Aufgabe 2 (ca. 30 Minuten):

Lösen Sie die algebraische Gleichung

$$z^4 + 4z^2 + 16 = 0$$

mit Hilfe einer geeigneten Substitution und zeichnen Sie die Lösungen in der Gaussschen Zahlenebene ein.

$$z^4 + 4z^2 + 16 = 0 \qquad z^2 = u \qquad \boxed{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$\rightarrow \quad u^2 + 4u + 16 = 0$$

$$u_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 16}}{2 \cdot 1} = \frac{-4 \pm \sqrt{16 - 64}}{2}$$

$$= \frac{-4 \pm \sqrt{-48}}{2} = \frac{-4 \pm 4i\sqrt{3}}{2} = \underline{-2 \pm 2i\sqrt{3}}$$

$$u_1 = -2 - 2i\sqrt{3} \qquad u_2 = -2 + 2i\sqrt{3}$$

$$[u = z^2]: \; z_1^2 = -2 - 2i\sqrt{3} \qquad z_2^2 = -2 + 2i\sqrt{3}$$

$\Rightarrow$ Polarform:

$$|u_1| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = 4$$

$$\arg(u_1) = \arctan\left(\frac{2\sqrt{3}}{-2}\right) = \arctan\left(-\sqrt{3}\right) = -\frac{2\pi}{3}$$

$$|u_2| = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = 4$$

$$\arg(u_2) = \arctan\left(\frac{-2\sqrt{3}}{-2}\right) = \arctan(\sqrt{3}) = \frac{\pi}{3}$$

$$z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{i\pi/3} \qquad z_2 = 2\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right) = 2e^{-i\frac{\pi}{3}}$$

$$z_3 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2e^{i\,2\pi/3} \qquad z_4 = 2\left(\cos -\frac{2\pi}{3} + i\sin -\frac{2\pi}{3}\right) = 2e^{-i\,2\pi/3}$$

$x_1 = r \cdot \cos\varphi = 1$ $\quad y_1 = r \cdot \sin\varphi = \sqrt{3} = 1{,}7321$

$x_2 = r \cdot \cos\varphi = 1$ $\quad y_2 = r \cdot \sin\varphi = -\sqrt{3} = -1{,}7321$

$x_3 = r \cdot \cos\varphi = -1$ $\quad y_3 = r \cdot \sin\varphi = \sqrt{3} = 1{,}7321$

$x_4 = r \cdot \cos\varphi = -1$ $\quad y_4 = r \cdot \sin\varphi = -\sqrt{3} = -1{,}7321$

$z_1 = 2e^{i\pi/3}$

$z_2 = 2e^{-i\pi/3}$

$z_3 = 2e^{i\frac{2\pi}{3}}$

$z_4 = 2e^{-i\frac{2\pi}{3}}$