Nonlinear difference subspace method of motor imagery EEG classification in Brain-Computer Interface

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SUPPLEMENTARY MATERIAL

Algorithm 1: K-SM classification

Input: Training data $\boldsymbol{F}_x^m \in \mathbb{R}^{2C \times N_m}$, *m*= {*1, 2, …M*}; Testing data $\boldsymbol{F}_y \in \mathbb{R}^{2C \times N_t}$; σ^2; Dimension v of nonlinear subspace.

Output: Class labels of $\boldsymbol{F}_y$.

1. Compute kernel matrix $\boldsymbol{K}^m \in \mathbb{R}^{N_m \times N_m}$ on training data $\boldsymbol{F}_x^m$.

2. Center the kernel matrix $\boldsymbol{K}^m$

$\widetilde{\boldsymbol{K}}^m = \boldsymbol{K}^m - \mathbb{1}_{N_m}\boldsymbol{K}^m - \boldsymbol{K}^m\mathbb{1}_{N_m} + \mathbb{1}_{N_m}\boldsymbol{K}^m\mathbb{1}_{N_m}$; where $\mathbb{1}_m \in \mathbb{R}^{N_m \times N_m}$ is a matrix of all elements as $\frac{1}{N_m}$.

3. Solve eigenvalue problem

$$\widetilde{\boldsymbol{K}}^m\boldsymbol{A}^m = \Delta\boldsymbol{A}^m$$

$\boldsymbol{A}^m = [\boldsymbol{a}_1^m, \boldsymbol{a}_2^m, \ldots\ldots \boldsymbol{a}_{N_m}^m] \in \mathbb{R}^{N_m \times N_m}$ is the eigenvector matrix of $\widetilde{\boldsymbol{K}}^m$, Δ is the diagonal matrix of eigenvalues of $\widetilde{\boldsymbol{K}}^m$ sorted in decreasing value.

4. for $i = 1{:}N_t$

- Compute the kernel matrix $\boldsymbol{K}_{test}^m \in \mathbb{R}^{N_m \times 1}$ as

$$\boldsymbol{K}_{test}^m = [k(\boldsymbol{f}_{x,1}^m, \boldsymbol{f}_{y,i}), k(\boldsymbol{f}_{x,2}^m, \boldsymbol{f}_{y,i}) \ldots\ldots k(\boldsymbol{f}_{x,N_m}^m, \boldsymbol{f}_{y,i})]^T$$

- Project test trial onto class-*m* subspace

$$\boldsymbol{p_1} = [\boldsymbol{a}_1^1, \boldsymbol{a}_2^1, \ldots\ldots \boldsymbol{a}_v^1]^T\boldsymbol{K}_{test}^m$$
$$\boldsymbol{p}_2 = [\boldsymbol{a}_1^2, \boldsymbol{a}_2^2, \ldots\ldots \boldsymbol{a}_v^2]^T\boldsymbol{K}_{test}^m$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\boldsymbol{p}_M = [\boldsymbol{a}_1^2, \boldsymbol{a}_2^2, \ldots\ldots \boldsymbol{a}_v^2]^T\boldsymbol{K}_{test}^m$$

- Classify test trial $\boldsymbol{f_{y,i}}$ to the subspace of maximum projection

$$Class(i) = \max_m(norm(\boldsymbol{p}_m))$$

5. end

Algorithm 2: K-DSM classification

Input: Training data $\boldsymbol{F}_x^m \in \mathbb{R}^{2C \times N_m}$, *m*= {*1, 2, …M*}; Testing data $\boldsymbol{F}_y \in \mathbb{R}^{2C \times N_t}$; σ^2; Dimension v of nonlinear subspaces.

Output: Class labels of $\boldsymbol{F}_y$.

1. Compute kernel matrix $\boldsymbol{K}^{m,m'} \in \mathbb{R}^{N_m \times N_{m'}}$ on training data $\boldsymbol{F}_x^m$and $\boldsymbol{F}_x^{m'}$,$\forall m \,\&\, m' = \{1, 2, \ldots M\}$. The elements of matrix $\boldsymbol{K}^{m,m'}$ are defined as $[k_{ij}]$, where $k_{ij} = k(\boldsymbol{f}_{x,i}^m, \boldsymbol{f}_{x,j}^{m'}) = (\emptyset(\boldsymbol{f}_{x,i}^m).\emptyset(\boldsymbol{f}_{x,j}^{m'}))$ is the dot product between the *i*^th^ point of $\boldsymbol{F}_x^m$ and *j*^th^ point of $\boldsymbol{F}_x^{m'}$ in the feature space.

