

Nonlinear difference subspace method of motor imagery EEG classification in Brain-Computer Interface

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SUPPLEMENTARY MATERIAL

Algorithm 1: K-SM classification

Input: Training data $\mathbf{F}_x^m \in \mathbb{R}^{2C \times N_m}$, $m = \{1, 2, \dots, M\}$; Testing data $\mathbf{F}_y \in \mathbb{R}^{2C \times N_t}$; σ^2 ; Dimension v of nonlinear subspace.

Output: Class labels of \mathbf{F}_y .

1. Compute kernel matrix $\mathbf{K}^m \in \mathbb{R}^{N_m \times N_m}$ on training data \mathbf{F}_x^m .

2. Center the kernel matrix \mathbf{K}^m

$$\tilde{\mathbf{K}}^m = \mathbf{K}^m - \mathbb{1}_{N_m} \mathbf{K}^m - \mathbf{K}^m \mathbb{1}_{N_m} + \mathbb{1}_{N_m} \mathbf{K}^m \mathbb{1}_{N_m}; \text{ where } \mathbb{1}_m \in \mathbb{R}^{N_m \times N_m} \text{ is a matrix of all elements as } \frac{1}{N_m}.$$

3. Solve eigenvalue problem

$$\tilde{\mathbf{K}}^m \mathbf{A}^m = \Delta \mathbf{A}^m$$

$\mathbf{A}^m = [\mathbf{a}_1^m, \mathbf{a}_2^m, \dots, \mathbf{a}_{N_m}^m] \in \mathbb{R}^{N_m \times N_m}$ is the eigenvector matrix of $\tilde{\mathbf{K}}^m$, Δ is the diagonal matrix of eigenvalues of $\tilde{\mathbf{K}}^m$ sorted in decreasing value.

4. for $i = 1: N_t$

- Compute the kernel matrix $\mathbf{K}_{test}^m \in \mathbb{R}^{N_m \times 1}$ as

$$\mathbf{K}_{test}^m = [k(\mathbf{f}_{x,1}^m, \mathbf{f}_{y,i}), k(\mathbf{f}_{x,2}^m, \mathbf{f}_{y,i}) \dots k(\mathbf{f}_{x,N_m}^m, \mathbf{f}_{y,i})]^T$$

- Project test trial onto class- m subspace

$$\mathbf{p}_1 = [\mathbf{a}_1^1, \mathbf{a}_2^1, \dots, \mathbf{a}_v^1]^T * \mathbf{K}_{test}^m$$

$$\mathbf{p}_2 = [\mathbf{a}_1^2, \mathbf{a}_2^2, \dots, \mathbf{a}_v^2]^T * \mathbf{K}_{test}^m$$

.....

$$\mathbf{p}_M = [\mathbf{a}_1^M, \mathbf{a}_2^M, \dots, \mathbf{a}_v^M]^T * \mathbf{K}_{test}^m$$

- Classify test trial $\mathbf{f}_{y,i}$ to the subspace of maximum projection

$$Class(i) = \max_m (norm(\mathbf{p}_m))$$

5. end

Algorithm 2: K-DSM classification

Input: Training data $\mathbf{F}_x^m \in \mathbb{R}^{2C \times N_m}$, $m = \{1, 2, \dots, M\}$; Testing data $\mathbf{F}_y \in \mathbb{R}^{2C \times N_t}$; σ^2 ; Dimension v of nonlinear subspaces.

Output: Class labels of \mathbf{F}_y .

1. Compute kernel matrix $\mathbf{K}^{m,m'} \in \mathbb{R}^{N_m \times N_{m'}}$ on training data \mathbf{F}_x^m and $\mathbf{F}_x^{m'}$, $\forall m \& m' = \{1, 2, \dots, M\}$. The elements of matrix $\mathbf{K}^{m,m'}$ are defined as $[k_{ij}]$, where $k_{ij} = k(\mathbf{f}_{x,i}^m, \mathbf{f}_{x,j}^{m'}) = (\phi(\mathbf{f}_{x,i}^m) \cdot \phi(\mathbf{f}_{x,j}^{m'}))$ is the dot product between the i^{th} point of \mathbf{F}_x^m and j^{th} point of $\mathbf{F}_x^{m'}$ in the feature space.

