Local Average Treatment Effects with Imperfect Compliance and Interference

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Abstract

This study analyzes the identification and estimation of causal effects in situations where units interact and treatment is endogenous due to imperfect compliance. In cases where units do not interact, monotonicity in potential treatments identifies local average treatment effects (LATE). When units interact, monotonicity can still apply, but additional restrictions on potential treatments, such as one-sided noncompliance or personalized encouragement, are typically required. This paper generalizes these restrictions into a weaker concept of monotonicity and provides a unified framework for this context. Direct and indirect LATEs are identified under strictly weaker restrictions on potential treatments compared to existing approaches, but with the assumption of an additional exclusion restriction for the endogenous treatment. A simple parametric estimator for causal effects is introduced, and its performance is evaluated through simulations. The estimator also assesses biases in existing methods when various underlying assumptions are violated. The estimation procedure is illustrated using an experimental study in Kenya, which provided access to a savings account.

Keywords: Causal Inference, Social Networks, Imperfect Compliance, Treatment effects.
JEL Codes: C13, C21, C36.

1 Introduction

The causal effects of a program are of great interest in various economic studies, and the potential outcome framework is a popular approach for analyzing these effects (e.g., Rubin (1974), Rubin (2005)). Traditionally, it is assumed that experimental units do not interact, meaning their potential outcomes are not influenced by other units' treatments. This assumption is known as the *Stable Unit Treatment Value Assumption* (SUTVA). Under this assumption, the causal effect is typically defined as the difference between potential outcomes in two distinct scenarios: when the unit is treated and when it is not. However, since only one of these scenarios is observable, the other remains counterfactual. This introduces the fundamental problem in identifying causal effects. A common assumption to address this problem is the 'ignorability' of the treatment, which means that the treatment is exogenously assigned. Consequently, the distribution of observed outcomes in treated and untreated groups can be used to recover the potential outcome distribution.

However, these assumptions are restrictive in many situations. First, as economic agents naturally interact with each other, SUTVA can be violated. This phenomenon is referred to as treatment interference. For instance, the treatments of other units can also have a causal effect due to the reflection problem (e.g., Manski (1993)) or general equilibrium effects (e.g., Heckman, Lochner, and Taber (1999), Munro, Wager, and Xu (2021)). Second, even if the treatment is fully randomized, the ignorability assumption can be violated if units do not perfectly comply with their assigned treatment. In other words, the distribution of treatment effects might vary according to different (non)compliance patterns.

This paper addresses the identification and estimation of treatment effects in the presence of interference and imperfect compliance. To begin with, consider a scenario where units interact. Each unit's potential outcome is written as a function of the all units' treatment. Differences between potential outcomes under different treatment statuses of others define indirect, or spillover, effects. However, analyzing such spillover effects is challenging due to the numerous counterfactual potential scenarios. For example, while there are only two potential outcomes when units do not interact, now there are 2^N potential outcomes for each unit when N units interact.

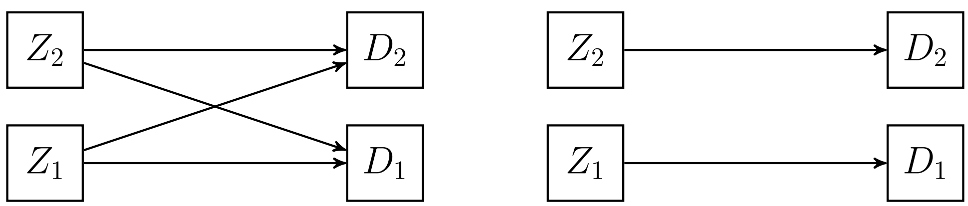
The conventional approach in the literature assumes that the treatment vector affects the potential outcome through a known function, which is often called *exposure map* or *effective treatment* in the literature (e.g., Manski (2013)). For instance, Leung (2020) shows that if interference occurs through an anonymous network and only within neighborhoods in distance 1, then only the unit's treatment and the number of treated and untreated neighbors determine the potential outcome. Studies in this literature measure spillover effects as the effects from changes in the exposure level (e.g., Aronow and Samii (2017), Leung (2020), Forastiere, Airoldi, and Mealli (2021), Cai, Janvry, and Sadoulet (2015), Vazquez-Bare (2023)). However, assuming researchers know a correctly specified exposure map is quite restrictive, and some discuss the potential misspecification of the exposure map (e.g., Sävje, Aronow, and Hudgens (2021), Leung (2022)).

On the other hand, if interference is not anonymous, it is difficult to expect a simple exposure map to exist. In such cases, only interactions among a small number of units can be analyzed to avoid the high dimensionality of potential scenarios, and some studies have focused on interactions between pairs of units (e.g., Vazquez-Bare (2022), Kormos, Lieli, and Huber (2023)). The baseline scenario in this paper also considers interactions between two non-anonymous units.

Next, imperfect compliance introduces various (non)compliance patterns, and understanding these patterns is essential to defining and identifying causal effects. Without interference, we can define two potential treatment take-ups as $D(1)$ and $D(0)$, for treated and untreated units, respectively. Then, each unit can be classified into 4 categories: *always-taker* ($D(1) = D(0) = 1$), *never-taker* ($D(1) = D(0) = 0$), *complier* ($D(1) > D(0)$), and *defier* ($D(1) < D(0)$). Notably, since always-takers and never-takers do not alter their treatment take-up in response to different treatment assignments, only the effects from compliers or defiers are identifable. One key strategy for identification is to exclude defiers, thereby focusing on the causal effects of compliers. The resulting causal effect is the Local Average Treatment Effect (LATE) on compliers (e.g., Imbens and Angrist (1994), Imbens and Rubin (1997), Imbens and Rubin (2010)). The exclusion of defiers is equivalently stated by the monotonicity assumption: $D(1) \geq D(0)$. I refer to this as *classical monotonicity* in this paper.

In more general scenarios where units interact and do not perfectly comply, an additional layer of interference must be considered: the interference in treatment take-up decisions. That is, the potential treatment is now influenced by the other's treatment assignments. This situation is depicted in panel (a) of Figure 1. Consider the case of two units, and let $D_i(z_i, z_j)$ be binary potential treatment take-up for unit $i \in \{1, 2\}$, where $(z_i, z_j) \in \{0, 1\}^2$ be two units' treatment assignments. Then, for each unit, there are 4 potential treatments, resulting in 16 (2^4) possible compliance patterns and a total of 256 (2^8) possible joint compliance configurations of two units, as noted by Kormos, Lieli, and Huber (2023). This complexity is significantly greater than the four compliance types in the classical case. Therefore, to define and identify meaningful causal effects, some or many compliance types need to be excluded, as in the strategy used in classical cases.

Figure 1: Interactions in Treatment Take-Up Decisions



(a) General Case (b) Personalized Encouragement

As compliance types are determined by potential treatments, studies in this literature have employed various restrictions on potential treatments to reduce complexity. Some studies assume no interaction in the treatment take-up decision (i.e,. $D_i(z_i, 1) = D_i(z_i, 0)$) as depicted in panel (b) of Figure 1. This is called *personalized encouragement*, and the classical monotonicity can be applied together with it (e.g., Kang and Imbens (2016), DiTraglia et al. (2023), Sánchez-Becerra (2021), Blackwell (2017)). When allowing interference in treatment take-up decisions, a number of studies assume *one-sided nomcompliance* (e.g., Vazquez-Bare (2022), DiTraglia et al. (2023), Sánchez-Becerra (2021)), which means only treated units have the opportunity to take the treatment or not (i.e., $D_i(0, 0) = D_i(0, 1) = 0$). Studies also assume an extended version of monotonicity. Recall that classical monotonicity imposes an ordering on the set $\{D(1), D(0)\}$. The natural generalization of this concept is imposing some ordering on the set of all potential treatments. For instance, in the two-units case, Vazquez-Bare (2022) imposes a total ordering on the set of all potential treatments as $D_i(1,1) \geq D_i(1,0) \geq D_i(0,1) \geq D_i(0,0)$. I refer to this as *total monotonicity* in this paper. A less restrictive concept of monotonicity assumes a partial ordering on potential treatments as $D_i(1,z) \geq D_i(0,z)$, for all z, which I call *marginal monotonicity* (e.g., Vazquez-Bare (2022), Imai, Jiang, and Malani (2021), Hoshino and Yanagi (2023), Ohnishi and Sabbaghi (2024)).

The aforementioned restrictions effectively exclude certain compliance types. For example, in the case of two units, there are only 5 possible marginal compliance types under total monotonicity, which further reduces to 3 when combined with one-sided noncompliance, as shown by Vazquez-Bare (2022). However, while classical monotonicity is interpreted as excluding defiers, total and marginal monotonicity might be difficult to interpret. Overall, these restrictions are not only challenging to verify, as they impose restrictions on potential treatments that are not always observable, but they can also be difficult to justify in certain situations because they impose many almost-sure inequalities on potential treatments. For example, when the probabilities of both events $D(1,1) \geq D(0,1)$ and $D(1,1) < D(0,1)$ are 0.5, total/marginal monotonicity fails.

This paper analyzes the identification of causal effects by significantly relaxing and generalizing the assumptions on potential treatments. The main idea is to transform the aforementioned assumptions into a weaker concept of monotonicity. The essence of these assumptions is imposing an almost-sure ordering on potential treatments. Therefore, I consider a minimal partial ordering on the set of potential treatments. For example, it may only assume that $D_i(0,1) \geq D_i(0,0)$ with probability 1, leaving distributions of $D_i(1,1)$ and $D_i(1,0)$ unrestricted.

The parameters of interest in this paper are the local average direct and indirect effects. I propose a general identification method for these parameters using the aforementioned weak monotonicity assumption. Initially, I demonstrate that the intention-to-treat (ITT) effects on outcomes can be represented as a linear combination of the parameters of interest. The causal parameters are then recovered by the coefficients of the ITT equation, provided there is an additional exclusion restriction that shifts the distribution of compliance types. I discuss several notable special cases of the general results, including identification under personalized encouragement, one-sided noncompliance, total/marginal monotonicity, and scenarios where none of these assumptions are fully satisfied. Additionally, I propose a parametric estimation procedure for these parameters, evaluate its performance using Monte Carlo simulations, and illustrate it with experimental data from studies conducted by Dupas, Keats, and Robinson (2017) and Dupas, Keats, and Robinson (2019).

This paper relates to the wide literature on estimating causal effects with interference. The mainstream approach to deal with interference involves designing experiments with two-stage randomization, as proposed by Hudgens and Halloran (2008). Two-stage randomization first randomly assigns the treatment rate at the group level and then assigns treatment within each group according to the assigned rate. Recent studies address imperfect compliance under this two-stage randomization (e.g., Kang and Imbens (2016), Blackwell (2017), DiTraglia et al. (2023), Sánchez-Becerra (2021), Imai, Jiang, and Malani (2021), Hoshino and Yanagi (2023)).

In contrast, some studies do not require two-stage randomization and are applicable to standard randomized experiments with perfect compliance (e.g., Leung (2020), Vazquez-Bare (2023)) as well as those with imperfect compliance (e.g., Vazquez-Bare (2022), Kormos, Lieli, and Huber (2023)). This paper aligns with this latter strand of literature and does not require two-stage randomization.

This paper primarily focuses on the interaction between two units. While social interactions generally involve multiple or many units, interactions between two units represent the most fundamental structure for understanding key features of social interactions. Additionally, a small number of units allows for analyzing non-anonymous networks. This setting is also applicable to various examples, including dating relationships (e.g., Milardo, Johnson, and Huston (1983)), married couples (e.g., Foos and De Rooij (2017), Vazquez-Bare (2022)), pairs of adolescents in their social cognition (e.g., Hermans et al. (2020)), pairs trading in finance[1] (e.g., Elliott, Van Der Hoek, and Malcolm (2005), Gatev, Goetzmann, and Rouwenhorst (2006), Vidyamurthy (2004)).

Therefore, this paper is most closely related to the works of Vazquez-Bare (2022) and Kormos, Lieli, and Huber (2023) where the main discussion focuses on the interaction

[1] An advanced trading strategy that involves opening one long position and one short position for two financial securities.

between two units. Vazquez-Bare (2022) assumes total monotonicity and one-sided noncompliance, while Kormos, Lieli, and Huber (2023) assumes only one-sided noncompliance to identify direct and indirect LATEs. In this paper, I relax both the total monotonicity and one-sided noncompliance assumptions. The cost of this generalization is the need for an additional exclusion restriction for the compliance types. These studies are also complementary in that Vazquez-Bare (2022) extends the model to the general N units case, and Kormos, Lieli, and Huber (2023) explicitly considers two treatments, adding an extra layer of heterogeneity. I provide a comparison of the proposed estimation in this paper and those from Vazquez-Bare (2022) by assessing bias when one-sided noncompliance or total monotonicity is violated via simulation.

In summary, this paper makes two key contributions. First, it proposes a unified framework for causal effects under interference and imperfect compliance by generalizing several existing restrictions on potential treatments—such as total monotonicity, one-sided noncompliance, and personalized encouragement—into a weaker concept of monotonicity. Second, it derives a general identification result that remains valid even when aforementioned assumptions are violated.

The structure of this paper is as follows: Section 2 introduces the baseline settings and notations, presents the weak concept of monotonicity, and defines the parameters of interest. Section 3 addresses the general identification results of the parameters and their special cases. Section 4 proposes a two-stage estimation procedure and derives its asymptotic properties. Section 5 evaluates the proposed estimator through Monte Carlo simulations. Section 6 provides an empirical illustration using experimental data from Dupas, Keats, and Robinson (2019). Section 7 concludes the paper.

2 Model

This section discusses the settings, assumptions, and parameters of interest. First, I introduce the notations and baseline setup of this paper. I examine various restrictions on potential treatments used in the literature, and then, I present a generalization of those restrictions in Assumption 2. Lastly, I define the parameters of interest in Definition 3.

2.1 The Basic Setup

Consider a population consisting of G independent groups. Within each group, there are two units denoted by $i = 1, 2$. For each unit i in group g, there are two binary random variables Z_{ig}, and D_{ig}. The variable Z_{ig} takes the value 1 if unit i in group g is assigned to the treatment group, while D_{ig} takes the value 1 if that unit actually takes up the treatment.

To simplify the notation, I omit the group index subscript g when it does not make confusion. Throughout this paper, I use $i \in \{1, 2\}$ to denote a generic unit index and $j = 3 - i$ to denote the index of the other unit. Let $\boldsymbol{Z} = (Z_1, Z_2)$ and $\boldsymbol{D} = (D_1, D_2)$ represent vectors of treatment assignments and treatment take-up statuses for a group. For unit $i \in \{1, 2\}$, I define $\boldsymbol{Z}_i = (Z_i, Z_j)$ and $\boldsymbol{D}_i = (D_i, D_j)$, where the first component corresponds to unit i's own variable, and the second component corresponds to the other unit's variable. Similarly, for a given treatment assignment $\boldsymbol{z} \in \{0, 1\}^2$ and treatment take-up status $\boldsymbol{d} \in \{0, 1\}^2$, I define $\boldsymbol{z}_i = (z_i, z_j)$ and $\boldsymbol{d}_i = (d_i, d_j)$. Next, for unit i, let $D_i(\boldsymbol{z}_i)$ denote the potential treatment take-up when the treatment assignment is given by $\boldsymbol{z}_i$, and let $Y_i(\boldsymbol{d}_i, \boldsymbol{z}_i)$ denote the potential outcome when the treatment assignment and treatment take-up are given by $\boldsymbol{z}_i$ and $\boldsymbol{d}_i$. Additionally, let $\boldsymbol{Y} = (Y_1, Y_2)$ be the vector of observed outcomes for both units. Lastly, let $\boldsymbol{X} = (X_1, X_2)$ represent a vector of covariates, including both unit-specific characteristics and group characteristics, and denote $\boldsymbol{X}_i = (X_i, X_j)$. To begin with, I consider Assumption 1.

Assumption 1.

(A) (Exclusion Restriction I) For each unit $i \in \{1, 2\}$ and for each $\boldsymbol{d} \in \{0, 1\}^2$,

$$Y_i(\boldsymbol{d}, \boldsymbol{z}) = Y_i(\boldsymbol{d}, \boldsymbol{z}') \quad \forall \boldsymbol{z}, \boldsymbol{z}' \in \{0, 1\}^2.$$

(B) (Conditional Independence) Treatment assignments $\boldsymbol{Z} = (Z_1, Z_2)$ are independent of potential outcomes and potential treatment take-ups, conditional on $\boldsymbol{X}$, i.e.,

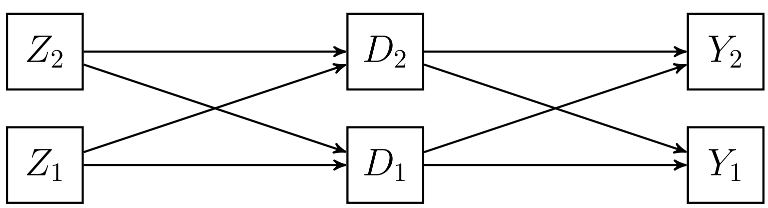
$$\left\{(Y_i(\boldsymbol{d}_i, \boldsymbol{z}_i), D_i(\boldsymbol{z}_i)) : (\boldsymbol{d}_i, \boldsymbol{z}_i) \in \{0, 1\}^4, i \in \{1, 2\}\right\} \perp \boldsymbol{Z} | \boldsymbol{X}.$$

(C) (Overlap) $\Pr(\boldsymbol{Z}_i = \boldsymbol{z} | \boldsymbol{X}) > 0$ with probability 1, for all $\boldsymbol{z} \in \{0, 1\}^2$ and $i \in \{1, 2\}$.

As often assumed in the literature, Assumptions 1-(A), (B) ensure that the treatment assignment $\boldsymbol{Z}$ is (conditionally) randomly assigned across groups, and thus serves as a valid instrument for the endogenous treatment $\boldsymbol{D}$. In particular, according to Assumption 1-(A), the potential outcome can be expressed solely as a function of the treatment take-up: $Y_i(\boldsymbol{d}_i) = Y_i(\boldsymbol{d}_i, \boldsymbol{z})$, for all $\boldsymbol{z} \in \{0,1\}^2$. Assumption 1-(C) is the standard overlap assumption that guarantees the existence of corresponding conditional expectations.

Given the setting, I explicitly examine two types of interactions between units. The first is the interaction in the treatment take-up decision, where each unit's potential treatment depends on both their own and others' treatment assignments ($D_i = D_i(\boldsymbol{z}_i) = D_i(z_i, z_j)$). The second is the spillover in potential outcomes, where each unit's potential outcome is determined by the treatment take-up status of both units ($Y_i = Y_i(\boldsymbol{d}_i) = Y_i(d_i, d_j)$). This situation is depicted in Figure 2. In the following subsection, we will discuss how each of these interactions complicates the problem.

