Assignment 3

Due: Thursday, February 25 (at the beginning of class)

Assume that $\Sigma = \{0, 1\}$ for every question on this assignment.

1. Suppose that $f : \Sigma^* \to \Sigma^*$ is a computable function with the property that

$$x \leq y \;\Rightarrow\; f(x) \leq f(y)$$

for all strings $x, y \in \Sigma^*$. (For two strings $u, v \in \Sigma^*$ we interpret $u \leq v$ to mean that either $u = v$ or u comes before v in the lexicographic ordering of Σ^*.)

Prove that $\mathrm{range}(f) = \{f(x) \,:\, x \in \Sigma^*\}$ is a decidable language.

2. Prove that the language

$$\{\langle D, k\rangle \,:\, D \text{ is an DFA, } k \in \mathbb{N}, \text{ and } D \text{ accepts at most } k \text{ strings over its alphabet}\}$$

is decidable.

For the purposes of this question you should assume that every DFA under consideration has an alphabet of the form $\Gamma = \{\tau_0, \ldots, \tau_{n-1}\}$ for some choice of $n \geq 1$ and a set of states $Q = \{q_0, \ldots, q_{m-1}\}$ for some choice of $m \geq 1$.

3. Prove that the language

$$\mathrm{INF}_{\mathrm{CFG}} = \{\langle G\rangle \,:\, G \text{ is a context-free grammar and } L(G) \text{ is infinite}\}$$

is decidable.

Along similar lines to the previous question, you should assume for this question that every context-free grammar under consideration has an alphabet of the form $\Gamma = \{\tau_0, \ldots, \tau_{n-1}\}$ for some choice of $n \geq 1$ and a set of variables $V = \{X_0, \ldots, X_{m-1}\}$ for some choice of $m \geq 1$.

4. Suppose that $A \subseteq \Sigma^*$ is a Turing-recognizable language. Define

$$B = \{x \in \Sigma^* \,:\, \text{there exists a string } y \in \Sigma^* \text{ such that } xy \in A\}\,.$$

Prove that B is Turing-recognizable.

5. Assume that A is an infinite Turing-recognizable language. Prove that there exists an infinite decidable language $B \subseteq A$.

6. [Bonus question] Suppose we have agreed on a reasonable encoding scheme for DTMs having the input alphabet Σ, whereby (i) every such DTM is encoded by at least one string over Σ, and (ii) every string over Σ is an encoding of some such DTM.

(One way to define such an encoding scheme would be to start with an ordinary encoding scheme for DTMs having the input alphabet Σ. Then, any string $w \in \Sigma^*$ that would normally not encode a valid DTM is interpreted as an encoding of the same DTM as the lexicographically first string after w that does encode a valid DTM with respect to the original scheme. You should not assume, however, that the encoding has necessarily been chosen in this way.)

Prove that there exists a DTM M with input alphabet Σ, and an encoding $\langle M\rangle \in \Sigma^*$ of M, with the following property: for M^{R} being the DTM encoded by the string $\langle M\rangle^{\mathrm{R}}$ it holds that $L(M^{\mathrm{R}}) = L(M)^{\mathrm{R}}$. In other words, by reversing the encoding $\langle M\rangle$, we obtain an encoding of a DTM accepting the reverse of the language $L(M)$.