subscript start from 0

Definition 1. *subseq(x, I) :=*

   subseq(x, I) := ϵ | I = {}

   subseq(x, I) := subseq(x, I \ i) x_i where i = max(I) | otherwise

Definition 2. *shuffle(x, y) := { w | w $\in$ perm(x y), subseq(w, A) = x, subseq(w, B) = y, A $\cap$ B = {}, A $\cup$ B = $Z_{|x|+|y|}$ }*

   shuffle(x) := x

   shuffle(x, y) := { w | w $\in$ perm(x y), subseq(w, A) = x, subseq(w, B) = y, A $\cap$ B = {}, A $\cup$ B = $Z_{|x|+|y|}$ }

   shuffle($\cdots$, w) := $\bigcup_{W \in shuffle(\cdots)}$ shuffle(W, w)

Lemma 1. *x $\in$ shuffle($\alpha_0^{W_0}$, $\cdots$) where | x $|_{\alpha_i}$ = W_i and $\bigcup\{\alpha_i\}$ = Σ*

Proof by Mathematical Induction. $\vDash$

   Induction   Assume x $\in$ shuffle($\alpha_0^{W_0}$, $\cdots$) where | x $|_{\alpha_i}$ = W_i and $\bigcup\{\alpha_i\}$ = Σ for | Σ |< $k + 1$

   Let x' $\in$ Σ' where | Σ | = k + 1, | x' $|_{\alpha_i}$ = W_i and $\bigcup\{\alpha_i\}$ = Σ'

   Let I_{α_k} = $\{i \mid i \in Z_k, x'_i = \alpha_k\}$, $I_{\hat{\alpha}_k}$ = $Z_{|x|} \setminus I_{\alpha_k}$

   Let y = subseq(x, $I_{\hat{\alpha}_k}$) $\subseteq$ shuffle($\alpha_0^{W_0}$, $\cdots$, $\alpha_{k-1}^{W_{k-1}}$), y' = subseq(x, I_{α_k})

   Then y $\in$ shuffle($\alpha_0^{W_0}$, $\cdots$, $\alpha_{k-1}^{W_{k-1}}$)

   Since x' $\in$ shuffle(y, y'), x' $\in$ shuffle($\alpha_0^{W_0}$, $\cdots$)

   Baseline

   x $\in$ shuffle(x) = shuffle($\alpha_0^{W_0}$) where | x $|_{\alpha_i}$ = W_i and $\bigcup\{\alpha_i\}$ = Σ for | Σ |= 1

   x $\in$ perm($\alpha_0^{W_0}$ $\alpha_1^{W_1}$) = shuffle($\alpha_0^{W_0}$, $\alpha_1^{W_1}$) where | x $|_{\alpha_i}$ = W_i and $\bigcup\{\alpha_i\}$ = Σ for | Σ |= 2

   Conclusion   x $\in$ shuffle($\alpha_0^{W_0}$, $\cdots$) where | x $|_{\alpha_i}$ = W_i and $\bigcup\{\alpha_i\}$ = Σ $\square$

Lemma 2. *shuffle(A B, C D) $\subseteq$ shuffle(A, C) shuffle(B, D)*