

Third Edition

# 100 STATISTICAL TESTS

Gopal K Kanji

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*3rd Edition*

**Gopal K. Kanji**

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# PREFACE

Some twenty years ago, it was only necessary to know about a dozen statistical tests in order to be a practising statistician, and these were all available in the few statistical textbooks that existed at that time. In recent years the number of tests has grown tremendously and, while modern books carry the more common tests, it is often quite difficult for a practising statistician quickly to turn up a reference to some of the less used but none the less important tests which are now in the literature. Accordingly, we have attempted to collect together information on most commonly used tests which are currently available and present it, together with a guide to further reading, to make a useful reference book for both the applied statistician and the everyday user of statistics. Naturally, any such compilation must omit some tests through oversight, and the author would be very pleased to hear from any reader about tests which they feel ought to have been included.

The work is divided into several sections. In the first we define a number of terms used in carrying out statistical tests, we define the thinking behind statistical testing and indicate how some of the tests can be linked together in an investigation. In the second section we give examples of test procedures and in the third we provide a list of all the 100 statistical tests. The fourth section classifies the tests under a variety of headings. This became necessary when we tried to arrange the tests in some logical sequence. Many such logical sequences are available and, to meet the possible needs of the reader, these cross-reference lists have been provided. The main part of the work describes most commonly used tests currently available to the working statistician. No attempts at proof are given, but an elementary knowledge of statistics should be sufficient to allow the reader to carry out the test. In every case the appropriate formulae are given and where possible we have used schematic diagrams to preclude any ambiguities in notation. Where there has been a conflict of notation between existing textbooks, we have endeavoured to use the most commonly accepted symbols. The next section provides a list of the statistical tables required for the tests followed by the tables themselves, and the last section provides references for further information.

Because we have brought together material which is spread over a large number of sources, we feel that this work will provide a handy reference source, not only for practising statisticians but also for teachers and students of statistics. We feel that no one can remember details of all the tests described here. We have tried to provide not only a memory jogger but also a first reference point for anyone coming across a particular test with which he or she is unfamiliar.

Lucidity of style and simplicity of expression have been our twin objectives, and every effort has been made to avoid errors. Constructive criticism and suggestions will help us in improving the book.



## COMMON SYMBOLS

Each test or method may have its own terminology and symbols but the following are commonly used by all statisticians.

$n$	number of observations (sample size)
$K$	number of samples (each having $n$ elements)
$\alpha$	level of significance
$\nu$	degrees of freedom
$\sigma$	standard deviation (population)
$s$	standard deviation (sample)
$\mu$	population mean
$\bar{x}$	sample mean
$\rho$	population correlation coefficient
$r$	sample correlation coefficient
$Z$	standard normal deviate

# INTRODUCTION TO THE BOOK

This book presents a collection of statistical tests which can help experimenters and researchers draw conclusions from a series of observational data. The main part of the book provides a one/two page summary of each of the most common statistical tests, complete with details of each test objective, the limitations (or assumptions) involved, a brief outline of the method, a worked example and the numerical calculation. At the start of the book there are more, detailed, worked examples of the nine most common tests. The information provides an ideal “memory jog” for statisticians, practitioners and other regular users of statistics who are competent statisticians but who need a sourcebook for precise details of some or all the various tests.

*100 Statistical Tests* lists 100 different inferential tests used to solve a variety of statistical problems. Each test is presented in an accurate, succinct format with a suitable formula. The reader can follow an example using the numerical calculation provided (without the arithmetic steps), refer to the needed table and review the statistical conclusion.

After a first introduction to statistical testing the second section of the book provides examples of the test procedures which are laid out clearly while the graphical display of critical regions are presented in a standard way.

The third section lists the objective of each of the tests described in the text. The next section gives a useful classification of the tests presented by the type of the tests: (a) for linear data: parametric classical tests, parametric tests, distribution free tests, sequential tests and (b) for circular data: parametric tests. This invaluable table also gives a concise summary of common statistical problem types and a list of tests which may be appropriate. The problem types are classified by the number of samples (1, 2 or  $k$  samples), whether parametric or non-parametric tests are required, and the area of interest (e.g. central tendency, distribution function, association).

The pages of the next section are devoted to the description of the 100 tests. Under each test, the object, limitation and the method of testing are presented followed by an example and the numerical calculation. The listings of limitations add to the comprehensive picture of each test. The descriptions of the methods are explained clearly. The examples cited in the tests help the reader grasp a clear understanding of the methods of testing.

The first of the following two sections gives the list of tables while second section displays 39 statistical tables many of which have accompanying diagrams illustrated in a standard way. This comprehensive list covers all the commonly used standard tables.

The book is concluded with references and index.

# INTRODUCTION TO STATISTICAL TESTING

Having collected together a number of tests, it is necessary to consider what can be tested, and we include here some very general remarks about the general problem of hypothesis testing. Students regard this topic as one full of pitfalls for the unwary, and even teachers and experienced statisticians have been known to misinterpret the conclusions of their analysis.

Broadly speaking there are two basic concepts to grasp before commencing. First, the tests are designed neither to prove nor to disprove hypotheses. We never set out to prove anything; our aim is to show that an idea is untenable as it leads to an unsatisfactorily small probability. Second, the hypothesis we are trying to disprove is always chosen to be the one in which there is no change; for example, there is no difference between the two population means, between the two samples, etc. This is why it is usually referred to as the null hypothesis,  $H_0$ . If these concepts were firmly held in mind, we believe that the subject of hypothesis testing would lose a lot of its mystique. (However, it is only fair to point out that some hypotheses are not concerned with such matters.)

To describe the process of hypothesis testing we feel that we cannot do better than follow the five-step method introduced by Neave (1976a):

**Step 1** Formulate the practical problem in terms of hypotheses. This can be difficult in some cases. We should first concentrate on what is called the alternative hypothesis,  $H_1$ , since this is the more important from the practical point of view. This should express the range of situations that we wish the test to be able to diagnose. In this sense, a positive test can indicate that we should take action of some kind. In fact, a better name for the alternative hypothesis would be the action hypothesis. Once this is fixed it should be obvious whether we carry out a one- or two-tailed test.

The null hypothesis needs to be very simple and represents the status quo, i.e. there is no difference between the processes being tested. It is basically a standard or control with which the evidence pointing to the alternative can be compared.

**Step 2** Calculate a statistic ( $T$ ), a function purely of the data. All good test statistics should have two properties: (a) they should tend to behave differently when  $H_0$  is true from when  $H_1$  is true; and (b) their probability distribution should be calculable under the assumption that  $H_0$  is true. It is also desirable that tables of this probability distribution should exist.

**Step 3** Choose a critical region. We must be able to decide on the kind of values of  $T$  which will most strongly point to  $H_1$  being true rather than  $H_0$  being true. Critical regions can be of three types: right-sided, so that we reject  $H_0$  if the test statistic is greater than or equal to some (right) critical value; left-sided, so that we reject  $H_0$  if the test statistic is less than or equal to some (left) critical value; both-sided, so that we reject  $H_0$  if the test statistic is *either* greater than or equal to the right critical value *or* less than or equal to the left critical value. A value of  $T$  lying in a suitably defined critical region will lead us to reject  $H_0$  in favour of  $H_1$ ; if  $T$  lies outside the critical region we do not reject  $H_0$ . We should never conclude by accepting  $H_0$ .

**Step 4** Decide the size of the critical region. This involves specifying how great a risk we are prepared to run of coming to an incorrect conclusion. We define the significance level or size of the test, which we denote by  $\alpha$ , as the risk we are prepared to take in rejecting  $H_0$  when it is in fact true. We refer to this as an error of the first

type or a Type I error. We usually set  $\alpha$  to between 1 and 10 per cent, depending on the severity of the consequences of making such an error.

We also have to contend with the possibility of not rejecting  $H_0$  when it is in fact false and  $H_1$  is true. This is an error of the second type or Type II error, and the probability of this occurring is denoted by  $\beta$ .

Thus in testing any statistical hypothesis, there are four possible situations which determine whether our decision is correct or in error. These situations are illustrated as follows:

		Situation	
		$H_0$ is true	$H_0$ is false
Conclusion	$H_0$ is not rejected	Correct decision	Type II error
	$H_0$ is rejected	Type I error	Correct decision

**Step 5** Many textbooks stop after step 4, but it is instructive to consider just where in the critical region the calculated value of  $T$  lies. If it lies close to the boundary of the critical region we may say that there is some evidence that  $H_0$  should be rejected, whereas if it is at the other end of the region we would conclude there was considerable evidence. In other words, the actual significance level of  $T$  can provide useful information beyond the fact that  $T$  lies in the critical region.

In general, the statistical test provides information from which we can judge the significance of the increase (or decrease) in any result. If our conclusion shows that the increase is not significant then it will be necessary to confirm that the experiment had a fair chance of establishing an increase had there been one present to establish.

In order to do this we generally turn to the power function of the test, which is usually computed before the experiment is performed, so that if it is insufficiently powerful then the design can be changed. The power function is the probability of detecting a genuine increase underlying the observed increase in the result, plotted as a function of the genuine increase, and therefore the experimental design must be chosen so that the probability of detecting the increase is high. Also the choice among several possible designs should be made in favour of the experiment with the highest power. For a given experiment testing a specific hypothesis, the power of the test is given by  $1 - \beta$ .

Having discussed the importance of the power function in statistical tests we would now like to introduce the concept of robustness. The term 'robust' was first introduced in 1953 to denote a statistical procedure which is insensitive to departures from the assumptions underlying the model on which it is based. Such procedures are in common use, and several studies of robustness have been carried out in the field of 'analysis of variance'. The assumptions usually associated with analysis of variance are that the errors in the measurements (a) are normally distributed, (b) are statistically independent and (c) have equal variances.

Most of the parametric tests considered in this book have made the assumption that the populations involved have normal distributions. Therefore a test should only be carried out when the normality assumption is not violated. It is also a necessary part of

the test to check the effect of applying these tests when the assumption of normality is violated.

In parametric tests the probability distribution of the test statistic under the null hypothesis can only be calculated by an additional assumption on the frequency distribution of the population. If this assumption is not true then the test loses its validity. However, in some cases the deviation of the assumption has only a minor influence on the statistical test, indicating a robust procedure. A parametric test also offers greater discrimination than the corresponding distribution-free test.

For the non-parametric test no assumption has to be made regarding the frequency distribution and therefore one can use estimates for the probability that any observation is greater than a predetermined value.

Neave (1976b) points out that it was the second constraint in step 2, namely that the probability distribution of the test statistic should be calculable, which led to the growth of the number of non-parametric tests. An inappropriate assumption of normality had often to be built into the tests. In fact, when comparing two samples, we need only look at the relative ranking of the sample members. In this way under  $H_0$  all the rank sequences are equally likely to occur, and so it became possible to generate any required significance level comparatively easily.

Two simple tests based on this procedure are the Wald–Wolfowitz number of runs test and the median test proposed by Mood, but these are both low in power. The Kolmogorov–Smirnov test has higher power but is more difficult to execute. A test which is extremely powerful and yet still comparatively easy to use is the Wilcoxon–Mann–Whitney test. Many others are described in later pages of this book.

# EXAMPLES OF TEST PROCEDURES

## Test 1 Z-test for a population mean (variance known)

### Hypotheses and alternatives

1.  $H_0: \mu = \mu_0$   
 $H_1: \mu \neq \mu_0$
2.  $H_0: \mu = \mu_0$   
 $H_1: \mu > \mu_0$

### Test statistics

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

$n$  is sample size

$\bar{x}$  is sample mean

$\sigma$  is population standard deviation

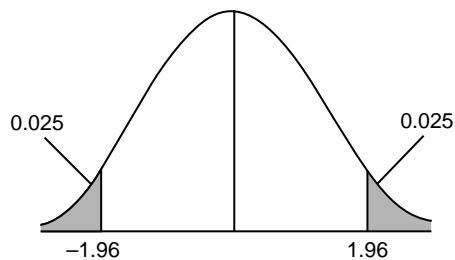
### When used

When the population variance  $\sigma^2$  is known and the population distribution is normal.

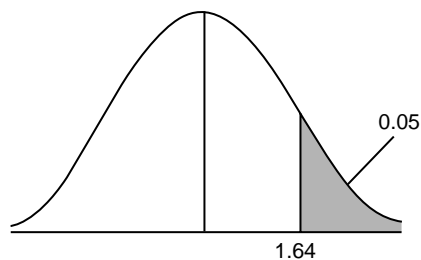
### Critical region

Using  $\alpha = 0.05$  [see Table 1]

1.



2.



### Data

$$\begin{aligned} H_0: \mu_0 &= 4.0 \\ n &= 9, \bar{x} = 4.6 \\ \sigma &= 1.0 \\ \therefore Z &= 1.8 \end{aligned}$$

### Conclusion

1. Do not reject  $H_0$  [see Table 1].
2. Reject  $H_0$

**Test 3 Z-test for two population means (variances known and unequal)****Hypotheses and alternatives**

1.  $H_0: \mu_1 - \mu_2 = \mu_0$   
 $H_1: \mu_1 - \mu_2 \neq \mu_0$
2.  $H_0: \mu_1 - \mu_2 = \mu_0$   
 $H_1: \mu_1 - \mu_2 > \mu_0$

**Test statistics**

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - \mu_0}{\left( \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \right)^{\frac{1}{2}}}$$

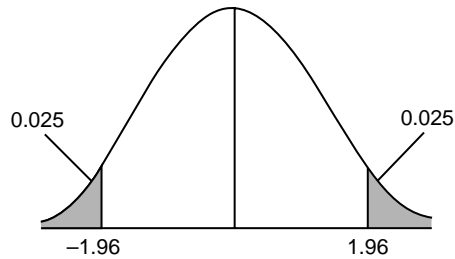
**When used**

When the variances of both populations,  $\sigma_1^2$  and  $\sigma_2^2$ , are known. Populations are normally distributed.

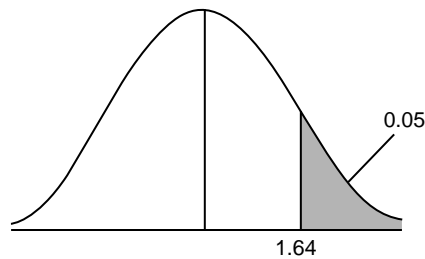
**Critical region**

Using  $\alpha = 0.05$  [see Table 1]

1.



2.

**Data**

$$\begin{aligned}
 H_0: \mu_1 - \mu_2 &= 0 \\
 n_1 &= 9, n_2 = 16 \\
 \bar{x}_1 &= 1.2, \bar{x}_2 = 1.7 \\
 \sigma_1^2 &= 1, \sigma_2^2 = 4 \\
 \therefore Z &= -0.832
 \end{aligned}$$

**Conclusion**

1. Do not reject  $H_0$ .
2. Do not reject  $H_0$ .

**Test 7 t-test for a population mean (variance unknown)**

**Hypotheses and alternatives**

1.  $H_0: \mu = \mu_0$   
 $H_1: \mu \neq \mu_0$
2.  $H_0: \mu = \mu_0$   
 $H_1: \mu > \mu_0$

**Test statistics**

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

where

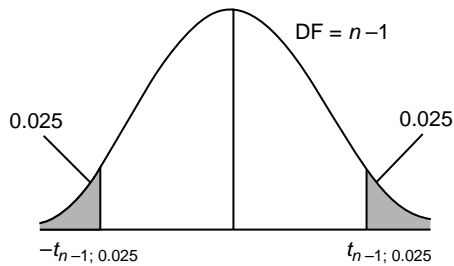
$$s^2 = \frac{\sum (x - \bar{x})^2}{n - 1}.$$

**When used**

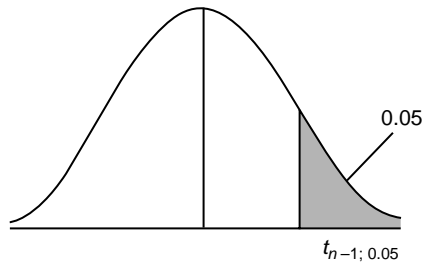
If  $\sigma^2$  is not known and the estimate  $s^2$  of  $\sigma^2$  is based on a small sample (i.e.  $n < 20$ ) and a normal population.

**Critical region and degrees of freedom**

1.



2.



**Data**

$$\begin{aligned} H_0: \mu_0 &= 4.0 \\ n &= 9, \bar{x} = 3.1 \\ s &= 1.0 \\ \therefore t &= -2.7 \end{aligned}$$

**Conclusion**

1.  $t_{8;0.025} = \pm 2.306$  [see Table 2].  
Reject  $H_0$ .
2.  $t_{8;0.05} = -1.860$  (left-hand side) [see Table 2].  
Reject  $H_0$ .



**Test 8 t-test for two population means (variance unknown but equal)****Hypotheses and alternative**

1.  $H_0: \mu_1 - \mu_2 = \mu_0$   
 $H_1: \mu_1 - \mu_2 \neq \mu_0$
2.  $H_0: \mu_1 - \mu_2 = \mu_0$   
 $H_1: \mu_1 - \mu_2 > \mu_0$

**Test statistics**

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s \left( \frac{1}{n_1} + \frac{1}{n_2} \right)^{\frac{1}{2}}}$$

where

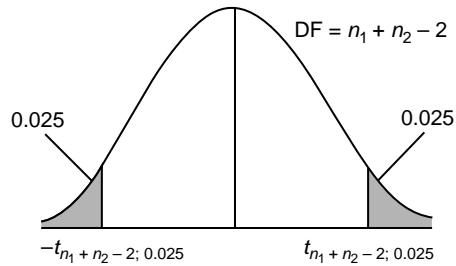
$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}.$$

**When used**

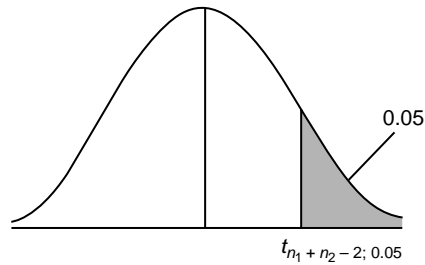
Given two samples from normal populations with equal variances  $\sigma^2$ .

**Critical region and degrees of freedom**

1.



2.

**Data**

$$\begin{aligned} H_0: \mu_1 - \mu_2 &= 0 \\ n_1 &= 16, n_2 = 16 \\ \bar{x}_1 &= 5.0, \bar{x}_2 = 4 \\ s &= 2.0 \\ \therefore t &= 1.414 \end{aligned}$$

**Conclusion**

1.  $t_{30;0.025} = \pm 2.042$  [see Table 2].  
Do not reject  $H_0$ .
2.  $t_{30;0.05} = 1.697$  [see Table 2].  
Do not reject  $H_0$ .

# Test 10 Method of paired comparisons

## Hypotheses and alternatives

1.  $H_0: \mu_d = 0$   
 $H_1: \mu_d \neq 0$
2.  $H_0: \mu_d = 0$   
 $H_1: \mu_d > 0$

## Test statistics

$$t = \frac{\bar{d} - \mu_d}{s/\sqrt{n}}$$

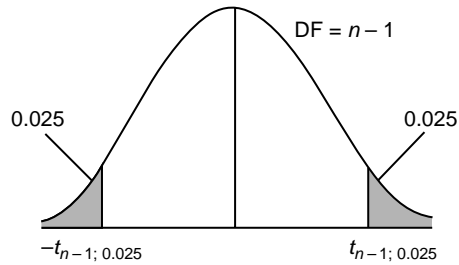
where  $d_i = x_i - y_i$ , the difference in the  $n$  paired observations.

## When used

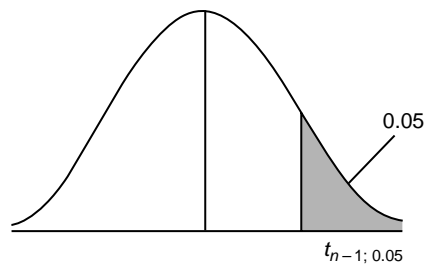
When an experiment is arranged so that each observation in one sample can be ‘paired’ with a value from the second sample and the populations are normally distributed.

## Critical region and degrees of freedom

1.



2.



## Data

$$\begin{aligned} n_1 &= 16, \bar{d} = 1.0 \\ s &= 1.0 \\ \therefore t &= 4.0 \end{aligned}$$

## Conclusion

1.  $t_{15;0.025} = \pm 2.131$  [see Table 2].  
Reject  $H_0$ .
2.  $t_{15;0.05} = 1.753$  [see Table 2].  
Reject  $H_0$ .

**Test 15  $\chi^2$ -test for a population variance****Hypotheses and alternatives**

1.  $H_0: \sigma^2 = \sigma_0^2$   
 $H_1: \sigma^2 \neq \sigma_0^2$
2.  $H_0: \sigma^2 = \sigma_0^2$   
 $H_1: \sigma^2 > \sigma_0^2$

**Test statistics**

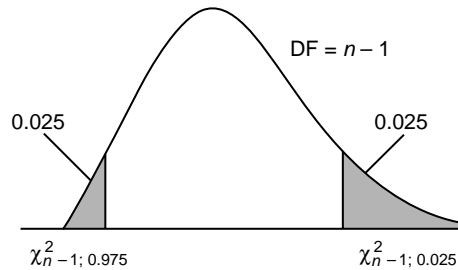
$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

**When used**

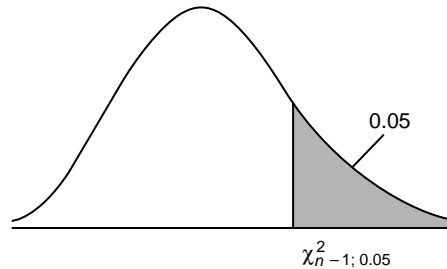
Given a sample from a normal population with unknown variance.

**Critical region and degrees of freedom**

1.



2.

**Data**

$$\begin{aligned} H_0: \sigma^2 &= 4.0 \\ n_1 &= 17, s^2 = 7.0 \\ \therefore \chi^2 &= 28.0 \end{aligned}$$

**Conclusion**

1.  $\chi^2_{16; 0.025} = 28.85$  [see Table 5].  
 $\therefore$  Do not reject  $H_0$ .
2.  $\chi^2_{16; 0.05} = 26.30$  [see Table 5].  
 $\therefore$  Reject  $H_0$ .

**Test 16 F-test for two population variances**

**Hypotheses and alternatives**

1.  $H_0: \sigma_1^2 = \sigma_2^2$   
 $H_1: \sigma_1^2 \neq \sigma_2^2$
2.  $H_0: \sigma_1^2 = \sigma_2^2$   
 $H_1: \sigma_1^2 > \sigma_2^2$

**Test statistics**

$$F = \frac{s_1^2}{s_2^2}, \quad (s_1^2 > s_2^2)$$

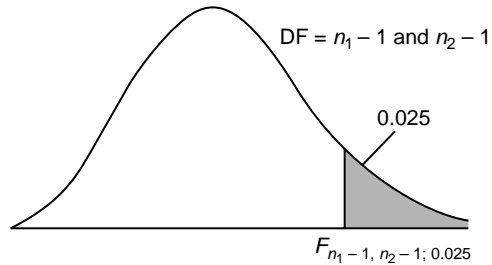
where  $s_1^2$  and  $s_2^2$  are sample variances.  
(If, in 2,  $s_1^2 < s_2^2$ , do not reject  $H_0$ .)

**When used**

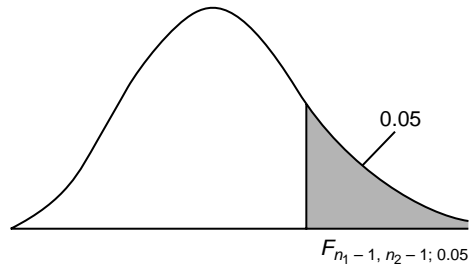
Given two sample with unknown variances  $\sigma_1^2$  and  $\sigma_2^2$  and normal populations.

**Critical region and degrees of freedom**

1.



2.



**Data**

$$\begin{aligned} H_0: \sigma_1^2 &= \sigma_2^2 \\ n_1 &= 11, n_2 = 16 \\ s_1^2 &= 6.0, s_2^2 = 3.0 \\ \therefore F &= 2.0 \end{aligned}$$

**Conclusion**

1.  $F_{10, 15; 0.025} = 3.06$ .  
Do not reject  $H_0$ .
2.  $F_{10, 15; 0.05} = 2.54$ . [see Table 3].  
Do not reject  $H_0$ .

**Test 37  $\chi^2$ -test for goodness of fit****Hypotheses and alternatives**

Goodness of fit for Poisson distribution with known mean  $\lambda$

**Test statistics**

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

$O_i$  is the  $i$ th observed frequency,  $i = 1$  to  $k$ ;  
 $E_i$  is expected frequency,  
 where  $E_i$  must be  $> 5$ .

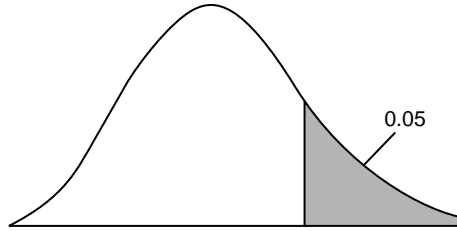
**When used**

To compare observed frequencies against those obtained under assumptions about the parent populations.

**Critical region and degrees of freedom**

Using  $\alpha = 0.05$  [see Table 5]

DF: variable, normally one less than the number of frequency comparisons ( $k$ ) in the summation in the test statistic.

**Data**

$H_0$ : Distribution.  
 Poisson with  $\lambda = 2$ .

$x_i$	$O_i$	$E_i$
0	10	13.5
1	27	27.0
2	30	27.0
3	19	18.0
4	8	9.0
$\geq 5$	6	5.5

$$\therefore \chi^2 = 1.45$$

**Conclusion**

$\nu = 5$ .

$\chi_{5;0.05}^2 = 11.07$  [see Table 5].

Do not reject  $H_0$ .

**Test 44**  $\chi^2$ -test for independence

**Hypotheses and alternatives**

Contingency table

**Test statistics**

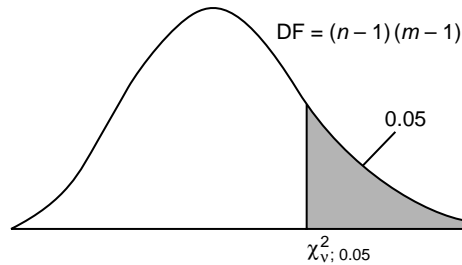
$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} \text{ [see Table 5]}$$

**When used**

Given a bivariate frequency table for attributes with  $m$  and  $n$  levels.

**Critical region and degrees of freedom**

Using  $\alpha = 0.05$  [see Table 5]



**Data**

Grade	Machine				Total
	I		II		
	$O_i$	$E_i$	$O_i$	$E_i$	
A	3	4	7	6	10
B	9	8	11	12	20
C	8	8	12	12	20
Total	20	20	30	30	50

$$\chi^2 = 0.625$$

**Conclusion**

$$\chi^2_{2; 0.05} = 5.99 \text{ [see Table 5].}$$

Do not reject  $H_0$ . The grades are independent of the machine.

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# CLASSIFICATION OF TESTS

	Test numbers		
	1 sample	2 samples	$K$ samples
<b>For linear data</b>			
<b>Parametric classical tests</b>			
for central tendency	1, 7, 19	2, 3, 8, 9, 10, 18	22, 26, 27, 28, 29, 30, 77, 78, 79, 80, 87
for proportion	4	5, 6, 25	—
for variability	15, 21, 24, 34	16, 17	31, 32, 86
for distribution functions	20, 33, 75, 88, 89, 94	—	76
for association	11, 12, 13, 81, 82	14, 23, 84, 92	85
for probability	83	—	—
<b>Parametric tests</b>			
for distribution function	35, 37	36, 39, 40	38, 41, 42, 43, 44
<b>Distribution-free tests</b>			
for central tendency	45, 47	46, 48, 50, 52, 93	51, 54, 55, 56, 57
for variability	—	53	—
for distribution functions	—	49	—
for association	58, 59	—	72, 73, 74
for randomness	63, 64, 65, 66, 67, 69, 70, 71	68	—
<b>Sequential tests</b>			
central tendency	60, 90	—	—
variability	61	—	—
for proportion	62	—	—
for ratio	91	—	—
	Test numbers		
	1 sample	2 samples	$K$ samples
<b>For circular data</b>			
<b>Parametric tests</b>			
for randomness	95	—	—
for distribution function	96	—	—
for central tendency	—	97, 98	—
for variability	—	99	100



# THE TESTS

## Test 1 Z-test for a population mean (variance known)

### Object

To investigate the significance of the difference between an assumed population mean  $\mu_0$  and a sample mean  $\bar{x}$ .

### Limitations

1. It is necessary that the population variance  $\sigma^2$  is known. (If  $\sigma^2$  is not known, see the  $t$ -test for a population mean (Test 7).)
2. The test is accurate if the population is normally distributed. If the population is not normal, the test will still give an approximate guide.

### Method

From a population with assumed mean  $\mu_0$  and known variance  $\sigma^2$ , a random sample of size  $n$  is taken and the sample mean  $\bar{x}$  calculated. The test statistic

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

may be compared with the standard normal distribution using either a one- or two-tailed test, with critical region of size  $\alpha$ .

### Example

For a particular range of cosmetics a filling process is set to fill tubs of face powder with 4 gm on average and standard deviation 1 gm. A quality inspector takes a random sample of nine tubs and weighs the powder in each. The average weight of powder is 4.6 gm. What can be said about the filling process?

A two-tailed test is used if we are concerned about over- and under-filling.

In this  $Z = 1.8$  and our acceptance range is  $-1.96 < Z < 1.96$ , so we do not reject the null hypothesis. That is, there is no reason to suggest, for this sample, that the filling process is not running on target.

On the other hand if we are only concerned about over-filling of the cosmetic then a one-tailed test is appropriate. The acceptance region is now  $Z < 1.645$ . Notice that we have fixed our probability, which determines our acceptance or rejection of the null hypothesis, at 0.05 (or 10 per cent) whether the test is one- or two-tailed. So now we reject the null hypothesis and can reasonably suspect that we are over-filling the tubs with cosmetic.

Quality control inspectors would normally take regular small samples to detect the departure of a process from its target, but the basis of this process is essentially that suggested above.

**Numerical calculation**

$$\mu_0 = 4.0, n = 9, \bar{x} = 4.6, \sigma = 1.0$$

$$Z = 1.8$$

Critical value  $Z_{0.05} = 1.96$  [Table 1].

$H_0: \mu = \mu_0, H_1: \mu \neq \mu_0$ . (Do not reject the null hypothesis  $H_0$ .)

$H_0: \mu = \mu_0, H_1: \mu > \mu_0$ . (Reject  $H_0$ .)

## Test 2 Z-test for two population means (variances known and equal)

### Object

To investigate the significance of the difference between the means of two populations.

### Limitations

1. Both populations must have equal variances and this variance  $\sigma^2$  must be known. (If  $\sigma^2$  is not known, see the  $t$ -test for two population means (Test 8).)
2. The test is accurate if the populations are normally distributed. If not normal, the test may be regarded as approximate.

### Method

Consider two populations with means  $\mu_1$  and  $\mu_2$ . Independent random samples of size  $n_1$  and  $n_2$  are taken which give sample means  $\bar{x}_1$  and  $\bar{x}_2$ . The test statistic

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma \left( \frac{1}{n_1} + \frac{1}{n_2} \right)^{\frac{1}{2}}}$$

may be compared with the standard normal distribution using either a one- or two-tailed test.

### Example

Two teams of financial sales persons are compared to see if it is likely that the instruction each has received could have led to differing success rates. A sample of nine transactions (which involves the whole team) yields an average success rate of 1.2. Similarly a sample of 16 transactions for the second team yields a success rate of 1.7. The variances for both teams are equal to 2.0750 (standard deviation 1.4405). The success rate is calculated using a range of output measures for a transaction.

If we are only interested to know of a difference between the two teams then a two-tailed test is appropriate. In this case we accept the null hypothesis and can assume that both teams are equally successful. This is because our acceptance region is  $-1.96 < Z < 1.96$  and we have computed a  $Z$  value, for this sample, of  $-0.833$ .

On the other hand, if we suspect that the first team had received better training than the second team we would use a one-tailed test.

For our example, here, this is certainly not the case since our  $Z$  value is negative. Our acceptance region is  $Z < 1.645$ . Since the performance is in the wrong direction we don't even need to perform a calculation. Notice that we are not doing all possible combination of tests so that we can find a significant result. Our test is based on our design of the 'experiment' or survey planned before we collect any data. Our data do not have a bearing on the form of the testing.



**Numerical calculation**

$$n_1 = 9, n_2 = 16, \bar{x}_1 = 1.2, \bar{x}_2 = 1.7, \sigma = 1.4405, \sigma^2 = 2.0750$$

$$Z = -0.833$$

Critical value  $Z_{0.05} = 1.96$  [Table 1].

$H_0: \mu_1 - \mu_2 = 0, H_1: \mu_1 - \mu_2 \neq 0$ . (Do not reject  $H_0$ .)

$H_1: \mu_1 - \mu_2 = 0, H_1: \mu_1 - \mu_2 > 0$ . (Do not reject  $H_0$ .)

### Test 3 Z-test for two population means (variances known and unequal)

#### Object

To investigate the significance of the difference between the means of two populations.

#### Limitations

1. It is necessary that the two population variances be known. (If they are not known, see the  $t$ -test for two population means (Test 9).)
2. The test is accurate if the populations are normally distributed. If not normal, the test may be regarded as approximate.

#### Method

Consider two populations with means  $\mu_1$  and  $\mu_2$  and variances  $\sigma_1^2$  and  $\sigma_2^2$ . Independent random samples of size  $n_1$  and  $n_2$  are taken and sample means  $\bar{x}_1$  and  $\bar{x}_2$  are calculated. The test statistic

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\left( \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \right)^{\frac{1}{2}}}$$

may be compared with the standard normal distribution using either a one- or two-tailed test.

#### Example

Brand A of a jumbo-sized pack of potato crisp is known to have a more variable weight than brand B of potato crisp. Population variances are  $0.000576 \text{ gm}^2$  and  $0.001089 \text{ gm}^2$ , respectively. The respective means for samples of size 13 and 8 are 80.02 gm and 79.98 gm.

Is there a difference between the two brands in terms of the weights of the jumbo packs? We do not have any pre-conceived notion of which brand might be 'heavier' so we use a two-tailed test. Our acceptance region is  $-1.96 < Z < 1.96$  and our calculated  $Z$  value of 2.98. We therefore reject our null hypothesis and can conclude that there is a difference with brand B yielding a heavier pack of crisps.

#### Numerical calculation

$$n_1 = 13, n_2 = 8, \bar{x}_1 = 80.02, \bar{x}_2 = 79.98, \sigma_1^2 = 0.000576, \sigma_2^2 = 0.001089$$

$$Z = 2.98$$

Critical value  $Z_{0.05} = 1.96$  [Table 1].

Reject the null hypothesis of no difference between means.

## Test 4 Z-test for a proportion (binomial distribution)

### Object

To investigate the significance of the difference between an assumed proportion  $p_0$  and an observed proportion  $p$ .

### Limitations

The test is approximate and assumes that the number of observations in the sample is sufficiently large (i.e.  $n \geq 30$ ) to justify the normal approximation to the binomial.

### Method

A random sample of  $n$  elements is taken from a population in which it is assumed that a proportion  $p_0$  belongs to a specified class. The proportion  $p$  of elements in the sample belonging to this class is calculated. The test statistic is

$$Z = \frac{|p - p_0| - 1/2n}{\left\{ \frac{p_0(1 - p_0)}{n} \right\}^{\frac{1}{2}}}$$

This may be compared with a standard normal distribution using either a one- or two-tailed test.

### Example

The pass rate for a national statistics test has been 0.5, or 50 per cent for some years. A random sample of 100 papers from independent (or non-college based) students yields a pass rate of 40 per cent. Does this show a significant difference? Our computed  $Z$  is  $-2.0$  and our acceptance region is  $-1.96 < Z < 1.96$ . So we reject the null hypothesis and conclude that there is a difference in pass rates. In this case, the independent students fare worse than those attending college. While we might have expected this, there are other possible factors that could point to either an increase or decrease in the pass rate. Our two-tailed test affirms our ignorance of the possible direction of a difference, if one exists.

### Numerical calculation

$$n = 100, p = 0.4, p_0 = 0.5$$

$$Z = -2.1$$

Critical value  $Z_{0.05} = \pm 1.96$  [Table 1].

Reject the null hypothesis of no difference in proportions.

## Test 5 Z-test for the equality of two proportions (binomial distribution)

### Object

To investigate the assumption that the proportions  $\pi_1$  and  $\pi_2$  of elements from two populations are equal, based on two samples, one from each population.

### Limitations

The test is approximate and assumes that the number of observations in the two samples is sufficiently large (i.e.  $n_1, n_2 \geq 30$ ) to justify the normal approximation to the binomial.

### Method

It is assumed that the populations have proportions  $\pi_1$  and  $\pi_2$  with the same characteristic. Random samples of size  $n_1$  and  $n_2$  are taken and respective proportions  $p_1$  and  $p_2$  calculated. The test statistic is

$$Z = \frac{(p_1 - p_2)}{\left\{ P(1 - P) \left( \frac{1}{n_1} + \frac{1}{n_2} \right) \right\}^{\frac{1}{2}}}$$

where

$$P = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2}.$$

Under the null hypothesis that  $\pi_1 = \pi_2$ ,  $Z$  is approximately distributed as a standard normal deviate and the resulting test may be either one- or two-tailed.

### Example

Two random samples are taken from two populations, which are two makes of clock mechanism produced in different factories. The first sample of size 952 yielded the proportion of clock mechanisms, giving accuracy not within fixed acceptable limits over a period of time, to be 0.325 per cent. The second sample of size 1168 yielded 5.73 per cent. What can be said about the two populations of clock mechanisms, are they significantly different? Again, we do not have any pre-conceived notion of whether one mechanism is better than the other, so a two-tailed test is employed.

With a  $Z$  value of  $-6.93$  and an acceptance region of  $-1.96 < Z < 1.96$ , we reject the null hypothesis and conclude that there is significant difference between the mechanisms in terms of accuracy. The second mechanism is significantly less accurate than the first.

### Numerical calculation

$$n_1 = 952, n_2 = 1168, p_1 = 0.00325, p_2 = 0.0573$$

$$Z = -6.93$$

Critical value  $Z_{0.05} = \pm 1.96$  [Table 1].

Reject the null hypothesis.

## Test 6 Z-test for comparing two counts (Poisson distribution)

### Object

To investigate the significance of the difference between two counts.

### Limitations

The test is approximate and assumes that the counts are large enough for the normal approximation to the Poisson to apply.

### Method

Let  $n_1$  and  $n_2$  be the two counts taken over times  $t_1$  and  $t_2$ , respectively. Then the two average frequencies are  $R_1 = n_1/t_1$  and  $R_2 = n_2/t_2$ . To test the assumption of equal average frequencies we use the test statistic

$$Z = \frac{(R_1 - R_2)}{\left(\frac{R_1}{t_1} + \frac{R_2}{t_2}\right)^{\frac{1}{2}}}.$$

This may be compared with a standard normal distribution using either a one-tailed or two-tailed test.

### Example

Two traffic roundabouts are compared for intensity of traffic at non-peak times with steady conditions. Roundabout one has 952/2 arrivals over 11 minutes and roundabout two has 1168/2 arrivals over 15 minutes. The arrival rates, per minute, are therefore 476/11 (43.27) and 584/15 (38.93) respectively.

What do these results say about the two arrival rates or frequency taken over the two time intervals? We calculate a Z value of 2.4 and have an acceptance region of  $-1.96 < Z < 1.96$ . So we reject the null hypothesis of no difference between the two rates. Roundabout one has an intensity of arrival significantly higher than roundabout two.

### Numerical calculation

$$n_1 = 952, n_2 = 1168, R_1 = \frac{n_1}{t_1} = 43.27, R_2 = \frac{n_2}{t_2} = 38.93$$

$$t_1 = 22, t_2 = 30$$

$$Z = \frac{(R_1 - R_2)}{\left(\frac{R_1}{t_1} + \frac{R_2}{t_2}\right)^{\frac{1}{2}}} = \frac{4.34}{(3.26)^{\frac{1}{2}}} = \frac{4.34}{1.81} = 2.40$$

Critical value  $Z_{0.05} = 1.96$  [Table 1].

Reject the null hypothesis of no difference between the counts.

## Test 7 *t*-test for a population mean (variance unknown)

### Object

To investigate the significance of the difference between an assumed population mean  $\mu_0$  and a sample mean  $\bar{x}$ .

### Limitations

1. If the variance of the population  $\sigma^2$  is known, a more powerful test is available: the *Z*-test for a population mean (Test 1).
2. The test is accurate if the population is normally distributed. If the population is not normal, the test will give an approximate guide.

### Method

From a population with assumed mean  $\mu_0$  and unknown variance, a random sample of size  $n$  is taken and the sample mean  $\bar{x}$  calculated as well as the sample standard deviation using the formula

$$s = \left\{ \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1} \right\}^{\frac{1}{2}}.$$

The test statistic is

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

which may be compared with Student's *t*-distribution with  $n - 1$  degrees of freedom. The test may be either one-tailed or two-tailed.

### Example

A sample of nine plastic nuts yielded an average diameter of 3.1 cm with estimated standard deviation of 1.0 cm. It is assumed from design and manufacturing requirements that the population mean of nuts is 4.0 cm. What does this say about the mean diameter of plastic nuts being produced? Since we are concerned about both under- and over-sized nuts (for different reasons) a two-tailed test is appropriate.

Our computed *t* value is  $-2.7$  and acceptance region  $-2.3 < t < 2.3$ . We reject the null hypothesis and accept the alternative hypothesis of a difference between the sample and population means. There is a significant difference (a drop in fact) in the mean diameters of plastic nuts (i.e. between the sample and population).

**Numerical calculation**

$$\mu_0 = 4, n = 9, \bar{x} = 3.1, s = 1.0, v = n - 1$$

$$t = -2.7$$

Critical value  $t_{8; 0.025} = \pm 2.3$  [Table 2].

$H_0: \mu = \mu_0, H_1: \mu \neq \mu_0$ . (Reject  $H_0$ .)

## Test 8 *t*-test for two population means (variances unknown but equal)

### Object

To investigate the significance of the difference between the means of two populations.

### Limitations

1. If the variance of the populations is known, a more powerful test is available: the *Z*-test for two population means (Test 2).
2. The test is accurate if the populations are normally distributed. If the populations are not normal, the test will give an approximate guide.

### Method

Consider two populations with means  $\mu_1$  and  $\mu_2$ . Independent random samples of size  $n_1$  and  $n_2$  are taken from which sample means  $\bar{x}_1$  and  $\bar{x}_2$  together with sums of squares

$$s_1^2 = \sum_{i=1}^{n_1} (x_i - \bar{x}_1)^2$$

and

$$s_2^2 = \sum_{i=1}^{n_2} (x_i - \bar{x}_2)^2$$

are calculated. The best estimate of the population variance is found as  $s^2 = [(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2]/(n_1 + n_2 - 2)$ . The test statistic is

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s \left( \frac{1}{n_1} + \frac{1}{n_2} \right)^{\frac{1}{2}}}$$

which may be compared with Student's *t*-distribution with  $n_1 + n_2 - 2$  degrees of freedom. The test may be either one-tailed or two-tailed.

### Example

Two snack foods are made and sold in 30 gm packets. Random samples of size 12 are taken from the production line of each snack food and means and variances obtained viz.: mean1 31.75 gm, variance1 112.25 gm<sup>2</sup>; mean2 28.67 gm, variance2 66.64 gm<sup>2</sup>. What can be said about the two production processes in relation to the weight of packets?

We use a two-tailed test and find that *t* is 0.798. Our acceptance region is  $-2.07 < t < 2.07$  and so we accept our null hypothesis. So we can conclude that the mean weight of packs from the two production lines is the same.



**Numerical calculation**

$$n_1 = 12, n_2 = 12, \bar{x}_1 = 31.75, \bar{x}_2 = 28.67, \nu = n_1 + n_2 - 2$$

$$s_1^2 = 112.25, s_2^2 = 66.64$$

$$s^2 = 89.445$$

$$t = 0.798, \nu = 12 + 12 - 2 = 22$$

Critical value  $t_{22; 0.025} = 2.07$  [Table 2].

Reject the alternative hypothesis.

## Test 9 *t*-test for two population means (variances unknown and unequal)

### Object

To investigate the significance of the difference between the means of two populations.

### Limitations

1. If the variances of the populations are known, a more powerful test is available: the *Z*-test for two population means (Test 3).
2. The test is approximate if the populations are normally distributed or if the sample sizes are sufficiently large.
3. The test should only be used to test the hypothesis  $\mu_1 = \mu_2$ .

### Method

Consider two populations with means  $\mu_1$  and  $\mu_2$ . Independent random samples of size  $n_1$  and  $n_2$  are taken from which sample means  $\bar{x}_1$  and  $\bar{x}_2$  and variances

$$s_1^2 = \frac{\sum_{i=1}^{n_1} (x_i - \bar{x}_1)^2}{n_1 - 1} \quad \text{and} \quad s_2^2 = \frac{\sum_{i=1}^{n_2} (x_i - \bar{x}_2)^2}{n_2 - 1}$$

are calculated. The test statistic is

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\left( \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^{\frac{1}{2}}}$$

which may be compared with Student's *t*-distribution with degrees of freedom given by

$$v = \left\lfloor \frac{\left\{ \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right\}^2}{\frac{s_1^4}{n_1^2(n_1 - 1)} + \frac{s_2^4}{n_2^2(n_2 - 1)}} \right\rfloor.$$

### Example

Two financial organizations are about to merge and, as part of the rationalization process, some consideration is to be made of service duplication. Two sales teams responsible for essentially identical products are compared by selecting samples from each and reviewing their respective profit contribution levels per employee over a period of two weeks. These are found to be 3166.00 and 2240.40 with estimated

variance of 6328.27 and 221 661.3 respectively. How do the two teams compare on performance?

We compute a  $t$  value of 5.72. Our acceptance region is  $-2.26 < t < 2.26$  so we reject the null hypothesis and accept the alternative. There is a significant difference between the two teams. Team 1 is more productive than team 2.

**Numerical calculation**

$$n_1 = 4, n_2 = 9, \bar{x}_1 = 3166.0, \bar{x}_2 = 2240.4, s_1^2 = 6328.67, s_2^2 = 221\,661.3$$

$$t = 5.72, \nu = 9 \text{ (rounded)}$$

Critical value  $t_{9; 0.025} = 2.26$  [Table 2].

Reject the null hypothesis.

## Test 10 *t*-test for two population means (method of paired comparisons)

### Object

To investigate the significance of the difference between two population means,  $\mu_1$  and  $\mu_2$ . No assumption is made about the population variances.

### Limitations

1. The observations for the two samples must be obtained in pairs. Apart from population differences, the observations in each pair should be carried out under identical, or almost identical, conditions.
2. The test is accurate if the populations are normally distributed. If not normal, the test may be regarded as approximate.

### Method

The differences  $d_i$  are formed for each pair of observations. If there are  $n$  such pairs of observations, we can calculate the variance of the differences by

$$s^2 = \sum_{i=1}^n \frac{(d_i - \bar{d})^2}{n-1}$$

Let the means of the samples from the two populations be denoted by  $\bar{x}_1$  and  $\bar{x}_2$ . Then the test statistic becomes

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - 0}{s/n^{1/2}}$$

which follows Student's *t*-distribution with  $n - 1$  degrees of freedom. The test may be either one-tailed or two-tailed.

### Example

To compare the efficacy of two treatments for a respiratory condition, ten patients are selected at random and the treatments are administered using an oral spray. The patients then perform a treadmill exercise until a maximum exercise rate is reached. The times for these are compared. A suitable period of time is ensured between treatments to prevent the effect of treatments to interact. Do the two treatments differ? In this case we do not expect one particular treatment to be superior to the other so a two-tailed test is used. We compute a *t* value of  $-0.11$  and have an acceptance region of  $-2.26 < t < 2.26$ . So we accept the null hypothesis of no difference between the two treatments. However, in such situations it is often the case that an improvement over an existing or original treatment is expected. Then a one-tailed test would be appropriate.

**Numerical calculation**

$$\bar{d} = \bar{x}_1 - \bar{x}_2 = -0.1, n = 10, v = n - 1, s = 2.9$$

$$t = -0.11, v = 9$$

Critical value  $t_{9;0.025} = 2.26$  [Table 2].

Do not reject the null hypothesis of no difference between means.

## Test 11 *t*-test of a regression coefficient

### Object

To investigate the significance of the regression coefficient of  $y$  on  $x$ .

### Limitations

The usual assumptions for regression should apply, namely:

1. the variable  $y$  follows a normal distribution for each value of  $x$ ;
2. the variance among the  $y$  values remains constant for any given values of  $x$ .

### Method

In order to estimate a linear regression of the form  $y = A + B(x - \bar{x})$ , a sample of  $n$  pairs of points  $(x_i, y_i)$  is required.  $B$  is called the regression coefficient, and to test the null hypothesis that this is equal to zero we first calculate the sample estimate

$$b = \frac{\sum x_i y_i - \frac{1}{n} \sum x_i \sum y_i}{\sum x_i^2 - \frac{1}{n} \left( \sum x_i \right)^2}.$$

The variance of the  $x$ s and the variance of the  $y$ s about the regression line are calculated as follows:

$$s_x^2 = \frac{\sum (x_i - \bar{x})^2}{n - 1} \quad \text{and} \quad s_{y \cdot x}^2 = \frac{\sum \{y_i - \bar{y} - b(x_i - \bar{x})\}^2}{n - 2}$$

where  $\bar{x}$  and  $\bar{y}$  are the means of the  $x$ s and  $y$ s, respectively. The test statistic becomes

$$t = \frac{bs_x}{s_{y \cdot x}} (n - 1)^{-\frac{1}{2}}$$

which follows Student's  $t$ -distribution with  $n - 2$  degrees of freedom. The test must be two-tailed since  $b$  may be positive or negative. However, the test may be one-tailed if the alternative hypothesis is directional.

### Example

In an investigation of the relationship between a reaction test, on a vehicle simulator, and a composite test a sample of 12 male subjects is selected. The composite test of reactions is a much cheaper alternative to a vehicle simulator test. A regression relationship is computed with regression coefficient 5.029 and  $t$  value 6.86. The acceptance region for the null hypothesis is  $-2.23 < t < 2.23$ . Since the computed  $t$  value lies outside the acceptance region we conclude that slope ( $b$  coefficient) is significantly greater than zero and a significant regression exists.

Notice that the test does not tell us how good a predictor  $x$  is of  $y$ , only that the regression is significant.

**Numerical calculation**

$$\sum x_i = 766, \sum y_i = 1700, \sum x_i^2 = 49\,068,$$

$$\sum y_i^2 = 246\,100, \sum x_i y_i = 109\,380$$

$$n = 12, \bar{x} = 68.83, \bar{y} = 141.67, \nu = n - 2$$

$$s_x^2 = 15.61, s_y^2 = 478.8, b = 5.029$$

$$s_{y \cdot x}^2 = 92.4$$

$$t = 6.86, \nu = 10$$

Critical value  $t_{10; 0.025} = \pm 2.23$  [Table 2].

Reject the null hypothesis.

## Test 12 *t*-test of a correlation coefficient

### Object

To investigate whether the difference between the sample correlation coefficient and zero is statistically significant.

### Limitations

It is assumed that the  $x$  and  $y$  values originate from a bivariate normal distribution, and that the relationship is linear. To test an assumed value of the population coefficient other than zero, refer to the  $Z$ -test for a correlation coefficient (Test 13).

### Method

Given a sample of  $n$  points  $(x_i, y_i)$  the correlation coefficient  $r$  is calculated from the formula

$$r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\left[ \sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2 \right]^{\frac{1}{2}}}$$

To test the null hypothesis that the population value of  $r$  is zero, the test statistic

$$t = \frac{r}{\sqrt{1 - r^2}} \cdot \sqrt{n - 2}$$

is calculated and this follows Student's  $t$ -distribution with  $n - 2$  degrees of freedom. The test will normally be two-tailed but in certain cases could be one-tailed.

### Example

In a study of the possible relationship between advertising on television and product preferences a panel of television viewers is selected. For two brands of toothpaste, one supermarket own brand and one popular brand, panel members were asked to score (on a scale from 1 to 20) their preference for each product. The correlation coefficient between brands was 0.32, which is modest but is it significantly greater than zero? The calculated  $t$  value is 1.35. Our acceptance region is  $-1.35 < t < 1.35$  so we accept the null hypothesis. So there is no association between the brands compared, which would suggest a clear preference for popular brand or own brand. Consumers are less likely to substitute own brand for popular brand when preferences appear not to be associated.

### Numerical calculation

$$n = 18, r = 0.32, v = n - 2$$

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} = \frac{0.32\sqrt{16}}{\sqrt{1-(0.32)^2}} = 1.35$$

Critical value  $t_{16;0.05} = 1.75$  [Table 2].

Do not reject the null hypothesis. *NB*: In this case the  $x$  and  $y$  variables are independent.



## Test 13 Z-test of a correlation coefficient

### Object

To investigate the significance of the difference between a correlation coefficient and a specified value  $\rho_0$ .

### Limitations

1. The  $x$  and  $y$  values originate from normal distributions.
2. The variance in the  $y$  values is independent of the  $x$  values.
3. The relationship is linear.

When these conditions cannot be met, the user should turn to the Kendall rank correlation test (Test 59).

### Method

With  $r$  as defined in the  $t$ -test of a correlation coefficient (Test 12), using the Fisher Z-transformation we have

$$Z_1 = \frac{1}{2} \log_e \left( \frac{1+r}{1-r} \right) = 1.1513 \log_{10} \left( \frac{1+r}{1-r} \right).$$

The distribution of  $Z_1$  is approximately normal with mean  $\mu_{Z_1}$  and standard deviation  $\sigma_{Z_1}$  where

$$\mu_{Z_1} = \frac{1}{2} \log_e \left( \frac{1+\rho_0}{1-\rho_0} \right) = 1.1513 \log_{10} \left( \frac{1+r}{1-r} \right)$$

$$\sigma_{Z_1} = \frac{1}{\sqrt{n-3}}.$$

The test statistic is now

$$Z = \frac{Z_1 - \mu_{Z_1}}{\sigma_{Z_1}}.$$

### Example

A market research company has assumed from previous research that the correlation between two brands in terms of consumer preference is 0.50. This value has a bearing on stocking levels in supermarkets since one brand will often substitute for another when the number on the shelves of one product runs out. A panel of 24 consumers produces a correlation on preference scores (based on a scale of 1 to 20) for the two brands of 0.75. Can we say that the correlation coefficient is at least 0.50? The Fisher Z-transformation value, calculated as 0.973, yields a test statistic of 1.94. The acceptance region is  $Z < 1.64$ . Since the calculated value is greater than the critical value we reject the null hypothesis. This means that the correlation coefficient is at least 0.50 and there is no need to re-evaluate supermarket stocking policy for these two products.

**Numerical calculation**

$$r = 0.75, \rho_0 = 0.50, n = 24$$

$$\mu_{Z_1} = 1.1513 \log_{10} 3 = 0.5493, Z_1 = 1.1513 \log_{10} \left( \frac{1 + 0.75}{1 - 0.75} \right)$$

$$= 0.9730 \text{ [Table 4]}$$

$$\sigma_{Z_1} = 0.2182$$

$$Z = \frac{Z_1 - \mu_{Z_1}}{\sigma_{Z_1}} = 1.94.$$

The critical value at  $\alpha = 0.10$  is 1.64 [Table 1].

The calculated value is greater than the critical value.

Reject the null hypothesis of no difference.

## Test 14 Z-test for two correlation coefficients

### Object

To investigate the significance of the difference between the correlation coefficients for a pair of variables occurring from two different samples and the difference between two specified values  $\rho_1$  and  $\rho_2$ .

### Limitations

1. The  $x$  and  $y$  values originate from normal distributions.
2. The variance in the  $y$  values is independent of the  $x$  values.
3. The relationships are linear.

### Method

Using the notation of the Z-test of a correlation coefficient, we form for the first sample

$$Z_1 = \frac{1}{2} \log_e \left( \frac{1 + r_1}{1 - r_1} \right) = 1.1513 \log_{10} \left( \frac{1 + r_1}{1 - r_1} \right)$$

which has mean  $\mu_{Z_1} = \frac{1}{2} \log_e [(1 + \rho_1)/(1 - \rho_1)]$  and variance  $\sigma_{Z_1} = 1/\sqrt{n_1 - 3}$ , where  $n_1$  is the size of the first sample;  $Z_2$  is determined in a similar manner. The test statistic is now

$$Z = \frac{(Z_1 - Z_2) - (\mu_{Z_1} - \mu_{Z_2})}{\sigma}$$

where  $\sigma = (\sigma_{Z_1}^2 + \sigma_{Z_2}^2)^{\frac{1}{2}}$ .  $Z$  is normally distributed with mean 0 and with variance 1.

### Example

A market research company is keen to categorize a variety of brands of potato crisp based on the correlation coefficients of consumer preferences. The market research company has found that if consumers' preferences for brands are similar then marketing programmes can be merged. Two brands of potato crisp are compared for two advertising regions. Panels are selected of sizes 28 and 35 for the two regions and correlation coefficients for brand preferences are 0.50 and 0.30 respectively. Are the two associations statistically different or can marketing programmes be merged? The calculated  $Z$  value is 0.8985 and the acceptance region for the null hypothesis is  $-1.96 < Z < 1.96$ . So we accept the null hypothesis and conclude that we can go ahead and merge the marketing programmes. This, of course, assumes that the correlation coefficient is a good measure to use for grouping market research programmes.

### Numerical calculation

$$n_1 = 28, n_2 = 35, r_1 = 0.50, r_2 = 0.30, \alpha = 0.05$$

$$Z_1 = 1.1513 \log_{10} \left( \frac{1 + r_1}{1 - r_1} \right) = 0.5493 \text{ [Table 4]}$$

$$Z_2 = 1.1513 \log_{10} \left( \frac{1+r_2}{1-r_2} \right) = 0.3095 \text{ [Table 4]}$$

$$\sigma = \left( \frac{1}{n_1 - 3} + \frac{1}{n_2 - 3} \right)^{\frac{1}{2}} = 0.2669$$

$$Z = \frac{0.5493 - 0.3095}{0.2669} = 0.8985$$

The critical value at  $\alpha = 0.05$  is 1.96 [Table 1].

Do not reject the null hypothesis.

## Test 15 $\chi^2$ -test for a population variance

### Object

To investigate the difference between a sample variance  $s^2$  and an assumed population variance  $\sigma_0^2$ .

### Limitations

It is assumed that the population from which the sample is drawn follows a normal distribution.

### Method

Given a sample of  $n$  values  $x_1, x_2, \dots, x_n$ , the values of

$$\bar{x} = \frac{\sum x_i}{n} \quad \text{and} \quad s^2 = \frac{\sum (x_i - \bar{x})^2}{n - 1}$$

are calculated. To test the null hypothesis that the population variance is equal to  $\sigma_0^2$  the test statistic  $(n - 1)s^2/\sigma_0^2$  will follow a  $\chi^2$ -distribution with  $n - 1$  degrees of freedom. The test may be either one-tailed or two-tailed.

### Example

A manufacturing process produces a fixed fluid injection into micro-hydraulic systems. The variability of the volume of injected fluid is critical and is set at 9 sq ml. A sample of 25 hydraulic systems yields a sample variance of 12 sq ml. Has the variability of the volume of fluid injected changed? The calculated chi-squared value is 32.0 and the 5 per cent critical value is 36.42. So we do not reject the null hypothesis of no difference. This means that we can still consider the variability to be set as required.

### Numerical calculation

$$\bar{x} = 70, \sigma_0^2 = 9, n = 25, s^2 = 12, v = 24$$

$$\chi^2 = (n - 1)s^2/\sigma_0^2 = \frac{24 \times 12}{9} = 32.0$$

$$\text{Critical value } \chi_{24; 0.05}^2 = 36.42 \text{ [Table 5].}$$

Do not reject the null hypothesis. The difference between the variances is not significant.

## Test 16 *F*-test for two population variances (variance ratio test)

### Object

To investigate the significance of the difference between two population variances.

### Limitations

The two populations should both follow normal distributions. (It is not necessary that they should have the same means.)

### Method

Given samples of size  $n_1$  with values  $x_1, x_2, \dots, x_{n_1}$  and size  $n_2$  with values  $y_1, y_2, \dots, y_{n_2}$  from the two populations, the values of

$$\bar{x} = \frac{\sum x_i}{n_1}, \quad \bar{y} = \frac{\sum y_i}{n_2}$$

and

$$s_1^2 = \frac{\sum (x_i - \bar{x})^2}{n_1 - 1}, \quad s_2^2 = \frac{\sum (y_i - \bar{y})^2}{n_2 - 1}$$

can be calculated. Under the null hypothesis that the variances of the two populations are equal the test statistic  $F = s_1^2/s_2^2$  follows the *F*-distribution with  $(n_1 - 1, n_2 - 1)$  degrees of freedom. The test may be either one-tailed or two-tailed.

### Example

Two production lines for the manufacture of springs are compared. It is important that the variances of the compression resistance (in standard units) for the two production lines are the same. Two samples are taken, one from each production line and variances are calculated. What can be said about the two population variances from which the two samples have been taken? Is it likely that they differ? The variance ratio statistic *F* is calculated as the ratio of the two variances and yields a value of  $0.36/0.087 = 4.14$ . The 5 per cent critical value for *F* is 5.41. We do not reject our null hypothesis of no difference between the two population variances. There is no significant difference between population variances.

### Numerical calculation

$$n_1 = 4, n_2 = 6, \sum x = 0.4, \sum x^2 = 0.30, s_1^2 = 0.087$$

$$\sum y = 0.06, \sum y^2 = 1.78, s_2^2 = 0.36$$

$$F_{3;5} = \frac{0.36}{0.087} = 4.14$$

Critical value  $F_{3.5;0.05} = 5.41$  [Table 3].

Do not reject the null hypothesis. The two population variances are not significantly different from each other.

## Test 17 *F*-test for two population variances (with correlated observations)

### Object

To investigate the difference between two population variances when there is correlation between the pairs of observations.

### Limitations

It is assumed that the observations have been performed in pairs and that correlation exists between the paired observations. The populations are normally distributed.

### Method

A random sample of size  $n$  yields the following pairs of observations  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ . The variance ratio  $F$  is calculated as in Test 16. Also the sample correlation  $r$  is found from

$$r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\left[ \sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2 \right]^{\frac{1}{2}}}.$$

The quotient

$$\gamma_F = \frac{F - 1}{[(F + 1)^2 - 4r^2F]^{\frac{1}{2}}}$$

provides a test statistic with degrees of freedom  $\nu = n - 2$ . The critical values for this test can be found in Table 6. Here the null hypothesis is  $\sigma_1^2 = \sigma_2^2$ , when the population correlation is not zero. Here  $F$  is greater than 1.

### Example

A researcher tests a sample panel of television viewers on their support for a particular issue prior to a focus group, during which the issue is discussed in some detail. The panel members are then asked the same questions after the discussion. The pre-discussion view is  $x$  and the post-discussion view is  $y$ . The question, here, is ‘has the focus group altered the variability of responses?’

We find the test statistic,  $F$ , is 0.796. Table 6 gives us a 5 per cent critical value of 0.811. For this test, since the calculated value is greater than the critical value, we do not reject the null hypothesis of no difference between variances. Hence the focus group has not altered the variability of responses.

**Numerical calculation**

$$n_1 = n_2 = 6, \sum x = 0.4, \sum x^2 = 0.30, s_1^2 = 0.087$$

$$\sum y = 0.06, \sum y^2 = 1.78, s_2^2 = 0.36, F = \frac{s_2^2}{s_1^2} = 4.14, r = 0.811$$

$$\begin{aligned} \gamma_F &= \frac{F - 1}{[(F + 1)^2 - 4r^2F]^{\frac{1}{2}}} = \frac{4.14 - 1}{[(5.14)^2 - 4r^2 \cdot 4.14]^{\frac{1}{2}}} \\ &= \frac{3.14}{[26.42 - 16.56 \times 0.658]^{\frac{1}{2}}} = 0.796 \end{aligned}$$

$$\alpha = 0.05, \nu = n - 2 = 4, r = 0.811 \text{ [Table 6].}$$

Hence do not reject the hypothesis of no difference between variances.

The null hypothesis  $\sigma_1^2 = \sigma_2^2$  has to be reflected when the value of the test-statistic equals or exceeds the critical value.



## Test 18 Hotelling's $T^2$ -test for two series of population means

### Object

To compare the results of two experiments, each of which yields a multivariate result. In other words, we wish to know if the mean pattern obtained from the first experiment agrees with the mean pattern obtained for the second.

### Limitations

All the variables can be assumed to be independent of each other and all variables follow a multivariate normal distribution. (The variables are usually correlated.)

### Method

Denote the results of the two experiments by subscripts A and B. For ease of description we shall limit the number of variables to three and we shall call these  $x$ ,  $y$  and  $z$ . The number of observations is denoted by  $n_A$  and  $n_B$  for the two experiments. It is necessary to solve the following three equations to find the statistics  $a$ ,  $b$  and  $c$ :

$$\begin{aligned} a[(xx)_A + (xx)_B] + b[(xy)_A + (xy)_B] + c[(xz)_A + (xz)_B] \\ = (n_A + n_B - 2)(\bar{x}_A - \bar{x}_B) \\ a[(xy)_A + (xy)_B] + b[(yy)_A + (yy)_B] + c[(yz)_A + (yz)_B] \\ = (n_A + n_B - 2)(\bar{y}_A - \bar{y}_B) \\ a[(xz)_A + (xz)_B] + b[(yz)_A + (yz)_B] + c[(zz)_A + (zz)_B] \\ = (n_A + n_B - 2)(\bar{z}_A - \bar{z}_B) \end{aligned}$$

where  $(xx)_A = \sum (x_A - \bar{x}_A)^2$ ,  $(xy)_A = \sum (x_A - \bar{x}_A)(y_A - \bar{y}_A)$ , and similar definitions exist for other terms.

Hotelling's  $T^2$  is defined as

$$T^2 = \frac{n_A n_B}{n_A + n_B} \cdot \{a(\bar{x}_A - \bar{x}_B) + b(\bar{y}_A - \bar{y}_B) + c(\bar{z}_A - \bar{z}_B)\}$$

and the test statistic is

$$F = \frac{n_A + n_B - p - 1}{p(n_A + n_B - 2)} T^2$$

which follows an  $F$ -distribution with  $(p, n_A + n_B - p - 1)$  degrees of freedom. Here  $p$  is the number of variables.

### Example

Two batteries of visual stimulus are applied in two experiments on young male and female volunteer students. A researcher wishes to know if the multivariate pattern of

responses is the same for males and females. The appropriate  $F$  statistic is computed as 3.60 and compared with the tabulated value of 4.76 [Table 3]. Since the computed  $F$  value is less than the critical  $F$  value the null hypothesis is of no difference between the two multivariate patterns of stimulus. So the males and females do not differ in their responses on the stimuli.

### Numerical calculation

$$n_A = 6, n_B = 4, DF = v = 6 + 4 - 4 = 6, \alpha = 0.05$$

$$(xx) = (xx)_A + (xx)_B = 19, (yy) = 30, (zz) = 18, (xy) = -6, v_1 = p = 3$$

$$(xz) = 1, (yz) = -7, \bar{x}_A = +7, \bar{x}_B = 4.5, \bar{y}_A = 8, \bar{y}_B = 6, \bar{z}_A = 6, \bar{z}_B = 5$$

The equations

$$19a - 6b + c = 20$$

$$-6a + 30b - 7c = 16$$

$$a - 7b + 18c = 8$$

are satisfied by  $a = 1.320, b = 0.972, c = 0.749$ . Thus

$$T^2 = \frac{6 \times 4}{10} \cdot (1.320 \times 2.5 + 0.972 \times 2 + 0.749 \times 1) = 14.38$$

$$F = \frac{6}{3 \times 8} \times 14.38 = 3.60$$

Critical value  $F_{3,6;0,0} = 4.76$  [Table 3].