2. Solve the eigenvalue problem

$$\mathbf{K}^{m,m} \mathbf{A}^m = \Delta \mathbf{A}^m$$

$\mathbf{A}^m = [\mathbf{a}_1^m, \mathbf{a}_2^m, \dots, \mathbf{a}_{N_m}^m] \in \mathbb{R}^{N_m \times v}$ are the eigenvectors corresponding to v largest eigenvalues of $\mathbf{K}^{m,m}$, Δ is the diagonal matrix of v eigenvalues of $\mathbf{K}^{m,m}$ sorted in decreasing value.

3. Compute $\mathbf{D}^{m,m'} \in \mathbb{R}^{v \times v} = \mathbf{A}^{mT} \mathbf{K}^{m,m'} \mathbf{A}^{m'}$

$$4. \text{ Compute } \mathbf{D} \in \mathbb{R}^{Mv \times Mv} = \begin{bmatrix} \mathbf{D}^{1,1} & \mathbf{D}^{1,2} & \dots & \mathbf{D}^{1,M} \\ \mathbf{D}^{2,1} & \mathbf{D}^{2,2} & \dots & \mathbf{D}^{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{D}^{M,1} & \mathbf{D}^{M,2} & \dots & \mathbf{D}^{M,M} \end{bmatrix}$$

5. Solve the eigenvalue problem

$$\mathbf{D} \mathbf{B} = \Lambda \mathbf{B}$$

$\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_{n_\emptyset}] \in \mathbb{R}^{Mv \times n_\emptyset}$ are the eigenvectors corresponding to least n_\emptyset eigenvalues of \mathbf{D} .

6. Project all class- m trials onto KDS:

- Compute kernel matrix $\mathbf{K}_{pj}^m \in \mathbb{R}^{MN_m \times N_m}$ to project class- m data onto KDS as

$$\mathbf{K}_{pj}^m = \begin{bmatrix} k(\mathbf{f}_{x,1}^1, \mathbf{f}_{x,1}^m) & k(\mathbf{f}_{x,1}^1, \mathbf{f}_{x,2}^m) & \dots & k(\mathbf{f}_{x,1}^1, \mathbf{f}_{x,N_m}^m) \\ \vdots & \vdots & \ddots & \vdots \\ k(\mathbf{f}_{x,N_1}^1, \mathbf{f}_{x,1}^m) & k(\mathbf{f}_{x,N_1}^1, \mathbf{f}_{x,2}^m) & \dots & k(\mathbf{f}_{x,N_1}^1, \mathbf{f}_{x,N_m}^m) \\ k(\mathbf{f}_{x,1}^2, \mathbf{f}_{x,1}^m) & k(\mathbf{f}_{x,1}^2, \mathbf{f}_{x,2}^m) & \dots & k(\mathbf{f}_{x,1}^2, \mathbf{f}_{x,N_m}^m) \\ \vdots & \vdots & \ddots & \vdots \\ k(\mathbf{f}_{x,N_2}^2, \mathbf{f}_{x,1}^m) & k(\mathbf{f}_{x,N_2}^2, \mathbf{f}_{x,2}^m) & \dots & k(\mathbf{f}_{x,N_2}^2, \mathbf{f}_{x,N_m}^m) \\ \vdots & \vdots & \ddots & \vdots \\ k(\mathbf{f}_{x,N_M}^M, \mathbf{f}_{x,1}^m) & k(\mathbf{f}_{x,N_M}^M, \mathbf{f}_{x,2}^m) & \dots & k(\mathbf{f}_{x,N_M}^M, \mathbf{f}_{x,N_m}^m) \end{bmatrix}$$

- The projected class- m data \mathbf{F}_x^m onto the KDS is

$$\mathbf{F}_{x_{KDS}}^m \in \mathbb{R}^{n_0 \times N_m} = \mathbf{B}^T * \begin{bmatrix} [\mathbf{A}^1]^T & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & [\mathbf{A}^2]^T & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & [\mathbf{A}^M]^T \end{bmatrix} * \mathbf{K}_{pj}^m$$

