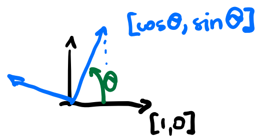
3D Transformations

Basic transformations

1. Create a matrix M1 that rotates 3D space around the x axis by 90 degrees.
2. Create a matrix M2 that rotates 3D space around the y axis by 90 degrees.
3. Create a matrix M3 that translates 3D space by 1 unit along the x axis.

① $M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

In general around x axis: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$



$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

② $M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

③ $M_3 = \left[\begin{array}{ccc|c} & I & & 1 \\ & & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

Composition of transformations

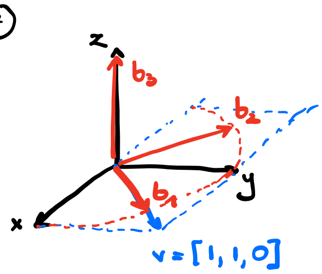
4. Consider two transformations:

>> First, translate 3D space by 1 unit along the x-axis and then rotate it around the y-axis by 90 degrees.
>> Second, rotate 3D space around the y-axis by 90 degrees and then translate it by 1 unit along the x-axis.

Do you expect the matrices representing the above transformations to be the same?

5. Consider the results of two multiplications: `M4 = M2 * M3` and `M5 = M3 * M2`. Do you expect to get the same results? Check computationally if your predictions were correct.
6. Consider two multiplications: `M4 * M5` and `M5 * M4`. Do you expect to get the same results? Check computationally if your predictions were correct.
7. Create a matrix `M6` that rotates a 3D space around the `(1, 1, 0)` vector by any given number of degrees.

⑦



$\mathcal{B} = \{b_1, b_2, b_3\}$ $\quad b_1 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]$ $\quad b_2 = [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]$ $\quad b_3 = \hat{z}$

$M(id)^{St}_{\mathcal{B}} \cdot M(r_x(\theta))^{\mathcal{B}}_{\mathcal{B}} \cdot M(id)^{\mathcal{B}}_{St}$

$\begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}^T =$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \\ - \,\|\, - \\ & \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -\frac{\cos\theta}{\sqrt{2}} & +\frac{\cos\theta}{\sqrt{2}} & -\sin\theta \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} & \cos\theta \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{\cos\theta}{2} & \frac{1}{2} - \frac{\cos\theta}{2} & + \frac{\sin\theta}{\sqrt{2}} \\ \frac{1}{2} - \frac{\cos\theta}{2} & \frac{1}{2} + \frac{\cos\theta}{2} & - \frac{\sin\theta}{\sqrt{2}} \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} & \cos\theta \end{bmatrix}$$

Different representations of transformations

Familirize yourself with the Rodrigues Formula: https://mathworld.wolfram.com/RodriguesRotationFormula.html.

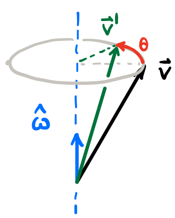
8. Use the Rodrigues Formula to create a matrix `M6` from the previous exercise. Compare the results and make sure they are the same.
9. Use the Rodrigues Formula to create a matrix `M7` that rotates 3D space around the (1, 1, 1) vector by 90 degrees.

If you want to know more about Rodrigues Formula and how it is derived you can check out these videos: https://www.youtube.com/watch?v=UaK2q22mMEg and https://www.youtube.com/watch?v=q-ESzg03mQc.

10. We have the following rotation matrix:

```
R = [[0.966496, -0.214612, 0.14081],
     [0.241415, 0.946393, -0.214612],
     [-0.0872034, 0.241415, 0.966496]]
```

What is the axis and the angle of rotation R? *Hint:* consider a vector parallel to the axis of rotation. How the matrix R should act on this vector?



$$\vec{v}' = \langle \vec{v}, \hat{\omega} \rangle \cdot \hat{\omega} + \vec{v}_\perp \cdot \cos\theta + (\hat{\omega} \times \vec{v}) \cdot \sin\theta =$$

the part of original vector $\vec{v}$ which is parallel to $\hat{\omega}$; this part will be the same after the rotation

the part of the original vector $\vec{v}$ which is orthogonal to $\hat{\omega}$ rotated by the angle θ

$$= \langle \vec{v}, \hat{\omega} \rangle \hat{\omega} + (\vec{v} - \langle \vec{v}, \hat{\omega} \rangle \hat{\omega}) \cos\theta + (\hat{\omega} \times \vec{v}) \cdot \sin\theta = (1 - \cos\theta) \langle \vec{v}, \hat{\omega} \rangle \hat{\omega} + \vec{v} \cos\theta + (\hat{\omega} \times \vec{v}) \sin\theta$$