Do not reject the null hypothesis.

## Test 19 Discriminant test for the origin of a $p$ -fold sample

### Object

To investigate the origin of one series of values for  $p$  random variates, when one of two markedly different populations may have produced that particular series.

### Limitations

This test provides a decision rule which is closely related to Hotelling's  $T^2$ -test (Test 18), hence is subject to the same limitations.

### Method

Using the notation of Hotelling's  $T^2$ -test, we may take samples from the two populations and obtain two quantities

$$\begin{aligned}D_A &= a\bar{x}_A + b\bar{y}_A + c\bar{z}_A \\D_B &= a\bar{x}_B + b\bar{y}_B + c\bar{z}_B\end{aligned}$$

for the two populations. From the series for which the origin has to be traced we can obtain a third quantity

$$D_S = a\bar{x}_S + b\bar{y}_S + c\bar{z}_S.$$

If  $D_A - D_S < D_B - D_S$  we say that the series belongs to population A, but if  $D_A - D_S > D_B - D_S$  we conclude that population B produced the series under consideration.

### Example

A discriminant function is produced for a collection of pre-historic dog bones. A new relic is found and the appropriate measurements are taken. There are two ancient populations of dog A or B to which the new bones could belong. To which population do the new bones belong? This procedure is normally performed by statistical computer software. The  $D_A$  and  $D_B$  values as well as the  $D_S$  value are computed. The  $D_S$  value is closer to  $D_A$  and so the new dog bone relic belongs to population A.

### Numerical calculation

$$a = 1.320, b = 0.972, c = 0.749$$

$$\bar{x}_A = 7, \bar{y}_A = 8, \bar{z}_A = 6, \bar{x}_B = 4.5, \bar{y}_B = 6, \bar{z}_B = 5$$

$$D_A = 1.320 \times 7 + 0.972 \times 8 + 0.749 \times 6 = 21.510$$

$$D_B = 1.320 \times 4.5 + 0.972 \times 6 + 0.749 \times 5 = 15.517$$

If  $\bar{x}_S = 6, \bar{y}_S = 6$  and  $\bar{z}_S = 7$ , then

$$D_S = 1.320 \times 6 + 0.972 \times 6 + 0.749 \times 7 = 18.995$$

$$D_A - D_S = 21.510 - 18.995 = 2.515$$

$$D_B - D_S = 15.517 - 18.995 = -3.478$$

$D_S$  lies closer to  $D_A$ .  $D_S$  belongs to population A.

## Test 20 Fisher's cumulant test for normality of a population

### Object

To investigate the significance of the difference between a frequency distribution based on a given sample and a normal frequency distribution with the same mean and the same variance.

### Limitations

The sample size should be large, say  $n > 50$ . If the two distributions do not have the same mean and the same variance then the  $w/s$ -test (Test 33) can be used.

### Method

Sample moments can be calculated by

$$M_r = \sum_{i=1}^n x_i^r \quad \text{or} \quad M_r = \sum_{i=1}^n x_i^n f_i$$

where the  $x_i$  are the interval midpoints in the case of grouped data and  $f_i$  is the frequency. The first four sample cumulants (Fisher's  $K$ -statistics) are

$$K_1 = \frac{M_1}{n}$$

$$K_2 = \frac{nM_2 - M_1^2}{n(n-1)}$$

$$K_3 = \frac{n^2M_3 - 3nM_2M_1 + 2M_1^3}{n(n-1)(n-2)}$$

$$K_4 = \frac{(n^3 + n^2)M_4 - 4(n^2 + n)M_3M_1 - 3(n^2 - n)M_2^2 + 12M_2M_1^2 - 6M_1^4}{n(n-1)(n-2)(n-3)}$$

To test for skewness the test statistic is

$$u_1 = \frac{K_3}{(K_2)^{\frac{3}{2}}} \times \left(\frac{n}{6}\right)^{\frac{1}{2}}$$

which should follow a standard normal distribution.

To test for kurtosis the test statistic is

$$u_2 = \frac{K_4}{(K_2)^2} \times \left(\frac{n}{24}\right)^{\frac{1}{2}}$$

which should follow a standard normal distribution.

A combined test can be obtained using the test statistic

$$\chi^2 = \left[ \frac{K_3}{(K_2)^{\frac{3}{2}}} \times \left( \frac{n}{6} \right)^{\frac{1}{2}} \right]^2 + \left[ \frac{K_4}{(K_2)^2} \times \left( \frac{n}{24} \right)^{\frac{1}{2}} \right]^2$$

which will approximately follow a  $\chi^2$ -distribution with two degrees of freedom.

### Example

#### Example A

A large sample of 190 component measurements yields the following calculations (see table). Do the sample data follow a normal distribution? The test for skewness is a  $u_1$  statistic of 0.473 and the critical value of the normal test statistic is 1.96. Since  $u_1$  is less than this critical value we do not reject the null hypothesis of no difference. So for skewness the data are similar to a normal distribution. For kurtosis we have  $u_2$  statistic of 0.474 and, again, a critical value of 1.96. So, again, we accept the null hypothesis; kurtosis is not significantly different from that of a normal distribution with the same mean and variance. The combined test gives a calculated chi-squared value 0.449 which is smaller than the 5 per cent critical value of 5.99. So we conclude that the data follow a normal distribution.

#### Example B

We calculate the values of skewness and kurtosis together with their respective standard deviations and produce:

$$u_1 = \text{skewness/s d} = 0.477$$

$$u_2 = \text{kurtosis/s d} = 0.480$$

Table 7 gives (for sample sizes 200 and 175) critical values for  $u_1$  of 0.282 to 0.301 and for  $u_2$  of 0.62 to 0.66. So, again, we accept the null hypothesis.

### Numerical calculation

#### Example A

$$\sum f = n = 190, \sum fx = 151, \sum fx^2 = 805,$$

$$\sum fx^3 = 1837, \sum fx^4 = 10753$$

$$\text{i.e. } M_1 = 151, M_2 = 805, M_3 = 1837, M_4 = 10753$$

$$K_2 = \frac{(190 \times 805) - (151)^2}{190 \times 189} = 3.624310$$

$$K_3 = \frac{(190)^2 \times 1837 - 3 \times 190 \times 805 \times 151 + 2(151)^3}{190 \times 189 \times 188} = 0.5799445$$

$$K_4 = \frac{2795421924}{190 \times 189 \times 188 \times 187} = 2.214280$$

**Test for skewness**

$$u_1 = \frac{0.579945}{3.62431\sqrt{3.624310}} \times 5.6273 = 0.08405 \times 5.6273 = 0.473$$

The critical value at  $\alpha = 0.05$  is 1.96.

Do not reject the null hypothesis [Table 1].

**Test for kurtosis**

$$u_2 = \frac{2.214279}{(3.62431)^2} \times \left(\frac{190}{24}\right)^{\frac{1}{2}} = 0.1686 \times 2.813657 = 0.474$$

The critical value at  $\alpha = 0.05$  is 1.96.

Do not reject the null hypothesis [Table 1].

**Combined test**

$$\chi^2 = (0.473)^2 + (0.474)^2 = 0.2237 + 0.2250 = 0.449$$

which is smaller than the critical value 5.99 [Table 5].

**Example B**

$$\begin{aligned} \text{Let skewness} = g_1 &= \frac{K_3}{K_2\sqrt{K_2}} = \frac{0.579945}{3.624310\sqrt{3.624310}} \\ &= 0.084052 \end{aligned}$$

$$\text{kurtosis} = g_2 = \frac{K_4}{K_2^2} = \frac{2.214279}{(3.624310)^2} = 0.168570$$

$$\begin{aligned} \text{standard deviation } \sigma(g_1) &= \sqrt{\frac{6n(n-1)}{(n-2)(n+1)(n+3)}} \\ &= \sqrt{\frac{6 \times 190 \times 189}{188 \times 191 \times 193}} = \sqrt{0.0310898} = 0.176323 \end{aligned}$$

$$\begin{aligned} \text{standard deviation } \sigma(g_2) &= \sqrt{\frac{24n(n-1)^2}{(n-3)(n-2)(n+3)(n+5)}} \\ &= \sqrt{\frac{24 \times 190 \times 189^2}{187 \times 188 \times 193 \times 195}} = 0.350872 \end{aligned}$$

$$\text{Here } u_1 = \frac{0.084052}{0.176323} = 0.477, u_2 = \frac{0.168570}{0.350872} = 0.480.$$

Critical values for  $g_1$  lie between 0.282 (for 200) and 0.301 (for 175) [Table 7].

The right-side critical value for  $g_2$  lies between 0.62 and 0.66 [Table 7].

Hence the null hypothesis should not be rejected.

## Test 21 Dixon's test for outliers

### Object

To investigate the significance of the difference between a suspicious extreme value and other values in the sample.

### Limitations

1. The sample size should be greater than 3.
2. The population which is being sampled is assumed normal.

### Method

Consider a sample of size  $n$ , where the sample is arranged with the suspect value in front, its nearest neighbour next and then the following values arranged in ascending (or descending) order. The order is determined by whether the suspect value is the largest or the smallest. Denoting the ordered series by  $x_1, x_2, \dots, x_n$ , the test statistic  $r$  where

$$\begin{aligned} r &= (x_2 - x_1)/(x_n - x_1) && \text{if } 3 < n \leq 7, \\ r &= (x_2 - x_1)/(x_{n-1} - x_1) && \text{if } 8 \leq n \leq 10, \\ r &= (x_3 - x_1)/(x_{n-1} - x_1) && \text{if } 11 \leq n \leq 13, \\ r &= (x_3 - x_1)/(x_{n-2} - x_1) && \text{if } 14 \leq n \leq 25. \end{aligned}$$

Critical values for  $r$  can be obtained from Table 8. The null hypothesis that the outlier belongs to the sample is rejected if the observed value of  $r$  exceeds the critical value.

### Example

As part of a quality control programmed/implementation small samples are taken, at regular intervals, for a number of processes. On several of these processes there is the potential for inaccuracies occurring in the measurements that are taken due to the complexity of the measuring process and the inexperience of the process workers. One such sample of size 4 is tested for potential outliers and the following are produced:  $x_1 = 326$ ,  $x_2 = 177$ ,  $x_3 = 176$ ,  $x_4 = 157$ .

Dixon's ratio yields  $r = 0.882$ .

The critical value at the 5 per cent level from Table 8 is 0.765, so the calculated value exceeds the critical value. We thus reject the null hypothesis that the outlier belongs to the sample. Thus we need to re-sample and measure again or only use three sample values in this case.

### Numerical calculation

$x_1 = 326$ ,  $x_2 = 177$ ,  $x_3 = 176$ ,  $x_4 = 157$ ,  $n = 4$

$$\text{Here } r = \frac{x_2 - x_1}{x_n - x_1} = \frac{177 - 326}{157 - 326} = 0.882$$

The critical value at  $\alpha = 0.05$  is 0.765 [Table 8].

The calculated value exceeds the critical value.

Hence reject the null hypothesis that the value  $x_1$  comes from the same population.

## Test 22 *F*-test for $K$ population means (analysis of variance)

### Object

To test the null hypothesis that  $K$  samples are from  $K$  populations with the same mean.

### Limitations

It is assumed that the populations are normally distributed and have equal variances. It is also assumed that the samples are independent of each other.

### Method

Let the  $j$ th sample contain  $n_j$  elements ( $j = 1, \dots, K$ ). Then the total number of elements is

$$N = \sum_{j=1}^K n_j$$

The  $i$ th element of the  $j$ th sample can be denoted by  $x_{ij}$  ( $i = 1, \dots, n_j$ ), and the mean of the  $j$ th sample becomes

$$x_{.j} = \sum_{i=1}^{n_j} x_{ij} / n_j$$

The variance of the observations with respect to their own sample means becomes

$$s_1^2 = \frac{\sum_j \sum_i (x_{ij} - x_{.j})^2}{N - K}$$

or equivalently, denoting the total sum of squares of all the observations as  $s_T^2$ ,

$$(s_T^2 - s_2^2) / (N - K)$$

with  $N - K$  degrees of freedom. Similarly, the variance of the sample means with respect to the grand mean becomes

$$s_2^2 = \frac{\sum_j n_j (x_{.j} - x_{..})^2}{K - 1}$$

where

$$x_{..} = \frac{1}{K} \sum_i \sum_j x_{ij}$$

and  $s_2^2$  has  $K - 1$  degrees of freedom.



The test statistic is  $F = s_2^2/s_1^2$ , which follows the  $F$ -distribution with  $(K - 1, N - K)$  degrees of freedom. A one-tailed test is carried out as it is necessary to ascertain whether  $s_2^2$  is larger than  $s_1^2$ .

### Example

A petroleum company tests three additives on its premium unleaded petrol to assess their effect on petrol consumption. The company uses a basic car of a particular make and model with cars randomly allocated to treatments (additives). An analysis of variance compares the effect of the additives on petrol consumption. Since the calculated  $F$  statistic at 37 is greater than the tabulated value of 4.26 the variance between additives is greater than the variance within additives. The additives have an effect on petrol consumption.

### Numerical calculation

$$K = 3, N = 12, n_1 = 3, n_2 = 5, n_3 = 4, \alpha = 0.05$$

$$\sum_{i=1}^{n_1} x_{i1} = 53.5, \sum_{i=1}^{n_2} x_{i2} = 102.5, \sum_{i=1}^{n_3} x_{i3} = 64.4$$

$$T = 53.5 + 102.5 + 64.4 = 220.4$$

$$x_1 = 17.83, x_2 = 20.50, x_3 = 16.10, x_{..} = T/N = 18.37$$

$$T^2/N = 4048.01$$

$$\begin{aligned} s_T^2 &= \left[ \left( \sum_{i=1}^{n_1} x_{i1}^2 + \sum_{i=1}^{n_2} x_{i2}^2 + \sum_{i=1}^{n_3} x_{i3}^2 \right) - \frac{T^2}{N} \right] \\ &= [(954.43 + 2105.13 + 1037.98) - 4048.01] \\ &= 4097.54 - 4048.01 \\ &= 49.53 \end{aligned}$$

$$s_2^2 = 44.17$$

$$\begin{aligned} F_{2,9} &= \frac{s_2^2/(K - 1)}{s_1^2/(N - K)} = \frac{s_2^2/(K - 1)}{(s_T^2 - s_2^2)/(N - K)} \\ &= \frac{44.17/2}{(49.53 - 44.17)/9} \simeq 37 \end{aligned}$$

Critical value  $F_{2,9;0.05} = 4.26$  [Table 3].

The calculated value is greater than the critical value.

The variance between the samples is significantly larger than the variance within the samples.

## Test 23 The Z-test for correlated proportions

### Object

To investigate the significance of the difference between two correlated proportions in opinion surveys. It can also be used for more general applications.

### Limitations

1. The same people are questioned both times on a yes–no basis.
2. The sample size must be quite large.

### Method

$N$  people respond to a yes–no question both before and after a certain stimulus. The following two-way table can then be built up:

		First poll	
		Yes	No
Second poll	Yes	$a$	$b$
	No	$c$	$d$
		$N$	

To decide whether the stimulus has produced a significant change in the proportion answering ‘yes’, we calculate the test statistic

$$Z = \frac{b - c}{N\sigma}$$

where

$$\sigma = \sqrt{\frac{(b + c) - (b - c)^2/N}{N(N - 1)}}.$$

### Example

Sampled panels of potential buyers of a financial product are asked if they might buy the product. They are then shown a product advertisement of 30 seconds duration and asked again if they would buy the product. Has the advertising stimulus produced a significant change in the proportion of the panel responding ‘yes’?

We have

		First poll	
		Yes	No
Second poll	Yes	30	15
	No	9	51

which yields the test statistic  $Z = 1.23$ . The 5 per cent critical value from the normal distribution is 1.96. Since 1.23 is less than 1.96 we do not reject the null hypothesis of no difference. The advertisement does not increase the proportion saying 'yes'. Notice that we have used a one-tailed test, here, because we are only interested in an increase, i.e. a positive effect of advertising.

### Numerical calculation

$$a = 30, b = 15, c = 9, d = 51, N = 105$$

The null hypothesis is that there is no apparent change due to the stimulus. The difference in proportion is

$$\frac{b - c}{N} = \frac{15}{105} - \frac{9}{105} = \frac{6}{105} = 0.0571$$

$$\sigma = \sqrt{\frac{(15 + 9) - (15 - 9)^2/105}{105 \times 104}} = 0.0465$$

$$Z = \frac{0.0571}{0.0465} = 1.23$$

The critical value at  $\alpha = 0.05$  is 1.96 [Table 1].

The calculated value is less than the critical value.

Do not reject the null hypothesis.

## Test 24 $\chi^2$ -test for an assumed population variance

### Object

To investigate the significance of the difference between a population variance  $\sigma^2$  and an assumed value  $\sigma_0^2$ .

### Limitations

It is assumed that the sample is taken from a normal population.

### Method

The sample variance

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1}$$

is calculated. The test statistic is then

$$\chi^2 = \frac{s^2}{\sigma_0^2} (n - 1)$$

which follows a  $\chi^2$ -distribution with  $n - 1$  degrees of freedom.

### Example

An engineering process has specified variance for a machined component of 9 square cm. A sample of 25 components is selected at random from the production and the mean value for a critical dimension on the component is measured at 71 cm with sample variance of 12 square cm. Is there a difference between variances? A calculated chi-squared value of 32 is less than the tabulated value of 36.4 suggesting no difference between variances.

### Numerical calculation

$$n = 25, \bar{x} = 71, s^2 = 12, \sigma_0^2 = 9$$

$$H_0: \sigma^2 = \sigma_0^2, H_1: \sigma^2 \neq \sigma_0^2$$

$$\chi^2 = 24 \times \frac{12}{9} = 32$$

Critical value  $\chi_{24; 0.05}^2 = 36.4$  [Table 5].

Do not reject the null hypothesis. The difference between the variances is not significant.

## Test 25 *F*-test for two counts (Poisson distribution)

### Object

To investigate the significance of the difference between two counted results (based on a Poisson distribution).

### Limitations

It is assumed that the counts satisfy a Poisson distribution and that the two samples were obtained under similar conditions.

### Method

Let  $\mu_1$  and  $\mu_2$  denote the means of the two populations and  $N_1$  and  $N_2$  the two counts. To test the hypothesis  $\mu_1 = \mu_2$  we calculate the test statistic

$$F = \frac{N_1}{N_2 + 1}$$

which follows the  $F$ -distribution with  $(2(N_2 + 1), 2N_1)$  degrees of freedom. When the counts are obtained over different periods of time  $t_1$  and  $t_2$ , it is necessary to compare the counting rates  $N_1/t_1$  and  $N_2/t_2$ . Hence the appropriate test statistic is

$$F = \frac{\frac{1}{t_1}(N_1 + 0.5)}{\frac{1}{t_2}(N_1 + 0.5)}$$

which follows the  $F$ -distribution with  $(2N_1 + 1, 2N_2 + 1)$  degrees of freedom.

### Example

Two automated kiln processes (producing baked plant pots) are compared over their standard cycle times, i.e. 4 hours. Kiln 1 produced 13 triggered process corrections and kiln 2 produced 3 corrections. What can we say about the two kiln mean correction rates, are they the same? The calculated  $F$  statistic is 3.25 and the critical value from Table 3 is 2.32. Since the calculated value exceeds the critical value we conclude that there is a statistical difference between the two counts. Kiln 1 has a higher error rate than kiln 2.

### Numerical calculation

$$N_1 = 13, N_2 = 3, t_1 = t_2$$

$$f_1 = 2(N_2 + 1) = 2(3 + 1) = 8, f_2 = 2N_1 = 2 \times 13 = 26$$

$$F = \frac{N_1}{N_2 + 1} = \frac{13}{3 + 1} = 3.25$$

Critical value  $F_{8,26;0.05} = 2.32$  [Table 3].

The calculated value exceeds the table value.

Hence reject the null hypothesis.

## Test 26 *F*-test for the overall mean of $K$ subpopulations (analysis of variance)

### Object

To investigate the significance of the difference between the overall mean of  $K$  subpopulations and an assumed value  $\mu_0$  for the population mean. Two different null hypotheses are tested; the first being that the  $K$  subpopulations have the same mean ( $\mu_1 = \mu_2 = \dots = \mu_K$ ) and the second that the overall mean is equal to the assumed value ( $\mu = \mu_0$ ).

### Limitations

The  $K$  samples from the subpopulations are independent of each other. The subpopulations should also be normally distributed and have the same variance.

### Methods

#### Method A

To test  $H_0 : \mu_1 = \mu_2 = \dots = \mu_K$ , we calculate the test statistic

$$F = \frac{s_1^2/(K-1)}{s_2^2/(N-K)}$$

where  $N$  is the total number of observations in the  $K$  samples,  $n_j$  is the number of observations in the  $j$ th sample,

$$\left. \begin{aligned} s_1^2 &= \sum_j n_j (x_{.j} - x_{..})^2 \\ s_2^2 &= \sum_i \sum_j (x_{ij} - x_{.j})^2 \end{aligned} \right\} \quad i = 1, \dots, n_j, \quad j = 1, \dots, K,$$

and  $x_{ij}$  is the  $i$ th observation in the  $j$ th sample

$$\begin{aligned} x_{.j} &= \frac{1}{n_j} \sum_i x_{ij} \\ x_{..} &= \frac{1}{N} \sum_i \sum_j x_{ij} \end{aligned}$$

The value of  $F$  should follow the  $F$ -distribution with  $(K-1, N-K)$  degrees of freedom.

#### Method B

To test  $H_0: \mu = \mu_0$ , we calculate the test statistic

$$F = \frac{N(x_{..} - \mu_0)^2}{s_1^2/(K-1)}$$

which should follow the  $F$ -distribution with  $(1, K-1)$  degrees of freedom.

**Example**

A nutritional researcher wishes to test the palatability of six different formulations of vitamin/mineral supplement which are added to children's food. They differ only in their taste. Are they equally palatable? Do they, overall, produce a given average consumption of food? In a trial, six groups each of five children are given the different formulations. The first calculated  $F$  value of 4.60 tests for equality of palatability. Since this exceeds the tabulated value of 2.62 the null hypothesis of no difference is rejected. The formulations do affect the palatability of the food eaten since different quantities are eaten. The second  $F$  value of 2.01, since it is less than the tabulated value of 6.61, suggests that the formulations, if used together over a period of time, will not affect consumption.

**Numerical calculation**

$$n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = 5, K = 6, N = 30, \mu_0 = 1500$$

$$x_{.1} = 1505, x_{.2} = 1528, x_{.3} = 1564, x_{.4} = 1498, x_{.5} = 1600, x_{.6} = 1470$$

$$\bar{x}_{..} = 9165/6 = 1527.5$$

$$s_1^2/5 = 11\,272, s_2^2/24 = 2451, s_T^2 = 3972$$

$$N(x_{..} - \mu_0)^2 = 22\,687.5$$

$$(a) F = 11\,272/2451 = 4.60.$$

Critical value  $F_{5,24;0.05} = 2.62$  [Table 3].

Reject the null hypothesis.

$$(b) F = 22\,687.5/11\,272 = 2.01.$$

Critical value  $F_{1,5;0.05} = 6.61$  [Table 3].

Do not reject the null hypothesis.

## Test 27 *F*-test for multiple comparison of contrasts between $K$ population means

### Object

This test is an extension of the preceding one, to investigate which particular set of mean values or linear combination of mean values shows differences with the other mean values.

### Limitations

As for the preceding test, with the addition that the comparisons to be examined should be decided on at or before the start of the analysis.

### Method

With the notation as before, we must define a contrast as a linear function of the means

$$\lambda = \sum_{j=1}^K a_j \mu_j$$

under the condition that  $\sum_{j=1}^K a_j = 0$ .

The test statistic becomes

$$F = \frac{1}{s^2} \frac{\left( \sum_{j=1}^K a_j \bar{x}_{.j} \right)^2}{\sum_{j=1}^K a_j^2 / n_j}$$

(see Test 26) which should follow the  $F$ -distribution with  $(1, N - K)$  degrees of freedom. Here,

$$s^2 = \frac{\sum_{j=1}^K \sum_{i=1}^{n_j} (x_{ij})^2 - \sum_{j=1}^K n_j \bar{x}_j^2}{N - K}.$$

For simple differences of the type  $\bar{x}_{.i} - \bar{x}_{.j}$  the test statistic becomes

$$F = \frac{(\bar{x}_{.i} - \bar{x}_{.j})^2}{s^2 \left( \frac{1}{n_i} + \frac{1}{n_j} \right)}$$



**Example**

Three different filters are used on an artificial lighting system for plant production. An agricultural researcher wishes to test whether any filter is better than the others in terms of the yield of a crop. He designs and conducts his experiment and performs the  $F$  test. He compares each plant yield with the others.

For comparing filter 1 with filter 2 his calculated  $F$  value is 0.646; for comparing filter 1 with filter 3 his  $F$  value is 36.78 and for comparing filter 2 with filter 3 his  $F$  value is 39.90. So it appears that filter 1 and filter 2 are similar in relation to plant growth but there is a difference between filter 1 and filter 3 and filter 2 and filter 3.

**Numerical calculation**

$$n_1 = 6, n_2 = 4, n_3 = 2, N = 12, K = 3, v_1 = 1, v_2 = 9$$

$$\bar{x}_{.1} = 2.070, \bar{x}_{.2} = 2.015, \bar{x}_{.3} = 2.595, \bar{x} = 2.139$$

$$\sum_i \sum_j x_{ij}^2 = 55.5195, \quad \sum n_j \bar{x}_j^2 = 55.41835$$

$$\lambda_1 = \mu_1 - \mu_2, \lambda_2 = \mu_1 - \mu_3, \lambda_3 = \mu_2 - \mu_3 \text{ (contrasts)}$$

$$\bar{x}_{.1} - \bar{x}_{.2} = 0.055$$

$$\bar{x}_{.1} - \bar{x}_{.3} = -0.525$$

$$\bar{x}_{.2} - \bar{x}_{.3} = -0.580$$

$$\frac{(x_{.1} - x_{.2})^2}{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \frac{(0.055)^2}{\left(\frac{1}{6} + \frac{1}{4}\right)} = 0.00726$$

$$\frac{(x_{.1} - x_{.3})^2}{\left(\frac{1}{n_1} + \frac{1}{n_3}\right)} = \frac{(-0.525)^2}{\left(\frac{1}{6} + \frac{1}{2}\right)} = 0.4134$$

$$\frac{(x_{.2} - x_{.3})^2}{\left(\frac{1}{n_2} + \frac{1}{n_3}\right)} = \frac{(-0.580)^2}{\left(\frac{1}{4} + \frac{1}{2}\right)} = 0.4485$$

$$s^2 = \frac{55.5195 - 55.41835}{9} = \frac{0.10115}{9} = 0.01124$$

$$F_1 = \frac{0.00726}{0.01124} = 0.646, F_2 = \frac{0.4134}{0.01124} = 36.78, F_3 = \frac{0.4485}{0.01124} = 39.90$$

Critical value  $F_{1,9;0.05} = 5.12$  [Table 3]. Both  $F_2$  and  $F_3$  are larger than 5.12. There is no significant difference between the means for group 1 and group 2, but there is a significant difference between group 3 and groups 1 and 2.

## Test 28 Tukey test for multiple comparison of $K$ population means (unequal sample sizes)

### Object

To investigate the significance of all possible differences between  $K$  population means.

### Limitations

The  $K$  populations are normally distributed with equal variances.

### Method

Consider samples of size  $n_1, n_2, \dots, n_K$  from the  $K$  populations. From Table 9 the critical values of  $q$  can be found using degrees of freedom

$$\nu = \left( \sum_{j=1}^K n_j \right) - K.$$

The total variance of the samples is now calculated from

$$s^2 = \frac{\sum_{j=1}^K (n_j - 1) \cdot s_j^2}{N - K}$$

where  $s_j^2$  is the variance of the  $j$ th sample and  $N$  is the total sample size. Finally, a limit  $W$  is calculated

$$W = \frac{qs}{n^{\frac{1}{2}}}$$

where  $q$  ( $= w/s$ ) is the Studentized range and

$$n = \frac{K}{\left( \frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_K} \right)}.$$

If this limit  $W$  is exceeded by the absolute difference between any two sample means, then the corresponding population means differ significantly.

### Example

Five different grades of grit used for agricultural purposes are produced. A filling machine fills small sacks with a nominal 500 gm weight, although more is usually dispensed. An agricultural merchant is concerned that the weight does not differ between grades. He uses Tukey's test to compare all five grades and produces a critical limit of  $W$  at 55.3. Since the largest difference grades is 35.8 and is less than 55.17 he concludes that the grades do not differ with respect to weight.

**Numerical calculation**

$$n_1 = n_2 = n_3 = n_4 = n_5 = 5, K = 5, N = 25$$

$$s_1^2 = 406.0, s_2^2 = 574.8, s_3^2 = 636.8, s_4^2 = 159.3, s_5^2 = 943.2$$

$$\bar{x}_1 = 534.0, \bar{x}_2 = 536.4, \bar{x}_3 = 562.6, \bar{x}_4 = 549.4, \bar{x}_5 = 526.8$$

$$s^2 = \frac{2720.1}{5} = 544.02, s = 23.32, \nu = 25 - 5 = 20$$

Critical value for  $q$  for  $K = 5$ , and  $\nu = 20$  at  $\alpha = 0.05$  is 5.29 [Table 9].

$$W = \frac{5.29 \times 23.32}{\sqrt{5}} = 55.17$$

The largest difference between the sample means is  $562.6 - 526.8 = 35.8$  which is less than 55.17. Hence the population means do not differ significantly.

## Test 29 The Link–Wallace test for multiple comparison of $K$ population means (equal sample sizes)

### Object

To investigate the significance of all possible differences between  $K$  population means.

### Limitations

1. The  $K$  populations are normally distributed with equal variances.
2. The  $K$  samples each contain  $n$  observations.

### Method

The test statistic is

$$K_L = \frac{nw(\bar{x})}{\sum_{i=1}^k w_i(x)}$$

where  $w_i(x)$  is the range of the  $x$  values for the  $i$ th sample

$w(\bar{x})$  is the range of the sample means

$n$  is the sample size.

The null hypothesis  $\mu_1 = \mu_2 = \dots = \mu_K$  is rejected if the observed value of  $K_L$  is larger than the critical value obtained from Table 10.

### Example

Three advertising theme tunes are compared using three panels to assess their pleasure, using a set of scales. The test statistic  $D$  is computed as 2.51, and then the three differences between the ranges of means are also calculated. Tunes 3 and 2 differ as do tunes 3 and 1, but tunes 1 and 2 do not.

### Numerical calculation

$$n = 8, w_1(x) = 7, w_2(x) = 6, w_3(x) = 4, K = 3$$

$$w_1(\bar{x}) = 4.750, w_2(\bar{x}) = 4.625, w_3(\bar{x}) = 7.750$$

$$K_L = \frac{nw(\bar{x})}{\sum_{i=1}^k w_i(x)} = \frac{8(7.750 - 4.625)}{7 + 6 + 4} = 1.47$$

Critical value  $K_{8,3;0.05} = 1.18$  [Table 10].

Reject the null hypothesis of equal means.

Using  $K_{8,3;0.05} = 1.18$ , the critical value for the sample mean differences is

$$D = \frac{1.18 \sum w_i(x)}{n} = \frac{1.18 \times 17}{8} = 2.51$$

Since  $w_1(\bar{x}) - w_2(\bar{x}) = 0.125$  (less than  $D$ )

$w_3(\bar{w}) - w_2(\bar{x}) = 3.125$  (greater than  $D$ )

$w_3(\bar{x}) - w_1(\bar{x}) = 3.00$  (greater than  $D$ )

reject the null hypothesis  $\mu_1 = \mu_3$  and  $\mu_2 = \mu_3$ .

## Test 30 Dunnett's test for comparing $K$ treatments with a control

### Object

To investigate the significance of the differences, when several treatments are compared with a control.

### Limitations

1. The  $K + 1$  samples, consisting of  $K$  treatments and one control, all have the same size  $n$ .
2. The samples are drawn independently from normally distributed populations with equal variances.

### Method

The variance within the  $K + 1$  groups is calculated from

$$S_W^2 = \frac{S_0^2 + S_1^2 + \cdots + S_K^2}{(K - 1)(n - 1)}$$

where  $S_0^2$  is the sum of squares of deviations from the group mean for the control group and  $S_j^2$  is a similar expression for the  $j$ th treatment group. The standard deviation of the differences between treatment means and control means is then

$$S(\bar{d}) = \sqrt{2S_W^2/n}$$

The quotients

$$D_j = \frac{\bar{x}_j - \bar{x}_0}{S(\bar{d})} \quad (j = 1, 2, \dots, K)$$

are found and compared with the critical values of  $|D_j|$  found from Table 11. If an observed value is larger than the tabulated value, one may conclude that the corresponding difference in means between treatment  $j$  and the control is significant.

### Example

Four new topical treatments for athlete's foot are compared with a control, which is the current accepted treatment. Patients are randomly allocated to each treatment and the number of days to clear up of the condition is the treatment outcome. Do the new treatments differ from the control? The critical value,  $D$ , from Table 11 is 2.23 and all standardized differences are compared. Treatment 1 shows a difference from control which is significant. This treatment results in a longer time to clear up.

**Numerical calculation**

$$K = 4, n = 10, \bar{x}_0 = 14.5, \bar{x}_1 = 25.0, \bar{x}_2 = 15.7, \bar{x}_3 = 18.1, \bar{x}_4 = 21.9$$

$$S_0^2 = 261.0, S_1^2 = 586.0, S_2^2 = 292.6, S_3^2 = 320.4, S_4^2 = 556.4$$

$$S_W^2 = \frac{261 + 586 + 292.6 + 320.4 + 556.4}{(4 - 1)(10 - 1)} = 74.681$$

$$S(\bar{d}) = \sqrt{2S_W^2/n} = \left( \frac{2 \times 74.681}{10} \right)^{\frac{1}{2}} = 3.865$$

Critical value  $D_{4,45;0.05} = 2.23$  [Table 11].

$$D_1 = 2.72, D_2 = 0.31, D_3 = 0.93, D_4 = 1.91$$

The value of  $D_1$  is larger than the critical value. Hence  $D_1$  is significantly different from the control value.

## Test 31 Bartlett's test for equality of $K$ variances

### Object

To investigate the significance of the differences between the variances of  $K$  normally distributed populations.

### Limitations

It is assumed that all the populations follow a normal distribution.

### Method

Samples are drawn from each of the populations. Let  $s_j^2$  denote the variance of a sample of  $n_j$  items from the  $j$ th population ( $j = 1, \dots, K$ ). The overall variance is defined by

$$s^2 = \frac{\sum_{j=1}^K (n_j - 1) \cdot s_j^2}{\sum_{j=1}^K (n_j - 1)}.$$

The test statistic is

$$B = \frac{2.30259}{C} \left\{ \sum (n_j - 1) \log s^2 - \sum (n_j - 1) \log s_j^2 \right\}$$

where

$$C = 1 + \frac{1}{3(K+1)} \left\{ \sum \frac{1}{(n_j - 1)} - \frac{1}{\sum (n_j - 1)} \right\}$$

and  $\log_e 10 = 2.30259$ .

### Case A ( $n_j > 6$ )

$B$  will approximate to a  $\chi^2$ -distribution with  $K - 1$  degrees of freedom. The null hypothesis of equal variances is rejected if  $B$  is larger than the critical value.

### Case B ( $n_j \leq 6$ )

The test statistic becomes  $BC = M$  and this should be referred to Table 12. When the value of  $M$  exceeds the tabulated value, the null hypothesis can be rejected.

### Example

A bank of four machining heads is compared in relation to the variability of the end machined components. A quality engineer has collected a randomized sequence of components and measured the relevant component dimensions. In his first test he has large samples and computes the test statistic  $B$ , which is 7.31. This is less than the critical value of 7.81 from Table 5. He so concludes that the variabilities for the four machine heads are the same.



In a subsequent test of equality of variances the engineer takes smaller samples which necessitate his use of tables for comparison. In this case his sample values produce an  $M$  value which is less than the tabulated value of 9.21 from Table 12. So again he concludes that the variances are the same.

### Numerical calculation

#### Example A

$$n_1 = 31, n_2 = 15, n_3 = 20, n_4 = 42, K = 4$$

$$s_1^2 = 5.47, s_2^2 = 4.64, s_3^2 = 11.47, s_4^2 = 11.29$$

$$s^2 = \frac{910}{104} = 8.75, \log s^2 = 0.94201, C = 1.01$$

$$\sum (n_j - 1) \log s^2 = 97.9690, \sum (n_j - 1) \log s_j^2 = 94.7630$$

$$B = \frac{1}{1.01} [2.30259(97.9690 - 94.7630)] = \frac{7.38}{1.01} = 7.31$$

Critical value  $\chi_{3;0.05}^2 = 7.81$  [Table 5].

Hence the null hypothesis is not rejected.

#### Example B

$$n_1 = 3, n_2 = 3, n_3 = 3, n_4 = 4$$

$$s_1^2 = 6.33, s_2^2 = 1.33, s_3^2 = 4.33, s_4^2 = 4.33$$

$$\text{Pooled variance } s^2 = \frac{(12.66 + 2.66 + 8.66 + 12.99)}{2 + 2 + 2 + 3} = 4.11$$

Further

$$\begin{aligned} M &= 2.30259\{9 \log 4.11 - 2 \log 6.33 - 2 \log 1.33 - 2 \log 4.33 - 3 \log 4.33\} \\ &= 2.30259\{9 \times 0.6138 - 2 \times 0.8014 - 2 \times 0.1239 - 2 \times 0.6365 \\ &\quad - 3 \times 0.6365\} = 1.131 \end{aligned}$$

$$C = \left\{ \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} - \frac{1}{9} \right\} = 1.7222$$

The critical value of  $M$  for  $\alpha = 0.05$ ,  $K = 4$  is 9.21 [Table 12]; even for  $C = 2.0$ . Do not reject the null hypothesis.

## Test 32 Hartley's test for equality of $K$ variances

### Object

To investigate the significance of the differences between the variances of  $K$  normally distributed populations.

### Limitations

1. The populations should be normally distributed.
2. The sizes of the  $K$  samples should be (approximately) equal.

### Method

Samples are drawn from each of the populations. The test statistic is  $F_{\max} = s_{\max}^2 / s_{\min}^2$  where  $s_{\max}^2$  is the largest of the  $K$  sample variances and  $s_{\min}^2$  is the smallest of the  $K$  sample variances.

Critical values of  $F_{\max}$  can be obtained from Table 13. If the observed ratio exceeds this critical value, the null hypothesis of equal variances should be rejected.

### Example

Four types of spring are tested for their response to a fixed weight since they are used to calibrate a safety shut-off device. It is important that the variability of responses is equal. Samples of responses to a weight on each spring are taken. The Hartley  $F$  statistic is calculated to be 2.59 and is compared with the critical tabulated value of 2.61 [Table 13]. Since the calculated statistic is less than the tabulated value the null hypothesis of equal variances is accepted.

### Numerical calculation

$$n_1 = n_2 = n_3 = n_4 = 30, K = 4$$

$$s_1^2 = 16.72, s_2^2 = 36.27, s_3^2 = 14.0, s_4^2 = 15.91$$

$$F = \frac{36.27}{14.0} = 2.59$$

The critical value of  $F_{\max}$ , at a 5 per cent level of significance, for  $n = 30$ ,  $K = 4$  is 2.61 [Table 13].

Hence do not reject the null hypothesis.

## Test 33 The $w/s$ -test for normality of a population

### Object

To investigate the significance of the difference between a frequency distribution based on a given sample and a normal frequency distribution.

### Limitations

This test is applicable if the sample is taken from a population with continuous distribution.

### Method

This is a much simpler test than Fisher's cumulant test (Test 20). The sample standard deviation ( $s$ ) and the range ( $w$ ) are first determined. Then the Studentized range  $q = w/s$  is found.

The test statistic is  $q$  and critical values are available for  $q$  from Table 14. If the observed value of  $q$  lies outside the two critical values, the sample distribution cannot be considered as a normal distribution.

### Example

For this test of normality we produce the ratio of sample range divided by sample standard deviation and compare with critical values from Table 14.

We have two samples for consideration. They are taken from two fluid injection processes. The two test statistics,  $q_1$  and  $q_2$ , are both within their critical values. Hence we accept the null hypothesis that both samples could have been taken from normal distributions. Such tests are particularly relevant to quality control situations.

### Numerical calculation

$$n_1 = 4, n_2 = 9, \bar{x}_1 = 3166, \bar{x}_2 = 2240.4, \alpha = 0.025$$

$$s_1^2 = 6328.67, s_2^2 = 221\,661.3, s_1 = 79.6, s_2 = 471$$

$$w_1 = 171, w_2 = 1333$$

$$q_1 = \frac{w_1}{s_1} = 2.15, q_2 = \frac{w_2}{s_2} = 2.83$$

Critical values for this test are:

for  $n_1 = 4$ , 1.93 and 2.44 [Table 14];

for  $n_2 = 9$ , 2.51 and 3.63 [Table 14].

Hence the null hypothesis cannot be rejected.

## Test 34 Cochran's test for variance outliers

### Object

To investigate the significance of the difference between one rather large variance and  $K - 1$  other variances.

### Limitations

1. It is assumed that the  $K$  samples are taken from normally distributed populations.
2. Each sample is of equal size.

### Method

The test statistic is

$$C = \frac{\text{largest of the } s_i^2}{\text{sum of all } s_i^2}$$

where  $s_i^2$  denotes the variance of the  $i$ th sample. Critical values of  $C$  are available from Table 15. The null hypothesis that the large variance does not differ significantly from the others is rejected if the observed value of  $C$  exceeds the critical value.

### Example

In a test for the equality of  $k$  means (analysis of variance) it is assumed that the  $k$  populations have equal variances. In this situation a quality control inspector suspects that errors in data recording have led to one variance being larger than expected. She performs this test to see if her suspicions are well founded and, therefore, if she needs to repeat sampling for this population (a machine process line). Her test statistic,  $C = 0.302$  and the 5 per cent critical value from Table 15 is 0.4241. Since the test statistic is less than the critical value she has no need to suspect data collection error since the largest variance is not statistically different from the others.

### Numerical calculation

$$s_1^2 = 26, s_2^2 = 51, s_3^2 = 40, s_4^2 = 24, s_5^2 = 28$$

$$n_1 = n_2 = n_3 = n_4 = n_5 = 10, K = 5, v = n - 1 = 9$$

$$C = \frac{51}{26 + 51 + 40 + 24 + 28} = 0.302$$

Critical value  $C_{9;0.05} = 0.4241$  [Table 15].

The calculated value is less than the critical value.

Do not reject the null hypothesis.

## Test 35 The Kolmogorov–Smirnov test for goodness of fit

### Object

To investigate the significance of the difference between an observed distribution and a specified population distribution.

### Limitations

This test is applicable when the population distribution function is continuous.

### Method

From the sample, the cumulative distribution  $S_n(x)$  is determined and plotted as a step function. The cumulative distribution  $F(x)$  of the assumed population is also plotted on the same diagram.

The maximum difference between these two distributions

$$D = |F - S_n|$$

provides the test statistic and this is compared with the value  $D(\alpha)$  obtained from Table 16.

If  $D > D_\alpha$  the null hypothesis that the sample came from the assumed population is rejected.

### Example

As part of the calibration of a traffic flow model, a traffic engineer has collected a large amount of data. In one part of his model he wishes to test an assumption that traffic arrival at a particular road intersection follows a Poisson model, with mean 7.6 arrivals per unit time interval. Can he reasonably assume that this assumption is true?

His test statistic, maximum  $D$  is 0.332 and the 5 per cent critical value from Table 16 is 0.028. So he cannot assume such an arrival distribution and must seek another to use in his traffic model. Without a good distributional fit his traffic model would not produce robust predictions of flow.

### Numerical calculation

To test the hypothesis that the data constitute a random sample from a Poisson population with mean 7.6.

$$F(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \lambda = 7.6, n = 3366$$

$$S_n(x_i) = \frac{\text{cu}(x_i)}{3366}, F(x_i) = \frac{e^{-7.6} 7.6^6}{6!} = 0.3646 \text{ etc.}$$

No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$x_i$	5	14	24	57	111	197	278	378	418	461	433	413	358	219
$\text{cu}(x_i)$	5	19	43	100	211	408	686	1064	1482	1943	2376	2789	3147	3366
$S_n(x_i)$	0.001	0.005	0.012	0.029	0.062	0.121	0.204	0.316	0.440	0.577	0.705	0.828	0.935	1.0
$F(x_i)$	0.004	0.009	0.055	0.125	0.231	0.365	0.510	0.648	0.765	0.854	0.915	0.954	0.976	0.989
$D$	0.003	0.014	0.043	0.096	0.169	0.244	0.306	0.332	0.325	0.277	0.210	0.126	0.041	0.011

$$\max D = 0.332, D_{14;0.01} = \frac{1.63}{\sqrt{3366}} = \frac{1.63}{58.01} = 0.028 \text{ where } D > D_\alpha \text{ [Table 16].}$$

The hypothesis may be rejected.

**Test 36 The Kolmogorov–Smirnov test for comparing two populations**

**Object**

To investigate the significance of the difference between two population distributions, based on two sample distributions.

**Limitations**

The best results are obtained when the samples are sufficiently large – say, 15 or over.

**Method**

Given samples of size  $n_1$  and  $n_2$  from the two populations, the cumulative distribution functions  $S_{n_1}(x)$  and  $S_{n_2}(y)$  can be determined and plotted. Hence the maximum value of the difference between the plots can be found and compared with a critical value obtained from Table 16. If the observed value exceeds the critical value the null hypothesis that the two population distributions are identical is rejected.

**Example**

A quality control engineer uses an empirical distribution as a calibration for a particular machining process. He finds that he has better results using this than any other theoretical model. He compares his sample via the Kolmogorov–Smirnov test statistic  $d = 0.053$ . Since this is less than the 5 per cent critical value of Table 16 he can assume that both samples came from the same population.

**Numerical calculation**

$n_1 - n_2 = n = 10$

Sample $x$	0.6	1.2	1.6	1.7	1.7	2.1	2.8	2.9	3.0	3.2
cu( $x$ )	0.6	1.8	3.4	5.1	6.8	8.9	11.7	14.6	17.6	20.8
Sample $y$	2.1	2.3	3.0	3.1	3.2	3.2	3.5	3.8	4.6	7.2
cu( $y$ )	2.1	4.4	7.4	10.5	13.7	16.9	20.4	24.2	28.8	36.0
$S_{n_1}(x)$	0.029	0.086	0.163	0.245	0.327	0.428	0.562	0.702	0.846	1.0
$S_{n_2}(y)$	0.058	0.122	0.205	0.291	0.380	0.469	0.566	0.672	0.800	1.00
Difference $D$	0.029	0.036	0.042	0.046	0.053	0.041	0.004	0.03	0.046	0

$\max D = 0.053$

Critical value  $D_{20;0.01} = 0.356$  [Table 16].

Do not reject the hypothesis. Both samples come from the same population.

## Test 37 The $\chi^2$ -test for goodness of fit

### Object

To investigate the significance of the differences between observed data arranged in  $K$  classes, and the theoretically expected frequencies in the  $K$  classes.

### Limitations

1. The observed and theoretical distributions should contain the same number of elements.
2. The division into classes must be the same for both distributions.
3. The expected frequency in each class should be at least 5.
4. The observed frequencies are assumed to be obtained by random sampling.

### Method

The test statistic is

$$\chi^2 = \sum_{i=1}^K \frac{(O_i - E_i)^2}{E_i}$$

where  $O_i$  and  $E_i$  represent the observed and theoretical frequencies respectively for each of the  $K$  classes. This statistic is compared with a value obtained from  $\chi^2$  tables with  $\nu$  degrees of freedom. In general,  $\nu = K - 1$ . However, if the theoretical distribution contains  $m$  parameters to be estimated from the observed data, then  $\nu$  becomes  $K - 1 - m$ . For example, to fit data to a normal distribution may require the estimation of the mean and variance from the observed data. In this case  $\nu$  would become  $K - 1 - 2$ .

If  $\chi^2$  is greater than the critical value we reject the null hypothesis that the observed and theoretical distributions agree.

### Example

The experiment of throwing a die can be regarded as a general application and numerous specific situations can be postulated. In this case we have six classes or outcomes, each of which is equally likely to occur. The outcomes could be time (10 minute intervals) of the hour for an event to occur, e.g. failure of a component, choice of supermarket checkout by customers, consumer preferences for one of six types of beer, etc. Since the calculated value is less than the critical value from the table it can be reasonably assumed that the die is not biased towards any side or number. That is, it is a fair die. The outcomes are equally likely.

### Numerical calculation

A die is thrown 120 times. Denote the observed number of occurrences of  $i$  by  $O_i$ ,  $i = 1, \dots, 6$ . Can we consider the die to be fair at the 5 per cent level of significance?

$$K = 6, \nu = 6 - 1$$

$$O_1 = 25, O_2 = 17, O_3 = 15, O_4 = 23, O_5 = 24, O_6 = 16$$



$$E_1 = 20, E_2 = 20, E_3 = 20, E_4 = 20, E_5 = 20, E_6 = 20$$

$$\chi^2 = \frac{25}{20} + \frac{9}{20} + \frac{25}{20} + \frac{9}{20} + \frac{16}{20} + \frac{16}{20} = 5.0$$

Critical value  $\chi^2_{5;0.05} = 11.1$  [Table 5].

The calculated value is less than the critical value.

Hence there are no indications that the die is not fair.

## Test 38 The $\chi^2$ -test for compatibility of $K$ counts

### Object

To investigate the significance of the differences between  $K$  counts.

### Limitations

The counts must be obtained under comparable conditions.

### Method

#### Method (a) Times for counts are all equal

Let the  $i$ th count be denoted by  $N_i$ . Then the null hypothesis is that  $N_i = \text{constant}$ , for  $i = 1, \dots, K$ . The test statistic is

$$\chi^2 = \sum_{i=1}^K \frac{(N_i - \bar{N})^2}{\bar{N}}$$

where  $\bar{N}$  is the mean of the  $K$  counts  $\sum_{i=1}^K N_i / K$ . This is compared with a value obtained from  $\chi^2$  tables with  $K - 1$  degrees of freedom. If  $\chi^2$  exceeds this value the null hypothesis is rejected.

#### Method (b) Times for counts are not all equal

Let the time to obtain the  $i$ th count be  $t_i$ . The test statistic becomes

$$\chi^2 = \sum_{i=1}^K \frac{(N_i - t_i \bar{R})^2}{t_i \bar{R}}$$

where  $\bar{R} = \sum N_i / \sum t_i$ . This is compared with Table 5 as above.

### Example

A lime producing rotary kiln is operated on a shift-based regime with four shift workers. A training system is to be adopted and it is desired to have some idea about how the workers operate in terms of out-of-control warning alerts. Do all the four shift workers operate the kiln in a similar way? The number of alerts over a full shift is recorded and the test statistic is calculated. Do the four workers operate the kiln in a similar way? The answer to this question has bearing on the sort of training that could be implemented for the workers.

The chi-squared test statistic is 13.6 and the 5 per cent critical value of the chi-squared distribution from Table 5 is 7.81. Since the calculated chi-squared value exceeds the 5 per cent critical value we reject the null hypothesis that the counts are effectively the same and conclude that the four counts are not consistent with each other. The four workers operate the kiln differently. Hence different training schemes may be appropriate.

**Numerical calculation**

$N_1 = 5, N_2 = 12, N_3 = 18, N_4 = 19, \bar{N} = 11, v = K - 1 = 4 - 1 = 3$

Using method (a),  $\chi^2 = 13.6$ .

The critical value  $\chi^2_{3;0.05} = 7.81$  [Table 5].

Hence reject the null hypothesis. The four counts are not consistent with each other.

## Test 39 Fisher's exact test for consistency in a $2 \times 2$ table

### Object

To investigate the significance of the differences between observed frequencies for two dichotomous distributions.

### Limitations

This test is applicable if the classification is dichotomous and the elements originate from two sources. It is usually applied when the number of elements is small or the expected frequencies are less than 5.

### Method

A  $2 \times 2$  contingency table is built up as follows:

	Class 1	Class 2	Total
Sample 1	$a$	$b$	$a + b$
Sample 2	$c$	$d$	$c + d$
Total	$a + c$	$b + d$	$n = a + b + c + d$

The test statistic is

$$\sum p = \frac{(a+b)!(c+d)!(a+c)!(b+d)!}{n!} \sum_i \frac{1}{a_i!b_i!c_i!d_i!}$$

where the summation is over all possible  $2 \times 2$  schemes with a cell frequency equal to or smaller than the smallest experimental frequency (keeping the row and column totals fixed as above).

If  $\sum p$  is less than the significance level chosen, we may reject the null hypothesis of independence between samples and classes, i.e. that the two samples have been drawn from one common population.

### Example

A medical officer has data from two groups of potential airline pilot recruits. Two different reaction tests have been used in selection of the potential recruits. He uses a more sophisticated test and finds the results given for the two samples. Class 1 represents quick reactions and class 2 represents less speedy reactions. Is there a difference between the two selection tests?

Since some cell frequencies are less than 5 the medical officer uses Fisher's exact test, which produces a probability of 0.156. This is greater than 0.05 (our usual level) so the medical officer does not reject the null hypothesis of no difference between the selection tests.

**Numerical calculation**

class 1 = accepted, class 2 = rejected, significance level  $\alpha = 0.05$

$a = 9, b = 2, c = 7, d = 6, a + b = 11, c + d = 13, a + c = 16, b + d = 8$

Two possible sets of data which deviate more from the null hypothesis are  $a = 10, b = 1, c = 6, d = 7$  and  $a = 11, b = 0, c = 5, d = 8$ .

We add the three probabilities of the three schemes according to hyper-geometric distributions. This gives

$$\sum p = \frac{11! 13! 16! 8!}{24!} \left\{ \frac{1}{2! 6! 7! 9!} + \frac{1}{1! 6! 7! 10!} + \frac{1}{0! 5! 8! 11!} \right\} = 0.156.$$

This is greater than 0.05. Do not reject the null hypothesis.

## Test 40 The $\chi^2$ -test for consistency in a $2 \times 2$ table

### Object

To investigate the significance of the differences between observed frequencies for two dichotomous distributions.

### Limitations

It is necessary that the two sample sizes are large enough. This condition is assumed to be satisfied if the total frequency is  $n > 20$  and if all the cell frequencies are greater than 3. When continuous distributions are applied to discrete values one has to apply Yates' correction for small sample sizes.

### Method

When two samples are each divided into two classes the following  $2 \times 2$  table can be built up:

	Class 1	Class 2	Total
Sample 1	$a$	$b$	$a + b$
Sample 2	$c$	$d$	$c + d$
Total	$a + c$	$b + d$	$n = a + b + c + d$

The test statistic is

$$\chi^2 = \frac{(n-1)(ad-bc)^2}{(a+b)(a+c)(b+d)(c+d)}.$$

This is compared with a value obtained from  $\chi^2$  tables with 1 degree of freedom. If  $\chi^2$  exceeds the critical value the null hypothesis of independence between samples and classes is rejected. In other words, the two samples were not drawn from one common population.

### Example

In this case the medical officer (in Test 39) has a larger sample and can use the chi-squared test.

He obtains a chi-squared value of 4.79 which is greater than the tabulated value of 2.7 from Table 5. Hence he rejects the null hypothesis and now concludes that the two selection methods do produce when compared with the sophisticated reaction test.

### Numerical calculation

$$a = 15, b = 85, c = 4, d = 77$$

$$a + b = 100, c + d = 81, a + c = 19, b + d = 162$$

$$\alpha = 0.10, \nu = 1, \chi_{1;0.10}^2 = 2.7, n = 181 \text{ [Table 5]}$$

$$\chi^2 = \frac{180(15 \times 77 - 4 \times 85)^2}{100 \times 81 \times 19 \times 162} = 4.79$$

Hence reject the null hypothesis,  $H_0$ .

## Test 41 The $\chi^2$ -test for consistency in a $K \times 2$ table

### Object

To investigate the significance of the differences between  $K$  observed frequency distributions with a dichotomous classification.

### Limitations

It is necessary that the  $K$  sample sizes are large enough. This is usually assumed to be satisfied if the cell frequencies are equal to 5.

### Method

When the observations in  $K$  samples are divided into two classes the following  $K \times 2$  table can be built up:

	Class 1	Class 2	Total
Sample 1	$x_1$	$n_1 - x_1$	$n_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Sample $i$	$x_i$	$n_i - x_i$	$n_i$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Sample $K$	$x_K$	$n_K - x_K$	$n_K$
Total	$x = \sum_{i=1}^K x_i$	$n - x$	$n = \sum_{i=1}^K n_i$

The test statistic is

$$\chi^2 = \frac{n^2}{x(n-x)} \left\{ \left( \sum_{i=1}^K \frac{x_i^2}{n_i} \right) - \left( \frac{x^2}{n} \right) \right\}.$$

This is compared with a value obtained from  $\chi^2$  tables with  $K - 1$  degrees of freedom. The null hypothesis of independence between samples and classes is rejected if  $\chi^2$  exceeds the critical value.

### Example

Our medical officer (as in Test 39) now has four different colour-blindness selection tests and wishes to see if they produce differences when compared with the recruitment standards. Her data produce a chi-squared value of 5.495 compared with the tabulated value of 5.99. She does not reject the null hypothesis and concludes that the colour blindness tests do not differ in their outcome.

**Numerical calculation**

$$x_{11} = 14, x_{12} = 22, x_{21} = 18, x_{22} = 16, x_{31} = 8, x_{32} = 2$$

(Here  $x_{i1} = x_1$  and  $x_{i2} = n_1 - x_1$ .)

$$n_1 = 36, n_2 = 34, n_3 = 10, n = \sum n_i = 80$$

$$\alpha = 0.05, \nu = 2, \chi^2_{2,0.05} = 5.99, n = 80, x = 40 \text{ [Table 5]}$$

$$\chi^2 = \frac{80^2}{40 \times 40} \left[ \frac{14^2}{36} + \frac{18^2}{34} + \frac{8^2}{10} \right] - \frac{40^2}{80} = 5.495$$

Hence do not reject the null hypothesis.



## Test 42 The Cochran test for consistency in an $n \times K$ table of dichotomous data

### Object

To investigate the significance of the differences between  $K$  treatments on the same  $n$  elements with a binomial distribution.

### Limitations

1. It is assumed that there are  $K$  series of observations on the same  $n$  elements.
2. The observations are dichotomous and the observations in the two classes are represented by 0 or 1.
3. The number of elements must be sufficiently large – say, greater than 10.

### Method

From the  $n \times K$  table, let  $R_i$  denote the row totals ( $i = 1, \dots, n$ ) and  $C_j$  denote the column totals ( $j = 1, \dots, K$ ). Let  $S$  denote the total score, i.e.  $S = \sum_i R_i = \sum_j C_j$ . The test statistic is

$$Q = \frac{K(K-1) \sum_j (C_j - \bar{C})^2}{KS - \sum_i R_i^2} \quad \text{where } \bar{C} = \frac{\sum_j C_j}{K}.$$

This approximately follows a  $\chi^2$ -distribution with  $K - 1$  degrees of freedom.

The null hypothesis that the  $K$  samples come from one common dichotomous distribution is rejected if  $Q$  is larger than the tabulated value.

### Example

A panel of expert judges assess whether each of four book cover formats is acceptable or not. Each book cover format, therefore, receives an acceptability score. The Cochran  $Q$  statistic is calculated as 12.51, which is larger than the tabulated chi-squared value of 7.81 [Table 5]. It seems that the book covers are not equally acceptable to the judges.

### Numerical calculation

$$K = 4, C_1 = 12, C_2 = 8, C_3 = 6, C_4 = 3, \sum C_i^2 = 253$$

$$\sum R_i = 29, \bar{C} = 7.25, \sum R_i^2 = 75$$

$$\alpha = 0.05, \nu = 3, \chi_{3;0.05}^2 = 7.81 \text{ [Table 5]}$$

$$Q = \frac{4(3 \times 253 - 29^2)}{4 \times 29 - 75} = \frac{513}{41} = 12.51$$

Hence reject the null hypothesis.

## Test 43 The $\chi^2$ -test for consistency in a $2 \times K$ table

### Object

To investigate the significance of the differences between two distributions based on two samples spread over  $K$  classes.

### Limitations

1. The two samples are sufficiently large.
2. The  $K$  classes when put together form a complete series.

### Method

The  $2 \times K$  table can be described symbolically by the following table:

	Class						Total
	1	2	...	$j$	...	$K$	
Sample 1	$n_{11}$	$n_{12}$	...	$n_{1j}$	...	$n_{1K}$	$N_1$
Sample 2	$n_{21}$	$n_{22}$	...	$n_{2j}$	...	$n_{2K}$	$N_2$
Total	$n_{.1}$	$n_{.2}$	...	$n_{.j}$	...	$n_{.K}$	$N_1 + N_2$

where  $n_{ij}$  represents the frequency of individuals in the  $i$ th sample in the  $j$ th class ( $i = 1, 2$  and  $j = 1, \dots, K$ ). Another table of expected frequencies is now calculated where the value in the  $i$ th row and  $j$ th column is

$$e_{ij} = \frac{N_i n_{.j}}{N_1 + N_2}.$$

The test statistic is

$$\chi^2 = \sum_{j=1}^K \frac{(n_{1j} - e_{1j})^2}{e_{1j}} + \sum_{j=1}^K \frac{(n_{2j} - e_{2j})^2}{e_{2j}}.$$

This is compared with the value obtained from a  $\chi^2$  table with  $K - 1$  degrees of freedom. If  $\chi^2$  exceeds this critical value, the null hypothesis that the two samples originate from two populations with the same distribution is rejected.

### Example

Our medical officer from Tests 39 and 40 now wishes to have a third (or middle) class which represents a reserve list of potential recruits who do not quite satisfy the stringent class1 requirements. She can still use the chi-squared test to compare the reaction selection tests. Her data produce a chi-squared value of 4.84, which is less than the tabulated critical value of 5.99. She concludes that the reaction tests do not differ when a third classification of an intermediate reaction is introduced.

**Numerical calculation**

$n_{11} = 50, n_{12} = 47, n_{13} = 56, n_{21} = 5, n_{22} = 14, n_{23} = 8$   
 $n_{.1} = 55, n_{.2} = 61, n_{.3} = 64, N_1 = 153, N_2 = 27, N = 180$   
 $e_{11} = 46.75, e_{12} = 51.85, e_{13} = 54.40, e_{21} = 8.25, e_{22} = 9.15, e_{23} = 9.60$   
 $\alpha = 0.05, \nu = (3 - 1)(2 - 1) = 2, \chi^2_{2;0.05} = 5.99$  [Table 5]

$$\chi^2 = \frac{3.25^2}{46.75} + \frac{(-4.85)^2}{51.85} + \frac{1.6^2}{54.40} + \frac{(-3.25)^2}{8.25} + \frac{4.85^2}{9.15} + \frac{(-1.6)^2}{9.60} = 4.84$$

Do not reject the null hypothesis.

## Test 44 The $\chi^2$ -test for independence in a $p \times q$ table

### Object

To investigate the difference in frequency when classified by one attribute after classification by a second attribute.

### Limitations

The sample should be sufficiently large. This condition will be satisfied if each cell frequency is greater than 5.

### Method

The sample, of size  $N$ , can be categorized into  $p$  classes by the first attribute and into  $q$  classes by the second. The frequencies of individuals in each classification can be shown symbolically by the table:

		First attribute						Total
		1	2	...	$i$	...	$p$	
Second attribute	1	$n_{11}$	$n_{21}$	...	$n_{i1}$	...	$n_{p1}$	$n_{.1}$
	2	$n_{12}$	$n_{22}$	...	$n_{i2}$	...	$n_{p2}$	$n_{.2}$
	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	$j$	$n_{1j}$	$n_{2j}$	...	$n_{ij}$	...	$n_{pj}$	$n_{.j}$
	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	$q$	$n_{1q}$	$n_{2q}$	...	$n_{iq}$	...	$n_{pq}$	$n_{.q}$
	Total	$n_{1.}$	$n_{2.}$	...	$n_{i.}$	...	$n_{p.}$	$N$

The test statistic is

$$\chi^2 = \sum_{i=1}^p \sum_{j=1}^q \frac{(n_{ij} - n_{i.}n_{.j}/N)^2}{n_{i.}n_{.j}/N}$$

which follows a  $\chi^2$  distribution with  $(p-1)(q-1)$  degrees of freedom. If  $\chi^2$  exceeds the critical value then the null hypothesis that the two attributes are independent of each other is rejected.

### Example

An educational researcher has collected data on parents' preferences for their children's education. There are three categories of parents' educational achievement level (1, low; 2, medium; 3, high) and two levels for preferences (1, yes; 2, no). She calculates a chi-squared value of 10.67 and compares this with a value of 5.99 from the table. She, therefore, rejects the null hypothesis and concludes that parents' preferences vary with their own educational achievement.

**Numerical calculation**

$n_{11} = 32, n_{21} = 12, n_{12} = 14, n_{22} = 22, n_{13} = 6, n_{23} = 9$   
 $n_{1\cdot} = 52, n_{2\cdot} = 43, n_{\cdot 1} = 44, n_{\cdot 2} = 36, n_{\cdot 3} = 15, N = 95$   
 $\alpha = 0.05, \nu = (3 - 1)(2 - 1) = 2, \chi^2_{2;0.05} = 5.99$  [Table 5]

$$\chi^2 = \frac{7.9^2}{24.1} + \frac{(-7.9)^2}{19.9} + \frac{(-5.7)^2}{19.7} + \frac{(5.7)^2}{16.3} + \frac{(-2.2)^2}{8.2} + \frac{2.2^2}{6.8} = 10.67$$

Hence reject the null hypothesis.

## Test 45 The sign test for a median

### Object

To investigate the significance of the difference between a population median and a specified value  $M_0$ .

### Limitations

It is assumed that the observations in the sample are independent of each other. Any sample values equal to  $M_0$  should be discarded from the sample.

### Method

A count is made of the number  $n_1$  of sample values exceeding  $M_0$ , and also of the number  $n_2$  below  $M_0$ . The null hypothesis is that the population median equals  $M_0$ . If the alternative hypothesis is that the population median does not equal  $M_0$  then the test statistic,  $T$ , is the smaller of  $n_1$  and  $n_2$  with  $n$  taken as the sum of  $n_1$  and  $n_2$ .

If the alternative hypothesis is that the population median is greater than  $M_0$ , then  $T = n_1$ . If the alternative hypothesis is that the population median is greater than  $M_0$ , then  $T = n_2$ . The null hypothesis is rejected if  $T$  is greater than the critical value obtained from Table 17.

### Example

It is assumed that the median value of a financial ratio is 0.28; this being the recycled material cost for new build domestic constructions. A random sample of ten new builds is taken and the ratios computed. Can it be assumed that the sample has been taken from a population of ratios with median 0.28? Since the calculated  $T$  value of 4 is less than the critical value of 7 (from Table 17) this assumption is accepted.

### Numerical calculation

Sample values  $x_1 = 0.28, x_2 = 0.18, x_3 = 0.24, x_4 = 0.30, x_5 = 0.40$   
 $x_6 = 0.36, x_7 = 0.15, x_8 = 0.42, x_9 = 0.23, x_{10} = 0.48$

Null hypothesis:  $H_0 = 0.28$

$n_1 = 5, n_2 = 4, T = 4, n = 5 + 4 = 9$

The critical value at  $\alpha = 0.05$  is 7 [Table 17].

Hence do not reject the null hypothesis.

**Test 46 The sign test for two medians (paired observations)**

**Object**

To investigate the significance of the difference between the medians of two distributions.

**Limitations**

The observations in the two samples should be taken in pairs, one from each distribution. Each one of a pair of observations should be taken under the same conditions, but it is not necessary that different pairs should be taken under similar conditions.