Figure 2: Two Layers of Interactions



2.2 Monotonicity and Compliance Types

Imperfect compliance can introduce endogeneity in treatment take-up. The distribution of potential outcomes can vary based on different compliance patterns. Therefore, understanding these compliance and non-compliance patterns is crucial for defining and analyzing causal effects. This section discusses various assumptions, such as monotonicity, and corresponding compliance types defined in different settings.

2.2.1 Classical Case: Single-Unit

The compliance pattern of each unit is determined by the distribution of their potential treatments. To illustrate, consider the case of a single-unit ($N = 1$). There are two potential treatment takeup statuses: when the unit is treated ($D(1)$), and when it is untreated ($D(0)$). Since each status is binary, there are 4 (2^2) possible compliance types for each unit. As summarized in Table 1, each unit is classified by *always-taker* if $D(1) = D(0) = 1$, *complier* if $D(1) = 1, D(0) = 0$, *defier* if $D(1) = 0, D(0) = 1$, and *never-taker* if $D(1) = D(0) = 0$.

Table 1: Compliance Types Without Spillovers on the Take-Up Decision

Type	$D(1)$, or $D(1,1) = D(1,0)$	$D(0)$, or $D(0,1) = D(0,0)$
Always-taker	1	1
Complier	1	0
Defier	0	1
Never-taker	0	0

Notes: The assumption $D_i(z_i, z_j) = D_i(z_i, z_j')$ for all $z_j, z_j' \in \{0, 1\}$ is referred to as personalized encouragement (Kang and Imbens (2016)), treatment exclusion restriction (Blackwell (2017)), or individualized offer response (DiTraglia et al. (2023)).

The classical approach to identifying causal effects is excluding certain compliance types by imposing restrictions on the distribution of potential treatments. Imbens and Angrist (1994) introduce the monotonicity assumption, $\Pr(D(1) \geq D(0)) = 1$, which excludes the possibility of units being defiers, and thereby allows the identification of the local average treatment effect among compliers. To distinguish the other type of monotonicities, I refer to this as *classical monotonicity* in this paper. A similar frequently employed restriction in this setting is one-sided noncompliance (OSN), which assumes that only units in the treatment group can decide whether to take the treatment, while units in the control group cannot, i.e., $\Pr(D(0) = 0) = 1$. Thus, OSN excludes not only defiers but also always-takers. Although verifying these restrictions will not be straightforward since we do not observe all potential treatment statuses, the OSN situation can be evident by the design of the experiment, such as providing non-transferable voucher for the treatment take-up.

2.2.2 Restrictions on Potential Treatments

Consider a more general case with two units in each group ($N = 2$), which is the primary focus of this paper. If treatment take-up responses are determined by both their own and the other unit's treatment assignments, there are 4 potential treatment statuses: $D(1,1)$, $D(1,0)$, $D(0,1)$, and $D(0,0)$. Given that each potential treatment is binary, there are 16 (2^4) possible compliance types for each unit. Furthermore, considering two units, there are a total of 256 (16^2) possible joint compliance types for both units as noted by Kormos, Lieli, and Huber (2023). However, as in classical cases, not all scenarios are of interest, and some compliance types can be excluded by imposing appropriate restrictions on the potential treatment distribution.

One can simply assume there is no interactions in the treatment take-up decision (e.g., Kang and Imbens (2016), Blackwell (2017), DiTraglia et al. (2023)), as illustrated in panel (b) of Figure 1. Then, as in the single-unit case, there are 4 compliance types for each unit, as summarized in Table 1, and the classical monotonicity can be applied correspondingly. However, this assumption is obviously strong, as it excludes any interference in the treatment take-up. For instance, it is violated when each unit is able to take the treatment if at least one unit in the pair is assigned to the treatment group.

Another approach is to extend classical monotonicity to cases with two units. For example, one can define an almost-sure ordering on the set $\mathcal{D} := \{(D_i(z_i, z_j), D_j(z_j, z_i)) : (z_i, z_j) \in \{0,1\}^2\}$. For example, Vazquez-Bare (2022) assumes the following ordering:

$$D_i(1,1) \geq D_i(1,0) \geq D_i(0,1) \geq D_i(0,0), \text{ with probability } 1, \tag{1}$$

for all $i \in \{1,2\}$. Because this is a almost-sure total ordering, I refer to this as *total monotonicity* to distinguish it from the other monotonicities used in this paper. Under total monotonicity, 9 out of 16 compliance types are excluded for each unit. The interpretation of those 5 compliance type according the total monotonicity (1) is summarized in Table 2. A weaker version of total monotonicity assumes partial ordering on the set $\mathcal{D}$. For instance, some studies use $D_i(1,z) \geq D_i(0,z)$ for each $z \in \{0,1\}$ (e.g., Imai, Jiang, and Malani (2021), Vazquez-Bare (2022) , Hoshino and Yanagi (2023), Ohnishi and Sabbaghi (2024)). I call this as *marginal monotonicity* in this paper. However, compared to the classical monotonicity that is equivalent to excluding the possibility of being a defier, the total/marginal monotonicity are stronger since they exclude many compliance patterns, and may be difficult to interpret.

As in the single-unit case, the one-sided noncompliance (OSN) assumption can be applicable in specific experimental designs, where it is extended by $D(0,1) = D(0,0) = 0$ with probability 1. This assumption also significantly reduces the number of possible compliance types, and when the OSN assumption and total monotonicity are combined, each unit has 3 possible compliance types as summarized in Table 2.

Table 2: Compliacne Types with Total Monotonicity or OSN

Type	$D(1,1)$	$D(1,0)$	$D(0,1)$	$D(0,0)$
Always taker (AT)	1	1	1	1
Social complier (SC)	1	1	1	0
Complier (C)	1	1	0	0
Cross defier (CD)	0	1	0	0
Group complier (GC)	1	0	0	0
Never taker (NT)	0	0	0	0

Notes: This table shows 6 possible compliance types under one-sided noncompliance (OSN) or total monotonicity. Each classification follows those used in Vazquez-Bare (2022), Hoshino and Yanagi (2023), and Kormos, Lieli, and Huber (2023). Under total monotonicity (1), the cross-defier is excluded, resulting in 5 compliance types. By contrast, under OSN ($D_i(0,z) = 0$ a.s., for $z \in \{0,1\}$), the always taker and social complier are excluded, resulting in 4 compliance types. Therefore, when combining OSN and total monotonicity, we have only 3 compliance types: complier, group complier, and never taker.

2.2.3 A Generalization of Monotonicity

While personalized encouragement, OSN, and total/marginal monotonicity facilitate the identification of specific causal effects, they are not universally applicable. Consider the following scenario for $i \in \{1,2\}$:

$$\Pr(D_i(0,0) = 0) = 1. \tag{2}$$

This means that if both units are in the control group, neither can take the treatment. On the other hand, units could take the treatment if at least one unit is treated. I refer to this situations as *Weak One-Sided Noncompliance* (WOSN). In this case, the distributions of $D_i(1,1)$, $D_i(1,0)$, and $D_i(0,1)$ remain unrestricted, which means there is no almost-sure ordering on these potential treatment statuses. Therefore, WOSN violates total/marginal monotonicity. In addition, while WOSN is implied by OSN, it does not necessarily imply OSN, as $D(0,1) = 1$ could occur with positive probability.

A key distinction of this study from existing literature is that cases such as WOSN are still of interest. Therefore, I focus on some ordering of potential treatments that holds only for specific treatment assignment scenarios, rather than for all or many possible scenarios. To this end, I first define *monotone pairs* as follows.

Definition 1 (Monotone Pair). Let $\mathcal{M}$ be a collection of triples $(\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{r})$, where $\boldsymbol{z}, \boldsymbol{z}' \in \{0,1\}^2$, and $\boldsymbol{r} \in \{-1,1\}^2$ such that $r_i(D_i(\boldsymbol{z}_i) - D_i(\boldsymbol{z}'_i)) \geq 0$ for all $i \in \{1,2\}$, i.e, $\mathcal{M} := \{(\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{r}) : r_i(D_i(\boldsymbol{z}_i) - D_i(\boldsymbol{z}'_i)) \geq 0,\ i \in \{1,2\}\}$. And define elements $\boldsymbol{m} \in \mathcal{M}$ as a *monotone pair* if it exists.

Any almost-sure ordering on potential treatment can be expressed using monotone pairs. For instance, an almost-sure equality such as $D_i(\boldsymbol{z}_i) = D_i(\boldsymbol{z}'_i)$ can be written as two monotone pairs, $\boldsymbol{m}_1 = (\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{r})$ and $\boldsymbol{m}_2 = (\boldsymbol{z}', \boldsymbol{z}, \boldsymbol{r})$, for a given $\boldsymbol{r}$.

The vector $\boldsymbol{r}$ specifies the direction of monotonicity. It will mostly be the same for both units, i.e., $(1,1)$ or $(-1,-1)$. However, the direction can be opposite for the two units. For instance, let $\boldsymbol{z} = (1,0)$ and $\boldsymbol{z}' = (0,1)$. If we impose $D_i(1,0) \geq D_i(0,1)$ for both units, then $\boldsymbol{z}$ and $\boldsymbol{z}'$ form a monotone pair, but the direction is flipped for unit 2 since $\boldsymbol{z}_2 = (0,1)$ and $\boldsymbol{z}'_2 = (1,0)$.

Note that any monotone pair $\boldsymbol{m} \in \mathcal{M}$ with the same direction, we can assume $r_1 = r_2 = 1$ without loss of generality since we can exchange the first and the second assignment to obtain another monotone pair with the opposite direction. As the following arguments are mostly about the monotone pairs with the same directions, I focus on the collection: $\mathcal{M}_1 = \{(\boldsymbol{z}, \boldsymbol{z}') : \boldsymbol{m} \in \mathcal{M}, r_1 = r_2 = 1\}$. By abusing notation, I call a member in $\mathcal{M}_1$ as also a monotone pair.

If there are no restrictions on the potential treatment distribution, then $\mathcal{M}$ or $\mathcal{M}_1$

are empty. However, since such cases are not of interest in this paper, I assume we have at least one monotone pair with the same direction. Thus, the monotonicity assumption used in this study is stated as follows.

Assumption 2 (Monotonicity)**.** $\mathcal{M}_1$ *is nonempty.*

The restrictions on potential treatments discussed in Section 2.2.2 can be represented by an appropriate set of monotone pairs. Also, any other set of restrictions on the ordering of potential treatments can be represented as monotone pairs. Therefore, Assumption 2 is not only a natural generalization of total/marginal monotonicity but also a general restriction on the potential treatment distribution. In particular, compared to total monotonicity (1), which imposes an almost-sure total ordering on the set $\mathcal{D}$, Assumption 2 focus on almost-sure *partial* ordering on $\mathcal{D}$. For each monotone pair, the corresponding concept of compliance types are defined as follows:

Definition 2 (Compliance Types)**.** Suppose $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{r})$ is a monotone pair. Then define a unit i as $\boldsymbol{m}$-always-taker if $D_i(\boldsymbol{z}_i) = D_i(\boldsymbol{z}_i') = 1$, $\boldsymbol{m}$-never-taker if $D_i(\boldsymbol{z}_i) = D_i(\boldsymbol{z}_i') = 0$, and $\boldsymbol{m}$-complier if $r_i(D_i(\boldsymbol{z}_i) - D_i(\boldsymbol{z}_i')) > 0$, conditional on $\boldsymbol{X}$ with probability 1, respectively.

Table 3 describes the monotone pairs induced by personalized encouragement with classical monotonicity, total monotonicity, one-sided noncompliance, and weak one-sided noncompliance. Under total monotonicity or personalized encouragement, all possible pairs $(\boldsymbol{z}, \boldsymbol{z}') \in \{0,1\}^4$ can be formed as monotone pairs for some direction. However, since one-sided noncompliance does not impose any ordering on $D(1,1), D(1,0)$, some treatment assignments cannot form a monotone pair, as shown in Panel (c) of Table 3. Thus, the collection $\mathcal{M}$ induced by one-sided noncompliance is smaller than that of total monotonicity, or personalized encouragement. Additionally, there are only two monotone pairs under weak one-sided noncompliance.

Type$_i$ in Table 3 present the interpretation of each $\boldsymbol{m}$-complier In Panel (a), a unit is complier (C) if $D_i(1) > D_i(0)$. In Panel (b), (c), and (d), the interpretation is from the classification in Table 2. That is, unit i is complier (C) if $D_i(1,0) > D_i(0,0)$, social complier (SC) if $D_i(0,1) > D_i(0,0)$, group complier (GC) if $D_i(1,1) > D_i(1,0)$, and

Table 3: Monotone Pairs

(a) Personalized Encouragement + Classical Monotonicity

$\boldsymbol{z}$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,0)$	$(1,0)$	$(0,1)$
$\boldsymbol{z}'$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\boldsymbol{r}$	(1,1), (-1,1)	(1,1), (1,-1)	(1,1)	(1,-1)	(1,1) (1,-1)	(1,1) (1,-1)
Type_1		C	C	C	C	
Type_2	C		C	C		C

(b) Total Monotonicity

$\boldsymbol{z}$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,0)$	$(1,0)$	$(0,1)$
$\boldsymbol{z}'$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\boldsymbol{r}$	(1,1)	(1,1)	(1,1)	(1,-1)	(1,1)	(1,1)
Type_1	GC	GC, C	GC, C, SC	C	C, SC	SC
Type_2	GC,C	GC	GC, C, SC	C	SC	C, SC

(c) One-Sided Nomcompliance

$\boldsymbol{z}$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,0)$	$(1,0)$	$(0,1)$
$\boldsymbol{z}'$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\boldsymbol{r}$			(1,1)	(1,-1)	(1,1), (1,-1)	(1,1), (1,-1)
Type_1			GC, or C	C, or CD	C, or CD	
Type_2			GC, or C	C, or CD		C, or CD

(d) Weak One-Sided Nomcompliance

$\boldsymbol{z}$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,0)$	$(1,0)$	$(0,1)$
$\boldsymbol{z}'$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,0)$
$\boldsymbol{r}$					(1,1)	(1,1)
Type_1					C	SC
Type_2					SC	C

Notes: Panel (a), (b), (c) and (d) show the monotone pairs $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{r})$ under personalized encouragement with classical monotonicity $D_i(1) \geq D_i(0)$, total monotonicity $D(1,1) \geq D(1,0) \geq D(0,1) \geq D(0,0)$, one-sided noncompliance $D(0,1) = D(0,0) = 0$, and weak one-sided noncompliance $D(0,0) = 0$, respectively. Type_i denotes the interpretation of $\boldsymbol{m}$-complier for each unit, following the definitions in Table 1 (for panel (a)) and Table 2 (for panel (b), (c), and (d)).

cross-defier (CD) if $D_i(0,1) > D_i(1,0)$.

In general, these interpretations are context-dependent and determined by the entire set $\mathcal{M}$ and specific values of $\boldsymbol{m}$. In particular, the interpretation of the $\boldsymbol{m}$-compliance type might differ between the two units unless $\boldsymbol{z}_1 = \boldsymbol{z}_2$. For example, consider a monotone pair $\boldsymbol{m}_1 = ((1,1),(1,0),(1,1))$ under total monotonicity in the first column of Panel (b). Unit 1 is an $\boldsymbol{m}_1$-complier if $D(1,1) > D(1,0)$, while unit 2 is an $\boldsymbol{m}_1$-complier if $D(1,1) > D(0,1)$. In this case, unit 1 is classified as a group complier (GC), and unit 2 is classified as either a group complier (GC) or a complier (C).

When there are more than two monotone pairs, it becomes possible to further divide the $\boldsymbol{m}$-complier into finer compliance types. For example, consider another monotone pairs $\boldsymbol{m}_2 = ((1,1),(0,1),(1,1))$ and $\boldsymbol{m}_4 = ((1,0),(0,1),(1,-1))$ under total monotonicity. Unit 2 is an $\boldsymbol{m}_2$-complier if it is a group complier (GC), and an $\boldsymbol{m}_4$-complier if it is a complier (C). Therefore, the $\boldsymbol{m}_1$-complier for unit 2 can be further divided into $\boldsymbol{m}_2$-complier and $\boldsymbol{m}_4$-complier.

2.3 Potential Outcomes and Parameters of Interest

The potential outcome is defined based on treatment take-up statuses of both units to explicitly account for interference in determining potential outcomes, thus violating SUTVA. Suppose the observed outcome is given by $Y_i = Y_i(\boldsymbol{D}_i) = Y_i(D_i, D_j)$. Then, it can be written as:

$$
\begin{aligned}
Y_i &= \sum_{\boldsymbol{d} \in \{0,1\}^2} \mathbb{1}\{\boldsymbol{D}_i = \boldsymbol{d}\} Y_i(\boldsymbol{d}) \\
&= Y_i(0,0) + \Delta_i Y_i(0) D_i + \Delta_j Y_i(0) D_j + \Delta^2 Y_i D_i D_j
\end{aligned} \tag{3}
$$

where $\Delta_i Y_i(d) = Y_i(1,d) - Y_i(0,d)$, $\Delta_j Y_i(d) = Y_i(d,1) - Y_i(d,0)$, for $d \in \{0,1\}$, and $\Delta^2 Y_i = Y_i(1,1) - Y_i(0,1) - Y_i(1,0) + Y_i(0,0) = \Delta_i Y_i(1) - \Delta_i Y_i(0)$. The $\Delta_i Y_i(d)$ is interpreted as the direct effect on unit i's outcome, in the sense that it represents the causal effect on the outcome resulting from changes in own treatment when the partner's treatment take-up is fixed at $d \in \{0,1\}$. Similarly, $\Delta_j Y_i(d)$ is interpreted as the indirect effect on unit i's outcome when its own treatment status is fixed at d.

The parameters of interest are the average local direct ($\Delta_i Y_i(d)$) and indirect ($\Delta_j Y_i(d)$)

effects. Suppose we have a monotone pair $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{r})$. Denote $\boldsymbol{z}_i = (z_i, z_{3-i})$, $\boldsymbol{z}'_i = (z'_i, z'_{3-i})$ for each $i \in \{1, 2\}$, and define $K_i^{\boldsymbol{m}} := r_i(D_i(\boldsymbol{z}_i) - D_i(\boldsymbol{z}'_i))$. Then, $K_i^{\boldsymbol{m}}$ is a binary random variable that takes a value of 1 if unit i is an $\boldsymbol{m}$-complier. The parameters of interest in this study are defined as follows.

