's Inequality allows us to bound the probability of a simultaneous event (the intersection) in terms of the probabilities of the individual events.

Example 1.2.10 (Bonferroni's Inequality) Bonferroni's Inequality is particularly useful when it is difficult (or even impossible) to calculate the intersection probability, but some idea of the size of this probability is desired. Suppose A and B are two events and each has probability .95. Then the probability that both will occur is bounded below by

$$P(A \cap B) \ge P(A) + P(B) - 1 = .95 + .95 - 1 = .90.$$

Note that unless the probabilities of the individual events are sufficiently large, the Bonferroni bound is a useless (but correct!) negative number. ||

We close this section with a theorem that gives some useful results for dealing with a collection of sets.

Theorem 1.2.11 *If P is a probability function, then*

a. $P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$ *for any partition* $C_1, C_2, \ldots$;

b. $P(\cup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} P(A_i)$ *for any sets* $A_1, A_2, \ldots$. *(Boole's Inequality)*

Proof: Since $C_1, C_2, \ldots$ form a partition, we have that $C_i \cap C_j = \emptyset$ for all $i \ne j$, and $S = \cup_{i=1}^{\infty} C_i$. Hence,

$$A = A \cap S = A \cap \left(\bigcup_{i=1}^{\infty} C_i \right) = \bigcup_{i=1}^{\infty} (A \cap C_i),$$