subscript start from 0

Definition 1. *subseq(x, I) :=*

subseq(x, I) $:= \epsilon \mid I = \{\}$

subseq(x, I) $:=$ *subseq(x,* $I \setminus i$*)* x_i *where* $i = max(I) \mid$ *otherwise*

Definition 2. *shuffle(x, y)* $:= \{\ w \mid w \in perm(x\ y),\ subseq(w,\ A) = x,\ subseq(w,\ B) = y,\ A \cap B = \{\},\ A \cup B = Z_{|x|+|y|}\ \}$

shuffle(x) $:= x$

shuffle(x, y) $:= \{\ w \mid w \in perm(x\ y),\ subseq(w,\ A) = x,\ subseq(w,\ B) = y,\ A \cap B = \{\},\ A \cup B = Z_{|x|+|y|}\ \}$

shuffle$(\cdots,\ w) := \bigcup_{W \in shuffle(\cdots)}$ *shuffle(W, w)*

Lemma 1. $x \in$ *shuffle*$(\alpha_0^{W_0}, \cdots)$ *where* $\mid x \mid_{\alpha_i} = W_i$ *and* $\bigcup\{\alpha_i\} = \Sigma$

Proof by Mathematical Induction. $\models$

Induction Assume x $\in$ shuffle$(\alpha_0^{W_0}, \cdots)$ where $\mid x \mid_{\alpha_i} = W_i$ and $\bigcup\{\alpha_i\} = \Sigma$ for $\mid \Sigma \mid < k+1$

Let x' $\in \Sigma'$ where $\mid \Sigma \mid$ = k + 1, $\mid x' \mid_{\alpha_i} = W_i$ and $\bigcup\{\alpha_i\} = \Sigma'$

Let $I_{\alpha_k} = \{i \mid i \in Z_k, x'_i = \alpha_k\}$, $I_{\hat{\alpha_k}} = Z_{|x|} \setminus I_{\alpha_k}$

Let y = subseq(x, $I_{\hat{\alpha_k}}$) $\subseteq$ shuffle$(\alpha_0^{W_0}, \cdots, \alpha_{k-1}^{W_{k-1}})$, y' = subseq(x, I_{α_k})

Then y $\in$ shuffle$(\alpha_0^{W_0}, \cdots, \alpha_{k-1}^{W_{k-1}})$

Since x' $\in$ shuffle(y, y'), x' $\in$ shuffle$(\alpha_0^{W_0}, \cdots)$

Baseline

x $\in$ shuffle(x) = shuffle$(\alpha_0^{W_0})$ where $\mid x \mid_{\alpha_i} = W_i$ and $\bigcup\{\alpha_i\} = \Sigma$ for $\mid \Sigma \mid = 1$

x $\in$ perm$(\alpha_0^{W_0}\ \alpha_1^{W_1})$ = shuffle$(\alpha_0^{W_0}, \alpha_1^{W_1})$ where $\mid x \mid_{\alpha_i} = W_i$ and $\bigcup\{\alpha_i\} = \Sigma$ for $\mid \Sigma \mid = 2$

Conclusion x $\in$ shuffle$(\alpha_0^{W_0}, \cdots)$ where $\mid x \mid_{\alpha_i} = W_i$ and $\bigcup\{\alpha_i\} = \Sigma$ $\square$

Lemma 2. *shuffle(A B, C D)* $\subseteq$ *shuffle(A, C) shuffle(B, D)*