University of Waterloo
CS 462 — Formal Languages and Parsing
Winter 2013
Problem Set 3

Distributed Tuesday, January 22 2013.

Due Tuesday, January 29 2013, in class.

All answers should be accompanied by proofs.

1. [10 marks] Must every primitive word of length > 1 have at least two unbordered conjugates? Prove or disprove.

   Yes they do.

   Proof by Contradiction. $\vDash$

   Let $w \in \sum^*$ and $\alpha \in \sum$ s.t. $\neg\exists v^i = w, \mid x \mid > 0, \mid x \mid_\alpha > 0$

   shuffle$(\cdots$, w) := $\bigcup_{W \in shuffle(\cdots)}$shuffle(W, w)

   $\square$

2. [10 marks] Consider the equation in words $x Ш y = z^2$, where Ш is the perfect shuffle. Describe all solutions to this equation. Hint: there are separate cases depending on whether $|x|, |y|$ are both even or both odd.

   (By "describe all solutions" we mean come up with a characterization of the solutions that is somewhat analogous to the descriptions in Theorems 2.3.2 and 2.3.3 of the course text, giving necessary and sufficient conditions for x, y, z.)

   if $|x| = |y| = 2 * i \vDash$

   Let x = xa xb and y = ya yb where $|xa| = |xb| = |ya| = |yb| = i$

   $x Ш y = xa Ш ya \ xb Ш yb = zz$

   $xa Ш ya = z = xb Ш yb$

   xa = xb and ya = yb

   if $|x| = |y| = 2 * i + 1 \vDash$

   Let x = xa xc xb and y = ya yc yb where $|xa| = |xb| = |ya| = |yb| = i and |xc| = |yc| = 1$

   $x Ш y = xa Ш ya$ xc yc $xb Ш yb =$ z z

   xa xc = yc yb and ya = xb

3. [10 marks] Call a word w *uneven* if every nonempty subword has the property that at least one letter appears an odd number of times. For example, `abac` is uneven.

   (a) [5 marks] Show that if w is an uneven word over an alphabet with k letters, then $|w| < 2^k$.

   (b) [5 marks] Prove that the bound in (a) is sharp, by exhibiting an uneven word of length $2^k - 1$ over every alphabet of size $k \geq 1$.

For (a)

Given W $\in \sum^*$ and $\mid \sum \mid$ = k

Assume $\mid W \mid \geq 2^k$

Let prefix(w, j) = w' where w' w" = w and $\mid w' \mid$ = j

Let $\bigcup_{i \in Z_{|\sum|}} \{\alpha_i\} = \sum$

Let par(w) = v where v_i = $\mid w \mid_{\alpha_i}$ mod 2

Let V = $\{par(prefix(w, j)) \mid j \in Z_{|W|+1} \setminus \{0\}\}$

if $\exists$ i where V_i = par(prefix(W, j)) = 0 then prefix(W, j) is an even word

otherwise $\exists$ i, j where $m \lneq n$, V_i = par(prefix(W, m)) = V_j = par(prefix(W, n)) because there $2^k - 1$ unique non-zero vector in the space and $\mid V \mid = \mid W \mid \geq 2^k$

Let prefix(W, m) z = prefix(W, n)

Then par(prefix(W, m)) par(z) = par(prefix(W, n))

Then par(z) = par(prefix(W, n)) - par(prefix(W, m)) = 0

Then z is an even word

In both cases, W end up contains an even word. Thus $\mid W \mid \not\leq 2^k$.

For (b)

Given $\mid \sum \mid$ = k

Let $\bigcup_{i \in Z_{|\sum|}} \{\alpha_i\} = \sum$

Let $w_0 = \alpha_0$

Let $w_{n+1} = w_n Ш \alpha_{n+1}^{|w_n|}\ \alpha_{n+1}$ for $n \geq 0$

By observation, $\mid w_0 \mid = 2^1 - 1$ and w_0 is uneven

Given $\mid w_n \mid = 2^n - 1$ and w_n is uneven

$\mid w_{n+1} \mid = \mid w_{n+1} \mid * 2 + 1 = 2^{n+1} - 1$

Let w be a subword of w_{n+1}

if $\mid w \mid$ = 1, then w is uneven otherwise $\mid w \mid \geq 2$, then $w \in perm(w'\alpha_{n+1}^i)$ where w' is a subword of w_n

Then par(w) = par(w') + $par(\alpha_{n+1}^{i})$

Then $par(w') \neq 0$ because w' is uneven

Then $par(w') + par(\alpha_{n+1}^{i}) \neq 0$ because $\mid w \mid_{\alpha_{n+1}} = 0$

Then w is uneven because $par(w) \neq 0$ In both cases, w is uneven. Therefore w_{n+1} is uneven

By induction, $\mid w_k \mid = 2^k - 1$ and w_k is uneven.