University of Waterloo
CS 798 — Mathematical Foundations of Computer Networking
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Assignment 2
Siwei Yang - 20258568

1. Problem 1

   > You are given a fair die and a biased die. Let X be a random variable representing the value shown face up when a die is rolled. For the fair die, $P(X = i) = \frac{1}{6}$, $i = 1, 2, \ldots, 6$; and for the biased die, $P(X = 1) = \frac{3}{4}$ and $P(X = i) = \frac{1}{20}$, $i = 2, 3, \ldots, 6$. You have no way of telling which die is biased. Suppose you select a die at random and upon rolling it, you get a 5, what is the probability that you have selected the biased die?

   Using the provided probability function, we know that for a unbiased die to roll out a 5, the probability is $\frac{1}{6}$; while the biased die can roll out 5 with a probability of $\frac{1}{20}$. Since the die is chosen at random, therefore it is equally likely to select either die. Thus, the probability of rolling out a 5 is

   $$0.5 * \frac{1}{6} + 0.5 * \frac{1}{20} \tag{1}$$

   However, we know that the result is 5. In this case, the probability that the biased die is selected is given as

   $$\frac{0.5 * \frac{1}{20}}{0.5 * \frac{1}{6} + 0.5 * \frac{1}{20}} = \frac{3}{13} \tag{2}$$

2. Problem 2

   > Let X be a random variable representing the number of times a coin is tossed until for the first time the same result appears twice in succession. It is assumed that the tosses are independent of each other, and for each toss, $P(head) = p$ and $P(tail) = q = 1 - p$.

   (a) Give an expression for $P(X = 6)$.

   There are only two ways for $X = 6$ to happen: alternating and ends with two tails or alternating and ends with two heads. And we know the probability for that is

   $$P(X = 6) = p^4 * (1-p)^2 + p^2 * (1-p)^4 = p^4 * q^2 + p^2 * q^4 \tag{3}$$

(b) For the special case $p = q = 0.5$, find the probability mass function of X.

From the reasoning above, it is easy to conclude that $P(X = 2) = 0.5 * 0.5 + 0.5 * 0.5 = 0.5$. And that $P(X = 2 + i) = P(X = 2) * 0.5^i$. So we can conclude this is a geometric series:

$$P(X = i) = 0.5^{i-1} \text{where } i \geq 2 \tag{4}$$

3. Problem 3

> X and Y are exponentially distributed random variables with parameters λ and μ, respectively. Suppose X and Y are independent. Derive an expression for $P(X > Y)$.

We know that $f_Y(x) = \mu * e^{-\mu * x}$, and $P(X > Y \mid Y = x) = P(X > x) = 1 - e^{-\lambda * x}$. Therefore, the cumulative probability is

$$P(X > Y) = \int_{-\infty}^{\infty} \mu * e^{-\mu * x} * (1 - e^{-\lambda * x}) \mathrm{d}x \tag{5}$$

4. Problem 4

> Let m and σ^2 be, respectively, the mean and variance of a random variable X. Express $E[(X - b)^2]$ as a function of b, m, and σ^2.

Consider the variance of X

$$\sigma^2 = E[(X - m)^2] = E[X^2 - 2 * m * X + m^2] = E[X^2] - m^2 \tag{6}$$

and consider $E[(X - b)^2]$

$$E[(X - b)^2] = E[X^2 - 2 * b * X + b^2] = E[X^2] - 2 * m * b + b^2 \tag{7}$$

therefore, we can substitute $\sigma^2 = E[X^2] - m^2$ to get

$$E[(X - b)^2] = (\sigma^2 + m^2) - 2 * m * b + b^2 = \sigma^2 + (m^2 - 2 * m * b + b^2) \tag{8}$$

5. Problem 5

> Suppose $Y = X_1 + X_2$ where X_1 and X_2 are independent random variables. X_1 has Poisson distribution with parameter λ_1 and X_2 has Poisson distribution with parameter λ_2.

(a) Show that Y also has Poisson distribution, with parameter $\lambda_1 + \lambda_2$. For Y to get a particular value k, X_1 and X_2 must get a pair of values that sum to k. Thus the formula

$$\begin{aligned} P_Y(k) &= \sum_{i=0}^{k} \frac{\lambda_1^i * e^{-\lambda_1}}{i!} * \frac{\lambda_2^{(k-i)} * e^{-\lambda_2}}{(k-i)!} \\ &= \sum_{i=0}^{k} \binom{k}{i} * \frac{\lambda_1^i * \lambda_2^{(k-i)}}{(\lambda_1+\lambda_2)^k} * \frac{(\lambda_1+\lambda_2)^k * e^{-(\lambda_1+\lambda_2)}}{k!} \\ &= \frac{(\lambda_1+\lambda_2)^k * e^{-(\lambda_1+\lambda_2)}}{k!} * \sum_{i=0}^{k} \binom{k}{i} * \frac{\lambda_1^i * \lambda_2^{(k-i)}}{(\lambda_1+\lambda_2)^k} \\ &= \frac{(\lambda_1+\lambda_2)^k * e^{-(\lambda_1+\lambda_2)}}{k!} * (\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2})^k \\ &= \frac{(\lambda_1+\lambda_2)^k * e^{-(\lambda_1+\lambda_2)}}{k!} \end{aligned} \tag{9}$$