7. Compute linear subspace of class- m projected data $\mathbf{F}_{x_{KDS}}^m$,

- Find covariance matrix \mathbf{R}_{ds}^m of $\mathbf{F}_{x_{KDS}}^m$
 - Solve the singular-value decomposition problem
- $$\mathbf{R}_{ds}^m = \mathbf{U}_{ds}^m * \mathbf{E}_{ds}^m * \mathbf{V}_{ds}^m$$

8. for $i = 1: N_t$

- Compute the kernel matrix $\mathbf{K}_{pj}^{test} \in \mathbb{R}^{MN_m \times 1}$ as
- $$[k(\mathbf{f}_{x,1}^1, \mathbf{f}_{y,i}), \dots, k(\mathbf{f}_{x,N_1}^1, \mathbf{f}_{y,i}) \dots k(\mathbf{f}_{x,1}^M, \mathbf{f}_{y,i}) \dots k(\mathbf{f}_{x,N_M}^M, \mathbf{f}_{y,i})]^T$$
- Project $\mathbf{f}_{y,i}$ onto KDS

$$\mathbf{f}_{y_{KDS},i} \in \mathbb{R}^{n_0 \times 1} = \mathbf{B}^T * \begin{bmatrix} [\mathbf{A}^1]^T & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & [\mathbf{A}^2]^T & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & [\mathbf{A}^M]^T \end{bmatrix} * \mathbf{K}_{pj}^{test}$$

- On KDS, project test trial $\mathbf{f}_{y_{KDS},i}$ onto the v_{DS} -dimensional class- m subspace

$$\mathbf{p}_1 = \mathbf{U}_{ds}^1(:, 1:v_{DS})^T * \mathbf{f}_{y_{KDS},i}$$

$$\mathbf{p}_2 = \mathbf{U}_{ds}^2(:, 1:v_{DS})^T * \mathbf{f}_{y_{KDS},i}$$

.....

$$\mathbf{p}_M = \mathbf{U}_{ds}^M(:, 1:v_{DS})^T * \mathbf{f}_{y_{KDS},i}$$

- Classify test trial $\mathbf{f}_{y,i}$ to the subspace of maximum projection

$$Class(i) = \max_m (norm(\mathbf{p}_m))$$

9. end

Algorithm 3: I-SM classification

Input: Training data $\mathbf{F}_x^m \in \mathbb{R}^{2C \times N_m}$, $m=\{1, 2, \dots, M\}$; Class labels $\mathbf{z} \in \mathbb{R}^{1 \times N}$ of the training data; Testing data $\mathbf{F}_y \in \mathbb{R}^{2C \times N_t}$; σ^2 ; Dimension v_m of class- m subspace.

Output: Class labels of \mathbf{F}_y .

1. Arrange matrix $\mathbf{F}_x = [\mathbf{F}_x^1 \mathbf{F}_x^2 \dots \mathbf{F}_x^M] \in \mathbb{R}^{2C \times N}$, where $N = N_1 + N_2 + \dots + N_M$.

2. Get embedding projection $\mathbf{P} = \mathbf{ISoP}(\mathbf{F}_x, \mathbf{z}, \alpha, \eta)$ using SD-ISoP Algorithm (given as **Algorithm 1** in paper).

3. Project training data \mathbf{F}_x^m onto the low-dimensional subspace

$$\mathbf{F}_{x_lowdim}^m = \mathbf{P} * \mathbf{F}_x^m$$

4. Compute linear subspace of class- m projected data $\mathbf{F}_{x_lowdim}^m$,

- Find covariance matrix \mathbf{R}^m of $\mathbf{F}_{x_lowdim}^m$
- Solve singular-value decomposition problem

$$\mathbf{R}^m = \mathbf{U}^m * \mathbf{E}^m * \mathbf{V}^m$$

5. for $i = 1: N_t$

6. Project test trial onto low dimensional space

$$\mathbf{f}_{y,i_lowdim} = \mathbf{P} * \mathbf{f}_{y,i}$$
 7. On low-dimensional space, project test trial \mathbf{f}_{y,i_lowdim} onto d -dimensional class- m subspace

