

University of Waterloo
CS 798 — Mathematical Foundations of Computer
Networking
Winter 2014
Assignment 2
Siwei Yang - 20258568

1. Problem 1

You are given a fair die and a biased die. Let X be a random variable representing the value shown face up when a die is rolled. For the fair die, $P(X = i) = \frac{1}{6}$, $i = 1, 2, \dots, 6$; and for the biased die, $P(X = 1) = \frac{3}{4}$ and $P(X = i) = \frac{1}{20}$, $i = 2, 3, \dots, 6$. You have no way of telling which die is biased. Suppose you select a die at random and upon rolling it, you get a 5, what is the probability that you have selected the biased die?

Using the provided probability function, we know that for a unbiased die to roll out a 5, the probability is $\frac{1}{6}$; while the biased die can roll out 5 with a probability of $\frac{1}{20}$. Since the die is chosen at random, therefore it is equally likely to select either die. Thus, the probability of rolling out a 5 is

$$0.5 * \frac{1}{6} + 0.5 * \frac{1}{20} \quad (1)$$

However, we know that the result is 5. In this case, the probability that the biased die is selected is given as

$$\frac{0.5 * \frac{1}{20}}{0.5 * \frac{1}{6} + 0.5 * \frac{1}{20}} = \frac{3}{13} \quad (2)$$

2. Problem 2

Let X be a random variable representing the number of times a coin is tossed until for the first time the same result appears twice in succession. It is assumed that the tosses are independent of each other, and for each toss, $P(head) = p$ and $P(tail) = q = 1 - p$.

- (a) Give an expression for $P(X = 6)$.

There are only two ways for $X = 6$ to happen: alternating and ends with two tails or alternating and ends with two heads. And we know the probability for that is

$$P(X = 6) = p^4 * (1 - p)^2 + p^2 * (1 - p)^4 = p^4 * q^2 + p^2 * q^4 \quad (3)$$

- (b) For the special case $p = q = 0.5$, find the probability mass function of X .

From the reasoning above, it is easy to conclude that $P(X = 2) = 0.5 * 0.5 + 0.5 * 0.5 = 0.5$. And that $P(X = 2 + i) = P(X = 2) * 0.5^i$. So we can conclude this is a geometric series:

$$P(X = i) = 0.5^{i-1} \text{ where } i \geq 2 \quad (4)$$

3. Problem 3

X and Y are exponentially distributed random variables with parameters λ and μ , respectively. Suppose X and Y are independent. Derive an expression for $P(X > Y)$.

We know that $f_Y(x) = \mu * e^{-\mu * x}$, and $P(X > Y | Y = x) = P(X > x) = 1 - e^{-\lambda * x}$. Therefore, the cumulative probability is

$$\begin{aligned} P(X > Y) &= \int_0^{\infty} \mu * e^{-\mu * x} * (1 - e^{-\lambda * x}) dx \\ &= \int_0^{\infty} \mu * e^{-\mu * x} dx + \int_0^{\infty} -\mu * e^{-(\mu + \lambda) * x} dx \\ &= 1 - \frac{\mu}{\mu + \lambda} \\ &= \frac{\lambda}{\mu + \lambda} \end{aligned} \quad (5)$$

4. Problem 4

Let m and σ^2 be, respectively, the mean and variance of a random variable X . Express $E[(X - b)^2]$ as a function of b , m , and σ^2 .

Consider the variance of X

$$\sigma^2 = E[(X - m)^2] = E[X^2 - 2 * m * X + m^2] = E[X^2] - m^2 \quad (6)$$

and consider $E[(X - b)^2]$

$$E[(X - b)^2] = E[X^2 - 2 * b * X + b^2] = E[X^2] - 2 * m * b + b^2 \quad (7)$$

therefore, we can substitute $\sigma^2 = E[X^2] - m^2$ to get

$$E[(X - b)^2] = (\sigma^2 + m^2) - 2 * m * b + b^2 = \sigma^2 + (m^2 - 2 * m * b + b^2) \quad (8)$$

5. Problem 5

Suppose $Y = X_1 + X_2$ where X_1 and X_2 are independent random variables. X_1 has Poisson distribution with parameter λ_1 and X_2 has Poisson distribution with parameter λ_2 .