2. Solve the eigenvalue problem

$$\boldsymbol{K}^{m,m}\boldsymbol{A}^m = \Delta\boldsymbol{A}^m$$

$\boldsymbol{A}^m = [\boldsymbol{a}_1^m, \boldsymbol{a}_2^m, \ldots\ldots \boldsymbol{a}_{v_m}^m] \in \mathbb{R}^{N_m \times v}$ are the eigenvectors corresponding to v largest eigenvalues of $\boldsymbol{K}^{m,m}$, Δ is the diagonal matrix of v eigenvalues of $\boldsymbol{K}^{m,m}$ sorted in decreasing value.

3. Compute $\boldsymbol{D}^{m,m'} \in \mathbb{R}^{v \times v} = \boldsymbol{A}^{m^T}\boldsymbol{K}^{m,m'}\boldsymbol{A}^{m'}$

4. Compute $\boldsymbol{D} \in \mathbb{R}^{Mv \times Mv} = \begin{bmatrix} \boldsymbol{D}^{1,1} & \boldsymbol{D}^{1,2} & . & \boldsymbol{D}^{1,M} \\ \boldsymbol{D}^{2,1} & \boldsymbol{D}^{2,2} & . & \boldsymbol{D}^{2,M} \\ . & . & . & . \\ \boldsymbol{D}^{M,1} & \boldsymbol{D}^{M,2} & . & \boldsymbol{D}^{M,M} \end{bmatrix}$

5. Solve the eigenvalue problem

$$\boldsymbol{DB} = \boldsymbol{\Lambda B}$$

$\boldsymbol{B}$= $[\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots\ldots \boldsymbol{b}_{n_\emptyset}] \in \mathbb{R}^{Mv \times n_\emptyset}$ are the eigenvectors corresponding to least $n_\emptyset$ eigenvalues of $\boldsymbol{D}$.

6. Project all class-*m* trials onto KDS:

- Compute kernel matrix $\boldsymbol{K}_{pj}^{m} \in \mathbb{R}^{MN_m \text{ x } N_m}$ to project class-*m* data onto KDS as

$$\boldsymbol{K}_{pj}^{m} = \begin{bmatrix} k(\boldsymbol{f}_{x,1}^{1}, \boldsymbol{f}_{x,1}^{m}) & k(\boldsymbol{f}_{x,1}^{1}, \boldsymbol{f}_{x,2}^{m}) & . & . & k(\boldsymbol{f}_{x,1}^{1}, \boldsymbol{f}_{x,N_m}^{m}) \\ . & . & . & . & . \\ k(\boldsymbol{f}_{x,N_1}^{1}, \boldsymbol{f}_{x,1}^{m}) & k(\boldsymbol{f}_{x,N_1}^{1}, \boldsymbol{f}_{x,2}^{m}) & . & . & k(\boldsymbol{f}_{x,1}^{1}, \boldsymbol{f}_{x,N_m}^{m}) \\ k(\boldsymbol{f}_{x,1}^{2}, \boldsymbol{f}_{x,1}^{m}) & k(\boldsymbol{f}_{x,1}^{2}, \boldsymbol{f}_{x,2}^{m}) & . & . & k(\boldsymbol{f}_{x,1}^{1}, \boldsymbol{f}_{x,N_m}^{m}) \\ . & . & . & . & . \\ k(\boldsymbol{f}_{x,N_2}^{2}, \boldsymbol{f}_{x,1}^{m}) & k(\boldsymbol{f}_{x,N_2}^{2}, \boldsymbol{f}_{x,2}^{m}) & . & . & k(\boldsymbol{f}_{x,1}^{1}, \boldsymbol{f}_{x,N_m}^{m}) \\ . & . & . & . & . \\ . & . & . & . & . \\ k(\boldsymbol{f}_{x,N_M}^{M}, \boldsymbol{f}_{x,1}^{m}) & k(\boldsymbol{f}_{x,N_M}^{M}, \boldsymbol{f}_{x,2}^{m}) & . & . & k(\boldsymbol{f}_{x,1}^{1}, \boldsymbol{f}_{x,N_m}^{m}) \end{bmatrix}$$

- The projected class-*m* data $\boldsymbol{F}_{x}^{m}$onto the KDS is

$$\boldsymbol{F}_{x_{KDS}}^{m} \in \mathbb{R}^{\,n_{\emptyset} \text{x } N_m} = B^T \begin{bmatrix} [\boldsymbol{A}^1]^T & \boldsymbol{0} & . & . & \boldsymbol{0} \\ \boldsymbol{0} & [\boldsymbol{A}^2]^T & . & . & \boldsymbol{0} \\ . & . & . & . & . \\ . & . & . & . & . \\ \boldsymbol{0} & \boldsymbol{0} & . & . & [\boldsymbol{A}^M]^T \end{bmatrix} \boldsymbol{K}_{pj}^{m}$$