Definition 3 (Parameters of Interest)**.** For each $i \in \{1, 2\}$, $j = 3 - i$, and $d \in \{0, 1\}$,

$$\begin{aligned} \delta_i^{\boldsymbol{m}}(d) &:= E[Y_i(1, d) - Y_i(0, d) | K_i^{\boldsymbol{m}} = 1] = E[\Delta_i Y_i(d) | K_i^{\boldsymbol{m}} = 1], \\ \theta_i^{\boldsymbol{m}}(d) &:= E[Y_i(d, 1) - Y_i(d, 0) | K_j^{\boldsymbol{m}} = 1] = E[\Delta_j Y_i(d) | K_j^{\boldsymbol{m}} = 1]. \end{aligned} \tag{4}$$

The term $\delta_i^{\boldsymbol{m}}(d)$ represents the local average *direct* effect, which is the average change in the outcome due to a unit's own treatment take-up when the unit is $\boldsymbol{m}$-complier and the other unit's take-up status is fixed at d. Similarly, $\theta_i^{\boldsymbol{m}}(d)$ represents the local average *indirect* treatment effect, capturing the average changes from the other unit's treatment take-up when the other unit is $\boldsymbol{m}$-complier, and own treatment status is fixed at d. Because the interpretations of $\boldsymbol{m}$-compliers are determined by an empirical context, the interpretations of these parameters are also context-specific.

As noted, multiple monotone pairs can imply finer and disjoint compliance types. In this case, the corresponding local average effects can also be expressed by a weighted averages of those of finer compliance types. See Appendix C for a detailed example.

3 Identification

This section discusses the identification of causal parameters defined in (4). The goal is to identify these parameters under weaker restrictions on the potential treatment than total/marginal monotonicity, one-sided noncompliance, and personalized encouragement.

I present that the intention-to-treat (ITT) effects on the outcome can be represented by a weighted average of the parameters, which I call the ITT equation. I then propose a general result showing that the parameters are identified as coefficients in the ITT equation, provided there is an additional exclusion restriction for the compliance types.

Furthermore, I analyze special cases. This includes cases under the restrictions discussed in Section 2.2.2 and cases where we don't need additional exclusion restrictions. These cases are particularly noteworthy because they include existing identification results in the literature, such as the local average treatment effects proposed in Imbens and Angrist (1994) with classical monotonicity, or Vazquez-Bare (2022) with total monotonicity and one-sided noncompliance.

3.1 Intention-to-Treat Effects on Outcomes

The intention-to-treat (ITT) effect on an outcome is defined as the difference between the average observed outcomes under two different treatment assignments. The following lemma summarizes the relationship between the ITT effect on the outcome, the causal effects ($\Delta_i Y_i$ and $\Delta_j Y_i$), and the compliance types ($K_i^{\boldsymbol{m}}$).

Lemma 1. *Suppose Assumptions 1 and 2 hold, and let $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{r}) \in \mathcal{M}$ be a monotone pair. Then, the conditional intention-to-treat (ITT) effect on the outcome of unit $i \in \{1, 2\}$ with respect to the monotone pair $\boldsymbol{m}$ is given by*

$$\begin{aligned} ITT_i^{\boldsymbol{m}}(\boldsymbol{X}) :&= E[Y_i | \boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}] - E[Y_i | \boldsymbol{Z}_i = \boldsymbol{z}_i', \boldsymbol{X}] \\ &= r_i E[K_i^{\boldsymbol{m}} \Delta_i Y_i(0) | \boldsymbol{X}] + r_j E[K_j^{\boldsymbol{m}} \Delta_j Y_i(0) | \boldsymbol{X}] + r_i r_j E[K_{ij}^{\boldsymbol{m}} \Delta^2 Y_i | \boldsymbol{X}], \end{aligned} \tag{5}$$

where $j = 3 - i$ is the other unit's index, $K_i^{\boldsymbol{m}} := r_i \left[D_i(\boldsymbol{z}_i) - D_i(\boldsymbol{z}_i')\right]$, and $K_{ij}^{\boldsymbol{m}} := r_i r_j \left[D_i(\boldsymbol{z}_i) D_j(\boldsymbol{z}_j) - D_i(\boldsymbol{z}_i') D_j(\boldsymbol{z}_j')\right]$.

This relationship directly follows from (3) and Assumption 1. The first two terms in (5) are related to the direct and indirect effects of $\boldsymbol{m}$-compliers, and the marginal distributions for each unit to be $\boldsymbol{m}$-compliers. Recall that $K_i^{\boldsymbol{m}}$ is a binary variable that indicates unit i being a $\boldsymbol{m}$-complier. Furthermore, because $E[D_i(\boldsymbol{z}_i)] = E[D_i | \boldsymbol{Z}_i = \boldsymbol{z}_i]$ from Assumption 1, the distribution of $K_i^{\boldsymbol{m}}$ is identified by $\Pr(K_i^{\boldsymbol{m}} = 1) = E[K_i^{\boldsymbol{m}}] = r_i(E[D_i | \boldsymbol{Z}_i = \boldsymbol{z}_i] - E[D_i | \boldsymbol{Z}_i = \boldsymbol{z}_i'])$, which is an ITT effect of D_i. By constrast, the last term in (5) is related to the joint distribution of compliance types for units, which is not generally identifiable. However, for monotone pairs with the same direction, i.e., $r_1 = r_2$, $K_{ij}^{\boldsymbol{m}}$ is also a binary variable, and its distribution is identified by an ITT effect of $D_i D_j$, similar to $K_i^{\boldsymbol{m}}$ as shown in Proposition 1.

Proposition 1 (Distribution of $\boldsymbol{m}$-Compliers)**.** *Suppose Assumptions 1 and 2 hold, and let* $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$*. Let* $\boldsymbol{T}$ *be a subset of the set of exogenous variables in* $\boldsymbol{X}$*. Then, the following conditional distributions are identified:*

$$
\begin{aligned}
P_i^{\boldsymbol{m}}(\boldsymbol{T}) &:= \Pr(K_i^{\boldsymbol{m}} = 1|\boldsymbol{T}) = E\left[E[D_i|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}] - E[D_i|\boldsymbol{Z}_i = \boldsymbol{z}_i', \boldsymbol{X}]|\, \boldsymbol{T}\right], \\
P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) &:= \Pr(K_{ij}^{\boldsymbol{m}} = 1|\boldsymbol{T}) = E\left[E[D_iD_j|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}] - E[D_iD_j|\boldsymbol{Z}_i = \boldsymbol{z}_i', \boldsymbol{X}]|\, \boldsymbol{T}\right],
\end{aligned}
\tag{6}
$$

In particular, the unconditional distributions $P_i^{\boldsymbol{m}} = \Pr(K_i^{\boldsymbol{m}} = 1)$ *and* $P_{ij}^{\boldsymbol{m}} = \Pr(K_{ij}^{\boldsymbol{m}} = 1)$ *are also identified by not conditioning on* $\boldsymbol{T}$ *in (6).*

Suppose we have a monotone pair $\boldsymbol{m} \in \mathcal{M}_1$, and define $\zeta_i^{\boldsymbol{m}} := E[\Delta^2 Y_i|K_{ij}^{\boldsymbol{m}} = 1]$. Then, by integrating over the distribution of $\boldsymbol{X}$ and applying law of iterative expectation on (5), we obtain the following expression:

$$
E[ITT_i^{\boldsymbol{m}}(\boldsymbol{X})] = \delta_i^{\boldsymbol{m}}(0)P_i^{\boldsymbol{m}} + \theta_i^{\boldsymbol{m}}(0)P_j^{\boldsymbol{m}} + \zeta_i^{\boldsymbol{m}}P_{ij}^{\boldsymbol{m}}. \tag{7}
$$

Therefore, the (unconditional) ITT effect of outcome is expressed as a weighted average of the coefficients $\delta_i^{\boldsymbol{m}}(0)$, $\theta_i^{\boldsymbol{m}}(0)$, and $\zeta_i^{\boldsymbol{m}}$. Specifically, the first two coefficients are the causal parameters of interest defined in (4). The last coefficient $\zeta_i^{\boldsymbol{m}}$ in the last term is not of primary interest at this point because interpreting the local average $\zeta_i^{\boldsymbol{m}}$ is not straightforward.[2] This is because the conditioning event occurs when both units are $\boldsymbol{m}$-compliers, or when one unit is an $\boldsymbol{m}$-complier and the other unit is an $\boldsymbol{m}$-always-taker. In Section 3.3, I discuss special cases when this coefficient has a clear interpretation, which results in the cancellation of the last term.

3.2 Identification with Additional Exclusion Restriction

Equation (7) can be seen as a single equation with three unknowns: $\delta_i^{\boldsymbol{m}}(0)$, $\theta_i^{\boldsymbol{m}}(0)$, and $\zeta_i^{\boldsymbol{m}}$. The main idea to recover these unknowns from the ITT equation (7) is to use exogenous variation in the distributions $P_i^{\boldsymbol{m}}$ and $P_{ij}^{\boldsymbol{m}}$ by introducing additional

[2]This coefficient can be considered a measure of the cross-derivative of the outcome with respect to the treatments of both units, effectively capturing the interaction effects of the treatments on the outcomes. This coefficient corresponds to the local average indirect effect (LAIE) as defined by Hoshino and Yanagi (2023).

exclusion restrictions on the endogenous treatment. Suppose there exists $\boldsymbol{T}$, a subset of exogenous variables in $\boldsymbol{X}$, that satisfies:

$$E[ITT_i^{\boldsymbol{m}}(\boldsymbol{X})|\boldsymbol{T}] = \delta_i^{\boldsymbol{m}}(0)P_i^{\boldsymbol{m}}(\boldsymbol{T}) + \theta_i^{\boldsymbol{m}}(0)P_j^{\boldsymbol{m}}(\boldsymbol{T}) + \zeta_i^{\boldsymbol{m}}P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}). \tag{8}$$

Then, since the conditional distributions $P_i^{\boldsymbol{m}}(\boldsymbol{T})$ and $P_{ij}^{\boldsymbol{m}}(\boldsymbol{T})$ are identified by Proposition 1, the parameters can be identified as linear coefficients in this ITT equation. Note that (8) is the conditional expectation version of (7), and the coefficients are conditional expectations conditioned on the compliance types. Therefore, the required condition for $\boldsymbol{T}$ is that it needs to be correlated with the compliance types to generate some exogenous variation in $P_i^{\boldsymbol{m}}(\boldsymbol{T})$ and $P_{ij}^{\boldsymbol{m}}(\boldsymbol{T})$, but it does not need to be correlated with the potential outcome once the compliance types are given. Since the compliance types are determined by the potential treatment take-up, $\boldsymbol{T}$ can be considered as an additional exclusion restriction. Before formally stating the required conditions, consider a simple example to illustrate this idea.

Example 1. Let $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$ be a monotone pair with the same direction. Suppose the exogenous variables $\boldsymbol{X}$ include a discrete random variable T that only takes values in $\{t_1, t_2, t_3\}$. Assume that T is correlated with the potential treatment take-up, and therefore correlated with $K_i^{\boldsymbol{m}}$, $K_j^{\boldsymbol{m}}$ and $K_{ij}^{\boldsymbol{m}}$. Also assume that T is independent of potential outcomes, once the distributions of potential treatment take-up $(D_i(\cdot), D_j(\cdot))$ are given. Then, by integrating (5) over the conditional distribution of $\boldsymbol{X}$ given T, we have ITT equation (8): for $\ell = 1, 2, 3$,

$$E[ITT_i^{\boldsymbol{m}}(\boldsymbol{X})|T = t_\ell] = \delta_i^{\boldsymbol{m}}(0)P_i^{\boldsymbol{m}}(t_\ell) + \theta_i^{\boldsymbol{m}}(0)P_j^{\boldsymbol{m}}(t_\ell) + \zeta_i^{\boldsymbol{m}}P_{ij}^{\boldsymbol{m}}(t_\ell),$$

In this example, the parameters can be recovered by

$$\begin{pmatrix} \delta_i^{\boldsymbol{m}}(0) \\ \theta_i^{\boldsymbol{m}}(0) \\ \zeta_i^{\boldsymbol{m}} \end{pmatrix} = \begin{pmatrix} P_i^{\boldsymbol{m}}(t_1) & P_j^{\boldsymbol{m}}(t_1) & P_{ij}^{\boldsymbol{m}}(t_1) \\ P_i^{\boldsymbol{m}}(t_2) & P_j^{\boldsymbol{m}}(t_2) & P_{ij}^{\boldsymbol{m}}(t_2) \\ P_i^{\boldsymbol{m}}(t_3) & P_j^{\boldsymbol{m}}(t_3) & P_{ij}^{\boldsymbol{m}}(t_3) \end{pmatrix}^{-1} \begin{pmatrix} E[ITT_i^{\boldsymbol{m}}(\boldsymbol{X})|T = t_1] \\ E[ITT_i^{\boldsymbol{m}}(\boldsymbol{X})|T = t_2] \\ E[ITT_i^{\boldsymbol{m}}(\boldsymbol{X})|T = t_3] \end{pmatrix}, \tag{9}$$

for each $i \in \{1, 2\}$, if the matrix including $P_i^{\boldsymbol{m}}(t_\ell)$ and $P_{ij}^{\boldsymbol{m}}(t_\ell)$ is invertible. $\square$

The following assumption formally states the required conditions for such additional exclusion restrictions in terms of appropriate mean independence.

Assumption 3 (Exclusion Restriction II)**.** *The exogenous variables* $\boldsymbol{X} = (X_1, X_2)$ *are divided into two parts:* $X_i = (W_i, T_i)$, $i \in \{1, 2\}$. *For the vector of exogenous variables* $\boldsymbol{T} = (T_1, T_2)$, *assume* $E[\Delta_i Y_i(0)|K_i^{\boldsymbol{m}}, \boldsymbol{T}] = E[\Delta_i Y_i(0)|K_i^{\boldsymbol{m}}]$, $E[\Delta_j Y_i(0)|K_j^{\boldsymbol{m}}, \boldsymbol{T}] = E[\Delta_j Y_i(0)|K_j^{\boldsymbol{m}}]$, *and* $E[\Delta^2 Y_i|K_{ij}^{\boldsymbol{m}}, \boldsymbol{T}] = E[\Delta^2 Y_i|K_{ij}^{\boldsymbol{m}}]$ *for each* $\boldsymbol{m} \in \mathcal{M}_1$.

As shown in Example 1, if $\boldsymbol{T}$ is a vector of exogenous variables satisfying Assumption 3, then we have (8) by integrating (5) over the conditional distribution of $\boldsymbol{X}$ given $\boldsymbol{T}$. The ITT equation (8) can be thought of as an identifying moment condition as summarized in Lemma 2.

Lemma 2. *Suppose Assumptions 1-3 hold. Let* $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$ *be a monotone pair, and*

$$\boldsymbol{\beta}_i^{\boldsymbol{m}} = \begin{pmatrix} \delta_i^{\boldsymbol{m}}(0) \\ \theta_i^{\boldsymbol{m}}(0) \\ \zeta_i^{\boldsymbol{m}} \end{pmatrix}, \quad \tilde{\boldsymbol{P}}_i^{\boldsymbol{m}}(\boldsymbol{T}) = \begin{pmatrix} P_i^{\boldsymbol{m}}(\boldsymbol{T}) \\ P_j^{\boldsymbol{m}}(\boldsymbol{T}) \\ P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) \end{pmatrix}, \quad \tilde{\boldsymbol{D}}_i = \begin{pmatrix} D_i \\ D_j \\ D_i D_j \end{pmatrix}.$$

Then, we have the following conditional moment restriction:

$$E\left[\tilde{\boldsymbol{Y}}^{\boldsymbol{m}} - \tilde{\boldsymbol{P}}^{\boldsymbol{m}}(\boldsymbol{T})\boldsymbol{\beta}^{\boldsymbol{m}}\middle|\boldsymbol{T}\right] = E\left[\omega^{\boldsymbol{m}}\left(\boldsymbol{Y} - \tilde{\boldsymbol{D}}\boldsymbol{\beta}^{\boldsymbol{m}}\right)\middle|\boldsymbol{T}\right] = 0, \tag{10}$$

where $\boldsymbol{\beta}^{\boldsymbol{m}} = (\boldsymbol{\beta}_1^{\boldsymbol{m}\prime}, \boldsymbol{\beta}_2^{\boldsymbol{m}\prime})'$, $\tilde{\boldsymbol{Y}}^{\boldsymbol{m}} = \omega^{\boldsymbol{m}}\boldsymbol{Y} = \omega^{\boldsymbol{m}}(Y_1, Y_2)'$, $\tilde{\boldsymbol{P}}^{\boldsymbol{m}}(\boldsymbol{T})$ *is the block diagonal matrix of* $\tilde{\boldsymbol{P}}_1^{\boldsymbol{m}}(\boldsymbol{T})'$ *and* $\tilde{\boldsymbol{P}}_2^{\boldsymbol{m}}(\boldsymbol{T})'$, $\tilde{\boldsymbol{D}}$ *is the block diagonal matrix of* $\tilde{\boldsymbol{D}}_1'$ *and* $\tilde{\boldsymbol{D}}_2'$, *and* [3]

$$\omega^{\boldsymbol{m}} = \frac{\mathbb{1}\{\boldsymbol{Z}_i = \boldsymbol{z}_i\}}{\Pr(\boldsymbol{Z}_i = \boldsymbol{z}_i|\boldsymbol{X})} - \frac{\mathbb{1}\{\boldsymbol{Z}_i = \boldsymbol{z}_i'\}}{\Pr(\boldsymbol{Z}_i = \boldsymbol{z}_i'|\boldsymbol{X})}. \tag{11}$$

The following Proposition 2 demonstrates that $\boldsymbol{\beta}^{\boldsymbol{m}}$ is identified using the standard instrumental variable (IV) estimand derived from the conditional moment in (10). Assumption 4 provides a sufficient condition for this identification.

[3]The weight $\omega^{\boldsymbol{m}}$ does not depend on the unit index, since $\mathbb{1}\{\boldsymbol{Z}_i = \boldsymbol{z}_i\} = \mathbb{1}\{\boldsymbol{Z}_j = \boldsymbol{z}_j\}$.

Assumption 4 (Identification I)**.** *There exists a matrix* $\boldsymbol{R}(\boldsymbol{T}) \in \mathbb{R}^{6\times 2}$ *of functions of* $\boldsymbol{T}$ *such that* $E\left[\boldsymbol{R}(\boldsymbol{T})\tilde{\boldsymbol{P}}^{\boldsymbol{m}}(\boldsymbol{T})\right] = E\left[\omega^{\boldsymbol{m}}\boldsymbol{R}(\boldsymbol{T})\tilde{\boldsymbol{D}}\right]$ *is nonsingular.*

Proposition 2. *Suppose Assumptions 1-4 hold, and let* $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$ *be a monotone pair. Then, the parameter is identified by* $\boldsymbol{\beta}^{\boldsymbol{m}} = E\left[\omega^{\boldsymbol{m}}\boldsymbol{R}(\boldsymbol{T})\tilde{\boldsymbol{D}}\right]^{-1} E\left[\omega^{\boldsymbol{m}}\boldsymbol{R}(\boldsymbol{T})\boldsymbol{Y}\right]$*.*

For Assumption 4 to hold, $\boldsymbol{T}$ must have at least 3 distinct values with positive probability. This is because the rank of the 6×6 matrix $\boldsymbol{R}(\boldsymbol{T})\tilde{\boldsymbol{P}}^{\boldsymbol{m}}(\boldsymbol{T})$ is at most 2 for a fixed $\boldsymbol{T}$. This rank condition for the exogenous variable $\boldsymbol{T}$ to have sufficient variation is intuitive, as shown in the Example 1.