It is not necessary to take readings provided the sign of the difference between two observations of a pair can be determined.

**Method**

The signs of the differences between each pair of observations are recorded. The test statistic,  $r$ , is the number of times that the least frequent sign occurs. If this is less than the critical value obtained from Table 18 the null hypothesis that the two population medians are equal is rejected.

**Example**

A quality engineer takes two samples from a production line, one before a maintenance modification and one after. Has the modification altered the median value of a critical measurement (standard units) from the production items? For each pair of values the production machine settings are the same. He obtains a value of  $r = 3$  and compares this with a value of 1 from Table 18. The maintenance has altered the median value since the critical value is less than the calculated value.

**Numerical calculation**

$x_i$	0.19	0.22	0.18	0.17	1.20	0.14	0.09	0.13	0.26	0.66
$y_i$	0.21	0.27	0.15	0.18	0.40	0.08	0.14	0.28	0.30	0.68
Sign	—	—	+	—	+	+	—	—	—	—

There are 3 plus signs. 7 minus signs,  $r = 3$   
 $n = 10, r_{10;0.10} = 1$  [Table 18].  
Do not reject the null hypothesis.

## Test 47 The signed rank test for a mean

### Object

To investigate the significance of the difference between a population mean and a specified value  $\mu_0$ .

### Limitations

This is a distribution-free test and requires a symmetrical population. The observations must be obtained randomly and independently from a continuous distribution.

### Method

From the sample values  $x_i$  determine the differences  $x_i - \mu_0$  and arrange them in ascending order irrespective of sign. Sample values equal to  $x_i - \mu_0 = 0$  are not included in the analysis.

Rank numbers are now assigned to the differences. Where ties occur among differences, the ranks are averaged among them. Then each rank number is given the sign of the corresponding difference  $x_i - \mu_0$ .

The sum of the ranks with a positive sign and the sum of the ranks with a negative sign are calculated. The test statistic  $T$  is the smaller of these two sums. Critical values of this statistic can be found from Table 17. When the value of  $T$  falls in the critical region, i.e. less than the tabulated values the null hypothesis that the population mean is equal to  $\mu_0$  is rejected.

### Example

The mean deposit rate (GBP per savings level) for a sample of ten investors is examined to see if mail advertising has altered this from a value of 0.28. The signed rank test is used and produces a  $T$  value of 17. Since this calculated value is greater than the tabulated value we do not reject the null hypotheses. It would appear that the advertising has not altered the mean deposit level.

### Numerical calculation

$$\mu_0 = 0.28, n = 10, \alpha = 0.05, T_{9;\alpha} = 7$$

Here  $n = 10 - 1 = 9$  (one zero).

$x_i$	0.28	0.18	0.24	0.30	0.40	0.36	0.15	0.42	0.23	0.48
$x_i - \mu_0$	0	-0.10	-0.04	+0.02	+0.12	+0.08	-0.13	+0.14	-0.05	+0.20
Signed rank	—	-5	-2	+1	+6	+4	-7	+8	-3	+9

Sum of plus ranks = 28, sum of minus ranks = 17,  $T = 17$

$T > T_{9;\alpha}$  [Table 17].

Hence do not reject the null hypothesis.



**Test 48    The signed rank test for two means (paired observations)**

**Object**

To investigate the significance of the difference between the means of two similarly shaped distributions.

**Limitations**

The observations in the two samples should be taken in pairs, one from each distribution. Each one of a pair of observations should be taken under the same conditions, but it is not necessary that different pairs should be taken under similar conditions. Any pair of observations giving equal values will be ignored in the analysis.

**Method**

The differences between pairs of observations are formed and these are ranked, irrespective of sign. Where ties occur, the average of the corresponding ranks is used. Then each rank is allocated the sign from the corresponding difference.

The sum of the ranks with a positive sign and the sum of the ranks with a negative sign are calculated. The test statistic  $T$  is the smaller of these two sums. Critical values of this statistic can be found from Table 19. When the value of  $T$  is less than the critical value, the null hypothesis of equal population means is rejected.

**Example**

A manually operated component punch produces two springs at each operation. It is desired to test if the mean component specification differs between the two springs. The sample of pairs of springs produces a signed rank test statistic,  $T$ , of 11, which is less than the tabulated value of 17. Hence the null hypothesis of no difference is rejected. The punch needs re-setting.

**Numerical calculation**

$x_i$	1.38	9.69	0.39	1.42	0.54	5.94	0.59	2.67	2.44	0.56	0.69	0.71	0.95	0.50
$y_i$	1.42	10.37	0.39	1.46	0.55	6.15	0.61	2.69	2.68	0.53	0.72	0.72	0.93	0.53
$x_i - y_i$	-0.04	-0.68	0	-0.04	-0.01	-0.21	-0.02	-0.02	-0.24	+0.03	-0.03	-0.01	+0.02	-0.03
Rank	-9.5	-13	0	-9.5	-1.5	-11	-4	-4	-12	+7	-7	-1.5	+4	-7

Minus signs = 11, plus signs = 2, rank for plus sign = 4 + 7 = 11  
 $T_{13;0.05} = 17$  [Table 19]  
Reject the null hypothesis.

## Test 49 The Wilcoxon inversion test ( $U$ -test)

### Object

To test if two random samples could have come from two populations with the same frequency distribution.

### Limitations

It is assumed that the two frequency distributions are continuous and that the two samples are random and independent.

### Method

Samples of size  $n_1$  and  $n_2$  are taken from the two populations. When the two samples are merged and arranged in ascending order, there will be a number of jumps (or inversions) from one series to the other. The smaller of the number of inversions and the number of non-inversions forms the test statistic,  $U$ . The null hypothesis of the same frequency distribution is rejected if  $U$  exceeds the critical value obtained from Table 20.

### Example

An educational researcher has two sets of adjusted reading scores for two sets of five pupils who have been taught by different methods. It is possible that the two samples could have come from the same population frequency distribution.

The collected data produce a calculated  $U$  value of 4. Since the sample  $U$  value equals the tabulated critical value the educational researcher rejects the null hypothesis of no difference. The data suggest that the two reading teaching methods produce different results.

### Numerical calculation

$$n_1 = 5, n_2 = 5, \alpha = 0.05$$

$x_i$	11.79	11.21	13.20	12.66	13.37
$y_i$	10.34	11.40	10.19	12.10	11.46

Rearrangement gives the following series

$$10.19, 10.34, \frac{11.21}{(2)}, 11.40, 11.46, \frac{11.79}{(4)}, 12.10, \frac{12.66}{(5)}, \frac{13.20}{(5)}, \frac{13.37}{(5)}$$

where underlined values come from the first row ( $x_i$ ). Below these underlines, the corresponding number of inversions, i.e. the number of times a  $y$ -value comes after an  $x$ -value, is given in parentheses.

The number of inversions is  $2 + 4 + 5 + 5 + 5 = 21$ .

The number of non-inversions is  $n_1 n_2 - 21 = 25 - 21 = 4$ .

The critical value at  $\alpha = 0.05$  is 4 [Table 20].

The sample value of  $U$  is equal to the critical value.

The null hypothesis may be rejected; alternatively, the experiment could be repeated by collecting a second set of data.

## Test 50 The median test of two populations

### Object

To test if two random samples could have come from two populations with the same frequency distribution.

### Limitations

The two samples are assumed to be reasonably large.

### Method

The median of the combined sample of  $n_1 + n_2$  elements is found. Then, for each series in turn, the number of elements above and below this median can be found and entered in a  $2 \times 2$  table of the form:

	Sample 1	Sample 2	Total
Left of median	$a$	$b$	$a + b$
Right of median	$c$	$d$	$c + d$
Total	$n_1 = a + c$	$n_2 = b + d$	$N = n_1 + n_2$

The test statistic is

$$\chi^2 = \frac{\{|ad - bc| - \frac{1}{2}N\}^2 N}{(a + b)(a + c)(b + d)(c + d)}.$$

If this value exceeds the critical value obtained from  $\chi^2$  tables with one degree of freedom, the null hypothesis of the same frequency distribution is rejected.

### Example

A housing officer has data relating to residents' assessment of their housing conditions in a small, isolated estate. Half of the houses in the estate are maintained by one maintenance company and the other half by another company. Do the repair regimes of the two companies produce similar results from the residents? Samples of 15 residents are taken from each half. The calculated chi-squared value is 0.53, which is less than the tabulated value of 3.84. The housing officer does not reject the null hypothesis and concludes that the two maintenance companies produce similar results from their repair regimes.

### Numerical calculation

$$a = 9, b = 6, c = 6, d = 9$$

$$a + b = a + c = b + d = c + d = 15$$

$$n_1 = 15, n_2 = 15, N = 30$$

$$\chi^2 = \frac{\{|9^2 - 6^2| - 15\}^2 \times 30}{15 \times 15 \times 15 \times 15} = \frac{8}{15} = 0.53$$

$$\chi_{1;0.05}^2 = 3.84 \text{ [Table 5]}$$

Do not reject the null hypothesis.

## Test 51 The median test of $K$ populations

### Object

To test if  $K$  random samples could have come from  $K$  populations with the same frequency distribution.

### Limitations

The  $K$  samples are assumed to be reasonably large – say, greater than 5.

### Method

The  $K$  samples are first amalgamated and treated as a single grand sample, of which the median is found. Then, for each of the  $K$  samples, the number of elements above and below this median can be found. These can be arranged in the form of a  $2 \times K$  table and then a  $\chi^2$ -test can be carried out.

	Sample						
	1	2	...	$j$	...	$K$	Total
Above median	$a_{11}$	$a_{12}$	...	$a_{1j}$	...	$a_{1K}$	A
Below median	$a_{21}$	$a_{22}$	...	$a_{2j}$	...	$a_{2K}$	B
Total	$a_1$	$a_2$	...	$a_j$	...	$a_K$	$N$

In this table,  $a_{1j}$  represents the number of elements above the median and  $a_{2j}$  the number of elements below the median in the  $j$ th sample ( $j = 1, 2, \dots, K$ ). Expected frequencies are calculated from

$$e_{1j} = \frac{Aa_j}{N} \quad \text{and} \quad e_{2j} = \frac{Ba_j}{N}.$$

The test statistic is

$$\chi^2 = \sum_{j=1}^K \frac{(a_{1j} - e_{1j})^2}{e_{1j}} + \sum_{j=1}^K \frac{(a_{2j} - e_{2j})^2}{e_{2j}}.$$

This is compared with a critical value from Table 5 with  $K - 1$  degrees of freedom. The null hypothesis that the  $K$  populations have the same frequency distribution is rejected if  $\chi^2$  exceeds the critical value.

### Example

The housing officer in test 50 has a larger estate which is maintained by five maintenance companies. He has sampled the residents receiving maintenance from each company in proportion to the number of houses each company maintains. The officer now produces a chi-squared value of 0.2041. Do the five maintenance companies differ in their effect on resident's assessment? The tabulated chi-squared value is 9.49, so the officer concludes that the standards of maintenance are the same.

**Numerical calculation**

	1	2	Sample			Total
	3	4	5			
Above median	20	30	25	40	30	145
Below median	25	35	30	45	32	167
Total	45	65	55	85	62	312

$$e_{11} = \frac{145 \times 45}{312} = 20.91 \quad e_{21} = \frac{167 \times 45}{312} = 24.08$$

$$e_{12} = 30.21 \quad e_{22} = 34.79$$

$$e_{13} = 25.56 \quad e_{23} = 29.44$$

$$e_{14} = 39.50 \quad e_{24} = 45.50$$

$$e_{15} = 28.81 \quad e_{25} = 33.19$$

$$\begin{aligned} \chi^2 &= 0.0396 + 0.0015 + 0.0123 + 0.0063 + 0.0491 \\ &\quad + 0.0351 + 0.0013 + 0.0107 + 0.0055 + 0.0427 \\ &= 0.2041 \end{aligned}$$

$$\chi_{4;0.05}^2 = 9.49 \text{ [Table 5].}$$

Hence do not reject the null hypothesis.

## Test 52 The Wilcoxon–Mann–Whitney rank sum test of two populations

### Object

To test if two random samples could have come from two populations with the same mean.

### Limitations

It is assumed that the two populations have continuous frequency distributions with the same shape and spread.

### Method

The results of the two samples  $x$  and  $y$  are combined and arranged in order of increasing size and given a rank number. In cases where equal results occur the mean of the available rank numbers is assigned. The rank sum  $R$  of the smaller sample is now found. Let  $N$  denote the size of the combined samples and  $n$  denote the size of the smaller sample.

A second quantity

$$R^1 = n(N + 1) - R$$

is now calculated. The values  $R$  and  $R^1$  are compared with critical values obtained from Table 21. If either  $R$  or  $R^1$  is less than the critical value the null hypothesis of equal means would be rejected.

*Note* If the samples are of equal size, then the rank sum  $R$  is taken as the smaller of the two rank sums which occur.

### Example

A tax inspector wishes to compare the means of two samples of expenses claims taken from the same company but separated by a period of time (the values have been adjusted to account for inflation). Are the mean expenses for the two periods the same? He calculates a test statistic,  $R^1$  of 103 and compares this with the tabulated value of 69. Since the calculated value is greater than the tabulated critical value he concludes that the mean expenses have not changed.

### Numerical calculation

											Total
$x$	50.5	37.5	49.8	56.0	42.0	56.0	50.0	54.0	48.0		
Rank	9	1	7	15.5	2	15.5	8	13	6		77
$y$	57.0	52.0	51.0	44.2	55.0	62.0	59.0	45.2	53.5	44.4	
Rank	17	11	10	3	14	19	18	5	12	4	113

$$R = 77, n_1 = 9, n_2 = 10, N = 19, R^1 = 9 \times 20 - 77 = 103$$

The critical value at  $\alpha = 0.05$  is 69 [Table 21].

Hence there is no difference between the two means.

## Test 53 The Siegel–Tukey rank sum dispersion test of two variances

### Object

To test if two random samples could have come from two populations with the same variance.

### Limitations

It is assumed that the two populations have continuous frequency distributions and that the sample sizes are not too small, e.g.  $n_1 + n_2 > 20$ .

### Method

The results of the two samples are combined and arranged in order of increasing size. Ranks are allocated according to the following scheme:

- The lowest value is ranked 1.
- The highest two values are ranked 2 and 3 (the largest value is given the value 2).
- The lowest two unranked values are ranked 4 and 5 (the smallest value is given the value 4).
- The highest two unranked values are ranked 6 and 7 (the largest value is given the value 6).

This procedure continues, working from the ends towards the centre, until no more than one unranked value remains. That is to say, if the number of values is odd, the middle value has no rank assigned to it.

Let  $n_1$  and  $n_2$  denote the sizes of the two samples and let  $n_1 \leq n_2$ . Let  $R_1$  be the rank sum of the series of size  $n_1$ . The test statistic is

$$Z = \frac{R_1 - n_1(n_1 + n_2 + 1)/2 + \frac{1}{2}}{\sqrt{n_1 n_2 (n_1 + n_2 + 1)/12}}.$$

This will approximately follow a standard normal distribution. The null hypothesis of equal variance is rejected if  $Z$  falls in the critical region.

### Example

A catering manager wants to know if two types of pre-prepared sauce give the same spread or variability of values. This is because he has to set his dispensers to a fixed value and an unusually large value will cause problems. He takes a sample of ten sauces of each type and compares them using the Siegel–Tukey rank sum dispersion test. He produces a  $Z$  value of  $-2.154$  which is outside the acceptance region [Table 1] of  $\pm 1.96$ . He rejects the null hypothesis of no difference and concludes, in this case, that sauce type  $y$  has greater dispersion than type  $x$ .

**Numerical calculation**

Combined rank assignment of two sample data  $x, y$ :

Sample	$x$	$y$	$y$	$y$	$y$	$y$	$x$	$x$	$y$	$x$
Value	2.4	2.9	3.3	3.6	4.2	4.9	6.1	7.3	7.3	8.5
Rank	1	4	5	8	9	12	13	16	17	20

Sample	$x$	$x$	$x$	$x$	$x$	$y$	$x$	$y$	$y$	$y$
Value	8.8	9.4	9.8	10.1	10.1	11.7	12.6	13.1	15.3	16.5
Rank	19	18	15	14	11	10	7	6	3	2

$$n_1 = n_2 = 10$$

$$R_x = 1 + 13 + 16 + 20 + 19 + 18 + 15 + 14 + 11 + 7 = 134$$

$$R_y = 4 + 5 + 8 + 9 + 12 + 17 + 10 + 6 + 3 + 2 = 76$$

$$\text{Hence } R_1 = 76$$

$$Z = \frac{76 - 10(10 + 10 + 1)/2 + \frac{1}{2}}{\sqrt{10 \cdot 10(10 + 10 + 1)/12}} = \frac{-28.5}{\sqrt{175}} = \frac{-28.5}{13.23} = -2.154$$

The critical values at  $\alpha = 0.05$  are  $-1.96$  and  $+1.96$  [Table 1].

Hence reject the null hypothesis.



## Test 54 The Kruskal–Wallis rank sum test of $K$ populations ( $H$ -test)

### Object

To test if  $K$  random samples could have come from  $K$  populations with the same mean.

### Limitations

Each sample size should be at least 5 in order for  $\chi^2$  to be used, though sample sizes need not be equal. The  $K$  frequency distributions should be continuous.

### Method

The  $K$  samples are combined and arranged in order of increasing size and given a rank number. Where ties occur the mean of the available rank numbers is used. The rank sum for each of the  $K$  samples is calculated.

Let  $R_j$  be the rank sum of the  $j$ th sample,  $n_j$  be the size of the  $j$ th sample, and  $N$  be the size of the combined sample. The test statistic is

$$H = \left\{ \frac{12}{N(N+1)} \sum_{j=1}^K \frac{R_j^2}{n_j} \right\} - 3(N+1).$$

This follows a  $\chi^2$ -distribution with  $K - 1$  degrees of freedom. The null hypothesis of equal means is rejected when  $H$  exceeds the critical value. Critical values of  $H$  for small sample sizes and  $K = 3, 4, 5$  are given in Table 22.

### Example

A cake preference score is a combination of four components, viz. tastes, appearance, smell and texture. The minimum score is 0 and the maximum 100. Three cake formulations are compared using these scores by three panels of accredited tasters. The results produce an  $H$  test statistic of 2.15. This is less than the tabulated value of 4.61 [Table 5]. The catering manager concludes the three cake formulations are equally preferred.

### Numerical calculation

Combined rank assignment of three sample data  $x_1, x_2, x_3$ :

Sample	$x_1$	$x_1$	$x_1$	$x_1$	$x_1$	$x_1$	$x_1$	$x_1$	$x_1$	$x_2$
Value	1.7	1.9	6.1	12.5	16.5	25.1	30.5	42.1	82.5	13.6
Rank	1	2	3	4	7	10.5	14	15	20	6

Sample	$x_2$	$x_2$	$x_2$	$x_2$	$x_2$	$x_3$	$x_3$	$x_3$	$x_3$	$x_3$
Value	19.8	25.2	46.2	46.2	61.1	13.4	20.9	25.1	29.7	46.9
Rank	8	12	16.5	16.5	19	5	9	10.5	13	18

$$R_1 = 76.5, R_2 = 78.0, R_3 = 55.5$$

$$H = \frac{12}{420} (2280.30) - 63 = 2.15, \chi^2_{2;0.10} = 4.61 \text{ [Table 5].}$$

Do not reject the hypothesis.

## Test 55 The rank sum difference test for the multiple comparison of $K$ population means

### Object

To test if  $K$  random samples came from populations with the same mean.

### Limitations

The  $K$  samples must have the same size, and the frequency distributions of the population are assumed continuous.

### Method

The  $K$  samples are combined and arranged in order of increasing size and then given a rank number. The highest raw value is assigned rank 1. For each sample the rank sum is determined.

To compare two population means the rank sums of the corresponding samples,  $R_i$  and  $R_j$ , are taken and the test statistic is  $R_i - R_j$ . Critical values of this test statistic can be obtained from Table 23. When  $R_i - R_j$  exceeds the critical value the null hypothesis of equal means is rejected.

### Example

A perfume manufacturer has four floral fragrances and wishes to compare each one against the others in a preference test. Selected perfume testers can give a perfume a score between 1 and 100. For each of these four fragrances four testers are used and the results are shown. The critical value from Table 23 is 34.6. Fragrances 1 and 2 and 1 and 3 are viewed as different, with fragrance 1 generally preferred.

### Numerical calculation

Sample			
1	2	3	4
70 (16)	12 (2)	10 (1)	29 (6)
52 (14)	18 (3)	43 (11)	31 (7)
51 (13)	35 (8)	28 (5)	41 (10)
67 (15)	36 (9)	26 (4)	44 (12)
$R_1 = 58$	$R_2 = 22$	$R_3 = 21$	$R_4 = 35$

$n = 4, K = 4$

The values in the brackets are the assigned rank numbers.

Here  $R_1 - R_2 = 58 - 22 = 36$      $R_2 - R_3 = 22 - 21 = 1$   
 $R_1 - R_3 = 58 - 21 = 37$      $R_2 - R_4 = 22 - 35 = -13$   
 $R_1 - R_4 = 58 - 35 = 23$      $R_3 - R_4 = 21 - 35 = -14$

The critical value at  $\alpha = 0.05$  is 34.6 [Table 23].

Hence samples 1 and 2 and samples 1 and 3 are significantly different.

## Test 56 The rank sum maximum test for the largest $K$ population means

### Object

To investigate the difference between the largest mean and the  $K - 1$  other population means.

### Limitations

It is assumed that the populations have continuous frequency distributions and that the  $K$  samples are of equal size  $n$ .

### Method

The  $K$  samples are merged together and rank numbers allocated to the  $K_n$  observations. The sum of the rank numbers of the observations belonging to a particular sample is formed. This is repeated for each sample and the test statistic is the largest of these rank sums. When the test statistic exceeds the critical value obtained from Table 24 the mean of the population generating the maximum rank sum is said to be significantly large.

### Example

As an alternative to Test 55 the perfume manufacturer uses the rank sum maximum test for the largest 4 population means. The largest  $R$  value is  $R_1$  at 58 which is greater than the tabulated value of 52. Hence fragrance 1 is significantly greater (in preference) than the other fragrances. This is a similar result to that found with Test 55.

### Numerical calculation

Combined rank assignment of four samples, i.e.  $K = 4$ ,  $n = 4$ .

Sample	$x_1$	$x_1$	$x_1$	$x_1$	$x_2$	$x_2$	$x_2$	$x_2$	$x_3$	$x_3$	$x_3$	$x_3$	$x_4$	$x_4$	$x_4$	$x_4$
Value	70	52	51	67	12	18	35	36	10	43	28	26	29	31	41	44
Rank	16	14	13	15	2	3	8	9	1	11	5	4	6	7	10	12

$$R_1 = 58, R_2 = 22, R_3 = 21, R_4 = 35$$

The critical value at  $\alpha = 0.05$  is 52 [Table 24].

The calculated value of  $R_1$  is greater than the critical value.

Hence the sample 1 is statistically significantly greater than the others.

**Test 57 The Steel test for comparing  $K$  treatments with a control**

**Object**

To test the null hypothesis that all treatments have the same effect as the control treatment.

**Limitations**

The  $K$  samples, one from each treatment and one from the control, should all be of the same size.

**Method**

Each of the treatment samples is compared with the control sample in turn. To test the  $j$ th sample, it is merged with the control sample and rank numbers are allocated to the  $2n$  observations. This provides two rank sums and the smallest of these is used as the test statistic if a two-tailed test is desired. To test the alternative hypothesis that treatment  $j$  has a smaller effect than the control treatment, the rank sum for the  $j$ th control sample forms the test statistic. In both cases, the null hypothesis that there is no difference between the  $j$ th treatment and the control is rejected if the test statistic is less than the critical value obtained from Table 25.

**Example**

Four different sprain relief creams are compared with controls. Treatments are allocated at random and each is compared with its control. The results show that rank sums for controls 1 and 4 are less than the critical tabulated value of 76 (Table 25). Hence treatment creams 1 and 4 are significant and hence more effective than placebo in relieving sprain effects.

**Numerical calculation**

$n = 10, K = 4$

Rank assignment and rank sums are as follows:

											Total
Control 1	1.5	1.5	3	4	6	7	8	11	13	14.5	69.5
Treatment 1	5	9	10	12	14.5	16	17	18	19	20	140.5
Control 2	2.5	2.5	5	7	8	9	12	15	18	19	98
Treatment 2	1	4	6	10.5	10.5	13	15	15	17	20	112
Control 3	1.5	1.5	3	5	8	9	10.5	14	15	18	85.5
Treatment 3	4	6.5	6.5	10.5	12	13	16.5	16.5	19	20	124.5
Control 4	1.5	1.5	3	5	6	7	9	12	15	16	76
Treatment 4	4	8	10	12	13	14	17	18	19	20	135

The critical value at  $\alpha = 0.05$  is 76 [Table 25].

Since control 1 and control 4 are less than or equal to the critical value, treatments 1 and 4 are significant.

## Test 58 The Spearman rank correlation test (paired observations)

### Object

To investigate the significance of the correlation between two series of observations obtained in pairs.

### Limitations

It is assumed that the two population distributions are continuous and that the observations  $x_i$  and  $y_i$  have been obtained in pairs.

### Method

The  $x_i$  observations are assigned the rank numbers  $1, 2, \dots, n$  in order of increasing magnitude. A similar procedure is carried out for all the  $y_i$  observations. For each pair of observations, the difference in the ranks,  $d_i$ , can be determined. The quantity  $R = \sum_{i=1}^n d_i^2$  is now calculated.

For large samples ( $n > 10$ ) the test statistic is

$$Z = \frac{6R - n(n^2 - 1)}{n(n+1)\sqrt{(n-1)}}$$

which may be compared with tables of the standard normal distribution. For small samples, the test statistic

$$r_s = 1 - \frac{6R}{n(n^2 - 1)}$$

must be compared with critical values obtained from Table 26. In both cases, if the experimental value lies in the critical region one has to reject the null hypothesis of no correlation between the two series.

### Example

A panel of consumers is asked to rate two brands of vegetarian sausage. It is hoped that advertising can be combined in a mail out to potential consumers. A small sample is taken and panel members are asked to rate each brand. The results produce a  $Z$  value of  $-2.82$ . The critical value for  $Z$  is  $1.64$  so the null hypothesis of zero correlation is rejected. Consumers tend to report similar preferences for the two brands of sausage.

### Numerical calculation

$d_i$ :  $0, -1, -2, 0, +3, -1, -1, +2, 0, 0, 2$

Hence  $R = 24$ ,  $n = 11$

$$Z = \frac{6 \times 24 - 11(11^2 - 1)}{11 \times 12\sqrt{10}} = \frac{144 - 1320}{132 \times 3.1623} = \frac{-1176}{417.42} = -2.82$$

The critical value of  $Z$  at  $\alpha = 0.05$  is  $1.64$  [Table 1].

Hence reject the null hypothesis.

**Test 59 The Kendall rank correlation test  
(paired observations)**

**Object**

To investigate the significance of the correlation between two series of observations obtained in pairs.

**Limitations**

It is assumed that the two population distributions are continuous and that the observations  $x_i$ , and  $y_i$ , have been obtained in pairs.

**Method**

The  $x_i$  observations are assigned the rank numbers  $1, 2, \dots, n$  in order of increasing magnitude. A similar procedure is carried out for all the  $y_i$  observations. Each of the possible pairs of rank numbers (there will be  $\frac{1}{2}n(n - 1)$  of these) is now examined. Each pair  $(x_i, y_i)$  will be compared successively and systematically with each other pair  $(x_j, y_j)$ . When  $x_i - x_j$  and  $y_i - y_j$  have the same sign a score of  $+1$  is obtained. When they have opposite signs a score of  $-1$  is obtained. When there is a difference of zero, no score is obtained. These scores are summed together and this sum is denoted  $S$ . In this manner we can work with observational results without having determined the rank numbers.

For large  $n$  ( $n > 10$ ),  $Z$  follows a normal distribution and hence the test statistic

$$Z = \frac{S}{\{n(n - 1)(2n + 5)/18\}^{\frac{1}{2}}}$$

may be compared with tables of the standard normal distribution. For small samples, critical values of  $S$  may be obtained from Table 27.

In both cases, if the experimental value lies in the critical region one has to reject the null hypothesis of no correlation between the two series.

**Example**

A tax inspector wishes to investigate whether there is any correlation between total investment incomes (£00's), obs1 and total additional income (£00's), obs 2. He has collected a sample of 10 tax forms and calculates an  $S$  value of 33. He compares this with the critical value of 21 obtained from Table 27. Since the calculated value is greater than the tabulated value he concludes that there is a significant correlation.

**Numerical calculation**

Observation 1	7.1	8.3	10.7	9.4	12.6	11.1	10.3	13.1	9.6	12.4
Observation 2	62	66	74	74	82	76	72	79	68	74
Plus scores	9	8	5	3	4	3	3	2	1	0
Minus scores	0	0	0	2	1	1	1	0	0	0

Total plus scores = 38, total minus scores = 5

$$S = 38 - 5 = 33, n = 10$$

Critical value  $S_{10;005} = 21$  [Table 27].

The calculated value is greater than the critical value.

Reject the null hypothesis.



## Test 60 The sequential test for a population mean (variance known)

### Object

To test the null hypothesis that the mean  $\mu$  of a population with known variance has the value  $\mu_0$  rather than the value  $\mu_1$ .

### Limitations

1. The observations can be obtained sequentially as necessary.
2. The observations are independent and follow a normal distribution with known variance  $\sigma^2$ .

### Method

First the Type I and Type II errors for the test must be fixed, say,  $\alpha$  and  $\beta$ . The test consists of plotting a sequential analysis chart. In this case, as the observations are obtained the cumulative value  $\sum_{i=1}^m (x_i - c)$  is plotted against the sample size to date,  $m$ . The constant  $c$  is chosen as a convenient value close to  $\frac{1}{2}(\mu_0 + \mu_1)$

On the chart are two boundary lines:

$$\sum_{i=1}^m (x_i - c) = \frac{\sigma^2}{\mu_1 - \mu_0} \log \left( \frac{1 - \beta}{\alpha} \right) + m \left( \frac{\mu_0 + \mu_1}{2} - c \right),$$

$$\sum_{i=1}^m (x_i - c) = \frac{\sigma^2}{\mu_1 - \mu_0} \log \left( \frac{\beta}{1 - \alpha} \right) + m \left( \frac{\mu_0 + \mu_1}{2} - c \right).$$

If the plot crosses the upper boundary the null hypothesis is rejected, and it will not be rejected if the plot crosses the lower boundary.

### Example

As part of a quality monitoring programme, measurements of a critical dimension of an automotive component are taken at regular intervals. The quality engineer uses a cu-sum and a sequential test for the process mean. His test is that the mean is constant at 8.30 units, i.e. the specified value rather than 8.33 units, when problems would occur. He produces a cu-sum chart and plots the sequential values upon it. What does he conclude about the process? He has three options, viz. reject the null hypothesis that the mean is 8.30, do not reject the null hypothesis that the mean is 8.30 or continue testing.

Since the lower boundary is crossed at observation 7 he accepts the null hypothesis and stops testing.

### Numerical calculation

Successive observations: 8.34, 8.29, 8.30, 8.31, 8.30, 8.32, 8.30

$\mu_0 = 8.30$ ,  $\mu_1 = 8.33$ ,  $\alpha = 0.05$ ,  $\beta = 0.05$

$\mu_1 - \mu_0 = 8.33 - 8.30 = 0.03$ ,  $\bar{\mu} = 8.315$

Let the standard deviation be 0.02.

$$\frac{\sigma^2}{\mu_1 - \mu_0} \log \left( \frac{\beta}{1 - \alpha} \right) = \frac{0.02^2}{0.03} \times \log \left( \frac{0.05}{0.95} \right) = -0.039$$

$$\frac{\sigma^2}{\mu_1 - \mu_0} \log \left( \frac{1 - \beta}{\alpha} \right) = +0.039$$

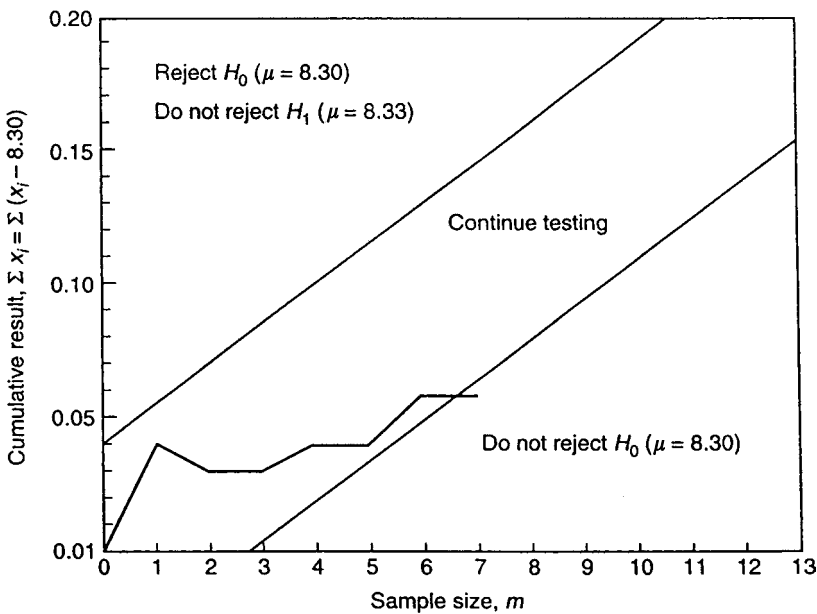
Critical boundary lines are

$$\sum x_i = -0.039 - 8.315m \quad \text{or} \quad \sum x'_i = -0.039 + 0.015m$$

and

$$\sum x_i = 0.039 - 8.315m \quad \text{or} \quad \sum x'_i = 0.039 + 0.015m$$

$m$	1	2	3	4	5	6	7	8	9	10
$x'_i$	0.04	-0.01	0.00	0.01	0.00	+0.02	0.00			
cu-sum	0.04	0.03	0.03	0.04	0.04	0.06	0.06			
$H_0$ boundary	-0.024	-0.009	0.006	0.021	0.036	0.051	0.066	0.081	0.096	0.111
$H_1$ boundary	0.054	0.069	0.084	0.099	0.114	0.129	0.144	0.159	0.174	0.189



## Test 61 The sequential test for a standard deviation (mean known)

### Object

To test the null hypothesis that the standard deviation  $\sigma$ , of a population with known mean, has the value  $\sigma_0$  rather than the value  $\sigma_1$ .

### Limitations

1. The observations can be obtained sequentially as necessary.
2. The observations are independent and come from a normal distribution with known mean  $\mu$ .

### Method

First the Type I and Type II errors for the test must be decided upon, say,  $\alpha$  and  $\beta$ . The test consists of plotting a sequential analysis chart. As the observations are obtained the cumulative value  $\sum_{i=1}^m (x_i - \mu)^2$  is plotted against the sample size to date,  $m$ .

On the chart are two boundary lines:

$$\sum_{i=1}^m (x_i - \mu)^2 = \frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2} \log\left(\frac{1 - \beta}{\alpha}\right) - m \frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2} \log\left(\frac{\sigma_0^2}{\sigma_1^2}\right),$$

$$\sum_{i=1}^m (x_i - \mu)^2 = \frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2} \log\left(\frac{\beta}{1 - \alpha}\right) - m \frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2} \log\left(\frac{\sigma_0^2}{\sigma_1^2}\right).$$

If the plot crosses the upper boundary, the null hypothesis is rejected; if the plot crosses the lower boundary, the null hypothesis is not rejected.

### Example

A quality engineer wants to set up a sequential test for a standard deviation. His process has a mean specification of 2 units and variance 4 units (standard deviation 2 units). He sets his Type I error at 0.15 and his Type II error at 0.25. He calculates his cumulative sum of squared deviations from the specified standard deviation of 2 units. If this cumulative sum lies within the range 8.37 to 37.90 he continues to sample. If the sum is less than 8.37 or greater than 37.90 he respectively accepts or rejects the null hypothesis.

### Numerical calculation

Consider a sample from  $N(2, \sigma^2)$  and  
 $H_0: \sigma_0^2 = 4$  against  $H_1: \sigma_1^2 = 6$ .  
 Let  $\alpha = 0.15$ ,  $\beta = 0.25$  and  $m = 10$ .

Then continue sampling if

$$\frac{24 \left[ 2 \log \left( \frac{0.25}{0.85} \right) + 10 \log \left( \frac{4}{6} \right) \right]}{6 - 4}$$

$$< \sum_{i=1}^m (x_i - 2)^2 < \frac{24 \left[ 2 \log \left( \frac{0.75}{0.15} \right) + 10 \log \left( \frac{4}{6} \right) \right]}{6 - 4}$$

$$\frac{24[-1.0628 + 1.76]}{2} < \sum_{i=1}^m (x_i - 2)^2 < \frac{24[1.398 + 1.76]}{2}$$

or

$$8.37 < \sum_{i=1}^m (x_i - 2)^2 < 37.90$$

Hence do not reject  $H_0$  if  $\sum_{i=1}^m (x_i - 2)^2 \leq 8.37$  and reject  $H_0$  if  $\sum_{i=1}^m (x_i - 2)^2 \geq 37.90$ .

## Test 62 The sequential test for a dichotomous classification

### Object

To test the null hypothesis that the parameter  $p$  of a population has the value  $p_0$  rather than the value  $p_1$ .

### Limitations

1. The observations can be obtained sequentially as necessary.
2. The observations are independent and follow a Bernoulli distribution.

### Method

This test is typically used in quality control when we wish to determine if the proportion defective in a sample falls below  $p_0$  (accept batch) or exceeds  $p_1$  (reject batch). First we need to decide on the Type I and Type II errors for the test, say  $\alpha$  and  $\beta$ .

The test consists of plotting a sequential analysis chart. As the observations are obtained the number of defective items  $r_m$  is plotted against the sample size to date,  $m$ . On the chart are two boundary lines:

$$r_m \left\{ \log \left( \frac{p_1}{p_0} \right) - \log \left( \frac{1-p_1}{1-p_0} \right) \right\} + m \log \left( \frac{1-p_1}{1-p_0} \right) = \log \left( \frac{\beta}{1-\alpha} \right)$$

$$r_m \left\{ \log \left( \frac{p_1}{p_0} \right) - \log \left( \frac{1-p_1}{1-p_0} \right) \right\} + m \log \left( \frac{1-p_1}{1-p_0} \right) = \log \left( \frac{1-\beta}{\alpha} \right).$$

If the plot crosses the upper boundary the null hypothesis is rejected; if the plot crosses the lower boundary the null hypothesis is not rejected.

### Example

A quality control engineer sets up a sequential test for the proportion defective in a sample from a large batch. If the proportion is below 0.10 he accepts the batch, but if the proportion is above 0.20 he rejects the batch, otherwise he continues to sample. After the 21st observation the plot of the number of defective items versus the sample number crosses the upper boundary line. This suggests that he should reject the null hypothesis ( $p = 0.10$ ) and accept the alternative hypothesis ( $p = 0.20$ ). The whole batch is therefore rejected.

### Numerical calculation

$H_0: p = p_0 = 0.10$  and  $H_1: p = p_1 = 0.20$

Let  $\alpha = 0.01$  and  $\beta = 0.05$ , and results are:

$a, a, a, r, a, r, a, a, r, a, a, r, r, a, r, r, a, r$

(where  $a$  = not defective and  $r$  = defective).

Then

$$\log \left( \frac{p_1}{p_0} \right) = \log \left( \frac{0.20}{0.10} \right) = 0.693$$

$$\log \left( \frac{1-p_1}{1-p_0} \right) = \log \left( \frac{0.80}{0.90} \right) = -0.118$$

$$\log \left( \frac{\beta}{1-\alpha} \right) = \log \left( \frac{0.05}{0.99} \right) = -2.986$$

$$\log \left( \frac{1-\beta}{\alpha} \right) = \log \left( \frac{0.95}{0.01} \right) = 4.554$$

Boundary lines are:

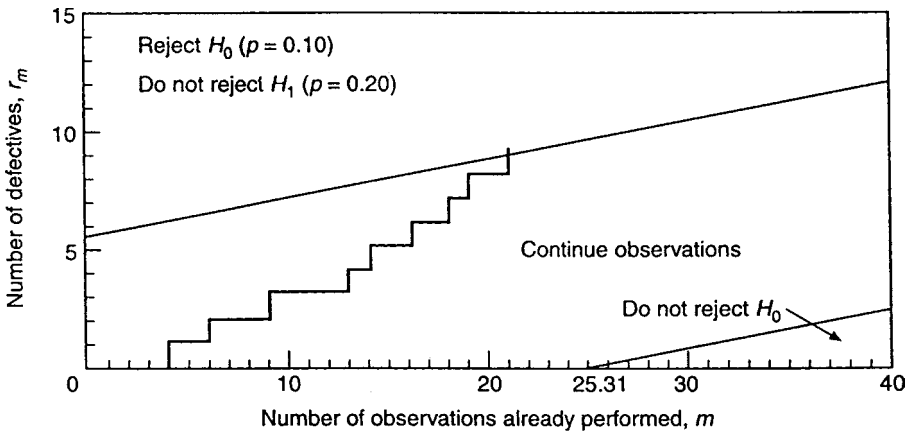
$$0.811r_m - 0.118m = -2.986$$

$$0.811r_m - 0.118m = 4.554.$$

If  $m = 0$ , the two boundary lines are  $r_{m_1} = -3.68$  and  $r_{m_2} = 5.62$ .

If  $m = 30$ , the two boundary lines are  $r_{m_1} = 0.68$  and  $r_{m_2} = 9.98$ .

The first line intersects the  $m$ -axis at  $m = 25.31$ . The sequential analysis chart is now as follows:



After the 21st observation we can conclude that the alternative hypothesis  $H_1$  may not be rejected. This means that  $p \geq 0.20$ . The percentage of defective elements is too large. The whole lot has to be rejected.

## Test 63 The adjacency test for randomness of fluctuations

### Object

To test the null hypothesis that the fluctuations in a series are random in nature.

### Limitations

It is assumed that the observations are obtained independently of each other and under similar conditions.

### Method

For a series of  $n$  terms,  $x_i$  ( $i = 1, \dots, n$ ), the test statistic is defined as

$$L = 1 - \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{2 \sum_{i=1}^n (x_i - \bar{x})^2}.$$

For  $n > 25$ , this approximately follows a normal distribution with mean zero and variance

$$\sqrt{\frac{(n-2)}{(n-1)(n+1)}}.$$

For  $n < 25$ , critical values for

$$D = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

are available in Table 28.

In both cases the null hypothesis is rejected if  $L$  exceeds the critical values.

### Example

An energy forecaster has produced a model of energy demand which she has fitted to some data for an industry sector over a standard time period. To assess the goodness of fit of the model she performs a test for randomness on the residuals from the model. If these are random then the model is a good fit to the data. She calculates the  $D$  statistic and compares it with the values from Table 28 of 1.37 and 2.63. Since  $D$  is less than the lower critical value she rejects the null hypothesis of randomness and concludes that the model is not a good fit to the data.

**Numerical calculation**

$$\sum x_i = 2081.94, \sum x_i^2 = 166\,736.9454$$

$$\sum_{i=1}^n (x_i - \bar{x})^2 = 26.4006, \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 = 31.7348, n = 25$$

$$D = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{31.7348}{26.4006} = 1.20$$

The critical values at  $\alpha = 0.05$  are 1.37 (lower limit) and 2.63 (upper limit) [Table 28].

The calculated value is less than the lower limit.

Hence the null hypothesis is to be rejected.



## Test 64 The serial correlation test for randomness of fluctuations

### Object

To test the null hypothesis that the fluctuations in a series have a random nature.

### Limitations

It is assumed that the observations are obtained independently of each other and under similar conditions.

### Method

The first serial correlation coefficient for a series of  $n$  terms,  $x_i$  ( $i = 1, \dots, n$ ), is defined as

$$r_1 = \frac{n}{n-1} \left\{ \frac{\sum_{i=1}^{n-1} (x_i - \bar{x})(x_{i+1} - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} \right\}$$

and this forms the test statistic.

For  $n \leq 30$ , critical values for  $r_1$  can be found from Table 29. For  $n > 30$ , the normal distribution provides a reasonable approximation. In both cases the null hypothesis is rejected if the test statistic exceeds the critical values.

### Example

A production line is tested for a systematic trend in the values of a measured characteristic of the components produced. A serial correlation test for randomness is used. If there is a significant correlation then the quality engineer will look for an assignable cause and so improve the resulting quality of components. He computes his first serial correlation as 0.585. The critical value from Table 29 is 0.276. So the correlation between successive components is significant.

### Numerical calculation

$x_i$ : 69.76, 67.88, 68.28, 68.48, 70.15, 71.25, 69.94, 71.82, 71.27, 68.79, 68.89, 69.70, 69.86, 68.35, 67.61, 67.64, 68.06, 68.72, 69.37, 68.18, 69.35, 69.72, 70.46, 70.94, 69.26, 70.20

$n = 26$ ,  $\sum x_i = 1804.38$ ,  $\bar{x} = 69.40$ ,  $\sum (x_i - \bar{x})^2 = 34.169$

$\sum x_{i+1} \cdot x_i = 125\,242.565$ ,  $\sum x_{i+1} \cdot x_i - (\sum x_i)^2 / n = 19.981$

$$r_1 = \frac{19.981}{34.169} = 0.585$$

The critical value at  $\alpha = 0.05$  is about 0.276 [Table 29].

Hence the null hypothesis is rejected; the correlation between successive observations is significant.

## Test 65 The turning point test for randomness of fluctuations

### Object

To test the null hypothesis that the variations in a series are independent of the order of the observations.

### Limitations

It is assumed that the number of observations,  $n$ , is greater than 15, and the observations are made under similar conditions.

### Method

The number of turning points, i.e. peaks and troughs, in the series is determined and this value forms the test statistic. For large  $n$ , it may be assumed to follow a normal distribution with mean  $\frac{2}{3}(n-2)$  and variance  $(16n-29)/90$ . If the test statistic exceeds the critical value, the null hypothesis is rejected.

### Example

An investment analyst wishes to examine a time series for a particular investment portfolio. She is especially keen to know if there are any turning points or if the series is effectively random in nature. She calculates her test statistic to be 1.31 which is less than the tabulated value of 1.96 [Table 1]. She concludes that the series is effectively random and no turning points can be detected.

### Numerical calculation

$p$  = peak,  $t$  = trough,  $n = 19$ ,  $\alpha = 0.05$

0.68;    0.34( $t$ );    0.62;    0.73( $p$ );    0.57;  
 0.32( $t$ );    0.58( $p$ );    0.34( $t$ );    0.59( $p$ );    0.56;  
 0.49;    0.17( $t$ );    0.30;    0.39;    0.42( $p$ );  
 0.41( $t$ );    0.46;    0.50;    0.45

$$\text{Mean} = \frac{2}{3} \times 17 = 11.3, \text{ variance} = \frac{16 \times 19 - 29}{90} = 3.05,$$

standard deviation = 1.75

$$\text{Test statistic} = \left| \frac{9 - 11.3}{1.75} \right| = 1.33$$

The critical value at  $\alpha = 0.05$  is 1.96 [Table 1].

Hence the departure from randomness is not significant.

## Test 66 The difference sign test for randomness in a sample

### Object

To test the null hypothesis that the fluctuations of a sample are independent of the order in the sequence.

### Limitations

It is assumed that the number of observations is large and that they have been obtained under similar conditions.

### Method

From the sequence of observations a sequence of successive differences is formed. The number of + signs,  $p$ , in this derived sequence forms the test statistic.

Let  $n$  be the initial sample size. For large  $n$ ,  $p$  may be assumed to follow a normal distribution with mean  $(n - 1)/2$  and variance  $(n + 1)/12$ . When the test statistic lies in the critical region the null hypothesis is rejected.

### Example

A quality engineer suspects that there is some systematic departure from randomness in machined component production lines. He uses the difference sign test for randomness to assess this. His test statistic of 4.54 is for his first sample of size 20 from production line 1. Since this value is greater than the tabulated value of 1.64 from Table 1 he concludes that there is a positive trend in this case. In the other samples from the other production lines he cannot reject the null hypothesis of randomness.

### Numerical calculation

$$n = 20, \alpha = 0.05$$

List	$S_1$	$S_2$	$S_3$	$S_4$	$S_5$
$p$	16	11	10	9	10

$$\text{Mean} = \frac{n - 1}{2} = \frac{19}{2} = 9.5, \text{ variance} = \frac{20 + 1}{12} = 1.75, \\ \text{standard deviation} = 1.32$$

$$p(S_1) = \frac{16 - 9.5 - 0.5}{1.32} = 4.54$$

The critical value at  $\alpha = 0.05$  is 1.64 [Table 1].

Reject the null hypothesis in this case.

However,  $p(S_2) = 0.76$ ,  $p(S_3) = 0.0$ ,  $p(S_4) = -0.76$ ,  $p(S_5) = 0$ .

Do not reject the null hypothesis in these cases, where a positive trend is not indicated.

## Test 67 The run test on successive differences for randomness in a sample

### Object

To test the null hypothesis that observations in a sample are independent of the order in the sequence.

### Limitations

It is necessary that the observations in the sample be obtained under similar conditions.

### Method

From the sequence of observations, a sequence of successive differences is formed, i.e. each observation has the preceding one subtracted from it. The number of runs of + and – signs in this sequence of differences,  $K$ , provides the test statistic.

Let  $n$  be the initial sample size. For  $5 \leq n \leq 40$ , critical values of  $K$  can be obtained from Table 30. For  $n > 40$ ,  $K$  may be assumed to follow a normal distribution with mean  $(2n - 1)/3$  and variance  $(16n - 29)/90$ . In both cases, when the test statistic lies in the critical region, the null hypothesis is rejected.

### Example

A quality engineer tests five production lines for systematic effects. He uses the run test on successive differences. He calculates the number of successive plus or minus signs for each line. He then compares these with the tabulated values of 9 and 17, from Table 30. For line A his number of runs is 7 which is less than the critical value, 9, so he rejects the null hypothesis of randomness. The values of 6 and 19 for lines C and D result in a similar conclusion. The test statistics for lines B and E do not lie in the critical region so he accepts the null hypothesis for these.

### Numerical calculation

Lists	A	B	C	D	E
Number ( $K$ ) of runs (plus and minus)	7	12	6	19	12

$$n = 20, \alpha = 0.05$$

The critical values are (left) 9 and (right) 17 [Table 30].

#### For cases A, C and D

$K(A) = 7$  and  $K(C) = 6$ , which are less than 9, and  $K(D) = 19$  which is greater than 17.

Hence reject the null hypothesis.

#### For cases B and E

Test statistics do not lie in the critical region [Table 30].

Do not reject the null hypothesis.

## Test 68 The run test for randomness of two related samples

### Object

To test the null hypothesis that the two samples have been randomly selected from the same population.

### Limitations

It is assumed that the two samples have been taken under similar conditions and that the observations are independent of each other.

### Method

The first sample of  $n_1$  elements are all given a + sign and the second sample of  $n_2$  elements are all given a - sign. The two samples are then merged and arranged in increasing order of magnitude (the allocated signs are to differentiate between the two samples and do not affect their magnitudes). A succession of values with the same sign, i.e. from the same sample, is called a run. The number of runs ( $K$ ) of the combined samples is found and is used to calculate the test statistic,  $Z$ . For  $n_1$  and  $n_2 \geq 10$ ,

$$Z = \frac{K - \mu_K + \frac{1}{2}}{\sigma_K}$$

can be compared with the standard normal distribution: here

$$\mu_K = \frac{2n_1n_2}{n_1 + n_2} + 1 \quad \text{and} \quad \sigma_K^2 = \frac{2n_1n_2(2n_1n_2 - n_1 - n_2)}{(n_1 + n_2)^2 \cdot (n_1 + n_2 - 1)}.$$

When the test statistic lies in the critical region, reject the null hypothesis.

### Example

A maintenance programme has been conducted on a plastic forming component production line. The supervisor responsible for the line wants to ensure that the maintenance has not altered the machine settings and so she performs the run test for randomness of two related samples. She collects two samples from the line, one before the maintenance and one after. The test statistic value is 0.23 which is outside the critical value of  $\pm 1.96$ . She concludes that the production line is running as usual.

### Numerical calculation

$n_1 = 10, n_2 = 10, K = 11, \alpha = 0.05$

Sample  $S_1$ : 26.3, 28.6, 25.4, 29.2, 27.6, 25.6, 26.4, 27.7, 28.2, 29.0

Sample  $S_2$ : 28.5, 30.0, 28.8, 25.3, 28.4, 26.5, 27.2, 29.3, 26.2, 27.5

$S_1$  and  $S_2$  are merged and arranged in increasing order of magnitude, and signs are allocated to obtain the number of runs  $K$ :

- + + - + + - - - + + + - - + - + + - -

$$\mu_K = \frac{2 \times 10 \times 10}{10 + 10} + 1 = \frac{200}{20} + 1 = 11$$

$$\sigma_K^2 = \frac{2 \times 10 \times 10(2 \times 10 \times 10 - 10 - 10)}{(10 + 10)^2(10 + 10 - 1)} = 4.74, \quad \sigma_K = 2.18$$

$$Z = \frac{11 - 11 + \frac{1}{2}}{2.18} = 0.23. \text{ Critical value at } \alpha = 0.05 \text{ is } 1.96 \text{ [Table 1].}$$

Hence do not reject the hypothesis.

## Test 69 The run test for randomness in a sample

### Object

To test the significance of the order of the observations in a sample.

### Limitations

It is necessary that the observations in the sample be obtained under similar conditions.

### Method

All the observations in the sample larger than the median value are given a + sign and those below the median are given a – sign. If there is an odd number of observations then the median observation is ignored. This ensures that the number of + signs ( $n$ ) is equal to the number of – signs. A succession of values with the same sign is called a run and the number of runs,  $K$ , of the sample in the order of selection is found. This forms the test statistic.

For  $n > 30$ , this test statistic can be compared with a normal distribution with mean  $n + 1$  and variance  $\frac{1}{2}n(2n - 2)/(2n - 1)$ . The test may be one- or two-tailed depending on whether we wish to test if  $K$  is too high, too low or possibly both.

For  $n < 30$ , critical values for  $K$  are provided in Table 31. In both cases the null hypothesis that the observations in the sample occurred in a random order is rejected if the test statistic lies in the critical region.

### Example

A quality control engineer has two similar processes, which produce dual threaded nuts. He suspects that there is some intermittent fault on atleast one process and so decides to test for randomness using the run test for randomness. In his first sample, from process A, he calculates the number of runs of the same sign to be 6. For his second process, B, he calculates the number of runs to be 11. The critical values are 9 and 19, from Table 31. Since for the process A, 6 is in the critical region, his suspicions for this process are well founded. Process B shows no departure from randomness.

### Numerical calculation

$$n_1 = n_2 = 13$$

### Sample A

81.02, 80.08, 80.05, 79.70, 79.13, 77.09, 80.09,  
 (+) (–) (–) (–) (–) (–) (–)  
 79.40, 80.56, 80.97, 80.17, 81.35, 79.64, 80.82, 81.26,  
 (–) (+) (+) (+) (+) (–) (+) (+)  
 80.75, 80.74, 81.59, 80.14, 80.75, 81.01, 79.09,  
 (+) (+) (+) (+) (+) (+) (–)  
 78.73, 78.45, 79.56, 79.80  
 (–) (–) (–) (–)

Median value = 80.12 and number of runs = 6.

**Sample B**

69.76, 67.88, 68.28, 68.48, 70.15, 71.25, 69.94,  
 (+) (−) (−) (−) (+) (+) (+)  
 71.82, 71.27, 69.70, 68.89, 69.24, 69.86, 68.35,  
 (+) (+) (+) (−) (−) (+) (−)  
 67.61, 67.64, 68.06, 68.72, 69.37, 68.18, 69.35,  
 (−) (−) (−) (−) (+) (−) (−)  
 69.72, 70.46, 70.94, 69.26, 70.20  
 (+) (+) (+) (−) (+)

Median value = 69.36 and number of runs = 11.

The critical values at  $\alpha = 0.10$  are (lower) 9 and (upper) 19 [Table 31].

For Sample A number of runs  $K = 6$  lies in the critical region. Hence reject the null hypothesis (i.e. the fluctuation is not random).

For Sample B number of runs  $K = 11$  does not lie in the critical region.

Do not reject the null hypothesis (i.e. the fluctuation may be considered to be random).



## Test 70 The Wilcoxon–Mann–Whitney rank sum test for the randomness of signs

### Object

To test that the occurrence of + and – signs in a sequence is random.

### Limitations

This is a distribution-free test, applicable if the observations are random and independent and the two frequency distributions are continuous.

### Method

Let  $n_1$  be the number of + or – signs, whichever is the larger,  $n_2$  be the number of opposite signs and  $N = n_1 + n_2$ . From the integers describing the natural order of the signs, the rank sum  $R$  of the smallest number of signs is determined. The value  $R' = n_2(N + 1) - R$  is calculated. The smaller of  $R$  and  $R'$  is used as the test statistic. If it is less than the critical value obtained from Table 21 the null hypothesis of random + and – signs is rejected.

### Example

A simple fuel monitoring system has a target fuel usage level and fuel use is determined at regular intervals. If the fuel use is higher or lower than the target value then a plus or minus sign is recorded. Departures from target on either side would signal a potential problem. An energy monitoring officer has recorded some data and uses the Wilcoxon–Mann–Witney rank sum test for randomness. He obtains a minimum rank sum of 29 and, since this lies in the critical region (Table 21), he concludes that he has a fuel usage problem.

### Numerical calculation

Successive observations in a sequence are coded with a plus or minus sign:

|   |   |   |   |   |   |   |   |   |    |    |    |    |    |
|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| + | + | + | – | + | + | + | + | – | –  | –  | +  | –  | –  |

$n_1 = 8, n_2 = 6$  (minus signs),  $N = 14$

Rank sum of minus signs =  $4 + 9 + 10 + 11 + 13 + 14 = 61$

$R' = 6(14 + 1) - 61 = 29$

The critical value at  $\alpha = 0.025$  is 29 [Table 21].

Reject the null hypothesis; alternatively, the experiment could be repeated.

## Test 71 The rank correlation test for randomness of a sample

### Object

To test that the fluctuations in a sample have a random nature. This test may be used to test the elements of a time series for the presence of a trend.

### Limitations

This is a distribution-free test, applicable if the observations occur in a natural sequence and have been obtained under similar or comparable conditions. It is sensitive to the occurrence of a positive or negative trend, and relatively insensitive to the occurrence of sudden jumps.

### Method

The observations are ranked in increasing order of magnitude  $R_i$ . The correlation between these rank and the integers representing the natural order of the observations is then calculated. This can be tested using the Spearman rank correlation test (Test 58) or the Kendall rank correlation test (Test 59). If the sample is larger than the  $T$  statistic  $T$  can be compared with the critical value of the normal distribution.

### Example

A merchandising manager observes the sales of a particular item of clothing across all her stores. She is looking for a discernable trend so that she can be pre-emptive of stock challenges. She produces a Spearman rank correlation between the natural order and the sorted data order of 0.771. Her  $T$  statistic is 3.36 which is in the critical region. She thus rejects the null hypothesis of randomness and is able to adjust production levels to account for this trend.

### Numerical calculation

|                 |    |     |     |     |    |     |     |     |     |     |
|-----------------|----|-----|-----|-----|----|-----|-----|-----|-----|-----|
| Order ( $x_i$ ) | 1  | 2   | 3   | 4   | 5  | 6   | 7   | 8   | 9   | 10  |
| Obs.            | 98 | 101 | 110 | 105 | 99 | 106 | 104 | 109 | 100 | 102 |
| Rank ( $y_i$ )  | 1  | 4   | 10  | 7   | 2  | 8   | 6   | 9   | 3   | 5   |

---

|                 |     |     |     |     |     |     |     |     |     |     |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Order ( $x_i$ ) | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  |
| Obs.            | 119 | 123 | 118 | 116 | 122 | 130 | 115 | 124 | 127 | 114 |
| Rank ( $y_i$ )  | 15  | 17  | 14  | 13  | 16  | 20  | 12  | 18  | 19  | 11  |

$$\sum (x_i - y_i)^2 = 304 = R$$

$$r_R = 1 - \frac{6R}{n(n^2 - 1)} = 0.771$$

$$T = \frac{6R - n(n^2 - 1)}{n(n + 1)\sqrt{n - 1}} = -3.36$$

The critical value at  $\alpha = 0.05$  is 1.96 [Table 1].

The calculated value is greater than the critical value.

Reject the null hypothesis.

**Test 72 The Wilcoxon–Wilcox test for comparison of multiple treatments of a series of subjects**

**Object**

To compare the significance of the difference in response for  $K$  treatments applied to  $n$  subjects.

**Limitations**

It is assumed that a subject’s response to one treatment is not affected by the same subject’s response to another treatment; and that the response distribution for each subject is continuous.

**Method**

The data are represented by a table of  $n$  rows and  $K$  columns. The rank numbers  $1, 2, \dots, K$  are assigned to each row and then the sum of the rank numbers for each column,  $R_j (j = 1, 2, \dots, K)$  is determined. A pair of treatments, say  $p$  and  $q$ , can now be compared by using as test statistic  $|R_p - R_q|$ . If this exceeds the critical value obtained from Table 32 the null hypothesis of equal effects of the  $p$  and  $q$  treatments is rejected.

**Example**

Six different ice cream flavours are compared by six tasters who assign a score (1 to 25) to each flavour. The food technologist wishes to compare each flavour with the others and uses the Wilcoxon–Wilcox test of multiple treatments. She finds that the rank sum difference for the flavours comparisons A–E, A–F and D–F are significant.

**Numerical calculation**

| Rank sums  |        |    |    |    |    |    | Rank sum differences $ R_p - R_q $ |   |    |    |     |     |
|------------|--------|----|----|----|----|----|------------------------------------|---|----|----|-----|-----|
| Serial no. | Sample |    |    |    |    |    |                                    | D | B  | C  | E   | F   |
|            | A      | B  | C  | D  | E  | F  |                                    |   |    |    |     |     |
| 1          | 1      | 5  | 3  | 2  | 4  | 6  | A 8                                | 2 | 15 | 17 | 19* | 25* |
| 2          | 1      | 3  | 6  | 2  | 4  | 5  | D 10                               |   | 13 | 15 | 17  | 23* |
| 3          | 2      | 3  | 4  | 1  | 5  | 6  | B 23                               |   |    | 2  | 4   | 10  |
| 4          | 1      | 4  | 3  | 2  | 6  | 5  | C 25                               |   |    |    | 2   | 8   |
| 5          | 2      | 5  | 3  | 1  | 4  | 6  | E 27                               |   |    |    |     | 6   |
| 6          | 1      | 3  | 6  | 2  | 4  | 5  |                                    |   |    |    |     |     |
| Rank sum   | 8      | 23 | 25 | 10 | 27 | 33 |                                    |   |    |    |     |     |

\* Exceeds critical value.

$K = 6, n = 6, \alpha = 0.05$ , critical value = 18.5 [Table 32].

## Test 73 Friedman's test for multiple treatment of a series of subjects

### Object

To investigate the significance of the differences in response for  $K$  treatments applied to  $n$  subjects.

### Limitations

It is assumed that a subject's response to one treatment is not affected by the same subject's response to another treatment; and that the response distribution for each subject is continuous.

### Method

The data can be represented by a table of  $n$  rows and  $K$  columns. In each row the rank numbers  $1, 2, \dots, K$  are assigned in order of increasing value. For each of the  $K$  columns the rank sum  $R_j (j = 1, 2, \dots, K)$  is determined.

The test statistic is

$$G = \frac{12}{nK(K+1)} \sum_{j=1}^K R_j^2 - 3n(K+1).$$

If this exceeds the critical  $\chi^2$  value obtained from Table 5 with  $K-1$  degrees of freedom, the null hypothesis that the effects of the  $K$  treatments are all the same is rejected.

If ties occur in the ranking procedure one has to assign the average rank member for each series of equal results. In this case the test statistic becomes

$$G = \frac{12(K-1)S}{nK^3 - D}$$

where  $S = \sum_{j=1}^K (R_j - \bar{R})^2$  and  $D = \sum f_i t_i^3$ .

### Example

Four different newspaper advertisement styles are compared to see if they produce the same effect on a panel of viewers/consumers. The different styles relate to size and position. There are 15 panel members and they rank each advertisement. The test statistic produced by this procedure is 12.51. The critical value from Table 5 is 7.81. So we conclude that the advertisement styles are not equally effective.

### Numerical calculation

$t_i$  is the size of the  $i$ th group of equal observations.

$$n = 15, K = 4, \bar{R} = \frac{n(K+1)}{2}$$

**Rank numbers (showing many ties)**

| Rows ( $n$ )    | Columns ( $K$ ) |       |       |       |
|-----------------|-----------------|-------|-------|-------|
|                 | $c_1$           | $c_2$ | $c_3$ | $c_4$ |
| 1               | 3.5             | 3.5   | 1.5   | 1.5   |
| 2               | 4.0             | 2.0   | 2.0   | 2.0   |
| 3               | 1.5             | 3.5   | 2.5   | 2.5   |
| 4               | 3.5             | 3.5   | 1.5   | 1.5   |
| 5               | 3.0             | 3.0   | 3.0   | 1.0   |
| 6               | 3.0             | 3.0   | 1.0   | 3.0   |
| 7               | 3.5             | 1.5   | 3.5   | 1.5   |
| 8               | 2.5             | 2.5   | 2.5   | 2.5   |
| 9               | 3.0             | 3.0   | 1.0   | 3.0   |
| 10              | 2.5             | 2.5   | 2.5   | 2.5   |
| 11              | 3.5             | 1.5   | 3.5   | 1.5   |
| 12              | 4.0             | 2.0   | 2.0   | 2.0   |
| 13              | 2.5             | 2.5   | 2.5   | 2.5   |
| 14              | 3.0             | 3.0   | 3.0   | 1.0   |
| 15              | 4.0             | 2.0   | 3.0   | 1.0   |
| $R_j$           | 47              | 39    | 35    | 29    |
| $\bar{R}$       | 37.5            | 37.5  | 37.5  | 37.5  |
| $R_j - \bar{R}$ | +9.5            | +1.5  | -2.5  | -8.5  |

$$S = \sum (R_j - \bar{R})^2 = 171$$

Here 1, 2, 3 and 4 are the size of the groups of equal observations and  $D = \sum f_i t_i^3$ .

| $t_i$ | $f_i$ | $f_i t_i$ | $f_i t_i^3$ |
|-------|-------|-----------|-------------|
| 1     | 7     | 7         | 7           |
| 2     | 10    | 20        | 80          |
| 3     | 7     | 21        | 189         |
| 4     | 3     | 12        | 192         |
| Total |       | 60        | 468         |

Hence  $D = 468$

$$G = \frac{12 \times (4 - 1) \times 171}{15 \times 4^3 - 468} = 12.51$$

The critical value is  $\chi_{3;005}^2 = 7.81$  [Table 5].  
Since  $G > 7.81$ , reject the null hypothesis.

## Test 74 The rank correlation test for agreement in multiple judgements

### Object

To investigate the significance of the correlation between  $n$  series of rank numbers, assigned by  $n$  members of a committee to  $K$  subjects.

### Limitations

This test can be applied if the judges decide independently and if the subjects show obvious differences in the quality being judged.

### Method

Let  $n$  judges give rank numbers to  $K$  subjects.

Compute  $S = nK(K^2 - 1)/12$  and  $S_D =$  the sum of squares of the differences between subjects' mean ranks and the overall mean rank. Let

$$D_1 = \frac{S_D}{n_1}, \quad D_2 = S - D_1, \quad S_1^2 = \frac{D_1}{K-1}, \quad S_2^2 = \frac{D_2}{K(n-1)}.$$

The test statistic is  $F = S_1^2/S_2^2$  which follows the  $F$ -distribution with  $(K-1, K(n-1))$  degrees of freedom. If this exceeds the critical value obtained from Table 3, the null hypothesis of agreement between the judgements is rejected.