Now, this defines a map $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $\varphi(\vec{v}) = \vec{v}'$. If possible, we would like to represent the formula as a matrix multiplication. For that purpose:

$$\hat{\omega} \times \vec{v} = \begin{Bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega_x & \omega_y & \omega_z \\ v_x & v_y & v_z \end{Bmatrix} = \begin{bmatrix} \omega_y v_z - \omega_z v_y \\ \omega_z v_x - \omega_x v_z \\ \omega_x v_y - \omega_y v_x \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\langle \vec{v},\hat{\omega}\rangle\hat{\omega} = (v_x\omega_x+v_y\omega_y+v_z\omega_z)\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}\omega_x^2 v_x + \omega_x\omega_y v_y + \omega_x\omega_z v_z\\ \omega_x\omega_y v_x + \omega_y^2 v_y + \omega_y\omega_z v_z\\ \omega_x\omega_z v_x + \omega_y\omega_z v_y + \omega_z^2 v_z\end{bmatrix} = \begin{bmatrix}\omega_x^2 & \omega_x\omega_y & \omega_x\omega_z\\ \omega_x\omega_y & \omega_y^2 & \omega_y\omega_z\\ \omega_x\omega_z & \omega_y\omega_z & \omega_z^2\end{bmatrix}\begin{bmatrix}v_x\\ v_y\\ v_z\end{bmatrix}$$

So:

$$\varphi(\vec{v}) = (1-\cos\theta)\begin{bmatrix}\omega_x^2 & \omega_x\omega_y & \omega_x\omega_z\\ \omega_x\omega_y & \omega_y^2 & \omega_y\omega_z\\ \omega_x\omega_z & \omega_y\omega_z & \omega_z^2\end{bmatrix}\vec{v} + \cos\theta\cdot I\cdot\vec{v} + \sin\theta\begin{bmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{bmatrix}\cdot\vec{v} =$$

The sum + + results in the formula for the rotation matrix $R_{\hat{\omega}}(\theta)$:

$$R_{\hat{\omega}}(\theta) = e^{\tilde{\omega}\theta} \quad (1)$$

$$= I + \tilde{\omega}\sin\theta + \tilde{\omega}^2(1-\cos\theta) \quad (2)$$

$$= \begin{bmatrix}\cos\theta + \omega_x^2(1-\cos\theta) & \omega_x\omega_y(1-\cos\theta) - \omega_z\sin\theta & \omega_y\sin\theta + \omega_x\omega_z(1-\cos\theta)\\ \omega_z\sin\theta + \omega_x\omega_y(1-\cos\theta) & \cos\theta + \omega_y^2(1-\cos\theta) & -\omega_x\sin\theta + \omega_y\omega_z(1-\cos\theta)\\ -\omega_y\sin\theta + \omega_x\omega_z(1-\cos\theta) & \omega_x\sin\theta + \omega_y\omega_z(1-\cos\theta) & \cos\theta + \omega_z^2(1-\cos\theta)\end{bmatrix}, \quad (3)$$

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$$\hat{\omega} = [1/\sqrt{2}, 1/\sqrt{2}, 0] \rightsquigarrow R_{\hat{\omega}}(\theta) = \begin{bmatrix}\frac{1+\cos\theta}{2} & \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}}\\ \frac{1-\cos\theta}{2} & \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}}\\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} & \cos\theta\end{bmatrix}$$

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$$R_{\hat{\omega}}(\theta) = \begin{bmatrix}\frac{1}{3} & \frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}+\frac{1}{3}\\ \frac{1}{\sqrt{3}}+\frac{1}{3} & \frac{1}{3} & -\frac{1}{\sqrt{3}}+\frac{1}{3}\\ -\frac{1}{\sqrt{3}}+\frac{1}{3} & \frac{1}{\sqrt{3}}+\frac{1}{3} & \frac{1}{3}\end{bmatrix} \rightsquigarrow \hat{\omega} = ? \text{ and } \theta = ?$$