However, even if $\boldsymbol{T}$ has sufficient variations, Assumption 4 is violated if $\tilde{\boldsymbol{P}}^{\boldsymbol{m}}(\boldsymbol{T})$ is not of full column rank with probability 1, e.g., when $P_i^{\boldsymbol{m}}(\boldsymbol{T})$, $P_j^{\boldsymbol{m}}(\boldsymbol{T})$, and $P_{ij}^{\boldsymbol{m}}(\boldsymbol{T})$ are linearly dependent. However, even in such cases, some parameters may still be identifiable, which will be discussed in the following subsection.

3.3 Special Cases

This subsection discusses about three special cases where Assumption 4 is violated due to the linear dependence of $P_i^{\boldsymbol{m}}, P_j^{\boldsymbol{m}}$, and $P_{ij}^{\boldsymbol{m}}$. Recall that the ITT equation (7) and (8) consist of three terms. The first special case involves the cancellation of the third interaction term, which can occur when there is an ordering on the potential treatments of the two units. The second special case is when the second term is zero because the potential treatment for a unit is almost surely equal for two different treatment assignments. The third special case occurs when the last two terms are canceled out, since one unit is almost surely an always-taker or never-taker. Assumption 4 is is appropriately relaxed in each of these special cases, and particularly in the third case, some parameters can be identified without Assumption 3.

3.3.1 Identification Under Cross-Monotonicity

Since the interaction term $\zeta_i^{\boldsymbol{m}} P_{ij}^{\boldsymbol{m}}(\boldsymbol{T})$ in (8) is relates to the joint compliance pattern, the first special case involves a scenario where the potential treatments of two units follow a specific order. The following lemma provides an observation for this special case.

Lemma 3. *Let $\boldsymbol{T}$ be a subset of the set exogenous variable $\boldsymbol{X}$, and $\boldsymbol{z} \in \{0,1\}^2$ be given. For each unit $i \in \{1,2\}$ and $j = 3-i$, $\Pr(D_i(\boldsymbol{z}_i) \geq D_j(\boldsymbol{z}_j)|\boldsymbol{T}) = 1$ if and only if $\Pr(D_j(\boldsymbol{z}_j) = D_i(\boldsymbol{z}_i)D_j(\boldsymbol{z}_j)|\boldsymbol{T}) = 1$.*

I refer to the condition $\Pr(D_i(\boldsymbol{z}_i) \geq D_j(\boldsymbol{z}_j)|\boldsymbol{T}) = 1$ as a *cross monotonicity* between $D_i(\boldsymbol{z}_i)$ and $D_j(\boldsymbol{z}_j)$ at the assignment $\boldsymbol{z}$. If $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$, and if cross monotonicity holds at both $\boldsymbol{z}$ and $\boldsymbol{z}'$, then Lemma 3 implies that:

$$K_{ij}^{\boldsymbol{m}} = D_i(\boldsymbol{z}_i)D_j(\boldsymbol{z}_j) - D_i(\boldsymbol{z}_i')D_j(\boldsymbol{z}_j') = D_j(\boldsymbol{z}_j) - D_j(\boldsymbol{z}_j') = K_j^{\boldsymbol{m}}, \tag{12}$$

conditional on $\boldsymbol{T}$ with probability 1. Thus we have $P_j^{\boldsymbol{m}}(\boldsymbol{T}) = P_{ij}^{\boldsymbol{m}}(\boldsymbol{T})$, and the coefficient $\zeta_i^{\boldsymbol{m}}$ is now interpreted as $\theta_i^{\boldsymbol{m}}(1) - \theta_i^{\boldsymbol{m}}(0)$, or $\delta_i^{\boldsymbol{m}}(1) - \delta_i^{\boldsymbol{m}}(0)$. Therefore, the interaction term in (8) is cancelled out, and we have a simpler moment condition as stated in Lemma 4.

Lemma 4. *Suppose Assumptions 1-3 hold. Let $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$ be a monotone pair satisfying $\Pr(D_i(\boldsymbol{z}_i) \geq D_j(\boldsymbol{z}_j)|\boldsymbol{T}) = \Pr(D_i(\boldsymbol{z}_i') \geq D_j(\boldsymbol{z}_j')|\boldsymbol{T}) = 1$, and*

$$\check{\boldsymbol{\beta}}_i^{\boldsymbol{m}} = \begin{pmatrix} \delta_i^{\boldsymbol{m}}(0) \\ \theta_i^{\boldsymbol{m}}(1) \end{pmatrix}, \quad \check{\boldsymbol{\beta}}_j^{\boldsymbol{m}} = \begin{pmatrix} \delta_j^{\boldsymbol{m}}(1) \\ \theta_j^{\boldsymbol{m}}(0) \end{pmatrix}, \quad \check{\boldsymbol{P}}_i^{\boldsymbol{m}}(\boldsymbol{T}) = \begin{pmatrix} P_i^{\boldsymbol{m}}(\boldsymbol{T}) \\ P_j^{\boldsymbol{m}}(\boldsymbol{T}) \end{pmatrix}, \quad \check{\boldsymbol{D}}_i = \begin{pmatrix} D_i \\ D_j \end{pmatrix},$$

Then, we have the following conditional moment restriction:

$$E\left[\tilde{\boldsymbol{Y}}^{\boldsymbol{m}} - \check{\boldsymbol{P}}^{\boldsymbol{m}}(\boldsymbol{T})\check{\boldsymbol{\beta}}^{\boldsymbol{m}}\middle|\boldsymbol{T}\right] = E\left[\omega^{\boldsymbol{m}}\left(\boldsymbol{Y} - \check{\boldsymbol{D}}\check{\boldsymbol{\beta}}^{\boldsymbol{m}}\right)\middle|\boldsymbol{T}\right] = 0, \tag{13}$$

where $\check{\boldsymbol{\beta}}^{\boldsymbol{m}} = (\check{\boldsymbol{\beta}}_i^{\boldsymbol{m}\prime}, \check{\boldsymbol{\beta}}_j^{\boldsymbol{m}\prime})'$, $\tilde{\boldsymbol{Y}}^{\boldsymbol{m}} = \omega^{\boldsymbol{m}}\boldsymbol{Y} = \omega^{\boldsymbol{m}}(Y_i, Y_j)'$, $\check{\boldsymbol{P}}^{\boldsymbol{m}}(\boldsymbol{T})$ is the block diagonal matrix of $\check{\boldsymbol{P}}_i^{\boldsymbol{m}}(\boldsymbol{T})'$ and $\check{\boldsymbol{P}}_j^{\boldsymbol{m}}(\boldsymbol{T})'$, $\check{\boldsymbol{D}}$ is the block diagonal matrix of $\check{\boldsymbol{D}}_i'$ and $\check{\boldsymbol{D}}_j'$.

Following Assumption 5 is a sufficient condition to identify $\check{\boldsymbol{\beta}}^{\boldsymbol{m}}$ from conditional moment (13) and Proposition 3 shows the IV estimand is identitifed correspondingly.

Assumption 5 (Identification II)**.** *There exists a matrix $\check{\boldsymbol{R}}(\boldsymbol{T}) \in \mathbb{R}^{4\times 2}$ of functions of $\boldsymbol{T}$ such that $E\left[\check{\boldsymbol{R}}(\boldsymbol{T})\check{\boldsymbol{P}}^{\boldsymbol{m}}(\boldsymbol{T})\right] = E\left[\omega^{\boldsymbol{m}}\check{\boldsymbol{R}}(\boldsymbol{T})\check{\boldsymbol{D}}\right]$ is nonsingular.*

Proposition 3. *Suppose Assumptions 1-3, and 5 hold, and let $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$ be a* *monotone pair satisfying* $\Pr(D_i(\boldsymbol{z}_i) \geq D_j(\boldsymbol{z}_j)|\boldsymbol{T}) = \Pr(D_i(\boldsymbol{z}'_i) \geq D_j(\boldsymbol{z}'_j)|\boldsymbol{T}) = 1$. *Then, the parameter is identified by* $\check{\boldsymbol{\beta}}^{\boldsymbol{m}} = E\left[\omega^{\boldsymbol{m}}\check{\boldsymbol{R}}(\boldsymbol{T})\check{\boldsymbol{D}}\right]^{-1} E\left[\omega^{\boldsymbol{m}}\check{\boldsymbol{R}}(\boldsymbol{T})\boldsymbol{Y}\right]$.

The following example illustrates a case when cross monotonicity is satisfied.

Remark 1 (Weak one-sided noncompliance with joint take-up)**.** Suppose the weak one-sided noncompliance (WOSN) in (2) holds: $\Pr(D_i(0,0) = 0|\boldsymbol{X}) = 1$ for all $i \in \{1,2\}$. Thus, for any $\boldsymbol{z} \in \{0,1\}^2$, $D_i(\boldsymbol{z}) \geq D_i(0,0) = 0$ with probability 1, and we have monotone pairs of $(\boldsymbol{z}, (0,0), (1,1))$. Consider two monotone pairs $\boldsymbol{m}_1 = ((1,0),(0,0),(1,1))$ and $\boldsymbol{m}_2 = ((0,1),(0,0),(1,1))$. Notice that for the treatment assignment $(0,0)$, cross monotonicity holds trivially to both units since $D_1(0,0) = D_2(0,0) = 0$ almost surely.

Additionally, suppose units can take the treatment jointly if at least one unit is treated. Consider the treatment assignment $(1,0)$ where only unit 1 receives the treatment. Then, unit 2 can take up the treatment only if both units take it jointly, i.e., $D_2(0,1) = 1$ implies $D_1(1,0) = 1$. Thus, we have cross monotonicity $D_2(0,1) \geq D_1(1,0)$ for the assignment $(1,0)$. The opposite holds for the assignment $(0,1)$. Therefore, we can apply Proposition 3 to identify $\delta_i^{\boldsymbol{m}}(0)$ and $\theta_i^{\boldsymbol{m}}(1)$ for both units for both monotone pairs $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$. $\square$

3.3.2 Identification When $P_j^{\boldsymbol{m}}(\boldsymbol{T}) = 0$

The second special case is when $P_j^{\boldsymbol{m}} = 0$, or, equivalently $D_j(\boldsymbol{z}) = D_j(\boldsymbol{z}')$, and hence Assumption 4 is violated. In this scenario, the second term in (8) becomes zero. The corresponding moment condition is summarized in Lemma 5.

Lemma 5. *Suppose Assumptions 1-3 hold. Let* $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$ *be a monotone pair satisfying* $\Pr(D_i(\boldsymbol{z}_i) = D_j(\boldsymbol{z}'_j)|\boldsymbol{T}) = 1$, *and*

$$\bar{\boldsymbol{\beta}}_i^{\boldsymbol{m}} = \begin{pmatrix} \delta_i^{\boldsymbol{m}}(0) \\ \zeta_i^{\boldsymbol{m}} \end{pmatrix}, \quad \bar{\boldsymbol{\beta}}_j^{\boldsymbol{m}} = \begin{pmatrix} \theta_j^{\boldsymbol{m}}(0) \\ \zeta_j^{\boldsymbol{m}} \end{pmatrix}, \quad \bar{\boldsymbol{P}}_i^{\boldsymbol{m}}(\boldsymbol{T}) = \begin{pmatrix} P_i^{\boldsymbol{m}}(\boldsymbol{T}) \\ P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) \end{pmatrix}, \quad \bar{\boldsymbol{D}}_i = \begin{pmatrix} D_i \\ D_i D_j \end{pmatrix},$$

Then, we have the following conditional moment restriction:

$$E\left[\tilde{\boldsymbol{Y}}^{\boldsymbol{m}} - \bar{\boldsymbol{P}}^{\boldsymbol{m}}(\boldsymbol{T})\bar{\boldsymbol{\beta}}^{\boldsymbol{m}}\middle|\boldsymbol{T}\right] = E\left[\omega^{\boldsymbol{m}}\left(\boldsymbol{Y} - \bar{\boldsymbol{D}}\bar{\boldsymbol{\beta}}^{\boldsymbol{m}}\right)\middle|\boldsymbol{T}\right] = 0, \tag{14}$$

where $\bar{\boldsymbol{\beta}}^{\boldsymbol{m}} = (\bar{\boldsymbol{\beta}}_i^{\boldsymbol{m}\prime}, \bar{\boldsymbol{\beta}}_j^{\boldsymbol{m}\prime})'$, $\tilde{\boldsymbol{Y}}^{\boldsymbol{m}} = \omega^{\boldsymbol{m}}\boldsymbol{Y} = \omega^{\boldsymbol{m}}(Y_i, Y_j)'$, $\bar{\boldsymbol{P}}^{\boldsymbol{m}}(\boldsymbol{T})$ *is the block diagonal matrix of* $\bar{\boldsymbol{P}}_i^{\boldsymbol{m}}(\boldsymbol{T})'$, *and* $\bar{\boldsymbol{P}}_j^{\boldsymbol{m}}(\boldsymbol{T})'$, $\bar{\boldsymbol{D}}$ *is the block diagonal matrix of* $\bar{\boldsymbol{D}}_i'$ *and* $\bar{\boldsymbol{D}}_j'$.

Following Assumption 6 is a sufficient condition to identify $\bar{\boldsymbol{\beta}}^{\boldsymbol{m}}$ from conditional moment (14) and Proposition 4 shows the IV estimand is identitifed correspondingly.

Assumption 6 (Identification II)**.** *There exists a matrix* $\bar{\boldsymbol{R}}(\boldsymbol{T}) \in \mathbb{R}^{4\times 2}$ *of functions of* $\boldsymbol{T}$ *such that* $E\left[\bar{\boldsymbol{R}}(\boldsymbol{T})\bar{\boldsymbol{P}}^{\boldsymbol{m}}(\boldsymbol{T})\right] = E\left[\omega^{\boldsymbol{m}}\bar{\boldsymbol{R}}(\boldsymbol{T})\bar{\boldsymbol{D}}\right]$ *is nonsingular.*

Proposition 4. *Suppose Assumptions 1-3, and 6 hold, and let* $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$ *be a monotone pair satisfying* $\Pr(D_i(\boldsymbol{z}_i) = D_j(\boldsymbol{z}_j')|\boldsymbol{T}) = 1$. *Then, the parameter is identified by* $\bar{\boldsymbol{\beta}}^{\boldsymbol{m}} = E\left[\omega^{\boldsymbol{m}}\bar{\boldsymbol{R}}(\boldsymbol{T})\bar{\boldsymbol{D}}\right]^{-1} E\left[\omega^{\boldsymbol{m}}\bar{\boldsymbol{R}}(\boldsymbol{T})\boldsymbol{Y}\right]$.

The personalized encouargement design with classical monotonicity is an example in that Proposition 4 can be appliled.

Remark 2 (Personalized Encouargement with Classical Monotonicity)**.** As demonstrated in Table 3, under personalized encouragement (i.e, units do not interact in treatment take-up decision), we have a monotone pair $\boldsymbol{m} = ((1,1),(0,1)) \in \mathcal{M}_1$, where $K_2^{\boldsymbol{m}} = 0$ since $D_2(1,1) = D_2(1,0)$. The same argument can apply to unit 1 for the monotone pair $\boldsymbol{m} = ((1,1),(1,0))$. $\square$

3.3.3 Identification without Assumption 3

Lastly, there are special situations where two out of three terms in (7) are cancelled out. Consider a monotone pair $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$ with $D_j(\boldsymbol{z}_j) = D_j(\boldsymbol{z}_j')$ with probability 1. Then we have:

$$K_{ij}^{\boldsymbol{m}} = K_i^{\boldsymbol{m}} D_j(\boldsymbol{z}_j) = K_i^{\boldsymbol{m}} D_j(\boldsymbol{z}_j'). \tag{15}$$

If unit j is almost surely a $\boldsymbol{m}$-never-taker, i.e., when $D_j(\boldsymbol{z}_j) = D_j(\boldsymbol{z}_j') = 1$ with probability 1, then we have $K_j^{\boldsymbol{m}} = K_{ij}^{\boldsymbol{m}} = 0$ with probability 1 from (15). It follows that $P_j^{\boldsymbol{m}} = P_{ij}^{\boldsymbol{m}} = 0$. Therefore, $\delta_i^{\boldsymbol{m}}(0)$ is a single unknown in (7), and directly recovered by the ratio of $E[ITT_i(\boldsymbol{X})]$ and $P_i^{\boldsymbol{m}}$ without Assumption 3. The opposite case occurs when unit j is almost surely a $\boldsymbol{m}$-always-taker, and $K_j^{\boldsymbol{m}} = K_{ij}^{\boldsymbol{m}} = 1$. Proposition 5 summarizes the identification in these cases.

Proposition 5. *Suppose Assumptions 1-2 hold, $P_i^{\boldsymbol{m}} > 0$ for $i \in \{1, 2\}$, and let $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$ be a monotone pair.*

(i) If j is $\boldsymbol{m}$-never-taker with probability 1, i.e., $\Pr(D_j(\boldsymbol{z}_j) = D_j(\boldsymbol{z}'_j) = 0) = 1$, then $\delta_i^{\boldsymbol{m}}(0) = E[\omega^{\boldsymbol{m}} Y_i]/P_i^{\boldsymbol{m}}$, and $\theta_j^{\boldsymbol{m}}(0) = E[\omega^{\boldsymbol{m}} Y_j]/P_i^{\boldsymbol{m}}$.

(ii) If j is $\boldsymbol{m}$-always-taker with probability 1, i.e., $\Pr(D_j(\boldsymbol{z}_j) = D_j(\boldsymbol{z}'_j) = 1) = 1$, then $\delta_i^{\boldsymbol{m}}(1) = E[\omega^{\boldsymbol{m}} Y_i]/P_i^{\boldsymbol{m}}$, and $\theta_j^{\boldsymbol{m}}(1) = E[\omega^{\boldsymbol{m}} Y_j]/P_i^{\boldsymbol{m}}$.

This proposition generalizes the identification of LATEs under one-sided noncompliance and LATE under SUTVA. The following two examples illustrate these situations.