### Example

A wine tasting panel is selected by asking a number of questions and also by tasting assessment. In one assessment three judges are compared for agreement. One of the judges is an expert taster. Ten wines are taken and the judges are asked to rank them on a particular taste criterion. Are the three judges consistent? The test statistic,  $F$  is 0.60, which is less than the tabulated value of 2.39. So the new panel members can be recruited.

### Numerical calculation

$n = 3, K = 10, v_1 = K - 1 = 9, v_2 = K(n - 1) = 20$

|             | Rank number |      |      |      |      |      |      |      |      |      | Total |
|-------------|-------------|------|------|------|------|------|------|------|------|------|-------|
|             | A           | B    | C    | D    | E    | F    | G    | H    | I    | J    |       |
| Judge 1     | 1           | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 55    |
| Judge 2     | 7           | 10   | 4    | 1    | 6    | 8    | 9    | 5    | 2    | 3    | 55    |
| Judge 3     | 9           | 6    | 10   | 3    | 5    | 4    | 7    | 8    | 2    | 1    | 55    |
| Total score | 17          | 18   | 17   | 8    | 16   | 18   | 23   | 21   | 13   | 14   | 165   |
| Mean        | 16.5        | 16.5 | 16.5 | 16.5 | 16.5 | 16.5 | 16.5 | 16.5 | 16.5 | 16.5 | 165   |
| Difference  | 0.5         | 1.5  | 0.5  | -8.5 | -0.5 | 1.5  | 6.5  | 4.5  | -3.5 | -2.5 |       |

$$S = \frac{3 \times 10(100 - 1)}{12} = 247.5, \quad S_D = 158.50,$$

$$D_1 = \frac{158.50}{3} = 52.83, \quad D_2 = S - D_1 = 247.50 - 52.83 = 194.67,$$

$$S_1^2 = \frac{52.83}{9} = 5.87, \quad S_2^2 = \frac{194.67}{10 \times 2} = \frac{194.67}{20} = 9.73$$

$$F = S_1^2/S_2^2 = \frac{5.87}{9.73} = 0.60$$

Critical value  $F_{9;20;0.05} = 2.39$  [Table 3].

Do not reject the null hypothesis.

## Test 75 A test for the continuous distribution of a random variable

### Object

To test a model for the distribution of a random variable of the continuous type.

### Limitations

This test is applicable if some known continuous distribution function is being tested. A partition of the random values into different sets must be available using the closed interval  $[0, 1]$ .

### Method

Let  $F(W)$  be the distribution function of  $W$  which we want to test. The null hypothesis is

$$H_0: F(W) = F_0(W)$$

where  $F_0(W)$  is some known continuous distribution function.

The test is based on the  $\chi^2$  statistic. In order to use this, we must partition the set of possible values of  $W$  into  $k$  (not necessarily equal) sets. Partition the interval  $[0, 1]$  into  $k$  sets such that  $0 = b_0 < \dots < b_k = 1$ . Let  $a_i = F_0^{-1}(b_i)$ ,  $i = 1, 2, \dots, k-1$ ,  $A_1 = [-\alpha, a_1]$ ,  $A_i = [-a_{i-1}, a_i]$ , for  $i = 2, 3, \dots, k-1$  and  $A_k = (a_{k-1}, \alpha)$ ;  $p_i = P(W \in A_i)$ ,  $i = 1, 2, \dots, k$ . Let  $Y_i$  denote the number of times the observed value of  $W$  belongs to  $A_i$ ,  $i = 1, 2, \dots, k$  in  $n$  independent repetitions of the experiment. Then  $Y_1, Y_2, \dots, Y_k$  have a multinomial distribution with parameters  $n, p_1, p_2, \dots, p_k$ . Let  $\pi_i = P(W \in A_i)$  when the distribution function of  $W$  is  $F_0(W)$ .

Then we test the hypothesis:

$$H_0^*: p_i = \pi_i, \quad i = 1, 2, \dots, k.$$

$H_0^*$  is rejected if the observed value of the  $\chi^2$  statistic

$$Q_{k-1} = \sum_{i=1}^{k-1} \frac{(Y_i - n\pi_i)^2}{n\pi_i}$$

is at least as great as  $C$ , where  $C$  is selected to yield the desired significance level.

### Example

A continuous distribution is tested by calculating  $Q_9$  which follows approximately a chi-squared distribution. The value of 4.0 is compared with 16.92 from Table 5. Since the calculated value is not in the critical region, the null hypothesis that the data follows the given continuous distribution is accepted.



### Numerical calculation

Let  $W$  denote the outcome of a random experiment. Let  $F(W)$  denote the distribution function of  $W$  and let

$$F_0(W) = \begin{cases} 0, & W < -1 \\ \frac{1}{2}(W^3 + 1), & -1 \leq W < 1 \\ 1, & W \geq 1. \end{cases}$$

The interval  $[-1, 1]$  can be partitioned into 10 sets of equal probability with the point  $b_i = i/10, i = 0, 1, \dots, 10$ .

If  $a_i = F^{-1}(b_i) = (2b_i - 1)^{\frac{1}{3}}, i = 1, 2, \dots, 9$  then the sets  $A_1 = [-1, a_1], A_2 = [a_1, a_2], \dots, A_{10} = [a_9, 1]$  will each have probability 0.1. If the random sample of size  $n = 50$  is observed then  $n\pi_i = 50 \times 0.1 = 5.0$ . Let the summary of the 50 observed values be

$$A_1 = 6, A_2 = 4, A_3 = 5, A_4 = 6, A_5 = 4, A_6 = 4, A_7 = 6, A_8 = 8, A_9 = 3, A_{10} = 4.$$

Then the calculated value of  $Q_9$  is

$$Q_9 = \frac{(6-5)^2}{5} + \frac{(4-5)^2}{5} + \dots + \frac{(4-5)^2}{5} = 4.0.$$

Critical value  $\chi_{9;0.05}^2 = 16.92$  [Table 5].

Hence do not reject the null hypothesis.

## Test 76 A test for the equality of multinomial distributions

### Object

To test the equality of  $h$  independent multinomial distributions.

### Limitations

If  $p_i$  is the probability of an item being assigned to the  $i$ th class, then this test is applicable if  $y_{ij}$  is the number of items occurring in the class associated with  $p_i$ .

### Method

Let  $p_{ij} = P(A_i)$ ,  $i = 1, 2, \dots, k$ ;  $j = 1, 2, \dots, h$ . It is required to test

$$H_0: p_{i1} = p_{i2} = \dots = p_{ih} = p_i, \quad i = 1, 2, \dots, k.$$

Carry out the  $j$ th experiment  $n_j$  times, making sure that the  $n_j$  instances are independent, and let  $Y_{1j}, Y_{2j}, \dots, Y_{kj}$  denote the frequencies of the respective events  $A_1, A_2, \dots, A_k$ . Then

$$Q = \sum_{j=1}^h \sum_{i=1}^k \frac{(Y_{ij} - n_j p_{ij})^2}{n_j p_{ij}}$$

has an approximate  $\chi^2$ -distribution with  $h(k - 1)$  degrees of freedom. Under  $H_0$  we estimate  $k - 1$  probabilities from

$$\hat{p}_i = \frac{\sum_{j=1}^h Y_{ij}}{\sum_{j=1}^h n_j}, \quad i = 1, 2, \dots, k - 1;$$

the estimate of  $p_k$  then follows from  $\hat{p}_k = 1 - \sum_{i=1}^{k-1} \hat{p}_i$ . Then

$$Q = \sum_{j=1}^h \sum_{i=1}^k \frac{(Y_{ij} - n_j \hat{p}_i)^2}{n_j \hat{p}_i}$$

has an approximate  $\chi^2$ -distribution with

$$h(k - 1) - (k - 1) = (h - 1)(k - 1)$$

degrees of freedom.

**Example**

An electronic allocation of visual stimuli to one of five categories, or grades ensures that the allocations are equally likely and not subject to any bias effects. An experiment to allocate 50 stimuli to 5 grades is repeated once. Are the allocations equally likely? The computed  $Q$  statistic of 5.18 is less than the tabulated value of 9.488 (Table 5) so there is no justification to suspect unequal allocation probabilities.

**Numerical calculation**

| Group | Grade |       |       |       |       | Total |
|-------|-------|-------|-------|-------|-------|-------|
|       | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ |       |
| 1     | 8     | 13    | 16    | 10    | 3     | 50    |
| 2     | 4     | 9     | 14    | 16    | 7     | 50    |

$$n = 50$$

$$P(A_1) = \frac{8+4}{100} = 0.12, P(A_2) = 0.22, P(A_3) = 0.30$$

$$P(A_4) = 0.26, P(A_5) = 0.10$$

Thus we have estimates of  $n_1P_{i1} = 6$ ,  $n_2P_{i2} = 11$ ,  $n_3P_{i3} = 15$ ,  $n_4P_{i4} = 13$  and  $n_5P_{i5} = 5$ , respectively.

Hence the computed value of  $Q$  is:

$$\begin{aligned}
 Q &= \frac{(8-6)^2}{6} + \frac{(13-11)^2}{11} + \frac{(16-15)^2}{15} + \frac{(10-13)^2}{13} + \frac{(3-5)^2}{5} \\
 &\quad + \frac{(4-6)^2}{6} + \frac{(9-11)^2}{11} + \frac{(14-15)^2}{15} + \frac{(16-13)^2}{13} + \frac{(7-5)^2}{5} \\
 &= 5.18
 \end{aligned}$$

The critical value is  $\chi^2_{4;0.05} = 9.488$  [Table 5].

The calculated value is less than the critical value. Do not reject the null hypothesis.

## Test 77 *F*-test for non-additivity

### Object

To test for non-additivity in a two-way classification.

### Limitations

This test is applicable if the observations are independently and normally distributed with constant variance.

### Method

In the two-way classification with one observation per cell (fixed effects model), we assume additivity (absence of interaction effects). In the case of any doubt about this additivity, Tukey proposed a test under the following set-up:

$$Y_{ij} = \mu + \alpha_i + \beta_j + \lambda\alpha_i\beta_j + e_{ij}$$

subject to the conditions that

$$\sum_i \alpha_i = \sum_j \beta_j = 0$$

and that the  $e_{ij}$  are independently  $N(0, \sigma^2)$ . Under this set-up, the interaction effect is represented by  $\lambda\alpha_i\beta_j$ , where

$$\sum_{\substack{i \\ \text{all } j}} \lambda\alpha_i\beta_j = \sum_{\substack{j \\ \text{all } i}} \lambda\alpha_i\beta_j = 0.$$

A test for non-additivity is obtained by a test for  $H_0: \lambda = 0$  or equivalently by a test  $H_0: E(Y_{ij}) = \mu + \alpha_i + \beta_j$  under this set-up. But this set-up does not conform to the Gauss–Markov model for  $E(Y_{ij})$  which are not linear in the parameter  $\mu, \alpha_i, \beta_j$  and  $\lambda$ . A set of unbiased estimators for,  $\mu, \alpha_i$  and  $\beta_j$  are:

$$\mu^* = Y_{00}, \quad \alpha_i^* = Y_{i0} - Y_{00}, \quad \beta_j^* = Y_{0j} - Y_{00}.$$

The least squares (unbiased) estimator of  $\lambda$  is obtained by minimizing

$$s_E^2 = \sum_i \sum_j (Y_{ij} - \mu - \alpha_i - \beta_j - \lambda\alpha_i\beta_j)^2$$

with respect to  $\lambda$  under the assumption that  $\alpha_i$  and  $\beta_j$  are known. Thus

$$\lambda^* = \frac{\sum_i \sum_j \alpha_i \beta_j Y_{ij}}{\sum_i \alpha_i^2 \sum_j \beta_j^2}.$$

Then the sum of squares due to interaction, i.e. due to  $\lambda^*$ , is given by

$$s_{\lambda^*}^2 = \frac{\left[ \sum_i \sum_j \alpha_i \beta_j Y_{ij} \right]^2}{\sum_i \alpha_i^2 \sum_j \beta_j^2}$$

with  $l$  degrees of freedom. The sum of squares due to non-additivity is given by

$$s_N^2 = \frac{\left[ \sum_i \sum_j \alpha_i^* \beta_j^* Y_{ij} \right]^2}{\sum_i \alpha_i^{*2} \sum_j \beta_j^{*2}}.$$

For all given  $\alpha_i^*$ ,  $\beta_j^*$ , for all  $i, j$ , we have that  $s_N^2/\sigma^2$  and  $(s_E^2 - s_N^2)/\sigma^2 = s_R^2$  are independent and have  $\chi^2$ -distribution with 1 and  $(p-1)(q-1)-1$  degrees of freedom, respectively.

We reject  $H_0: \lambda = 0$  at level  $\alpha$  if the variance ratio for non-additivity is too large, i.e. if

$$[(p-1)(q-1)-1] \frac{s_N^2}{s_R^2} > F_{1, (p-1)(q-1)-1; \alpha}$$

and fail to reject otherwise.

### Example

A thermal bond is tested to assess whether the resultant strength of the bond is the result of the combination of the main effects of temperature and pressure only. That is, no temperature/pressure interaction exists. There are four levels of temperature and three levels of pressure. The  $F$  test statistic for non-additivity is 0.2236 which is compared with the critical tabulated value of 6.61 [Table 3]. Since the calculated  $F$  is less than the critical value the assumption of no interaction is upheld.

### Numerical calculation

| $i \backslash j$ | 1  | 2 | 3   | 4    | $Y_{i0}$ | $\bar{Y}_{i0}$ |
|------------------|----|---|-----|------|----------|----------------|
| 1                | 14 | 2 | 1   | 2    | 19       | 4.75           |
| 2                | 2  | 0 | 2   | 2    | 6        | 1.5            |
| 3                | 2  | 1 | 5   | 0    | 8        | 2              |
| $Y_{0j}$         | 18 | 3 | 8   | 4    | 33       |                |
| $\bar{Y}_{0j}$   | 6  | 1 | 2.7 | 1.33 |          |                |

$$\mu = Y_{00} = \frac{33}{12} = 2.75$$

$$\alpha_1^* = 19 - 2.75 = 16.25, \alpha_2^* = 3.25, \alpha_3^* = 5.25$$

$$\beta_1^* = 18 - 2.75 = 15.25, \beta_2^* = 0.25, \beta_3^* = 5.25, \beta_4^* = 1.25$$

$$\lambda^* = \frac{4023.75}{79\,229.788} = 0.05079, s_E^2 = 4568.38, s_N^2 = 204.35$$

$$\text{Let } \sigma^2 = 16, \text{ then } \frac{s_E^2}{\sigma^2} = 285.52 \text{ and } \frac{s_N^2}{\sigma^2} = 12.77, s_R^2 = 272.75.$$

$$\text{Hence } F = \frac{12.77/1}{285.52/5} = 0.2236.$$

Critical value  $F_{1.5; 0.52} = 6.61$  [Table 3].

We do not reject the null hypothesis  $\lambda = 0$ .

## Test 78 *F*-test for testing main effects and interaction effects in a two-way classification

### Object

To test the main effects and interaction effects for the case of a two-way classification with an equal number of observations per cell.

### Limitations

This test is applicable if the error in different measurements is normally distributed; if the relative size of these errors is unrelated to any factor of the experiment; and if the different measurements themselves are independent.

### Method

Suppose we have  $n$  observations per cell of the two-way table, the observations being  $Y_{ijk}$ ,  $i = 1, 2, \dots, p$  (level of  $A$ );  $j = 1, 2, \dots, q$  (level of  $B$ ) and  $k = 1, 2, \dots, r$ . We use the model:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk}$$

subject to the conditions that

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_{\substack{j \\ \text{all } j}} (\alpha\beta)_{ij} = \sum_{\substack{i \\ \text{all } i}} (\alpha\beta)_{ij} = 0$$

and that the  $e_{ij}$  are independently  $N(0, \sigma^2)$ . Here  $(\alpha\beta)_{ij}$  is the interaction effect due to simultaneous occurrence of the  $i$ th level of  $A$  and the  $j$ th level of  $B$ . We are interested in testing:

$$H_{AB}: \text{all } (\alpha\beta)_{ij} = 0,$$

$$H_A: \text{all } \alpha_i = 0,$$

$$H_B: \text{all } \beta_j = 0.$$

Under the present set-up, the sum of squares due to the residual is given by

$$s_E^2 = \sum_i \sum_j \sum_k (Y_{ijk} - Y_{ij0})^2,$$

with  $rpq - pq$  degrees of freedom, and the interaction sum of squares due to  $H_{AB}$  is

$$r \sum_i \sum_j (\widehat{\alpha\beta})_{ij}^2,$$

with  $(p-1)(q-1)$  degrees of freedom, where

$$(\widehat{\alpha\beta})_{ij} = Y_{ij0} - Y_{i00} - Y_{0j0} + Y_{000}$$

and this is also called the sum of squares due to the interaction effects.

Denoting the interaction and error mean squares by  $\bar{s}_{AB}^2$  and  $\bar{s}_E^2$  respectively. The null hypothesis  $H_{AB}$  is tested at the  $\alpha$  level of significance by rejecting  $H_{AB}$  if

$$\frac{\bar{s}_{AB}^2}{\bar{s}_E^2} > F_{(p-1)(q-1), pq(r-1); \alpha}$$

and failing to reject it otherwise.

For testing  $H_A: a_i = \sigma$  for all  $i$ , the restricted residual sum of squares is

$$\begin{aligned} s_1^2 &= \sum_i \sum_j \sum_k (Y_{ijk} - Y_{ij0} + Y_{i00} - Y_{000})^2 \\ &= s_E^2 + rq \sum_i (Y_{i00} - Y_{000})^2, \end{aligned}$$

with  $rpq - pq + p - 1$  degrees of freedom, and

$$s_A^2 = rq \sum_i (Y_{i00} - Y_{000})^2,$$

with  $p-1$  degrees of freedom. With notation analogous to that for the test for  $H_{AB}$ , the test for  $H_A$  is then performed at level  $\alpha$  by rejecting  $H_A$  if

$$\frac{\bar{s}_A^2}{\bar{s}_E^2} > F_{(p-1), pq(r-1); \alpha}$$

and failing to reject it otherwise. The test for  $H_B$  is similar.

### Example

An experiment is conducted in which a crop yield is compared for three different levels of pesticide spray and three different levels of anti-fungal seed treatment. There are four replications of the experiment at each level combination. Do the different levels of pesticide spray and anti-fungal treatment effect crop yield and is there a significant interaction? The ANOVA table yields  $F$  ratios that are all below the appropriate  $F$  value from Table 3 so the experiment has yielded no significant effects and the experimenter needs to find more successful treatments.



**Numerical calculation**

| A | I  | B  |  | III | Table of means  |      |      |      |                 |
|---|----|----|--|-----|-----------------|------|------|------|-----------------|
|   |    | II |  |     |                 |      |      |      |                 |
| 1 | 95 | 60 |  | 86  |                 |      |      |      |                 |
|   | 85 | 90 |  | 77  |                 |      |      |      |                 |
|   | 74 | 80 |  | 75  |                 |      |      |      |                 |
|   | 74 | 70 |  | 70  |                 |      |      |      |                 |
| 2 | 90 | 89 |  | 83  | 1               | 82   | 75   | 77   | 78.0            |
|   | 80 | 90 |  | 70  | 2               | 86   | 89   | 75   | 83.3            |
|   | 92 | 91 |  | 75  | 3               | 80   | 78   | 85   | 81.0            |
|   | 82 | 86 |  | 72  |                 |      |      |      |                 |
| 3 | 70 | 68 |  | 74  | $\bar{Y}_{.j.}$ | 82.7 | 80.7 | 79.0 | $\bar{Y}_{i..}$ |
|   | 80 | 73 |  | 86  |                 |      |      |      |                 |
|   | 85 | 78 |  | 91  |                 |      |      |      |                 |
|   | 85 | 93 |  | 89  |                 |      |      |      |                 |

$$s_A^2 = 3 \times 4 \times \sum_i (\bar{Y}_{i..} - \bar{Y}_{...})^2 = 3 \times 4 \times 14.13 = 169.56$$

$$s_B^2 = 3 \times 4 \times \sum_j (\bar{Y}_{.j.} - \bar{Y}_{...})^2 = 12 \times 6.86 = 82.32$$

$$s_{AB}^2 = 4 \sum_i \sum_j (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2 = 4 \times 140.45 = 561.80$$

$$s_E^2 = \sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{ij.})^2 = 1830.0$$

**ANOVA table**

| Source | SS      | DF | MS     | F ratio |
|--------|---------|----|--------|---------|
| A      | 169.56  | 2  | 84.78  | 1.25    |
| B      | 82.32   | 2  | 41.16  | 0.61    |
| AB     | 561.80  | 4  | 140.45 | 2.07    |
| Error  | 1830.00 | 27 | 67.78  |         |

Critical values  $F_{2, 27}(0.05) = 3.35$  [Table 3],

$F_{4, 27}(0.05) = 2.73$  [Table 3].

Hence we do not reject any of the three hypotheses.

## Test 79 *F*-test for testing main effects in a two-way classification

### Object

To test the main effects in the case of a two-way classification with unequal numbers of observations per cell.

### Limitations

This test is applicable if the error in different measurements is normally distributed; if the relative size of these errors is unrelated to any factor of the experiment; and if the different measurements themselves are independent.

### Method

We consider the case of testing the null hypothesis

$$H_A: \alpha_i = 0 \quad \text{for all } i \quad \text{and} \quad H_B: \beta_j = 0 \quad \text{for all } j$$

under additivity. Under  $H_A$ , the model is:

$$Y_{ijk} = \mu + \beta_j + e_{ijk},$$

with the  $e_{ijk}$  independently  $N(0, \sigma^2)$ . The residual sum of squares (SS) under  $H_A$  is

$$s_2^2 = \sum_i \sum_j \sum_k Y_{ijk}^2 - \sum_j C_j^2 / n_{.j}.$$

with  $n - q$  degrees of freedom, where  $n_{.j}(\mu + \beta_j) + \sum_i n_{ij}\alpha_i = C_j$  and  $n_{ij}$  is the number of observations in the  $(i, j)$ th cell and  $\sum_j n_{ij} = n_{i.}$  and  $\sum_i n_{ij} = n_{.j}$ . the adjusted SS due to  $A$  is

$$SS_{A^*} = s_2^2 - s_1^2 = \sum_i \left( R_i - \sum_j p_{ij} C_j \right) \hat{\alpha}_i$$

with  $p - 1$  degrees of freedom, where  $n_{i.}(\mu + \alpha_i) + \sum_j n_{ij}\beta_j = R_i$ ,  $p_{ij} = n_{ij}/n_{.j}$ .

Under additivity, the test statistic for  $H_A$  is

$$\frac{SS_{A^*}}{s_1^2} \frac{n - p - q + 1}{p - 1},$$

which, under  $H_A$ , has the  $F$ -distribution with  $(p - 1, n - p - q + 1)$  degrees of freedom. Similarly, the test statistic for  $H_B$  is

$$\frac{SS_{B^*}}{s_1^2} \frac{n - p - q + 1}{q - 1},$$

which, under  $H_B$ , has the  $F$ -distribution with  $(q - 1, n - p - q + 1)$  degrees of freedom; where  $SS_{B^*} = \sum_j (C_j - \sum_i q_{ij} R_i) \hat{\beta}_j$  is the adjusted SS due to  $B$ , with  $q - 1$  degrees of freedom.

### Example

Three different chelating methods ( $A$ ) are used on three grades of vitamin supplement ( $B$ ). The availability of vitamin is tested by a standard timed-release method. Since some of the tests failed there are unequal cell numbers. An appropriate analysis of variance is conducted, so that the sums of squares are adjusted accordingly. Here chelating method produce significantly different results but no interaction is indicated. However, Grade of vitamin has indicated no differences.

### Numerical calculation

| $B$   | $A$      |          |          | Total |
|-------|----------|----------|----------|-------|
|       | 1        | 2        | 3        |       |
| 1     |          | 22       | 60       | 369   |
|       |          |          | (126)    |       |
|       | 88       | 26       | 66       |       |
|       | (172)    | (71)     | [0.2857] |       |
| 2     | 84       | 23       |          | 538   |
|       | [0.2500] | [0.4286] |          |       |
|       | 108      |          | 82       |       |
|       | 98       | 10       | 54       |       |
| 3     | (308)    | (34)     | (196)    | 394   |
|       | 102      | 24       | 60       |       |
|       | [0.3750] | [0.2857] | [0.4286] |       |
|       |          |          |          |       |
| 3     | 108      | 20       | 50       | 394   |
|       | 80       | (36)     | (82)     |       |
|       | (276)    | 16       | 32       |       |
|       | [0.3750] | [0.2857] | [0.2857] |       |
| Total | 756      | 141      | 404      | 1301  |

#### Note

1. Values in parentheses are the totals.
2. Values in brackets are the ratio of the number of observations divided by the column total number of observations, e.g. the first column has  $2/8 = 0.25$ .

$T = 1301$ , and  $T_2 = \text{observation sum of squares} = 100\,021$

CF (correction factor) =  $\frac{T^2}{N} = 76\,936.41$

Total SS =  $T_2 - \frac{T^2}{N} = 23\,084.59$

$$SS \text{ between cells} = \frac{1}{2}(172)^2 + \frac{1}{3}(71)^2 + \cdots + \frac{1}{2}(82)^2 - CF = 21\,880.59$$

$$SS \text{ within cells (error)} = \text{total } SS - SS \text{ between cells} = 1204.00$$

$$SS_A \text{ unadjusted} = \frac{1}{8}(756)^2 + \frac{1}{7}(141)^2 + \frac{1}{7}(404)^2 - CF = 20\,662.30$$

$$SS_B \text{ unadjusted} = \frac{1}{7}(369)^2 + \frac{1}{8}(538)^2 + \frac{1}{7}(394)^2 - CF = 872.23$$

$$C_{11} = 7 - 2(0.2500) - 3(0.4286) - 2(0.2857) = 4.6428$$

$$C_{12} = -5.0178, Q_1 = 369 - 756(0.2500) - 141(0.4286) \\ - 404(0.2857) = 4.145$$

$$Q_2 = 41.062, Q_3 = -45.2065, \hat{\alpha}_1 = 0.8341, \hat{\alpha}_2 = 5.4146, \hat{\alpha}_3 = -6.2487$$

$$SS_B \text{ adjusted} = Q_1\hat{\alpha}_1 + Q_2\hat{\alpha}_2 + Q_3\hat{\alpha}_3 = 508.27$$

$$SS_A \text{ adjusted} = SS_B \text{ adjusted} + SS_A \text{ unadjusted} \\ - SS_B \text{ unadjusted} = 20\,298.34$$

$$SS \text{ interaction} = SS \text{ between cells} - SS_B \text{ adjusted} \\ - SS_A \text{ unadjusted} = 710.02$$

**ANOVA table**

| Source                | DF | SS        | MS        | F ratio |
|-----------------------|----|-----------|-----------|---------|
| SSA adjusted          | 2  | 20 298.34 | 10 149.17 | 109.58  |
| SSB adjusted          | 2  | 508.27    | 254.14    | 2.744   |
| Interaction <i>AB</i> | 4  | 710.02    | 177.50    | 1.92    |
| Error                 | 13 | 1 204.00  | 92.62     |         |

Critical values  $F_{2,13;0.05} = 3.81$  [Table 3]

$F_{4,13;0.05} = 3.18$  [Table 3]

The main effects of *A* is significantly different, whereas the main effect *B* and interaction between *A* and *B* are not significant.

## Test 80 *F*-test for nested or hierarchical classification

### Object

To test for nestedness in the case of a nested or hierarchical classification.

### Limitations

This test is applicable if the error in different measurements is normally distributed; if the relative size of these errors is unrelated to any factor of the experiment; and if the different measurements themselves are independent.

### Method

In the case of a nested classification, the levels of factor *B* will be said to be nested with the levels of factor *A* if any level of *B* occurs with only a single level of *A*. This means that if *A* has *p* levels, then the *q* levels of *B* will be grouped into *p* mutually exclusive and exhaustive groups, such that the *i*th group of levels of *B* occurs only with the *i*th level of *A* in the observations. Here we shall only consider two-factor nesting, where the number of levels of *B* associated with the *i*th level of *A* is *q<sub>i</sub>*, i.e. we consider the case where there are  $\sum_i q_i$  levels of *B*.

For example, consider a chemical experiment where factor *A* stands for the method of analysing a chemical, there being *p* different methods. Factor *B* may represent the different analysts, there being *q<sub>i</sub>* analysts associated with the *i*th method.

The *j*th analyst performs the *n<sub>ij</sub>* experiments allotted to him. The corresponding fixed effects model is:

$$Y_{ijk} = \mu + \alpha_i + \beta_{ij} + e_{ijk}, \quad \begin{array}{l} i = 1, 2, \dots, p; \quad j = 1, 2, \dots, q_i; \\ k = 1, 2, \dots, n_{ij}, \end{array}$$

$$\sum_{i=1}^p n_i \alpha_i = \sum_{\text{all } j} n_{ij} \beta_j = 0$$

$$n_i = \sum_j n_{ij}, \quad n = \sum_i n_i \text{ and } e_{ijk} \text{ are independently } N(0, \sigma^2).$$

We are interested in testing  $H_A: \alpha_i = 0$ , for all *i*, and  $H_B: \beta_{ij} = 0$ , for all *i, j*.

The residual sum of squares is given by

$$s_E^2 = \sum_i \sum_j \sum_k (Y_{ijk} - Y_{ij0})^2$$

with  $\sum_{ij} (n_{ij} - 1)$  degrees of freedom; the sums of squares due respectively to *A* and *B* are

$$s_A^2 = \sum_i n_i (Y_{i00} - Y_{000})^2$$

with  $p - 1$  degrees of freedom, and

$$s_B^2 = \sum_i \sum_j n_{ij}(Y_{ij0} - Y_{i00})$$

with  $\sum_i (q_i - 1)$  degrees of freedom.

To perform the tests for  $H_A$  and  $H_B$  we first require the mean squares,  $\bar{s}_E^2$ ,  $\bar{s}_A^2$  and  $\bar{s}_B^2$ , corresponding to these sums of squares; we then calculate  $\bar{s}_A^2/\bar{s}_E^2$  to test  $H_A$  and  $\bar{s}_B^2/\bar{s}_E^2$  to test  $H_B$ , each of which, under the respective null hypothesis, follows the  $F$ -distribution with appropriate degrees of freedom.

Nested models are frequently used in sample survey investigations.

### Example

An educational researcher wishes to establish the relative contribution from the teachers and schools towards pupils' reading scores. She has collected data relating to twelve teachers (three in each of four schools). The analysis of variance table produces an  $F$  ratio of 1.46 which is less than the critical value of 2.10 from Table 3. So the differences between teachers are not significant. The differences between schools are, however, significant since the calculated  $F$  ratio of 6.47 is greater than the critical value of 4.07. Why the schools should be different is another question.

### Numerical calculation

Scores of pupils from three teachers in each of four schools are shown in the following table.

|               | Schools |      |       |         |       |       |         |       |       |         |       |       |      |
|---------------|---------|------|-------|---------|-------|-------|---------|-------|-------|---------|-------|-------|------|
|               | I       |      |       | II      |       |       | III     |       |       | IV      |       |       |      |
|               | Teacher |      |       | Teacher |       |       | Teacher |       |       | Teacher |       |       |      |
|               | 1       | 2    | 3     | 1       | 2     | 3     | 1       | 2     | 3     | 1       | 2     | 3     |      |
|               | 44      | 39   | 39    | 51      | 48    | 44    | 46      | 45    | 43    | 42      | 45    | 39    |      |
|               | 41      | 37   | 36    | 49      | 43    | 43    | 43      | 40    | 41    | 39      | 40    | 38    |      |
|               | 39      | 35   | 33    | 45      | 42    | 42    | 41      | 38    | 39    | 38      | 37    | 35    |      |
|               | 36      | 35   | 31    | 44      | 40    | 39    | 40      | 38    | 37    | 36      | 37    | 35    |      |
|               | 35      | 34   | 28    | 40      | 37    | 37    | 36      | 35    | 34    | 34      | 32    | 35    |      |
|               | 32      | 30   | 26    | 40      | 34    | 36    | 34      | 34    | 33    | 31      | 32    | 29    |      |
| Teacher total | 227     | 210  | 193   | 269     | 244   | 241   | 240     | 230   | 227   | 220     | 223   | 211   |      |
| Mean          | 37.83   | 35.0 | 32.17 | 44.83   | 40.67 | 40.17 | 40.0    | 38.33 | 37.83 | 36.67   | 37.17 | 35.17 |      |
| School total  | 630     |      |       | 754     |       |       | 697     |       |       | 654     |       |       | 2735 |
| Mean          | 35.00   |      |       | 41.89   |       |       | 38.72   |       |       | 36.34   |       |       |      |

$$T = 2735$$

$$\text{CF (correction factor)} = \frac{2735^2}{72} = 103\,892.01$$

$$\text{Total sum of squares} = 105\,637.00 - 103\,892.01 = 1744.99$$

Between-schools sum of squares

$$= \frac{630^2}{18} + \frac{754^2}{18} + \frac{697^2}{18} + \frac{654^2}{18} - CF = 493.60$$

Between teachers (within school) sum of squares

$$= \frac{227^2}{6} + \frac{210^2}{6} + \frac{193^2}{6} + \frac{630^2}{18} + \text{similar terms for schools II, III and IV} = 203.55$$

$$\text{Within-group sum of squares} = 1744.99 - 493.60 - 203.55 = 1047.84.$$

**ANOVA table**

|                        | DF | SS      | Mean square |
|------------------------|----|---------|-------------|
| Schools                | 3  | 493.60  | 164.53      |
| Teachers within school | 8  | 203.55  | 25.44       |
| Pupils within teachers | 60 | 1047.84 | 17.46       |
| Total                  | 71 | 1744.99 |             |

Teacher differences:

$$F = \frac{25.44}{17.46} = 1.46$$

Critical value  $F_{8,60;0.05} = 2.10$  [Table 3].

The calculated value is less than the critical value.

Hence the differences between teachers are not significant.

School differences:

$$F = \frac{164.53}{25.44} = 6.47$$

Critical value  $F_{3,8;0.05} = 4.07$  [Table 3].

The calculated value is greater than the critical value.

Hence the differences between schools are significant.

## Test 81 *F*-test for testing regression

### Object

To test the presence of regression of variable  $Y$  on the observed value  $X$ .

### Limitations

For given  $X$ , the  $Y$ s are normally and independently distributed. The error terms are normally and independently distributed with mean zero.

### Method

Suppose, corresponding to each value  $X_i$  ( $i = 1, 2, \dots, p$ ) of the independent random variable  $X$ , we have a corresponding array of observations  $Y_{ij}$  ( $j = 1, 2, \dots, n_i$ ) on the dependent variable  $Y$ . Using the model:

$$Y_{ij} = \mu_i + e_{ij}, \quad i = 1, 2, \dots, p, \quad j = 1, 2, \dots, n_i,$$

where the  $e_{ij}$  are independently  $N(0, \sigma^2)$ , we are interested in testing  $H_0$ : all  $\mu_i$  are equal, against  $H_1$ : not all  $\mu_i$  are equal. ' $H_0$  is true' implies the absence of regression of  $Y$  on  $X$ . Then the sums of squares are given by

$$s_B^2 = \sum_i n_i (Y_{i0} - Y_{00})^2, \quad s_E^2 = \sum_i \sum_j (Y_{ij} - Y_{i0})^2.$$

Denoting the corresponding mean squares by  $\bar{s}_B^2$  and  $\bar{s}_E^2$  respectively, then, under  $H_0$ ,  $F = \bar{s}_B^2 / \bar{s}_E^2$  follows the  $F$ -distribution with  $(p - 1, n - p)$  degrees of freedom.

### Example

It is desired to test for the presence of regression (i.e. non-zero slope) in comparing an independent variable  $X$ , with a dependent variable  $Y$ .

A small-scale experiment is set up to measure perceptions on a simple dimension ( $Y$ ) to a visual stimulus ( $X$ ). The results test for the presence of a regression of  $Y$  on  $X$ . The experiment is repeated three times at two levels of  $X$ .

Since the calculated  $F$  value of 24 is larger than the critical  $F$  value from Table 3, the null hypothesis is rejected, indicating the presence of regression.

### Numerical calculation

|          |           |  |
|----------|-----------|--|
| $Y_{ij}$ |           |  |
| $X_1$    | 1   2   3 | $n_1 = 3$ , where $Y_{i0} = \sum_{j=1}^{n_i} \frac{Y_{ij}}{n_i}, \quad i = 1, 2, \dots, p$ |
| $X_2$    | 7   5   6 | $n_2 = 3$ , and $Y_{00} = \sum_i \sum_j \frac{Y_{ij}}{n}, \quad n = \sum n_i$              |



$$\text{Hence } Y_{10} = \frac{1+2+3}{3} = 2, Y_{20} = \frac{7+5+6}{3} = 6,$$

$$Y_{00} = \frac{1+2+3+7+5+6}{6} = 4$$

$$s_B^2 = 3(2-4)^2 + 3(6-4)^2 = 24$$

$$s_E^2 = (1-2)^2 + (2-2)^2 + (3-2)^2 + (7-6)^2 + (5-6)^2 + (6-6)^2 = 4$$

$$\bar{s}_B^2 = 24/1 = 24, \bar{s}_E^2 = 4/4 = 1, F = 24/1 = 24$$

Critical value  $F_{1,4;0.05} = 7.71$  [Table 3].

Hence reject the null hypothesis, indicating the presence of regression.

## Test 82 *F*-test for testing linearity of regression

### Object

To test the linearity of regression between an  $X$  variable and a  $Y$  variable.

### Limitations

For given  $X$ , the  $Y$ s are normally and independently distributed. The error terms are normally and independently distributed with mean zero.

### Method

Once the relationship between  $X$  and  $Y$  is established using Test 81, we would further like to know whether the regression is linear or not. Under the same set-up as Test 81, we are interested in testing:

$$H_0: \mu_i = \alpha + \beta X_i, \quad i = 1, 2, \dots, n,$$

Under  $H_0$ ,

$$s_E^2 = \sum_i (y_i - \bar{y}) - b^2 \sum_i n_i (x_i - \bar{x})^2,$$

with  $n - 2$  degrees of freedom, and the sum of squares due to regression

$$s_R^2 = b^2 \sum_i n_i (x_i - \bar{x})^2,$$

with 1 degree of freedom. The ratio of mean squares

$F = s_R^2 / s_E^2$  is used to test  $H_0$  with  $(1, n - 2)$  degrees of freedom.

### Example

In a chemical reaction the quantity of plastic polymer ( $Y$ ) is measured at each of four levels of an enzyme additive ( $X$ ). The experiment is repeated three times at each level of  $X$  to enable a test of linearity of regression to be performed.

The data produce an  $F$  value of 105.80 and this is compared with the critical  $F$  value of 4.96 from Table 3. Since the critical value is exceeded we conclude that there is a significant regression.

### Numerical calculation

| $i$   | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   |
|-------|------|------|------|------|------|------|------|------|------|------|------|------|
| $x_i$ | 150  | 150  | 150  | 200  | 200  | 200  | 250  | 250  | 250  | 300  | 300  | 300  |
| $y_i$ | 77.4 | 76.7 | 78.2 | 84.1 | 84.5 | 83.7 | 88.9 | 89.2 | 89.7 | 94.8 | 94.7 | 95.9 |

$n = 12, n - 2 = 10$ . For  $\beta = 0$ , test  $H_0: \beta = 0$  against  $H_1: \beta \neq 0$

The total sum of squares is

$$\sum y_i^2 - \left( \sum y_i \right)^2 / n = 513.1167,$$

and

$$s_R^2 = b \frac{\left( \sum x_i y_i - \frac{1}{n} \sum x_i \sum y_i \right)^2}{\sum x_i^2 - \frac{1}{n} (\sum x_i)^2} = 509.10,$$

$$s_E^2 = 4.0117, \bar{s}_R^2 = 42.425, \bar{s}_E^2 = 0.401, F = 105.80$$

Critical value  $F_{1,10;0.05} = 4.96$  [Table 3].

Hence reject the null hypothesis and conclude that  $\beta \neq 0$ .

## Test 83 Z-test for the uncertainty of events

### Object

To test the significance of the reduction of uncertainty of past events.

### Limitations

Unlike sequential analyses, this test procedure requires a probability distribution of a variable.

### Method

It is well known that the reduction of uncertainty by knowledge of past events is the basic concept of sequential analysis. The purpose here is to test the significance of this reduction of uncertainty using the statistic

$$Z = \frac{P(B_{+k}|A) - P(B)}{\sqrt{\frac{P(B)[1 - P(B)][1 - P(A)]}{(n - k)P(A)}}$$

where  $P(A)$  = probability of  $A$ ,  $P(B)$  = probability of  $B$  and  $P(B_{+k}|A) = P(B|A)$  at lag  $k$ .

### Example

An economic researcher wishes to test for the reduction of uncertainty of past events. He notes that following a financial market crash (event  $A$ ) a particular economic index rises (event  $B$ ). His test statistic of  $Z = 2.20$  is greater than the tabulated value of 1.96 from Table 1. This is a significant result allowing him to claim a reduction of uncertainty for events  $A$  and  $B$ .

### Numerical calculation

Consider a sequence of  $A$  and  $B$ :  $AA\ BA\ BA\ BB\ AB\ AB$

$$n = 12, \quad k = 1, \quad P(A) = \frac{6}{12} = 0.5, \quad P(B) = \frac{6}{12} = 0.5$$

We note that  $A$  occurs six times and that of these six times  $B$  occurs immediately after  $A$  five times. Given that  $A$  just occurred we have

$$P(B|A) \text{ at lag one} = P(B_{+1}|A) = \frac{5}{6} = 0.83.$$

Therefore the test statistic is

$$Z = \frac{0.83 - 0.50}{\sqrt{\frac{0.50(1 - 0.50)(1 - 0.50)}{(12 - 1)(0.50)}}} = 2.20.$$

The critical value at  $\alpha = 0.05$  is 1.96 [Table 1].

The calculated value is greater than the critical value.

Hence it is significant.

## Test 84 Z-test for comparing sequential contingencies across two groups using the 'log odds ratio'

### Object

To test the significance of the difference in sequential connections across groups.

### Limitations

This test is applicable when a logit transformation can be used and  $2 \times 2$  contingency tables are available.

### Method

Consider a person's antecedent behaviour ( $W_t$ ) taking one of the two values:

$$W_t = \begin{cases} 1 & \text{for negative effect} \\ 0 & \text{for positive effect.} \end{cases}$$

Let us use  $H_t + 1$ , a similar notation, for the spouse's consequent behaviour. A fundamental distinction may be made between measures of association in contingency tables which are either sensitive or insensitive to the marginal (row) totals. A measure that is invariant to the marginal total is provided by the so-called logit transformation. The logit is defined by:

$$\text{logit}(P) = \log_e \frac{P}{1 - P}.$$

We can now define a statistic  $\beta$  as follows:

$$\beta = \text{logit}[P_r(H_{t+k} = 1 | W_t = 1)] - \text{logit}[P_r(H_{t+k} = 1 | W_t = 0)].$$

Hence  $\beta$  is known as the logarithm of the 'odds ratio' which is the cross product ratio in a  $2 \times 2$  contingency table, i.e. if we have a table in which first row is  $(a, b)$  and second row is  $(c, d)$  then

$$\beta = \log \left( \frac{ad}{bc} \right).$$

In order to test whether  $\beta$  is different across groups we use the statistic

$$Z = \frac{\beta_1 - \beta_2}{\sqrt{\sum \left( \frac{1}{f_i} \right)}}$$

where  $f_i$  is the frequency in the  $i$ th cell and  $Z$  is the standard normal variate, i.e.  $N(0, 1)$ .

**Example**

A social researcher wishes to test a hypothesis concerning the behaviour of adult couples. She compares a man's behaviour with a consequent spouse's behaviour for couples in financial distress and for those not in financial distress. A log-odds ratio test is used. In this case the  $Z$  value of 1.493 is less than the critical value of 1.96 from Table 1. She concludes that there is insufficient evidence to suggest financial distress affects couples' behaviour in the way she hypothesizes.

**Numerical calculation**

| $H_t$ | Distressed couples<br>$W_{t+1}$ |     | Non-distressed couples<br>$W_{t+1}$ |    |
|-------|---------------------------------|-----|-------------------------------------|----|
|       | 1                               | 0   | 1                                   | 0  |
| 1     | 76                              | 100 | 80                                  | 63 |
| 0     | 79                              | 200 | 43                                  | 39 |

$$\beta_1 = \log_e \frac{76 \times 200}{79 \times 100} = 0.654; \quad \beta_2 = \log_e \frac{80 \times 39}{43 \times 63} = 0.141$$

$$Z = \frac{0.654 - 0.141}{\sqrt{\frac{1}{76} + \frac{1}{79} + \frac{1}{100} + \frac{1}{200} + \frac{1}{80} + \frac{1}{43} + \frac{1}{63} + \frac{1}{39}}} = 1.493$$

The critical value at  $\alpha = 0.05$  is 1.96 [Table 1].

The calculated value is less than the critical value.

Hence it is not significant and the null hypothesis (that  $\beta$  is not different across groups) cannot be rejected.

## Test 85 *F*-test for testing the coefficient of multiple regression

### Object

A multiple linear regression model is used in order to test whether the population value of each multiple regression coefficient is zero.

### Limitations

This test is applicable if the observations are independent and the error term is normally distributed with mean zero.

### Method

Let  $X_1, X_2, \dots, X_k$  be  $k$  independent variables and  $X_{1i}, X_{2i}, \dots, X_{ki}$  be their fixed values, corresponding to dependent variables  $Y_i$ . We consider the model:

$$Y_i = \beta_0 + \beta_1 X'_{1i} + \dots + \beta_k X'_{ki} + e_i$$

where  $X'_{ji} = X_{ji} - \bar{X}_j$  and the  $e_i$  are independently  $N(0, \sigma^2)$ .

We are interested in testing whether the population value of each multiple regression coefficient is zero. We want to test:

$$H_0: \beta_1 = \beta_2 = \dots = \beta_k = 0 \quad \text{against} \quad H_1: \text{not all } \beta_k = 0$$

for  $k = 1, 2, \dots, p - 1$ , where  $p$  is the number of parameters. The error sum of squares is

$$s_E^2 = \sum_i (Y_i - \bar{Y})^2 - \sum_j b_j \sum_i Y_i X'_{ji}$$

with  $n - k - 1$  degrees of freedom, where  $b_j$  is the least-squares estimator of  $\beta_j$ .

The sum of squares due to  $H_0$  is

$$s_H^2 = \sum_j b_j \sum_i Y_i X'_{ji}$$

with  $k$  degrees of freedom. Denoting the corresponding mean squares by  $\bar{s}_E^2$  and  $\bar{s}_H^2$  respectively, then, under  $H_0$ ,  $F = \bar{s}_H^2 / \bar{s}_E^2$  follows the  $F$ -distribution with  $(k, n - k - 1)$  degrees of freedom and can be used for testing  $H_0$ .

The appropriate decision rule is: if the calculated  $F \leq F_{p-1, n-p; 0.05}$ , do not reject  $H_0$ ; if the calculated  $F > F_{p-1, n-p; 0.05}$ , reject  $H_0$ .

### Example

In an investigation of the strength of concrete ( $Y$ ), a number of variables were measured ( $X_1, X_2, \dots, X_k$ ) and a multiple regression analysis performed. The global  $F$  test is a

test of whether any of the  $X$  variables is significant (i.e. any of the  $\beta_i$  coefficients,  $i = 1, \dots, k$  are non-zero).

The calculated  $F$  value of 334.35 is greater than the tabulated  $F$  value of 3.81 [Table 3]. So at least one of the  $X$  variables is useful in the prediction of  $Y$ .

### Numerical calculation

$$n = 16, p = 3, v = p - 1, v_2 = n - p$$

Critical value  $F_{2, 13; 0.05} = 3.81$  [Table 3].

From the computer output of a certain set of data

$$\bar{s}_H^2 = 96.74439, \bar{s}_E^2 = 0.28935$$

$$F = 96.74439 / 0.28935 = 334.35$$

Hence reject the null hypothesis.



## Test 86 *F*-test for variance of a random effects model

### Object

To test for variance in a balanced random effects model of random variables.

### Limitations

This test is applicable if the random variables are independently and normally distributed with mean zero.

### Method

Let the random variable  $Y_{ij\dots m}$  for a balanced case be such that:

$$Y_{ij\dots m} = \mu + a_i + b_{ij} + c_{ijk} + \dots + e_{ijk\dots m}$$

where  $\mu$  is a constant and the random variables  $a_i, b_{ij}, c_{ijk}, \dots, e_{ijk\dots m}$  are completely independent and  $a_i \sim N(0, \sigma_a^2), \dots, b_{ij} \sim N(0, \sigma_b^2), c_{ijk} \sim N(0, \sigma_c^2), \dots, e_{ijk\dots m} \sim N(0, \sigma_e^2)$ .

Then the test for  $H_0 = \sigma_i^2 = \sigma_j^2 (i \neq j)$  against  $H_1 = \sigma_i^2 > \sigma_j^2$  is given by  $s_i^2/s_j^2$  which is distributed as  $(\sigma_i^2/\sigma_j^2)F$  and follows the  $F$ -distribution with  $(f_i, f_j)$  degrees of freedom. Here  $s_i^2$  and  $s_j^2$  are the estimates of  $\sigma_i^2$  and  $\sigma_j^2$ .

### Example

An electronic engineer wishes to test for company and sample electrical static differences for a component used in a special process. He selects three companies at random and also two samples. He collects four measurements of static from selected components. Are the companies all the same with respect to electrical resistance of the particular component? His analysis of variance calculation produces an  $F$  value of 1.279. He compares this with the tabulated  $F$  value of 9.55 [Table 3] and concludes that there is no evidence to suggest company differences.

### Numerical calculation

$$s_1^2 = 5.69, s_2^2 = 4.45, f_1 = 2, f_2 = 3$$

$$H_0: \sigma_1^2 = \sigma_2^2; H_1: \sigma_1^2 > \sigma_2^2$$

$$F = \frac{s_1^2}{s_2^2} = \frac{5.69}{4.45} = 1.279$$

Critical value  $F_{2,3;0.05} = 9.55$  [Table 3]

Hence we do not reject the null hypothesis  $H_0$ .

$$f_1 = n_1 - 1 \quad \text{Here: } n_1 = 3, n_2 = 2$$

$$f_2 = 2n_2 - 1$$

## Test 87 *F*-test for factors *A* and *B* and an interaction effect

### Object

To test for the homogeneity of factors *A* and *B* and the absence of an interaction effect.

### Limitations

This test is applicable if the random variables and interaction effects are jointly normal with mean zero and the error terms are independently normally distributed with mean zero.

### Method

Let the factor having fixed levels be labelled *A* and represented by the columns of the table and let the randomly sampled factor be *B* and represented by the rows. Let

$$Y_{ijk} = \mu + \alpha_i + b_k + c_{jk} + e_{ijk}$$

where

- $\alpha_i$  is the fixed effect of the treatment indicated by the column *i*;
- $b_k$  = random variable associated with the *k*th row;
- $c_{jk}$  = random interaction effect operating on the (*j*, *k*)th cell; and
- $e_{ijk}$  = random error associated with observation *i* in the (*j*, *k*)th cell.

We make the following assumptions:

1.  $b_k$  and  $c_{jk}$  are jointly normal with mean zero and with variance  $\sigma_B^2$  and  $\sigma_{AB}^2$ , respectively;
2.  $e_{ijk}$  are normally distributed with mean zero and variance  $\sigma_E^2$ ;
3.  $e_{ijk}$  are independent of  $b_k$  and  $c_{jk}$ ;
4.  $e_{ijk}$  are independent of each other.

Denoting the column, interaction, row and error mean squares by  $\bar{s}_C^2$ ,  $\bar{s}_I^2$ ,  $\bar{s}_R^2$  and  $\bar{s}_E^2$  respectively, then  $\bar{s}_C^2/\bar{s}_I^2 \sim F_{c-1, (r-1)(c-1)}$  provides an appropriate test for the column effects, i.e.  $H_0: \sigma_A^2 = 0$ ;  $\bar{s}_R^2/\bar{s}_E^2 \sim F_{r-1, rc(n-1)}$  provides a test for  $H_0: \sigma_B^2 = 0$ ; and  $\bar{s}_I^2/\bar{s}_E^2 \sim F_{(r-1)(c-1), r(n-1)}$  provides a test for  $H_0: \sigma_{AB}^2 = 0$ .

### Example

An educational researcher has collected data on pupils' performance in relation to three tasks; six classes are compared. The analysis of variance suggests that neither classroom nor task factors is significant but their interaction is significant.

**Numerical calculation**

In the following table the values are a combination of classroom and task and represent the independent performance of two subjects in each cell.

| Classroom | Tasks              |                     |                     | Total |
|-----------|--------------------|---------------------|---------------------|-------|
|           | I                  | II                  | III                 |       |
| 1         | 7.8<br>8.7 (16.5)  | 11.1<br>12.0 (23.1) | 11.7<br>10.0 (21.7) | 61.3  |
| 2         | 8.0<br>9.2 (17.2)  | 11.3<br>10.6 (21.9) | 9.8<br>11.9 (21.7)  | 60.8  |
| 3         | 4.0<br>6.9 (10.9)  | 9.8<br>10.1 (19.9)  | 11.7<br>12.6 (24.3) | 55.1  |
| 4         | 10.3<br>9.4 (19.7) | 11.4<br>10.5 (21.9) | 7.9<br>8.1 (16.0)   | 57.6  |
| 5         | 9.3<br>10.6 (19.9) | 13.0<br>11.7 (24.7) | 8.3<br>7.9 (16.2)   | 60.8  |
| 6         | 9.5<br>9.8 (19.3)  | 12.2<br>12.3 (24.5) | 8.6<br>10.5 (19.1)  | 62.9  |
| Total     | 103.5              | 136.0               | 119.0               | 358.5 |

$$CF \text{ (correction factor)} = 358.5^2/36 = 3570.0625$$

$$TSS = 7.8^2 + \dots + 10.5^2 - (3570.0625) = 123.57$$

The sums of squares are given by

$$s_C^2 = \frac{(103.5)^2 + (136.0)^2 + (119.0)^2}{12} - CF = 44.04$$

$$s_R^2 = \frac{61.3^2 + \dots + 62.9^2}{6} - CF = 6.80$$

$$s_E^2 = 7.8^2 + \dots + 10.5^2 - \frac{16.5^2 + \dots + 19.1^2}{2} = 14.54$$

$$s_I^2 = 123.57 - 44.04 - 6.8 - 14.54 = 58.19$$

**ANOVA table**

| Source      | DF | SS     | MS    | <i>F</i> |
|-------------|----|--------|-------|----------|
| Column      | 2  | 44.04  | 22.02 | 3.78     |
| Row         | 5  | 6.80   | 1.36  | 1.68     |
| Interaction | 10 | 58.19  | 5.82  | 7.19     |
| Error       | 18 | 14.54  | 0.81  |          |
| Total       | 35 | 123.57 |       |          |

Critical values  $F_{10,18;0.05} = 2.41$  [Table 3]

$$F_{2,10;0.05} = 4.10$$

$$F_{5,18;0.05} = 2.77$$

Hence  $H_0: \sigma_{AB} = 0$  is rejected.

$H_0: \sigma_A^2 = 0$  is not rejected.

$H_0: \sigma_B^2 = 0$  is not rejected.

## Test 88 Likelihood ratio test for the parameter of a rectangular population

### Object

To test for one of the parameters of a rectangular population with probability density function

$$f_0(X) = \begin{cases} \frac{1}{2\beta}, & \alpha - \beta \leq X \leq \alpha + \beta \\ 0, & \text{otherwise.} \end{cases}$$

### Limitations

This test is applicable if the observations are a random sample from a rectangular distribution.

### Method

Let  $X_1, X_2, \dots, X_n$  be a random sample from the above rectangular population. We are interested in testing  $H_0: \alpha = 0$  against  $H_1: \alpha \neq 0$ . Then, the likelihood ratio test criterion for testing  $H_0$  is:

$$\lambda = \left( \frac{X_{(n)} - X_{(1)}}{2Z} \right)^n = \left( \frac{R}{2Z} \right)^n,$$

where  $R$  is the sample range and  $Z = \max[-X_{(1)}, X_{(n)}]$ . Then the asymptotic distribution of  $2 \log_e \lambda$  is  $\chi^2_2$ .

### Example

An electronic component test profile follows a rectangular distribution at stepped input levels. An electronic engineer wishes to test a particular component type and collects his data. He uses a likelihood ratio test. He obtains a value of his test statistic of  $-4.2809$  and compares this with his tabulated value. He thus rejects the null hypothesis that one parameter is zero.

### Numerical calculation

Consider  $(-0.2, -0.3, -0.4, 0.4, 0.3, 0.5)$  as a random sample from a rectangular distribution. Here,

$$n = 6$$

$$R = 0.5 - (-0.2) = 0.7, Z = \max[0.4, 0.5] = 0.5$$

$$\lambda = \left( \frac{R}{2Z} \right)^6 = (0.7)^6 = 0.1176, \quad 2 \log_e \lambda = -4.2809$$

Critical value  $\chi^2_{2;0.05} = 5.99$  [Table 5].

Hence we reject the null hypothesis that  $\alpha = 0$ .

## Test 89 Uniformly most powerful test for the parameter of an exponential population

### Object

To test the parameter ( $\theta$ ) of the exponential population with probability density function

$$f(X, \theta) = \theta e^{-\theta X}, \quad X > 0.$$

### Limitations

The test is applicable if the observations are a random sample from an exponential distribution.

### Method

Let  $X_1, X_2, \dots, X_n$  be a random sample from an exponential distribution with parameter  $\theta$ . Let our null hypothesis be  $H_0: \theta = \theta_0$  against the alternative  $H_1: \theta = \theta_1$  ( $\theta_1 \neq \theta_0$ ).

**Case (a)**  $\theta_1 > \theta_0$ : The most powerful critical region is given by:

$$W_0 = \left[ X = \sum_i X_i \leq \chi_{2n; 1-\alpha}^2 / 2\theta_0 \right].$$

Since  $W_0$  is independent of  $\theta_1$ , so  $W_0$  is uniformly most powerful for testing  $H_0: \theta = \theta_0$  against  $H_1: \theta > \theta_0$ .

**Case (b)**  $\theta_1 < \theta_0$ : The most powerful critical region is given by:

$$W_1 = \left[ X = \sum_i X_i > \chi_{2n; \alpha}^2 / 2\theta_0 \right].$$

Again  $W_1$  is independent of  $\theta_1$  and so it is also uniformly most powerful for testing  $H_0: \theta = \theta_0$  against  $H_1: \theta < \theta_0$ .

### Example

An electronic component is tested for an exponential failure distribution with a given parameter. The given example produces a critical region for the test of parameter equal to 1 against the alternative hypothesis that it equals 2.

### Numerical calculation

Let us consider a sample of size 2 from the population  $f(X|\theta) = \theta e^{-\theta X}, X > 0$ . Consider testing  $H_0: \theta = 1$  against  $H_1: \theta = 2$ , i.e.  $\theta > \theta_0$ . The critical region is

$$\begin{aligned} W &= \left\{ \mathbf{X} / \sum X_i \leq \chi_{0.95, 4/2}^2 \right\} \\ &= \left\{ \mathbf{X} / \sum X_i \leq 0.71/2 \right\} \text{ [Table 5]} \\ &= \left\{ \mathbf{X} / \sum X_i \leq 0.36 \right\} \end{aligned}$$

## Test 90 Sequential test for the parameter of a Bernoulli population

### Object

To test the parameter of the Bernoulli population by the sequential method.

### Limitations

This test is applicable if the observations are independent and identically follow the Bernoulli distribution.

### Method

Let  $X_1, X_2, \dots, X_m$  be independent and identically distributed random variables having the distribution with probability density function (PDF)

$$f_0^{(x)} = \begin{cases} \theta^x (1 - \theta)^{1-x}, & X = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

where  $0 < \theta < 1$ . We want to test  $H_0: \theta = \theta_0$  against  $H_1: \theta = \theta_1$ .

We fail to reject  $H_0$  if  $S_m \leq a_m$ , and we reject  $H_0$  if  $S_m \geq r_m$ . We continue sampling, i.e. taking observations, if  $a_m < S_m < r_m$ , where  $S_m = \sum_{i=1}^m X_i$  and

$$a_m = \frac{\log \frac{\beta}{1 - \alpha}}{\log \left( \frac{\theta_1}{\theta_0} \right) - \log \left( \frac{1 - \theta_1}{1 - \theta_0} \right)} + \frac{m \log \left( \frac{1 - \theta_0}{1 - \theta_1} \right)}{\log \left( \frac{\theta_1}{\theta_0} \right) - \log \left( \frac{1 - \theta_1}{1 - \theta_0} \right)}$$

$$r_m = \frac{\log \frac{1 - \beta}{\alpha}}{\log \left( \frac{\theta_1}{\theta_0} \right) - \log \left( \frac{1 - \theta_1}{1 - \theta_0} \right)} + \frac{m \log \left( \frac{1 - \theta_0}{1 - \theta_1} \right)}{\log \left( \frac{\theta_1}{\theta_0} \right) - \log \left( \frac{1 - \theta_1}{1 - \theta_0} \right)}.$$

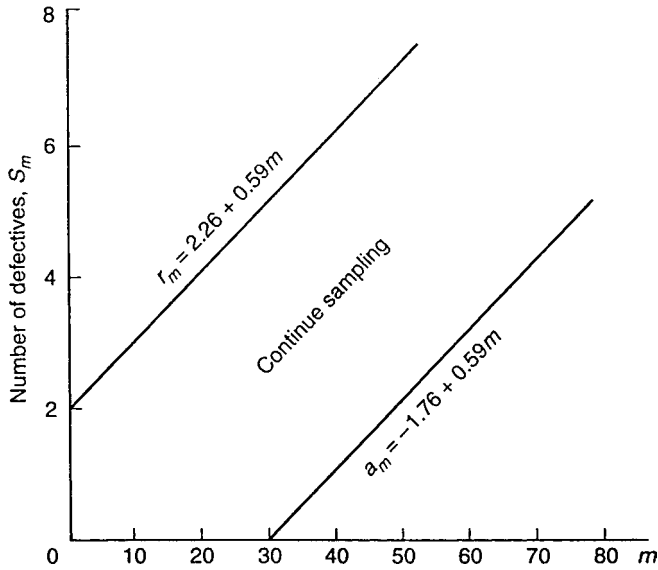
### Example

Same as Numerical calculation.

### Numerical calculation

While dealing with the sampling of manufactured products,  $\theta$  may be looked upon as the true proportion of defectives under a new production process. A manufacturer may be willing to adopt the new process if  $\theta \leq \theta_0$  and will reject it if  $\theta \geq \theta_1$  and he may not be decisive if  $\theta_0 < \theta < \theta_1$ . To reach a decision, he may use a sequential sampling plan, taking one item at each stage and at random. Here  $S_m$  will be the number of defectives up to the  $m$ th stage,  $a_m$  the corresponding acceptance number and  $r_m$  the

rejection number. If  $\theta_0 = 0.04$  and  $\theta_1 = 0.08$ ,  $\alpha = 0.15$ ,  $\beta = 0.25$ , then the graphical representation of the sequential plan is:



As soon as  $(m, S_m)$  lies on or below the line for  $a_m$  or on or above the line for  $r_m$ , sampling is to be stopped; the new process is to be considered in the former case and rejected in the latter.



## Test 91 Sequential probability ratio test

### Object

A sequential test for the ratio between the mean and the standard deviation of a normal population where both are unknown.

### Limitations

This test is applicable if the observations are normally distributed with unknown mean and variance.

### Method

Let  $X \sim N(\mu, \sigma^2)$ , where both  $\mu$  and  $\sigma^2$  are unknown. We want to find a sequential test for testing  $H_0: \mu/\sigma = r_0$  against  $H_1: \mu/\sigma = r_1$ . The sequential probability ratio test procedure is as follows.

1. Continue sampling if  $b_n < t_n < a_n$ , where

$$t_n = \frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n X_i^2}} = \frac{\sum_{i=1}^n y_i}{\sqrt{\sum_{i=1}^n y_i^2}}$$

where  $y_i = X_i/|X_i|, i = 1, 2, \dots, n$ ,

$$a_n = \log \frac{1 - \beta}{\alpha} \quad \text{and} \quad b_n = \log \frac{\beta}{1 - \alpha}.$$

2. Fail to reject  $H_0$  if  $t_n \leq b_n$  and reject  $H_0$  if  $t_n \geq a_n$ .

### Example

A useful measure is the ratio of mean divided by standard deviation since it is independent of measured units. Here we set up a sequential test for the ratio equal to 0.2 versus the alternative that it is equal to 0.4. The rule is that if the test statistic is less than  $\log(7/15)$  do not reject the null hypothesis and if the test statistic is greater than  $\log(13/5)$  then reject the null hypothesis; otherwise continue to sample.

### Numerical calculation

Consider a sample from  $N(\mu, \sigma^2)$  where  $\mu$  and  $\sigma$  are both unknown. Then we want a sequential test for testing  $H_0: \mu/\sigma = 0.2$  against  $H_1: \mu/\sigma = 0.4$ .

Let  $\alpha = 0.25, \beta = 0.35, a_n = \log(0.65/0.25), b_n = \log(0.35/0.75)$ .

If  $\log 7/15 < t_n < \log 13/5$  then continue sampling.

If  $t_n \leq \log 7/15$  do not reject  $H_0$ , and if  $t_n \geq \log 13/5$  reject  $H_0$ .

## Test 92 Durbin–Watson test

### Object

To test whether the error terms in a regression model are autocorrelated.

### Limitations

This test is applicable if the autocorrelation parameter and error terms are independently normally distributed with mean zero and variance  $s^2$ .

### Method

This test is based on the first-order autoregressive error model  $\varepsilon_t = \varphi\varepsilon_{t-1} + u_t$  where  $\varphi$  is the autocorrelation parameter and the  $u_t$ , are independently normally distributed with zero mean and variance  $\sigma^2$ . When one is concerned with positive autocorrelation the alternatives are given as follows:

$$H_0: \varphi \leq 0 \quad H_1: \varphi > 0.$$

Here  $H_0$  implies that error terms are uncorrelated or negatively correlated, while  $H_1$  implies that they are positively autocorrelated. This test is based on the difference between adjacent residuals  $\varepsilon_t - \varepsilon_{t-1}$  and is given by

$$d = \frac{\sum_{t=2}^n (e_t - e_{t-1})^2}{\sum_{t=1}^n e_t^2}$$

where  $e_t$  is the regression residual for period  $t$ , and  $n$  is the number of time periods used in fitting the regression model.

When the error terms are positively autocorrelated the adjacent residuals will tend to be of similar magnitude and the numerator of the test statistic  $d$  will be small. If the error terms are either not correlated or negatively correlated  $e_t$  and  $e_{t-1}$  will tend to differ and the numerator of the test statistic will be larger. The exact action limit for this test is difficult to calculate and the test is used with lower bound  $d_L$  and the upper bound  $d_U$ . When the statistic  $d$  is less than the lower bound  $d_L$ , we conclude that positive autocorrelation is present. Similarly, when the test statistic exceeds the upper bound  $d_U$ , we conclude that positive autocorrelation is not present. When  $d_L < d < d_U$ , the test is inconclusive.

### Example

Data on the sales (£m) of a large company ( $Y$ ) compared with its sector total ( $X$ ) have been collected over a five-year period. In order to test the significance of the regression of  $Y$  on  $X$ , and to obtain confidence intervals on predictions, it is necessary to perform a test of serial correlation on the error term or residuals. The Durbin–Watson test statistic

( $d$ ) is equal to 0.4765 and is less than the lower  $d$  value ( $d_L$ ) of 1.20 [Table 33]. So the residuals are positively autocorrelated. An appropriate adjustment to the error sum of squares is necessary.

