Assignment 4

Due: Thursday, March 18 (at the beginning of class)

Let $\Sigma = \{0,1\}$ and assume that all languages (and classes of languages) considered in this assignment are over the alphabet Σ. As usual, you should assume that a reasonable encoding scheme has been fixed whereby every DTM M is encoded by at least one string $\langle M \rangle$ (which, for this assignment, is a string over Σ).

1. Recall that the language INF is defined as follows:

$$\text{INF} = \{\langle M \rangle \,:\, M \text{ is a DTM and } L(M) \text{ is infinite}\}\,.$$

Prove that there exists a decidable language $B \subseteq \Sigma^*$ such that

$$\text{INF} = \{x \in \Sigma^* \,:\, (\forall y \in \Sigma^*)(\exists z \in \Sigma^*)[\langle x, y, z \rangle \in B]\}\,.$$

2. For each string $x \in \Sigma^*$ define a language $B_x \subseteq \Sigma^*$ as

$$B_x = \{\langle M \rangle \,:\, M \text{ is a DTM for which } x \in L(M)\}\,.$$

Prove that $B_x \leq_m B_y$ for every choice of strings $x, y \in \Sigma^*$.

3. Prove that there does not exist a DTM M with the following property: for every DTM K that halts on all inputs, M accepts $\langle K \rangle$ if and only if $L(K)$ is infinite.

4. Prove that there exist languages B and C such that these two properties simultaneously hold:
   (a) $B \leq_T C$
   (b) $B \not\leq_m C$ and $B \not\leq_m \overline{C}$.

5. Let $f : \mathbb{N} \to \mathbb{N}\backslash\{0\}$ be an arbitrary function, and let $B, C \subseteq \Sigma^*$ be languages such that the symmetric difference $B \triangle C$ is finite. Prove the following logical equivalences:

$$\begin{aligned} B \in \text{DTIME}(f) &\Leftrightarrow C \in \text{DTIME}(f), \\ B \in \text{NTIME}(f) &\Leftrightarrow C \in \text{NTIME}(f), \\ B \in \text{DSPACE}(f) &\Leftrightarrow C \in \text{DSPACE}(f), \\ B \in \text{NSPACE}(f) &\Leftrightarrow C \in \text{NSPACE}(f). \end{aligned}$$

Hint: once you have the right idea, all of the equivalences will follow—you should not have to consider them separately.