Remark 3 (One-sided Noncompliance)**.** As demonstrated in Table 2, the one-sided noncompliance implies a monotone pair $\boldsymbol{m}_1 = ((1, 0), (0, 0)) \in \mathcal{M}_1$, where unit 2 is a $\boldsymbol{m}_1$-never-taker with probability 1 (since $D_2(0, 1) = D_2(0, 0) = 0$ with probability 1). The same argument can apply to unit 1 for the monotone pair $\boldsymbol{m}_2 = ((0, 1), (0, 0)) \in \mathcal{M}_1$. Therefore, assuming one-sided noncompliance implies that one unit is almost surely an $\boldsymbol{m}$-never-taker. Thus, Proposition 4 generalizes the identification result proposed in Vazquez-Bare (2022), and this is a strict generalization since total monotonicity is not necessary. □

Remark 4 (Identification without Interactions)**.** Suppose there is no interaction in treatment take-up decision as shown in Remark 2. Then, for a monotone pair $\boldsymbol{m} = ((1, 1), (0, 1)) \in \mathcal{M}_1$, we have $K_j^{\boldsymbol{m}} = 0$, and therefore the second term in (7) is zero. Additionally, if there is no spillover in outcomes (i.e., $Y_i(d, 1) = Y_i(d, 0)$ a.s.), then we have (i) $\Delta^2 Y_i = 0$; (ii) $\delta_i^{\boldsymbol{m}}(0) = \delta_i^{\boldsymbol{m}}(1)$; (iii) $\theta_i^{\boldsymbol{m}}(0) = \theta_i^{\boldsymbol{m}}(1) = 0$. Therefore, (i) implies $\zeta_i^{\boldsymbol{m}} = 0$, thus the last term in (7) is zero. (ii) and (iii) imply that Part (i) and (ii) of Proposition 5 are identical, and the resulting local direct effect reduces to the classical LATE estimand proposed in Imbens and Angrist (1994). □

4 Estimation

In this section, I propose a two-stage procedure for estimating direct and indirect local average treatment effects discussed in previous sections. Let $\boldsymbol{V}_g = (\boldsymbol{Y}_g, \boldsymbol{Z}_g, \boldsymbol{D}_g, \boldsymbol{X}_g)$ be observed data for $g = 1, ..., G$. Conditional moment (10) derived from Lemma 2 and Assumption 4 imply an unconditional moment: $E\left[\omega^{\boldsymbol{m}} \boldsymbol{R}(\boldsymbol{T}_g)\left(\boldsymbol{Y}_g - \tilde{\boldsymbol{D}}_g \boldsymbol{\beta}^{\boldsymbol{m}}\right)\right] = 0.$[4,5]

Suppose we have a monotone pair $\boldsymbol{m} \in \mathcal{M}_1$, and I omit the $\boldsymbol{m}$ superscript for simplicity in this section. The optimal choice of 6×2 matrix $\boldsymbol{R}(\cdot)$ of instruments for the efficient IV estimator is $\boldsymbol{R}(\boldsymbol{T}_g) = -\tilde{\boldsymbol{P}}'(\boldsymbol{T}_g)\boldsymbol{S}^{-1}(\boldsymbol{T}_g)$, where $\tilde{\boldsymbol{P}}(\boldsymbol{T}_g) = E[\omega_g \tilde{\boldsymbol{D}}_g | \boldsymbol{T}_g]$, $S(\boldsymbol{T}_g) := E[\omega_g^2 \boldsymbol{\varepsilon}_g \boldsymbol{\varepsilon}_g' | \boldsymbol{T}_g]$, and $\boldsymbol{\varepsilon}_g := \boldsymbol{Y}_g - \tilde{\boldsymbol{D}}_g \boldsymbol{\beta}$.[6] The efficient IV estimator is implemented via a two-stage procedure. In the first-stage, the weight ω_g and the instrument $\boldsymbol{R}(\boldsymbol{T}_g)$ are estimated, and then $\boldsymbol{\beta}$ is estimated in the second stage using the estimated weight and instruments. I consider a parametric first stage estimation. First, the propensity score is estimated using the following parametric model with parameter $\boldsymbol{\gamma} \in \Gamma \subset \mathbb{R}^{k_\gamma}$, and a function $q : (\boldsymbol{z}, \boldsymbol{X}, \boldsymbol{\gamma}) \mapsto (0, 1]$:

$$\Pr(\boldsymbol{Z}_g = \boldsymbol{z} | \boldsymbol{X}_g; \boldsymbol{\gamma}) = q(\boldsymbol{z}, \boldsymbol{X}_g, \boldsymbol{\gamma}).$$

The weight ω corresponding to the parameter γ is defined as:

$$\omega_g(\boldsymbol{\gamma}) = \frac{\mathbb{1}\{\boldsymbol{Z}_g = \boldsymbol{z}\}}{q(\boldsymbol{z}, \boldsymbol{X}_g, \boldsymbol{\gamma})} - \frac{\mathbb{1}\{\boldsymbol{Z}_g = \boldsymbol{z}'\}}{q(\boldsymbol{z}', \boldsymbol{X}_g, \boldsymbol{\gamma})}.$$

Using a consistent estimator $\hat{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$, and some feasible instrument $\tilde{\boldsymbol{R}}(\boldsymbol{T}_g)$, the first-stage IV estimator of $\boldsymbol{\beta}$ is given by:

$$\tilde{\boldsymbol{\beta}} = \left[\frac{1}{G}\sum_{g=1}^{G} \omega_g(\hat{\boldsymbol{\gamma}}) \tilde{\boldsymbol{R}}(\boldsymbol{T}_g) \tilde{\boldsymbol{D}}_g\right]^{-1} \frac{1}{G}\sum_{g=1}^{G} \omega_g(\hat{\boldsymbol{\gamma}}) \tilde{\boldsymbol{R}}(\boldsymbol{T}_g) \boldsymbol{Y}_g.$$

[4]The same argument can be used to derive estimator for the special cases in Proposition 3 and 4, by replacing $\tilde{\boldsymbol{D}}^{\boldsymbol{m}}$ to $\check{\boldsymbol{D}}^{\boldsymbol{m}}$ and $\bar{\boldsymbol{D}}^{\boldsymbol{m}}$, respectively. Estimator corresponding to Proposition 5 is proposed by Vazquez-Bare (2022).

[5]If there are multiple monotone pairs, then we can stack all conditional moment conditions and derive the estimator similarly for efficiency gain.

[6]This choice of instruments is optimal in the sence that it minimizes the asymptotic variance (e.g., Hansen (1985), Chamberlain (1987)).

Let $\tilde{\boldsymbol{\varepsilon}}_g := \boldsymbol{Y}_g - \tilde{\boldsymbol{D}}_g\tilde{\boldsymbol{\beta}}$ be the first stage residual. The optimal matrix of instruments is estimated by the parametric model $\boldsymbol{R}(\boldsymbol{T}_g, \boldsymbol{\phi}) := -\tilde{\boldsymbol{P}}'(\boldsymbol{T}_g, \boldsymbol{\phi})\boldsymbol{S}(\boldsymbol{T}_g, \boldsymbol{\phi})^{-1}$ with the parameter $\boldsymbol{\phi} \in \Phi \subset \mathbb{R}^{k_\phi}$,[7] where $\boldsymbol{S}(\boldsymbol{T}_g, \boldsymbol{\phi}) := E[\omega_g(\hat{\boldsymbol{\gamma}})^2\tilde{\boldsymbol{\varepsilon}}_g\tilde{\boldsymbol{\varepsilon}}_g'|\boldsymbol{T}_g; \boldsymbol{\phi}]$ and $\tilde{\boldsymbol{P}}(\boldsymbol{T}_g, \boldsymbol{\phi}) := E[\omega_g(\hat{\boldsymbol{\gamma}})\tilde{\boldsymbol{D}}_g|\boldsymbol{T}_g; \boldsymbol{\phi}]$ are parametric models for the conditonal means. In the second-stage, the parameter $\boldsymbol{\beta}$ is estimated by:

$$\hat{\boldsymbol{\beta}} = \left[\frac{1}{G}\sum_{g=1}^{G}\omega_g(\hat{\boldsymbol{\gamma}})\boldsymbol{R}(\boldsymbol{T}_g, \hat{\boldsymbol{\phi}})\tilde{\boldsymbol{D}}_g\right]^{-1}\frac{1}{G}\sum_{g=1}^{G}\omega_g(\hat{\boldsymbol{\gamma}})\boldsymbol{R}(\boldsymbol{T}_g, \hat{\boldsymbol{\phi}})\boldsymbol{Y}_g.$$

For the first-stage parameters, I assume they are estimated by asymptotically bounded as stated in Assumption 7.

Assumption 7 (First-Stage)**.** *The first-stage parameter $\boldsymbol{\gamma}$ and $\boldsymbol{\phi}$ are consistently estimated by $\hat{\boldsymbol{\gamma}}$ and $\hat{\boldsymbol{\phi}}$. Specifically, $\hat{\boldsymbol{\gamma}}$ satisfies*

$$\sqrt{G}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0) = \frac{1}{\sqrt{G}}\sum_{g=1}^{G}\psi_\gamma(\boldsymbol{V}_g, \boldsymbol{\gamma}_0) + o_p(1), \quad E\left[\|\psi_\gamma(\boldsymbol{V}_g, \boldsymbol{\gamma}_0)\psi_\gamma(\boldsymbol{V}_g, \boldsymbol{\gamma}_0)'\|\right] < \infty,$$

with $E[\psi_\gamma(\boldsymbol{V}_g, \boldsymbol{\gamma}_0)] = 0$.

As noted by Wooldridge (2010), the first-stage estimation of instrument does not affect the asymptotic variance of the second-stage estimation of $\boldsymbol{\beta}$. However, the asymptotic variance need to be adjusted to account for the first-stage estimation error of $\Pr(\boldsymbol{Z}_g = \boldsymbol{z}|\boldsymbol{X}_g)$. Assumption 8 in Appendix A.2 lists the required regularity conditions for consistency and asymptotic normality of the estimator $\hat{\boldsymbol{\beta}}$, and Proposition 6 states the asymptotic properties of the proposed estimator.

Proposition 6. *Under Assumptions 1-4, 7, and regularity conditions (Assumption 8), the two-stage estimator $\hat{\boldsymbol{\beta}}$ is a consistent estimator for $\boldsymbol{\beta}_0$, and*

$$\sqrt{G}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \xrightarrow{d} N(0, \boldsymbol{\Sigma}),$$

[7]The parameter $\boldsymbol{\phi}$ includes the parameters in the conditional expectations for optimal instrument as well as first-stage estimators $\hat{\boldsymbol{\gamma}}$, and $\tilde{\boldsymbol{\beta}}$.

where $\boldsymbol{\Sigma} = E\left[\psi_\beta(\boldsymbol{V}_g, \boldsymbol{\beta}_0)\psi_\beta(\boldsymbol{V}_g, \boldsymbol{\beta}_0)'\right]$*, and*

$$\begin{aligned}
\psi_\beta(\boldsymbol{V}_g, \boldsymbol{\beta}_0) &= \boldsymbol{A}^{-1}\left[\omega_g \boldsymbol{R}(\boldsymbol{T}_g)(\boldsymbol{Y}_g - \tilde{\boldsymbol{D}}_g\boldsymbol{\beta}_0) + \boldsymbol{B}\psi_\gamma(\boldsymbol{V}_g, \boldsymbol{\gamma}_0)\right], \\
\boldsymbol{A} &= E\left[\tilde{\boldsymbol{P}}'(\boldsymbol{T}_g)\boldsymbol{S}^{-1}(\boldsymbol{T}_g)\tilde{\boldsymbol{P}}(\boldsymbol{T}_g)\right], \\
\boldsymbol{B} &= E\left[\boldsymbol{R}(\boldsymbol{T}_g)(\boldsymbol{Y}_g - \tilde{\boldsymbol{D}}_g\boldsymbol{\beta}_0)\frac{\partial \omega_g(\boldsymbol{\gamma}_0)}{\partial \gamma'}\right].
\end{aligned}$$

Furthermore, the plug-in estimator of asymptotic variance is consistent:

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{G}\sum_{g=1}^{G} \hat{\psi}_\beta(\boldsymbol{V}_g, \hat{\boldsymbol{\beta}})\hat{\psi}_\beta(\boldsymbol{V}_g, \hat{\boldsymbol{\beta}})' \xrightarrow{p} \boldsymbol{\Sigma},$$

where $\hat{\psi}_\gamma$ *is the empirical influence function of* ψ_γ*, and,*

$$\begin{aligned}
\hat{\psi}_\beta(\boldsymbol{V}_g, \hat{\boldsymbol{\beta}}) &= \hat{\boldsymbol{A}}^{-1}\left[\omega_g(\hat{\boldsymbol{\gamma}})\boldsymbol{R}(\boldsymbol{T}_g, \hat{\boldsymbol{\phi}})(\boldsymbol{Y}_g - \tilde{\boldsymbol{D}}_g\hat{\boldsymbol{\beta}}_0) + \hat{\boldsymbol{B}}\psi_\gamma(\boldsymbol{V}_g, \hat{\boldsymbol{\gamma}})\right], \\
\hat{\boldsymbol{A}} &= \frac{1}{G}\sum_{g=1}^{G} \omega_g(\hat{\boldsymbol{\gamma}})\boldsymbol{R}(\boldsymbol{T}_g, \hat{\boldsymbol{\phi}})\tilde{\boldsymbol{D}}_g, \\
\hat{\boldsymbol{B}} &= \frac{1}{G}\sum_{g=1}^{G} \boldsymbol{R}(\boldsymbol{T}_g, \hat{\boldsymbol{\phi}})(\boldsymbol{Y}_g - \tilde{\boldsymbol{D}}_g\hat{\boldsymbol{\beta}})\frac{\partial \omega_g(\hat{\boldsymbol{\gamma}})}{\partial \boldsymbol{\gamma}'}.
\end{aligned}$$

5 Simulation

This section examines the estimation procedure described in Section 4 using simulations. Three distinct designs are established based on the restrictions on potential treatment discussed in Section 2.2.2, including total monotonicity, one-sided noncompliance, and personalized encouragement. Additionally, a design is considered where only weak one-sided noncompliance (as discussed in Remark 1) holds, but none of the restrictions from the previous three designs are satisfied.

5.1 Data Generating Process and Designs

First, exogenous variables $X_i = (T_i, W_i)$ are generated by uniform and standard normal distributions, respectively: $T_i \sim \text{Uniform}(0,1)$, and $W_i \sim N(0,1)$ for $i \in \{1,2\}$. Treatment assignments, take-ups, and outcomes are generated as follows.

5.1.1 Treatment Assignments, Take-up and Designs

Treatment assignment and potential treatments are generated by the following binary response models:

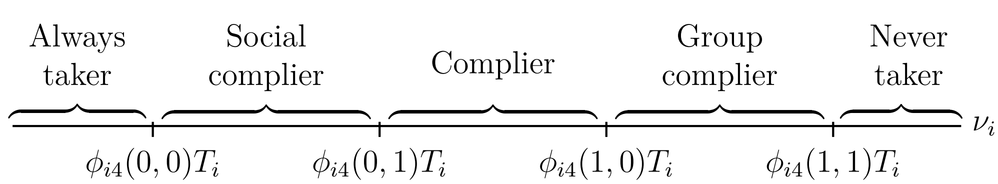
$$Z_i = \mathbb{1}\{\eta_i \leq \gamma_{i1} + W_i\gamma_{i2} + W_j\gamma_{i3} + T_i\gamma_{i4} + T_j\gamma_{i5}\}, \tag{16}$$

$$D_i(z_i, z_j) = \mathbb{1}\{\nu_i \leq \phi_{i1}(z_i, z_j) + \phi_{i2}(z_i, z_j)W_i + \phi_{i3}(z_i, z_j)W_j + \phi_{i4}(z_i, z_j)T_i\}, \tag{17}$$

where η_i and ν_i are standard normal error terms. I set $\gamma_{i0} = 1$ and $\gamma_{i2} = \gamma_{i3} = \gamma_{i4} = \gamma_{i5} = 0.1$ for both units $i = 1, 2$. Therefore, the treatment is randomly assigned conditional on both units' exogenous characteristics $\boldsymbol{X}$. Then, I consider four designs based on the values of ϕ, which determine the distribution of potential treatment and compliance types.

Design 1 represents a situation where total monotonicity holds. If $\phi_{i1}(\boldsymbol{z})$, $\phi_{i2}(\boldsymbol{z})$, and $\phi_{i3}(\boldsymbol{z})$ are fixed for all $\boldsymbol{z} \in \{0,1\}^2$, then the ordering of potential treatments follows from the ordering on $\phi_{i4}(\boldsymbol{z})$. Specifically, if $\phi_{i4}(1,1) \geq \phi_{i4}(1,0) \geq \phi_{i4}(0,1) \geq \phi_{i4}(0,0)$ for both units, then the DGP satisfies total monotonicity. Compliance types in this case are summarized in Table 2, and they are classified by the realization of ν_1, as illustrated in Figure 3.

Figure 3: Five Compliacne Types under Total Monotonicity



Design 2 represents a situation where both total monotonicity and one-sided non-compliance (OSN) hold. OSN is implemented by setting both $D_i(0,1) = D_i(0,0) = 0$

Table 4: Coefficients for Designs

Design	(z_i, z_j)	Unit 1 ϕ_{11}	ϕ_{12}	ϕ_{13}	ϕ_{14}	Unit 2 ϕ_{21}	ϕ_{22}	ϕ_{23}	ϕ_{24}
1 (TM)	$(1, 1)$	-1	0.1	0.1	7	-1.5	0.2	0.2	8
	$(1, 0)$	-1	0.1	0.1	4	-1.5	0.2	0.2	4
	$(0, 1)$	-1	0.1	0.1	1	-1.5	0.2	0.2	2
	$(0, 0)$	-1	0.1	0.1	0	-1.5	0.2	0.2	0
2 (TM+OSN)	$(1, 1)$	-1	0.1	0.1	7	-1.5	0.2	0.2	8
	$(1, 0)$	-1	0.1	0.1	4	-1.5	0.2	0.2	4
	$(0, 1)$	-1000	0.1	0.1	1	-1000	0.2	0.2	2
	$(0, 0)$	-1000	0.1	0.1	0	-1000	0.2	0.2	0
3 (PE+CM)	$(1, 1)$	-1	0.1	0.1	7	-1.5	0.2	0.2	8
	$(1, 0)$	-1	0.1	0.1	7	-1.5	0.2	0.2	8
	$(0, 1)$	-1	0.1	0.1	1	-1.5	0.2	0.2	2
	$(0, 0)$	-1	0.1	0.1	1	-1.5	0.2	0.2	2
4 (WOSN)	$(1, 1)$	-1	0.2	0.1	7	-1.5	0.4	0.2	8
	$(1, 0)$	-1	0.1	0.2	4	-1.5	0.2	0.4	4
	$(0, 1)$	-1	0.2	0.2	1	-1.5	0.4	0.4	2
	$(0, 0)$	-1000	0.1	0.1	0	-1000	0.2	0.2	0

Notes: Potential treatment is generated by (17). Design 1 satisfies total monotonicity (TM). Design 2 satisfies total monotonicity (TM) and one-sided noncompliance (OSN). Design 3 satisfies personalized encouragement (PE) and the classical monotonicity (CM). Design 4 satisfies weak one-sided noncompliance (WOSN).