### Numerical calculation

$$n = 20, \alpha = 0.05$$

| Quarter- $t$ | Company sale<br>$Y_i$ | Industry sales<br>$X_i$ |
|--------------|-----------------------|-------------------------|
| 1            | 77.044                | 746.512                 |
| 2            | 78.613                | 762.345                 |
| 3            | 80.124                | 778.179                 |
| 4            | —                     | —                       |
| —            | —                     | —                       |
| —            | —                     | —                       |
| —            | —                     | —                       |
| —            | —                     | —                       |
| 20           | 102.481               | 1006.882                |

A computer run of a regression package provided us with the value of the test statistic  $d = 0.4765$ :  $d_L = 1.20$  and  $d_U = 1.41$  from Table 33. Since  $d = 0.4765 < 1.20$ , the error terms are positively autocorrelated.

## Test 93 Duckworth's test for comparing the medians of two populations

### Object

A quick and easy test for comparing the medians of two populations which could be used for a wide range of  $m$  and  $n$  observations.

### Limitations

This is not a powerful test but it is easy to use and the table values can be easily obtained. It works only on the largest and the smallest values of the observations from different populations.

### Method

Consider the smallest observation from the  $X$  population and the largest from the  $Y$  population. Then the test statistic,  $D$ , is the sum of the overlaps, the number of  $X$  observations that are smaller than the smallest  $Y$ , plus the number of  $Y$  observations that are larger than the largest  $X$ . If either  $3 + 4n/3 \leq m \leq 2n$  or vice versa we subtract 1 from  $D$ . Under these circumstances, the table of critical values consists of the three numbers; 7, 10 and 13. If  $D \leq 7$  we reject the hypothesis of equal medians at  $\alpha = 0.05$ .

### Example

Two groups of workers are compared in terms of their daily rates of pay. Are they significantly different? We use Duckworth's test since the median is an appropriate measure of central tendency for an income variable. The test statistic is 5 which is less than 7 and so not significant. We have no reason to assume any difference in rates of pay between the two groups.

### Numerical calculation

$$m = n = 12$$

|      |      |      |      |      |      |      |      |      |      |      |      |
|------|------|------|------|------|------|------|------|------|------|------|------|
| 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   |
| 66.3 | 68.3 | 68.5 | 69.2 | 70.0 | 70.1 | 70.4 | 70.9 | 71.1 | 71.2 | 72.1 | 72.1 |
| X    | X    | X    | Y    | Y    | X    | X    | Y    | X    | X    | Y    | Y    |
| 13   | 14   | 15   | 16   | 17   | 18   | 19   | 20   | 21   | 22   | 23   | 24   |
| 72.1 | 72.7 | 72.8 | 73.3 | 73.6 | 74.1 | 74.2 | 74.6 | 74.7 | 74.8 | 75.5 | 75.8 |
| X    | X    | Y    | Y    | X    | X    | Y    | Y    | Y    | X    | Y    | Y    |

We note that there are three  $X$  observations below all the  $Y$  observations and two  $Y$ s above all the  $X$ s. The total is  $D = 5$ , which is less than 7 and so not significant.

## Test 94 $\chi^2$ -test for a suitable probabilistic model

### Object

Many experiments yield a set of data, say  $X_1, X_2, \dots, X_n$ , and the experimenter is often interested in determining whether the data can be treated as the observed values of the random sample  $X_1, X_2, \dots, X_n$  from a given distribution. That is, would this proposed distribution be a reasonable probabilistic model for these sample items?

### Limitations

This test is applicable if both distributions have the same interval classification and the same number of elements. The observed data are observed by random sampling.

### Method

Let  $X_1$  denote the number of heads that occur when coins are tossed at random, under the assumptions that the coins are independent and the probability of heads for each coin has a binomial distribution. An experiment resulted in certain observed values at  $Y_i$  corresponding to 0, 1, 2, 3 and 4 heads.

Let  $A_1 = \{0\}$ ,  $A_2 = \{1\}$ ,  $A_3 = \{2\}$ ,  $A_4 = \{3\}$ ,  $A_5 = \{4\}$  be the corresponding heads and if  $\pi_i = P(X \in A_i)$  when  $X$  is  $B(4, \frac{1}{2})$ , then we have

$$\pi_1 = \pi_5 = \binom{4}{0} \left(\frac{1}{2}\right)^4 = 0.0625$$

$$\pi_2 = \pi_4 = \binom{4}{1} \left(\frac{1}{2}\right)^4 = 0.25$$

$$\pi_3 = \binom{4}{2} \left(\frac{1}{2}\right)^4 = 0.375.$$

If  $\alpha = 0.05$ , then the null hypothesis

$$H_0: p_1 = \pi_1, p_2 = \pi_2, p_3 = \pi_3, p_4 = \pi_4, p_5 = \pi_5$$

is rejected if the calculated value is greater than the tabulated value using

$$\left[ q_{k-1} = \sum_{i=1}^k \frac{(y_i - n\pi_i)^2}{n\pi_i} \right] \sim \chi_{k-1}^2.$$

### Example

We wish to test whether a binomial distribution is a good model for an application area. A full group of prisoner trainees consists of up to five members. Each full group must have two escorts if five members turn up for training. If up to three turn up for training then only one escort is needed. Data is collected over 100 group turnouts. Is a binomial

model a good fit? The calculated chi-squared statistic of 4.47 is less than the critical value of 9.49 [Table 5] so a binomial model is an acceptable model for this data.

### Numerical calculation

In this case  $y_1 = 7$ ,  $y_2 = 18$ ,  $y_3 = 40$ ,  $y_4 = 31$  and  $y_5 = 4$ . The computed value is

$$\begin{aligned} q_4 &= \frac{(7 - 6.25)^2}{6.25} + \frac{(18 - 25)^2}{25} + \frac{(40 - 37.5)^2}{37.5} + \frac{(31 - 25)^2}{25} \\ &\quad + \frac{(4 - 6.25)^2}{6.25} = 4.47. \end{aligned}$$

Critical value  $\chi_4^2(0.05) = 9.49$  [Table 5].

Hence the hypothesis is not rejected. Thus the data support the hypothesis that  $B(4, \frac{1}{2})$  is a reasonable probabilistic model for  $X$ .

## Test 95 V-test (modified Rayleigh)

### Object

To test whether the observed angles have a tendency to cluster around a given angle indicating a lack of randomness in the distribution.

### Limitations

For grouped data the length of the mean vector must be adjusted, and for axial data all angles must be doubled.

### Method

Given a random sample of  $n$  angular values  $\Phi_1, \Phi_2, \dots, \Phi_n$  and a given theoretical direction determined by an angle  $\theta_0$ , then the test statistic for the test of randomness is:

$$V = (2n)^{\frac{1}{2}} \vartheta$$

where  $\vartheta = r \cos(\bar{\Phi} - \theta_0)$  and  $r$  is the length of the mean vector

$$r = \frac{1}{n} + \left[ \left( \sum \cos \Phi_i \right)^2 + \left( \sum \sin \Phi_i \right)^2 \right] = (\bar{x}^2 + \bar{y}^2)^{\frac{1}{2}}$$

$$\bar{\Phi} = \begin{cases} \arctan \left( \frac{\bar{y}}{\bar{x}} \right) & \text{if } \bar{x} > 0 \\ 180^\circ + \arctan \left( \frac{\bar{y}}{\bar{x}} \right) & \text{if } \bar{x} < 0. \end{cases}$$

If  $V$  is greater than or equal to the critical  $V(\alpha)$ , the null hypothesis, that the parent population is uniformly distributed (randomness), is rejected.

### Example

A radar screen produces a series of traces; angles from a centre are measured. Do these cluster around a value of 265 degrees? The calculated  $V$  statistic is 3.884, which is greater than the critical value of 2.302 [Table 34]. So the angles are not random and do cluster.

### Numerical calculation

$$n = 15$$

$$\Phi_1 = 250^\circ, \Phi_2 = 275^\circ, \Phi_3 = 285^\circ, \Phi_4 = 285^\circ, \Phi_5 = 290^\circ, \Phi_6 = 290^\circ, \\ \Phi_7 = 295^\circ, \Phi_8 = 300^\circ, \Phi_9 = 305^\circ, \Phi_{10} = 310^\circ, \Phi_{11} = 315^\circ, \Phi_{12} = 320^\circ, \\ \Phi_{13} = 330^\circ, \Phi_{14} = 330^\circ, \Phi_{15} = 5^\circ, \theta_0 = 265^\circ$$

$$\bar{x} = \frac{1}{n} (\cos \Phi_1 + \dots + \cos \Phi_{15}) = \frac{7.287}{15} = 0.4858$$

$$\bar{y} = \frac{1}{n}(\sin \Phi_1 + \cdots + \sin \Phi_{15}) = \frac{-11.367}{15} = -0.7578$$

$$r = (\bar{x}^2 + \bar{y}^2)^{\frac{1}{2}} = (0.4858^2 + 0.7578^2)^{\frac{1}{2}} = 0.9001$$

$$\bar{\Phi} = \arctan\left(\frac{-0.7578}{0.4858}\right) = -57.3^\circ, \text{ which is equivalent to } 303^\circ.$$

$$\vartheta = r \cos(\bar{\Phi} - \theta_0) = 0.9001 \times \cos(303^\circ - 265^\circ) = 0.9001 \times 0.7880 = 0.7093$$

$$V = (2 \times 15)^{\frac{1}{2}} \times 0.7093 = 5.4772 \times 0.7092 = 3.885$$

Critical value  $V_{15;0.01} = 2.302$  [Table 34].

Hence reject the null hypothesis of randomness.



## Test 96 Watson's $U_n^2$ -test

### Object

To test whether the given distribution fits a random sample of angular values.

### Limitations

This test is suitable for both unimodal and the multimodal cases. The test is very practical if a computer program is available. It can be used as a test for randomness.

### Method

Given a random sample of  $n$  angular values  $\Phi_1, \Phi_2, \dots, \Phi_n$  rearranged in ascending order:  $\Phi_1 \leq \Phi_2 \leq \dots \leq \Phi_n$ . Suppose  $F(\Phi)$  is the distribution function of the given theoretical distribution and let

$$V_i = F(\Phi_i), \quad i = 1, 2, \dots, n$$

$$\bar{V} = \frac{1}{n} \sum V_i \quad \text{and} \quad C_i = 2i - 1.$$

Then the test statistic is:

$$U_n^2 = \sum_{i=1}^n V_i^2 - \sum_{i=1}^n \left( \frac{C_i V_i}{n} \right) + n \left[ \frac{1}{3} - \left( \bar{V} - \frac{1}{2} \right)^2 \right].$$

If the sample value of  $U_n^2$  exceeds the critical value the null hypothesis is rejected. Otherwise the fit is satisfactory.

### Example

A particle atomizer produces traces on a filter paper, which is calibrated on an angular scale. Are the particles equally spaced on an angular scale? The distribution is one of equal angles and the calculated  $U$  squared statistic is 0.1361. This is smaller than the critical value of 0.184 [Table 35] so the null hypothesis of no difference from the theoretical distribution is accepted.

### Numerical calculation

$n = 13, F(\Phi) = \Phi/360^\circ, \Phi_1 = 20^\circ, \Phi_2 = 135^\circ, \Phi_3 = 145^\circ, \Phi_4 = 165^\circ,$   
 $\Phi_5 = 170^\circ, \Phi_6 = 200^\circ, \Phi_7 = 300^\circ, \Phi_8 = 325^\circ, \Phi_9 = 335^\circ, \Phi_{10} = 350^\circ,$   
 $\Phi_{11} = 350^\circ, \Phi_{12} = 350^\circ, \Phi_{13} = 355^\circ$

$V_i = \Phi_i/13, i = 1, \dots, 13$

$\sum V_i = 8.8889, \sum V_i^2 = 7.2310, \sum C_i V_i/n = 10.9893, \bar{V} = 0.68376$

$U_n^2 = 7.2310 - 10.9893 + 13 \left[ \frac{1}{3} - (0.18376)^2 \right] = 0.1361$

Critical value  $U_{13;0.05}^2 = 0.184$  [Table 35].

Do not reject the null hypothesis. The sample comes from the given theoretical distribution.

## Test 97 Watson's $U^2$ -test

### Object

To test whether two samples from circular observations differ significantly from each other with regard to mean direction or angular variance.

### Limitation

Both samples must come from a continuous distribution. In the case of grouping the class interval should not exceed  $5^\circ$ .

### Method

Given two random samples of  $n$  and  $m$  circular observations  $\Phi_1, \Phi_2, \dots, \Phi_n$  and  $\Psi_1, \Psi_2, \dots, \Psi_m$ , let  $n + m = N$  and  $d_1, d_2, \dots, d_N (k = 1, 2, \dots, N)$  be the differences between the sample distribution function, and let  $\bar{d}$  denote the mean of the  $N$  differences. Then the test statistic is given by:

$$U^2 = \frac{nm}{N^2} \left[ \sum_{k=1}^N d_k^2 - \frac{1}{N} \left( \sum_{k=1}^N d_k \right)^2 \right].$$

If  $U^2 > U^2(\alpha)$ , reject the null hypothesis.

### Example

Two prototype machines produce two angular displacement scales. Are they essentially the same with regard to mean direction and angular variance? The calculated  $U$  squared statistic is 0.261. Since this is greater than the critical value of 0.185 [Table 36] the null hypothesis of no difference is rejected. The two prototype machines differ.

### Numerical calculation

$$n = 8, m = 10, \sum_{k=1}^{16} d_k = 6.525, \sum_{k=1}^{16} d_k^2 = 3.422$$

$$U^2 = \frac{80}{18^2} \left( 3.422 - \frac{6.525 \times 6.525}{18} \right) = 0.261$$

Critical value  $U_{8,10;0.05}^2 = 0.185$  [Table 36].

The calculated value is greater than the critical value.

Reject the hypothesis. The two samples deviate significantly from each other.

## Test 98 Watson–Williams test

### Object

To test whether the mean angles of two independent circular observations differ significantly from each other.

### Limitations

Samples are drawn from a von Mises distribution and the concentration parameter  $k$  ( $>2$ ) must have the same value in each population.

### Method

Given two independent random samples of  $n$  and  $m$  circular observations  $\Phi_1, \Phi_2, \dots, \Phi_n$  and  $\Psi_1, \Psi_2, \dots, \Psi_m$ , for each sample, calculate the components of the resultant vectors

$$\begin{aligned} C_1 &= \sum_{i=1}^n \cos \Phi_i & S_1 &= \sum_{i=1}^n \sin \Phi_i \\ C_2 &= \sum_{i=1}^m \cos \Psi_i & S_2 &= \sum_{i=1}^m \sin \Psi_i \end{aligned}$$

with the resultant lengths

$$R_1 = (C_1^2 + S_1^2)^{\frac{1}{2}} \quad R_2 = (C_2^2 + S_2^2)^{\frac{1}{2}}.$$

The directions of the resultant vectors are given by  $\bar{\Phi}$  and  $\bar{\Psi}$ . For the combined sample, the components of the resultant vector are

$$C = C_1 + C_2 \quad \text{and} \quad S = S_1 + S_2.$$

Hence, the length of the resultant vector is

$$R = (C^2 + S^2)^{\frac{1}{2}}.$$

To test the unknown mean angles of the population use the test statistic

$$F = g(N - 2) \frac{R_1 + R_2 - R}{N - (R_1 + R_2)}$$

where  $N = n + m$  and  $g = 1 - 3/8\hat{k}$ , with  $k$  determined from

$$\bar{R} = \frac{R_1 + R_2}{N}$$

and Table 37. Reject the null hypothesis if the calculated value  $F$  is greater than the critical value  $F_{1, N-2}$ .

**Example**

A trainee-building surveyor has calibrated two angular measuring devices. Do they produce similar results? She takes ten measurements from each device and then uses the Watson–Williams test to compare them. Her  $F$  test statistic is 8.43, which is greater than the critical value of 8.29 [Table 3]. So the null hypothesis of no difference between the samples is rejected suggesting that the two devices have been calibrated differently.

**Numerical calculation**

$$n = 10, m = 10, N = 20, v_1 = 1, v_2 = N - 2$$

$$C_1 = 9.833, \quad C_2 = 9.849, \quad C = 19.682$$

$$S_1 = -1.558, \quad S_2 = 0.342, \quad S = -1.216$$

$$R_1 = 9.956, \quad R_2 = 9.854, \quad R = 19.721$$

$$\bar{R} = 0.991 \text{ for } \hat{k} \text{ greater than } 10, g = 1$$

$$\hat{k} = 50.241 \text{ [Table 37]}$$

$$F = \frac{18 \times 0.089}{0.190} = 8.432$$

Critical value  $F_{1,18;0.01} = 8.29$  [Table 3].

Reject the null hypothesis.

Hence the mean directions differ significantly.

## Test 99 Mardia–Watson–Wheeler test

### Object

To test whether two independent random samples from circular observations differ significantly from each other regarding mean angle, angular variance or both.

### Limitation

There are no ties between the samples and the populations have a continuous circular distribution.

### Method

Consider two independent samples of  $n$  and  $m$  circular observations  $\Phi_1, \Phi_2, \dots, \Phi_n$  and  $\Psi_1, \Psi_2, \dots, \Psi_m$ . We observe the order in which the random samples are arranged and then alter the space between successive sample points in such a way that all these spaces become the same size. Having spaced the sample points equally, the sample points are then ranked. Let  $r_1, r_2, \dots, r_n$  be the ranks of the first sample and  $\beta_i = r_i \delta$  ( $i = 1, \dots, n$ ) be the angles, where  $\delta$  and  $N = n + m$  are known as uniform scores. Then the resultant vector of the first sample has components

$$C_1 = \sum \cos \beta_i \quad S_1 = \sum \sin \beta_i$$

and the length of the resultant vector is

$$R_1 = (C^2 + S^2)^{\frac{1}{2}}.$$

The test statistic is given by

$$B = R_1^2.$$

If  $B > B_\alpha$ , reject the null hypothesis. When  $N > 17$ , the quantity  $\chi^2 = 2(N-1)R_1^2/nm$  has a  $\chi^2$ -distribution with 2 degrees of freedom if  $H_0$  is true.

### Example

A new improved boat navigation system is compared with the old one. Is the way they work similar in terms of the observations taken? A sailor uses the Mardia–Watson–Wheeler test and computes a  $B$  statistic of 3.618. This is smaller than the tabulated value of 9.47 [Table 37] so he concludes that there is no difference between the two systems.

### Numerical calculation

$n = 6, m = 4, N = 10, \delta = 360^\circ/10 = 36^\circ, \alpha = 0.05$

Here  $m$  is the smaller sample size.

For the first sample:

$r_1 = 1, r_2 = 2, r_3 = 3, r_4 = 4, r_5 = 8, r_6 = 9$

$$\beta_1 = 36^\circ, \beta_2 = 72^\circ, \beta_3 = 108^\circ, \beta_4 = 144^\circ, \beta_5 = 288^\circ, \beta_6 = 324^\circ$$

$$C_1 = 1.118, S_1 = 1.539, R_1 = 1.902$$

$$B = 3.618$$

Critical value  $B_{N,m;\alpha} = B_{10,4;0.05} = 9.47$  [Table 38].

The calculated value  $B$  is less than the critical value  $B_\alpha$ .

Hence there are no significant differences between the samples. (The second sample would lead to the same result.)

## Test 100 Harrison–Kanji–Gadsden test (analysis of variance for angular data)

### Object

To test whether the treatment effects of the  $q$  independent random samples from von Mises populations differ significantly from each other.

### Limitations

1. Samples are drawn from a von Mises population.
2. The concentration parameter  $k$  has the same value for each sample.
3.  $k$  must be at least 2.

### Method

Given a one-way classification situation with  $\mu_0$ ,  $\beta_j$  and  $e_{ij}$  denoting the overall mean direction, treatment effect and random error variation respectively, then

$$\Phi_{ij} = \mu_0 + \beta_j + e_{ij}, \quad i = 1, \dots, p; \quad j = 1, 2, \dots, q$$

where each observation  $\Phi_{ij}$  is an independent observation from a von Mises distribution with mean  $\mu_0 + \beta_j$  and concentration parameter  $k$ .

For the one-way situation, the components of variation are similarly

$$k \left[ N - \frac{R^2}{N} \right] = k \left[ \sum_{j=1}^q \left( \frac{R_j^2}{N_j} - \frac{R^2}{N} \right) \right] + k \left[ N - \sum_{j=1}^q \left( \frac{R_j^2}{N_j} \right) \right]$$

$$(\text{Total variation}) = (\text{Between variation}) + (\text{Residual variation})$$

and the test statistic for a large value of  $k$  is given by

$$F'_{q-1, N-q} = \beta \left[ \frac{(N-q) \left( \sum_{j=1}^q \left( \frac{R_j^2}{N_j} \right) - \frac{R^2}{N} \right)}{(q-1) \left\{ N - \sum_{j=1}^q \left( \frac{R_j^2}{N_j} \right) \right\}} \right]$$

where  $\sum_{i=1}^p \sum_{j=1}^q \cos(\Phi_{ij} - \hat{\mu}_0) = R$ , and  $\bar{\Phi}_j$  are the  $j$ th mean angles with corresponding resultant length  $r_j$ .

Let  $x_i$  and  $y_j$  be the rectangular components of  $r_j$ . Then:

$$r_{..} = \left[ \left( \frac{1}{q} \sum_{j=1}^q r_j \cos \bar{\Phi}_j \right)^2 + \left( \frac{1}{q} \sum_{j=1}^q r_j \sin \bar{\Phi}_j \right)^2 \right]^{\frac{1}{2}}$$

and  $R = Nr_{..}$ ,  $R_j = \sum_{i=1}^q \cos(\Phi_{ij} - \bar{\Phi}_j)$ ,

$$\bar{\Phi}_j = \arctan\left(\frac{\bar{y}}{\bar{x}}\right), \quad r_{.j} = [(\bar{x}^2 + \bar{y}^2)]^{\frac{1}{2}}$$

where

$$\bar{x} = \frac{1}{q}[\cos \Phi_{1j} + \cos \Phi_{2j} + \cdots + \cos \Phi_{qj}]$$

$$\bar{y} = \frac{1}{q}[\sin \Phi_{1j} + \sin \Phi_{2j} + \cdots + \sin \Phi_{qj}]$$

$\hat{k}$  is found by calculating the resultant  $\bar{R}$  and using Table 37. Here  $\bar{R} = R/N$ .

### Example

Five different magnification systems are compared for their effect on automatic angular differentiation. A flight simulation researcher uses the Harrison–Kanji–Gadsden test and computes the  $F$  statistic as 1.628. This is less than the tabulated value of 2.37 [Table 3]. So the researcher concludes that all five magnification systems are equally effective.

### Numerical calculation

| Magnification    |  |
|------------------|--|
| 100              | 82, 71, 85, 89, 78, 77, 74, 71, 68, 83, 72, 73, 81, 65, 62, 90, 92, 80, 77, 93, 75, 80, 69, 74, 77, 75, 71, 82, 84, 79, 78, 81, 89, 79, 82, 81, 85, 76, 71, 80, 94, 68, 72, 70, 59, 80, 86, 98, 82, 73 |
| 200              | 75, 74, 71, 63, 83, 74, 82, 78, 87, 87, 82, 71, 60, 66, 63, 85, 81, 78, 80, 89, 82, 82, 92, 80, 81, 74, 90, 78, 73, 72, 80, 59, 64, 78, 73, 70, 79, 79, 77, 81, 72, 76, 69, 73, 75, 84, 81, 51, 76, 88 |
| 400              | 70, 76, 79, 86, 77, 86, 77, 90, 88, 82, 84, 70, 87, 61, 71, 89, 72, 90, 74, 88, 82, 68, 83, 75, 90, 79, 89, 78, 74, 73, 71, 80, 83, 89, 68, 81, 47, 88, 69, 76, 71, 67, 76, 90, 84, 70, 80, 77, 93, 89 |
| 1200             | 78, 90, 72, 91, 73, 79, 82, 87, 78, 83, 74, 82, 85, 75, 67, 72, 78, 88, 89, 71, 73, 77, 90, 82, 80, 81, 89, 87, 78, 73, 78, 86, 73, 84, 68, 75, 70, 89, 54, 80, 90, 88, 81, 82, 88, 82, 75, 79, 83, 82 |
| $400 \times 1.3$ | 88, 69, 64, 78, 71, 68, 54, 80, 73, 72, 65, 73, 93, 84, 80, 49, 78, 82, 95, 69, 87, 83, 52, 79, 85, 67, 82, 84, 87, 83, 88, 79, 83, 77, 78, 89, 75, 72, 88, 78, 62, 86, 89, 74, 71, 73, 84, 56, 77, 71 |

$$q = 5, N_{.1} = N_{.2} = \cdots = N_{.q} = 50, N = 250$$

$$\nu_1 = q - 1 = 4, \nu_2 = N - q = 245$$

$$R = 238.550, \sum_{j=1}^q \left[ \frac{R_j^2}{N_{.j}} \right] = 228.1931, \frac{R^2}{N} = 227.6244$$



Between variation = 0.5687, within variation = 21.8069

$$F'_{q-1, N-q} = \beta \left[ \frac{0.142175}{0.089007} \right] = \beta(1.597337)$$

$$\hat{k} = 11.02, \quad \frac{1}{\beta} = 1 - \frac{1}{5\hat{k}} - \frac{1}{10\hat{k}^2} \quad \text{or} \quad \beta = 1.01959,$$

where  $\bar{R} = \frac{R}{N} = 0.954$  [Table 37]

Modified  $F' = \beta \times F_{4,245} = 1.01959 \times 1.597337$  or  $F'_{4,245} = 1.628$

Critical value  $F_{4,245;0.05} = 2.37$  [Table 3].

Hence there are no significant differences between the treatments.

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TABLES

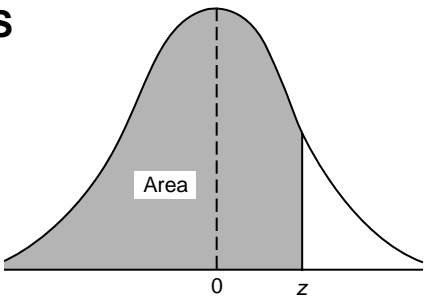


Table 1 The normal curve

(a) Area under the normal curve

| z    | 0.00   | 0.01   | 0.02   | 0.03   | 0.04   | 0.05   | 0.06   | 0.07   | 0.08   | 0.09   |
|------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| −3.4 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |
| −3.3 | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| −3.2 | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| −3.1 | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| −3.0 | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| −2.9 | 0.0019 | 0.0018 | 0.0017 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| −2.8 | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| −2.7 | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| −2.6 | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| −2.5 | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| −2.4 | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| −2.3 | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| −2.2 | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0124 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| −2.1 | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| −2.0 | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| −1.9 | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| −1.8 | 0.0359 | 0.0352 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| −1.7 | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| −1.6 | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| −1.5 | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| −1.4 | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0722 | 0.0708 | 0.0694 | 0.0681 |
| −1.3 | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| −1.2 | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| −1.1 | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| −1.0 | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| −0.9 | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| −0.8 | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| −0.7 | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| −0.6 | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| −0.5 | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| −0.4 | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| −0.3 | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| −0.2 | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| −0.1 | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| −0.0 | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |

Table 1(a) continued

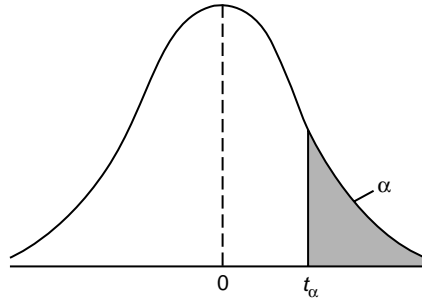
| $z$ | 0.00   | 0.01   | 0.02   | 0.03   | 0.04   | 0.05   | 0.06   | 0.07   | 0.08   | 0.09   |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9278 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.1 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.2 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| 3.3 | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| 3.4 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |

Source: Walpole and Myers, 1989

**(b) critical values for a standard normal distribution**

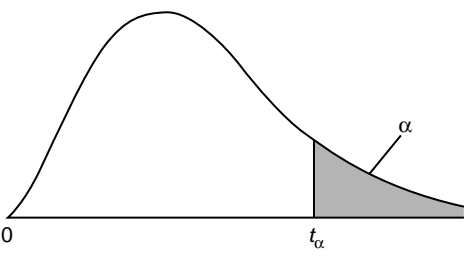
The normal distribution is symmetrical with respect to  $\mu = 0$ .

| Level of significance $\alpha$ |           | $z$  |
|--------------------------------|-----------|------|
| Two-sided                      | One-sided |      |
| 0.001                          | 0.0005    | 3.29 |
| 0.002                          | 0.001     | 3.09 |
| 0.0026                         | 0.0013    | 3.00 |
| 0.01                           | 0.05      | 2.58 |
| 0.02                           | 0.01      | 2.33 |
| 0.0456                         | 0.0228    | 2.00 |
| 0.05                           | 0.025     | 1.96 |
| 0.10                           | 0.05      | 1.64 |
| 0.20                           | 0.10      | 1.28 |
| 0.318                          | 0.159     | 1.00 |

**Table 2 Critical values of the *t*-distribution**

| $\nu$    | Level of significance $\alpha$ |       |        |        |        |
|----------|--------------------------------|-------|--------|--------|--------|
|          | 0.10                           | 0.05  | 0.025  | 0.01   | 0.005  |
| 1        | 3.078                          | 6.314 | 12.706 | 31.821 | 63.657 |
| 2        | 1.886                          | 2.920 | 4.303  | 6.965  | 9.925  |
| 3        | 1.638                          | 2.353 | 3.182  | 4.541  | 5.841  |
| 4        | 1.533                          | 2.132 | 2.776  | 3.747  | 4.604  |
| 5        | 1.476                          | 2.015 | 2.571  | 3.365  | 4.032  |
| 6        | 1.440                          | 1.943 | 2.447  | 3.143  | 3.707  |
| 7        | 1.415                          | 1.895 | 2.365  | 2.998  | 3.499  |
| 8        | 1.397                          | 1.860 | 2.306  | 2.896  | 3.355  |
| 9        | 1.383                          | 1.833 | 2.262  | 2.821  | 3.250  |
| 10       | 1.372                          | 1.812 | 2.228  | 2.764  | 3.169  |
| 11       | 1.363                          | 1.796 | 2.201  | 2.718  | 3.106  |
| 12       | 1.356                          | 1.782 | 2.179  | 2.681  | 3.055  |
| 13       | 1.350                          | 1.771 | 2.160  | 2.650  | 3.012  |
| 14       | 1.345                          | 1.761 | 2.145  | 2.624  | 2.977  |
| 15       | 1.341                          | 1.753 | 2.131  | 2.602  | 2.947  |
| 16       | 1.337                          | 1.746 | 2.120  | 2.583  | 2.921  |
| 17       | 1.333                          | 1.740 | 2.110  | 2.567  | 2.898  |
| 18       | 1.330                          | 1.734 | 2.101  | 2.552  | 2.878  |
| 19       | 1.328                          | 1.729 | 2.093  | 2.539  | 2.861  |
| 20       | 1.325                          | 1.725 | 2.086  | 2.528  | 2.845  |
| 21       | 1.323                          | 1.721 | 2.080  | 2.518  | 2.831  |
| 22       | 1.321                          | 1.717 | 2.074  | 2.508  | 2.819  |
| 23       | 1.319                          | 1.714 | 2.069  | 2.500  | 2.807  |
| 24       | 1.318                          | 1.711 | 2.064  | 2.492  | 2.797  |
| 25       | 1.316                          | 1.708 | 2.060  | 2.485  | 2.787  |
| 26       | 1.315                          | 1.706 | 2.056  | 2.479  | 2.779  |
| 27       | 1.314                          | 1.703 | 2.052  | 2.473  | 2.771  |
| 28       | 1.313                          | 1.701 | 2.048  | 2.467  | 2.763  |
| 29       | 1.311                          | 1.699 | 2.045  | 2.462  | 2.756  |
| 30       | 1.310                          | 1.697 | 2.042  | 2.457  | 2.750  |
| $\infty$ | 1.282                          | 1.645 | 1.960  | 2.326  | 2.576  |

Source: Fisher and Yates, 1974



**Table 3    Critical values of the *F*-distribution**

**Level of significance  $\alpha = 0.05$**

| $\nu_2 \backslash \nu_1$ | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     |
|--------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1                        | 161.4 | 199.5 | 215.7 | 224.6 | 230.2 | 234.0 | 236.8 | 238.9 | 240.5 |
| 2                        | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.35 | 19.37 | 19.38 |
| 3                        | 10.13 | 9.55  | 9.28  | 9.12  | 9.01  | 8.94  | 8.89  | 8.85  | 8.81  |
| 4                        | 7.71  | 6.94  | 6.59  | 6.39  | 6.26  | 6.16  | 6.09  | 6.04  | 6.00  |
| 5                        | 6.61  | 5.79  | 5.41  | 5.19  | 5.05  | 4.95  | 4.88  | 4.82  | 4.77  |
| 6                        | 5.99  | 5.14  | 4.76  | 4.53  | 4.39  | 4.28  | 4.21  | 4.15  | 4.10  |
| 7                        | 5.59  | 4.74  | 4.35  | 4.12  | 3.97  | 3.87  | 3.79  | 3.73  | 3.68  |
| 8                        | 5.32  | 4.46  | 4.07  | 3.84  | 3.69  | 3.58  | 3.50  | 3.44  | 3.39  |
| 9                        | 5.12  | 4.26  | 3.86  | 3.63  | 3.48  | 3.37  | 3.29  | 3.23  | 3.18  |
| 10                       | 4.96  | 4.10  | 3.71  | 3.48  | 3.33  | 3.22  | 3.14  | 3.07  | 3.02  |
| 11                       | 4.84  | 3.98  | 3.59  | 3.36  | 3.20  | 3.09  | 3.01  | 2.95  | 2.90  |
| 12                       | 4.75  | 3.89  | 3.49  | 3.26  | 3.11  | 3.00  | 2.91  | 2.85  | 2.80  |
| 13                       | 4.67  | 3.81  | 3.41  | 3.18  | 3.03  | 2.92  | 2.83  | 2.77  | 2.71  |
| 14                       | 4.60  | 3.74  | 3.34  | 3.11  | 2.96  | 2.85  | 2.76  | 2.70  | 2.65  |
| 15                       | 4.54  | 3.68  | 3.29  | 3.06  | 2.90  | 2.79  | 2.71  | 2.64  | 2.59  |
| 16                       | 4.49  | 3.63  | 3.24  | 3.01  | 2.85  | 2.74  | 2.66  | 2.59  | 2.54  |
| 17                       | 4.45  | 3.59  | 3.20  | 2.96  | 2.81  | 2.70  | 2.61  | 2.55  | 2.49  |
| 18                       | 4.41  | 3.55  | 3.16  | 2.93  | 2.77  | 2.66  | 2.58  | 2.51  | 2.46  |
| 19                       | 4.38  | 3.52  | 3.13  | 2.90  | 2.74  | 2.63  | 2.54  | 2.48  | 2.42  |
| 20                       | 4.35  | 3.49  | 3.10  | 2.87  | 2.71  | 2.60  | 2.51  | 2.45  | 2.39  |
| 21                       | 4.32  | 3.47  | 3.07  | 2.84  | 2.68  | 2.57  | 2.49  | 2.42  | 2.37  |
| 22                       | 4.30  | 3.44  | 3.05  | 2.82  | 2.66  | 2.55  | 2.46  | 2.40  | 2.34  |
| 23                       | 4.28  | 3.42  | 3.03  | 2.80  | 2.64  | 2.53  | 2.44  | 2.37  | 2.32  |
| 24                       | 4.26  | 3.40  | 3.01  | 2.78  | 2.62  | 2.51  | 2.42  | 2.36  | 2.30  |
| 25                       | 4.24  | 3.39  | 2.99  | 2.76  | 2.60  | 2.49  | 2.40  | 2.34  | 2.28  |
| 26                       | 4.23  | 3.37  | 2.98  | 2.74  | 2.59  | 2.47  | 2.39  | 2.32  | 2.27  |
| 27                       | 4.21  | 3.35  | 2.96  | 2.73  | 2.57  | 2.46  | 2.37  | 2.31  | 2.25  |
| 28                       | 4.20  | 3.34  | 2.95  | 2.71  | 2.56  | 2.45  | 2.36  | 2.29  | 2.24  |
| 29                       | 4.18  | 3.33  | 2.93  | 2.70  | 2.55  | 2.43  | 2.35  | 2.28  | 2.22  |
| 30                       | 4.17  | 3.32  | 2.92  | 2.69  | 2.53  | 2.42  | 2.33  | 2.27  | 2.21  |
| 40                       | 4.08  | 3.23  | 2.84  | 2.61  | 2.45  | 2.34  | 2.25  | 2.18  | 2.12  |
| 60                       | 4.00  | 3.15  | 2.76  | 2.53  | 2.37  | 2.25  | 2.17  | 2.10  | 2.04  |
| 120                      | 3.92  | 3.07  | 2.68  | 2.45  | 2.29  | 2.17  | 2.09  | 2.02  | 1.96  |
| $\infty$                 | 3.84  | 3.00  | 2.60  | 2.37  | 2.21  | 2.10  | 2.01  | 1.94  | 1.88  |

**Table 3** continuedLevel of significance  $\alpha = 0.05$ 

| $\nu_2 \backslash \nu_1$ | 10    | 12    | 15    | 20    | 24    | 30    | 40    | 60    | 120   | $\infty$ |
|--------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| 1                        | 241.9 | 243.9 | 245.9 | 248.0 | 249.1 | 250.1 | 251.1 | 252.2 | 253.3 | 254.3    |
| 2                        | 19.40 | 19.41 | 19.43 | 19.45 | 19.45 | 19.46 | 19.47 | 19.48 | 19.49 | 19.50    |
| 3                        | 8.79  | 8.74  | 8.70  | 8.66  | 8.64  | 8.62  | 8.59  | 8.57  | 8.55  | 8.53     |
| 4                        | 5.96  | 5.91  | 5.86  | 5.80  | 5.77  | 5.75  | 5.72  | 5.69  | 5.66  | 5.63     |
| 5                        | 4.74  | 4.68  | 4.62  | 4.56  | 4.53  | 4.50  | 4.46  | 4.43  | 4.40  | 4.36     |
| 6                        | 4.06  | 4.00  | 3.94  | 3.87  | 3.84  | 3.81  | 3.77  | 3.74  | 3.70  | 3.67     |
| 7                        | 3.64  | 3.57  | 3.51  | 3.44  | 3.41  | 3.38  | 3.34  | 3.30  | 3.27  | 3.23     |
| 8                        | 3.35  | 3.28  | 3.22  | 3.15  | 3.12  | 3.08  | 3.04  | 3.01  | 2.97  | 2.93     |
| 9                        | 3.14  | 3.07  | 3.01  | 2.94  | 2.90  | 2.86  | 2.83  | 2.79  | 2.75  | 2.71     |
| 10                       | 2.98  | 2.91  | 2.85  | 2.77  | 2.74  | 2.70  | 2.66  | 2.62  | 2.58  | 2.54     |
| 11                       | 2.85  | 2.79  | 2.72  | 2.65  | 2.61  | 2.57  | 2.53  | 2.49  | 2.45  | 2.40     |
| 12                       | 2.75  | 2.69  | 2.62  | 2.54  | 2.51  | 2.47  | 2.43  | 2.38  | 2.34  | 2.30     |
| 13                       | 2.67  | 2.60  | 2.53  | 2.46  | 2.42  | 2.38  | 2.34  | 2.30  | 2.25  | 2.21     |
| 14                       | 2.60  | 2.53  | 2.46  | 2.39  | 2.35  | 2.31  | 2.27  | 2.22  | 2.18  | 2.13     |
| 15                       | 2.54  | 2.48  | 2.40  | 2.33  | 2.29  | 2.25  | 2.20  | 2.16  | 2.11  | 2.07     |
| 16                       | 2.49  | 2.42  | 2.35  | 2.28  | 2.24  | 2.19  | 2.15  | 2.11  | 2.06  | 2.01     |
| 17                       | 2.45  | 2.38  | 2.31  | 2.23  | 2.19  | 2.15  | 2.10  | 2.06  | 2.01  | 1.96     |
| 18                       | 2.41  | 2.34  | 2.27  | 2.19  | 2.15  | 2.11  | 2.06  | 2.02  | 1.97  | 1.92     |
| 19                       | 2.38  | 2.31  | 2.23  | 2.16  | 2.11  | 2.07  | 2.03  | 1.98  | 1.93  | 1.88     |
| 20                       | 2.35  | 2.28  | 2.20  | 2.12  | 2.08  | 2.04  | 1.99  | 1.95  | 1.90  | 1.84     |
| 21                       | 2.32  | 2.25  | 2.18  | 2.10  | 2.05  | 2.01  | 1.96  | 1.92  | 1.87  | 1.81     |
| 22                       | 2.30  | 2.23  | 2.15  | 2.07  | 2.03  | 1.98  | 1.94  | 1.89  | 1.84  | 1.78     |
| 23                       | 2.27  | 2.20  | 2.13  | 2.05  | 2.01  | 1.96  | 1.91  | 1.86  | 1.81  | 1.76     |
| 24                       | 2.25  | 2.18  | 2.11  | 2.03  | 1.98  | 1.94  | 1.89  | 1.84  | 1.79  | 1.73     |
| 25                       | 2.24  | 2.16  | 2.09  | 2.01  | 1.96  | 1.92  | 1.87  | 1.82  | 1.77  | 1.71     |
| 26                       | 2.22  | 2.15  | 2.07  | 1.99  | 1.95  | 1.90  | 1.85  | 1.80  | 1.75  | 1.69     |
| 27                       | 2.20  | 2.13  | 2.06  | 1.97  | 1.93  | 1.88  | 1.84  | 1.79  | 1.73  | 1.67     |
| 28                       | 2.19  | 2.12  | 2.04  | 1.96  | 1.91  | 1.87  | 1.82  | 1.77  | 1.71  | 1.65     |
| 29                       | 2.18  | 2.10  | 2.03  | 1.94  | 1.90  | 1.85  | 1.81  | 1.75  | 1.70  | 1.64     |
| 30                       | 2.16  | 2.09  | 2.01  | 1.93  | 1.89  | 1.84  | 1.79  | 1.74  | 1.68  | 1.62     |
| 40                       | 2.08  | 2.00  | 1.92  | 1.84  | 1.79  | 1.74  | 1.69  | 1.64  | 1.58  | 1.51     |
| 60                       | 1.99  | 1.92  | 1.84  | 1.75  | 1.70  | 1.65  | 1.59  | 1.53  | 1.47  | 1.39     |
| 120                      | 1.91  | 1.83  | 1.75  | 1.66  | 1.61  | 1.55  | 1.50  | 1.43  | 1.35  | 1.25     |
| $\infty$                 | 1.83  | 1.75  | 1.67  | 1.57  | 1.52  | 1.46  | 1.39  | 1.32  | 1.22  | 1.00     |



Table 3 continued

Level of significance  $\alpha = 0.01$

| $\nu_2 \backslash \nu_1$ | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     |
|--------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1                        | 4052  | 4999  | 5403  | 5625  | 5764  | 5859  | 5928  | 5981  | 6022  |
| 2                        | 98.50 | 99.00 | 99.17 | 99.25 | 99.30 | 99.33 | 99.36 | 99.37 | 99.39 |
| 3                        | 34.12 | 30.82 | 29.46 | 28.71 | 28.24 | 27.91 | 27.67 | 27.49 | 27.35 |
| 4                        | 21.20 | 18.00 | 16.69 | 15.98 | 15.52 | 15.21 | 14.98 | 14.80 | 14.66 |
| 5                        | 16.26 | 13.27 | 12.06 | 11.39 | 10.97 | 10.67 | 10.46 | 10.29 | 10.16 |
| 6                        | 13.75 | 10.92 | 9.78  | 9.15  | 8.75  | 8.47  | 8.26  | 8.10  | 7.98  |
| 7                        | 12.25 | 9.55  | 8.45  | 7.85  | 7.46  | 7.19  | 6.99  | 6.84  | 6.72  |
| 8                        | 11.26 | 8.65  | 7.59  | 7.01  | 6.63  | 6.37  | 6.18  | 6.03  | 5.91  |
| 9                        | 10.56 | 8.02  | 6.99  | 6.42  | 6.06  | 5.80  | 5.61  | 5.47  | 5.35  |
| 10                       | 10.04 | 7.56  | 6.55  | 5.99  | 5.64  | 5.39  | 5.20  | 5.06  | 4.94  |
| 11                       | 9.65  | 7.21  | 6.22  | 5.67  | 5.32  | 5.07  | 4.89  | 4.74  | 4.63  |
| 12                       | 9.33  | 6.93  | 5.95  | 5.41  | 5.06  | 4.82  | 4.64  | 4.50  | 4.39  |
| 13                       | 9.07  | 6.70  | 5.74  | 5.21  | 4.86  | 4.62  | 4.44  | 4.30  | 4.19  |
| 14                       | 8.86  | 6.51  | 5.56  | 5.04  | 4.69  | 4.46  | 4.28  | 4.14  | 4.03  |
| 15                       | 8.68  | 6.36  | 5.42  | 4.89  | 4.56  | 4.32  | 4.14  | 4.00  | 3.89  |
| 16                       | 8.53  | 6.23  | 5.29  | 4.77  | 4.44  | 4.20  | 4.03  | 3.89  | 3.78  |
| 17                       | 8.40  | 6.11  | 5.18  | 4.67  | 4.34  | 4.10  | 3.93  | 3.79  | 3.68  |
| 18                       | 8.29  | 6.01  | 5.09  | 4.58  | 4.25  | 4.01  | 3.84  | 3.71  | 3.60  |
| 19                       | 8.18  | 5.93  | 5.01  | 4.50  | 4.17  | 3.94  | 3.77  | 3.63  | 3.52  |
| 20                       | 8.10  | 5.85  | 4.94  | 4.43  | 4.10  | 3.87  | 3.70  | 3.56  | 3.46  |
| 21                       | 8.02  | 5.78  | 4.87  | 4.37  | 4.04  | 3.81  | 3.64  | 3.51  | 3.40  |
| 22                       | 7.95  | 5.72  | 4.82  | 4.31  | 3.99  | 3.76  | 3.59  | 3.45  | 3.35  |
| 23                       | 7.88  | 5.66  | 4.76  | 4.26  | 3.94  | 3.71  | 3.54  | 3.41  | 3.30  |
| 24                       | 7.82  | 5.61  | 4.72  | 4.22  | 3.90  | 3.67  | 3.50  | 3.36  | 3.26  |
| 25                       | 7.77  | 5.57  | 4.68  | 4.18  | 3.85  | 3.63  | 3.46  | 3.32  | 3.22  |
| 26                       | 7.72  | 5.53  | 4.64  | 4.14  | 3.82  | 3.59  | 3.42  | 3.29  | 3.18  |
| 27                       | 7.68  | 5.49  | 4.60  | 4.11  | 3.78  | 3.56  | 3.39  | 3.26  | 3.15  |
| 28                       | 7.64  | 5.45  | 4.57  | 4.07  | 3.75  | 3.53  | 3.36  | 3.23  | 3.12  |
| 29                       | 7.60  | 5.42  | 4.54  | 4.04  | 3.73  | 3.50  | 3.33  | 3.20  | 3.09  |
| 30                       | 7.56  | 5.39  | 4.51  | 4.02  | 3.70  | 3.47  | 3.30  | 3.17  | 3.07  |
| 40                       | 7.31  | 5.18  | 4.31  | 3.83  | 3.51  | 3.29  | 3.12  | 2.99  | 2.89  |
| 60                       | 7.08  | 4.98  | 4.13  | 3.65  | 3.34  | 3.12  | 2.95  | 2.82  | 2.72  |
| 120                      | 6.85  | 4.79  | 3.95  | 3.48  | 3.17  | 2.96  | 2.79  | 2.66  | 2.56  |
| $\infty$                 | 6.63  | 4.61  | 3.78  | 3.32  | 3.02  | 2.80  | 2.64  | 2.51  | 2.41  |

Table 3 continued

Level of significance  $\alpha = 0.01$ 

| $\nu_1 \backslash \nu_2$ | 10    | 12    | 15    | 20    | 24    | 30    | 40    | 60    | 120   | $\infty$ |
|--------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| 1                        | 6056  | 6106  | 6157  | 6209  | 6235  | 6261  | 6287  | 6313  | 6339  | 6366     |
| 2                        | 99.40 | 99.42 | 99.43 | 99.45 | 99.46 | 99.47 | 99.47 | 99.48 | 99.49 | 99.50    |
| 3                        | 27.23 | 27.05 | 26.87 | 26.69 | 26.60 | 26.50 | 26.41 | 26.32 | 26.22 | 26.13    |
| 4                        | 14.55 | 14.37 | 14.20 | 14.02 | 13.39 | 13.84 | 13.75 | 13.65 | 13.56 | 13.46    |
| 5                        | 10.05 | 9.89  | 9.72  | 9.55  | 9.47  | 9.38  | 9.29  | 9.20  | 9.11  | 9.02     |
| 6                        | 7.87  | 7.72  | 7.56  | 7.40  | 7.31  | 7.23  | 7.14  | 7.06  | 6.97  | 6.88     |
| 7                        | 6.62  | 6.47  | 6.31  | 6.16  | 6.07  | 5.99  | 5.91  | 5.82  | 5.74  | 5.65     |
| 8                        | 5.81  | 5.67  | 5.52  | 5.36  | 5.28  | 5.20  | 5.12  | 5.03  | 4.95  | 4.86     |
| 9                        | 5.26  | 5.11  | 4.96  | 4.81  | 4.73  | 4.65  | 4.57  | 4.48  | 4.40  | 4.31     |
| 10                       | 4.85  | 4.71  | 4.56  | 4.41  | 4.33  | 4.25  | 4.17  | 4.08  | 4.00  | 3.91     |
| 11                       | 4.54  | 4.40  | 4.25  | 4.10  | 4.02  | 3.94  | 3.86  | 3.78  | 3.69  | 3.60     |
| 12                       | 4.30  | 4.16  | 4.01  | 3.86  | 3.78  | 3.70  | 3.62  | 3.54  | 3.45  | 3.36     |
| 13                       | 4.10  | 3.96  | 3.82  | 3.66  | 3.59  | 3.51  | 3.43  | 3.34  | 3.25  | 3.17     |
| 14                       | 3.94  | 3.80  | 3.66  | 3.51  | 3.43  | 3.35  | 3.27  | 3.18  | 3.09  | 3.00     |
| 15                       | 3.80  | 3.67  | 3.52  | 3.37  | 3.29  | 3.21  | 3.13  | 3.05  | 2.96  | 2.87     |
| 16                       | 3.69  | 3.55  | 3.41  | 3.26  | 3.18  | 3.10  | 3.02  | 2.93  | 2.84  | 2.75     |
| 17                       | 3.59  | 3.46  | 3.31  | 3.16  | 3.08  | 3.00  | 2.92  | 2.83  | 2.75  | 2.65     |
| 18                       | 3.51  | 3.37  | 3.23  | 3.08  | 3.00  | 2.92  | 2.84  | 2.75  | 2.66  | 2.57     |
| 19                       | 3.43  | 3.30  | 3.15  | 3.00  | 2.92  | 2.84  | 2.76  | 2.67  | 2.58  | 2.49     |
| 20                       | 3.37  | 3.23  | 3.09  | 2.94  | 2.86  | 2.78  | 2.69  | 2.61  | 2.52  | 2.42     |
| 21                       | 3.31  | 3.17  | 3.03  | 2.88  | 2.80  | 2.72  | 2.64  | 2.55  | 2.46  | 2.36     |
| 22                       | 3.26  | 3.12  | 2.98  | 2.83  | 2.75  | 2.67  | 2.58  | 2.50  | 2.40  | 2.31     |
| 23                       | 3.21  | 3.07  | 2.93  | 2.78  | 2.70  | 2.62  | 2.54  | 2.45  | 2.35  | 2.26     |
| 24                       | 3.17  | 3.03  | 2.89  | 2.74  | 2.66  | 2.58  | 2.49  | 2.40  | 2.31  | 2.21     |
| 25                       | 3.13  | 2.99  | 2.85  | 2.70  | 2.62  | 2.54  | 2.45  | 2.36  | 2.27  | 2.17     |
| 26                       | 3.09  | 2.96  | 2.81  | 2.66  | 2.58  | 2.50  | 2.42  | 2.33  | 2.23  | 2.13     |
| 27                       | 3.06  | 2.93  | 2.78  | 2.63  | 2.55  | 2.47  | 2.38  | 2.29  | 2.20  | 2.10     |
| 28                       | 3.03  | 2.90  | 2.75  | 2.60  | 2.52  | 2.44  | 2.35  | 2.26  | 2.17  | 2.06     |
| 29                       | 3.00  | 2.87  | 2.73  | 2.57  | 2.49  | 2.41  | 2.33  | 2.23  | 2.14  | 2.03     |
| 30                       | 2.98  | 2.84  | 2.70  | 2.55  | 2.47  | 2.39  | 2.30  | 2.21  | 2.11  | 2.01     |
| 40                       | 2.80  | 2.66  | 2.52  | 2.37  | 2.29  | 2.20  | 2.11  | 2.02  | 1.92  | 1.80     |
| 60                       | 2.63  | 2.50  | 2.35  | 2.20  | 2.12  | 2.03  | 1.94  | 1.84  | 1.73  | 1.60     |
| 120                      | 2.47  | 2.34  | 2.19  | 2.03  | 1.95  | 1.86  | 1.76  | 1.66  | 1.53  | 1.38     |
| $\infty$                 | 2.32  | 2.18  | 2.04  | 1.88  | 1.79  | 1.70  | 1.59  | 1.47  | 1.32  | 1.00     |

Source: Pearson and Hartley, 1970

**Table 4** Fisher z-transformation

$$z(r) = \frac{1}{2} \log_e \left( \frac{1+r}{1-r} \right) = \tanh^{-1} r = 1.1513 \log_{10} \left( \frac{1+r}{1-r} \right)$$

| <i>z</i> | 0      | 1      | 2      | 3      | 4      | 5      | 6      | 7      | 8      | 9      |
|----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0      | 0.0000 | 0.0100 | 0.0200 | 0.0300 | 0.0400 | 0.0500 | 0.0601 | 0.0701 | 0.0802 | 0.0902 |
| 0.1      | 0.1003 | 0.1104 | 0.1206 | 0.1307 | 0.1409 | 0.1511 | 0.1614 | 0.1717 | 0.1820 | 0.1923 |
| 0.2      | 0.2027 | 0.2132 | 0.2237 | 0.2341 | 0.2448 | 0.2554 | 0.2661 | 0.2769 | 0.2877 | 0.2986 |
| 0.3      | 0.3095 | 0.3205 | 0.3316 | 0.3428 | 0.3541 | 0.3654 | 0.3769 | 0.3884 | 0.4001 | 0.4118 |
| 0.4      | 0.4236 | 0.4356 | 0.4477 | 0.4599 | 0.4722 | 0.4847 | 0.4973 | 0.5101 | 0.5230 | 0.5361 |
| 0.5      | 0.5493 | 0.5627 | 0.5763 | 0.5901 | 0.6042 | 0.6184 | 0.6328 | 0.6475 | 0.6625 | 0.6777 |
| 0.6      | 0.6931 | 0.7089 | 0.7250 | 0.7414 | 0.7582 | 0.7753 | 0.7928 | 0.8107 | 0.8291 | 0.8480 |
| 0.7      | 0.8673 | 0.8872 | 0.9076 | 0.9287 | 0.9505 | 0.9730 | 0.9962 | 1.0203 | 1.0454 | 1.0714 |
| 0.8      | 1.0986 | 1.1270 | 1.1568 | 1.1881 | 1.2212 | 1.2562 | 1.2933 | 1.3331 | 1.3758 | 1.4219 |
| 0.90     | 1.4722 | 1.4775 | 1.4828 | 1.4882 | 1.4937 | 1.4992 | 1.5047 | 1.5103 | 1.5160 | 1.5217 |
| 0.91     | 1.5275 | 1.5334 | 1.5393 | 1.5453 | 1.5513 | 1.5574 | 1.5636 | 1.5698 | 1.5762 | 1.5826 |
| 0.92     | 1.5890 | 1.5956 | 1.6022 | 1.6089 | 1.6157 | 1.6226 | 1.6296 | 1.6366 | 1.6438 | 1.6510 |
| 0.93     | 1.6584 | 1.6658 | 1.6734 | 1.6811 | 1.6888 | 1.6967 | 1.7047 | 1.7129 | 1.7211 | 1.7295 |
| 0.94     | 1.7380 | 1.7467 | 1.7555 | 1.7645 | 1.7736 | 1.7828 | 1.7923 | 1.8019 | 1.8117 | 1.8216 |
| 0.95     | 1.8318 | 1.8421 | 1.8527 | 1.8635 | 1.8745 | 1.8857 | 1.8972 | 1.9090 | 1.9210 | 1.9333 |
| 0.96     | 1.9459 | 1.9588 | 1.9721 | 1.9857 | 1.9996 | 2.0139 | 2.0287 | 2.0439 | 2.0595 | 2.0756 |
| 0.97     | 2.0923 | 2.1095 | 2.1273 | 2.1457 | 2.1649 | 2.1847 | 2.2054 | 2.2269 | 2.2494 | 2.2729 |
| 0.98     | 2.2976 | 2.3235 | 2.3507 | 2.3796 | 2.4101 | 2.4427 | 2.4774 | 2.5147 | 2.5550 | 2.5987 |
| 0.99     | 2.6467 | 2.6995 | 2.7587 | 2.8257 | 2.9031 | 2.9945 | 3.1063 | 3.2504 | 3.4534 | 3.8002 |

Source: Neave, 1978

**Table 5 Critical values for the  $\chi^2$ -distribution**Columns  $a$  denote the lower boundaries or the left-sided critical values.Columns  $b$  denote the upper boundaries or the right-sided critical values.

|           | Level of significance $\alpha$ |        |                     |        |                     |        |                     |        |
|-----------|--------------------------------|--------|---------------------|--------|---------------------|--------|---------------------|--------|
| Two-sided | 0.20                           |        | 0.10                |        | 0.05                |        | 0.01                |        |
| One-sided | 0.10                           |        | 0.05                |        | 0.025               |        | 0.005               |        |
| $\nu$     | $a$                            | $b$    | $a$                 | $b$    | $a$                 | $b$    | $a$                 | $b$    |
| 1         | 0.016                          | 2.71   | 39.10 <sup>-4</sup> | 3.84   | 98.10 <sup>-5</sup> | 5.02   | 16.10 <sup>-5</sup> | 6.63   |
| 2         | 0.21                           | 4.61   | 0.10                | 5.99   | 0.05                | 7.38   | 0.02                | 9.21   |
| 3         | 0.58                           | 6.25   | 0.35                | 7.81   | 0.22                | 9.35   | 0.11                | 11.34  |
| 4         | 1.06                           | 7.78   | 0.71                | 9.49   | 0.48                | 11.14  | 0.30                | 13.28  |
| 5         | 1.61                           | 9.24   | 1.15                | 11.07  | 0.83                | 12.83  | 0.55                | 15.09  |
| 6         | 2.20                           | 10.64  | 1.64                | 12.59  | 1.24                | 14.45  | 0.87                | 16.81  |
| 7         | 2.83                           | 12.02  | 2.17                | 14.07  | 1.69                | 16.01  | 1.24                | 18.48  |
| 8         | 3.49                           | 13.36  | 2.73                | 15.51  | 2.18                | 17.53  | 1.65                | 20.09  |
| 9         | 4.17                           | 14.68  | 3.33                | 16.92  | 2.70                | 19.02  | 2.09                | 21.67  |
| 10        | 4.87                           | 15.99  | 3.94                | 18.31  | 3.25                | 20.48  | 2.56                | 23.21  |
| 11        | 5.58                           | 17.28  | 4.57                | 19.68  | 3.82                | 21.92  | 3.05                | 24.73  |
| 12        | 6.30                           | 18.55  | 5.23                | 21.03  | 4.40                | 23.34  | 3.57                | 26.22  |
| 13        | 7.04                           | 19.81  | 5.89                | 22.36  | 5.01                | 24.74  | 4.11                | 27.69  |
| 14        | 7.79                           | 21.06  | 6.57                | 23.68  | 5.63                | 26.12  | 4.66                | 29.14  |
| 15        | 8.55                           | 22.31  | 7.26                | 25.00  | 6.26                | 27.49  | 5.23                | 30.58  |
| 16        | 9.31                           | 23.54  | 7.96                | 26.30  | 6.91                | 28.85  | 5.81                | 32.00  |
| 17        | 10.09                          | 24.77  | 8.67                | 27.59  | 7.56                | 30.19  | 6.41                | 33.41  |
| 18        | 10.86                          | 25.99  | 9.39                | 28.87  | 8.23                | 31.53  | 7.01                | 34.81  |
| 19        | 11.65                          | 27.20  | 10.12               | 30.14  | 8.91                | 32.85  | 7.63                | 36.19  |
| 20        | 12.44                          | 28.41  | 10.85               | 31.41  | 9.59                | 34.17  | 8.26                | 37.57  |
| 21        | 13.24                          | 29.62  | 11.59               | 32.67  | 10.28               | 35.48  | 8.90                | 38.93  |
| 22        | 14.04                          | 30.81  | 12.34               | 33.92  | 10.98               | 36.78  | 9.54                | 40.29  |
| 23        | 14.85                          | 32.01  | 13.09               | 35.17  | 11.69               | 38.08  | 10.20               | 41.64  |
| 24        | 15.66                          | 33.20  | 13.85               | 36.42  | 12.40               | 39.36  | 10.86               | 42.98  |
| 25        | 16.47                          | 34.38  | 14.61               | 37.65  | 13.12               | 40.65  | 11.52               | 44.31  |
| 26        | 17.29                          | 35.56  | 15.38               | 38.89  | 13.84               | 41.92  | 12.20               | 45.64  |
| 27        | 18.11                          | 36.74  | 16.15               | 40.11  | 14.57               | 43.19  | 12.88               | 46.96  |
| 28        | 18.94                          | 37.92  | 16.93               | 41.34  | 15.31               | 44.46  | 13.56               | 48.28  |
| 29        | 19.77                          | 39.09  | 17.71               | 42.56  | 16.05               | 45.72  | 14.26               | 49.59  |
| 30        | 20.60                          | 40.26  | 18.49               | 43.77  | 16.79               | 46.98  | 14.95               | 50.89  |
| 40        | 29.05                          | 51.81  | 26.51               | 55.76  | 24.43               | 59.34  | 22.16               | 63.69  |
| 50        | 37.69                          | 63.17  | 34.76               | 67.50  | 32.36               | 71.42  | 29.71               | 76.15  |
| 60        | 46.46                          | 74.40  | 43.19               | 79.08  | 40.48               | 83.30  | 37.48               | 88.38  |
| 70        | 55.33                          | 85.53  | 51.74               | 90.53  | 48.76               | 95.02  | 45.44               | 100.43 |
| 80        | 64.28                          | 96.58  | 60.39               | 101.88 | 57.15               | 106.63 | 53.54               | 112.33 |
| 90        | 73.29                          | 107.57 | 69.13               | 113.15 | 65.65               | 118.14 | 61.75               | 124.12 |
| 100       | 82.36                          | 118.50 | 77.93               | 124.34 | 74.22               | 129.56 | 70.06               | 135.81 |

**Table 6 Critical values of  $r$  for the correlation test with  $\rho = 0$** The distribution is symmetrical with respect to  $\rho = 0$ .

|               | Level of significance $\alpha$ |       |        |        |        |
|---------------|--------------------------------|-------|--------|--------|--------|
| Two-sided     | 0.10                           | 0.05  | 0.02   | 0.01   | 0.001  |
| One-sided     | 0.05                           | 0.025 | 0.01   | 0.005  | 0.0005 |
| $\nu = n - 2$ |                                |       |        |        |        |
| 1             | 0.988                          | 0.997 | 0.9995 | 0.9999 | 1.000  |
| 2             | 0.900                          | 0.950 | 0.980  | 0.990  | 0.999  |
| 3             | 0.805                          | 0.878 | 0.934  | 0.959  | 0.991  |
| 4             | 0.729                          | 0.811 | 0.882  | 0.917  | 0.974  |
| 5             | 0.669                          | 0.754 | 0.833  | 0.874  | 0.951  |
| 6             | 0.622                          | 0.707 | 0.789  | 0.834  | 0.925  |
| 7             | 0.582                          | 0.666 | 0.750  | 0.798  | 0.898  |
| 8             | 0.549                          | 0.632 | 0.716  | 0.765  | 0.872  |
| 9             | 0.521                          | 0.602 | 0.685  | 0.735  | 0.847  |
| 10            | 0.497                          | 0.576 | 0.658  | 0.708  | 0.823  |
| 11            | 0.476                          | 0.553 | 0.634  | 0.684  | 0.801  |
| 12            | 0.458                          | 0.532 | 0.612  | 0.661  | 0.780  |
| 13            | 0.441                          | 0.514 | 0.592  | 0.641  | 0.760  |
| 14            | 0.426                          | 0.497 | 0.574  | 0.623  | 0.742  |
| 15            | 0.412                          | 0.482 | 0.558  | 0.606  | 0.725  |
| 16            | 0.400                          | 0.468 | 0.542  | 0.590  | 0.708  |
| 17            | 0.389                          | 0.456 | 0.528  | 0.575  | 0.693  |
| 18            | 0.378                          | 0.444 | 0.516  | 0.561  | 0.679  |
| 19            | 0.369                          | 0.433 | 0.503  | 0.549  | 0.665  |
| 20            | 0.360                          | 0.423 | 0.492  | 0.537  | 0.652  |
| 22            | 0.344                          | 0.404 | 0.472  | 0.515  | 0.629  |
| 24            | 0.330                          | 0.388 | 0.453  | 0.496  | 0.607  |
| 25            | 0.323                          | 0.381 | 0.445  | 0.487  | 0.597  |
| 30            | 0.296                          | 0.349 | 0.409  | 0.449  | 0.554  |
| 35            | 0.275                          | 0.325 | 0.381  | 0.418  | 0.519  |
| 40            | 0.257                          | 0.304 | 0.358  | 0.372  | 0.490  |
| 45            | 0.243                          | 0.288 | 0.338  | 0.372  | 0.415  |
| 50            | 0.231                          | 0.273 | 0.322  | 0.354  | 0.443  |
| 55            | 0.220                          | 0.261 | 0.307  | 0.338  | 0.424  |
| 60            | 0.211                          | 0.250 | 0.295  | 0.325  | 0.408  |
| 65            | 0.203                          | 0.240 | 0.284  | 0.312  | 0.393  |
| 70            | 0.195                          | 0.232 | 0.274  | 0.302  | 0.380  |
| 75            | 0.189                          | 0.224 | 0.264  | 0.292  | 0.368  |
| 80            | 0.183                          | 0.217 | 0.256  | 0.283  | 0.357  |
| 85            | 0.178                          | 0.211 | 0.249  | 0.275  | 0.347  |
| 90            | 0.173                          | 0.205 | 0.242  | 0.267  | 0.338  |
| 95            | 0.168                          | 0.200 | 0.236  | 0.260  | 0.329  |
| 100           | 0.164                          | 0.195 | 0.230  | 0.254  | 0.321  |
| 125           | 0.147                          | 0.174 | 0.206  | 0.228  | 0.288  |
| 150           | 0.134                          | 0.159 | 0.189  | 0.208  | 0.264  |
| 175           | 0.124                          | 0.148 | 0.174  | 0.194  | 0.248  |
| 200           | 0.116                          | 0.138 | 0.164  | 0.181  | 0.235  |
| 300           | 0.095                          | 0.113 | 0.134  | 0.148  | 0.188  |
| 500           | 0.074                          | 0.088 | 0.104  | 0.115  | 0.148  |
| 1000          | 0.052                          | 0.062 | 0.073  | 0.081  | 0.104  |
| 2000          | 0.037                          | 0.044 | 0.056  | 0.058  | 0.074  |

Source: De Jonge, 1963–4

**Table 7 Critical values of  $g_1$  and  $g_2$  for Fisher's cumulant test**

The  $g_1$  distribution may be considered as symmetrical with respect to 0;

the  $g_2$  distribution has to be considered as asymmetrical.