- (a) Show that Y also has Poisson distribution, with parameter $\lambda_1 + \lambda_2$.
 For Y to get a particular value k , X_1 and X_2 must get a pair of values that sum to k . Thus the formula

$$\begin{aligned}
 P_Y(k) &= \sum_{i=0}^k \frac{\lambda_1^i * e^{-\lambda_1}}{i!} * \frac{\lambda_2^{(k-i)} * e^{-\lambda_2}}{(k-i)!} \\
 &= \sum_{i=0}^k \binom{k}{i} * \frac{\lambda_1^i * \lambda_2^{(k-i)}}{(\lambda_1 + \lambda_2)^k} * \frac{(\lambda_1 + \lambda_2)^k * e^{-(\lambda_1 + \lambda_2)}}{k!} \\
 &= \frac{(\lambda_1 + \lambda_2)^k * e^{-(\lambda_1 + \lambda_2)}}{k!} * \sum_{i=0}^k \binom{k}{i} * \frac{\lambda_1^i * \lambda_2^{(k-i)}}{(\lambda_1 + \lambda_2)^k} \quad (9) \\
 &= \frac{(\lambda_1 + \lambda_2)^k * e^{-(\lambda_1 + \lambda_2)}}{k!} * \left(\frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2} \right)^k \\
 &= \frac{(\lambda_1 + \lambda_2)^k * e^{-(\lambda_1 + \lambda_2)}}{k!}
 \end{aligned}$$

Therefore, Y is a random variable having Poisson distribution, with parameter $\lambda_1 + \lambda_2$.

- (b) Using the results in (a), show that the conditional distribution $P(X_1 = k | Y = n)$ is Binomial.

$$\begin{aligned}
 P(X_1 = k | Y = n) &= \frac{P(X_1 = k, Y = n)}{P(Y = n)} \\
 &= \frac{\lambda_1^k * e^{-\lambda_1}}{k!} * \frac{\lambda_2^{(n-k)} * e^{-\lambda_2}}{(n-k)!} * \frac{n!}{(\lambda_1 + \lambda_2)^n * e^{-(\lambda_1 + \lambda_2)}} \\
 &= \binom{n}{k} * \frac{\lambda_1^k * \lambda_2^{(n-k)}}{(\lambda_1 + \lambda_2)^n} * \frac{(\lambda_1 + \lambda_2)^n * e^{-(\lambda_1 + \lambda_2)}}{n!} * \frac{n!}{(\lambda_1 + \lambda_2)^n * e^{-(\lambda_1 + \lambda_2)}} \\
 &= \binom{n}{k} * \frac{\lambda_1^k * \lambda_2^{(n-k)}}{(\lambda_1 + \lambda_2)^n} \\
 &= \binom{n}{k} * \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k * \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{n-k} \quad (10)
 \end{aligned}$$

This clearly shows that $P(X_1 = k | Y = n)$ is Binomial.

6. Problem 6

X and Y are continuous random variables, both uniformly distributed between 0 and L. Suppose X and Y are independent. Let $D = |X - Y|$. Derive an expression for $E[D]$.

First of all, we know that D has range $(0, L)$

$$E[D] = \int_{x=0}^L x * f_D(x) dx \quad (11)$$

and for $D = x$, it could be $X - Y = x$ or $Y - X = x$. And, the possible value of the greater one between X and Y lies in (x, L) . Therefore

$$\begin{aligned} f_D(x) &= \int_{X=x}^L \frac{1}{L} dX + \int_{Y=x}^L \frac{1}{L} dY \\ &= 2 * \frac{L-x}{L} * \frac{1}{L} \end{aligned} \quad (12)$$

Thus, we have

$$\begin{aligned} E[D] &= \int_{x=0}^L x * f_D(x) dx \\ &= \int_{x=0}^L \frac{2}{L^2} * (L-x) * x dx \\ &= \frac{2}{L^2} * \left(\frac{L^3}{2} - \frac{L^3}{3} \right) \\ &= \frac{L}{3} \end{aligned} \quad (13)$$

7. Problem 7

Suppose $Y = X_1 + X_2 + \dots + X_N$ where X_1, X_2, \dots, X_N are independent and identically distributed random variables with mean $E[X]$ and variance $var(X)$. N , the number of X_i 's in the sum, is also a random variable; the mean and variance of N are $E[N]$ and $var(N)$, respectively. Derive analytic expressions for $E[Y]$ and $var(Y)$.

$$E[Y] = \sum_{j=0}^{\infty} (P(N=j) * \sum_{i=1}^j X_i) \quad (14)$$

since X_i 's are identical, we can change all X_i 's to X_1 in the equation

$$E[Y] = \sum_{j=0}^{\infty} (P(N=j) * j * X_1) = X_1 * \sum_{j=0}^{\infty} (P(N=j) * j) \quad (15)$$

which can be further reduce to

$$E[Y] = X_1 * E[N] = E[X] * E[N] \quad (16)$$

for $\text{var}(Y)$, we have

$$\text{var}(Y) = \sum_{j=0}^{\infty} \left(P(N=j) * \left(\sum_{i=1}^j X_i - E[X] * E[N] \right)^2 \right) \quad (17)$$

