The Sturm-Liouville Equation of Two-Body Dynamical System

Siwei Luo*
The University of Illinois at Chicago

We introduce a Sturm-Liouville equation derived from Newton's law to describe the dynamical system of the two-body system under the influence of torque. In the paper, we present conic sections and the Bohr model are solutions of the equation. Moreover, the logarithm spiral can be the solution as well and the complex solutions are able to form the basis of Hilbert space.

Keywords: Sturm-Liouville problem, Two-body system

I. INTRODUCTION

Sturm-Liouville equations are ubiquitous in modern physics. Many second-order differential equations arise from different phenomena can be written as Sturm-Liouville type equations. Some famous examples are Bessel equation, Legendre equation and so forth. Based on the Sturm-Liouville theory, their eigenvalues can be ordered and eigenvectors enjoy the property of orthogonality.[1, 2] More often, they reveal the nature of oscillation and spectra of some dynamical system.

The two-body dynamical system has been studied extensively in physics, such as celestial orbit, Rutherford experiment, Bohr model and so forth. Based on the Newtonian mechanism, the trajectory of a particle in a two-body system is a conic section when angular momentum is constant. However, the evolution of dynamical system under the influence of torque is not fully understood yet, which is of great interest and complexity. We introduce a Sturm-Liouville equation derived from the Newtonian equation to describe the evolution of the system under the more general circumstances.

In this paper, we present that conic sections and Bohr models are solutions of the proposed Sturm-Liouville equation. Moreover, the orbit of the two-body system can also be logarithm spirals or even more complicated solutions.

Along with celestial orbits, the hydrogen atom may also be an interesting objective to be studied in this framework because of its relatively simple structure. Its orbital angular momentum is not a constant when the hydrogen atom absorbs or releases photon which carries angular momentum.[3, 4] The spectra of this two-body system, like a fingerprint, is essential for the hydrogen element identification. And it is related to the Sturm-Liouville equation.

II. TWO BODY DYNAMICAL SYSTEM

A. Newtonian equations

Recall that the two fundamental equations of the two-body system are Eqn.1 and Eqn.2 respectively.[6]

$$m\ddot{r} - m\dot{\theta}^2 r = -\frac{c}{r^2}, \dot{} = \frac{\partial}{\partial t} \tag{1}$$

$$\frac{\partial L}{\partial t} - \frac{\partial L}{\partial x}\frac{\partial x}{\partial t} = T \tag{2}$$

where c is a constant; m is the particle's mass; r is the radius; θ is angular displacement; t is time; L is the orbital angular momentum; T is the torque.

In principle, the Newtonian equation is a deterministic problem that can be solved with given boundary and initial condition. To derive the Sturm-Liouville equation, we need substitute r and $\dot{\theta}$ in the Eqn.1 with $r = 1/u$ and $\dot{\theta} = Lu^2/m$. It is essential not to presume the torque is zero in this calculation.

$$\dot{r} = -\frac{\dot{u}}{u^2} = -\frac{\dot{\theta}}{u^2}\frac{\partial u}{\partial \theta} = -\frac{L}{m}\frac{\partial u}{\partial \theta} \tag{3}$$

$$\ddot{r} = -\frac{L}{m}\frac{\partial^2 u}{\partial \theta^2}\dot{\theta} - \frac{1}{m}\frac{\partial L}{\partial \theta}\frac{\partial u}{\partial \theta}\dot{\theta} = -\frac{L^2u^2}{m^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{Lu^2}{m^2}\frac{\partial L}{\partial \theta}\frac{\partial u}{\partial \theta} \tag{4}$$

$$\dot{\theta}^2 r = \frac{L^2u^3}{m^2} \tag{5}$$

After simplification, the differential equation Eqn.1 reads:

$$L^2\frac{d^2u}{d\theta^2} + L\frac{dL}{d\theta}\frac{du}{d\theta} + L^2u - mc = 0 \tag{6}$$

* siuluosiwei@gmail.com

B. Sturm-Liouville equation of two-body system

Let us write the Eqn.6 in the form of a Sturm-Liouville equation,

$$\frac{d}{d\theta}(L\frac{du}{d\theta}) + Lu = \frac{mc}{L} \tag{7}$$

Now, this equation is extended to more general situations that orbital angular momentum has not to be a constant. While, the results from the classical mechanism when orbital angular momentum is a constant still solves Eqn.7. So it is straightforward to check the following propositions.