The columns  $a$  denote the lower boundaries or left-sided critical values.

The columns  $b$  denote the upper boundaries or right-sided critical values.

| g1        |                                |       | g2    |      |       |      |
|-----------|--------------------------------|-------|-------|------|-------|------|
|           | Level of significance $\alpha$ |       |       |      |       |      |
| Two-sided | 0.10                           | 0.02  | 0.10  |      | 0.02  |      |
| One-sided | 0.05                           | 0.01  | 0.05  |      | 0.01  |      |
| $n$       |                                |       | $a$   | $b$  | $a$   | $b$  |
| 50        | 0.550                          | 0.812 | —     | —    | —     | —    |
| 75        | 0.454                          | 0.664 | —     | —    | —     | —    |
| 100       | 0.395                          | 0.576 | −0.62 | 1.53 | −0.80 | 1.53 |
| 125       | 0.354                          | 0.514 | −0.57 | 1.34 | −0.74 | 1.34 |
| 150       | 0.324                          | 0.469 | −0.53 | 1.22 | −0.69 | 1.22 |
| 175       | 0.301                          | 0.434 | −0.50 | 1.11 | −0.66 | 1.11 |
| 200       | 0.282                          | 0.406 | −0.47 | 1.04 | −0.62 | 1.04 |
| 250       | 0.253                          | 0.362 | −0.44 | 0.91 | −0.57 | 0.91 |
| 300       | 0.231                          | 0.331 | −0.40 | 0.82 | −0.53 | 0.82 |
| 350       | 0.214                          | 0.306 | −0.37 | 0.75 | −0.49 | 0.75 |
| 400       | 0.201                          | 0.286 | −0.35 | 0.69 | −0.48 | 0.69 |
| 450       | 0.189                          | 0.270 | −0.33 | 0.65 | −0.44 | 0.65 |
| 500       | 0.180                          | 0.256 | −0.32 | 0.62 | −0.42 | 0.62 |
| 550       | 0.171                          | 0.244 | −0.30 | 0.59 | −0.41 | 0.59 |
| 600       | 0.163                          | 0.234 | −0.29 | 0.55 | −0.39 | 0.55 |
| 650       | 0.157                          | 0.225 | −0.28 | 0.53 | −0.38 | 0.53 |
| 700       | 0.151                          | 0.215 | −0.27 | 0.51 | −0.37 | 0.51 |
| 750       | 0.146                          | 0.208 | −0.26 | 0.49 | −0.35 | 0.49 |
| 800       | 0.142                          | 0.202 | −0.25 | 0.47 | −0.34 | 0.47 |
| 850       | 0.138                          | 0.196 | −0.25 | 0.46 | −0.33 | 0.46 |
| 900       | 0.134                          | 0.190 | −0.24 | 0.44 | −0.33 | 0.44 |
| 950       | 0.130                          | 0.185 | −0.23 | 0.43 | −0.32 | 0.43 |
| 1000      | 0.127                          | 0.180 | −0.23 | 0.42 | −0.31 | 0.42 |

Source: Geary and Pearson, n.d.; Bennett and Franklin, 1961

**Table 8 Critical values for the Dixon test of outliers**

| Test<br>Statistic                          | <i>n</i> | Level of significance $\alpha$ |       |       |       |       |       |       |
|--|----------|--------------------------------|-------|-------|-------|-------|-------|-------|
|  |          | 0.30                           | 0.20  | 0.10  | 0.05  | 0.02  | 0.01  | 0.005 |
| $r_{10} = \frac{x_2 - x_1}{x_n - x_1}$     | 3        | 0.684                          | 0.781 | 0.886 | 0.941 | 0.976 | 0.988 | 0.994 |
|  | 4        | 0.471                          | 0.560 | 0.679 | 0.765 | 0.846 | 0.889 | 0.926 |
|  | 5        | 0.373                          | 0.451 | 0.557 | 0.642 | 0.729 | 0.780 | 0.821 |
|  | 6        | 0.318                          | 0.386 | 0.482 | 0.560 | 0.644 | 0.698 | 0.740 |
|  | 7        | 0.281                          | 0.344 | 0.434 | 0.507 | 0.586 | 0.637 | 0.680 |
| $r_{11} = \frac{x_2 - x_1}{x_{n-1} - x_1}$ | 8        | 0.318                          | 0.385 | 0.479 | 0.554 | 0.631 | 0.683 | 0.725 |
|  | 9        | 0.288                          | 0.352 | 0.441 | 0.512 | 0.587 | 0.635 | 0.677 |
|  | 10       | 0.265                          | 0.325 | 0.409 | 0.477 | 0.551 | 0.597 | 0.639 |
| $r_{21} = \frac{x_3 - x_1}{x_{n-1} - x_1}$ | 11       | 0.391                          | 0.442 | 0.517 | 0.576 | 0.638 | 0.679 | 0.713 |
|  | 12       | 0.370                          | 0.419 | 0.490 | 0.546 | 0.605 | 0.642 | 0.675 |
|  | 13       | 0.351                          | 0.399 | 0.467 | 0.521 | 0.578 | 0.615 | 0.649 |
| $r_{22} = \frac{x_3 - x_1}{x_{n-2} - x_1}$ | 14       | 0.370                          | 0.421 | 0.492 | 0.546 | 0.602 | 0.641 | 0.674 |
|  | 15       | 0.353                          | 0.402 | 0.472 | 0.525 | 0.579 | 0.616 | 0.647 |
|  | 16       | 0.338                          | 0.386 | 0.454 | 0.507 | 0.559 | 0.595 | 0.624 |
|  | 17       | 0.325                          | 0.373 | 0.438 | 0.490 | 0.542 | 0.577 | 0.605 |
|  | 18       | 0.314                          | 0.361 | 0.424 | 0.475 | 0.527 | 0.561 | 0.589 |
|  | 19       | 0.304                          | 0.350 | 0.412 | 0.462 | 0.514 | 0.547 | 0.575 |
|  | 20       | 0.295                          | 0.340 | 0.401 | 0.450 | 0.502 | 0.535 | 0.562 |
|  | 21       | 0.287                          | 0.331 | 0.391 | 0.440 | 0.491 | 0.524 | 0.551 |
|  | 22       | 0.280                          | 0.323 | 0.382 | 0.430 | 0.481 | 0.514 | 0.541 |
|  | 23       | 0.274                          | 0.316 | 0.374 | 0.421 | 0.472 | 0.505 | 0.532 |
|  | 24       | 0.268                          | 0.310 | 0.367 | 0.413 | 0.464 | 0.497 | 0.524 |
|  | 25       | 0.262                          | 0.304 | 0.360 | 0.406 | 0.457 | 0.489 | 0.516 |

Source: Dixon and Massey, 1957

**Table 9 Critical values of the Studentized range for multiple comparison**

| Level of significance $\alpha = 0.05$ |        |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
|---------------------------------------|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $K$<br>$\nu_2$                        | 2      | 3     | 4     | 5     | 6     | 7     | 8     | 9     | 10    | 11    | 12    | 13    | 14    | 15    | 16    | 17    | 18    | 19    | 20    |
| 1                                     | 17.969 | 26.98 | 32.82 | 37.08 | 40.41 | 43.12 | 45.40 | 47.36 | 49.07 | 50.59 | 51.96 | 53.20 | 54.33 | 55.36 | 56.32 | 57.22 | 58.04 | 58.83 | 59.56 |
| 2                                     | 6.085  | 8.33  | 9.80  | 10.88 | 11.74 | 12.44 | 13.03 | 13.54 | 13.99 | 14.39 | 14.75 | 15.08 | 15.38 | 15.65 | 15.91 | 16.14 | 16.37 | 16.57 | 16.77 |
| 3                                     | 4.501  | 5.91  | 6.82  | 7.50  | 8.04  | 8.48  | 8.85  | 9.18  | 9.46  | 9.72  | 9.95  | 10.15 | 10.35 | 10.52 | 10.69 | 10.84 | 10.98 | 11.11 | 11.24 |
| 4                                     | 3.926  | 5.04  | 5.76  | 6.29  | 6.71  | 7.05  | 7.35  | 7.60  | 7.83  | 8.03  | 8.21  | 8.37  | 8.52  | 8.66  | 8.79  | 8.91  | 9.03  | 9.13  | 9.23  |
| 5                                     | 3.635  | 4.60  | 5.22  | 5.67  | 6.03  | 6.33  | 6.58  | 6.80  | 6.99  | 7.17  | 7.32  | 7.47  | 7.60  | 7.72  | 7.83  | 7.93  | 8.03  | 8.12  | 8.21  |
| 6                                     | 3.460  | 4.34  | 4.90  | 5.30  | 5.63  | 5.90  | 6.12  | 6.32  | 6.49  | 6.65  | 6.79  | 6.92  | 7.03  | 7.14  | 7.24  | 7.34  | 7.43  | 7.51  | 7.59  |
| 7                                     | 3.344  | 4.16  | 4.68  | 5.06  | 5.36  | 5.61  | 5.82  | 6.00  | 6.16  | 6.30  | 6.43  | 6.55  | 6.66  | 6.76  | 6.85  | 6.94  | 7.02  | 7.10  | 7.17  |
| 8                                     | 3.261  | 4.04  | 4.53  | 4.89  | 5.17  | 5.40  | 5.60  | 5.77  | 5.92  | 6.05  | 6.18  | 6.29  | 6.39  | 6.48  | 6.57  | 6.65  | 6.73  | 6.80  | 6.87  |
| 9                                     | 3.199  | 3.95  | 4.41  | 4.76  | 5.02  | 5.24  | 5.43  | 5.59  | 5.74  | 5.87  | 5.98  | 6.09  | 6.19  | 6.28  | 6.36  | 6.44  | 6.51  | 6.58  | 6.64  |
| 10                                    | 3.151  | 3.88  | 4.33  | 4.65  | 4.91  | 5.12  | 5.30  | 5.46  | 5.60  | 5.72  | 5.83  | 5.93  | 6.03  | 6.11  | 6.19  | 6.27  | 6.34  | 6.40  | 6.47  |
| 11                                    | 3.113  | 3.82  | 4.26  | 4.57  | 4.82  | 5.03  | 5.20  | 5.35  | 5.49  | 5.61  | 5.71  | 5.81  | 5.90  | 5.98  | 6.06  | 6.13  | 6.20  | 6.27  | 6.33  |
| 12                                    | 3.081  | 3.77  | 4.20  | 4.51  | 4.75  | 4.95  | 5.12  | 5.27  | 5.39  | 5.51  | 5.61  | 5.71  | 5.80  | 5.88  | 5.95  | 6.02  | 6.09  | 6.15  | 6.21  |
| 13                                    | 3.055  | 3.73  | 4.15  | 4.45  | 4.69  | 4.88  | 5.05  | 5.19  | 5.32  | 5.43  | 5.53  | 5.63  | 5.71  | 5.79  | 5.85  | 5.93  | 5.99  | 6.05  | 6.11  |
| 14                                    | 3.033  | 3.70  | 4.11  | 4.41  | 4.64  | 4.83  | 4.99  | 5.13  | 5.25  | 5.36  | 5.46  | 5.55  | 5.64  | 5.71  | 5.79  | 5.85  | 5.91  | 5.97  | 6.03  |
| 15                                    | 3.014  | 3.67  | 4.08  | 4.37  | 4.59  | 4.78  | 4.94  | 5.08  | 5.20  | 5.31  | 5.40  | 5.49  | 5.57  | 5.65  | 5.72  | 5.78  | 5.85  | 5.90  | 5.96  |
| 16                                    | 2.998  | 3.65  | 4.05  | 4.33  | 4.56  | 4.74  | 4.90  | 5.03  | 5.15  | 5.26  | 5.35  | 5.44  | 5.52  | 5.59  | 5.66  | 5.73  | 5.79  | 5.84  | 5.90  |
| 17                                    | 2.984  | 3.63  | 4.02  | 4.30  | 4.52  | 4.70  | 4.86  | 4.99  | 5.11  | 5.21  | 5.31  | 5.39  | 5.47  | 5.54  | 5.61  | 5.67  | 5.73  | 5.79  | 5.84  |
| 18                                    | 2.971  | 3.61  | 4.00  | 4.28  | 4.49  | 4.67  | 4.82  | 4.96  | 5.07  | 5.17  | 5.27  | 5.35  | 5.43  | 5.50  | 5.57  | 5.63  | 5.69  | 5.74  | 5.79  |
| 19                                    | 2.960  | 3.59  | 3.98  | 4.25  | 4.47  | 4.65  | 4.79  | 4.92  | 5.04  | 5.14  | 5.23  | 5.31  | 5.39  | 5.46  | 5.53  | 5.59  | 5.65  | 5.70  | 5.75  |
| 20                                    | 2.950  | 3.58  | 3.96  | 4.23  | 4.45  | 4.62  | 4.77  | 4.90  | 5.01  | 5.11  | 5.20  | 5.28  | 5.36  | 5.43  | 5.49  | 5.55  | 5.61  | 5.66  | 5.71  |
| 21                                    | 2.941  | 3.56  | 3.94  | 4.21  | 4.43  | 4.60  | 4.74  | 4.87  | 4.98  | 5.08  | 5.17  | 5.25  | 5.33  | 5.40  | 5.46  | 5.52  | 5.58  | 5.62  | 5.67  |
| 22                                    | 2.933  | 3.55  | 3.93  | 4.20  | 4.41  | 4.58  | 4.72  | 4.85  | 4.96  | 5.05  | 5.15  | 5.23  | 5.30  | 5.37  | 5.43  | 5.49  | 5.55  | 5.59  | 5.64  |
| 23                                    | 2.926  | 3.54  | 3.91  | 4.18  | 4.39  | 4.56  | 4.70  | 4.83  | 4.94  | 5.03  | 5.12  | 5.20  | 5.27  | 5.34  | 5.40  | 5.46  | 5.52  | 5.57  | 5.62  |
| 24                                    | 2.919  | 3.53  | 3.90  | 4.17  | 4.37  | 4.54  | 4.68  | 4.81  | 4.92  | 5.01  | 5.10  | 5.18  | 5.25  | 5.32  | 5.38  | 5.44  | 5.49  | 5.55  | 5.59  |



Table 9 continued

| $\nu_2 \backslash K$ | 2     | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   | 13   | 14   | 15   | 16   | 17   | 18   | 19   | 20   |
|----------------------|-------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 25                   | 2.913 | 3.52 | 3.89 | 4.16 | 4.36 | 4.52 | 4.66 | 4.79 | 4.90 | 4.99 | 5.08 | 5.16 | 5.23 | 5.30 | 5.36 | 5.42 | 5.48 | 5.52 | 5.57 |
| 26                   | 2.907 | 3.51 | 3.88 | 4.14 | 4.34 | 4.51 | 4.65 | 4.78 | 4.89 | 4.97 | 5.06 | 5.14 | 5.21 | 5.28 | 5.34 | 5.40 | 5.46 | 5.50 | 5.55 |
| 27                   | 2.902 | 3.51 | 3.87 | 4.13 | 4.33 | 4.50 | 4.63 | 4.76 | 4.87 | 4.96 | 5.04 | 5.12 | 5.19 | 5.26 | 5.32 | 5.38 | 5.43 | 5.48 | 5.53 |
| 28                   | 2.897 | 3.50 | 3.86 | 4.12 | 4.32 | 4.48 | 4.62 | 4.75 | 4.86 | 4.94 | 5.03 | 5.11 | 5.18 | 5.24 | 5.30 | 5.36 | 5.42 | 5.46 | 5.51 |
| 29                   | 2.892 | 3.49 | 3.85 | 4.11 | 4.31 | 4.47 | 4.61 | 4.73 | 4.84 | 4.93 | 5.01 | 5.09 | 5.16 | 5.23 | 5.29 | 5.35 | 5.40 | 5.44 | 5.49 |
| 30                   | 2.888 | 3.49 | 3.85 | 4.10 | 4.30 | 4.46 | 4.60 | 4.72 | 4.82 | 4.92 | 5.00 | 5.08 | 5.15 | 5.21 | 5.27 | 5.33 | 5.38 | 5.43 | 5.47 |
| 31                   | 2.884 | 3.48 | 3.83 | 4.09 | 4.29 | 4.45 | 4.59 | 4.71 | 4.82 | 4.91 | 4.99 | 5.07 | 5.14 | 5.20 | 5.26 | 5.32 | 5.37 | 5.41 | 5.46 |
| 32                   | 2.881 | 3.48 | 3.83 | 4.09 | 4.28 | 4.44 | 4.58 | 4.70 | 4.81 | 4.89 | 4.98 | 5.06 | 5.13 | 5.19 | 5.24 | 5.30 | 5.35 | 5.40 | 5.45 |
| 33                   | 2.877 | 3.47 | 3.82 | 4.08 | 4.27 | 4.44 | 4.57 | 4.69 | 4.80 | 4.88 | 4.97 | 5.04 | 5.11 | 5.17 | 5.23 | 5.29 | 5.34 | 5.39 | 5.44 |
| 34                   | 2.874 | 3.47 | 3.82 | 4.07 | 4.27 | 4.43 | 4.56 | 4.68 | 4.79 | 4.87 | 4.96 | 5.03 | 5.10 | 5.16 | 5.22 | 5.28 | 5.33 | 5.37 | 5.42 |
| 35                   | 2.871 | 3.46 | 3.81 | 4.07 | 4.26 | 4.42 | 4.55 | 4.67 | 4.78 | 4.86 | 4.95 | 5.02 | 5.09 | 5.15 | 5.21 | 5.27 | 5.32 | 5.36 | 5.41 |
| 36                   | 2.868 | 3.46 | 3.81 | 4.06 | 4.25 | 4.41 | 4.55 | 4.66 | 4.77 | 4.85 | 4.94 | 5.01 | 5.08 | 5.14 | 5.20 | 5.26 | 5.31 | 5.35 | 5.40 |
| 37                   | 2.865 | 3.45 | 3.80 | 4.05 | 5.25 | 4.41 | 4.54 | 4.65 | 4.76 | 4.84 | 4.93 | 5.00 | 5.08 | 5.14 | 5.19 | 5.25 | 5.30 | 5.34 | 5.39 |
| 38                   | 2.863 | 3.45 | 3.80 | 4.05 | 4.24 | 4.40 | 4.53 | 4.64 | 4.75 | 4.84 | 4.92 | 5.00 | 5.07 | 5.13 | 5.18 | 5.24 | 5.29 | 5.33 | 5.38 |
| 39                   | 2.861 | 3.44 | 3.79 | 4.04 | 4.24 | 4.40 | 4.53 | 4.64 | 4.75 | 4.83 | 4.92 | 4.99 | 5.06 | 5.12 | 5.17 | 5.23 | 5.28 | 5.32 | 5.37 |
| 40                   | 2.858 | 3.44 | 3.79 | 4.04 | 4.23 | 4.39 | 4.52 | 4.63 | 4.73 | 4.82 | 4.90 | 4.98 | 5.04 | 5.11 | 5.16 | 5.22 | 5.27 | 5.22 | 5.36 |
| 50                   | 2.841 | 3.41 | 3.76 | 4.00 | 4.19 | 4.34 | 4.47 | 4.58 | 4.69 | 4.76 | 4.85 | 4.92 | 4.99 | 5.05 | 5.10 | 5.15 | 5.20 | 5.24 | 5.29 |
| 60                   | 2.829 | 3.40 | 3.74 | 3.98 | 4.16 | 4.31 | 4.44 | 4.55 | 4.65 | 4.73 | 4.81 | 4.88 | 4.94 | 5.00 | 5.06 | 5.11 | 5.15 | 5.20 | 5.24 |
| 120                  | 2.800 | 3.36 | 3.68 | 3.92 | 4.10 | 4.24 | 4.36 | 4.47 | 4.56 | 4.64 | 4.71 | 4.78 | 4.84 | 4.90 | 4.95 | 5.00 | 5.04 | 5.09 | 5.13 |
| $\infty$             | 2.772 | 3.31 | 3.63 | 3.86 | 4.03 | 4.17 | 4.29 | 4.39 | 4.47 | 4.55 | 4.62 | 4.68 | 4.74 | 4.80 | 4.85 | 4.89 | 4.93 | 4.97 | 5.01 |

Table 9 continued

| Level of significance $\alpha = 0.01$ |        |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
|---------------------------------------|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\nu_2 \backslash K$                  | 2      | 3     | 4     | 5     | 6     | 7     | 8     | 9     | 10    | 11    | 12    | 13    | 14    | 15    | 16    | 17    | 18    | 19    | 20    |
| 1                                     | 90.025 | 135.0 | 164.3 | 185.6 | 202.2 | 215.8 | 227.2 | 237.0 | 245.6 | 253.2 | 260.0 | 266.2 | 271.8 | 277.0 | 281.8 | 286.3 | 290.4 | 294.3 | 298.0 |
| 2                                     | 14.036 | 19.02 | 22.29 | 24.72 | 26.63 | 28.20 | 29.53 | 30.68 | 31.69 | 32.59 | 33.40 | 34.13 | 34.81 | 35.43 | 36.00 | 36.53 | 37.03 | 37.50 | 37.95 |
| 3                                     | 8.260  | 10.62 | 12.17 | 13.33 | 14.24 | 15.00 | 15.64 | 16.20 | 16.69 | 17.13 | 17.53 | 17.89 | 18.22 | 18.52 | 18.81 | 19.07 | 19.32 | 19.55 | 19.77 |
| 4                                     | 6.511  | 8.12  | 9.17  | 9.96  | 10.58 | 11.10 | 11.55 | 11.93 | 12.27 | 12.57 | 12.84 | 13.09 | 13.32 | 13.53 | 13.73 | 13.91 | 14.08 | 14.24 | 14.40 |
| 5                                     | 5.702  | 6.98  | 7.80  | 8.42  | 8.91  | 9.32  | 9.67  | 9.97  | 10.24 | 10.48 | 10.70 | 10.89 | 11.08 | 11.24 | 11.40 | 11.55 | 11.68 | 11.81 | 11.93 |
| 6                                     | 5.243  | 6.33  | 7.08  | 7.56  | 7.97  | 8.32  | 8.61  | 8.87  | 9.10  | 9.30  | 9.48  | 9.65  | 9.81  | 9.95  | 10.08 | 10.21 | 10.32 | 10.43 | 10.54 |
| 7                                     | 4.949  | 5.92  | 6.54  | 7.01  | 7.37  | 7.68  | 7.94  | 8.17  | 8.37  | 8.55  | 8.71  | 8.86  | 9.00  | 9.12  | 9.24  | 9.35  | 9.46  | 9.55  | 9.65  |
| 8                                     | 4.745  | 5.64  | 6.20  | 6.62  | 6.96  | 7.24  | 7.47  | 7.68  | 7.86  | 8.03  | 8.18  | 8.31  | 8.44  | 8.55  | 8.66  | 8.76  | 8.85  | 8.94  | 9.03  |
| 9                                     | 4.596  | 5.43  | 5.96  | 6.35  | 6.66  | 6.91  | 7.13  | 7.33  | 7.49  | 7.65  | 7.78  | 7.91  | 8.03  | 8.13  | 8.23  | 8.33  | 8.41  | 8.49  | 8.57  |
| 10                                    | 4.482  | 5.27  | 5.77  | 6.14  | 6.43  | 6.67  | 6.87  | 7.05  | 7.21  | 7.36  | 7.49  | 7.60  | 7.71  | 7.81  | 7.91  | 7.99  | 8.08  | 8.15  | 8.23  |
| 11                                    | 4.392  | 5.15  | 5.62  | 5.97  | 6.25  | 6.48  | 6.67  | 6.84  | 6.99  | 7.13  | 7.25  | 7.36  | 7.46  | 7.56  | 7.65  | 7.73  | 7.81  | 7.88  | 7.95  |
| 12                                    | 4.320  | 5.05  | 5.50  | 5.84  | 6.10  | 6.32  | 6.51  | 6.67  | 6.81  | 6.94  | 7.06  | 7.17  | 7.26  | 7.36  | 7.44  | 7.52  | 7.59  | 7.66  | 7.73  |
| 13                                    | 4.260  | 4.96  | 5.40  | 5.73  | 5.98  | 6.19  | 6.37  | 6.53  | 6.67  | 6.79  | 6.90  | 7.01  | 7.10  | 7.19  | 7.27  | 7.35  | 7.42  | 7.48  | 7.55  |
| 14                                    | 4.210  | 4.89  | 5.32  | 5.63  | 5.88  | 6.08  | 6.26  | 6.41  | 6.54  | 6.66  | 6.77  | 6.87  | 6.96  | 7.05  | 7.13  | 7.20  | 7.27  | 7.33  | 7.39  |
| 15                                    | 4.167  | 4.84  | 5.25  | 5.56  | 5.80  | 5.99  | 6.16  | 6.31  | 6.44  | 6.55  | 6.66  | 6.76  | 6.84  | 6.93  | 7.00  | 7.07  | 7.14  | 7.20  | 7.26  |
| 16                                    | 4.131  | 4.79  | 5.19  | 5.49  | 5.72  | 5.92  | 6.08  | 6.22  | 6.35  | 6.46  | 6.56  | 6.66  | 6.74  | 6.82  | 6.90  | 6.97  | 7.03  | 7.09  | 7.15  |
| 17                                    | 4.099  | 4.74  | 5.14  | 5.43  | 5.66  | 5.85  | 6.01  | 6.15  | 6.27  | 6.38  | 6.48  | 6.57  | 6.66  | 6.73  | 6.81  | 6.87  | 6.94  | 7.00  | 7.05  |
| 18                                    | 4.071  | 4.70  | 5.09  | 5.38  | 5.60  | 5.79  | 5.94  | 6.08  | 6.20  | 6.31  | 6.41  | 6.50  | 6.58  | 6.65  | 6.73  | 6.79  | 6.85  | 6.91  | 6.97  |
| 19                                    | 4.045  | 4.67  | 5.05  | 5.33  | 5.55  | 5.73  | 5.89  | 6.02  | 6.14  | 6.25  | 6.34  | 6.43  | 6.51  | 6.58  | 6.65  | 6.72  | 6.78  | 6.84  | 6.89  |
| 20                                    | 4.024  | 4.64  | 5.02  | 5.29  | 5.51  | 5.69  | 5.84  | 5.97  | 6.09  | 6.19  | 6.28  | 6.37  | 6.45  | 6.52  | 6.59  | 6.65  | 6.71  | 6.77  | 6.82  |
| 21                                    | 4.004  | 4.61  | 4.99  | 5.26  | 5.47  | 5.65  | 5.80  | 5.92  | 6.04  | 6.14  | 6.24  | 6.32  | 6.39  | 6.47  | 6.53  | 6.59  | 6.65  | 6.70  | 6.76  |
| 22                                    | 3.986  | 4.58  | 4.96  | 5.22  | 5.43  | 5.61  | 5.76  | 5.88  | 6.00  | 6.10  | 6.19  | 6.27  | 6.35  | 6.42  | 6.48  | 6.54  | 6.60  | 6.65  | 6.70  |
| 23                                    | 3.970  | 4.56  | 4.93  | 5.20  | 5.40  | 5.57  | 5.72  | 5.84  | 5.96  | 6.06  | 6.15  | 6.23  | 6.30  | 6.37  | 6.43  | 6.49  | 6.55  | 6.60  | 6.65  |
| 24                                    | 3.955  | 4.55  | 4.91  | 5.17  | 5.37  | 5.54  | 5.69  | 5.81  | 5.92  | 6.02  | 6.11  | 6.19  | 6.26  | 6.33  | 6.39  | 6.45  | 6.51  | 6.56  | 6.61  |

Table 9 continued

| $\nu_2 \backslash K$ | 2     | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   | 13   | 14   | 15   | 16   | 17   | 18   | 19   | 20   |
|----------------------|-------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 25                   | 3.942 | 4.52 | 4.89 | 5.15 | 5.34 | 5.51 | 5.66 | 5.78 | 5.89 | 5.99 | 6.07 | 6.15 | 6.22 | 6.29 | 6.35 | 6.41 | 6.47 | 6.52 | 6.57 |
| 26                   | 3.930 | 4.50 | 4.87 | 5.12 | 5.32 | 5.49 | 5.63 | 5.75 | 5.86 | 5.95 | 6.04 | 6.12 | 6.19 | 6.26 | 6.32 | 6.38 | 6.43 | 6.48 | 6.53 |
| 27                   | 3.918 | 4.49 | 4.85 | 5.10 | 5.30 | 5.46 | 5.61 | 5.72 | 5.83 | 5.93 | 6.01 | 6.09 | 6.16 | 6.22 | 6.28 | 6.34 | 6.40 | 6.45 | 6.50 |
| 28                   | 3.908 | 4.47 | 4.83 | 5.08 | 5.28 | 5.44 | 5.58 | 5.70 | 5.80 | 5.90 | 5.98 | 6.06 | 6.13 | 6.19 | 6.25 | 6.31 | 6.37 | 6.42 | 6.47 |
| 29                   | 3.889 | 4.45 | 4.80 | 5.05 | 5.24 | 5.40 | 5.54 | 5.65 | 5.76 | 5.85 | 5.93 | 6.01 | 6.08 | 6.14 | 6.20 | 6.26 | 6.31 | 6.36 | 6.41 |
| 30                   | 3.889 | 4.45 | 4.80 | 5.05 | 5.24 | 5.40 | 5.54 | 5.65 | 5.76 | 5.85 | 5.93 | 6.01 | 6.08 | 6.14 | 6.20 | 6.26 | 6.31 | 6.36 | 6.41 |
| 31                   | 3.881 | 4.44 | 4.79 | 5.03 | 5.22 | 5.38 | 5.52 | 5.63 | 5.74 | 5.83 | 5.91 | 5.99 | 6.06 | 6.12 | 6.18 | 6.23 | 6.29 | 6.34 | 6.38 |
| 32                   | 3.873 | 4.43 | 4.78 | 5.02 | 5.21 | 5.37 | 5.50 | 5.61 | 5.72 | 5.81 | 5.89 | 5.97 | 6.03 | 6.09 | 6.16 | 6.21 | 6.26 | 6.31 | 6.36 |
| 33                   | 3.865 | 4.42 | 4.76 | 5.01 | 5.19 | 5.35 | 5.48 | 5.59 | 5.70 | 5.79 | 5.87 | 5.95 | 6.01 | 6.07 | 6.13 | 6.19 | 6.24 | 6.29 | 6.34 |
| 34                   | 3.859 | 4.41 | 4.75 | 4.99 | 5.18 | 5.34 | 5.47 | 5.58 | 5.68 | 5.77 | 5.86 | 5.93 | 5.99 | 6.05 | 6.12 | 6.17 | 6.22 | 6.27 | 6.31 |
| 35                   | 3.852 | 4.41 | 4.74 | 4.98 | 5.16 | 5.33 | 5.45 | 5.56 | 5.67 | 5.76 | 5.84 | 5.91 | 5.98 | 6.04 | 6.10 | 6.15 | 6.20 | 6.25 | 6.29 |
| 36                   | 3.846 | 4.40 | 4.73 | 4.97 | 5.15 | 5.31 | 5.44 | 5.55 | 5.65 | 5.74 | 5.82 | 5.90 | 5.96 | 6.02 | 6.08 | 6.13 | 6.18 | 6.23 | 6.28 |
| 37                   | 3.841 | 4.39 | 4.72 | 4.96 | 5.14 | 5.30 | 5.43 | 5.54 | 5.64 | 5.73 | 5.81 | 5.88 | 5.94 | 6.00 | 6.06 | 6.12 | 6.17 | 6.22 | 6.26 |
| 38                   | 3.835 | 4.38 | 4.72 | 4.95 | 5.13 | 5.29 | 5.41 | 5.52 | 5.62 | 5.72 | 5.80 | 5.87 | 5.93 | 5.99 | 6.05 | 6.10 | 6.15 | 6.20 | 6.24 |
| 39                   | 3.830 | 4.38 | 4.71 | 4.94 | 5.12 | 5.28 | 5.40 | 5.51 | 5.62 | 5.70 | 5.78 | 5.85 | 5.91 | 5.97 | 6.03 | 6.08 | 6.13 | 6.18 | 6.23 |
| 40                   | 3.825 | 4.37 | 4.70 | 4.93 | 5.11 | 5.26 | 5.39 | 5.50 | 5.60 | 5.69 | 5.76 | 5.83 | 5.90 | 5.96 | 6.02 | 6.07 | 6.12 | 6.16 | 6.21 |
| 50                   | 3.787 | 4.32 | 4.64 | 4.86 | 5.04 | 5.19 | 5.30 | 5.41 | 5.51 | 5.59 | 5.67 | 5.74 | 5.80 | 5.86 | 5.91 | 5.96 | 6.01 | 6.06 | 6.09 |
| 60                   | 3.762 | 4.28 | 4.59 | 4.82 | 4.99 | 5.13 | 5.25 | 5.36 | 5.45 | 5.53 | 5.60 | 5.67 | 5.73 | 5.78 | 5.84 | 5.89 | 5.93 | 5.97 | 6.01 |
| 120                  | 3.702 | 4.20 | 4.50 | 4.71 | 4.87 | 5.01 | 5.12 | 5.21 | 5.30 | 5.37 | 5.44 | 5.50 | 5.56 | 5.61 | 5.66 | 5.71 | 5.75 | 5.79 | 5.83 |
| $\infty$             | 3.643 | 4.12 | 4.40 | 4.60 | 4.76 | 4.88 | 4.99 | 5.08 | 5.16 | 5.23 | 5.29 | 5.35 | 5.40 | 5.45 | 5.49 | 5.54 | 5.57 | 5.61 | 5.65 |

Source: Sachs, 1972

**Table 10 Critical values of  $K$  for the Link–Wallace test**  
**Level of significance  $\alpha = 0.05$**

| $n \backslash K$ | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   | 13   | 14   | 15   | 16    | 17    | 18    | 19    | 20    | 30    | 40    | 50    |
|------------------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|-------|-------|-------|-------|-------|-------|-------|-------|
| 2                | 3.43 | 2.35 | 1.74 | 1.39 | 1.15 | 0.99 | 0.87 | 0.77 | 0.70 | 0.63 | 0.58 | 0.54 | 0.50 | 0.47 | 0.443 | 0.418 | 0.396 | 0.376 | 0.158 | 0.245 | 0.187 | 0.151 |
| 3                | 1.90 | 1.44 | 1.14 | 0.94 | 0.80 | 0.70 | 0.62 | 0.56 | 0.51 | 0.47 | 0.43 | 0.40 | 0.38 | 0.35 | 0.335 | 0.317 | 0.301 | 0.287 | 0.274 | 0.189 | 0.146 | 0.119 |
| 4                | 1.62 | 1.25 | 1.01 | 0.84 | 0.72 | 0.63 | 0.57 | 0.51 | 0.47 | 0.43 | 0.40 | 0.37 | 0.35 | 0.33 | 0.310 | 0.294 | 0.279 | 0.266 | 0.254 | 0.177 | 0.136 | 0.112 |
| 5                | 1.53 | 1.19 | 0.96 | 0.81 | 0.70 | 0.61 | 0.55 | 0.50 | 0.45 | 0.42 | 0.39 | 0.36 | 0.34 | 0.32 | 0.303 | 0.287 | 0.273 | 0.260 | 0.249 | 0.173 | 0.134 | 0.110 |
| 6                | 1.50 | 1.17 | 0.95 | 0.80 | 0.69 | 0.61 | 0.55 | 0.49 | 0.45 | 0.42 | 0.39 | 0.36 | 0.34 | 0.32 | 0.302 | 0.287 | 0.273 | 0.260 | 0.249 | 0.174 | 0.135 | 0.110 |
| 7                | 1.49 | 1.17 | 0.95 | 0.80 | 0.69 | 0.61 | 0.55 | 0.50 | 0.45 | 0.42 | 0.39 | 0.36 | 0.34 | 0.32 | 0.304 | 0.289 | 0.275 | 0.262 | 0.251 | 0.175 | 0.136 | 0.111 |
| 8                | 1.49 | 1.18 | 0.96 | 0.81 | 0.70 | 0.62 | 0.55 | 0.50 | 0.46 | 0.42 | 0.39 | 0.37 | 0.35 | 0.33 | 0.308 | 0.292 | 0.278 | 0.265 | 0.254 | 0.178 | 0.138 | 0.113 |
| 9                | 1.50 | 1.19 | 0.97 | 0.82 | 0.71 | 0.62 | 0.56 | 0.51 | 0.47 | 0.43 | 0.40 | 0.37 | 0.35 | 0.33 | 0.312 | 0.297 | 0.282 | 0.269 | 0.258 | 0.180 | 0.140 | 0.115 |
| 10               | 1.52 | 1.20 | 0.98 | 0.83 | 0.72 | 0.63 | 0.57 | 0.52 | 0.47 | 0.44 | 0.41 | 0.38 | 0.36 | 0.34 | 0.317 | 0.301 | 0.287 | 0.274 | 0.262 | 0.183 | 0.142 | 0.117 |
| 11               | 1.54 | 1.22 | 0.99 | 0.84 | 0.73 | 0.64 | 0.58 | 0.52 | 0.48 | 0.44 | 0.41 | 0.38 | 0.36 | 0.34 | 0.322 | 0.306 | 0.291 | 0.278 | 0.266 | 0.186 | 0.145 | 0.119 |
| 12               | 1.56 | 1.23 | 1.01 | 0.85 | 0.74 | 0.65 | 0.58 | 0.53 | 0.49 | 0.45 | 0.42 | 0.39 | 0.37 | 0.35 | 0.327 | 0.311 | 0.296 | 0.282 | 0.270 | 0.189 | 0.147 | 0.121 |
| 13               | 1.58 | 1.25 | 1.02 | 0.86 | 0.75 | 0.66 | 0.59 | 0.54 | 0.49 | 0.46 | 0.42 | 0.40 | 0.37 | 0.35 | 0.332 | 0.316 | 0.300 | 0.287 | 0.274 | 0.192 | 0.149 | 0.122 |
| 14               | 1.60 | 1.26 | 1.03 | 0.87 | 0.76 | 0.67 | 0.60 | 0.55 | 0.50 | 0.46 | 0.43 | 0.40 | 0.38 | 0.36 | 0.337 | 0.320 | 0.305 | 0.291 | 0.279 | 0.195 | 0.152 | 0.124 |
| 15               | 1.62 | 1.28 | 1.05 | 0.89 | 0.77 | 0.68 | 0.61 | 0.55 | 0.51 | 0.47 | 0.44 | 0.41 | 0.38 | 0.36 | 0.342 | 0.325 | 0.310 | 0.295 | 0.283 | 0.198 | 0.154 | 0.126 |
| 16               | 1.64 | 1.30 | 1.06 | 0.90 | 0.78 | 0.69 | 0.62 | 0.56 | 0.52 | 0.48 | 0.44 | 0.41 | 0.39 | 0.37 | 0.348 | 0.330 | 0.314 | 0.300 | 0.287 | 0.201 | 0.156 | 0.128 |
| 17               | 1.66 | 1.32 | 1.08 | 0.91 | 0.79 | 0.70 | 0.63 | 0.57 | 0.52 | 0.48 | 0.45 | 0.42 | 0.39 | 0.37 | 0.352 | 0.335 | 0.319 | 0.304 | 0.291 | 0.204 | 0.158 | 0.130 |
| 18               | 1.68 | 1.33 | 1.09 | 0.92 | 0.80 | 0.71 | 0.64 | 0.58 | 0.53 | 0.49 | 0.46 | 0.43 | 0.40 | 0.38 | 0.357 | 0.339 | 0.323 | 0.308 | 0.295 | 0.207 | 0.161 | 0.132 |
| 19               | 1.70 | 1.35 | 1.10 | 0.93 | 0.81 | 0.72 | 0.64 | 0.59 | 0.54 | 0.50 | 0.46 | 0.43 | 0.41 | 0.38 | 0.362 | 0.344 | 0.327 | 0.312 | 0.299 | 0.210 | 0.163 | 0.134 |
| 20               | 1.72 | 1.36 | 1.12 | 0.95 | 0.82 | 0.73 | 0.65 | 0.59 | 0.54 | 0.50 | 0.47 | 0.44 | 0.41 | 0.39 | 0.367 | 0.348 | 0.332 | 0.317 | 0.303 | 0.212 | 0.165 | 0.135 |
| 30               | 1.92 | 1.52 | 1.24 | 1.05 | 0.91 | 0.81 | 0.73 | 0.66 | 0.60 | 0.56 | 0.52 | 0.49 | 0.46 | 0.43 | 0.408 | 0.387 | 0.369 | 0.352 | 0.337 | 0.237 | 0.184 | 0.151 |
| 40               | 2.08 | 1.66 | 1.35 | 1.14 | 0.99 | 0.88 | 0.79 | 0.72 | 0.66 | 0.61 | 0.57 | 0.53 | 0.50 | 0.47 | 0.444 | 0.422 | 0.402 | 0.384 | 0.367 | 0.258 | 0.201 | 0.165 |
| 50               | 2.23 | 1.77 | 1.45 | 1.22 | 1.06 | 0.94 | 0.85 | 0.77 | 0.71 | 0.65 | 0.61 | 0.57 | 0.53 | 0.50 | 0.476 | 0.453 | 0.431 | 0.412 | 0.394 | 0.277 | 0.216 | 0.177 |
| 100              | 2.81 | 2.23 | 1.83 | 1.55 | 1.34 | 1.19 | 1.07 | 0.97 | 0.89 | 0.83 | 0.77 | 0.72 | 0.67 | 0.64 | 0.60  | 0.573 | 0.546 | 0.521 | 0.499 | 0.351 | 0.273 | 0.224 |
| 200              | 3.61 | 2.88 | 2.35 | 1.99 | 1.73 | 1.53 | 1.38 | 1.25 | 1.15 | 1.06 | 0.99 | 0.93 | 0.87 | 0.82 | 0.78  | 0.74  | 0.70  | 0.67  | 0.64  | 0.454 | 0.353 | 0.290 |
| 500              | 5.15 | 4.10 | 3.35 | 2.84 | 2.47 | 2.19 | 1.97 | 1.79 | 1.64 | 1.52 | 1.42 | 1.32 | 1.24 | 1.17 | 1.11  | 1.06  | 1.01  | 0.96  | 0.92  | 0.65  | 0.504 | 0.414 |
| 1000             | 6.81 | 5.43 | 4.44 | 3.77 | 3.28 | 2.90 | 2.61 | 2.37 | 2.18 | 2.22 | 1.88 | 1.76 | 1.65 | 1.56 | 1.47  | 1.40  | 1.33  | 1.27  | 1.22  | 0.86  | 0.669 | 0.549 |

Table 10 continued

Level of significance  $\alpha = 0.01$ 

| $V_2 \backslash K$ | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   | 13   | 14   | 15   | 16   | 17   | 18    | 19    | 20    | 30    | 40    | 50    |
|--------------------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|-------|-------|-------|-------|-------|-------|
| 2                  | 7.92 | 4.32 | 2.84 | 2.10 | 1.66 | 1.38 | 1.17 | 1.02 | 0.91 | 0.82 | 0.74 | 0.68 | 0.63 | 0.58 | 0.54 | 0.51 | 0.480 | 0.454 | 0.430 | 0.285 | 0.214 | 0.172 |
| 3                  | 3.14 | 2.12 | 1.57 | 1.25 | 1.04 | 0.89 | 0.78 | 0.69 | 0.62 | 1.57 | 0.52 | 0.48 | 0.45 | 0.42 | 0.39 | 0.37 | 0.352 | 0.334 | 0.318 | 0.217 | 0.165 | 0.134 |
| 4                  | 2.48 | 1.74 | 1.33 | 1.08 | 0.91 | 0.78 | 0.69 | 0.62 | 0.56 | 0.51 | 0.47 | 0.44 | 0.41 | 0.38 | 0.36 | 0.34 | 0.323 | 0.307 | 0.293 | 0.200 | 0.153 | 0.125 |
| 5                  | 2.24 | 1.60 | 1.24 | 1.02 | 0.86 | 0.75 | 0.66 | 0.59 | 0.54 | 0.49 | 0.46 | 0.42 | 0.40 | 0.37 | 0.35 | 0.33 | 0.314 | 0.299 | 0.285 | 0.196 | 0.151 | 0.123 |
| 6                  | 2.14 | 1.55 | 1.21 | 0.99 | 0.85 | 0.74 | 0.65 | 0.59 | 0.53 | 0.49 | 0.45 | 0.42 | 0.39 | 0.37 | 0.35 | 0.33 | 0.313 | 0.298 | 0.284 | 0.196 | 0.151 | 0.123 |
| 7                  | 2.10 | 1.53 | 1.20 | 0.99 | 0.84 | 0.73 | 0.65 | 0.59 | 0.53 | 0.49 | 0.45 | 0.42 | 0.39 | 0.37 | 0.35 | 0.33 | 0.314 | 0.299 | 0.286 | 0.198 | 0.152 | 0.124 |
| 8                  | 2.09 | 1.53 | 1.20 | 0.99 | 0.85 | 0.74 | 0.66 | 0.59 | 0.54 | 0.49 | 0.46 | 0.43 | 0.40 | 0.37 | 0.35 | 0.33 | 0.318 | 0.303 | 0.289 | 0.200 | 0.154 | 0.126 |
| 9                  | 2.09 | 1.54 | 1.21 | 1.00 | 0.85 | 0.75 | 0.66 | 0.60 | 0.54 | 0.50 | 0.46 | 0.43 | 0.40 | 0.38 | 0.36 | 0.34 | 0.322 | 0.307 | 0.293 | 0.200 | 0.156 | 0.127 |
| 10                 | 2.10 | 1.55 | 1.22 | 1.01 | 0.86 | 0.76 | 0.67 | 0.61 | 0.55 | 0.51 | 0.47 | 0.44 | 0.41 | 0.38 | 0.36 | 0.34 | 0.327 | 0.311 | 0.297 | 0.206 | 0.159 | 0.129 |
| 11                 | 2.11 | 1.56 | 1.23 | 1.02 | 0.87 | 0.76 | 0.68 | 0.61 | 0.56 | 0.51 | 0.48 | 0.44 | 0.42 | 0.39 | 0.37 | 0.35 | 0.332 | 0.316 | 0.302 | 0.209 | 0.161 | 0.132 |
| 12                 | 2.13 | 1.58 | 1.25 | 1.04 | 0.89 | 0.78 | 0.69 | 0.62 | 0.57 | 0.52 | 0.48 | 0.45 | 0.42 | 0.40 | 0.37 | 0.35 | 0.337 | 0.321 | 0.306 | 0.213 | 0.164 | 0.134 |
| 13                 | 2.15 | 1.60 | 1.26 | 1.05 | 0.90 | 0.79 | 0.70 | 0.63 | 0.58 | 0.53 | 0.49 | 0.46 | 0.43 | 0.40 | 0.38 | 0.36 | 0.342 | 0.326 | 0.311 | 0.216 | 0.166 | 0.136 |
| 14                 | 2.18 | 1.62 | 1.28 | 1.06 | 0.91 | 0.80 | 0.71 | 0.64 | 0.58 | 0.54 | 0.50 | 0.46 | 0.43 | 0.41 | 0.39 | 0.36 | 0.347 | 0.330 | 0.316 | 0.219 | 0.169 | 0.138 |
| 15                 | 2.20 | 1.63 | 1.30 | 1.08 | 0.92 | 0.81 | 0.72 | 0.65 | 0.59 | 0.54 | 0.50 | 0.47 | 0.44 | 0.41 | 0.39 | 0.37 | 0.352 | 0.335 | 0.320 | 0.222 | 0.171 | 0.140 |
| 16                 | 2.22 | 1.65 | 1.31 | 1.09 | 0.93 | 0.82 | 0.73 | 0.66 | 0.60 | 0.55 | 0.51 | 0.48 | 0.45 | 0.42 | 0.40 | 0.38 | 0.357 | 0.340 | 0.325 | 0.226 | 0.174 | 0.142 |
| 17                 | 2.25 | 1.67 | 1.33 | 1.10 | 0.95 | 0.83 | 0.74 | 0.67 | 0.61 | 0.56 | 0.52 | 0.48 | 0.45 | 0.43 | 0.40 | 0.38 | 0.362 | 0.345 | 0.329 | 0.229 | 0.176 | 0.144 |
| 18                 | 2.27 | 1.69 | 1.34 | 1.12 | 0.96 | 0.84 | 0.75 | 0.68 | 0.62 | 0.57 | 0.53 | 0.49 | 0.46 | 0.43 | 0.41 | 0.39 | 0.367 | 0.350 | 0.334 | 0.232 | 0.179 | 0.146 |
| 19                 | 2.30 | 1.71 | 1.36 | 1.13 | 0.97 | 0.85 | 0.76 | 0.68 | 0.62 | 0.57 | 0.53 | 0.50 | 0.46 | 0.44 | 0.41 | 0.39 | 0.372 | 0.354 | 0.338 | 0.235 | 0.181 | 0.148 |
| 20                 | 2.32 | 1.73 | 1.38 | 1.14 | 0.98 | 0.86 | 0.77 | 0.69 | 0.63 | 0.58 | 0.54 | 0.50 | 0.47 | 0.44 | 0.42 | 0.40 | 0.376 | 0.359 | 0.343 | 0.238 | 0.184 | 0.150 |
| 30                 | 2.59 | 1.95 | 1.54 | 1.27 | 1.09 | 0.96 | 0.85 | 0.77 | 0.70 | 0.65 | 0.60 | 0.56 | 0.52 | 0.49 | 0.46 | 0.44 | 0.419 | 0.399 | 0.381 | 0.266 | 0.205 | 0.168 |
| 40                 | 2.80 | 2.11 | 1.66 | 1.38 | 1.18 | 1.04 | 0.93 | 0.84 | 0.76 | 0.70 | 0.65 | 0.61 | 0.57 | 0.54 | 0.51 | 0.48 | 0.456 | 0.435 | 0.415 | 0.289 | 0.223 | 0.183 |
| 50                 | 2.99 | 2.25 | 1.78 | 1.48 | 1.27 | 1.11 | 0.99 | 0.90 | 0.82 | 0.75 | 0.70 | 0.65 | 0.61 | 0.57 | 0.54 | 0.51 | 0.489 | 0.466 | 0.446 | 0.310 | 0.240 | 0.196 |
| 100                | 3.74 | 2.83 | 2.24 | 1.86 | 1.60 | 1.40 | 1.25 | 1.13 | 1.03 | 0.95 | 0.88 | 0.82 | 0.77 | 0.73 | 0.69 | 0.65 | 0.62  | 0.590 | 0.564 | 0.539 | 0.304 | 0.248 |
| 200                | 4.79 | 3.63 | 2.88 | 2.39 | 2.06 | 1.81 | 1.61 | 1.46 | 1.33 | 1.23 | 1.14 | 1.06 | 0.99 | 0.94 | 0.88 | 0.84 | 0.80  | 0.76  | 0.73  | 0.507 | 0.392 | 0.320 |
| 500                | 6.81 | 5.16 | 4.10 | 3.41 | 2.93 | 2.58 | 2.30 | 2.08 | 1.90 | 1.75 | 1.62 | 1.52 | 1.42 | 1.34 | 1.26 | 1.20 | 1.14  | 1.09  | 1.04  | 0.73  | 0.560 | 0.458 |
| 1000               | 9.01 | 6.83 | 5.42 | 4.52 | 3.88 | 3.41 | 3.05 | 2.76 | 2.52 | 2.32 | 2.15 | 2.01 | 1.88 | 1.77 | 1.68 | 1.59 | 1.51  | 1.44  | 1.38  | 0.96  | 0.743 | 0.608 |

Source: Sachs, 1972

**Table 11 Critical values for the Dunnett test**  
**Level of significance  $\alpha = 0.01$**

| $\nu \backslash K$ | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    |
|--------------------|------|------|------|------|------|------|------|------|
| 5                  | 3.90 | 4.21 | 4.43 | 4.60 | 4.73 | 4.85 | 4.94 | 5.03 |
| 6                  | 3.61 | 3.88 | 4.07 | 4.21 | 4.33 | 4.43 | 4.51 | 4.59 |
| 7                  | 3.42 | 3.66 | 3.83 | 3.96 | 4.07 | 4.15 | 4.23 | 4.30 |
| 8                  | 3.29 | 3.51 | 3.67 | 3.79 | 3.88 | 3.96 | 4.03 | 4.09 |
| 9                  | 3.19 | 3.40 | 3.55 | 3.66 | 3.75 | 3.82 | 3.89 | 3.94 |
| 10                 | 3.11 | 3.31 | 3.45 | 3.56 | 3.64 | 3.71 | 3.78 | 3.83 |
| 11                 | 3.06 | 3.25 | 3.38 | 3.48 | 3.56 | 3.63 | 3.69 | 3.74 |
| 12                 | 3.01 | 3.19 | 3.32 | 3.42 | 3.50 | 3.56 | 3.62 | 3.67 |
| 13                 | 2.97 | 3.15 | 3.27 | 3.37 | 3.44 | 3.51 | 3.56 | 3.61 |
| 14                 | 2.94 | 3.11 | 3.23 | 3.32 | 3.40 | 3.46 | 3.51 | 3.56 |
| 15                 | 2.91 | 3.08 | 3.20 | 3.29 | 3.36 | 3.42 | 3.47 | 3.52 |
| 16                 | 2.88 | 3.05 | 3.17 | 3.26 | 3.33 | 3.39 | 3.44 | 3.48 |
| 17                 | 2.86 | 3.03 | 3.14 | 3.23 | 3.30 | 3.36 | 3.41 | 3.43 |
| 18                 | 2.84 | 3.01 | 3.12 | 3.21 | 3.27 | 3.33 | 3.38 | 3.42 |
| 20                 | 2.81 | 2.97 | 3.08 | 3.17 | 3.23 | 3.29 | 3.34 | 3.38 |
| 24                 | 2.77 | 2.92 | 3.03 | 3.11 | 3.17 | 3.22 | 3.27 | 3.31 |
| 30                 | 2.72 | 2.87 | 2.97 | 3.05 | 3.11 | 3.16 | 3.21 | 3.24 |
| 40                 | 2.68 | 2.82 | 2.92 | 2.99 | 3.05 | 3.10 | 3.14 | 3.18 |
| 60                 | 2.64 | 2.78 | 2.87 | 2.94 | 3.00 | 3.04 | 3.08 | 3.12 |
| 120                | 2.60 | 2.73 | 2.82 | 2.89 | 2.94 | 2.99 | 3.03 | 3.06 |
| $\infty$           | 2.56 | 2.68 | 2.77 | 2.84 | 2.89 | 2.93 | 2.97 | 3.00 |

**Level of significance  $\alpha = 0.05$**

| $\nu \backslash K$ | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    |
|--------------------|------|------|------|------|------|------|------|------|
| 5                  | 2.44 | 2.68 | 2.85 | 2.98 | 3.08 | 3.16 | 3.24 | 3.30 |
| 6                  | 2.34 | 2.56 | 2.71 | 2.83 | 2.92 | 3.00 | 3.07 | 3.12 |
| 7                  | 2.27 | 2.48 | 2.62 | 2.73 | 2.82 | 2.89 | 2.95 | 3.01 |
| 8                  | 2.22 | 2.42 | 2.55 | 2.66 | 2.74 | 2.81 | 2.87 | 2.92 |
| 9                  | 2.18 | 2.37 | 2.50 | 2.60 | 2.68 | 2.75 | 2.81 | 2.86 |
| 10                 | 2.15 | 2.34 | 2.47 | 2.56 | 2.64 | 2.70 | 2.76 | 2.81 |
| 11                 | 2.13 | 2.31 | 2.44 | 2.53 | 2.60 | 2.67 | 2.72 | 2.77 |
| 12                 | 2.11 | 2.29 | 2.41 | 2.50 | 2.58 | 2.64 | 2.69 | 2.74 |
| 13                 | 2.09 | 2.27 | 2.39 | 2.48 | 2.55 | 2.61 | 2.66 | 2.71 |
| 14                 | 2.08 | 2.25 | 2.37 | 2.46 | 2.53 | 2.59 | 2.64 | 2.69 |
| 15                 | 2.07 | 2.24 | 2.36 | 2.44 | 2.51 | 2.57 | 2.62 | 2.67 |
| 16                 | 2.06 | 2.23 | 2.34 | 2.43 | 2.50 | 2.56 | 2.61 | 2.65 |
| 17                 | 2.05 | 2.22 | 2.33 | 2.42 | 2.49 | 2.54 | 2.59 | 2.64 |
| 18                 | 2.04 | 2.21 | 2.32 | 2.41 | 2.48 | 2.53 | 2.58 | 2.62 |
| 20                 | 2.03 | 2.19 | 2.30 | 2.39 | 2.46 | 2.51 | 2.56 | 2.60 |
| 24                 | 2.01 | 2.17 | 2.28 | 2.36 | 2.43 | 2.48 | 2.53 | 2.57 |
| 30                 | 1.99 | 2.15 | 2.25 | 2.33 | 2.40 | 2.45 | 2.50 | 2.54 |
| 40                 | 1.97 | 2.13 | 2.23 | 2.31 | 2.37 | 2.42 | 2.47 | 2.51 |
| 60                 | 1.95 | 2.10 | 2.21 | 2.28 | 2.35 | 2.39 | 2.44 | 2.48 |
| 120                | 1.93 | 2.08 | 2.18 | 2.26 | 2.32 | 2.37 | 2.41 | 2.45 |
| $\infty$           | 1.92 | 2.06 | 2.16 | 2.23 | 2.29 | 2.34 | 2.38 | 2.46 |

Source: De Jonge, 1963–4

**Table 12 Critical values of  $M$  for the Bartlett test**

$C =$  constant values.

| Level of significance $\alpha = 0.01$ |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
|---------------------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $K \backslash C_i$                    | 0.0   | 0.5   | 1.0   | 1.5   | 2.0   | 2.5   | 3.0   | 3.5   | 4.0   | 4.5   | 5.0   | 6.0   | 7.0   | 8.0   | 9.0   | 10.0  |
| 3                                     | 9.21  | 9.92  | 10.47 | 10.78 | 10.81 | 10.50 | 9.83  | —     | —     | —     | —     | —     | —     | —     | —     | —     |
| 4                                     | 11.34 | 11.95 | 12.46 | 12.86 | 13.11 | 13.18 | 13.03 | 12.65 | 12.03 | —     | —     | —     | —     | —     | —     | —     |
| 5                                     | 13.28 | 13.81 | 14.30 | 14.71 | 15.03 | 15.25 | 15.34 | 15.28 | 15.06 | 14.66 | 14.07 | —     | —     | —     | —     | —     |
| 6                                     | 15.09 | 15.58 | 16.03 | 16.44 | 16.79 | 17.07 | 17.27 | 17.37 | 17.37 | 17.24 | 16.98 | 16.03 | —     | —     | —     | —     |
| 7                                     | 16.81 | 17.27 | 17.70 | 18.10 | 18.46 | 18.77 | 19.02 | 19.21 | 19.32 | 19.35 | 19.28 | 18.84 | 17.92 | —     | —     | —     |
| 8                                     | 18.48 | 18.91 | 19.32 | 19.71 | 20.07 | 20.39 | 20.67 | 20.90 | 21.08 | 21.20 | 21.35 | 21.13 | 20.64 | 19.76 | —     | —     |
| 9                                     | 20.09 | 20.50 | 20.90 | 21.28 | 21.64 | 21.97 | 22.26 | 22.52 | 22.74 | 22.91 | 23.03 | 23.10 | 22.91 | 22.41 | 21.56 | —     |
| 10                                    | 21.67 | 22.06 | 22.45 | 22.82 | 23.17 | 23.50 | 23.80 | 24.08 | 24.32 | 24.52 | 24.69 | 24.90 | 24.90 | 24.66 | 24.15 | 23.33 |
| 11                                    | 23.21 | 23.59 | 23.97 | 24.33 | 24.67 | 25.00 | 25.31 | 25.59 | 25.85 | 26.08 | 26.28 | 26.57 | 26.70 | 26.65 | 26.38 | 25.86 |
| 12                                    | 24.72 | 25.10 | 25.46 | 25.81 | 26.15 | 26.48 | 26.79 | 27.08 | 27.35 | 27.59 | 27.81 | 28.16 | 28.39 | 28.46 | 28.37 | 28.07 |
| 13                                    | 26.22 | 26.58 | 26.93 | 27.28 | 27.62 | 27.94 | 28.25 | 28.54 | 28.81 | 29.07 | 29.30 | 29.70 | 29.99 | 30.16 | 30.19 | 30.06 |
| 14                                    | 27.69 | 28.04 | 28.39 | 28.73 | 29.06 | 29.38 | 29.69 | 29.98 | 30.26 | 30.52 | 30.77 | 31.19 | 31.53 | 31.77 | 31.89 | 31.88 |
| 15                                    | 29.14 | 29.39 | 29.83 | 30.16 | 30.49 | 30.80 | 30.11 | 31.40 | 31.68 | 31.95 | 32.20 | 32.66 | 33.03 | 33.32 | 33.51 | 33.59 |

Table 12 continued

Level of significance  $\alpha = 0.05$ 

| $K \backslash C_i$ | 0.0   | 0.5   | 1.0   | 1.5   | 2.0   | 2.5   | 3.0   | 3.5   | 4.0   | 4.5   | 5.0   | 6.0   | 7.0   | 8.0   | 9.0   | 10.0  |
|--------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 3                  | 5.99  | 6.47  | 5.89  | 7.20  | 7.38  | 7.39  | 7.22  | —     | —     | —     | —     | —     | —     | —     | —     | —     |
| 4                  | 7.81  | 8.24  | 8.63  | 8.96  | 9.21  | 9.38  | 9.43  | 9.37  | 9.18  | —     | —     | —     | —     | —     | —     | —     |
| 5                  | 9.49  | 9.88  | 10.24 | 10.57 | 10.86 | 11.08 | 11.24 | 11.32 | 11.31 | 11.21 | 11.02 | —     | —     | —     | —     | —     |
| 6                  | 11.07 | 11.43 | 11.78 | 12.11 | 12.40 | 12.65 | 12.86 | 13.01 | 13.11 | 13.14 | 13.10 | 12.78 | —     | —     | —     | —     |
| 7                  | 12.59 | 12.94 | 13.27 | 13.59 | 13.88 | 14.15 | 14.38 | 14.58 | 14.73 | 14.83 | 14.88 | 14.81 | 14.49 | —     | —     | —     |
| 8                  | 14.07 | 14.40 | 14.72 | 15.03 | 15.32 | 15.60 | 15.84 | 16.06 | 16.25 | 16.40 | 16.51 | 16.60 | 16.49 | 16.16 | —     | —     |
| 9                  | 15.51 | 15.83 | 16.14 | 16.44 | 16.73 | 17.01 | 17.26 | 17.49 | 17.70 | 17.88 | 18.03 | 16.22 | 18.26 | 18.12 | 17.79 | —     |
| 10                 | 16.92 | 17.23 | 17.54 | 17.83 | 18.12 | 18.39 | 18.65 | 18.89 | 19.11 | 19.31 | 19.48 | 19.75 | 19.89 | 19.89 | 19.73 | 19.40 |
| 11                 | 18.31 | 18.61 | 18.91 | 19.20 | 19.48 | 19.76 | 20.02 | 20.26 | 20.49 | 20.70 | 20.89 | 21.21 | 21.42 | 21.52 | 21.49 | 21.32 |
| 12                 | 19.68 | 19.97 | 20.26 | 20.55 | 20.83 | 21.10 | 21.36 | 21.61 | 21.84 | 22.06 | 22.27 | 22.62 | 22.88 | 23.06 | 23.12 | 23.07 |
| 13                 | 21.03 | 21.32 | 21.60 | 21.89 | 22.16 | 22.43 | 22.69 | 22.94 | 23.18 | 23.40 | 23.62 | 23.99 | 24.30 | 24.53 | 24.66 | 24.70 |
| 14                 | 22.36 | 22.65 | 22.93 | 23.21 | 23.48 | 23.75 | 24.01 | 24.26 | 24.50 | 24.73 | 24.95 | 25.34 | 25.68 | 25.95 | 26.14 | 26.25 |
| 15                 | 23.68 | 23.97 | 24.24 | 24.52 | 24.79 | 25.05 | 25.31 | 25.56 | 25.80 | 26.04 | 26.26 | 26.67 | 27.03 | 27.33 | 27.56 | 27.73 |

Source: Merrington and Thompson, 1946



**Table 13 Critical values for the Hartley test (right-sided)**

**Level of significance  $\alpha = 0.01$**

| $n - 1 \backslash K$ | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   |
|----------------------|------|------|------|------|------|------|------|------|------|------|------|
| 2                    | 199  | 448  | 729  | 1036 | 1362 | 1705 | 2063 | 2432 | 2813 | 3204 | 3605 |
| 3                    | 47.5 | 85   | 120  | 151  | 184  | 216* | 249* | 281* | 310* | 337* | 361* |
| 4                    | 23.2 | 37   | 49   | 59   | 69   | 79   | 89   | 97   | 106  | 113  | 120  |
| 5                    | 14.9 | 22   | 28   | 33   | 38   | 42   | 46   | 50   | 54   | 57   | 60   |
| 6                    | 11.1 | 15.5 | 19.1 | 22   | 25   | 27   | 30   | 32   | 34   | 36   | 37   |
| 7                    | 8.89 | 12.1 | 14.5 | 16.5 | 18.4 | 20   | 22   | 23   | 24   | 26   | 27   |
| 8                    | 7.50 | 9.9  | 11.7 | 13.2 | 14.5 | 15.8 | 16.9 | 17.9 | 18.9 | 19.8 | 21   |
| 9                    | 6.54 | 8.5  | 9.9  | 11.1 | 12.1 | 13.1 | 13.9 | 14.7 | 15.3 | 16.0 | 16.6 |
| 10                   | 5.85 | 7.4  | 8.6  | 9.6  | 10.4 | 11.1 | 11.8 | 12.4 | 12.9 | 13.4 | 13.9 |
| 12                   | 4.91 | 6.1  | 6.9  | 7.6  | 8.2  | 8.7  | 9.1  | 9.5  | 9.9  | 10.2 | 10.6 |
| 15                   | 4.07 | 4.9  | 5.5  | 6.0  | 6.4  | 6.7  | 7.1  | 7.3  | 7.5  | 7.8  | 8.0  |
| 20                   | 3.32 | 3.8  | 4.3  | 4.6  | 4.9  | 5.1  | 5.3  | 5.5  | 5.6  | 5.8  | 5.9  |
| 30                   | 2.63 | 3.0  | 3.3  | 3.4  | 3.6  | 3.7  | 3.8  | 3.9  | 4.0  | 4.1  | 4.2  |
| 60                   | 1.96 | 2.2  | 2.3  | 2.4  | 2.4  | 2.5  | 2.5  | 2.6  | 2.6  | 2.7  | 2.7  |
| $\infty$             | 1.0  | 1.0  | 1.0  | 1.0  | 1.0  | 1.0  | 1.0  | 1.0  | 1.0  | 1.0  | 1.0  |

\* The unit digit is uncertain.