7. Compute linear subspace of class-*m* projected data $\boldsymbol{F}_{x_{KDS}}^{m}$,

- Find covariance matrix ${\boldsymbol{R}_{\boldsymbol{ds}}}^{m}$of $\boldsymbol{F}_{x_{KDS}}^{m}$
- Solve the singular-value decomposition problem:

$${\boldsymbol{R}_{\boldsymbol{ds}}}^{m} = {\boldsymbol{U}_{\boldsymbol{ds}}}^{m}\, {\boldsymbol{E}_{\boldsymbol{ds}}}^{m}\, {\boldsymbol{V}_{\boldsymbol{ds}}}^{m}$$

8. for $i = 1{:}\, N_t$

- Compute the kernel matrix $\boldsymbol{K}_{pj}^{test} \in \mathbb{R}^{MN_m \text{ x } 1}$ as

$$[k(\boldsymbol{f}_{x,1}^{1}, \boldsymbol{f}_{y,i}), \dots\ k(\boldsymbol{f}_{x,N_1}^{1}, \boldsymbol{f}_{y,i}) \dots .\, k(\boldsymbol{f}_{x,1}^{M}, \boldsymbol{f}_{y,i}) ..\, k(\boldsymbol{f}_{x,N_M}^{M}, \boldsymbol{f}_{y,i})]^T$$

- Project $\boldsymbol{f}_{y,i}$ onto KDS

$$\boldsymbol{f}_{y_{KDS},i} \in \mathbb{R}^{\,n_{\emptyset} \text{x } 1} = B^T \begin{bmatrix} [\boldsymbol{A}^1]^T & \boldsymbol{0} & . & . & \boldsymbol{0} \\ \boldsymbol{0} & [\boldsymbol{A}^2]^T & . & . & \boldsymbol{0} \\ . & . & . & . & . \\ . & . & . & . & . \\ \boldsymbol{0} & \boldsymbol{0} & . & . & [\boldsymbol{A}^M]^T \end{bmatrix} \boldsymbol{K}_{pj}^{test}$$

- On KDS, project test trial $\boldsymbol{f}_{y_{KDS},i}$ onto the v_{DS}-dimensional class-*m* subspace

$$\boldsymbol{p}_{\boldsymbol{1}} = \boldsymbol{U}_{ds}^{1}(:,1{:}\, v_{DS})^T \boldsymbol{f}_{y_{KDS},i}$$
$$\boldsymbol{p}_{2} = \boldsymbol{U}_{ds}^{2}(:,1{:}\, v_{DS})^T \boldsymbol{f}_{y_{KDS},i}$$
$$\dots\dots\dots\dots$$
$$\boldsymbol{p}_{M} = \boldsymbol{U}_{ds}^{M}(:,1{:}\, v_{DS})^T \boldsymbol{f}_{y_{KDS},i}$$

- Classify test trial $\boldsymbol{f}_{\boldsymbol{y},\boldsymbol{i}}$ to the subspace of maximum projection

$$Class(i) = \max_{m}(norm(\boldsymbol{p}_m))$$

9. end

Algorithm 3: I-SM classification

Input: Training data $\boldsymbol{F}_{x}^{m} \in \mathbb{R}^{2C \text{ x } N_m}$, *m*={1, 2, …*M*}; Class labels $\mathbf{z} \in \mathbb{R}^{1 \text{ xN}}$ of the training data; Testing data $\boldsymbol{F}_{y} \in \mathbb{R}^{2C \text{ x } N_t}$; σ^2; Dimension v_m of class-*m* subspace.

Output: Class labels of $\boldsymbol{F}_{y}$.

1. Arrange matrix $\boldsymbol{F}_{x} = [\boldsymbol{F}_{x}^{1}\, \boldsymbol{F}_{x}^{2} \dots \dots \boldsymbol{F}_{x}^{M}] \in \mathbb{R}^{2C \text{ x } N}$, where $N = N_1 + N_2 + \cdots .\, N_M$.

2. Get embedding projection $\boldsymbol{P} = \boldsymbol{ISoP}(\boldsymbol{F}_{x}, \boldsymbol{z}, \alpha, \eta)$ using SD-ISoP Algorithm (given as **Algorithm 1** in paper).