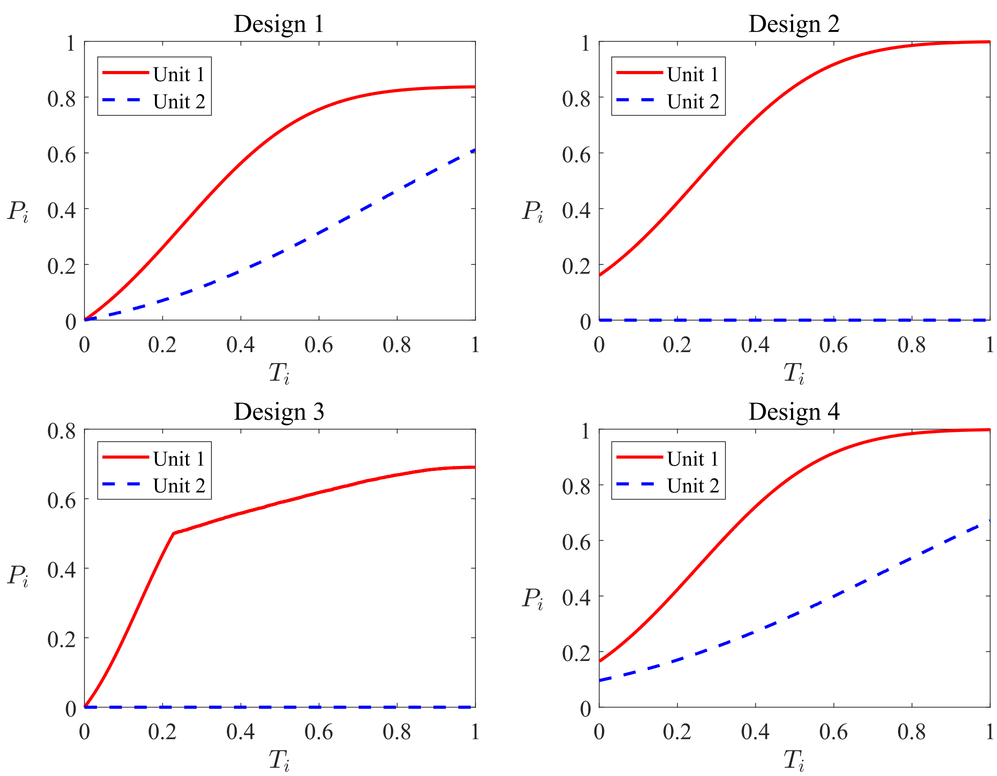
with probability 1. This is numerically implemented by setting the constant term $\phi_{i1}(\boldsymbol{z})$ to a large negative number for all $\boldsymbol{z} \in \{0, 1\}^2$.

Design 3 represents a situation under personalized encouragement ($D_i(d, 1) = D_i(d, 0)$ for $d \in \{0, 1\}$) with classical monotonicity ($D_i(1, \cdot) \geq D_i(0, \cdot)$). Personalized encouragement is implemented by $\phi_{ik}(z_i, 0) = \phi_{ik}(z_i, 1)$ for all k, for the potential treatment not depend on the other's treatment assignment. Classical monotonicity is implied by setting $\phi_{i4}(1, \cdot) \geq \phi_{i4}(0, \cdot)$ for both units $i = 1, 2$.

Design 4 represents a situation where neither total monotonicity, one-sided noncompliance, nor personalized encouragement are satisfied. In this design, $\phi_{ik}(\boldsymbol{z})$ for $k = 1, 2, 3$ are neither fixed nor satisfy certain monotonicity. Thus, the almost sure ordering of potential treatments is not guaranteed. Here, $D_i(0, 1)$ can be 1 with positive probability, and hence one-sided noncompliance does not hold. The only restriction on potential treatments is $D_i(0,0) = D_j(0,0) = 0$ with probability 1. Consequently, this design represents weak one-sided noncompliance.

Table 4 shows the coefficient values for each design. All four designs have two common monotone pairs $\boldsymbol{m}_1 = ((1,0),(0,0))$ and $\boldsymbol{m}_2 = ((0,1),(0,0))$ in $\mathcal{M}_1$. In the simulations, I focus on the monotone pair $\boldsymbol{m}_1$ only. Figure 4 describes the distribution of $\boldsymbol{m}_1$-compliers as function of T_i for each design.

Figure 4: Distributions of $\boldsymbol{m}_1$-compliers



Notes: Each plot describes $P_i := \Pr(K_i^{\boldsymbol{m}_1} = 1 | T_i) = \Pr(D_i(\boldsymbol{z}_i) > D_i(\boldsymbol{z}_i') | T_i)$, where $\boldsymbol{m}_1 = ((1,0),(0,0))$ (i.e., $\boldsymbol{z}_1 = (1,0), \boldsymbol{z}_2 = (0,1), \boldsymbol{z}_1' = \boldsymbol{z}_2' = (0,0)$). For design 2 and 3, $D_2(0,1) = D_2(0,0)$ with probability 1. Hence, $P_2 = \Pr(K_2^{\boldsymbol{m}_1} = 1 | \boldsymbol{T}_i) = 0$.

5.1.2 Outcomes

To consider the heterogeneity of potential outcomes in different compliance types, with respect to the $\boldsymbol{m}_1$-monotone pair, the potential outcome is generated by $Y_i(d_i, d_j) =$

$Y_i^D(d_i) + Y_i^I(d_j) + W_i + 0.5W_j$, where $Y_i^p(d)|K_i^{\boldsymbol{m}_1} \sim N(K_i^{\boldsymbol{m}_1}\bar{Y}_i^p(d), 1)$, for $p \in \{D, I\}$, $i \in \{1,2\}$, and $d \in \{0,1\}$. This data-generating process implies that the potential outcome is additively separable with respect to the potential treatment statuses of two units. This additive separability guarantees that $\Delta_i Y_i(1) = \Delta_i Y_i(0)$, or, $\Delta^2 Y_i = 0$. Hence, the direct and indirect effects do not depend on the other unit's treatment take-up status, and the coefficient $\zeta_i^{\boldsymbol{m}}$ in the last term in (7) and (8) is zero.[8]

For designs 1 and 4, the estimator based on the conditional moment (10) is used to estimate $\delta_i^{\boldsymbol{m}_1}(0), \theta_i^{\boldsymbol{m}_1}(0)$ for $i \in \{1,2\}$. For design 2, unit 2 is $\boldsymbol{m}_1$-never-taker with probability 1, i.e., $D_2(\boldsymbol{z}_2) = D_2(\boldsymbol{z}_2') = 0$. Therefore, $\delta_1^{\boldsymbol{m}_1}(0)$ and $\theta_2^{\boldsymbol{m}_1}(0)$ are identitifed by Proposition 5. For design 3, $D_2(\boldsymbol{z}_2) = D_2(\boldsymbol{z}_2')$, and therefore $\delta_1^{\boldsymbol{m}_1}(0)$ and $\theta_2^{\boldsymbol{m}_1}(0)$ are identitifed by Proposition 4. Thus, for designs 2 and 3, I estimate those identified parameters using the same estimation procedure, applying the moment conditions (14), (13), respectively.

5.2 Simulation Results

Table 5 presents the simulation results for Design 4. See Tables B.1-B.3 in the Appendix B.1 for the results of Designs 1-3. The first two columns use true propensity scores (corresponding to the true γs), while the last two columns estimate propensity scores in the first stage. "Probit" and "Linear" refer to estimating the optimal instrument ($P_i^{\boldsymbol{m}_1}(\boldsymbol{T})$) using the probit model and linear probability model, respectively.

Design 4 is a case where neither total monotonicity nor one-sided noncompliance hold. In this case, the mean squared error (MSE) of the estimators decreases in proportion to G^{-1} for all four methods, which verifies consistency as stated in Proposition 6. Looking at the coverage rate, inference based on plug-in standard error appears valid. It is more efficiently estimated when propensity scores are estimated rather than using actual propensity scores, as the MSE is smaller in the last two columns compared to the first two. This aligns with previous findings in the literature (e.g., Hahn (1998), Hirano, Imbens, and Ridder (2003)). Since the estimation of the instrument does not affect the limiting distribution, there is no significant difference between the results using probit

[8]This restriction is not necessary to simulate the estimation procedure. See table (B.7) in the Appendix B.2 for the result from a design that does not satisfy additive separability of outcomes.

Table 5: Simulation of Design 4

		Use true ω		Estimate ω	
	G	Linear	Probit	Linear	Probit
MSE	200	6439.37	320433.95	5123.69	39365.94
	400	1034.09	1424.85	729.9	813.82
	500	776.92	862.87	563.95	609.16
	800	434.99	450.69	318.93	343.06
	1,000	359.97	365.36	260.69	260.47
MAE	200	68.75	110.27	61.47	76.27
	400	44.87	47.77	37.84	39.02
	500	39.43	41.09	33.58	34.42
	800	30.29	30.68	25.82	26.05
	1,000	27.55	27.57	23.41	23.31
Coverage	200	0.95	0.91	0.94	0.91
	400	0.94	0.92	0.94	0.92
	500	0.95	0.93	0.95	0.93
	800	0.95	0.94	0.95	0.94
	1,000	0.95	0.94	0.95	0.94

Notes: This table presents simulation results for B = 10,000 replications. Column G denotes the number of independent groups. The mean squared error (MSE) is calculated by $\sum_{b=1}^{B}\|\hat{\beta}_b - \beta_0\|^2/B$, where $\hat{\beta}_b = (\hat{\delta}_1(0), \hat{\theta}_1(0), \hat{\delta}_2(0), \hat{\theta}_2(0))'$ is the vector of estimates in the b^{th} replication, and $\beta_0 = (\delta_1(0), \theta_1(0), \delta_2(0), \theta_2(0))'$ is the true vector of parameters. The actual parameter values are set as $\delta_1(0) = 20$, $\theta_1(0) = 10$, $\delta_2(0) = 30$, and $\theta_2(0) = 15$. The mean absolute error (MAE) is calculated by $\sum_{b=1}^{B}\sum_{k=1}^{4}|\hat{\beta}_{kb} - \beta_{0k}|/(4B)$. Coverage computes the minimum 95% coverage rate among the four estimates, i.e., $\min_{1\leq k\leq 4}\sum_{b=1}^{B}\mathbb{1}\{\hat{\beta}_{kb} - 1.96SE(\hat{\beta}_{kb}) \leq \beta_{0k} \leq \hat{\beta}_{kb} + 1.96SE(\hat{\beta}_{kb})\}/B$. The first two columns ("Use true ω") use the true propensity score $\Pr(\boldsymbol{Z}_i = \boldsymbol{z}|\boldsymbol{X})$ for the weight ω, while the last two columns ("Estimate ω") estimate the propensity score and hence ω. "Linear" and "Probit" denote that the optimal instrument $P_i^{\boldsymbol{m_1}}(\boldsymbol{T})$ is estimated by the linear probability model and probit model, respectively.

and linear models for estimating $P_i^{\boldsymbol{m}}(\boldsymbol{T})$. However, as the linear model appears slightly more efficient, I report the results of the estimator with first-stage propensity score estimation and a linear probability model for estimating instruments in all subsequent simulation analyses.

Table 6 provides more detailed information, including the mean, median, MSE, and coverage rate for each parameter in Design 4. It shows that both the mean and median converge to the actual values, and coverage rate converges to the correct size. See

Tables B.4-B.5 in the Appendix B.1 for the same results of Designs 1-3.

Table 6: Simulation of Design 4 for Each Parameter

		Design 4			
	G	$\delta_1(0)$	$\theta_1(0)$	$\delta_2(0)$	$\theta_2(0)$
Mean	200	19.8	9.82	29	14.99
	400	19.73	10.46	30.08	14.84
	500	19.87	10.05	29.79	14.94
	800	19.92	9.91	29.81	14.92
	1,000	19.95	9.99	29.71	15.05
Median	200	20.19	11.19	33.94	14.52
	400	19.93	10.84	32.28	14.55
	500	20.12	10.39	31.36	14.79
	800	20.05	10.11	30.79	14.85
	1,000	20.05	10.29	30.74	14.89
MSE	200	249.59	1220.86	3069.35	583.89
	400	49.16	200.53	398.22	81.99
	500	38.37	156.22	306.9	62.46
	800	21.9	86.9	173.52	36.61
	1,000	17.44	71.56	142.63	29.07
Coverage	200	0.97	0.98	0.94	0.98
	400	0.97	0.97	0.94	0.97
	500	0.96	0.97	0.95	0.97
	800	0.96	0.96	0.95	0.96
	1,000	0.96	0.96	0.95	0.96

Notes: This table presents simulation results for $B = 10{,}000$ replications. Column G denotes the number of groups. Mean is $\sum_{b=1}^{B} \hat{\beta}_b/B$, where $\hat{\beta}_b = (\hat{\delta}_1(0), \hat{\theta}_1(0), \hat{\delta}_2(0), \hat{\theta}_2(0))'$ is the vector of estimates in b^{th} replication, and $\beta_0 = (\delta_1(0), \theta_1(0), \delta_2(0), \theta_2(0))' = (20, 10, 30, 15)'$ is the true values of parameters. Med is the median of $\hat{\beta}_b$ among B simulated estimates. The mean squared error (MSE) is calculated by $\sum_{b=1}^{B} \|\hat{\beta}_b - \beta_0\|^2/B$. Coverage computes the minimum 95% coverage rate among the four estimates, i.e., $\min_{1\leq k\leq 4} \sum_{b=1}^{B} \mathbb{1}\{\hat{\beta}_{kb} - 1.96SE(\hat{\beta}_{kb}) \leq \beta_{0k} \leq \hat{\beta}_{kb} + 1.96SE(\hat{\beta}_{kb})\}/B$.

Table 7 compares MSE, mean absolute errror (MAE), and coverage rates of all four designs. It illustrates that the proposed estimators perform well under all four designs based on various restrictions on potential treatment as the MSEs decrease in proportion to G^{-1}, and the coverage rates seem to converge to 95%.

For Design 3, the parameters are identified by Proposition 4. However, since the coefficient ζ_i of the interaction term is zero by construction, the parameters in Design 3

Table 7: Simulation of Designs 1-4

	G	Design 1	Design 2	Design 3	Design 4
MSE	200	3094.09	31.01	105.09	5123.69
	400	741.76	14.18	46.45	729.9
	500	679.73	11.34	38.01	563.95
	800	288.04	6.97	22.48	318.93
	1,000	225.69	5.52	17.49	260.69
MAE	200	58.62	6.17	11.02	61.47
	400	37.05	4.18	7.47	37.84
	500	32.31	3.74	6.79	33.58
	800	24.53	2.94	5.26	25.82
	1,000	21.95	2.61	4.65	23.41
Coverage	200	0.97	0.93	0.95	0.94
	400	0.97	0.94	0.95	0.94
	500	0.96	0.94	0.95	0.95
	800	0.96	0.94	0.95	0.95
	1,000	0.96	0.94	0.95	0.95

Notes: This table presents simulation results for B = 10,000 replications. Column G denotes the number of independent groups. The mean squared error (MSE) is calculated by $\sum_{b=1}^{B} \|\hat{\beta}_b - \beta_0\|^2/B$, where $\hat{\beta}_b = (\hat{\delta}_1(0), \hat{\theta}_1(0), \hat{\delta}_2(0), \hat{\theta}_2(0))'$ is the vector of estimates in the b^{th} replication, and $\beta_0 = (\delta_1(0), \theta_1(0), \delta_2(0), \theta_2(0))'$ is the true vector of parameters. The actual parameter values are set as $\delta_1(0) = 20$, $\theta_1(0) = 10$, $\delta_2(0) = 30$, and $\theta_2(0) = 15$. The mean absolute error (MAE) is calculated by $\sum_{b=1}^{B} \sum_{k=1}^{4} |\hat{\beta}_{kb} - \beta_{0k}|/(4B)$. Coverage computes the minimum 95% coverage rate among the four estimates, i.e., $\min_{1 \leq k \leq 4} \sum_{b=1}^{B} \mathbb{1}\{\hat{\beta}_{kb} - 1.96SE(\hat{\beta}_{kb}) \leq \beta_{0k} \leq \hat{\beta}_{kb} + 1.96SE(\hat{\beta}_{kb})\}/B$. For all designs, the propensity score and the weight ω is estimated by probit model, and the optimal instrument $P_i^{\boldsymbol{m}_1}$ is estimated by linear probability model.

can also be identified using the ratio of the ITT effect and the distribution of compliance types, similar to those in Design 2, which is identified by Proposition 5. In these cases, for the monotone pair $\boldsymbol{m}_1$, the direct effects for unit 1 and indirect effects for unit 2 can also be estimated using the IV estimator proposed by Vazquez-Bare (2022).

Table 8 compares the mean absolute biases (MAE) of two estimators. The estimator denoted by R is the estimator proposed in Section 4. The estimator denoted by V is the estimator proposed by Vazquez-Bare (2022), which is equivalent to the IV estimator for the estimands in Proposition 5. It is noteworthy that for Designs 1 and 4, where total monotonicity or one-sided noncompliance is violated, the V estimator is no longer valid.

Indeed, the results for designs 1 and 4 reveal the bias of the V estimator. In Design 1, the bias arises from the violation of one-sided noncompliance, while in Design 4, it reflects bias from the violation of both total monotonicity and one-sided noncompliance. For Designs 2 and 3, both estimators R and V behave quite similarly, and the biases seem to converge to zero.

Table 8: Simulation of Designs and Bias When (TM) and (OSN) Are Violated

		Design 1		Design 4		Design 2		Design 3	
	G	$\delta_1(0)$	$\theta_2(0)$	$\delta_1(0)$	$\theta_2(0)$	$\delta_1(0)$	$\theta_2(0)$	$\delta_1(0)$	$\theta_2(0)$
R	200	0.44	0.04	0.2	0.01	0.24	0.19	0.7	0.53
	400	0.13	0.04	0.27	0.16	0.12	0.09	0.25	0.19
	500	0.17	0.07	0.13	0.06	0.1	0.08	0.13	0.1
	800	0.03	0.04	0.08	0.08	0.07	0.05	0.09	0.07
	1,000	0.11	0.12	0.05	0.05	0.04	0.03	0.11	0.08
	2,000	0.07	0.04	0	0.04	0.04	0.03	0.05	0.04
	5,000	0	0.02	0.01	0	0.02	0.02	0	0
V	200	4.63	13.97	4.47	14.07	0.42	0.31	0.61	0.45
	400	4.67	13.94	4.7	14.32	0.18	0.14	0.28	0.21
	500	4.68	14.04	4.71	14.4	0.15	0.11	0.27	0.2
	800	4.7	14.05	4.69	14.32	0.08	0.06	0.14	0.1
	1,000	4.57	13.82	4.75	14.38	0.07	0.05	0.07	0.05
	2,000	4.64	13.98	4.78	14.39	0.02	0.01	0.03	0.02
	5,000	4.66	13.98	4.79	14.39	0	0	0.04	0.03

Notes: This table presents simulation results for $B = 10{,}000$ replications. Column G denotes the number of groups. Each column shows mean absolute bias that is computed by $\frac{1}{B}\sum_{b=1}^{B}|\hat{\beta}_{kb} - \beta_{0k}|$ for each parameters. R denotes the proposed estimator using the additional exclusion restriction $\boldsymbol{T}$. V denotes the IV estimator proposed by Vazquez-Bare (2022), with inverse propensity score weighting to account for the conditioning covariates. To compare estimates for the two estimators, only direct effects for unit 1 ($\delta_1(0)$) and indirect effects for unit 2 ($\theta_2(0)$) are reported. The actual parameter values are $\delta_1(0) = 20$ and $\theta_2(0) = 15$. Designs 2 and 3 satisfy both total monotonicity (TM) and one-sided noncompliance (OSN). Design 1 only satisfies (TM). Neither condition is satisfied in Design 4.