Table 13 continued

Level of significance  $\alpha = 0.05$ 

| $n - 1 \backslash K$ | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   |
|----------------------|------|------|------|------|------|------|------|------|------|------|------|
| 2                    | 39.0 | 87.5 | 142  | 202  | 266  | 333  | 403  | 475  | 550  | 626  | 704  |
| 3                    | 15.4 | 27.8 | 39.2 | 50.7 | 62.0 | 72.9 | 83.5 | 93.9 | 104  | 114  | 124  |
| 4                    | 9.60 | 15.5 | 20.6 | 25.2 | 29.5 | 33.6 | 37.5 | 41.1 | 44.6 | 48.0 | 51.4 |
| 5                    | 7.15 | 10.8 | 13.7 | 16.3 | 18.7 | 20.8 | 22.9 | 24.7 | 26.5 | 28.2 | 29.9 |
| 6                    | 5.82 | 8.38 | 10.4 | 12.1 | 13.7 | 15.0 | 16.3 | 17.5 | 18.6 | 19.7 | 20.7 |
| 7                    | 4.99 | 6.94 | 8.44 | 9.70 | 10.8 | 11.8 | 12.7 | 13.5 | 14.3 | 15.1 | 15.8 |
| 8                    | 4.43 | 6.00 | 7.18 | 8.12 | 9.03 | 9.78 | 10.5 | 11.1 | 11.7 | 12.2 | 12.7 |
| 9                    | 4.03 | 5.34 | 6.31 | 7.11 | 7.80 | 8.41 | 8.95 | 9.45 | 9.91 | 10.3 | 10.7 |
| 10                   | 3.72 | 4.85 | 5.67 | 6.34 | 6.92 | 7.42 | 7.87 | 8.28 | 8.66 | 9.01 | 9.34 |
| 12                   | 3.28 | 4.16 | 4.79 | 5.30 | 5.72 | 6.09 | 6.42 | 6.72 | 7.00 | 7.25 | 7.48 |
| 15                   | 2.86 | 3.54 | 4.01 | 4.37 | 4.68 | 4.95 | 5.19 | 5.40 | 5.59 | 5.77 | 5.93 |
| 20                   | 2.46 | 2.95 | 3.29 | 3.54 | 3.76 | 3.94 | 4.10 | 4.24 | 4.37 | 4.49 | 4.59 |
| 30                   | 2.07 | 2.40 | 2.61 | 2.78 | 2.91 | 3.02 | 3.12 | 3.21 | 3.29 | 3.36 | 3.39 |
| 60                   | 1.67 | 1.85 | 1.96 | 2.04 | 2.11 | 2.17 | 2.22 | 2.26 | 2.30 | 2.33 | 2.36 |
| $\infty$             | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

Source: De Jonge, 1963–4

**Table 14 Critical values of  $w/s$  for the normality test**

Columns  $a$  denote the lower boundaries or the left-sided critical values.  
Columns  $b$  denote the upper boundaries or the right-sided critical values.

| $n$  | Level of significance $\alpha$ |       |       |       |       |       |       |       |       |       |       |       |
|------|--------------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|      | 0.000                          |       | 0.005 |       | 0.01  |       | 0.025 |       | 0.05  |       | 0.10  |       |
|      | $a$                            | $b$   | $a$   | $b$   | $a$   | $b$   | $a$   | $b$   | $a$   | $b$   | $a$   | $b$   |
| 3    | 1.732                          | 2.000 | 1.735 | 2.000 | 1.737 | 2.000 | 1.745 | 2.000 | 1.758 | 1.999 | 1.782 | 1.997 |
| 4    | 1.732                          | 2.449 | 1.82  | 2.447 | 1.87  | 2.445 | 1.93  | 2.439 | 1.98  | 2.429 | 2.04  | 2.409 |
| 5    | 1.826                          | 2.828 | 1.98  | 2.813 | 2.02  | 2.803 | 2.09  | 2.782 | 2.15  | 2.753 | 2.22  | 2.712 |
| 6    | 1.826                          | 3.162 | 2.11  | 3.115 | 2.15  | 3.095 | 2.22  | 3.056 | 2.28  | 3.012 | 2.37  | 2.949 |
| 7    | 1.871                          | 3.464 | 2.22  | 3.369 | 2.26  | 3.338 | 2.33  | 3.282 | 2.40  | 3.222 | 2.49  | 3.143 |
| 8    | 1.871                          | 3.742 | 2.31  | 3.585 | 2.35  | 3.543 | 2.43  | 3.471 | 2.50  | 3.399 | 2.59  | 3.308 |
| 9    | 1.897                          | 4.000 | 2.39  | 3.772 | 2.44  | 3.720 | 2.51  | 3.634 | 2.59  | 3.552 | 2.68  | 3.449 |
| 10   | 1.897                          | 4.243 | 2.46  | 3.935 | 2.51  | 3.875 | 2.59  | 3.777 | 2.67  | 3.685 | 2.76  | 3.57  |
| 11   | 1.915                          | 4.472 | 2.53  | 4.079 | 2.58  | 4.012 | 2.66  | 3.903 | 2.74  | 3.80  | 2.84  | 3.68  |
| 12   | 1.915                          | 4.690 | 2.59  | 4.208 | 2.64  | 4.134 | 2.72  | 4.02  | 2.80  | 3.91  | 2.90  | 3.78  |
| 13   | 1.927                          | 4.899 | 2.64  | 4.325 | 2.70  | 4.244 | 2.78  | 4.12  | 2.86  | 4.00  | 2.96  | 3.87  |
| 14   | 1.927                          | 5.099 | 2.70  | 4.431 | 2.75  | 4.34  | 2.83  | 4.21  | 2.92  | 4.09  | 3.02  | 3.95  |
| 15   | 1.936                          | 5.292 | 2.74  | 4.53  | 2.80  | 4.44  | 2.88  | 4.29  | 2.97  | 4.17  | 3.07  | 4.02  |
| 16   | 1.936                          | 5.477 | 2.79  | 4.62  | 2.84  | 4.52  | 2.93  | 4.37  | 3.01  | 4.24  | 3.12  | 4.09  |
| 17   | 1.944                          | 5.657 | 2.83  | 4.70  | 2.88  | 4.60  | 2.97  | 4.44  | 3.06  | 4.31  | 3.17  | 4.15  |
| 18   | 1.944                          | 5.831 | 2.87  | 4.78  | 2.92  | 4.67  | 3.01  | 4.51  | 3.10  | 4.37  | 3.21  | 4.21  |
| 19   | 1.949                          | 6.000 | 2.90  | 4.85  | 2.96  | 4.74  | 3.05  | 4.56  | 3.14  | 4.43  | 3.25  | 4.27  |
| 20   | 1.949                          | 6.164 | 2.94  | 4.91  | 2.99  | 4.80  | 3.09  | 4.63  | 3.18  | 4.49  | 3.29  | 4.32  |
| 25   | 1.961                          | 6.93  | 3.09  | 5.19  | 3.15  | 5.06  | 3.24  | 4.87  | 3.34  | 4.71  | 3.45  | 4.53  |
| 30   | 1.966                          | 7.62  | 3.21  | 5.40  | 3.27  | 5.26  | 3.37  | 5.06  | 3.47  | 4.89  | 3.59  | 4.70  |
| 35   | 1.972                          | 8.25  | 3.32  | 5.57  | 3.38  | 5.42  | 3.48  | 5.21  | 3.58  | 5.04  | 3.70  | 4.84  |
| 40   | 1.975                          | 8.83  | 3.41  | 5.71  | 3.47  | 5.56  | 3.57  | 5.34  | 3.67  | 5.16  | 3.79  | 4.96  |
| 45   | 1.978                          | 9.38  | 3.49  | 5.83  | 3.55  | 5.67  | 3.66  | 5.45  | 3.75  | 5.26  | 3.88  | 5.06  |
| 50   | 1.980                          | 9.90  | 3.56  | 5.93  | 3.62  | 5.77  | 3.73  | 5.54  | 3.83  | 5.35  | 3.95  | 5.14  |
| 55   | 1.982                          | 10.39 | 3.62  | 6.02  | 3.69  | 5.86  | 3.80  | 5.63  | 3.90  | 5.43  | 4.02  | 5.22  |
| 60   | 1.983                          | 10.86 | 3.68  | 6.10  | 3.75  | 5.94  | 3.86  | 5.70  | 3.96  | 5.51  | 4.08  | 5.29  |
| 65   | 1.985                          | 11.31 | 3.74  | 6.17  | 3.80  | 6.01  | 3.91  | 5.77  | 4.01  | 5.57  | 4.14  | 5.35  |
| 70   | 1.986                          | 11.75 | 3.79  | 6.24  | 3.85  | 6.07  | 3.96  | 5.83  | 4.06  | 5.63  | 4.19  | 5.41  |
| 75   | 1.987                          | 12.17 | 3.83  | 6.30  | 3.90  | 6.13  | 4.01  | 5.88  | 4.11  | 5.68  | 4.24  | 5.46  |
| 80   | 1.987                          | 12.57 | 3.88  | 6.35  | 3.94  | 6.18  | 4.05  | 5.93  | 4.16  | 5.73  | 4.28  | 5.51  |
| 85   | 1.988                          | 12.96 | 3.92  | 6.40  | 3.99  | 6.23  | 4.09  | 5.98  | 4.20  | 5.78  | 4.33  | 5.56  |
| 90   | 1.989                          | 13.34 | 3.96  | 6.45  | 4.02  | 6.27  | 4.13  | 6.03  | 4.24  | 5.82  | 4.36  | 5.60  |
| 95   | 1.990                          | 13.71 | 3.99  | 6.49  | 4.06  | 6.32  | 4.17  | 6.07  | 4.27  | 5.86  | 4.40  | 5.64  |
| 100  | 1.990                          | 14.07 | 4.03  | 6.53  | 4.10  | 6.36  | 4.21  | 6.11  | 4.31  | 5.90  | 4.44  | 5.68  |
| 150  | 1.993                          | 17.26 | 4.32  | 6.82  | 4.38  | 6.64  | 4.48  | 6.39  | 4.59  | 6.18  | 4.72  | 5.96  |
| 200  | 1.995                          | 19.95 | 4.53  | 7.01  | 4.59  | 6.84  | 4.68  | 6.60  | 4.78  | 6.39  | 4.90  | 6.15  |
| 500  | 1.998                          | 31.59 | 5.06  | 7.60  | 5.13  | 7.42  | 5.25  | 7.15  | 5.47  | 6.94  | 5.49  | 6.72  |
| 1000 | 1.999                          | 44.70 | 5.50  | 7.99  | 5.57  | 7.80  | 5.68  | 7.54  | 5.79  | 7.33  | 5.92  | 7.11  |

Source: Sachs, 1972

**Table 15 Critical values for the Cochran test for variance outliers**

Degrees of freedom  $\nu = n - 1$ .

**Level of significance  $\alpha = 0.01$**

| $K \backslash \nu_x$ | 1      | 2      | 3      | 4      | 5      | 6      | 7      | 8      | 9      | 10     | 16     | 36     | 144    | $\infty$ |
|----------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|----------|
| 2                    | 0.9999 | 0.9950 | 0.9794 | 0.9586 | 0.9373 | 0.9172 | 0.8988 | 0.8823 | 0.8674 | 0.8539 | 0.7949 | 0.7067 | 0.6062 | 0.5000   |
| 3                    | 0.9933 | 0.9423 | 0.8831 | 0.8335 | 0.7933 | 0.7606 | 0.7335 | 0.7107 | 0.6912 | 0.6743 | 0.6059 | 0.5153 | 0.4230 | 0.3333   |
| 4                    | 0.9676 | 0.8643 | 0.7814 | 0.7212 | 0.6761 | 0.6410 | 0.6129 | 0.5897 | 0.5702 | 0.5536 | 0.4884 | 0.4057 | 0.3251 | 0.2500   |
| 5                    | 0.9279 | 0.7885 | 0.6957 | 0.6329 | 0.5875 | 0.5531 | 0.5259 | 0.5037 | 0.4854 | 0.4697 | 0.4094 | 0.3351 | 0.2644 | 0.2000   |
| 6                    | 0.8828 | 0.7218 | 0.6258 | 0.5635 | 0.5195 | 0.4866 | 0.4608 | 0.4401 | 0.4229 | 0.4084 | 0.3529 | 0.2858 | 0.2229 | 0.1667   |
| 7                    | 0.8376 | 0.6644 | 0.5685 | 0.5080 | 0.4659 | 0.4347 | 0.4105 | 0.3911 | 0.3751 | 0.3616 | 0.3105 | 0.2494 | 0.1929 | 0.1429   |
| 8                    | 0.7945 | 0.6152 | 0.5209 | 0.4627 | 0.4226 | 0.3932 | 0.3704 | 0.3522 | 0.3373 | 0.3248 | 0.2779 | 0.2214 | 0.1700 | 0.1250   |
| 9                    | 0.7544 | 0.5727 | 0.4810 | 0.4251 | 0.3870 | 0.3592 | 0.3378 | 0.3207 | 0.3067 | 0.2950 | 0.2514 | 0.1992 | 0.1521 | 0.1111   |
| 10                   | 0.7175 | 0.5358 | 0.4469 | 0.3934 | 0.3572 | 0.3308 | 0.3106 | 0.2945 | 0.2813 | 0.2704 | 0.2297 | 0.1811 | 0.1376 | 0.1000   |
| 12                   | 0.6528 | 0.4751 | 0.3919 | 0.3428 | 0.3099 | 0.2861 | 0.2680 | 0.2535 | 0.2419 | 0.2320 | 0.1961 | 0.1535 | 0.1157 | 0.0833   |
| 15                   | 0.5747 | 0.4069 | 0.3317 | 0.2882 | 0.2593 | 0.2386 | 0.2228 | 0.2104 | 0.2002 | 0.1918 | 0.1612 | 0.1251 | 0.0934 | 0.0667   |
| 20                   | 0.4799 | 0.3297 | 0.2654 | 0.2288 | 0.2048 | 0.1877 | 0.1748 | 0.1646 | 0.1567 | 0.1501 | 0.1248 | 0.0960 | 0.0709 | 0.0500   |
| 24                   | 0.4247 | 0.2871 | 0.2295 | 0.1970 | 0.1759 | 0.1608 | 0.1495 | 0.1406 | 0.1338 | 0.1283 | 0.1060 | 0.0810 | 0.0595 | 0.0417   |
| 30                   | 0.3632 | 0.2412 | 0.1913 | 0.1635 | 0.1454 | 0.1327 | 0.1232 | 0.1157 | 0.1100 | 0.1054 | 0.0867 | 0.0658 | 0.0480 | 0.0333   |
| 40                   | 0.2940 | 0.1915 | 0.1508 | 0.1281 | 0.1135 | 0.1033 | 0.0957 | 0.0898 | 0.0853 | 0.0816 | 0.0668 | 0.0503 | 0.0363 | 0.0250   |
| 60                   | 0.2151 | 0.1371 | 0.1069 | 0.0902 | 0.0796 | 0.0722 | 0.0668 | 0.0625 | 0.0594 | 0.0567 | 0.0461 | 0.0344 | 0.0245 | 0.0167   |
| 120                  | 0.1225 | 0.0759 | 0.0585 | 0.0489 | 0.0429 | 0.0387 | 0.0357 | 0.0334 | 0.0316 | 0.0302 | 0.0242 | 0.0178 | 0.0125 | 0.0083   |
| $\infty$             | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0        |

Table 15 continued

Level of significance  $\alpha = 0.05$ 

| $\begin{matrix} \nu_x \\ K \end{matrix}$ | 1      | 2      | 3      | 4      | 5      | 6      | 7      | 8      | 9      | 10     | 16     | 36     | 144    | $\infty$ |
|--|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|----------|
| 2  | 0.9985 | 0.9750 | 0.9392 | 0.9057 | 0.8772 | 0.8534 | 0.8332 | 0.8159 | 0.8010 | 0.7880 | 0.7341 | 0.6602 | 0.5813 | 0.5000   |
| 3  | 0.9669 | 0.8709 | 0.7977 | 0.7457 | 0.7071 | 0.6771 | 0.6530 | 0.6333 | 0.6167 | 0.6025 | 0.5466 | 0.4748 | 0.4031 | 0.3333   |
| 4  | 0.9065 | 0.7679 | 0.6841 | 0.6287 | 0.5895 | 0.5598 | 0.5365 | 0.5175 | 0.5017 | 0.4884 | 0.4366 | 0.3720 | 0.3093 | 0.2500   |
| 5  | 0.8412 | 0.6838 | 0.5981 | 0.5441 | 0.5065 | 0.4783 | 0.4564 | 0.4387 | 0.4241 | 0.4118 | 0.3645 | 0.3066 | 0.2513 | 0.2000   |
| 6  | 0.7808 | 0.6161 | 0.5321 | 0.4803 | 0.4447 | 0.4184 | 0.3980 | 0.3817 | 0.3682 | 0.3568 | 0.3135 | 0.2612 | 0.2119 | 0.1667   |
| 7  | 0.7271 | 0.5612 | 0.4800 | 0.4307 | 0.3974 | 0.3726 | 0.3535 | 0.3384 | 0.3259 | 0.3154 | 0.2756 | 0.2278 | 0.1833 | 0.1429   |
| 8  | 0.6798 | 0.5157 | 0.4377 | 0.3910 | 0.3595 | 0.3362 | 0.3185 | 0.3043 | 0.2926 | 0.2829 | 0.2462 | 0.2022 | 0.1616 | 0.1250   |
| 9  | 0.6385 | 0.4775 | 0.4027 | 0.3584 | 0.3286 | 0.3067 | 0.2901 | 0.2768 | 0.2659 | 0.2568 | 0.2226 | 0.1820 | 0.1446 | 0.1111   |
| 10                                       | 0.6020 | 0.4450 | 0.3733 | 0.3311 | 0.3029 | 0.2823 | 0.2666 | 0.2541 | 0.2439 | 0.2353 | 0.2032 | 0.1655 | 0.1308 | 0.1000   |
| 12                                       | 0.5410 | 0.3924 | 0.3264 | 0.2880 | 0.2624 | 0.2439 | 0.2299 | 0.2187 | 0.2098 | 0.2020 | 0.1737 | 0.1403 | 0.1100 | 0.0833   |
| 15                                       | 0.4709 | 0.3346 | 0.2758 | 0.2419 | 0.2195 | 0.2034 | 0.1911 | 0.1815 | 0.1736 | 0.1671 | 0.1429 | 0.1144 | 0.0889 | 0.0667   |
| 20                                       | 0.3894 | 0.2705 | 0.2205 | 0.1921 | 0.1735 | 0.1602 | 0.1501 | 0.1422 | 0.1357 | 0.1303 | 0.1108 | 0.0879 | 0.0675 | 0.0500   |
| 24                                       | 0.3434 | 0.2354 | 0.1907 | 0.1656 | 0.1493 | 0.1374 | 0.1286 | 0.1216 | 0.1160 | 0.1113 | 0.0942 | 0.0743 | 0.0567 | 0.0417   |
| 30                                       | 0.2929 | 0.1980 | 0.1593 | 0.1377 | 0.1237 | 0.1137 | 0.1061 | 0.1002 | 0.0958 | 0.0921 | 0.0771 | 0.0604 | 0.0457 | 0.0333   |
| 40                                       | 0.2370 | 0.1576 | 0.1259 | 0.1082 | 0.0968 | 0.0887 | 0.0827 | 0.0780 | 0.0745 | 0.0713 | 0.0595 | 0.0462 | 0.0347 | 0.0250   |
| 60                                       | 0.1737 | 0.1131 | 0.0895 | 0.0765 | 0.0682 | 0.0623 | 0.0583 | 0.0552 | 0.0520 | 0.0497 | 0.0411 | 0.0316 | 0.0234 | 0.0167   |
| 120                                      | 0.0998 | 0.0632 | 0.0495 | 0.0419 | 0.0371 | 0.0337 | 0.0312 | 0.0292 | 0.0279 | 0.0266 | 0.0218 | 0.0165 | 0.0120 | 0.0083   |
| $\infty$                                 | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0        |

Source: Dixon and Massey, 1957

**Table 16 Critical values of  $D$  for the Kolmogorov–Smirnov one-sample test** $D$  = maximum values of the differences.

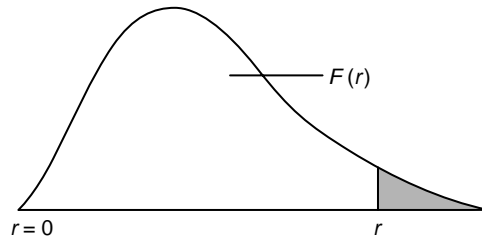
| $n$     | Level of significance $\alpha$ |                         |                         |                         |                         |
|---------|--------------------------------|-------------------------|-------------------------|-------------------------|-------------------------|
|         | 0.20                           | 0.15                    | 0.10                    | 0.05                    | 0.01                    |
| 1       | 0.900                          | 0.925                   | 0.950                   | 0.975                   | 0.995                   |
| 2       | 0.684                          | 0.726                   | 0.776                   | 0.842                   | 0.929                   |
| 3       | 0.565                          | 0.597                   | 0.642                   | 0.708                   | 0.823                   |
| 4       | 0.494                          | 0.525                   | 0.564                   | 0.624                   | 0.733                   |
| 5       | 0.446                          | 0.474                   | 0.510                   | 0.565                   | 0.669                   |
| 6       | 0.410                          | 0.436                   | 0.470                   | 0.521                   | 0.618                   |
| 7       | 0.381                          | 0.405                   | 0.438                   | 0.486                   | 0.577                   |
| 8       | 0.358                          | 0.381                   | 0.411                   | 0.457                   | 0.543                   |
| 9       | 0.339                          | 0.360                   | 0.388                   | 0.432                   | 0.514                   |
| 10      | 0.322                          | 0.342                   | 0.368                   | 0.410                   | 0.490                   |
| 11      | 0.307                          | 0.326                   | 0.352                   | 0.391                   | 0.468                   |
| 12      | 0.295                          | 0.313                   | 0.338                   | 0.375                   | 0.450                   |
| 13      | 0.284                          | 0.302                   | 0.325                   | 0.361                   | 0.433                   |
| 14      | 0.274                          | 0.292                   | 0.314                   | 0.349                   | 0.418                   |
| 15      | 0.266                          | 0.283                   | 0.304                   | 0.338                   | 0.404                   |
| 16      | 0.258                          | 0.274                   | 0.295                   | 0.328                   | 0.392                   |
| 17      | 0.250                          | 0.266                   | 0.286                   | 0.318                   | 0.381                   |
| 18      | 0.244                          | 0.259                   | 0.278                   | 0.309                   | 0.371                   |
| 19      | 0.237                          | 0.252                   | 0.272                   | 0.301                   | 0.363                   |
| 20      | 0.231                          | 0.246                   | 0.264                   | 0.294                   | 0.356                   |
| 25      | 0.21                           | 0.22                    | 0.24                    | 0.27                    | 0.32                    |
| 30      | 0.19                           | 0.20                    | 0.22                    | 0.24                    | 0.29                    |
| 35      | 0.18                           | 0.19                    | 0.21                    | 0.23                    | 0.27                    |
| Over 35 | $\frac{1.07}{\sqrt{n}}$        | $\frac{1.14}{\sqrt{n}}$ | $\frac{1.22}{\sqrt{n}}$ | $\frac{1.36}{\sqrt{n}}$ | $\frac{1.63}{\sqrt{n}}$ |

Source: Massey, 1951

**Table 17 Critical values of  $T$  for the sign test**

| Level of significance $\alpha$ |              |               |              |               | Level of significance $\alpha$ |              |               |              |               |
|--------------------------------|--------------|---------------|--------------|---------------|--------------------------------|--------------|---------------|--------------|---------------|
| Two-sided<br>One-sided         | 0.10<br>0.05 | 0.05<br>0.025 | 0.02<br>0.01 | 0.01<br>0.005 | Two-sided<br>One-sided         | 0.10<br>0.05 | 0.05<br>0.025 | 0.02<br>0.01 | 0.01<br>0.005 |
| $n$                            |              |               |              |               | $n$                            |              |               |              |               |
| 1                              | –            | –             | –            | –             | 31                             | 11           | 13            | 15           | 17            |
| 2                              | –            | –             | –            | –             | 32                             | 12           | 14            | 16           | 16            |
| 3                              | –            | –             | –            | –             | 33                             | 11           | 13            | 15           | 17            |
| 4                              | –            | –             | –            | –             | 34                             | 12           | 14            | 16           | 16            |
| 5                              | 5            | –             | –            | –             | 35                             | 11           | 13            | 15           | 17            |
| 6                              | 6            | 6             | –            | –             | 36                             | 12           | 14            | 16           | 18            |
| 7                              | 7            | 7             | 7            | –             | 37                             | 11           | 13            | 17           | 17            |
| 8                              | 6            | 8             | 8            | 8             | 38                             | 12           | 14            | 16           | 18            |
| 9                              | 7            | 7             | 9            | 9             | 39                             | 13           | 15            | 17           | 17            |
| 10                             | 8            | 8             | 10           | 10            | 40                             | 12           | 14            | 16           | 18            |
| 11                             | 7            | 9             | 9            | 11            | 45                             | 13           | 15            | 17           | 19            |
| 12                             | 8            | 8             | 10           | 10            | 46                             | 14           | 16            | 18           | 20            |
| 13                             | 7            | 9             | 11           | 11            | 49                             | 13           | 15            | 19           | 19            |
| 14                             | 8            | 10            | 10           | 12            | 50                             | 14           | 16            | 18           | 20            |
| 15                             | 9            | 9             | 11           | 11            | 55                             | 15           | 17            | 19           | 21            |
| 16                             | 8            | 10            | 12           | 12            | 56                             | 14           | 16            | 18           | 20            |
| 17                             | 9            | 9             | 11           | 13            | 59                             | 15           | 17            | 19           | 21            |
| 18                             | 8            | 10            | 12           | 12            | 60                             | 14           | 18            | 20           | 22            |
| 19                             | 9            | 11            | 11           | 13            | 65                             | 15           | 17            | 21           | 23            |
| 20                             | 10           | 10            | 12           | 14            | 66                             | 16           | 18            | 20           | 22            |
| 21                             | 9            | 11            | 13           | 13            | 69                             | 15           | 19            | 23           | 25            |
| 22                             | 10           | 12            | 12           | 14            | 70                             | 16           | 18            | 22           | 24            |
| 23                             | 9            | 11            | 13           | 15            | 75                             | 17           | 19            | 23           | 25            |
| 24                             | 10           | 12            | 14           | 14            | 76                             | 16           | 20            | 22           | 24            |
| 25                             | 11           | 11            | 13           | 15            | 79                             | 17           | 19            | 23           | 25            |
| 26                             | 10           | 12            | 14           | 14            | 80                             | 16           | 20            | 22           | 24            |
| 27                             | 11           | 13            | 13           | 15            | 89                             | 17           | 21            | 23           | 27            |
| 28                             | 10           | 12            | 14           | 16            | 90                             | 18           | 20            | 24           | 26            |
| 29                             | 11           | 13            | 15           | 15            | 99                             | 19           | 21            | 25           | 27            |
| 30                             | 10           | 12            | 14           | 16            | 100                            | 18           | 22            | 26           | 28            |

Source: Wijvekate, 1962



**Table 18 Critical values of  $r$  for the sign test for paired observations**

$N$  is the total number of equally probable dichotomous events,  $r$  the smaller of the number of events of either kind. If  $r \leq r_{N;\alpha}$  then there are too few events of one kind at the  $1 - \alpha$  confidence level. For large values of  $N$ ,  $r$  is approximately distributed as  $z$ . Use  $\mu = N/2$ ,  $\sigma = \sqrt{N}/2$  and  $Z_\alpha = (\mu - r_{N;\alpha})/\sigma$ .

| $N$ | $r_{N;0.10}$ | $r_{N;0.05}$ | $r_{N;0.01}$ | $r_{N;0.005}$ | $r_{N;0.001}$ |
|-----|--------------|--------------|--------------|---------------|---------------|
| 8   | 1            | 0            | 0            |               |               |
| 10  | 1            | 1            | 0            | 0             |               |
| 12  | 2            | 2            | 1            | 0             | 0             |
| 14  | 3            | 2            | 1            | 0             | 0             |
| 16  | 4            | 3            | 1            | 0             | 0             |
| 18  | 5            | 4            | 3            | 2             | 1             |
| 20  | 5            | 5            | 3            | 3             | 2             |
| 22  | 6            | 5            | 4            | 4             | 3             |
| 25  | 7            | 7            | 5            | 5             | 4             |
| 30  | 10           | 9            | 7            | 6             | 5             |
| 35  | 12           | 11           | 9            | 8             | 7             |
| 40  | 14           | 13           | 11           | 10            | 9             |
| 45  | 16           | 15           | 13           | 12            | 11            |
| 50  | 18           | 17           | 15           | 15            | 13            |
| 55  | 20           | 19           | 17           | 17            | 15            |

Source: Dixon and Massey, 1957



**Table 19 Critical values of  $T$  for the signed rank test for paired differences**

$n$  is the number of pairs of observations.

| n  | Level of significance $\alpha$ |      |       |           |      | n  | Level of significance $\alpha$ |      |       |           |      |
|----|--------------------------------|------|-------|-----------|------|----|--------------------------------|------|-------|-----------|------|
|    | Two-sided                      |      |       | One-sided |      |    | Two-sided                      |      |       | One-sided |      |
|    | 0.05                           | 0.01 | 0.001 | 0.05      | 0.01 |    | 0.05                           | 0.01 | 0.001 | 0.05      | 0.01 |
| 6  | 0                              |      |       | 2         |      | 36 | 208                            | 171  | 130   | 227       | 185  |
| 7  | 2                              |      |       | 3         | 0    | 37 | 221                            | 182  | 140   | 241       | 198  |
| 8  | 3                              | 0    |       | 5         | 1    | 38 | 235                            | 194  | 150   | 256       | 211  |
| 9  | 5                              | 1    |       | 8         | 3    | 39 | 249                            | 207  | 161   | 271       | 224  |
| 10 | 8                              | 3    |       | 10        | 5    | 40 | 264                            | 220  | 172   | 286       | 238  |
| 11 | 10                             | 5    | 0     | 13        | 7    | 41 | 279                            | 233  | 183   | 302       | 252  |
| 12 | 13                             | 7    | 1     | 17        | 9    | 42 | 294                            | 247  | 195   | 319       | 266  |
| 13 | 17                             | 9    | 2     | 21        | 12   | 43 | 310                            | 261  | 207   | 336       | 281  |
| 14 | 21                             | 12   | 4     | 25        | 15   | 44 | 327                            | 276  | 220   | 353       | 296  |
| 15 | 25                             | 15   | 6     | 30        | 19   | 45 | 343                            | 291  | 233   | 371       | 312  |
| 16 | 29                             | 19   | 8     | 35        | 23   | 46 | 361                            | 307  | 246   | 389       | 328  |
| 17 | 34                             | 23   | 11    | 41        | 27   | 47 | 378                            | 322  | 260   | 407       | 345  |
| 18 | 40                             | 27   | 14    | 47        | 32   | 48 | 396                            | 339  | 274   | 426       | 362  |
| 19 | 46                             | 32   | 18    | 53        | 37   | 49 | 415                            | 355  | 289   | 446       | 379  |
| 20 | 52                             | 37   | 21    | 60        | 43   | 50 | 434                            | 373  | 304   | 466       | 397  |
| 21 | 58                             | 42   | 25    | 67        | 49   | 51 | 453                            | 390  | 319   | 486       | 416  |
| 22 | 65                             | 48   | 30    | 75        | 55   | 52 | 473                            | 408  | 335   | 507       | 434  |
| 23 | 73                             | 54   | 35    | 83        | 62   | 53 | 494                            | 427  | 351   | 529       | 454  |
| 24 | 81                             | 61   | 40    | 91        | 69   | 54 | 514                            | 445  | 368   | 550       | 473  |
| 25 | 89                             | 68   | 45    | 100       | 76   | 55 | 536                            | 465  | 385   | 573       | 493  |
| 26 | 98                             | 75   | 51    | 110       | 84   | 56 | 557                            | 484  | 402   | 595       | 514  |
| 27 | 107                            | 83   | 57    | 119       | 92   | 57 | 579                            | 504  | 420   | 618       | 535  |
| 28 | 116                            | 91   | 64    | 130       | 101  | 58 | 602                            | 525  | 438   | 642       | 556  |
| 29 | 126                            | 100  | 71    | 140       | 110  | 59 | 625                            | 546  | 457   | 666       | 578  |
| 30 | 137                            | 109  | 78    | 151       | 120  | 60 | 648                            | 567  | 476   | 690       | 600  |
| 31 | 147                            | 118  | 86    | 163       | 130  | 61 | 672                            | 589  | 495   | 715       | 623  |
| 32 | 159                            | 128  | 94    | 175       | 140  | 62 | 697                            | 611  | 515   | 741       | 646  |
| 33 | 170                            | 138  | 102   | 187       | 151  | 63 | 721                            | 634  | 535   | 767       | 669  |
| 34 | 182                            | 148  | 111   | 200       | 162  | 64 | 747                            | 657  | 556   | 793       | 693  |
| 35 | 195                            | 159  | 120   | 213       | 173  | 65 | 772                            | 681  | 577   | 820       | 718  |

Source: Sachs, 1972

**Table 20 Critical values of  $U$  for the Wilcoxon inversion test** $n_1$  = number of elements in the largest sample; $n_2$  = number of elements in the smallest sample.

|           |           | Level of significance $\alpha$ |       |      |       |           |           | Level of significance $\alpha$ |       |      |       |
|-----------|-----------|--------------------------------|-------|------|-------|-----------|-----------|--------------------------------|-------|------|-------|
| Two-sided | One-sided | 0.10                           | 0.05  | 0.02 | 0.01  | Two-sided | One-sided | 0.10                           | 0.05  | 0.02 | 0.01  |
|           |           | 0.05                           | 0.025 | 0.01 | 0.005 |           |           | 0.05                           | 0.025 | 0.01 | 0.005 |
| $n_1$     | $n_2$     |                                |       |      |       | $n_1$     | $n_2$     |                                |       |      |       |
| 3         | 3         | 0                              | —     | —    | —     | 9         | 2         | 1                              | 0     | —    | —     |
| 4         | 3         | 0                              | —     | —    | —     | 9         | 3         | 4                              | 2     | 1    | 0     |
| 4         | 4         | 1                              | 0     | —    | —     | 9         | 4         | 6                              | 4     | 3    | 1     |
|           |           |                                |       |      |       | 9         | 5         | 9                              | 7     | 5    | 3     |
| 5         | 2         | 0                              | —     | —    | —     | 9         | 6         | 12                             | 10    | 7    | 5     |
| 5         | 3         | 1                              | 0     | —    | —     | 9         | 7         | 15                             | 12    | 9    | 7     |
| 5         | 4         | 2                              | 1     | 0    | —     | 9         | 8         | 18                             | 15    | 12   | 9     |
| 5         | 5         | 4                              | 2     | 1    | 0     | 9         | 9         | 21                             | 17    | 14   | 11    |
|           |           |                                |       |      |       |           |           |                                |       |      |       |
| 6         | 2         | 0                              | —     | —    | —     | 10        | 2         | 1                              | 0     |      |       |
| 6         | 3         | 2                              | 1     | —    | —     | 10        | 3         | 4                              | 3     | 1    | 0     |
| 6         | 4         | 3                              | 2     | 1    | 0     | 10        | 4         | 7                              | 5     | 3    | 2     |
| 6         | 5         | 5                              | 3     | 2    | 1     | 10        | 5         | 11                             | 8     | 6    | 4     |
| 6         | 6         | 7                              | 5     | 3    | 2     | 10        | 6         | 14                             | 11    | 8    | 6     |
|           |           |                                |       |      |       | 10        | 7         | 17                             | 14    | 11   | 9     |
| 7         | 2         | 0                              | —     | —    | —     | 10        | 8         | 20                             | 17    | 13   | 11    |
| 7         | 3         | 2                              | 1     | 0    | —     | 10        | 9         | 24                             | 20    | 16   | 13    |
| 7         | 4         | 4                              | 3     | 1    | 0     | 10        | 10        | 27                             | 23    | 19   | 16    |
| 7         | 5         | 6                              | 5     | 3    | 1     |           |           |                                |       |      |       |
| 7         | 6         | 8                              | 6     | 4    | 3     |           |           |                                |       |      |       |
| 7         | 7         | 11                             | 8     | 6    | 4     |           |           |                                |       |      |       |
|           |           |                                |       |      |       |           |           |                                |       |      |       |
| 8         | 2         | 1                              | 0     | —    | —     |           |           |                                |       |      |       |
| 8         | 3         | 3                              | 2     | 0    | —     |           |           |                                |       |      |       |
| 8         | 4         | 5                              | 4     | 2    | 1     |           |           |                                |       |      |       |
| 8         | 5         | 8                              | 6     | 4    | 2     |           |           |                                |       |      |       |
| 8         | 6         | 10                             | 8     | 6    | 4     |           |           |                                |       |      |       |
| 8         | 7         | 13                             | 10    | 7    | 6     |           |           |                                |       |      |       |
| 8         | 8         | 15                             | 13    | 9    | 7     |           |           |                                |       |      |       |

Source: Wijvekatte, 1962

**Table 21 Critical values of the smallest rank sum for the Wilcoxon–Mann–Whitney test**

$n_1$  = number of elements in the largest sample;  
 $n_2$  = number of elements in the smallest sample.

| Level of significance $\alpha$ |       |              |              |               |               | Level of significance $\alpha$ |       |              |              |               |               |
|--------------------------------|-------|--------------|--------------|---------------|---------------|--------------------------------|-------|--------------|--------------|---------------|---------------|
| Two-sided<br>One-sided         |       | 0.20<br>0.10 | 0.10<br>0.05 | 0.05<br>0.025 | 0.01<br>0.005 | Two-sided<br>One-sided         |       | 0.20<br>0.10 | 0.10<br>0.05 | 0.05<br>0.025 | 0.01<br>0.005 |
| $n_1$                          | $n_2$ |              |              |               |               | $n_1$                          | $n_2$ |              |              |               |               |
| 3                              | 2     | 3            | –            | –             | –             | 10                             | 6     | 38           | 35           | 32            | 27            |
| 3                              | 3     | 7            | 6            | –             | –             | 10                             | 7     | 49           | 45           | 42            | 37            |
| 4                              | 2     | 3            | –            | –             | –             | 10                             | 8     | 60           | 56           | 53            | 47            |
| 4                              | 3     | 7            | 6            | –             | –             | 10                             | 9     | 73           | 69           | 65            | 58            |
| 4                              | 4     | 13           | 11           | 10            | –             | 10                             | 10    | 87           | 82           | 78            | 71            |
| 5                              | 2     | 4            | 3            | –             | –             | 11                             | 1     | 1            | –            | –             | –             |
| 5                              | 3     | 8            | 7            | 6             | –             | 11                             | 2     | 6            | 4            | 3             | –             |
| 5                              | 4     | 14           | 12           | 11            | –             | 11                             | 3     | 13           | 11           | 9             | 6             |
| 5                              | 5     | 20           | 19           | 17            | 15            | 11                             | 4     | 21           | 18           | 16            | 12            |
|                                |       |              |              |               |               | 11                             | 5     | 30           | 27           | 24            | 20            |
| 6                              | 2     | 4            | 3            | –             | –             | 11                             | 6     | 40           | 37           | 34            | 28            |
| 6                              | 3     | 9            | 8            | 7             | –             | 11                             | 7     | 51           | 47           | 44            | 38            |
| 6                              | 4     | 15           | 13           | 12            | 10            | 11                             | 8     | 63           | 59           | 55            | 49            |
| 6                              | 5     | 22           | 20           | 18            | 16            | 11                             | 9     | 76           | 72           | 68            | 61            |
| 6                              | 6     | 30           | 28           | 26            | 13            | 11                             | 10    | 91           | 86           | 81            | 73            |
|                                |       |              |              |               |               | 11                             | 11    | 106          | 100          | 96            | 87            |
| 7                              | 2     | 4            | 3            | –             | –             |                                |       |              |              |               |               |
| 7                              | 3     | 10           | 8            | 7             | –             | 12                             | 1     | 1            | –            | –             | –             |
| 7                              | 4     | 16           | 14           | 13            | 10            | 12                             | 2     | 7            | 5            | 4             | –             |
| 7                              | 5     | 23           | 21           | 20            | 16            | 12                             | 3     | 14           | 11           | 10            | 7             |
| 7                              | 6     | 32           | 29           | 27            | 24            | 12                             | 4     | 22           | 19           | 17            | 13            |
| 7                              | 7     | 41           | 39           | 36            | 32            | 12                             | 5     | 32           | 28           | 26            | 21            |
|                                |       |              |              |               |               | 12                             | 6     | 42           | 38           | 35            | 30            |
| 8                              | 2     | 5            | 4            | 3             | –             | 12                             | 7     | 54           | 49           | 46            | 40            |
| 8                              | 3     | 11           | 9            | 8             | –             | 12                             | 8     | 66           | 62           | 58            | 51            |
| 8                              | 4     | 17           | 15           | 14            | 11            | 12                             | 9     | 80           | 75           | 71            | 63            |
| 8                              | 5     | 25           | 23           | 21            | 17            | 12                             | 10    | 94           | 89           | 84            | 76            |
| 8                              | 6     | 34           | 31           | 29            | 25            | 12                             | 11    | 110          | 104          | 99            | 90            |
| 8                              | 7     | 44           | 41           | 38            | 34            | 12                             | 12    | 127          | 120          | 115           | 105           |
| 8                              | 8     | 55           | 51           | 49            | 43            |                                |       |              |              |               |               |
|                                |       |              |              |               |               | 13                             | 1     | –            | –            | –             | –             |
| 9                              | 1     | 1            | –            | –             | –             | 13                             | 2     | 7            | 5            | 4             | –             |
| 9                              | 2     | 5            | 4            | 3             | –             | 13                             | 3     | 15           | 12           | 10            | 7             |
| 9                              | 3     | 11           | 9            | 8             | 6             | 13                             | 4     | 23           | 20           | 18            | 14            |
| 9                              | 4     | 19           | 16           | 14            | 11            | 13                             | 5     | 33           | 30           | 27            | 22            |
| 9                              | 5     | 27           | 24           | 22            | 18            | 13                             | 6     | 44           | 40           | 37            | 31            |
| 9                              | 6     | 36           | 33           | 31            | 26            | 13                             | 7     | 56           | 52           | 48            | 44            |
| 9                              | 7     | 46           | 43           | 40            | 35            | 13                             | 8     | 69           | 64           | 60            | 53            |
| 9                              | 8     | 58           | 54           | 51            | 45            | 13                             | 9     | 83           | 78           | 73            | 65            |
| 9                              | 9     | 70           | 66           | 62            | 56            | 13                             | 10    | 98           | 92           | 88            | 79            |
|                                |       |              |              |               |               | 13                             | 11    | 114          | 108          | 103           | 93            |
| 10                             | 1     | 1            | –            | –             | –             | 13                             | 12    | 131          | 125          | 119           | 109           |
| 10                             | 2     | 6            | 4            | 3             | –             | 13                             | 13    | 149          | 142          | 136           | 125           |
| 10                             | 3     | 12           | 10           | 9             | 6             |                                |       |              |              |               |               |
| 10                             | 4     | 20           | 17           | 15            | 12            |                                |       |              |              |               |               |
| 10                             | 5     | 28           | 26           | 23            | 19            |                                |       |              |              |               |               |

Table 21 continued

| Level of significance $\alpha$ |       |              |              |               |               | Level of significance $\alpha$ |       |              |              |               |               |
|--------------------------------|-------|--------------|--------------|---------------|---------------|--------------------------------|-------|--------------|--------------|---------------|---------------|
| Two-sided<br>One-sided         |       | 0.20<br>0.10 | 0.10<br>0.05 | 0.05<br>0.025 | 0.01<br>0.005 | Two-sided<br>One-sided         |       | 0.20<br>0.10 | 0.10<br>0.05 | 0.05<br>0.025 | 0.01<br>0.005 |
| $n_1$                          | $n_2$ |              |              |               |               | $n_1$                          | $n_2$ |              |              |               |               |
| 14                             | 1     | 1            | —            | —             | —             | 17                             | 4     | 28           | 25           | 21            | 16            |
| 14                             | 2     | 7            | 5            | 4             | —             | 17                             | 5     | 40           | 35           | 32            | 25            |
| 14                             | 3     | 16           | 13           | 11            | 7             | 17                             | 6     | 52           | 47           | 43            | 36            |
| 14                             | 4     | 25           | 21           | 19            | 14            | 17                             | 7     | 66           | 61           | 56            | 47            |
| 14                             | 5     | 35           | 31           | 28            | 22            | 17                             | 8     | 81           | 75           | 70            | 60            |
| 14                             | 6     | 46           | 42           | 38            | 32            | 17                             | 9     | 97           | 90           | 84            | 74            |
| 14                             | 7     | 59           | 54           | 50            | 43            | 17                             | 10    | 113          | 106          | 100           | 89            |
| 14                             | 8     | 72           | 67           | 62            | 54            | 17                             | 11    | 131          | 123          | 117           | 105           |
| 14                             | 9     | 86           | 81           | 76            | 67            | 17                             | 12    | 150          | 142          | 135           | 122           |
| 14                             | 10    | 102          | 96           | 91            | 81            | 17                             | 13    | 170          | 161          | 154           | 140           |
| 14                             | 11    | 118          | 112          | 106           | 96            | 17                             | 14    | 190          | 182          | 174           | 159           |
| 14                             | 12    | 136          | 129          | 123           | 112           | 17                             | 15    | 212          | 203          | 195           | 180           |
| 14                             | 13    | 154          | 147          | 141           | 129           | 17                             | 16    | 235          | 225          | 217           | 201           |
| 14                             | 14    | 174          | 166          | 160           | 147           | 17                             | 17    | 259          | 249          | 240           | 223           |
| 15                             | 1     | 1            | —            | —             | —             | 18                             | 1     | 1            | —            | —             | —             |
| 15                             | 2     | 8            | 6            | 4             | —             | 18                             | 2     | 9            | 7            | 5             | —             |
| 15                             | 3     | 16           | 13           | 11            | 8             | 18                             | 3     | 19           | 15           | 13            | 8             |
| 15                             | 4     | 26           | 22           | 20            | 15            | 18                             | 4     | 30           | 26           | 22            | 16            |
| 15                             | 5     | 37           | 33           | 29            | 23            | 18                             | 5     | 42           | 37           | 33            | 26            |
| 15                             | 6     | 48           | 44           | 40            | 33            | 18                             | 6     | 55           | 49           | 45            | 37            |
| 15                             | 7     | 61           | 56           | 52            | 44            | 18                             | 7     | 69           | 63           | 58            | 49            |
| 15                             | 8     | 75           | 69           | 65            | 56            | 18                             | 8     | 84           | 77           | 72            | 62            |
| 15                             | 9     | 90           | 84           | 79            | 69            | 18                             | 9     | 100          | 93           | 87            | 76            |
| 15                             | 10    | 106          | 99           | 94            | 84            | 18                             | 10    | 117          | 110          | 103           | 92            |
| 15                             | 11    | 123          | 116          | 110           | 99            | 18                             | 11    | 135          | 127          | 121           | 108           |
| 15                             | 12    | 141          | 133          | 127           | 115           | 18                             | 12    | 155          | 146          | 139           | 125           |
| 15                             | 13    | 159          | 152          | 145           | 133           | 18                             | 13    | 175          | 166          | 158           | 144           |
| 15                             | 14    | 179          | 171          | 164           | 151           | 18                             | 14    | 196          | 187          | 179           | 163           |
| 15                             | 15    | 200          | 192          | 184           | 171           | 18                             | 15    | 218          | 208          | 200           | 184           |
|                                |       |              |              |               |               | 18                             | 16    | 242          | 231          | 222           | 206           |
| 16                             | 1     | 1            | —            | —             | —             | 18                             | 17    | 266          | 255          | 246           | 228           |
| 16                             | 2     | 8            | 6            | 4             | —             | 18                             | 18    | 291          | 280          | 270           | 252           |
| 16                             | 3     | 17           | 14           | 12            | 8             |                                |       |              |              |               |               |
| 16                             | 4     | 27           | 24           | 21            | 15            | 19                             | 1     | 2            | 1            | —             | —             |
| 16                             | 5     | 38           | 34           | 30            | 24            | 19                             | 2     | 10           | 7            | 5             | 3             |
| 16                             | 6     | 50           | 46           | 42            | 34            | 19                             | 3     | 20           | 16           | 13            | 9             |
| 16                             | 7     | 64           | 58           | 54            | 46            | 19                             | 4     | 31           | 27           | 23            | 17            |
| 16                             | 8     | 78           | 72           | 67            | 58            | 19                             | 5     | 43           | 38           | 34            | 27            |
| 16                             | 9     | 93           | 87           | 82            | 72            | 19                             | 6     | 57           | 51           | 46            | 38            |
| 16                             | 10    | 109          | 103          | 97            | 86            | 19                             | 7     | 71           | 65           | 60            | 50            |
| 16                             | 11    | 127          | 120          | 113           | 102           | 19                             | 8     | 87           | 80           | 74            | 64            |
| 16                             | 12    | 145          | 138          | 131           | 119           | 19                             | 9     | 103          | 96           | 90            | 78            |
| 16                             | 13    | 165          | 156          | 150           | 130           | 19                             | 10    | 121          | 113          | 107           | 94            |
| 16                             | 14    | 185          | 176          | 169           | 155           | 19                             | 11    | 139          | 131          | 124           | 111           |
| 16                             | 15    | 206          | 197          | 190           | 175           | 19                             | 12    | 159          | 150          | 143           | 129           |
| 16                             | 16    | 229          | 219          | 211           | 196           | 19                             | 13    | 180          | 171          | 163           | 147           |
|                                |       |              |              |               |               | 19                             | 14    | 202          | 192          | 182           | 168           |
| 17                             | 1     | 1            | —            | —             | —             | 19                             | 15    | 224          | 214          | 205           | 189           |
| 17                             | 2     | 9            | 6            | 5             | —             | 19                             | 16    | 248          | 237          | 228           | 210           |
| 17                             | 3     | 18           | 15           | 12            | 8             |                                |       |              |              |               |               |

Source: Natrella, 1963

**Table 22 The Kruskal–Wallis test**

Critical region:  $H \geq$  tabulated value.

| $K = 3$      |                 |                 | $K = 4$      |                 |                 | $K = 5$      |                 |                 |
|--------------|-----------------|-----------------|--------------|-----------------|-----------------|--------------|-----------------|-----------------|
| Sample sizes | $\alpha = 0.05$ | $\alpha = 0.01$ | Sample sizes | $\alpha = 0.05$ | $\alpha = 0.01$ | Sample sizes | $\alpha = 0.05$ | $\alpha = 0.01$ |
| 2 2 2        | –               | –               | 2 2 1 1      | –               | –               | 2 2 1 1 1    | –               | –               |
| 3 2 1        | –               | –               | 2 2 2 1      | 5.679           | –               | 2 2 2 1 1    | 6.750           | –               |
| 3 2 2        | 4.714           | –               | 2 2 2 2      | 6.167           | 6.667           | 2 2 2 2 1    | 7.133           | 7.533           |
| 3 3 1        | 5.143           | –               |              |                 |                 | 2 2 2 2 2    | 7.418           | 8.291           |
| 3 3 2        | 5.361           | –               | 3 1 1 1      | –               | –               |              |                 |                 |
| 3 3 3        | 5.600           | 7.200           | 3 2 1 1      | –               | –               | 3 1 1 1 1    | –               | –               |
|              |                 |                 | 3 2 2 1      | 5.833           | –               | 3 2 1 1 1    | 6.583           | –               |
| 4 2 1        | –               | –               | 3 2 2 2      | 6.333           | 7.133           | 3 2 2 1 1    | 6.800           | 7.600           |
| 4 2 2        | 5.333           | –               | 3 3 1 1      | 6.333           | –               | 3 2 2 2 1    | 7.309           | 8.127           |
| 4 3 1        | 5.208           | –               | 3 3 2 1      | 6.244           | 7.200           | 3 2 2 2 2    | 7.682           | 8.682           |
| 4 3 2        | 5.444           | 6.444           | 3 3 2 2      | 6.527           | 7.636           | 3 3 1 1 1    | 7.111           | –               |
| 4 3 3        | 5.791           | 6.745           | 3 3 3 1      | 6.600           | 7.400           | 3 3 2 1 1    | 7.200           | 8.073           |
| 4 4 1        | 4.967           | 6.667           | 3 3 3 2      | 6.727           | 8.015           | 3 3 2 2 1    | 7.591           | 8.576           |
| 4 4 2        | 5.455           | 7.036           | 3 3 3 3      | 7.000           | 8.538           | 3 3 2 2 2    | 7.910           | 9.115           |
| 4 4 3        | 5.598           | 7.144           |              |                 |                 | 3 3 3 1 1    | 7.576           | 8.424           |
| 4 4 4        | 5.692           | 7.654           | 4 1 1 1      | –               | –               | 3 3 3 2 1    | 7.769           | 9.051           |
|              |                 |                 | 4 2 1 1      | 5.833           | –               | 3 3 3 2 2    | 8.044           | 9.505           |
| 5 2 1        | 5.000           | –               | 4 2 2 1      | 6.133           | 7.000           | 3 3 3 3 1    | 8.000           | 9.451           |
| 5 2 2        | 5.160           | 6.533           | 4 2 2 2      | 6.545           | 7.391           | 3 3 3 3 2    | 8.200           | 9.876           |
| 5 3 1        | 4.960           | –               | 4 3 1 1      | 6.178           | 7.067           | 3 3 3 3 3    | 8.333           | 10.20           |
| 5 3 2        | 5.251           | 6.909           | 4 3 2 1      | 6.309           | 7.455           |              |                 |                 |
| 5 3 3        | 5.648           | 7.079           | 4 3 2 2      | 6.621           | 7.871           |              |                 |                 |
| 5 4 1        | 4.985           | 6.955           | 4 3 3 1      | 6.545           | 7.758           |              |                 |                 |
| 5 4 2        | 5.273           | 7.205           | 4 3 3 2      | 6.795           | 8.333           |              |                 |                 |
| 5 4 3        | 5.656           | 7.445           | 4 3 3 3      | 6.984           | 8.659           |              |                 |                 |
| 5 4 4        | 5.657           | 7.760           | 4 4 1 1      | 5.945           | 7.909           |              |                 |                 |
| 5 5 1        | 5.127           | 7.309           | 4 4 2 1      | 6.386           | 7.909           |              |                 |                 |
| 5 5 2        | 5.338           | 7.338           | 4 4 2 2      | 6.731           | 8.346           |              |                 |                 |
| 5 5 3        | 5.705           | 7.578           | 4 4 3 1      | 6.635           | 8.231           |              |                 |                 |
| 5 5 4        | 5.666           | 7.823           | 4 4 3 2      | 6.874           | 8.621           |              |                 |                 |
| 5 5 5        | 5.780           | 8.000           | 4 4 3 3      | 7.038           | 8.876           |              |                 |                 |
|              |                 |                 | 4 4 4 1      | 6.725           | 8.588           |              |                 |                 |
| 6 1 1        | –               | –               | 4 4 4 2      | 6.957           | 8.871           |              |                 |                 |
| 6 2 1        | 4.822           | –               | 4 4 4 3      | 7.142           | 9.075           |              |                 |                 |
|              |                 |                 | 4 4 4 4      | 7.235           | 9.287           |              |                 |                 |
| 6 2 2        | 5.345           | 6.655           |              |                 |                 |              |                 |                 |
| 6 3 1        | 4.855           | 6.873           |              |                 |                 |              |                 |                 |
| 6 3 2        | 5.348           | 6.970           |              |                 |                 |              |                 |                 |
| 6 3 3        | 5.615           | 7.410           |              |                 |                 |              |                 |                 |
| 6 4 1        | 4.947           | 7.106           |              |                 |                 |              |                 |                 |
| 6 4 2        | 5.340           | 7.340           |              |                 |                 |              |                 |                 |
| 6 4 3        | 5.610           | 7.500           |              |                 |                 |              |                 |                 |
| 6 4 4        | 5.681           | 7.795           |              |                 |                 |              |                 |                 |
| 6 5 1        | 4.990           | 7.182           |              |                 |                 |              |                 |                 |
| 6 5 2        | 5.338           | 7.376           |              |                 |                 |              |                 |                 |
| 6 5 3        | 5.602           | 7.590           |              |                 |                 |              |                 |                 |
| 6 5 4        | 5.661           | 7.936           |              |                 |                 |              |                 |                 |
| 6 5 5        | 5.729           | 8.028           |              |                 |                 |              |                 |                 |
| 6 6 1        | 4.945           | 7.121           |              |                 |                 |              |                 |                 |
| 6 6 2        | 5.410           | 7.467           |              |                 |                 |              |                 |                 |
| 6 6 3        | 5.625           | 7.725           |              |                 |                 |              |                 |                 |
| 6 6 4        | 5.724           | 8.000           |              |                 |                 |              |                 |                 |
| 6 6 5        | 5.765           | 8.124           |              |                 |                 |              |                 |                 |
| 6 6 6        | 5.801           | 8.222           |              |                 |                 |              |                 |                 |
| 7 7 7        | 5.819           | 8.378           |              |                 |                 |              |                 |                 |
| 8 8 8        | 5.805           | 8.465           |              |                 |                 |              |                 |                 |

Source: Neave, 1978

**Table 23 Critical values for the rank sum difference test (two-sided)****Level of significance  $\alpha = 0.01$** 

| $n \backslash K$ | 3     | 4     | 5     | 6      | 7      | 8      | 9      | 10     |
|------------------|-------|-------|-------|--------|--------|--------|--------|--------|
| 1                | 4.1   | 5.7   | 7.3   | 8.9    | 10.5   | 12.2   | 13.9   | 15.6   |
| 2                | 10.9  | 15.3  | 19.7  | 24.3   | 28.9   | 33.6   | 38.3   | 43.1   |
| 3                | 19.5  | 27.5  | 35.7  | 44.0   | 52.5   | 61.1   | 69.8   | 78.6   |
| 4                | 29.7  | 41.9  | 54.5  | 67.3   | 80.3   | 93.6   | 107.0  | 120.6  |
| 5                | 41.2  | 58.2  | 75.8  | 93.6   | 111.9  | 130.4  | 149.1  | 168.1  |
| 6                | 53.9  | 76.3  | 99.3  | 122.8  | 146.7  | 171.0  | 195.7  | 220.6  |
| 7                | 67.6  | 95.8  | 124.8 | 154.4  | 184.6  | 215.2  | 246.3  | 277.7  |
| 8                | 82.4  | 116.8 | 152.2 | 188.4  | 225.2  | 262.6  | 300.6  | 339.0  |
| 9                | 98.1  | 139.2 | 181.4 | 224.5  | 268.5  | 313.1  | 358.4  | 404.2  |
| 10               | 114.7 | 162.8 | 212.2 | 262.7  | 314.2  | 366.5  | 419.5  | 473.1  |
| 11               | 132.1 | 187.6 | 244.6 | 302.9  | 362.2  | 422.6  | 483.7  | 545.6  |
| 12               | 150.4 | 213.5 | 278.5 | 344.9  | 412.5  | 481.2  | 551.0  | 621.4  |
| 13               | 169.4 | 240.6 | 313.8 | 388.7  | 464.9  | 542.4  | 621.0  | 700.5  |
| 14               | 189.1 | 268.7 | 350.5 | 434.2  | 519.4  | 606.0  | 693.8  | 782.6  |
| 15               | 209.6 | 297.8 | 388.5 | 481.3  | 575.8  | 671.9  | 769.3  | 867.7  |
| 16               | 230.7 | 327.9 | 427.9 | 530.1  | 634.2  | 740.0  | 847.3  | 955.7  |
| 17               | 252.5 | 359.0 | 468.4 | 580.3  | 694.4  | 810.2  | 927.8  | 1046.5 |
| 18               | 275.0 | 391.0 | 510.2 | 632.1  | 756.4  | 882.6  | 1010.6 | 1140.0 |
| 19               | 298.1 | 423.8 | 553.1 | 685.4  | 820.1  | 957.0  | 1095.8 | 1236.2 |
| 20               | 321.8 | 457.6 | 597.2 | 740.0  | 885.5  | 1033.3 | 1183.3 | 1334.9 |
| 21               | 346.1 | 492.2 | 642.4 | 796.0  | 952.6  | 1111.6 | 1273.0 | 1436.0 |
| 22               | 371.0 | 527.6 | 688.7 | 853.4  | 1021.3 | 1191.8 | 1364.8 | 1539.7 |
| 23               | 396.4 | 563.8 | 736.0 | 912.1  | 1091.5 | 1273.8 | 1458.8 | 1645.7 |
| 24               | 422.4 | 600.9 | 784.4 | 972.1  | 1163.4 | 1357.6 | 1554.8 | 1754.0 |
| 25               | 449.0 | 638.7 | 833.8 | 1033.3 | 1236.7 | 1443.2 | 1652.8 | 1864.6 |

**Level of significance  $\alpha = 0.05$** 

| $n \backslash K$ | 3     | 4     | 5     | 6     | 7     | 8     | 9     | 10    |
|------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1                | 3.3   | 4.7   | 6.1   | 7.5   | 9.0   | 10.5  | 12.0  | 13.5  |
| 2                | 8.8   | 12.6  | 16.5  | 20.5  | 24.7  | 28.9  | 33.1  | 37.4  |
| 3                | 15.7  | 22.7  | 29.9  | 37.3  | 44.8  | 52.5  | 60.3  | 68.2  |
| 4                | 23.9  | 34.6  | 45.6  | 57.0  | 68.6  | 80.4  | 92.4  | 104.6 |
| 5                | 33.1  | 48.1  | 63.5  | 79.3  | 95.5  | 112.0 | 128.8 | 145.8 |
| 6                | 43.3  | 62.9  | 83.2  | 104.0 | 125.3 | 147.0 | 169.1 | 191.4 |
| 7                | 54.4  | 79.1  | 104.6 | 130.8 | 157.6 | 184.9 | 212.8 | 240.9 |
| 8                | 66.3  | 96.4  | 127.6 | 159.6 | 192.4 | 225.7 | 259.7 | 294.1 |
| 9                | 78.9  | 114.8 | 152.0 | 190.2 | 229.3 | 269.1 | 309.6 | 350.6 |
| 10               | 92.3  | 134.3 | 177.8 | 222.6 | 268.4 | 315.0 | 362.4 | 410.5 |
| 11               | 106.3 | 154.8 | 205.0 | 256.6 | 309.4 | 363.2 | 417.9 | 473.3 |
| 12               | 120.9 | 176.2 | 233.4 | 292.2 | 352.4 | 413.6 | 476.0 | 539.1 |
| 13               | 136.2 | 198.5 | 263.0 | 329.3 | 397.1 | 466.2 | 536.5 | 607.7 |

**Table 23** continued

**Level of significance  $\alpha = 0.05$**

| $n \backslash K$ | 3     | 4     | 5     | 6     | 7      | 8      | 9      | 10     |
|------------------|-------|-------|-------|-------|--------|--------|--------|--------|
| 14               | 152.1 | 221.7 | 293.8 | 367.8 | 443.6  | 520.8  | 599.4  | 679.0  |
| 15               | 168.6 | 245.7 | 325.7 | 407.8 | 491.9  | 577.4  | 664.6  | 752.8  |
| 16               | 185.6 | 270.6 | 358.6 | 449.1 | 541.7  | 635.9  | 732.0  | 829.2  |
| 17               | 203.1 | 296.2 | 392.6 | 491.7 | 593.1  | 696.3  | 801.5  | 907.9  |
| 18               | 221.2 | 322.6 | 427.6 | 535.5 | 646.1  | 758.5  | 873.1  | 989.0  |
| 19               | 239.8 | 349.7 | 463.6 | 580.6 | 700.5  | 822.4  | 946.7  | 1072.4 |
| 20               | 258.8 | 377.6 | 500.5 | 626.9 | 756.4  | 888.1  | 1022.3 | 1158.1 |
| 21               | 278.4 | 406.1 | 538.4 | 674.4 | 813.7  | 955.4  | 1099.8 | 1245.9 |
| 22               | 298.4 | 435.3 | 577.2 | 723.0 | 872.3  | 1024.3 | 1179.1 | 1335.7 |
| 23               | 318.9 | 464.2 | 616.9 | 772.7 | 932.4  | 1094.8 | 1260.3 | 1427.7 |
| 24               | 339.8 | 495.8 | 657.4 | 823.5 | 993.7  | 1166.8 | 1343.2 | 1521.7 |
| 25               | 361.1 | 527.0 | 698.8 | 875.4 | 1056.3 | 1240.4 | 1427.9 | 1616.6 |

**Level of significance  $\alpha = 0.10$**

| $n \backslash K$ | 3     | 4     | 5     | 6     | 7     | 8      | 9      | 10     |
|------------------|-------|-------|-------|-------|-------|--------|--------|--------|
| 1                | 2.9   | 4.2   | 5.5   | 6.8   | 8.2   | 9.6    | 11.1   | 12.5   |
| 2                | 7.6   | 11.2  | 14.9  | 18.7  | 22.5  | 26.5   | 30.5   | 34.5   |
| 3                | 13.8  | 20.2  | 26.9  | 33.9  | 40.9  | 48.1   | 55.5   | 63.0   |
| 4                | 20.9  | 30.9  | 41.2  | 51.8  | 62.6  | 73.8   | 85.1   | 96.5   |
| 5                | 29.0  | 42.9  | 57.2  | 72.1  | 87.3  | 102.8  | 118.6  | 134.6  |
| 6                | 37.9  | 56.1  | 75.0  | 94.5  | 114.4 | 134.8  | 155.6  | 176.6  |
| 7                | 47.6  | 70.5  | 94.3  | 118.8 | 144.0 | 169.6  | 195.8  | 222.3  |
| 8                | 58.0  | 86.0  | 115.0 | 145.0 | 175.7 | 207.0  | 239.0  | 271.4  |
| 9                | 69.1  | 102.4 | 137.0 | 172.8 | 209.4 | 246.8  | 284.9  | 323.6  |
| 10               | 80.8  | 119.8 | 160.3 | 202.2 | 245.1 | 288.9  | 333.5  | 378.8  |
| 11               | 93.1  | 138.0 | 184.8 | 233.1 | 282.6 | 333.1  | 384.6  | 436.8  |
| 12               | 105.9 | 157.1 | 210.4 | 265.4 | 321.8 | 379.3  | 438.0  | 497.5  |
| 13               | 119.3 | 177.0 | 237.1 | 299.1 | 362.7 | 427.6  | 493.7  | 560.8  |
| 14               | 133.2 | 197.7 | 264.8 | 334.1 | 405.1 | 477.7  | 551.6  | 626.6  |
| 15               | 147.6 | 219.1 | 293.6 | 370.4 | 449.2 | 529.6  | 611.6  | 694.8  |
| 16               | 162.5 | 241.3 | 323.3 | 407.9 | 494.7 | 583.3  | 673.6  | 765.2  |
| 17               | 177.9 | 264.2 | 353.9 | 446.6 | 541.6 | 638.7  | 737.6  | 837.9  |
| 18               | 193.7 | 287.7 | 385.5 | 486.5 | 590.0 | 695.7  | 803.4  | 912.8  |
| 19               | 210.0 | 311.9 | 417.9 | 527.5 | 639.7 | 754.3  | 871.2  | 989.7  |
| 20               | 226.7 | 336.7 | 451.2 | 569.5 | 690.7 | 814.5  | 940.7  | 1068.8 |
| 21               | 243.8 | 362.2 | 485.4 | 612.6 | 743.0 | 876.2  | 1012.0 | 1149.8 |
| 22               | 261.3 | 388.2 | 520.4 | 656.8 | 796.6 | 939.4  | 1085.0 | 1232.7 |
| 23               | 279.2 | 414.9 | 556.1 | 702.0 | 851.4 | 1004.1 | 1159.7 | 1317.6 |
| 24               | 297.5 | 442.2 | 592.7 | 748.1 | 907.4 | 1070.2 | 1236.0 | 1404.3 |
| 25               | 316.2 | 470.0 | 630.0 | 795.3 | 964.6 | 1137.6 | 1314.0 | 1492.9 |

