This homework is counted 10% of the whole course. There are total 35 marks.

You are allowed to discuss with others and use any references, but if you do so please list your collaborators and cite your references for each question. This will not affect your marks. In any case, you must write your own solutions (meaning that after you understand the solutions either by your own or from discussions with others or from references, you must write the solutions in your own words without any help from others and without any help from the references). Not writing your own solutions or not listing your collaborators or not citing your references may be considered plagiarism.

If you have any questions or comments please let me (chi@cse.cuhk.edu.hk) and/or your TA Chiu (tckwok@cse.cuhk.edu.hk) know.

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1. **Bipartite Matching**

   (10 marks) Consider the following linear program for the bipartite perfect matching problem, where $x(e)$ is a variable for an edge e and $\delta(v)$ is the set of edges incident on v.

   $$\begin{aligned} \sum_{e \in \delta(v)} x(e) &= 1 \quad \forall v \in V \\ x(e) &\geq 0 \end{aligned}$$

   Suppose you are given a feasible fractional solution to this linear program. Show that you can obtain an integral solution (i.e. $x(e) \in \{0,1\}$) in $\tilde{O}(m)$ time where m is the number of edges in the graph.

   (Hint: Extend the approach for regular bipartite perfect matching.)

2. **Minimum s-t Cut**

   (10 marks) Consider the following linear program for the minimum s-t cut problem on an undirected graph, where there is a variable $x(v)$ for each vertex v.

   $$\begin{aligned} \min \sum_{uv \in E} |x(u) - x(v)| \\ x(s) = 1 \\ x(t) = 0 \end{aligned}$$

   Give a polynomial time algorithm to find an integral solution achieveing the same objective value as the optimal value of this linear program, and prove its correctness.

3. **Small World**

(15 marks) Generalize the analysis in L14 and prove that the expected number of steps in the decentralized search algorithm is $O(\log^2(n))$ in the 2-dimensional setting, where n is the number of nodes in a 2-dimensional grid (i.e. the grid points are from $(1,1)$ to $(\sqrt{n}, \sqrt{n})$ and there are edges between adjacent nodes), and for each node v there is one random edge vw whose probability is proportional to $1/d(v,w)^2$ where $d(v,w)$ is the distance between v and w in the grid.

Furthermore, prove that the expected number of steps by the decentralized search algorithm in the 2-dimensional setting is $\Omega(n^{1/3})$ for most pairs of nodes, when each vertex chooses a uniform random vertex to be its random neighbor.