Therefore, Y is a random variable having Poisson distribution, with parameter $\lambda_1 + \lambda_2$.

(b) Using the results in (a), show that the conditional distribution $P(X_1 = k|Y = n)$ is Binomial.

$$\begin{aligned} P(X_1 = k|Y = n) &= \frac{P(X_1 = k, Y = n)}{P(Y = n)} \\ &= \frac{\lambda_1^k * e^{-\lambda_1}}{k!} * \frac{\lambda_2^{(n-k)} * e^{-\lambda_2}}{(n-k)!} * \frac{n!}{(\lambda_1+\lambda_2)^n * e^{-(\lambda_1+\lambda_2)}} \\ &= \binom{n}{k} * \frac{\lambda_1^k * \lambda_2^{(n-k)}}{(\lambda_1+\lambda_2)^n} * \frac{(\lambda_1+\lambda_2)^n * e^{-(\lambda_1+\lambda_2)}}{n!} * \frac{n!}{(\lambda_1+\lambda_2)^n * e^{-(\lambda_1+\lambda_2)}} \\ &= \binom{n}{k} * \frac{\lambda_1^k * \lambda_2^{(n-k)}}{(\lambda_1+\lambda_2)^n} \\ &= \binom{n}{k} * (\frac{\lambda_1}{\lambda_1+\lambda_2})^k * (\frac{\lambda_2}{\lambda_1+\lambda_2})^{n-k} \end{aligned} \tag{10}$$

This clearly shows that $P(X_1 = k|Y = n)$ is Binomial.

6. Problem 6

> X and Y are continuous random variables, both uniformly distributed between 0 and L. Suppose X and Y are independent. Let $D = |X - Y|$. Derive an expression for $E[D]$.

First of all, we know that D has range $(0, L)$

$$E[D] = \int_{x=0}^{L} x * f_D(x) \mathrm{d}x \tag{11}$$

and for $D = x$, it could be $X - Y = x$ or $Y - X = x$. And, the possible value of the greater one between X and Y lies in (x, L). Therefore

$$\begin{aligned} f_D(x) &= \int_{X=x}^{L} \frac{1}{L} \mathrm{d}X + \int_{Y=x}^{L} \frac{1}{L} \mathrm{d}Y \\ &= 2 * \frac{L - x}{L} * \frac{1}{L} \end{aligned} \tag{12}$$

Thus, we have

$$\begin{aligned} E[D] &= \int_{x=0}^{L} x * f_D(x) \mathrm{d}x \\ &= \int_{x=0}^{L} \frac{2}{L^2} * (L - x) * x \mathrm{d}x \\ &= \frac{2}{L^2} * (\frac{L^3}{2} - \frac{L^3}{3}) \\ &= \frac{L}{3} \end{aligned} \tag{13}$$

7. Problem 7

> Suppose $Y = X_1 + X_2, \cdots + X_N$ where $X_1, X_2, \ldots, X_N$ are independent and identically distributed random variables with mean $E[X]$ and variance $var(X)$. N, the number of $X_i's$ in the sum, is also a random variable; the mean and variance of N are $E[N]$ and $var(N)$, respectively. Derive analytic expressions for $E[Y]$ and $var(Y)$.

$$E[Y] = \sum_{j=0}^{\infty} (P(N = j) * \sum_{i=1}^{j} X_i) \tag{14}$$

since $X_i''s$ are identical, we can change all $X_i's$ to X_1 in the equation

$$E[Y] = \sum_{j=0}^{\infty} (P(N = j) * j * X_1) = X_1 * \sum_{j=0}^{\infty} (P(N = j) * j) \tag{15}$$

which can be further reduce to

$$E[Y] = X_1 * E[N] = E[X] * E[N] \tag{16}$$

for var(Y), we have

$$var(Y) = \sum_{j=0}^{\infty} \Big(P(N = j) * (\sum_{i=1}^{j} X_i - E[X] * E[N])^2 \Big) \tag{17}$$