Source: Sachs, 1972

**Table 24 Critical values for the rank sum maximum test**

| $n$ | $K$ | Level of significance $\alpha$ |      |      |       |
|-----|-----|--------------------------------|------|------|-------|
|     |     | 0.10                           | 0.05 | 0.01 | 0.001 |
| 3   | 3   | 22                             | 23   |      |       |
|     | 4   | 30                             | 31   |      |       |
|     | 5   | 38                             | 39   |      |       |
|     | 6   | 46                             | 48   | 50   |       |
| 4   | 2   | 24                             | 25   |      |       |
|     | 3   | 37                             | 38   | 41   |       |
|     | 4   | 50                             | 52   | 55   |       |
|     | 5   | 63                             | 66   | 70   | 73    |
|     | 6   | 77                             | 80   | 85   | 89    |
| 5   | 2   | 35                             | 37   | 39   |       |
|     | 3   | 55                             | 57   | 61   | 64    |
|     | 4   | 75                             | 78   | 83   | 87    |
|     | 5   | 95                             | 98   | 105  | 111   |
|     | 6   | 115                            | 119  | 127  | 134   |
| 6   | 2   | 49                             | 51   | 54   |       |
|     | 3   | 77                             | 79   | 85   | 90    |
|     | 4   | 104                            | 108  | 115  | 122   |
|     | 5   | 133                            | 138  | 149  | 161   |
|     | 6   | 161                            | 167  | 180  | 196   |
| 7   | 2   | 65                             | 68   | 72   | 76    |
|     | 3   | 102                            | 105  | 112  | 119   |
|     | 4   | 138                            | 144  | 154  | 167   |
|     | 5   | 176                            | 182  | 196  | 212   |
|     | 6   | 213                            | 221  | 237  | 257   |
| 8   | 2   | 84                             | 87   | 92   | 97    |
|     | 3   | 130                            | 135  | 144  | 156   |
|     | 4   | 177                            | 183  | 197  | 212   |
|     | 5   | 225                            | 233  | 249  | 269   |
|     | 6   | 273                            | 282  | 302  | 326   |

*Source:* Sachs, 1970



**Table 25   Critical values for the Steel test**  
**One-sided testing**

| <i>n</i> | $\alpha$ | Number of samples <i>K</i> |     |     |     |     |     |     |     |
|----------|----------|----------------------------|-----|-----|-----|-----|-----|-----|-----|
|          |          | 2                          | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
| 4        | 0.05     | 11                         | 10  | 10  | 10  | 10  | –   | –   | –   |
|          | 0.01     | –                          | –   | –   | –   | –   | –   | –   | –   |
| 5        | 0.05     | 18                         | 17  | 17  | 16  | 16  | 16  | 16  | 15  |
|          | 0.01     | 15                         | –   | –   | –   | –   | –   | –   | –   |
| 6        | 0.05     | 27                         | 26  | 25  | 25  | 24  | 24  | 24  | 23  |
|          | 0.01     | 23                         | 22  | 21  | 21  | –   | –   | –   | –   |
| 7        | 0.05     | 37                         | 36  | 35  | 35  | 34  | 34  | 33  | 33  |
|          | 0.01     | 32                         | 31  | 30  | 30  | 29  | 29  | 29  | 29  |
| 8        | 0.05     | 49                         | 48  | 47  | 46  | 46  | 45  | 45  | 44  |
|          | 0.01     | 43                         | 42  | 41  | 40  | 40  | 40  | 39  | 39  |
| 9        | 0.05     | 63                         | 62  | 61  | 60  | 59  | 59  | 58  | 58  |
|          | 0.01     | 56                         | 55  | 54  | 53  | 52  | 52  | 51  | 51  |
| 10       | 0.05     | 79                         | 77  | 76  | 75  | 74  | 74  | 73  | 72  |
|          | 0.01     | 71                         | 69  | 68  | 67  | 66  | 66  | 65  | 65  |
| 11       | 0.05     | 97                         | 95  | 93  | 92  | 91  | 90  | 90  | 89  |
|          | 0.01     | 87                         | 85  | 84  | 83  | 82  | 81  | 81  | 80  |
| 12       | 0.05     | 116                        | 114 | 112 | 111 | 110 | 109 | 108 | 108 |
|          | 0.01     | 105                        | 103 | 102 | 100 | 99  | 99  | 98  | 98  |
| 13       | 0.05     | 138                        | 135 | 133 | 132 | 130 | 129 | 129 | 128 |
|          | 0.01     | 125                        | 123 | 121 | 120 | 119 | 118 | 117 | 117 |
| 14       | 0.05     | 161                        | 158 | 155 | 154 | 153 | 152 | 151 | 150 |
|          | 0.01     | 147                        | 144 | 142 | 141 | 140 | 139 | 138 | 137 |
| 15       | 0.05     | 186                        | 182 | 180 | 178 | 177 | 176 | 175 | 174 |
|          | 0.01     | 170                        | 167 | 165 | 164 | 162 | 161 | 160 | 160 |
| 16       | 0.05     | 213                        | 209 | 206 | 204 | 203 | 201 | 200 | 199 |
|          | 0.01     | 196                        | 192 | 190 | 188 | 187 | 186 | 185 | 184 |
| 17       | 0.05     | 241                        | 237 | 234 | 232 | 231 | 229 | 228 | 227 |
|          | 0.01     | 223                        | 219 | 217 | 215 | 213 | 212 | 211 | 210 |
| 18       | 0.05     | 272                        | 267 | 264 | 262 | 260 | 259 | 257 | 256 |
|          | 0.01     | 252                        | 248 | 245 | 243 | 241 | 240 | 239 | 238 |
| 19       | 0.05     | 304                        | 299 | 296 | 294 | 292 | 290 | 288 | 287 |
|          | 0.01     | 282                        | 278 | 275 | 273 | 271 | 270 | 268 | 267 |
| 20       | 0.05     | 339                        | 333 | 330 | 327 | 325 | 323 | 322 | 320 |
|          | 0.01     | 315                        | 310 | 307 | 305 | 303 | 301 | 300 | 299 |

**Table 25** continued**Two-sided testing**

| $n$ | $\alpha$ | Number of samples $K$ |     |     |     |     |     |     |     |
|-----|----------|-----------------------|-----|-----|-----|-----|-----|-----|-----|
|     |          | 2                     | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
| 4   | 0.05     | 10                    | —   | —   | —   | —   | —   | —   | —   |
|     | 0.01     | —                     | —   | —   | —   | —   | —   | —   | —   |
| 5   | 0.05     | 16                    | 16  | 16  | 15  | —   | —   | —   | —   |
|     | 0.01     | —                     | —   | —   | —   | —   | —   | —   | —   |
| 6   | 0.05     | 25                    | 24  | 23  | 23  | 22  | 22  | 22  | 21  |
|     | 0.01     | 21                    | —   | —   | —   | —   | —   | —   | —   |
| 7   | 0.05     | 35                    | 33  | 33  | 32  | 32  | 31  | 31  | 30  |
|     | 0.01     | 30                    | 29  | 28  | 28  | —   | —   | —   | —   |
| 8   | 0.05     | 46                    | 45  | 44  | 43  | 43  | 42  | 42  | 41  |
|     | 0.01     | 41                    | 40  | 39  | 38  | 38  | 37  | 37  | 37  |
| 9   | 0.05     | 60                    | 58  | 57  | 56  | 55  | 55  | 54  | 54  |
|     | 0.01     | 53                    | 52  | 51  | 50  | 49  | 49  | 49  | 48  |
| 10  | 0.05     | 75                    | 73  | 72  | 71  | 70  | 69  | 69  | 68  |
|     | 0.01     | 68                    | 66  | 65  | 64  | 63  | 62  | 62  | 62  |
| 11  | 0.05     | 92                    | 90  | 88  | 87  | 86  | 85  | 85  | 84  |
|     | 0.01     | 84                    | 82  | 80  | 79  | 78  | 78  | 77  | 77  |
| 12  | 0.05     | 111                   | 108 | 107 | 105 | 104 | 103 | 103 | 102 |
|     | 0.01     | 101                   | 99  | 97  | 96  | 95  | 94  | 94  | 93  |
| 13  | 0.05     | 132                   | 129 | 127 | 125 | 124 | 123 | 122 | 121 |
|     | 0.01     | 121                   | 118 | 116 | 115 | 114 | 113 | 112 | 112 |
| 14  | 0.05     | 154                   | 151 | 149 | 147 | 145 | 144 | 144 | 143 |
|     | 0.01     | 142                   | 139 | 137 | 135 | 134 | 133 | 132 | 132 |
| 15  | 0.05     | 179                   | 175 | 172 | 171 | 169 | 168 | 167 | 166 |
|     | 0.01     | 165                   | 162 | 159 | 158 | 156 | 155 | 154 | 154 |
| 16  | 0.05     | 205                   | 201 | 196 | 196 | 194 | 193 | 192 | 191 |
|     | 0.01     | 189                   | 186 | 184 | 182 | 180 | 179 | 178 | 177 |
| 17  | 0.05     | 233                   | 228 | 225 | 223 | 221 | 219 | 218 | 217 |
|     | 0.01     | 216                   | 212 | 210 | 208 | 206 | 205 | 204 | 203 |
| 18  | 0.05     | 263                   | 258 | 254 | 252 | 250 | 248 | 247 | 246 |
|     | 0.01     | 244                   | 240 | 237 | 235 | 233 | 232 | 231 | 230 |
| 19  | 0.05     | 294                   | 289 | 285 | 283 | 280 | 279 | 277 | 276 |
|     | 0.01     | 274                   | 270 | 267 | 265 | 262 | 261 | 260 | 259 |
| 20  | 0.05     | 328                   | 322 | 318 | 315 | 313 | 311 | 309 | 308 |
|     | 0.01     | 306                   | 302 | 298 | 296 | 293 | 292 | 290 | 289 |

Source: De Jonge, 1963–4

**Table 26 Critical values of  $r_s$  for the Spearman rank correlation test**

| $n$ | Level of significance $\alpha$ |        |        |        |        |        |
|-----|--------------------------------|--------|--------|--------|--------|--------|
|     | 0.001                          | 0.005  | 0.010  | 0.025  | 0.050  | 0.100  |
| 4   | —                              | —      | —      | —      | 0.8000 | 0.8000 |
| 5   | —                              | —      | 0.9000 | 0.9000 | 0.8000 | 0.7000 |
| 6   | —                              | 0.9429 | 0.8857 | 0.8286 | 0.7714 | 0.6000 |
| 7   | 0.9643                         | 0.8929 | 0.8571 | 0.7450 | 0.6786 | 0.5357 |
| 8   | 0.9286                         | 0.8571 | 0.8095 | 0.6905 | 0.5952 | 0.4762 |
| 9   | 0.9000                         | 0.8167 | 0.7667 | 0.6833 | 0.5833 | 0.4667 |
| 10  | 0.8667                         | 0.7818 | 0.7333 | 0.6364 | 0.5515 | 0.4424 |
| 11  | 0.8455                         | 0.7545 | 0.7000 | 0.6091 | 0.5273 | 0.4182 |
| 12  | 0.8182                         | 0.7273 | 0.6713 | 0.5804 | 0.4965 | 0.3986 |
| 13  | 0.7912                         | 0.6978 | 0.6429 | 0.5549 | 0.4780 | 0.3791 |
| 14  | 0.7670                         | 0.6747 | 0.6220 | 0.5341 | 0.4593 | 0.3626 |
| 15  | 0.7464                         | 0.6536 | 0.6000 | 0.5179 | 0.4429 | 0.3500 |
| 16  | 0.7265                         | 0.6324 | 0.5824 | 0.5000 | 0.4265 | 0.3382 |
| 17  | 0.7083                         | 0.6152 | 0.5637 | 0.4853 | 0.4118 | 0.3260 |
| 18  | 0.6904                         | 0.5975 | 0.5480 | 0.4716 | 0.3994 | 0.3148 |
| 19  | 0.6737                         | 0.5825 | 0.5333 | 0.4579 | 0.3895 | 0.3070 |
| 20  | 0.6586                         | 0.5684 | 0.5203 | 0.4451 | 0.3789 | 0.2977 |
| 21  | 0.6455                         | 0.5545 | 0.5078 | 0.4351 | 0.3688 | 0.2909 |
| 22  | 0.6318                         | 0.5426 | 0.4963 | 0.4241 | 0.3597 | 0.2829 |
| 23  | 0.6186                         | 0.5306 | 0.4852 | 0.4150 | 0.3518 | 0.2767 |
| 24  | 0.6070                         | 0.5200 | 0.4748 | 0.4061 | 0.3435 | 0.2704 |
| 25  | 0.5962                         | 0.5100 | 0.4654 | 0.3977 | 0.3362 | 0.2646 |
| 26  | 0.5856                         | 0.5002 | 0.4564 | 0.3894 | 0.3299 | 0.2588 |
| 27  | 0.5757                         | 0.4915 | 0.4481 | 0.3822 | 0.3236 | 0.2540 |
| 28  | 0.5660                         | 0.4828 | 0.4401 | 0.3749 | 0.3175 | 0.2490 |
| 29  | 0.5567                         | 0.4744 | 0.4320 | 0.3685 | 0.3113 | 0.2443 |
| 30  | 0.5479                         | 0.4665 | 0.4251 | 0.3620 | 0.3059 | 0.2400 |

Source: Sachs, 1972

**Table 27 Critical values of S for the Kendall rank correlation test**

|     | Level of significance $\alpha$ |               |              |               |
|-----|--------------------------------|---------------|--------------|---------------|
|     | 0.10<br>One-sided              | 0.05<br>0.025 | 0.02<br>0.01 | 0.05<br>0.005 |
| $n$ |                                |               |              |               |
| 4   | 6                              | —             | —            | —             |
| 5   | 8                              | 10            | 10           | —             |
| 6   | 11                             | 13            | 13           | 15            |
| 7   | 13                             | 15            | 17           | 19            |
| 8   | 16                             | 18            | 20           | 22            |
| 9   | 18                             | 20            | 24           | 26            |
| 10  | 21                             | 23            | 27           | 29            |
| 11  | 23                             | 27            | 31           | 33            |
| 12  | 26                             | 30            | 36           | 38            |
| 13  | 28                             | 34            | 40           | 44            |
| 14  | 33                             | 37            | 43           | 47            |
| 15  | 35                             | 41            | 49           | 53            |
| 16  | 38                             | 46            | 52           | 58            |
| 17  | 42                             | 50            | 58           | 64            |
| 18  | 45                             | 53            | 63           | 69            |
| 19  | 49                             | 57            | 67           | 75            |
| 20  | 52                             | 62            | 72           | 80            |
| 21  | 56                             | 66            | 78           | 86            |
| 22  | 61                             | 71            | 83           | 91            |
| 23  | 65                             | 75            | 89           | 99            |
| 24  | 68                             | 80            | 94           | 104           |
| 25  | 72                             | 86            | 100          | 110           |
| 26  | 77                             | 91            | 107          | 117           |
| 27  | 81                             | 95            | 113          | 125           |
| 28  | 86                             | 100           | 118          | 130           |
| 29  | 90                             | 106           | 126          | 138           |
| 30  | 95                             | 111           | 131          | 145           |
| 31  | 99                             | 117           | 137          | 151           |
| 32  | 104                            | 122           | 144          | 160           |
| 33  | 108                            | 128           | 152          | 166           |
| 34  | 113                            | 133           | 157          | 175           |
| 35  | 117                            | 139           | 165          | 181           |
| 36  | 122                            | 146           | 172          | 190           |
| 37  | 128                            | 152           | 178          | 198           |
| 38  | 133                            | 157           | 185          | 205           |
| 39  | 139                            | 163           | 193          | 213           |
| 40  | 144                            | 170           | 200          | 222           |

Source: De Jonge, 1963–4

**Table 28 Critical values of  $D$  for the adjacency test**

Columns  $a$  denote the lower boundaries or the left-sided critical values.  
Columns  $b$  denote the upper boundaries or the right-sided critical values.

|           | Level of significance $\alpha$ |      |      |      |
|-----------|--------------------------------|------|------|------|
|           | 0.10                           |      | 0.02 |      |
| Two-sided | 0.05                           |      | 0.01 |      |
| One-sided |                                |      |      |      |
| $n$       | $a$                            | $b$  | $a$  | $b$  |
| 4         | 0.78                           | 3.22 | 0.63 | 3.37 |
| 5         | 0.82                           | 3.18 | 0.54 | 3.46 |
| 6         | 0.89                           | 3.11 | 0.56 | 3.44 |
| 7         | 0.94                           | 3.06 | 0.61 | 3.39 |
| 8         | 0.98                           | 3.02 | 0.66 | 3.34 |
| 9         | 1.02                           | 2.98 | 0.71 | 3.29 |
| 10        | 1.06                           | 2.94 | 0.75 | 3.25 |
| 11        | 1.10                           | 2.90 | 0.79 | 3.21 |
| 12        | 1.13                           | 2.87 | 0.83 | 3.17 |
| 15        | 1.21                           | 2.79 | 0.92 | 3.08 |
| 20        | 1.30                           | 2.70 | 1.04 | 2.96 |
| 25        | 1.37                           | 2.63 | 1.13 | 2.87 |

Source: Hart, 1942

**Table 29 Critical values of  $r$  for the serial correlation test**

Columns  $a$  denote the lower boundaries or the left-sided critical values.  
Columns  $b$  denote the upper boundaries or the right-sided critical values.

|           | Level of significance $\alpha$ |       |        |       |
|-----------|--------------------------------|-------|--------|-------|
|           | 0.10                           |       | 0.02   |       |
| Two-sided | 0.05                           |       | 0.01   |       |
| One-sided |                                |       |        |       |
| $n$       | $a$                            | $b$   | $a$    | $b$   |
| 5         | -0.753                         | 0.253 | -0.798 | 0.297 |
| 6         | -0.708                         | 0.345 | -0.863 | 0.447 |
| 7         | -0.674                         | 0.370 | -0.799 | 0.510 |
| 8         | -0.625                         | 0.371 | -0.764 | 0.531 |
| 9         | -0.593                         | 0.366 | -0.737 | 0.533 |
| 10        | -0.564                         | 0.360 | -0.705 | 0.525 |
| 11        | -0.539                         | 0.353 | -0.679 | 0.515 |
| 12        | -0.516                         | 0.348 | -0.655 | 0.505 |
| 13        | -0.497                         | 0.341 | -0.634 | 0.495 |
| 14        | -0.479                         | 0.335 | -0.615 | 0.485 |
| 15        | -0.462                         | 0.328 | -0.597 | 0.475 |
| 20        | -0.399                         | 0.328 | -0.524 | 0.432 |
| 25        | -0.356                         | 0.276 | -0.473 | 0.398 |
| 30        | -0.325                         | 0.257 | -0.433 | 0.370 |

Source: Anderson, 1942

**Table 30 Critical values for the run test on successive differences**

Columns  $a$  denote the lower boundaries or the left-sided critical values.  
Columns  $b$  denote the upper boundaries or the right-sided critical values.

|     | Level of significance $\alpha$ |     |      |     |
|-----|--------------------------------|-----|------|-----|
|     | 0.02                           |     | 0.10 |     |
|     | 0.01                           |     | 0.05 |     |
| $n$ | $a$                            | $b$ | $a$  | $b$ |
| 5   | —                              | —   | 1    | —   |
| 6   | 1                              | —   | 1    | —   |
| 7   | 1                              | —   | 2    | —   |
| 8   | 2                              | —   | 2    | —   |
| 9   | 2                              | —   | 3    | 8   |
| 10  | 3                              | —   | 3    | 9   |
| 11  | 3                              | —   | 4    | 10  |
| 12  | 4                              | —   | 4    | 11  |
| 13  | 4                              | —   | 5    | 12  |
| 14  | 5                              | 13  | 6    | 12  |
| 15  | 5                              | 14  | 6    | 13  |
| 16  | 6                              | 15  | 7    | 14  |
| 17  | 6                              | 16  | 7    | 15  |
| 18  | 7                              | 17  | 8    | 15  |
| 19  | 7                              | 17  | 8    | 16  |
| 20  | 8                              | 18  | 9    | 17  |
| 21  | 8                              | 19  | 10   | 18  |
| 22  | 9                              | 20  | 10   | 18  |
| 23  | 10                             | 21  | 11   | 19  |
| 24  | 10                             | 21  | 11   | 20  |
| 25  | 11                             | 22  | 12   | 21  |
| 26  | 11                             | 23  | 13   | 21  |
| 27  | 12                             | 24  | 13   | 22  |
| 28  | 12                             | 24  | 14   | 23  |
| 29  | 13                             | 25  | 14   | 24  |
| 30  | 13                             | 26  | 15   | 24  |
| 31  | 14                             | 27  | 16   | 25  |
| 32  | 15                             | 27  | 16   | 26  |
| 33  | 15                             | 28  | 17   | 27  |
| 34  | 16                             | 29  | 17   | 27  |
| 35  | 16                             | 30  | 18   | 28  |
| 36  | 17                             | 30  | 19   | 29  |
| 37  | 18                             | 31  | 19   | 29  |
| 38  | 18                             | 32  | 20   | 30  |
| 39  | 19                             | 33  | 20   | 31  |
| 40  | 19                             | 33  | 21   | 32  |

Source: De Jonge, 1963–4

**Table 31 Critical values for the run test (equal sample sizes)**

Columns *a* denote the lower boundaries or the left-sided critical values.  
Columns *b* denote the upper boundaries or the right-sided critical values.

| Two-sided<br>One-sided | Level of significance $\alpha$ |          |          |          |          |          |          |          |
|------------------------|--------------------------------|----------|----------|----------|----------|----------|----------|----------|
|                        | 0.10                           |          | 0.05     |          | 0.02     |          | 0.01     |          |
|                        | 0.05                           |          | 0.025    |          | 0.01     |          | 0.005    |          |
| $n_1 = n_2$            | <i>a</i>                       | <i>b</i> | <i>a</i> | <i>b</i> | <i>a</i> | <i>b</i> | <i>a</i> | <i>b</i> |
| 5                      | 3                              | 9        |          |          | 2        | 10       |          |          |
| 6                      | 3                              | 11       |          |          | 2        | 12       |          |          |
| 7                      | 4                              | 12       |          |          | 3        | 13       |          |          |
| 8                      | 5                              | 13       |          |          | 4        | 14       |          |          |
| 9                      | 6                              | 14       |          |          | 4        | 16       |          |          |
| 10                     | 6                              | 16       |          |          | 5        | 17       |          |          |
| 11                     | 7                              | 17       | 7        | 16       | 6        | 18       | 5        | 18       |
| 12                     | 8                              | 18       | 7        | 18       | 7        | 19       | 6        | 19       |
| 13                     | 9                              | 19       | 8        | 19       | 7        | 21       | 7        | 20       |
| 14                     | 10                             | 20       | 9        | 20       | 8        | 22       | 7        | 22       |
| 15                     | 11                             | 21       | 10       | 21       | 9        | 23       | 8        | 23       |
| 16                     | 11                             | 23       | 11       | 22       | 10       | 24       | 9        | 24       |
| 17                     | 12                             | 24       | 11       | 24       | 10       | 26       | 10       | 25       |
| 18                     | 13                             | 25       | 12       | 25       | 11       | 27       | 10       | 27       |
| 19                     | 14                             | 26       | 13       | 26       | 12       | 28       | 11       | 28       |
| 20                     | 15                             | 27       | 14       | 27       | 13       | 29       | 12       | 29       |
| 21                     | 16                             | 28       |          |          | 14       | 30       |          |          |
| 22                     | 17                             | 29       |          |          | 14       | 32       |          |          |
| 23                     | 17                             | 31       |          |          | 15       | 33       |          |          |
| 24                     | 18                             | 32       |          |          | 16       | 34       |          |          |
| 25                     | 19                             | 33       | 18       | 33       | 17       | 35       | 16       | 35       |
| 26                     | 20                             | 34       |          |          | 18       | 36       |          |          |
| 27                     | 21                             | 35       |          |          | 19       | 37       |          |          |
| 28                     | 22                             | 36       |          |          | 19       | 39       |          |          |
| 29                     | 23                             | 37       |          |          | 20       | 40       |          |          |
| 30                     | 24                             | 38       | 22       | 39       | 21       | 41       | 20       | 41       |
| 35                     | 28                             | 43       | 27       | 44       | 25       | 46       | 24       | 47       |
| 40                     | 33                             | 48       | 31       | 50       | 30       | 51       | 29       | 52       |
| 45                     | 37                             | 54       | 36       | 55       | 34       | 57       | 33       | 58       |
| 50                     | 42                             | 59       | 40       | 61       | 38       | 63       | 37       | 64       |
| 55                     | 46                             | 65       | 45       | 66       | 43       | 68       | 42       | 69       |
| 60                     | 51                             | 70       | 49       | 72       | 47       | 74       | 46       | 75       |
| 65                     | 56                             | 75       | 54       | 77       | 52       | 79       | 50       | 81       |
| 70                     | 60                             | 81       | 58       | 83       | 56       | 85       | 55       | 86       |
| 75                     | 65                             | 86       | 63       | 88       | 61       | 90       | 59       | 92       |
| 80                     | 70                             | 91       | 68       | 93       | 65       | 96       | 64       | 97       |
| 85                     | 74                             | 97       | 72       | 99       | 70       | 101      | 68       | 103      |
| 90                     | 79                             | 102      | 77       | 104      | 74       | 107      | 73       | 108      |
| 95                     | 84                             | 107      | 82       | 109      | 79       | 112      | 77       | 114      |
| 100                    | 88                             | 117      | 80       | 115      | 84       | 113      | 82       | 119      |

Source: Dixon and Massey, 1957

**Table 32 Critical values for the Wilcoxon–Wilcox test (two-sided)****Level of significance  $\alpha = 0.01$** 

| $n \backslash K$ | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|------------------|------|------|------|------|------|------|------|------|
| 1                | 4.1  | 5.7  | 7.3  | 8.9  | 10.5 | 12.2 | 13.9 | 15.6 |
| 2                | 5.8  | 8.0  | 10.3 | 12.6 | 14.9 | 17.3 | 19.7 | 22.1 |
| 3                | 7.1  | 9.8  | 12.6 | 15.4 | 18.3 | 21.2 | 24.1 | 27.0 |
| 4                | 8.2  | 11.4 | 14.6 | 17.8 | 21.1 | 24.4 | 27.8 | 31.2 |
| 5                | 9.2  | 12.7 | 16.3 | 19.9 | 23.6 | 27.3 | 31.1 | 34.9 |
| 6                | 10.1 | 13.9 | 17.8 | 21.8 | 25.8 | 29.9 | 34.1 | 38.2 |
| 7                | 10.9 | 15.0 | 19.3 | 23.5 | 27.9 | 32.3 | 36.8 | 41.3 |
| 8                | 11.7 | 16.1 | 20.6 | 25.2 | 29.8 | 34.6 | 39.3 | 44.2 |
| 9                | 12.4 | 17.1 | 21.8 | 26.7 | 31.6 | 36.6 | 41.7 | 46.8 |
| 10               | 13.0 | 18.0 | 23.0 | 28.1 | 33.4 | 38.6 | 44.0 | 49.4 |
| 11               | 13.7 | 18.9 | 24.1 | 29.5 | 35.0 | 40.5 | 46.1 | 51.8 |
| 12               | 14.3 | 19.7 | 25.2 | 30.8 | 36.5 | 42.3 | 48.2 | 54.1 |
| 13               | 14.9 | 20.5 | 26.2 | 32.1 | 38.0 | 44.0 | 50.1 | 56.3 |
| 14               | 15.4 | 21.3 | 27.2 | 33.3 | 39.5 | 45.7 | 52.0 | 58.4 |
| 15               | 16.0 | 22.0 | 28.2 | 34.5 | 40.8 | 47.3 | 53.9 | 60.5 |
| 16               | 16.5 | 22.7 | 29.1 | 35.6 | 42.2 | 48.9 | 55.6 | 62.5 |
| 17               | 17.0 | 23.4 | 30.0 | 36.7 | 43.5 | 50.4 | 57.3 | 64.4 |
| 18               | 17.5 | 24.1 | 30.9 | 37.8 | 44.7 | 51.8 | 59.0 | 66.2 |
| 19               | 18.0 | 24.8 | 31.7 | 38.8 | 46.0 | 53.2 | 60.6 | 68.1 |
| 20               | 18.4 | 25.4 | 32.5 | 39.8 | 47.2 | 54.6 | 62.2 | 69.8 |
| 21               | 18.9 | 26.0 | 33.4 | 40.9 | 48.3 | 56.0 | 63.7 | 71.6 |
| 22               | 19.3 | 26.7 | 34.1 | 41.7 | 49.5 | 57.3 | 65.2 | 73.2 |
| 23               | 19.8 | 27.3 | 34.9 | 42.7 | 50.6 | 58.6 | 66.7 | 74.9 |
| 24               | 20.2 | 27.8 | 35.7 | 43.6 | 51.7 | 59.8 | 68.1 | 76.5 |
| 25               | 20.6 | 28.4 | 36.4 | 44.5 | 52.7 | 61.1 | 69.5 | 78.1 |

**Level of significance  $\alpha = 0.05$** 

| $n \backslash K$ | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|------------------|------|------|------|------|------|------|------|------|
| 1                | 3.3  | 4.7  | 6.1  | 7.5  | 9.0  | 10.5 | 12.0 | 13.5 |
| 2                | 4.7  | 6.6  | 8.6  | 10.7 | 12.7 | 14.8 | 17.0 | 19.2 |
| 3                | 5.7  | 8.1  | 10.6 | 13.1 | 15.6 | 18.2 | 20.8 | 23.5 |
| 4                | 6.6  | 9.4  | 12.2 | 15.1 | 18.0 | 21.0 | 24.0 | 27.1 |
| 5                | 7.4  | 10.5 | 13.6 | 16.9 | 20.1 | 23.5 | 26.9 | 30.3 |
| 6                | 8.1  | 11.5 | 14.9 | 18.5 | 22.1 | 25.7 | 29.4 | 33.2 |
| 7                | 8.8  | 12.4 | 16.1 | 19.9 | 23.9 | 27.8 | 31.8 | 35.8 |
| 8                | 9.4  | 13.3 | 17.3 | 21.3 | 25.5 | 29.7 | 34.0 | 38.3 |
| 9                | 9.9  | 14.1 | 18.3 | 22.6 | 27.0 | 31.5 | 36.0 | 40.6 |
| 10               | 10.5 | 14.8 | 19.3 | 23.8 | 28.5 | 33.2 | 38.0 | 42.8 |
| 11               | 11.0 | 15.6 | 20.2 | 25.0 | 29.9 | 34.8 | 39.8 | 44.9 |
| 12               | 11.5 | 16.2 | 21.1 | 26.1 | 31.2 | 36.4 | 41.6 | 46.9 |
| 13               | 11.9 | 16.9 | 22.0 | 27.2 | 32.5 | 37.9 | 43.3 | 48.8 |
| 14               | 12.4 | 17.5 | 22.8 | 28.2 | 33.7 | 39.3 | 45.0 | 50.7 |
| 15               | 12.8 | 18.2 | 23.6 | 29.2 | 34.9 | 40.7 | 46.5 | 52.5 |



**Table 32** continued

**Level of significance  $\alpha = 0.05$**

| $n \backslash K$ | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|------------------|------|------|------|------|------|------|------|------|
| 16               | 13.3 | 18.8 | 24.4 | 30.2 | 36.0 | 42.0 | 48.1 | 54.2 |
| 17               | 13.7 | 19.3 | 25.2 | 31.1 | 37.1 | 43.3 | 49.5 | 55.9 |
| 18               | 14.1 | 19.9 | 25.9 | 32.0 | 38.2 | 44.5 | 51.0 | 57.5 |
| 19               | 14.4 | 20.4 | 26.6 | 32.9 | 39.3 | 45.8 | 52.4 | 59.0 |
| 20               | 14.8 | 21.0 | 27.3 | 33.7 | 40.3 | 47.0 | 53.7 | 60.6 |
| 21               | 15.2 | 21.5 | 28.0 | 34.6 | 41.3 | 48.1 | 55.1 | 62.1 |
| 22               | 15.5 | 22.0 | 28.6 | 35.4 | 42.3 | 49.2 | 56.4 | 63.5 |
| 23               | 15.9 | 22.5 | 29.3 | 36.2 | 43.2 | 50.3 | 57.6 | 65.0 |
| 24               | 16.2 | 23.0 | 29.9 | 36.9 | 44.1 | 51.4 | 58.9 | 66.4 |
| 25               | 16.6 | 23.5 | 30.5 | 37.7 | 45.0 | 52.5 | 60.1 | 67.7 |

**Level of significance  $\alpha = 0.10$**

| $n \backslash K$ | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|------------------|------|------|------|------|------|------|------|------|
| 1                | 2.9  | 4.2  | 5.5  | 6.8  | 8.2  | 9.6  | 11.1 | 12.5 |
| 2                | 4.1  | 5.9  | 7.8  | 9.7  | 11.6 | 13.6 | 15.6 | 17.7 |
| 3                | 5.0  | 7.2  | 9.5  | 11.9 | 14.2 | 16.7 | 19.1 | 21.7 |
| 4                | 5.8  | 8.4  | 11.0 | 13.7 | 16.5 | 19.3 | 22.1 | 25.0 |
| 5                | 6.5  | 9.4  | 12.3 | 15.3 | 18.4 | 21.5 | 24.7 | 28.0 |
| 6                | 7.1  | 10.2 | 13.5 | 16.8 | 20.2 | 23.6 | 27.1 | 30.6 |
| 7                | 7.7  | 11.1 | 14.5 | 18.1 | 21.8 | 25.5 | 29.3 | 33.1 |
| 8                | 8.2  | 11.8 | 15.6 | 19.4 | 23.3 | 27.2 | 31.3 | 35.4 |
| 9                | 8.7  | 12.5 | 16.5 | 20.5 | 24.7 | 28.9 | 33.2 | 37.5 |
| 10               | 9.2  | 13.2 | 17.4 | 21.7 | 26.0 | 30.4 | 35.0 | 39.5 |
| 11               | 9.6  | 13.9 | 18.2 | 22.7 | 27.3 | 31.9 | 36.7 | 41.5 |
| 12               | 10.1 | 14.5 | 19.0 | 23.7 | 28.5 | 33.4 | 38.3 | 43.3 |
| 13               | 10.5 | 15.1 | 19.8 | 24.7 | 29.7 | 34.7 | 39.9 | 45.1 |
| 14               | 10.9 | 15.7 | 20.6 | 25.6 | 30.8 | 36.0 | 41.4 | 46.8 |
| 15               | 11.2 | 16.2 | 21.3 | 26.5 | 31.9 | 37.3 | 42.8 | 48.4 |
| 16               | 11.6 | 16.7 | 22.0 | 27.4 | 32.9 | 38.5 | 44.2 | 50.0 |
| 17               | 12.0 | 17.2 | 22.7 | 28.2 | 33.9 | 39.7 | 45.6 | 51.5 |
| 18               | 12.3 | 17.7 | 23.3 | 29.1 | 34.9 | 40.9 | 46.9 | 53.0 |
| 19               | 12.6 | 18.2 | 24.0 | 29.9 | 35.9 | 42.0 | 48.2 | 54.5 |
| 20               | 13.0 | 18.7 | 24.6 | 30.6 | 36.8 | 43.1 | 49.4 | 55.9 |
| 21               | 13.3 | 19.2 | 25.2 | 31.4 | 37.7 | 44.1 | 50.7 | 57.3 |
| 22               | 13.6 | 19.6 | 25.8 | 32.1 | 38.6 | 45.2 | 51.9 | 58.6 |
| 23               | 13.9 | 20.1 | 26.4 | 32.8 | 39.5 | 46.2 | 53.0 | 60.0 |
| 24               | 14.2 | 20.5 | 26.9 | 33.6 | 40.3 | 47.2 | 54.2 | 61.2 |
| 25               | 14.5 | 20.9 | 27.5 | 34.2 | 41.1 | 48.1 | 55.3 | 62.5 |

Source: Sachs, 1972

**Table 33 Durbin–Watson test bounds**

$d_L$  denotes the lower boundary or left-sided critical values.

$d_U$  denotes the upper boundary or right-sided critical values.

Example: for  $n = 20$ ,  $\alpha = 0.01$ , and two independent variables,

$d_L = 0.86$  and  $d_U = 1.27$ .

**Level of significance  $\alpha = 0.05$**

| $n$ | Number of independent variables ( $p - 1$ ) |       |       |       |       |       |       |       |       |       |
|-----|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|     | 1   |       | 2     |       | 3     |       | 4     |       | 5     |       |
|     | $d_L$                                       | $d_U$ | $d_L$ | $d_U$ | $d_L$ | $d_U$ | $d_L$ | $d_U$ | $d_L$ | $d_U$ |
| 15  | 1.08  | 1.36  | 0.95  | 1.54  | 0.82  | 1.75  | 0.69  | 1.97  | 0.56  | 2.21  |
| 16  | 1.10  | 1.37  | 0.98  | 1.54  | 0.86  | 1.73  | 0.74  | 1.93  | 0.62  | 2.15  |
| 17  | 1.13  | 1.38  | 1.02  | 1.54  | 0.90  | 1.71  | 0.78  | 1.90  | 0.67  | 2.10  |
| 18  | 1.16  | 1.39  | 1.05  | 1.53  | 0.93  | 1.69  | 0.82  | 1.87  | 0.71  | 2.06  |
| 19  | 1.18  | 1.40  | 1.08  | 1.53  | 0.97  | 1.68  | 0.86  | 1.85  | 0.75  | 2.02  |
| 20  | 1.20  | 1.41  | 1.10  | 1.54  | 1.00  | 1.68  | 0.90  | 1.83  | 0.79  | 1.99  |
| 21  | 1.22  | 1.42  | 1.13  | 1.54  | 1.03  | 1.67  | 0.93  | 1.81  | 0.83  | 1.96  |
| 22  | 1.24  | 1.43  | 1.15  | 1.54  | 1.05  | 1.66  | 0.96  | 1.80  | 0.86  | 1.94  |
| 23  | 1.26  | 1.44  | 1.17  | 1.54  | 1.08  | 1.66  | 0.99  | 1.79  | 0.90  | 1.92  |
| 24  | 1.27  | 1.45  | 1.19  | 1.55  | 1.10  | 1.66  | 1.01  | 1.78  | 0.93  | 1.90  |
| 25  | 1.29  | 1.45  | 1.21  | 1.55  | 1.12  | 1.66  | 1.04  | 1.77  | 0.95  | 1.89  |
| 26  | 1.30  | 1.46  | 1.22  | 1.55  | 1.14  | 1.65  | 1.06  | 1.76  | 0.98  | 1.88  |
| 27  | 1.32  | 1.47  | 1.24  | 1.56  | 1.16  | 1.65  | 1.08  | 1.76  | 1.01  | 1.86  |
| 28  | 1.33  | 1.48  | 1.26  | 1.56  | 1.18  | 1.65  | 1.10  | 1.75  | 1.03  | 1.85  |
| 29  | 1.34  | 1.48  | 1.27  | 1.56  | 1.20  | 1.65  | 1.12  | 1.74  | 1.05  | 1.84  |
| 30  | 1.35  | 1.49  | 1.28  | 1.57  | 1.21  | 1.65  | 1.14  | 1.74  | 1.07  | 1.83  |
| 31  | 1.36  | 1.50  | 1.30  | 1.57  | 1.23  | 1.65  | 1.16  | 1.74  | 1.09  | 1.83  |
| 32  | 1.37  | 1.50  | 1.31  | 1.57  | 1.24  | 1.65  | 1.18  | 1.73  | 1.11  | 1.82  |
| 33  | 1.38  | 1.51  | 1.32  | 1.58  | 1.26  | 1.65  | 1.19  | 1.73  | 1.13  | 1.81  |
| 34  | 1.39  | 1.51  | 1.33  | 1.58  | 1.27  | 1.65  | 1.21  | 1.73  | 1.15  | 1.81  |
| 35  | 1.40  | 1.52  | 1.34  | 1.58  | 1.28  | 1.65  | 1.22  | 1.73  | 1.16  | 1.80  |
| 36  | 1.41  | 1.52  | 1.35  | 1.59  | 1.29  | 1.65  | 1.24  | 1.73  | 1.18  | 1.80  |
| 37  | 1.42  | 1.53  | 1.36  | 1.59  | 1.31  | 1.66  | 1.25  | 1.72  | 1.19  | 1.80  |
| 38  | 1.43  | 1.54  | 1.37  | 1.59  | 1.32  | 1.66  | 1.26  | 1.72  | 1.21  | 1.79  |
| 39  | 1.43  | 1.54  | 1.38  | 1.60  | 1.33  | 1.66  | 1.27  | 1.72  | 1.22  | 1.79  |
| 40  | 1.44  | 1.54  | 1.39  | 1.60  | 1.34  | 1.66  | 1.27  | 1.72  | 1.23  | 1.79  |
| 45  | 1.48  | 1.57  | 1.43  | 1.62  | 1.38  | 1.67  | 1.34  | 1.72  | 1.29  | 1.78  |
| 50  | 1.50  | 1.59  | 1.46  | 1.63  | 1.42  | 1.67  | 1.38  | 1.72  | 1.34  | 1.77  |
| 55  | 1.53  | 1.60  | 1.49  | 1.64  | 1.45  | 1.68  | 1.41  | 1.72  | 1.38  | 1.77  |
| 60  | 1.55  | 1.62  | 1.51  | 1.65  | 1.48  | 1.69  | 1.44  | 1.73  | 1.41  | 1.77  |
| 65  | 1.57  | 1.63  | 1.54  | 1.66  | 1.50  | 1.70  | 1.47  | 1.73  | 1.44  | 1.77  |
| 70  | 1.58  | 1.64  | 1.55  | 1.67  | 1.52  | 1.70  | 1.49  | 1.74  | 1.46  | 1.77  |
| 75  | 1.60  | 1.65  | 1.57  | 1.68  | 1.54  | 1.71  | 1.51  | 1.74  | 1.49  | 1.77  |
| 80  | 1.61  | 1.66  | 1.59  | 1.69  | 1.56  | 1.72  | 1.53  | 1.74  | 1.51  | 1.77  |
| 85  | 1.62  | 1.67  | 1.60  | 1.70  | 1.57  | 1.72  | 1.55  | 1.75  | 1.52  | 1.77  |
| 90  | 1.63  | 1.68  | 1.61  | 1.70  | 1.59  | 1.73  | 1.57  | 1.75  | 1.54  | 1.78  |
| 95  | 1.64  | 1.69  | 1.62  | 1.71  | 1.60  | 1.73  | 1.58  | 1.75  | 1.56  | 1.78  |
| 100 | 1.65  | 1.69  | 1.63  | 1.72  | 1.61  | 1.74  | 1.59  | 1.76  | 1.57  | 1.78  |

**Table 33** continued**Level of significance  $\alpha = 0.01$** 

| <i>n</i> | Number of independent variables ( $p - 1$ ) |       |       |       |       |       |       |       |       |       |
|----------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|          | 1   |       | 2     |       | 3     |       | 4     |       | 5     |       |
|          | $d_L$                                       | $d_U$ | $d_L$ | $d_U$ | $d_L$ | $d_U$ | $d_L$ | $d_U$ | $d_L$ | $d_U$ |
| 15       | 0.81  | 1.07  | 0.70  | 1.25  | 0.59  | 1.46  | 0.49  | 1.70  | 0.39  | 1.96  |
| 16       | 0.84  | 1.09  | 0.74  | 1.25  | 0.63  | 1.44  | 0.53  | 1.66  | 0.44  | 1.90  |
| 17       | 0.87  | 1.10  | 0.77  | 1.25  | 0.67  | 1.43  | 0.57  | 1.63  | 0.48  | 1.85  |
| 18       | 0.90  | 1.12  | 0.80  | 1.26  | 0.71  | 1.42  | 0.61  | 1.60  | 0.52  | 1.80  |
| 19       | 0.93  | 1.13  | 0.83  | 1.26  | 0.74  | 1.41  | 0.65  | 1.58  | 0.56  | 1.77  |
| 20       | 0.95  | 1.15  | 0.86  | 1.27  | 0.77  | 1.41  | 0.68  | 1.57  | 0.60  | 1.74  |
| 21       | 0.97  | 1.16  | 0.89  | 1.27  | 0.80  | 1.41  | 0.72  | 1.55  | 0.63  | 1.71  |
| 22       | 1.00  | 1.17  | 0.91  | 1.28  | 0.83  | 1.40  | 0.75  | 1.54  | 0.66  | 1.69  |
| 23       | 1.02  | 1.19  | 0.94  | 1.29  | 0.86  | 1.40  | 0.77  | 1.53  | 0.70  | 1.67  |
| 24       | 1.04  | 1.20  | 0.96  | 1.30  | 0.88  | 1.41  | 0.80  | 1.53  | 0.72  | 1.66  |
| 25       | 1.05  | 1.21  | 0.98  | 1.30  | 0.90  | 1.41  | 0.83  | 1.52  | 0.75  | 1.65  |
| 26       | 1.07  | 1.22  | 1.00  | 1.31  | 0.93  | 1.41  | 0.85  | 1.52  | 0.78  | 1.64  |
| 27       | 1.09  | 1.23  | 1.02  | 1.32  | 0.95  | 1.41  | 0.88  | 1.51  | 0.81  | 1.63  |
| 28       | 1.10  | 1.24  | 1.04  | 1.32  | 0.97  | 1.41  | 0.90  | 1.51  | 0.83  | 1.62  |
| 29       | 1.12  | 1.25  | 1.05  | 1.33  | 0.99  | 1.42  | 0.92  | 1.51  | 0.85  | 1.61  |
| 30       | 1.13  | 1.26  | 1.07  | 1.34  | 1.01  | 1.42  | 0.94  | 1.51  | 0.88  | 1.61  |
| 31       | 1.15  | 1.27  | 1.08  | 1.34  | 1.02  | 1.42  | 0.96  | 1.51  | 0.90  | 1.60  |
| 32       | 1.16  | 1.28  | 1.10  | 1.35  | 1.04  | 1.43  | 0.98  | 1.51  | 0.92  | 1.60  |
| 33       | 1.17  | 1.29  | 1.11  | 1.36  | 1.05  | 1.43  | 1.00  | 1.51  | 0.94  | 1.59  |
| 34       | 1.18  | 1.30  | 1.13  | 1.36  | 1.07  | 1.43  | 1.01  | 1.51  | 0.95  | 1.59  |
| 35       | 1.19  | 1.31  | 1.14  | 1.37  | 1.08  | 1.44  | 1.03  | 1.51  | 0.97  | 1.59  |
| 36       | 1.21  | 1.32  | 1.15  | 1.38  | 1.10  | 1.44  | 1.04  | 1.51  | 0.99  | 1.59  |
| 37       | 1.22  | 1.32  | 1.16  | 1.38  | 1.11  | 1.45  | 1.06  | 1.51  | 1.00  | 1.59  |
| 38       | 1.23  | 1.33  | 1.18  | 1.39  | 1.12  | 1.45  | 1.07  | 1.52  | 1.02  | 1.58  |
| 39       | 1.24  | 1.34  | 1.19  | 1.39  | 1.14  | 1.45  | 1.09  | 1.52  | 1.03  | 1.58  |
| 40       | 1.25  | 1.34  | 1.20  | 1.40  | 1.15  | 1.46  | 1.10  | 1.52  | 1.05  | 1.58  |
| 45       | 1.29  | 1.38  | 1.24  | 1.42  | 1.20  | 1.48  | 1.16  | 1.53  | 1.11  | 1.58  |
| 50       | 1.32  | 1.40  | 1.28  | 1.45  | 1.24  | 1.49  | 1.20  | 1.54  | 1.16  | 1.59  |
| 55       | 1.36  | 1.43  | 1.32  | 1.47  | 1.28  | 1.51  | 1.25  | 1.55  | 1.21  | 1.59  |
| 60       | 1.38  | 1.45  | 1.35  | 1.48  | 1.32  | 1.52  | 1.28  | 1.56  | 1.25  | 1.60  |
| 65       | 1.41  | 1.47  | 1.38  | 1.50  | 1.35  | 1.53  | 1.31  | 1.57  | 1.28  | 1.61  |
| 70       | 1.43  | 1.49  | 1.40  | 1.52  | 1.37  | 1.55  | 1.34  | 1.58  | 1.31  | 1.61  |
| 75       | 1.45  | 1.50  | 1.42  | 1.53  | 1.39  | 1.56  | 1.37  | 1.59  | 1.34  | 1.62  |
| 80       | 1.47  | 1.52  | 1.44  | 1.54  | 1.42  | 1.57  | 1.39  | 1.60  | 1.36  | 1.62  |
| 85       | 1.48  | 1.53  | 1.46  | 1.55  | 1.43  | 1.58  | 1.41  | 1.60  | 1.39  | 1.63  |
| 90       | 1.50  | 1.54  | 1.47  | 1.56  | 1.45  | 1.59  | 1.43  | 1.61  | 1.41  | 1.64  |
| 95       | 1.51  | 1.55  | 1.49  | 1.57  | 1.47  | 1.60  | 1.45  | 1.62  | 1.42  | 1.64  |
| 100      | 1.52  | 1.56  | 1.50  | 1.58  | 1.48  | 1.60  | 1.46  | 1.63  | 1.44  | 1.65  |

Source: Durbin and Watson, 1951

**Table 34 Modified Rayleigh test (V-test)**

| <i>n</i> | Level of significance $\alpha$ |        |        |        |        |        |
|----------|--------------------------------|--------|--------|--------|--------|--------|
|          | 0.10                           | 0.05   | 0.01   | 0.005  | 0.001  | 0.0001 |
| 5        | 1.3051                         | 1.6524 | 2.2505 | 2.4459 | 2.7938 | 3.0825 |
| 6        | 1.3009                         | 1.6509 | 2.2640 | 2.4695 | 2.8502 | 3.2114 |
| 7        | 1.2980                         | 1.6499 | 2.2734 | 2.4858 | 2.8886 | 3.2970 |
| 8        | 1.2958                         | 1.6492 | 2.2803 | 2.4978 | 2.9164 | 3.3578 |
| 9        | 1.2942                         | 1.6484 | 2.2856 | 2.5070 | 2.9375 | 3.4034 |
| 10       | 1.2929                         | 1.6482 | 2.2899 | 2.5143 | 2.9540 | 3.4387 |
| 11       | 1.2918                         | 1.6479 | 2.2933 | 2.5201 | 2.9672 | 3.4669 |
| 12       | 1.2909                         | 1.6476 | 2.2961 | 2.5250 | 2.9782 | 3.4899 |
| 13       | 1.2902                         | 1.6474 | 2.2985 | 2.5290 | 2.9873 | 3.5091 |
| 14       | 1.2895                         | 1.6472 | 2.3006 | 2.5325 | 2.9950 | 3.5253 |
| 15       | 1.2890                         | 1.6470 | 2.3023 | 2.5355 | 3.0017 | 3.5392 |
| 16       | 1.2885                         | 1.6469 | 2.3039 | 2.5381 | 3.0075 | 3.5512 |
| 17       | 1.2881                         | 1.6467 | 2.3052 | 2.5404 | 3.0126 | 3.5617 |
| 18       | 1.2877                         | 1.6466 | 2.3064 | 2.5424 | 3.0171 | 3.5710 |
| 19       | 1.2874                         | 1.6465 | 2.3075 | 2.5442 | 3.0211 | 3.5792 |
| 20       | 1.2871                         | 1.6464 | 2.3085 | 2.5458 | 3.0247 | 3.5866 |
| 21       | 1.2868                         | 1.6464 | 2.3093 | 2.5473 | 3.0279 | 3.5932 |
| 22       | 1.2866                         | 1.6463 | 2.3101 | 2.5486 | 3.0308 | 3.5992 |
| 23       | 1.2864                         | 1.6462 | 2.3108 | 2.5498 | 3.0335 | 3.6047 |
| 24       | 1.2862                         | 1.6462 | 2.3115 | 2.5509 | 3.0359 | 3.6096 |
| 25       | 1.2860                         | 1.6461 | 2.3121 | 2.5519 | 3.0382 | 3.6142 |
| 26       | 1.2858                         | 1.6461 | 2.3127 | 2.5529 | 3.0402 | 3.6184 |
| 27       | 1.2856                         | 1.6460 | 2.3132 | 2.5538 | 3.0421 | 3.6223 |
| 28       | 1.2855                         | 1.6460 | 2.3136 | 2.5546 | 3.0439 | 3.6258 |
| 29       | 1.2853                         | 1.6459 | 2.3141 | 2.5553 | 3.0455 | 3.6292 |
| 30       | 1.2852                         | 1.6459 | 2.3145 | 2.5560 | 3.0471 | 3.6323 |
| 40       | 1.2843                         | 1.6456 | 2.3175 | 2.5610 | 3.0580 | 3.6545 |
| 50       | 1.2837                         | 1.6455 | 2.3193 | 2.5640 | 3.0646 | 3.6677 |
| 60       | 1.2834                         | 1.6454 | 2.3205 | 2.5660 | 3.0689 | 3.6764 |
| 70       | 1.2831                         | 1.6453 | 2.3213 | 2.5674 | 3.0720 | 3.6826 |
| 100      | 1.2826                         | 1.6452 | 2.3228 | 2.5699 | 3.0775 | 3.6936 |
| 500      | 1.2818                         | 1.6449 | 2.3256 | 2.5747 | 3.0877 | 3.7140 |
| 1000     | 1.2817                         | 1.6449 | 2.3260 | 2.5752 | 3.0890 | 3.7165 |

Source: Batschelet, 1981; original provided by W.T. Keeton

**Table 35**   **Watson's  $U_n^2$ -test**

|          | Level of significance $\alpha$ |       |       |       |       |
|----------|--------------------------------|-------|-------|-------|-------|
| $n$      | 0.10                           | 0.05  | 0.025 | 0.01  | 0.005 |
| 2        | 0.143                          | 0.155 | 0.161 | 0.164 | 0.165 |
| 3        | 0.145                          | 0.173 | 0.194 | 0.213 | 0.224 |
| 4        | 0.146                          | 0.176 | 0.202 | 0.233 | 0.252 |
| 5        | 0.148                          | 0.177 | 0.205 | 0.238 | 0.262 |
| 6        | 0.149                          | 0.179 | 0.208 | 0.243 | 0.269 |
| 7        | 0.149                          | 0.180 | 0.210 | 0.247 | 0.274 |
| 8        | 0.150                          | 0.181 | 0.211 | 0.250 | 0.278 |
| 9        | 0.150                          | 0.182 | 0.212 | 0.252 | 0.281 |
| 10       | 0.150                          | 0.182 | 0.213 | 0.254 | 0.283 |
| 12       | 0.150                          | 0.183 | 0.215 | 0.256 | 0.287 |
| 14       | 0.151                          | 0.184 | 0.216 | 0.258 | 0.290 |
| 16       | 0.151                          | 0.184 | 0.216 | 0.259 | 0.291 |
| 18       | 0.151                          | 0.184 | 0.217 | 0.259 | 0.292 |
| 20       | 0.151                          | 0.185 | 0.217 | 0.261 | 0.293 |
| 30       | 0.152                          | 0.185 | 0.219 | 0.263 | 0.296 |
| 40       | 0.152                          | 0.186 | 0.219 | 0.264 | 0.298 |
| 50       | 0.152                          | 0.186 | 0.220 | 0.265 | 0.299 |
| 100      | 0.152                          | 0.186 | 0.221 | 0.266 | 0.301 |
| $\infty$ | 0.152                          | 0.187 | 0.221 | 0.267 | 0.302 |

*Source:* Batschelet, 1981; adapted from Stephens, 1964

**Table 36** Watson's two-sample  $U^2$ -test $n$  and  $m$  are sample sizes.

| $n$      | $m$      | Level of significance $\alpha$ |       |       |       |       |       |       |       |       |
|----------|----------|--------------------------------|-------|-------|-------|-------|-------|-------|-------|-------|
|          |          | 0.50                           | 0.20  | 0.10  | 0.05  | 0.02  | 0.01  | 0.005 | 0.002 | 0.001 |
| 5        | 5        | 0.089                          | 0.161 | 0.225 | 0.225 |       |       |       |       |       |
| 5        | 6        | 0.085                          | 0.133 | 0.182 | 0.242 |       |       |       |       |       |
| 5        | 7        | 0.086                          | 0.128 | 0.171 | 0.200 | 0.257 |       |       |       |       |
| 5        | 8        | 0.085                          | 0.131 | 0.165 | 0.215 | 0.269 |       |       |       |       |
| 5        | 9        | 0.080                          | 0.124 | 0.159 | 0.191 | 0.280 | 0.280 |       |       |       |
| 5        | 10       | 0.084                          | 0.124 | 0.161 | 0.196 | 0.241 | 0.289 | 0.289 |       |       |
| 5        | 11       | 0.081                          | 0.124 | 0.156 | 0.190 | 0.229 | 0.297 | 0.297 |       |       |
| 5        | 12       | 0.078                          | 0.124 | 0.155 | 0.186 | 0.226 | 0.261 | 0.304 |       |       |
|          |          |                                |       |       |       |       |       |       |       |       |
| 6        | 6        | 0.088                          | 0.132 | 0.171 | 0.206 | 0.264 |       |       |       |       |
| 6        | 7        | 0.081                          | 0.121 | 0.154 | 0.194 | 0.282 | 0.282 |       |       |       |
| 6        | 8        | 0.083                          | 0.127 | 0.161 | 0.196 | 0.246 | 0.298 | 0.298 |       |       |
| 6        | 9        | 0.082                          | 0.126 | 0.156 | 0.193 | 0.232 | 0.262 | 0.311 |       |       |
| 6        | 10       | 0.077                          | 0.126 | 0.156 | 0.190 | 0.231 | 0.248 | 0.323 | 0.323 |       |
| 6        | 11       | 0.078                          | 0.121 | 0.157 | 0.187 | 0.225 | 0.262 | 0.289 | 0.333 |       |
| 6        | 12       | 0.080                          | 0.124 | 0.155 | 0.183 | 0.226 | 0.259 | 0.275 | 0.343 | 0.343 |
|          |          |                                |       |       |       |       |       |       |       |       |
| 7        | 7        | 0.079                          | 0.135 | 0.158 | 0.199 | 0.251 | 0.304 | 0.304 |       |       |
| 7        | 8        | 0.079                          | 0.120 | 0.156 | 0.182 | 0.225 | 0.272 | 0.322 |       |       |
| 7        | 9        | 0.079                          | 0.122 | 0.156 | 0.182 | 0.222 | 0.255 | 0.291 | 0.339 |       |
| 7        | 10       | 0.077                          | 0.123 | 0.155 | 0.187 | 0.227 | 0.262 | 0.277 | 0.353 | 0.353 |
| 7        | 11       | 0.077                          | 0.122 | 0.155 | 0.184 | 0.221 | 0.253 | 0.281 | 0.323 | 0.366 |
| 7        | 12       | 0.076                          | 0.122 | 0.154 | 0.186 | 0.226 | 0.252 | 0.276 | 0.308 | 0.377 |
|          |          |                                |       |       |       |       |       |       |       |       |
| 8        | 8        | 0.078                          | 0.125 | 0.156 | 0.184 | 0.226 | 0.250 | 0.296 | 0.344 |       |
| 8        | 9        | 0.078                          | 0.123 | 0.155 | 0.186 | 0.226 | 0.258 | 0.283 | 0.363 | 0.363 |
| 8        | 10       | 0.078                          | 0.122 | 0.155 | 0.185 | 0.222 | 0.249 | 0.280 | 0.336 | 0.380 |
| 8        | 11       | 0.077                          | 0.122 | 0.154 | 0.184 | 0.225 | 0.252 | 0.280 | 0.319 | 0.353 |
| 8        | 12       | 0.077                          | 0.121 | 0.156 | 0.185 | 0.223 | 0.252 | 0.281 | 0.317 | 0.340 |
|          |          |                                |       |       |       |       |       |       |       |       |
| 9        | 9        | 0.077                          | 0.125 | 0.155 | 0.187 | 0.225 | 0.266 | 0.286 | 0.340 | 0.384 |
| 9        | 10       | 0.076                          | 0.122 | 0.154 | 0.186 | 0.226 | 0.254 | 0.287 | 0.321 | 0.361 |
| 9        | 11       | 0.076                          | 0.121 | 0.154 | 0.185 | 0.225 | 0.255 | 0.281 | 0.317 | 0.341 |
| 9        | 12       | 0.077                          | 0.122 | 0.154 | 0.185 | 0.226 | 0.254 | 0.280 | 0.316 | 0.340 |
|          |          |                                |       |       |       |       |       |       |       |       |
| 10       | 10       | 0.075                          | 0.123 | 0.155 | 0.185 | 0.225 | 0.255 | 0.283 | 0.317 | 0.345 |
| 10       | 11       | 0.076                          | 0.122 | 0.154 | 0.186 | 0.224 | 0.255 | 0.279 | 0.317 | 0.341 |
| 10       | 12       | 0.076                          | 0.121 | 0.153 | 0.185 | 0.225 | 0.255 | 0.282 | 0.316 | 0.341 |
|          |          |                                |       |       |       |       |       |       |       |       |
| $\infty$ | $\infty$ | 0.071                          | 0.117 | 0.152 | 0.187 | 0.233 | 0.268 | 0.304 | 0.350 | 0.385 |

Source: Batschelet, 1981; adapted from Zar, 1974

**Table 37** Maximum likelihood estimate  $\hat{k}$  for given  $\bar{R}$  in the von Mises caseFor the solution  $k = A^{-1}(\rho)$ , replace  $\hat{k}$  by  $k$ ,  $\bar{R}$  by  $\rho$ .

| $\bar{R}$ | $\hat{k}$ | $\bar{R}$ | $\hat{k}$ | $\bar{R}$ | $\hat{k}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.00      | 0.00000   | 0.35      | 0.74783   | 0.70      | 2.01363   |
| 0.01      | 0.02000   | 0.36      | 0.77241   | 0.71      | 2.07685   |
| 0.02      | 0.04001   | 0.37      | 0.79730   | 0.72      | 2.14359   |
| 0.03      | 0.06003   | 0.38      | 0.82253   | 0.73      | 2.21425   |
| 0.04      | 0.08006   | 0.39      | 0.84812   | 0.74      | 2.28930   |
|           |           |           |           |           |           |
| 0.05      | 0.10013   | 0.40      | 0.87408   | 0.75      | 2.36930   |
| 0.06      | 0.12022   | 0.41      | 0.90043   | 0.76      | 2.45490   |
| 0.07      | 0.14034   | 0.42      | 0.92720   | 0.77      | 2.54686   |
| 0.08      | 0.16051   | 0.43      | 0.95440   | 0.78      | 2.64613   |
| 0.09      | 0.18073   | 0.44      | 0.98207   | 0.79      | 2.75382   |
|           |           |           |           |           |           |
| 0.10      | 0.20101   | 0.45      | 1.01022   | 0.80      | 2.87129   |
| 0.11      | 0.22134   | 0.46      | 1.03889   | 0.81      | 3.00020   |
| 0.12      | 0.24175   | 0.47      | 1.06810   | 0.82      | 3.14262   |
| 0.13      | 0.26223   | 0.48      | 1.09788   | 0.83      | 3.30114   |
| 0.14      | 0.28279   | 0.49      | 1.12828   | 0.84      | 3.47901   |
|           |           |           |           |           |           |
| 0.15      | 0.30344   | 0.50      | 1.15932   | 0.85      | 3.68041   |
| 0.16      | 0.32419   | 0.51      | 1.19105   | 0.86      | 3.91072   |
| 0.17      | 0.34503   | 0.52      | 1.22350   | 0.87      | 4.17703   |
| 0.18      | 0.36599   | 0.53      | 1.25672   | 0.88      | 4.48876   |
| 0.19      | 0.38707   | 0.54      | 1.29077   | 0.89      | 4.85871   |
|           |           |           |           |           |           |
| 0.20      | 0.40828   | 0.55      | 1.32570   | 0.90      | 5.3047    |
| 0.21      | 0.42962   | 0.56      | 1.36156   | 0.91      | 5.8522    |
| 0.22      | 0.45110   | 0.57      | 1.39842   | 0.92      | 6.5394    |
| 0.23      | 0.47273   | 0.58      | 1.43635   | 0.93      | 7.4257    |
| 0.24      | 0.49453   | 0.59      | 1.47543   | 0.94      | 8.6104    |
|           |           |           |           |           |           |
| 0.25      | 0.51649   | 0.60      | 1.51574   | 0.95      | 10.2716   |
| 0.26      | 0.53863   | 0.61      | 1.55738   | 0.96      | 12.7661   |
| 0.27      | 0.56097   | 0.62      | 1.60044   | 0.97      | 16.9266   |
| 0.28      | 0.58350   | 0.63      | 1.64506   | 0.98      | 25.2522   |
| 0.29      | 0.60625   | 0.64      | 1.69134   | 0.99      | 50.2421   |
|           |           |           |           |           |           |
| 0.30      | 0.62922   | 0.65      | 1.73945   | 1.00      | $\infty$  |
| 0.31      | 0.65242   | 0.66      | 1.78953   |           |           |
| 0.32      | 0.67587   | 0.67      | 1.84177   |           |           |
| 0.33      | 0.69958   | 0.68      | 1.89637   |           |           |
| 0.34      | 0.72356   | 0.69      | 1.95357   |           |           |

Source: Mardia, 1972

**Table 38 Mardia–Watson–Wheeler test** $n_1$  = smaller of the two sample sizes  $n_1, n_2$ ;  $n = n_1 + n_2$ .

| $n$ | $n_1$ | Level of significance $\alpha$ |       |       |       |
|-----|-------|--------------------------------|-------|-------|-------|
|     |       | 0.001                          | 0.01  | 0.05  | 0.10  |
| 8   | 4     |                                |       |       | 6.83  |
| 9   | 3     |                                |       |       | 6.41  |
|     | 4     |                                |       | 8.29  | 4.88  |
| 10  | 3     |                                |       |       | 6.85  |
|     | 4     |                                |       | 9.47  | 6.24  |
|     | 5     |                                |       | 10.47 | 6.85  |
| 11  | 3     |                                |       | 7.20  | 5.23  |
|     | 4     |                                |       | 10.42 | 7.43  |
|     | 5     |                                | 12.34 | 8.74  | 6.60  |
| 12  | 3     |                                |       | 7.46  | 5.73  |
|     | 4     |                                | 11.20 | 8.46  | 7.46  |
|     | 5     |                                | 13.93 | 10.46 | 7.46  |
|     | 6     |                                | 14.93 | 11.20 | 7.46  |
| 13  | 3     |                                |       | 7.68  | 6.15  |
|     | 4     |                                | 11.83 | 9.35  | 7.03  |
|     | 5     |                                | 15.26 | 10.15 | 7.39  |
|     | 6     |                                | 17.31 | 10.42 | 8.04  |
| 14  | 3     |                                |       | 7.85  | 6.49  |
|     | 4     |                                | 12.34 | 9.30  | 7.60  |
|     | 5     |                                | 16.39 | 10.30 | 7.85  |
|     | 6     | 19.20                          | 15.59 | 12.21 | 7.94  |
|     | 7     | 20.20                          | 16.39 | 11.65 | 8.85  |
| 15  | 3     |                                |       | 7.99  | 6.78  |
|     | 4     |                                | 12.78 | 8.74  | 7.91  |
|     | 5     | 17.35                          | 14.52 | 10.36 | 7.91  |
|     | 6     | 20.92                          | 17.48 | 11.61 | 9.12  |
|     | 7     | 22.88                          | 16.14 | 11.57 | 9.06  |
| 16  | 3     |                                |       | 8.11  | 5.83  |
|     | 4     |                                | 13.14 | 9.44  | 7.38  |
|     | 5     | 18.16                          | 15.55 | 10.44 | 9.03  |
|     | 6     | 22.43                          | 16.98 | 11.54 | 9.11  |
|     | 7     | 25.27                          | 18.16 | 12.66 | 9.78  |
| 17  | 3     |                                | 8.21  | 7.23  | 6.14  |
|     | 4     | 13.44                          | 11.76 | 9.74  | 7.64  |
|     | 5     | 18.86                          | 16.44 | 11.03 | 8.76  |
|     | 6     | 23.73                          | 17.76 | 12.21 | 9.41  |
|     | 7     | 27.40                          | 17.98 | 12.63 | 10.11 |
|     | 8     | 29.37                          | 19.11 | 13.36 | 10.15 |

Source: Mardia, 1972



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