$$\mathbf{p}_1 = \mathbf{U}^1(:, 1:d)^T * \mathbf{f}_{y,i_lowdim}$$

$$\dots\dots\dots$$

$$\mathbf{p}_M = \mathbf{U}^M(:, 1:d)^T * \mathbf{f}_{y,i_lowdim}$$
 8. Classify test trial $\mathbf{f}_{y,i}$ to the subspace of maximum projection

$$Class(i) = \max_m(norm(\mathbf{p}_m))$$
 9. end
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Algorithm 5: I-DSM based MI classification.

Input: Training data $\mathbf{F}_x^m \in \mathbb{R}^{2C \times N_m}$, $m=\{1, 2, \dots, M\}$; ISoP projection matrix $\mathbf{P} \in \mathbb{R}^{d_{isop} \times 2C}$; Testing data $\mathbf{F}_y \in \mathbb{R}^{2C \times N_t}$; σ^2 ; dimension of subspace d .

Output: Class labels of \mathbf{F}_y .

1. Project class- m training data \mathbf{F}_x^m onto the ISOMap space

$$\mathbf{F}_{x_iso}^m = \mathbf{P} * \mathbf{F}_x^m$$
 2. Compute linear subspace of class- m projected data $\mathbf{F}_{x_iso}^m$,
 - Find covariance matrix \mathbf{R}^m of $\mathbf{F}_{x_iso}^m$
 - Solve singular-value decomposition problem

$$\mathbf{R}^m = \mathbf{U}^m * \mathbf{E}^m * \mathbf{V}^m$$
 3. Compute the sum of the subspace matrix

$$\mathbf{S} = \sum_{j=1}^d \mathbf{U}^1(:, j) * \mathbf{U}^1(:, j)^T + \sum_{j=1}^d \mathbf{U}^2(:, j) * \mathbf{U}^2(:, j)^T + \dots\dots\dots + \sum_{j=1}^d \mathbf{U}^M(:, j) * \mathbf{U}^M(:, j)^T$$
 4. Solve singular-value decomposition problem

$$\mathbf{S} = \mathbf{U}_{ds} * \mathbf{E}_{ds} * \mathbf{V}_{ds}$$
 5. Construct the difference subspace $\mathbf{D} = \mathbf{U}_{ds}(:, find(diag(\mathbf{E}_{ds}) < 1))$ by considering only those eigenvectors of \mathbf{S} whose eigenvalues are less than 1.
 6. Project training data $\mathbf{F}_{x_iso}^m$ onto the difference subspace

$$\mathbf{F}_{x_ds}^m = \mathbf{D}^T * \mathbf{F}_{x_iso}^m$$
 7. Compute linear subspace of class- m on DS projected data $\mathbf{F}_{x_ds}^m$,
 - Find covariance matrix \mathbf{R}_{ds}^m of $\mathbf{F}_{x_ds}^m$
 - Solve singular-value decomposition problem

$$\mathbf{R}_{ds}^m = \mathbf{U}_{ds}^m * \mathbf{E}_{ds}^m * \mathbf{V}_{ds}^m$$
 8. for $i = 1:N_t$
 9. Project test trial onto ISOMap space and then onto DS

$$\mathbf{f}_{y,i_iso} = \mathbf{P} * \mathbf{f}_{y,i}$$

$$\mathbf{f}_{y,i_ds} = \mathbf{D}^T * \mathbf{f}_{y,i_iso}$$
 10. On DS, project test trial \mathbf{f}_{y,i_ds} onto d -dimensional class- m subspace

$$\mathbf{p}_1 = \mathbf{U}_{ds}^1(:, 1:d)^T * \mathbf{f}_{y,i_ds}$$

$$\dots\dots\dots$$

$$\mathbf{p}_M = \mathbf{U}_{ds}^M(:, 1:d)^T * \mathbf{f}_{y,i_ds}$$
 11. Classify test trial $\mathbf{f}_{y,i}$ to the subspace of maximum projection

$$Class(i) = \max_m(norm(\mathbf{p}_m))$$
 12. end
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