6 Empirical Illustration

In this section, I apply the proposed estimation method to data from the study conducted by Dupas, Keats, and Robinson (2017) and Dupas, Keats, and Robinson (2019). The dataset involves a randomized experiment carried out in the rural areas of Kenya's

Busia District in the Western Province from 2009 to 2012. In the sampled areas, banking services are limited because banks are primarily located in major towns, and opening a savings account incurs some costs. As a result, most people did not have savings accounts at the beginning of the experiment and typically kept their cash at home. The experiment aimed to assess the impact of providing a savings account on economic behaviors. The main findings in Dupas, Keats, and Robinson (2019) indicate that access to savings accounts reduces households' dependence on others in their financial network. The sample consists of 885 households, including 399 female-headed and 486 dual-headed households. In this section, only dual-headed households are analyzed to consider the interactions between spouses.

Let individuals 1 and 2 denote the female and male household heads, respectively. The experiment randomly distributed non-transferable vouchers for opening free savings accounts to individuals based on their region (around three market centers) and occupation. The treatment assignment ($\boldsymbol{Z}$) is defined as 1 if the individual received the voucher, and 0 if they did not. The first round of the experiment began in 2010, and vouchers were offered after this round. However, because most individuals had not yet opened an account by round 2, I consider rounds 3-6 as post-treatment periods. Treatment take-up ($\boldsymbol{D}$) is defined as 1 if the individual opens a savings account by redeeming the voucher, and 0 if they do not. Thus, by the construction of the treatment take-up, this experiment satisfies weak one-sided noncompliance, as if both spouses do not have the voucher, then neither can open the account.

Table 9: Treatment Assignments and Take-up

			Take-up			
Female	Male	# of HH	Both	Female only	Male only	None
Treatment	Treatment	150 (33.6%)	78	19	26	27
Treatment	Control	117 (26.2%)	9	68	0	40
Control	Treatment	104 (23.3%)	3	0	68	33
Control	Control	76 (17%)	0	0	0	76
Total		447	90	87	94	176

Notes: First two columns show the treatment assignment statuses of both individuals. Out of the 486 dual-headed households, one household is excluded due to non-response in the first round. Consequently, the total number of households with both individuals treated is 161, only the female treated is 126, only the male treated is 116, and neither treated is 82.

Table 9 summarizes the number of households in different treatment assignments and their treatment take-up statuses. The treatment was randomized at the individual level, and among 447 households available at round 3, 76 (17%) received no vouchers, 150 (33.6%) received vouchers for both individuals, 117 (26.2%) received vouchers for females only, and 104 (23.3%) received vouchers for males only. Individuals who received the voucher could open an account for free. They also had the option to open a joint account with their spouse. Thus, one-sided noncompliance was not satisfied in this experiment. For instance, an individual without a voucher could open an account if their spouse had the voucher and wanted to open the account jointly. In total, 521 individuals received the voucher, and the redemption rate for the vouchers was 69.3%. Among households where only one individual was treated, 5.4% opened joint accounts.

The weak one-sided noncompliance in this experiments implies two monotone pairs: $\boldsymbol{m}_1 = ((1,0),(0,0))$ and $\boldsymbol{m}_2 = ((0,1),(0,0))$ in $\mathcal{M}_1$.[9] Similar to the classification of compliance types in Table 2, I define each individual as a *complier* if $D_i(1,0) \geq D_i(0,0)$ with probability 1, and a *social complier* if $D_i(0,1) \geq D_i(0,0)$ with probability 1. Therefore, if the female is an $\boldsymbol{m}_1$-complier, she is a complier; if the male is an $\boldsymbol{m}_1$-complier, he is a social complier. Thus, the $\boldsymbol{m}_1$-direct effect for the female represents the direct local average treatment effect when she is a complier, and the $\boldsymbol{m}_1$-indirect effect represents the indirect local average effect when her spouse is a social complier. A similar interpretation applies to $\boldsymbol{m}_2$. Table 10 shows the estimated distribution of being each $\boldsymbol{m}$-compliance type.

Table 10: Distribution of Compliance Types

	$\boldsymbol{m}_1$		$\boldsymbol{m}_2$	
	Complier	Never-Taker	Complier	Never-Taker
Female	0.59	0.41	0.03	0.97
Male	0.05	0.95	0.68	0.32

Notes: This table shows the probability of unit i being an $\boldsymbol{m}$-complier and an $\boldsymbol{m}$-never taker. For each monotone pair, there is no $\boldsymbol{m}$-always-taker because $D_1(0,0) = D_2(0,0) = 0$ with probability 1 in this experiment. Therefore, the probability of being a $\boldsymbol{m}$-complier is $P_i := \Pr(K_i^{\boldsymbol{m}} = 1)$, and that of being a never-taker is $1 - P_i$.

[9] As noted in Remark 1, since $D(0,0) = 0$, the experiment implies $D(1,0) \geq D(0,0)$ and $D(0,1) \geq D(0,0)$ for both individuals with probability 1.

For individual-level outcomes, Dupas, Keats, and Robinson (2019) find that providing the vouchers has a significant positive intention-to-treat effect on making deposits and withdrawals. Hence, I use these two extensive margin responses as outcome variables in this application. The authors also pointed out that the values of animals and durable goods significantly determine treatment take-up, based on regression analysis.

Table 11 replicates the regressions for treatment take-up and individual-level outcomes. It also shows that the values of animals and durable goods ('value') are significant determinants of opening a savings account. Additionally, I find that participation in a qualitative survey ('sqs$_i$'), conducted after round 1, and the housing index ('h-index'), which indicates if the walls are cement or the roofs are iron or the floors are cement, also determine treatment take-up but do not affect the outcomes of interest. These variables are used as exclusion restrictions for treatment take-up, and hence for the distribution of $\boldsymbol{m}$-compliers.

Table 11: Determinants of Treatment Take-up

Variable	D_1	D_2	D_1D_2	Deposit$_1$	Deposit$_2$	Withdrawal$_1$	Withdrawal$_2$
sqs$_1$	-0.045	0.076*	0.024	-0.002	0.01	-0.003	-0.004
	(0.03)	(0.032)	(0.025)	(0.016)	(0.019)	(0.007)	(0.012)
sqs$_2$	0.038	0.147***	0.058*	-0.008	0.011	0.011	0.009
	(0.03)	(0.031)	(0.025)	(0.016)	(0.018)	(0.007)	(0.012)
sqs$_1$×sqs$_2$	0.099*	-0.244***	-0.037	0.023	-0.002	-0.004	0.001
	(0.043)	(0.044)	(0.035)	(0.023)	(0.026)	(0.009)	(0.017)
h-index	-0.245***	-0.106*	-0.14***	0.009	0.049	-0.01	0.022
	(0.044)	(0.045)	(0.036)	(0.024)	(0.027)	(0.01)	(0.017)
value	0.031**	0.034**	0.032***	0.008	0.006	0.002	0.005
	(0.012)	(0.012)	(0.01)	(0.006)	(0.007)	(0.003)	(0.005)
N	1,885	1,885	1,885	1,885	1,885	1,885	1,885

Notes: Control variables include age, education, region, indicators of membership in ROSCA, indicators of using mobile money, household size, and round-fixed effects for both the female and male heads. Additionally, the dependent variable at the baseline period is controlled for in the last four columns to replicate the specification used in Dupas, Keats, and Robinson (2019). *sqs* is the indicator of participation in the qualitative survey conducted after round 1. *h-index* indicates the type of residence. *value* represents the value of animals and durable goods. Standard errors are reported in parentheses. *,**,*** denote the significance levels at 10%, 5%, and 1%, respectively.

Table 12 and Table 13 present the estimation of direct and indirect local average treatment effects corresponding to each monotone pair, $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$. The columns labeled "V" use the V estimator, which is the IV estimator corresponding to the identification in Proposition 4. For the monotone pair $\boldsymbol{m}_1$, the direct effect for unit 1 and the indirect effect for unit 2 are estimable, while it is the opposite for the monotone pair $\boldsymbol{m}_2$. The column labeled "R" uses the IV estimator proposed in Section 4 using additional exclusion restrictions as listed above. Age, education, market center, and mobile money usage for both males and females are used as control variables.

Table 12: Make Deposit

		$\boldsymbol{m}_1$		$\boldsymbol{m}_2$	
Unit	Effects	R	V	R	V
Female	Direct	0.11^{**}	0.08^{**}	-0.25	
		(0.05)	(0.03)	(0.22)	
	Indirect	-0.35		0.03	-0.002
		(0.36)		(0.03)	(0.03)
Male	Direct	0.43		0.12^{***}	0.1^{***}
		(0.4)		(0.05)	(0.04)
	Indirect	-0.06	0.001	-0.06	
		(0.05)	(0.03)	(0.53)	

Notes: The dependent variable is 1 if the individual made at least one deposit. "Direct" denotes the local average treatment effects $E[Y_i(1,0) - Y_i(0,0)|K_i^{\boldsymbol{m}} = 1]$, and "Indirect" denotes $E[Y_i(0,1) - Y_i(0,0)|K_j^{\boldsymbol{m}} = 1]$. Plug-in clustered standard errors are reported in parentheses. $^{*},^{**},^{***}$ denote the significance levels at 10%, 5%, and 1%, respectively.

The identification in Proposition 4 is valid under a one-sided noncompliance situation. Therefore, the V estimation would be biased as one-sided noncompliance does not hold in the experiment. However, as only about 5% opened the joint account, the extent of the violation of one-sided noncompliance is small. This results in both estimations being similar in Table 12 and Table 13 for estimable parameters. The instrument is estimated by linear probability model.[10]

By using two monotone pairs $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, we can infer that the outcomes for both female and male heads primarily stem from direct effects. From the R-estimation, the effects are about an 11% increase for making deposits and a 6% increase for making

[10]See Table B.8 and Table B.9 in Appendix B.3 for the probit first stage.

Table 13: Make Withdrawal

Unit	Effects	$\boldsymbol{m}_1$ R	$\boldsymbol{m}_1$ V	$\boldsymbol{m}_2$ R	$\boldsymbol{m}_2$ V
Female	Direct	0.06**	0.04***	0.28	
		(0.03)	(0.02)	(0.23)	
	Indirect	-0.17		-0.005	0.01
		(0.24)		(0.004)	(0.01)
Male	Direct	0.16		0.06**	0.05*
		(0.34)		(0.03)	(0.03)
	Indirect	0.001	0.01	-0.44	
		(0.04)	(0.03)	(0.65)	

Notes: The dependent variable is 1 if the individual made at least one withdrawal. "Direct" denotes the local average treatment effects $E[Y_i(1,0) - Y_i(0,0)|K_i^{\boldsymbol{m}} = 1]$, and "Indirect" denotes $E[Y_i(0,1) - Y_i(0,0)|K_j^{\boldsymbol{m}} = 1]$. Plug-in clustered standard errors are reported in parentheses. *,**,*** denote the significance levels at 10%, 5%, and 1%, respectively.

withdrawals. These effects are smaller in the V estimation for the estimable parameters, which can be interpreted as bias resulting from the violation of one-sided noncompliance. According to both estimates, the indirect effects are not significantly estimated. This indicates that there is no evidence that an individual's banking behaviors are affected by their spouse's opening of a savings account.

7 Conclusion

This study analyzes the identification and estimation of causal effects in situations where units interact with imperfect compliance. Traditionally, such scenarios require restrictions on the distribution of potential treatments, often through monotonicity assumptions or one-sided noncompliance. This study relaxes these restrictions, introducing a general concept of monotonicity and providing a unified framework to analyze endogenous treatments in this context. The study proposes the identification of causal effects (local average direct and indirect effects) under weak monotonicity and an additional exclusion restriction for endogenous treatment, without relying on (total) monotonicity or one-sided noncompliance. The results also explain previous findings in the literature as special cases. A two-stage estimation procedure is proposed and its performance is verified using Monte Carlo simulations and an experimental dataset from Dupas, Keats, and Robinson (2017) and Dupas, Keats, and Robinson (2019).

This paper focuses on scenarios where interactions occur between two units within all groups. This is relevant for contexts such as married couples or pair trading in finance. However, this assumption may be too restrictive in cases where units do not interact in some groups. Furthermore, the analysis is limited to interactions between two units. Applying this framework to accommodate a general N units will not be straightforward and will require additional restrictions. Extending this approach to accommodate more complex scenarios will be a valuable direction for future research.

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A Appendix

A.1 Proofs for Section 3

Proof of Lemma 1. Let $i \in \{1,2\}$ be given, and $j = 3 - i$. From (3), we have

$$Y_i = Y_i(0,0) + \Delta_i Y_i(0) D_i + \Delta_j Y_i(0) D_j + \Delta^2 Y_i D_i D_j.$$

Let $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{r}) = ((z_1, z_2), (z_1', z_2'), (r_1, r_2))$ be a monotone pair, and denote $\boldsymbol{z}_i = (z_i, z_j)$. Then, we have

$$\begin{aligned}
E[Y_i|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}] &= E[Y_i(0,0)|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}] \\
&\quad + E[\Delta_i Y_i(0) D_i|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}] + E[\Delta_j Y_i(0) D_j|\boldsymbol{Z}_j = \boldsymbol{z}_j, \boldsymbol{X}] \\
&\quad + E\left[\Delta^2 Y_i D_i D_j|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}\right] \\
&= E[Y_i(0,0)|\boldsymbol{X}] \\
&\quad + E[\Delta_i Y_i(0) D_i(\boldsymbol{z}_i)|\boldsymbol{X}] + E[\Delta_j Y_i(0) D_j(\boldsymbol{z}_j)|\boldsymbol{X}] \\
&\quad + E\left[\Delta^2 Y_i D_i(\boldsymbol{z}_i) D_j(\boldsymbol{z}_j)|\boldsymbol{X}\right],
\end{aligned}$$

By Assumption 1-(B) Note that $r_i^2 = 1$ because $r_i \in \{-1, 1\}$. Therefore,

$$\begin{aligned}
&E[Y_i|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}] - E[Y_i|\boldsymbol{Z}_i = \boldsymbol{z}_i', \boldsymbol{X}] \\
&= E[\Delta_i Y_i(0)(D_i(\boldsymbol{z}_i) - D_i(\boldsymbol{z}_i'))|\boldsymbol{X}] + E[\Delta_j Y_i(0)(D_j(\boldsymbol{z}_j) - D_j(\boldsymbol{z}_j'))|\boldsymbol{X}] \\
&\qquad + E\left[\Delta^2 Y_i(D_i(\boldsymbol{z}_i) D_j(\boldsymbol{z}_j) - D_i(\boldsymbol{z}_i') D_j(\boldsymbol{z}_j'))|\boldsymbol{X}\right]. \\
&= r_i E[\Delta_i Y_i(0) r_i(D_i(\boldsymbol{z}_i) - D_i(\boldsymbol{z}_i'))|\boldsymbol{X}] + r_j E[\Delta_j Y_i(0) r_j(D_j(\boldsymbol{z}_j) - D_j(\boldsymbol{z}_j'))|\boldsymbol{X}] \\
&\qquad + r_i r_j E\left[\Delta^2 Y_i(r_i D_i(\boldsymbol{z}_i) r_j D_j(\boldsymbol{z}_j) - r_i D_i(\boldsymbol{z}_i') r_j D_j(\boldsymbol{z}_j'))|\boldsymbol{X}\right].
\end{aligned}$$

Thus, we have equation (5) with $K_i^{\boldsymbol{m}} = r_i(D_i(\boldsymbol{z}_i) - D_i(\boldsymbol{z}_i'))$ and $K_{ij}^{\boldsymbol{m}} = r_i D_i(\boldsymbol{z}_i) r_j D_j(\boldsymbol{z}_j) - r_i D_i(\boldsymbol{z}_i') r_j D_j(\boldsymbol{z}_j')$. □

Proof of Proposition 1. Let $i \in \{1,2\}$, $j = 3 - i$, $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$, and $\boldsymbol{T}$ be a subset of exogenous variables in $\boldsymbol{X}$. Then, both $K_i^{\boldsymbol{m}} = D_i(\boldsymbol{z}_i) - D_i(\boldsymbol{z}_i')$, and

$K_{ij}^{\boldsymbol{m}} = D_i(\boldsymbol{z}_i)D_j(\boldsymbol{z}_j) - D_i(\boldsymbol{z}_i')D_j(\boldsymbol{z}_j')$ are binary. Thus, we have:

$$\begin{aligned}
P_i^{\boldsymbol{m}}(\boldsymbol{T}) &= E[E[K_i^{\boldsymbol{m}}|\boldsymbol{X}]|\boldsymbol{T}] \\
&= E[E[D_i(\boldsymbol{z}_i)|\boldsymbol{X}] - E[D_i(\boldsymbol{z}_i')|\boldsymbol{X}]|\boldsymbol{T}] \\
&= E[E[D_i|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}] - E[D_i|\boldsymbol{Z}_i = \boldsymbol{z}_i', \boldsymbol{X}]|\boldsymbol{T}], \\
P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) &= E[E[K_{ij}^{\boldsymbol{m}}|\boldsymbol{X}]|\boldsymbol{T}] \\
&= E[E[D_i(\boldsymbol{z}_i)D_j(\boldsymbol{z}_j)|\boldsymbol{X}] - E[D_i(\boldsymbol{z}_i')D_j(\boldsymbol{z}_j')|\boldsymbol{X}]|\boldsymbol{T}] \\
&= E[E[D_iD_j|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}] - E[D_iD_j|\boldsymbol{Z}_i = \boldsymbol{z}_i', \boldsymbol{X}]|\boldsymbol{T}],
\end{aligned}$$

since potential treatments are independent of $\boldsymbol{Z}_i$ conditional on $\boldsymbol{X}$ by Assumption 1-(B). □