Proposition 1. *Conic section(Kepler orbit) are solutions of Eqn.7 when orbital angular momentum is a constant.*

First, conic sections are the solutions of the equation, when the orbital angular momentum is constant, the Eqn.7 can be written as:

$$\frac{d^2u}{d\theta^2} + u = \frac{mc}{L^2} \tag{8}$$

And the solutions is:

$$u = \frac{mc}{L^2}(1 + ecos(\theta)) \tag{9}$$

where, e is eccentricity.

If $\frac{du}{d\theta} = 0$, particularly, then $L^2u = mc$ is the third Kepler's law.[6]

Proposition 2. *The Bohr orbits are solutions of Eqn.7 when orbital angular momentum $L = n\hbar$.*

It is worth to check that the Bohr model is also the solution to the equation.

$$L^2\frac{d^2u}{d\theta^2} + L\frac{dL}{d\theta}\frac{du}{d\theta} + (L^2 - \frac{m_e e^2}{4\pi\varepsilon_0 u})u = 0 \tag{10}$$

$$L = n\hbar \tag{11}$$

where,$\hbar$ is the reduced Plank constant.

From this result, the allowed orbits in the Bohr model[7] are special cases of the equations and can be achieved when orbital angular momentum of electron equals to $n\hbar$. Then, the solutions are:

$$r_n = \frac{1}{u_n} = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2}n^2 \tag{12}$$

C. Spiral solutions and orthogonal basis

The most intriguing part of this study is to see that some extraordinary orbits can be solutions of the two-body system. It is not that much obvious to get spiral solutions. Because, compared with other Sturm-Liouville type equations like Bessel equation, Legendre equation, the Eqn.7 actually has two unknown variables L and u. Furthermore, L and u are not independent of each other. Therefore, to solve this equation, one may need to guess and construct L and u simultaneously. Without loss of generality, in the following discussion let m = 1 in Eqn.7.

Proposition 3. *Logarithm spiral can be solutions of Eqn.7.*

First, let $u = c_1e^{-\theta}$, $L = c_2e^{\frac{\theta}{2}}$, the Eqn.7 reads:

$$\frac{1}{2}c_1c_2e^{-\frac{\theta}{2}} + c_1c_2e^{-\frac{\theta}{2}} = \frac{c}{c_2}e^{-\frac{\theta}{2}} \tag{13}$$

$$\frac{3}{2}c_1c_2e^{-\frac{\theta}{2}} = \frac{c}{c_2}e^{-\frac{\theta}{2}} \tag{14}$$

As long as the coefficients satisfy the following condition:

$$\frac{3}{2}c_1c_2^2 = c \tag{15}$$

the logarithm spiral is the solution of the equation.

Proposition 4. *The sequence $u_n = c_ne^{-ik\theta}, k \in \mathbb{Z}$ forms the orthogonal basis of Hilbert space.*

Similarly, let $u = c_1e^{-i\theta}$, $L = c_2e^{i\frac{\theta}{2}}$, the Eqn.7 reads:

$$-\frac{1}{2}c_1c_2e^{-i\frac{\theta}{2}} + c_1c_2e^{-i\frac{\theta}{2}} = \frac{c}{c_2}e^{-i\frac{\theta}{2}} \tag{16}$$

$$\frac{1}{2}c_1c_2e^{-i\frac{\theta}{2}} = \frac{c}{c_2}e^{-i\frac{\theta}{2}} \tag{17}$$

$$\frac{1}{2}c_1c_2^2 = c \tag{18}$$

By the observation of these two solutions, let $u = c_1e^{-ik\theta}$, $L = c_2e^{ik\frac{\theta}{2}}$, the Eqn.7 reads:

$$-\frac{1}{2}c_1c_2k^2e^{-ik\frac{\theta}{2}} + c_1c_2e^{-ik\frac{\theta}{2}} = \frac{c}{c_2}e^{-ik\frac{\theta}{2}} \tag{19}$$

$$(1 - \frac{k^2}{2})c_1c_2^2 = c \tag{20}$$

Significantly, the sequence $u_n = c_ne^{-ik\theta}, k \in \mathbb{Z}$ forms the orthogonal basis of Hilbert space.[13]

III. CONCLUSION

We extend our study of the two-body dynamical system to a more general situation that some torque is exerted on the system. The Sturm-Liouville type equation arises naturally from the study of the two-body dynamical system under nonzero torque. We have observed some fascinating orbits and patterns can solve the equation. More importantly, it will help understand the spectra and resonance of the dynamical system.

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