3. Project training data $\boldsymbol{F}_{x}^{m}$ onto the low-dimensional subspace

$$\boldsymbol{F}_{x\_lowdim}^{m} = P\boldsymbol{F}_{x}^{m}$$

4. Compute linear subspace of class-*m* projected data $\boldsymbol{F}_{x\_lowdim}^{m}$,

- Find covariance matrix $\boldsymbol{R}^{m}$of $\boldsymbol{F}_{x\_lowdim}^{m}$
- Solve singular-value decomposition problem

$$\boldsymbol{R}^{m} = \boldsymbol{U}^{m}\, \boldsymbol{E}^{m}\, \boldsymbol{V}^{m}$$

5. for $i = 1{:}\, N_t$

6. Project test trial onto low dimensional space

$$\boldsymbol{f}_{y,i\_lowdim} = P\boldsymbol{f}_{y,i}$$

7. On low-dimensional space, project test trial $\boldsymbol{f}_{y,i\_lowdim}$ onto d-dimensional class-m subspace

$$\boldsymbol{p}_1 = U^1(:,1:d)^T \boldsymbol{f}_{y,i\_lowdim}$$

$$\dots\dots\dots\dots\dots$$

$$\boldsymbol{p}_M = U^M(:,1:d)^T \boldsymbol{f}_{y,i\_lowdim}$$

8. Classify test trial $\boldsymbol{f}_{y,i}$ to the subspace of maximum projection

$$Class(i) = {}^{max}_{m}(norm(\boldsymbol{p}_m))$$

9. end

Algorithm 5: I-DSM based MI classification.

Input: Training data $\boldsymbol{F}_x^m \in \mathbb{R}^{2C \times N_m}$, m={1, 2, …M}; ISoP projection matrix $\boldsymbol{P} \in \mathbb{R}^{d_{isop} \times 2C}$; Testing data $\boldsymbol{F}_y \in \mathbb{R}^{2C \times N_t}$; σ^2; dimension of subspace d.

Output: Class labels of $\boldsymbol{F}_y$.

1. Project class-m training data $\boldsymbol{F}_x^m$ onto the ISOMap space

$$\boldsymbol{F}_{x\_iso}^m = P\boldsymbol{F}_x^m$$

2. Compute linear subspace of class-m projected data $\boldsymbol{F}_{x\_iso}^m$,

- Find covariance matrix $\boldsymbol{R}^m$of $\boldsymbol{F}_{x\_iso}^m$
- Solve singular-value decomposition problem

$$\boldsymbol{R}^m = \boldsymbol{U}^m \boldsymbol{E}^m \boldsymbol{V}^m$$

3. Compute the sum of the subspace matrix

$$S = \sum_{j=1}^{d} \boldsymbol{U}^1(:,j)\boldsymbol{U}^1(:,j)^T + \sum_{j=1}^{d} \boldsymbol{U}^2(:,j)\boldsymbol{U}^2(:,j)^T + \cdots \dots \dots \dots . + \sum_{j=1}^{d} \boldsymbol{U}^M(:,j)\boldsymbol{U}^M(:,j)^T$$

4. Solve singular-value decomposition problem

$$\boldsymbol{S} = \boldsymbol{U}_{ds}\boldsymbol{E}_{ds}\boldsymbol{V}_{ds}$$

5. Construct the difference subspace $D = U_{ds}(:, find(diag(E_{ds}) < 1)$ by considering only those eigenvectors of **S** whose eigenvalues are less than 1.

6. Project training data $\boldsymbol{F}_{x\_iso}^m$ onto the difference subspace

$$\boldsymbol{F}_{x\_ds}^m = D^T \boldsymbol{F}_{x\_iso}^m$$

7. Compute linear subspace of class-m on DS projected data $\boldsymbol{F}_{x\_ds}^m$,

- Find covariance matrix ${\boldsymbol{R}_{ds}}^m$of $\boldsymbol{F}_{x\_ds}^m$
- Solve singular-value decomposition problem

$$\boldsymbol{R}_{ds}^m = \boldsymbol{U}_{ds}^m\, \boldsymbol{E}_{ds}^m\, \boldsymbol{V}_{ds}^m$$

8. for $i = 1: N_t$

9. Project test trial onto ISOMap space and then onto DS

$$\boldsymbol{f}_{y,i\_iso} = P\boldsymbol{f}_{y,i}$$

$$\boldsymbol{f}_{y,i\_ds} = D^T \boldsymbol{f}_{y,i\_iso}$$

10. On DS, project test trial $\boldsymbol{f}_{y,i\_ds}$ onto d-dimensional class-m subspace

$$\boldsymbol{p}_1 = \boldsymbol{U}_{ds}^1(:,1:d)^T \boldsymbol{f}_{y,i\_ds}$$

$$\dots\dots\dots\dots\dots$$

$$\boldsymbol{p}_M = \boldsymbol{U}_{ds}^M(:,1:d)^T \boldsymbol{f}_{y,i\_ds}$$

11. Classify test trial $\boldsymbol{f}_{y,i}$ to the subspace of maximum projection

$$Class(i) = {}^{max}_{m}(norm(\boldsymbol{p}_m))$$

12. end