Assignment 5

Due: Thursday, April 1 (at the beginning of class)

Let $\Sigma = \{0,1\}$ and assume that all languages and classes of languages considered in this assignment are over the alphabet Σ. Also assume that the word "polynomial" means "non-constant polynomial with nonnegative integer coefficients."

1. Suppose that $A \subseteq \Sigma^*$ is NP-complete, $B \subseteq \Sigma^*$ is in P, $A \cap B = \varnothing$, and $A \cup B \neq \Sigma^*$. Prove that $A \cup B$ is NP-complete.

2. For every language $A \subseteq \Sigma^*$ and function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n) \geq n+1$ for all $n \in \mathbb{N}$, define a new language
$$\mathrm{pad}(A,f) = \left\{ x01^{f(|x|)-|x|-1} \;:\; x \in A \right\}.$$
The way to think about the language $\mathrm{pad}(A,f)$ is that it "pads" strings $x \in A$ with a tail of the form 01^m, which has the effect of artificially increasing the length of strings in the language: for each $x \in A$ there is a unique corresponding string $x01^m \in \mathrm{pad}(A,f)$ whose length is $f(|x|)$.

   (a) Prove that for any polynomial p with $p(n) \geq n+1$ we have
$$A \in \text{DTIME}\left(2^{p(n)}\right) \quad \Leftrightarrow \quad \mathrm{pad}(A,p) \in \text{DTIME}(2^n).$$

   (b) Prove that for any polynomial p with $p(n) \geq n+1$ we have
$$A \in \text{PSPACE} \quad \Leftrightarrow \quad \mathrm{pad}(A,p) \in \text{PSPACE}.$$

   (c) Use the time-hierarchy theorem, along with parts (a) and (b), to prove
$$\text{PSPACE} \neq \bigcup_{k \geq 1} \text{DTIME}\left(2^{k \cdot n}\right).$$

   (d) Repeat (b) and (c) with NP in place of PSPACE.

3. Prove that the following two statements are equivalent (i.e., that each one implies the other).

   Statement 1: For every language $B \in \text{P}$ and every polynomial p, there exists a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ such that
$$\left(\exists y \in \Sigma^{p(|x|)}\right) [\langle x, y \rangle \in B] \quad \Leftrightarrow \quad \langle x, f(x) \rangle \in B$$

   Statement 2: P = NP.

4. [Bonus problem] Prove that if there exists a set $S \subseteq \mathbb{N}$ for which the language $\{1^n : n \in S\}$ is NP-complete, then P = NP. (Hint: thinking about the previous problem may help with this one.)