since $X_i''s$ are identical, we can change all $X_i's$ to X_1 in the equation

$$\begin{aligned} var(Y) &= \sum_{j=0}^{\infty} \Big(P(N = j) * (j * X_1 - E[X] * E[N])^2 \Big) \\ &= \sum_{j=0}^{\infty} \Big(P(N = j) * (j^2 * X_1^2 - 2 * j * X_1 * E[X] * E[N] + E[X]^2 * E[N]^2) \Big) \\ &= \sum_{j=0}^{\infty} \Big(P(N = j) * (j^2 * X_1^2 - 2 * j * X_1 * E[X] * E[N]) \Big) + E[X]^2 * E[N]^2 \end{aligned} \tag{18}$$

then we move on to solve smaller parts of $\sum_{j=0}^{\infty} \Big(P(N = j) * (j^2 * X_1^2) \Big)$ and $\sum_{j=0}^{\infty} \Big(P(N = j) * (-2 * j * X_1 * E[X] * E[N]) \Big)$. Since $X_i's$ and N are independent, we have

$$\begin{aligned} \sum_{j=0}^{\infty} \Big(P(N = j) * (j^2 * X_1^2) \Big) &= X_1^2 * \sum_{j=0}^{\infty} P(N = j) * j^2 \\ &= X_1^2 * N^2 \end{aligned} \tag{19}$$

and we know $var(X) = (X_1 - E[X])^2 = X_1^2 - 2 * X_1 * E[X] + E[X]^2 = X_1^2 - E[X]^2$ and similarly $var(N) = N^2 - E[N]^2$, therefore we can substitute into the equation above to get

$$\sum_{j=0}^{\infty} \Big(P(N = j) * (j^2 * X_1^2) \Big) = (var(X) + E[X]^2) * (var(N) * E[N]^2) \tag{20}$$

$$\begin{aligned} \sum_{j=0}^{\infty} \Big(P(N = j) * (-2 * j * X_1 * E[X] * E[N]) \Big) &= -2 * X_1 * E[X] * E[N] * \sum_{j=0}^{\infty} P(N = j) * j \\ &= -2 * X_1 * E[X] * E[N]^2 \\ &= -2 * E[X]^2 * E[N]^2 \end{aligned} \tag{21}$$

So, combining all results above, we have

$$\begin{aligned} var(Y) &= (var(X) + E[X]^2) * (var(N) + E[N]^2) - E[X]^2 * E[N]^2 \\ &= var(X) * var(N) + var(X) * E[N]^2 + var(N) * E[X]^2 \end{aligned} \tag{22}$$

8. Problem 8

Suppose $X_1 + X_2, \cdots + X_N$ are independent random variables.

   (a) Let $T = max(X_1 + X_2, \cdots + X_N)$. Express $F_T(x)$ as a function of $F_{X_i}(x)$, $i = 1, 2, \ldots, n$. [Hint: Interpret $P(T \leq x)$.]
   Since $F_T(x)$ is the probability that $T \leq x$, then we know it happens only if all $X_i's \leq x$. And since all $X_i's$ are independent, therefore the probability that all $X_i's \leq x$ is $\prod_{i=1}^{N} F_{X_i}(x)$

   $$F_T(x) = \prod_{i=1}^{N} F_{X_i}(x) \tag{23}$$

   (b) Let $T = mix(X_1 + X_2, \cdots + X_N)$. Express $F_T(x)$ as a function of $F_{X_i}(x)$, $i = 1, 2, \ldots, n$. [Hint: Interpret $P(T > x)$.] Show further that if each of the $X_i's$ has exponential distribution, T is also exponential.
   Since $F_T(x)$ is the probability that $T \leq x$, $1 - F_T(x)$ is the probability that $T > x$ and we know it happens only if all $X_i's > x$. And since all $X_i's$ are independent, therefore the probability that all $X_i's > x$ is $\prod_{i=1}^{N} 1 - F_{X_i}(x)$

   $$1 - F_T(x) = \prod_{i=1}^{N} 1 - F_{X_i}(x) \tag{24}$$

   simplify to

   $$F_T(x) = 1 - \prod_{i=1}^{N} 1 - F_{X_i}(x) \tag{25}$$

   Now, assume each of $X_i's$ has exponential distribution with $F_{X_i}(x) = 1 - e^{-\lambda_i * x}$, then we will have

   $$\begin{aligned} F_T(x) &= 1 - \prod_{i=1}^{N} 1 - F_{X_i}(x) \\ &= 1 - \prod_{i=1}^{N} e^{-\lambda_i * x} \\ &= 1 - e^{-x * \sum_{i=1}^{N} \lambda_i} \end{aligned} \tag{26}$$

   Therefore, T also has a exponential distribution with parameter $\sum_{i=1}^{N} \lambda_i$.