- $\hat{\omega}$:

vectors parallel to the axis of rotation should not change during rotation $\rightsquigarrow$ we are looking for eigenvectors with the eigenvalue $\lambda = 1$: $R_{\hat{\omega}}(\theta)\vec{v} = \vec{v}$

$$(R_{\hat{\omega}}(\theta) - I)\vec{v} = 0 \rightsquigarrow \begin{bmatrix}-\frac{2}{3} & \frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}+\frac{1}{3}\\ \frac{1}{\sqrt{3}}+\frac{1}{3} & -\frac{2}{3} & -\frac{1}{\sqrt{3}}+\frac{1}{3}\end{bmatrix}\begin{bmatrix}v_x\\ v_y\\ v_z\end{bmatrix} = \vec{0} \rightsquigarrow \ldots \rightsquigarrow \vec{v} = [1,1,1] \rightsquigarrow \hat{\omega} = \frac{1}{\sqrt{3}}[1,1,1]$$

- θ:

$$\left.\begin{aligned}&[R_{\hat{\omega}}(\theta)]_{11} = \frac{1}{3} \Leftrightarrow \cos\theta + \frac{1}{3}(1-\cos\theta) = \frac{1}{3} \Leftrightarrow \frac{2}{3}\cos\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2} \vee \theta = \frac{3\pi}{2}\\ &[R_{\hat{\omega}}(\theta)]_{21} = \frac{1}{\sqrt{3}}+\frac{1}{3} \Leftrightarrow \frac{1}{\sqrt{3}}\sin\theta + \frac{1}{3} = \frac{1}{\sqrt{3}}+\frac{1}{3} \Leftrightarrow \sin\theta = 1\end{aligned}\right\} \rightsquigarrow \theta = \frac{\pi}{2}$$

Camera Callibration and Stereo

Intrinsic camera parameters

11. Determine the intrinsic camera matrix of a pinhole cameras which has focal length of 1 and optical center at `(300.5, 300.5)`.
12. Find 2D coordinates of the projection of the 3D point `(10, 10, 5)` onto an image captured by the camera from the previous exercise.

Basic triangulation

13. Imagine that you have two cameras like in the previous exercises. The optical axes of both cameras are parallel. The axis of the second camera is displaced by `(x=1, y=0, z=0)` relative to the first camera. Estimate the distance from the first camera to a point whose coordinates are `(303, 303)` on an image captured by the first camera and `(298, 303)` on the image captured by the second camera.

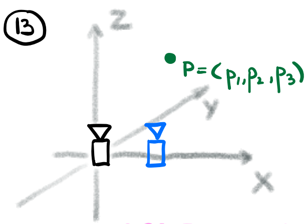
(11)

$$K = \begin{bmatrix} f_x & 0 & o_x \\ 0 & f_y & o_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 300.5 \\ 0 & 1 & 300.5 \\ 0 & 0 & 1 \end{bmatrix}$$

(12)

$$K \cdot \begin{bmatrix} 10 \\ 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 1512.5 \\ 1512.5 \\ 5 \end{bmatrix} \equiv \begin{bmatrix} 302.5 \\ 302.5 \\ 1 \end{bmatrix}$$

(13)



$$\rightsquigarrow \quad K \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \equiv \begin{bmatrix} 303 \\ 303 \\ 1 \end{bmatrix} \quad \text{and} \quad K \cdot \begin{bmatrix} p_1 - 1 \\ p_2 \\ p_3 \end{bmatrix} \equiv \begin{bmatrix} 298 \\ 303 \\ 1 \end{bmatrix} \rightsquigarrow$$

$$\rightsquigarrow \begin{bmatrix} p_1 + 300.5p_3 \\ p_2 + 300.5p_3 \\ p_3 \end{bmatrix} \equiv \begin{bmatrix} 303 \\ 303 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p_1 + 300.5p_3 - 1 \\ p_2 + 300.5p_3 \\ p_3 \end{bmatrix} \equiv \begin{bmatrix} 298 \\ 303 \\ 1 \end{bmatrix} \rightsquigarrow$$

$$\rightsquigarrow \begin{cases} \frac{p_1}{p_3} = 2.5 \\ \frac{p_2}{p_3} = 2.5 \end{cases} \quad \text{and} \quad \begin{cases} \frac{p_1 - 1}{p_3} = -2.5 \\ \frac{p_2}{p_3} = 2.5 \end{cases} \rightsquigarrow \begin{cases} p_1 = 2.5p_3 \\ p_2 = 2.5p_3 \\ 2.5p_3 - 1 = -2.5p_3 \end{cases} \rightsquigarrow P = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{5}\right)$$

distance to the 1st camera