since X_i'' s are identical, we can change all X_i' s to X_1 in the equation

$$\begin{aligned} \text{var}(Y) &= \sum_{j=0}^{\infty} \left(P(N=j) * (j * X_1 - E[X] * E[N])^2 \right) \\ &= \sum_{j=0}^{\infty} \left(P(N=j) * (j^2 * X_1^2 - 2 * j * X_1 * E[X] * E[N] + E[X]^2 * E[N]^2) \right) \\ &= \sum_{j=0}^{\infty} \left(P(N=j) * (j^2 * X_1^2 - 2 * j * X_1 * E[X] * E[N]) \right) + E[X]^2 * E[N]^2 \end{aligned} \quad (18)$$

then we move on to solve smaller parts of $\sum_{j=0}^{\infty} \left(P(N=j) * (j^2 * X_1^2) \right)$ and $\sum_{j=0}^{\infty} \left(P(N=j) * (-2 * j * X_1 * E[X] * E[N]) \right)$. Since X_i' s and N are independent, we have

$$\begin{aligned} \sum_{j=0}^{\infty} \left(P(N=j) * (j^2 * X_1^2) \right) &= X_1^2 * \sum_{j=0}^{\infty} P(N=j) * j^2 \\ &= X_1^2 * N^2 \end{aligned} \quad (19)$$

and we know $\text{var}(X) = (X_1 - E[X])^2 = X_1^2 - 2 * X_1 * E[X] + E[X]^2 = X_1^2 - E[X]^2$ and similarly $\text{var}(N) = N^2 - E[N]^2$, therefore we can substitute into the equation above to get

$$\sum_{j=0}^{\infty} \left(P(N=j) * (j^2 * X_1^2) \right) = (\text{var}(X) + E[X]^2) * (\text{var}(N) + E[N]^2) \quad (20)$$

$$\begin{aligned} \sum_{j=0}^{\infty} \left(P(N=j) * (-2 * j * X_1 * E[X] * E[N]) \right) &= -2 * X_1 * E[X] * E[N] * \sum_{j=0}^{\infty} P(N=j) * j \\ &= -2 * X_1 * E[X] * E[N]^2 \\ &= -2 * E[X]^2 * E[N]^2 \end{aligned} \quad (21)$$

So, combining all results above, we have

$$\begin{aligned} \text{var}(Y) &= (\text{var}(X) + E[X]^2) * (\text{var}(N) + E[N]^2) - E[X]^2 * E[N]^2 \\ &= \text{var}(X) * \text{var}(N) + \text{var}(X) * E[N]^2 + \text{var}(N) * E[X]^2 \end{aligned} \quad (22)$$

8. Problem 8

Suppose $X_1 + X_2, \dots, X_N$ are independent random variables.

- (a) Let $T = \max(X_1 + X_2, \dots, X_N)$. Express $F_T(x)$ as a function of $F_{X_i}(x)$, $i = 1, 2, \dots, n$. [Hint: Interpret $P(T \leq x)$.]

Since $F_T(x)$ is the probability that $T \leq x$, then we know it happens only if all $X'_i \leq x$. And since all X'_i s are independent, therefore the probability that all $X'_i \leq x$ is $\prod_{i=1}^N F_{X_i}(x)$

$$F_T(x) = \prod_{i=1}^N F_{X_i}(x) \quad (23)$$

- (b) Let $T = \min(X_1 + X_2, \dots, X_N)$. Express $F_T(x)$ as a function of $F_{X_i}(x)$, $i = 1, 2, \dots, n$. [Hint: Interpret $P(T > x)$.] Show further that if each of the X'_i s has exponential distribution, T is also exponential.

Since $F_T(x)$ is the probability that $T \leq x$, $1 - F_T(x)$ is the probability that $T > x$ and we know it happens only if all $X'_i > x$. And since all X'_i s are independent, therefore the probability that all $X'_i > x$ is $\prod_{i=1}^N 1 - F_{X_i}(x)$

$$1 - F_T(x) = \prod_{i=1}^N 1 - F_{X_i}(x) \quad (24)$$

simplify to

$$F_T(x) = 1 - \prod_{i=1}^N 1 - F_{X_i}(x) \quad (25)$$

Now, assume each of X'_i s has exponential distribution with $F_{X_i}(x) = 1 - e^{-\lambda_i * x}$, then we will have

$$\begin{aligned} F_T(x) &= 1 - \prod_{i=1}^N 1 - F_{X_i}(x) \\ &= 1 - \prod_{i=1}^N e^{-\lambda_i * x} \\ &= 1 - e^{-x * \sum_{i=1}^N \lambda_i} \end{aligned} \quad (26)$$

Therefore, T also has an exponential distribution with parameter $\sum_{i=1}^N \lambda_i$.