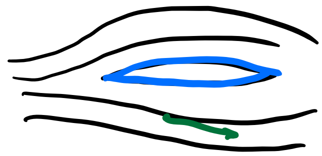
Dynamical Systems:

x - state of the system

$$\dot{x} = f(x)$$



Linearization around fixed points:

$$f(\bar{x}) = 0$$

$$\Delta x = x - \bar{x}$$

$$\left.\begin{aligned}\Delta\dot{x} = f(\bar{x} + \Delta x) = \underbrace{f(\bar{x})}_{=0} + D_{\bar{x}}f \cdot \Delta x + \text{higher order terms}\end{aligned}\right\} \dot{x} = Ax$$

small

System 1

Consider the following 1-dimensional system that can be used to model population growth x is the population size, P_{max} is population limit above which the environment becomes resource scarse.

$$\dot{x} = f(x) = x(P_{max} - x)$$

state: $x \in \mathbb{R}$

$$f(x) = x(P_{max} - x)$$

fixed points:

$$f(\bar{x}) = 0 \iff x = 0 \quad \vee \quad x = P_{max}$$

Linearization:

$$f'(x) = P_{max} - 2x$$

i) around $\bar{x} = 0$

$$\Delta x = x - \bar{x} = x$$

$$D_{\bar{x}}f \left\{ f'(\bar{x}) = P_{max} \right.$$

$$\dot{x} = \Delta\dot{x} = P_{max} \cdot \Delta x = x$$

ii) around $\bar{x} = P_{max}$

$$\Delta x = x - P_{max}$$

$$D_{\bar{x}}f \left\{ f'(P_{max}) = -P_{max} \right.$$

$$\dot{x} = \Delta\dot{x} = -P_{max} \cdot \Delta x = -P_{max} \cdot (x - P_{max})$$

System 2

Damped pendulum (δ is the damping coefficient), where θ denotes the angle.

$$\ddot{\theta} = -\sin(\theta) - \delta\dot{\theta}$$

Denote

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

$$\dot{x} = \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_2 \\ -\sin(x_1) - \delta x_2 \end{pmatrix}$$

which is non-linear due to the $\sin$ function.

state: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} \in \mathbb{R}^2$

$$f(\vec{x}) = \begin{pmatrix} x_2 \\ -\sin x_1 - \delta x_2 \end{pmatrix}$$

fixed points:

$$f(\vec{x}) = 0 \iff \begin{cases} x_1 = k\pi & , k \in \mathbb{Z} \\ x_2 = 0 \end{cases} \iff \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \vee \vec{x} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

linearization:

$$D_x f = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -\delta \end{bmatrix}$$

i) $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\dot{\vec{x}} = \Delta\dot{\vec{x}} = D_{\bar{x}} f \cdot \Delta\vec{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

ii) $\bar{x} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$

$$\dot{\vec{x}} = \Delta\dot{\vec{x}} = D_{\bar{x}} f \cdot \Delta\vec{x} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix} \begin{bmatrix} x_1 - \pi \\ x_2 \end{bmatrix}$$

System 3

In the following system you can assume $-\pi \leq \theta \leq \pi$

$$\dot{r} = r^2 - r$$

$$\dot{\theta} = \sin^2(\theta/2)$$

state: $\vec{x} = \begin{pmatrix} r \\ \theta \end{pmatrix}$

$$f(\vec{x}) = \begin{pmatrix} r^2 - r \\ \sin^2\left(\frac{\theta}{2}\right) \end{pmatrix}$$

fixed points:

$$f(\vec{x}) = 0 \Leftrightarrow \begin{cases} r(r-1) = 0 \\ \sin^2\left(\frac{\theta}{2}\right) = 0 \end{cases} \Leftrightarrow \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \vee \vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

linearization:

$$D_x f = \begin{bmatrix} 2r-1 & 0 \\ 0 & \cancel{2}\sin\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{2}\right)\cdot\frac{1}{\cancel{2}} \end{bmatrix}$$

I) $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\dot{\vec{x}} = \Delta\dot{\vec{x}} = D_{\bar{x}} f \cdot \Delta\bar{x} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} r \\ \theta \end{bmatrix}$$

II) $\bar{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\dot{\vec{x}} = \Delta\dot{\vec{x}} = D_{\bar{x}} f \cdot \Delta\bar{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} r-1 \\ \theta \end{bmatrix}$$

System 4

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(3-x-2y) \\ y(2-x-y) \end{pmatrix}$$

state: $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$f(\vec{x}) = \begin{pmatrix} x(3-x-2y) \\ y(2-x-y) \end{pmatrix}$$

fixed points:

$$f(\vec{x}) = 0 \Leftrightarrow \vec{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \vee \vec{x}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \vee \vec{x}_3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \vee \vec{x}_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

linearization:

$$D_x f = \begin{bmatrix} 3-2x-2y & -2x \\ -2y & 2-2x-2y \end{bmatrix}$$

$$f(\vec{x}) = \begin{bmatrix} 3x - x^2 - 2xy \\ 2y - 2xy - y^2 \end{bmatrix}$$

I) $\vec{x}_1$

$$\dot{\vec{x}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

II) $\vec{x}_2$

$$\dot{\vec{x}} = \begin{bmatrix} -1 & 0 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y-2 \end{bmatrix}$$

III) $\vec{x}_3$

$$\vec{x} = \begin{bmatrix} -3 & -6 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x-3 \\ y \end{bmatrix}$$

IV) $\vec{x}_4$

$$\vec{x} = \begin{bmatrix} -1 & -2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$