Proof of Lemma 2. The ITT effects on outcomes and treatment take-ups are expressed by weighted averages by using the weight defined in (11). Observe that for any $\boldsymbol{z} \in \{0,1\}^2$,

$$\begin{aligned}
E[\mathbb{1}\{\boldsymbol{Z}_i = \boldsymbol{z}\}Y_i|\boldsymbol{X}] &= E[Y_i|\boldsymbol{Z}_i = \boldsymbol{z}, \boldsymbol{X}]\Pr(\boldsymbol{Z}_i = \boldsymbol{z}|\boldsymbol{X}), \\
E[\mathbb{1}\{\boldsymbol{Z}_i = \boldsymbol{z}\}D_i|\boldsymbol{X}] &= E[D_i|\boldsymbol{Z}_i = \boldsymbol{z}, \boldsymbol{X}]\Pr(\boldsymbol{Z}_i = \boldsymbol{z}|\boldsymbol{X}), \\
E[\mathbb{1}\{\boldsymbol{Z}_i = \boldsymbol{z}\}D_iD_j|\boldsymbol{X}] &= E[D_iD_j|\boldsymbol{Z}_i = \boldsymbol{z}, \boldsymbol{X}]\Pr(\boldsymbol{Z}_i = \boldsymbol{z}|\boldsymbol{X}),
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
E[ITT_i^{\boldsymbol{m}}(\boldsymbol{X})|\boldsymbol{T}] &= E\left[E[Y_i|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}] - E[Y_i|\boldsymbol{Z}_i = \boldsymbol{z}_i', \boldsymbol{X}]\right|\boldsymbol{T}] \\
&= E[E[\omega^{\boldsymbol{m}}Y_i|\boldsymbol{X}]|\boldsymbol{T}] \\
&= E[\omega^{\boldsymbol{m}}Y_i|\boldsymbol{T}],
\end{aligned}$$

by law of iterative expectation. Similarly, from Proposition 1, we have

$$\begin{aligned}
P_i^{\boldsymbol{m}}(\boldsymbol{T}) &= E\left[E[D_i|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}] - E[D_i|\boldsymbol{Z}_i = \boldsymbol{z}_i', \boldsymbol{X}]|\, \boldsymbol{T}\right], \\
&= E[E[\omega^{\boldsymbol{m}} D_i|\boldsymbol{X}]|\boldsymbol{T}], \\
&= E[\omega^{\boldsymbol{m}} D_i|\boldsymbol{T}], \\
P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) &= E\left[E[D_i D_j|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{X}] - E[D_i D_j|\boldsymbol{Z}_i = \boldsymbol{z}_i', \boldsymbol{X}]|\, \boldsymbol{T}\right], \\
&= E[E[\omega^{\boldsymbol{m}} D_i D_j|\boldsymbol{X}]|\boldsymbol{T}], \\
&= E[\omega^{\boldsymbol{m}} D_i D_j|\boldsymbol{T}].
\end{aligned}$$

Next, by stacking (8) for both units, we have

$$E\left[\left.\begin{pmatrix} ITT_i^{\boldsymbol{m}}(\boldsymbol{X}) \\ ITT_j^{\boldsymbol{m}}(\boldsymbol{X}) \end{pmatrix}\right| \boldsymbol{T}\right] = \begin{pmatrix} P_i^{\boldsymbol{m}}(\boldsymbol{T}) & P_j^{\boldsymbol{m}}(\boldsymbol{T}) & P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) & 0 & 0 & 0 \\ 0 & 0 & 0 & P_j^{\boldsymbol{m}}(\boldsymbol{T}) & P_i^{\boldsymbol{m}}(\boldsymbol{T}) & P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) \end{pmatrix} \begin{pmatrix} \delta_i^{\boldsymbol{m}}(0) \\ \theta_i^{\boldsymbol{m}}(1) \\ \zeta_i^{\boldsymbol{m}} \\ \delta_j^{\boldsymbol{m}}(1) \\ \theta_j^{\boldsymbol{m}}(0) \\ \zeta_j^{\boldsymbol{m}} \end{pmatrix}.$$

Using the notations defined in Lemma 2, it can be written as

$$E\left[\tilde{\boldsymbol{Y}}^{\boldsymbol{m}}|\boldsymbol{T}\right] = \tilde{\boldsymbol{P}}^{\boldsymbol{m}}(\boldsymbol{T})\boldsymbol{\beta}^{\boldsymbol{m}} = E\left[\tilde{\boldsymbol{D}}^{\boldsymbol{m}}|\boldsymbol{T}\right]\boldsymbol{\beta}^{\boldsymbol{m}},$$

thus we have (10). □

Proof of Lemma 3. Because potential treatment statuses are binary random variables, $D_i(\boldsymbol{z}_i) \leq 1$ with probability 1. Therefore,

$$D_i(\boldsymbol{z}_i)D_j(\boldsymbol{z}_j) \leq D_j(\boldsymbol{z}_j), \tag{A.1}$$

with probability 1. Suppose $D_i(\boldsymbol{z}_i) \geq D_j(\boldsymbol{z}_j)$ with probability 1. Then,

$$D_i(\boldsymbol{z}_i)D_j(\boldsymbol{z}_j) \geq D_j^2(\boldsymbol{z}_j) = D_j(\boldsymbol{z}_j), \tag{A.2}$$

with probability 1. Thus, we have $D_i(\boldsymbol{z}_i)D_j(\boldsymbol{z}_j) = D_j(\boldsymbol{z}_j)$ with probability 1.

Conversely, assume $D_i(\boldsymbol{z}_i)D_j(\boldsymbol{z}_j) = D_j(\boldsymbol{z}_j)$ with probability 1. Suppose $D_i(\boldsymbol{z}_i) < D_j(\boldsymbol{z}_j)$ with positive probability, i.e., the event $E = \{D_i(\boldsymbol{z}_i) = 0, D_j(\boldsymbol{z}_j) = 1\}$ occurs with positive probability, i.e., the event $D_j(\boldsymbol{z}_j) \neq D_i(\boldsymbol{z}_i)D_j(\boldsymbol{z}_j)$ occurs with positive probability. The desired result is from applying the above argument conditional on any subset $\boldsymbol{T}$ of the set of exogenous variables $\boldsymbol{X}$ □

Proof of Lemma 4. Under the given Assumptions, we have $K_{ij}^{\boldsymbol{m}} = K_j^{\boldsymbol{m}}$ as shown in (12). Thus, $P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) = P_j^{\boldsymbol{m}}(\boldsymbol{T})$, and

$$\begin{aligned}
\zeta_i^{\boldsymbol{m}} &= E[\Delta^2 Y_i | K_{ij}^{\boldsymbol{m}} = 1] \\
&= E[(Y_i(1,1) - Y_i(0,1)) - (Y_i(1,0) - Y_i(0,0)) | K_{ij}^{\boldsymbol{m}} = 1] \\
&= E[\Delta_j Y_i(1) - \Delta_j Y_i(0) | K_{ij}^{\boldsymbol{m}} = 1] \\
&= \theta_i(1) - \theta_i(0).
\end{aligned}$$

It follows that $\zeta_i^{\boldsymbol{m}} P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) = (\theta_i(1) - \theta_i(0))P_j^{\boldsymbol{m}}(\boldsymbol{T})$, and (8) becomes

$$\begin{aligned}
E[ITT_i^{\boldsymbol{m}}(\boldsymbol{X})|\boldsymbol{T}] &= \delta_i^{\boldsymbol{m}}(0)P_i^{\boldsymbol{m}}(\boldsymbol{T}) + \theta_i^{\boldsymbol{m}}(0)P_j^{\boldsymbol{m}}(\boldsymbol{T}) + \zeta_i^{\boldsymbol{m}} P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) \\
&= \delta_i^{\boldsymbol{m}}(0)P_i^{\boldsymbol{m}}(\boldsymbol{T}) + \theta_i^{\boldsymbol{m}}(1)P_j^{\boldsymbol{m}}(\boldsymbol{T}) \qquad \text{(A.3)}
\end{aligned}$$

By stacking (A.3) for both units,

$$E\left[\left.\begin{pmatrix} ITT_i^{\boldsymbol{m}}(\boldsymbol{X}) \\ ITT_j^{\boldsymbol{m}}(\boldsymbol{X}) \end{pmatrix}\right| \boldsymbol{T}\right] = \begin{pmatrix} P_i^{\boldsymbol{m}}(\boldsymbol{T}) & P_j^{\boldsymbol{m}}(\boldsymbol{T}) & 0 & 0 \\ 0 & 0 & P_j^{\boldsymbol{m}}(\boldsymbol{T}) & P_i^{\boldsymbol{m}}(\boldsymbol{T}) \end{pmatrix} \begin{pmatrix} \delta_i^{\boldsymbol{m}}(0) \\ \theta_i^{\boldsymbol{m}}(1) \\ \delta_j^{\boldsymbol{m}}(1) \\ \theta_j^{\boldsymbol{m}}(0) \end{pmatrix}.$$

Using the notations defined in Lemma 4, we have (13). □

Proof of Lemma 5. Under the given Assumptions, we have $K_j^{\boldsymbol{m}} = 0$. Thus, (8) becomes

$$E[ITT_i^{\boldsymbol{m}}(\boldsymbol{X})|\boldsymbol{T}] = \delta_i^{\boldsymbol{m}}(0)P_i^{\boldsymbol{m}}(\boldsymbol{T}) + \zeta_i^{\boldsymbol{m}}P_{ij}^{\boldsymbol{m}}(\boldsymbol{T})$$
$$E[ITT_j^{\boldsymbol{m}}(\boldsymbol{X})|\boldsymbol{T}] = \theta_j^{\boldsymbol{m}}(0)P_i^{\boldsymbol{m}}(\boldsymbol{T}) + \zeta_j^{\boldsymbol{m}}P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) \tag{A.4}$$

By stacking (A.4) for both units,

$$E\left[\left.\begin{pmatrix} ITT_i^{\boldsymbol{m}}(\boldsymbol{X}) \\ ITT_j^{\boldsymbol{m}}(\boldsymbol{X}) \end{pmatrix}\right| \boldsymbol{T}\right] = \begin{pmatrix} P_i^{\boldsymbol{m}}(\boldsymbol{T}) & P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) & 0 & 0 \\ 0 & 0 & P_j^{\boldsymbol{m}}(\boldsymbol{T}) & P_{ij}^{\boldsymbol{m}}(\boldsymbol{T}) \end{pmatrix} \begin{pmatrix} \delta_i^{\boldsymbol{m}}(0) \\ \zeta_i^{\boldsymbol{m}} \\ \theta_j^{\boldsymbol{m}}(0) \\ \zeta_j^{\boldsymbol{m}} \end{pmatrix}.$$

Using the notations defined in Lemma 5, we have (14). □

Proof of Proposition 2, 3, and 4. Assumptions 1-3 imply conditional moment (10) by Lemma 2. Let $\boldsymbol{R}(\boldsymbol{T})$ be a matrix of functions of $\boldsymbol{T}$ satisfying Assumption 4. Then, we have the unconditional moment $E[\boldsymbol{R}(\boldsymbol{T})(\tilde{\boldsymbol{Y}}^{\boldsymbol{m}} - \tilde{\boldsymbol{D}}^{\boldsymbol{m}}\boldsymbol{\beta}^{\boldsymbol{m}})] = 0$, or, equivalently, $E[\boldsymbol{R}(\boldsymbol{T})\tilde{\boldsymbol{D}}^{\boldsymbol{m}}]\boldsymbol{\beta}^{\boldsymbol{m}} = E[\boldsymbol{R}(\boldsymbol{T})\tilde{\boldsymbol{Y}}^{\boldsymbol{m}}]$. Because $E[\boldsymbol{R}(\boldsymbol{T})\tilde{\boldsymbol{D}}^{\boldsymbol{m}}]$ is nonsingular by Assumption 4, resulting in $\boldsymbol{\beta}^{\boldsymbol{m}} = E[\boldsymbol{R}(\boldsymbol{T})\tilde{\boldsymbol{D}}^{\boldsymbol{m}}]^{-1}E[\boldsymbol{R}(\boldsymbol{T})\tilde{\boldsymbol{Y}}^{\boldsymbol{m}}]$ in Proposition 2.

The results in Proposition 3 and 4 follows by the same argument by replacing $(\tilde{\boldsymbol{D}}^{\boldsymbol{m}}, \boldsymbol{\beta}^{\boldsymbol{m}})$ with $(\check{\boldsymbol{D}}^{\boldsymbol{m}}, \check{\boldsymbol{\beta}}^{\boldsymbol{m}})$ and $(\bar{\boldsymbol{D}}^{\boldsymbol{m}}, \bar{\boldsymbol{\beta}}^{\boldsymbol{m}})$, respectively. □

Proof of Proposition 5. Recall (15):

$$K_{ij}^{\boldsymbol{m}} = K_i^{\boldsymbol{m}}D_j(\boldsymbol{z}_j) = K_i^{\boldsymbol{m}}D_j(\boldsymbol{z}_j'). \tag{A.5}$$

If unit j is $\boldsymbol{m}$-never-taker, then we have $K_{ij}^{\boldsymbol{m}} = K_j^{\boldsymbol{m}} = 0$. Thus, last two terms in (7) become zero, and (7) becomes

$$E[ITT_i^{\boldsymbol{m}}(\boldsymbol{X})] = \delta_i^{\boldsymbol{m}}(0)P_i^{\boldsymbol{m}}, \quad E[ITT_j^{\boldsymbol{m}}(\boldsymbol{X})] = \theta_j^{\boldsymbol{m}}(0)P_i^{\boldsymbol{m}}.$$

Thus, we have the result in Part (i). On the other hand, if unit j is $\boldsymbol{m}$-always-taker, then we have $K_{ij}^{\boldsymbol{m}} = K_i^{\boldsymbol{m}}$, and

$$\begin{aligned}
\zeta_i^{\boldsymbol{m}} &= E[\Delta^2 Y_i | K_{ij}^{\boldsymbol{m}} = 1] \\
&= E[(Y_i(1,1) - Y_i(0,1)) - (Y_i(1,0) - Y_i(0,0)) | K_i^{\boldsymbol{m}} = 1] \\
&= E[\Delta_i Y_i(1) - \Delta_i Y_i(0) | K_i^{\boldsymbol{m}} = 1] \\
&= \delta_i(1) - \delta_i(0).
\end{aligned}$$

Similarly, we have $\zeta_j^{\boldsymbol{m}} = \theta_j(1) - \theta_j(0)$. Therefore, (7) becomes

$$E[ITT_i^{\boldsymbol{m}}(\boldsymbol{X})] = \delta_i^{\boldsymbol{m}}(1) P_i^{\boldsymbol{m}}, \quad E[ITT_j^{\boldsymbol{m}}(\boldsymbol{X})] = \theta_j^{\boldsymbol{m}}(1) P_i^{\boldsymbol{m}}.$$

Thus, we have the result in Part (ii). $\square$

A.2 Regularity Conditions

Following Assumption lists required regularity conditions for asymptotic properties in Proposition 6.

Assumption 8 (Regularity Conditions)**.**

1. *$\{\boldsymbol{V}_g : 1 \leq g \leq G\}$ are independently and identically distributed.*

2. *$E[\|\boldsymbol{Y}_g\|^4] < \infty$.*

3. *$\boldsymbol{S}(\boldsymbol{T}_g)$ is positive definite and $E[\tilde{\boldsymbol{P}}'(\boldsymbol{T}_g)\boldsymbol{S}^{-1}(\boldsymbol{T}_g)\tilde{\boldsymbol{P}}(\boldsymbol{T}_g)]$ is nonsingular.*

4. *$\boldsymbol{\beta} \in \boldsymbol{\mathcal{B}} \subset \mathbb{R}^k$, $\boldsymbol{\mathcal{B}}$ is compact. And $\boldsymbol{\gamma}_0 \in \Gamma$, and $\boldsymbol{\phi}_0 \in \Phi$ are interior points.*

5. *(i) There exists q_0 such that $0 < q_0 \leq \inf_{\gamma \in \mathcal{N}_\gamma} q(\boldsymbol{z}, \boldsymbol{X}_g, \boldsymbol{\gamma})$ with probability 1, for all $\boldsymbol{z} \in \{0,1\}^2$, for some neighborhood $\mathcal{N}_\gamma$ of $\boldsymbol{\gamma}_0$, (ii) q is continuously differentiable in $\boldsymbol{\gamma}$ with probability 1, and (iii) $E\left[\sup_{\gamma \in} \left\| \frac{\partial q(\boldsymbol{z}, \boldsymbol{X}_g, \boldsymbol{\gamma})}{\gamma'} \right\|^2\right] < \infty$ for all $\boldsymbol{z} \in \{0,1\}^2$.*

6. *(i) There exists λ_0 such that $0 < \lambda_0 \leq \inf_{\phi \in \mathcal{N}_\phi} \{$minimum eigenvalue of $\boldsymbol{S}(\boldsymbol{T}_g, \boldsymbol{\phi})\}$ with probability 1, for some neighborhood $\mathcal{N}_\phi$ of $\boldsymbol{\phi}_0$, (ii) $\boldsymbol{S}(\boldsymbol{T}_g, \boldsymbol{\phi})$, $\tilde{\boldsymbol{P}}(\boldsymbol{T}_g, \boldsymbol{\phi})$ are*

continuously differentiable in $\boldsymbol{\phi}$ *with probability 1, and (iii)* $E\left[\sup_{\phi\in\mathcal{N}_\phi}\left\|\frac{\partial \boldsymbol{S}(\boldsymbol{T}_g,\boldsymbol{\phi})}{\partial \boldsymbol{\phi}'}\right\|^2\right] < \infty$, $E\left[\sup_{\phi\in\mathcal{N}_\phi}\left\|\frac{\partial \tilde{\boldsymbol{P}}(\boldsymbol{T}_g,\boldsymbol{\phi})}{\partial \boldsymbol{\phi}'}\right\|^2\right] < \infty$.

8.1 assumes the population consists of i.i.d. groups. 8.2 assumes the existence of the fourth moment. 8.3 is a specialization of Assumption 4 to the optimal matrix of instruments. 8.5 and 8.6 assume that the functions used in the parametric model are smooth and bounded.

A.3 Proof of Proposition 6

The following lemma is Lemma 4.3 in Newey and McFadden (1994), and will be used to prove Proposition 6.

Lemma 6. *Let* $\boldsymbol{V}_g$ *be a random vector whose support is* $\mathcal{V}$ *and* $\ell : \mathcal{V} \times \Phi \to \mathbb{R}^M$ *be a vector of real valued functions that is integrable with respect to the distribution of* $\boldsymbol{V}_g$ *at each point* $\phi \in \Phi \subset \mathbb{R}^K$*. Define followings:*

$$L_G(\phi) = \frac{1}{G}\sum_{g=1}^{G}\ell(\boldsymbol{V}_g, \phi), \quad L(\phi) = E[\ell(\boldsymbol{V}_g, \phi)].$$

Suppose (a) $\{\boldsymbol{V}_g\}$ *is independently and identically distributed; (b)* $\hat{\phi} \xrightarrow{p} \phi_0$*,* ϕ_0*; (c)* $\ell(\boldsymbol{v}, \phi)$ *is continuous at* ϕ_0 *for all* $\boldsymbol{v} \in \mathcal{V}$*; (d) For some neighborhood* $\mathcal{N}$ *of* ϕ_0*, we have* $E\left[\sup_{\phi\in\mathcal{N}}\|\ell(\boldsymbol{V}_g, \phi)\|\right] < \infty$*. Then,* $L(\phi)$ *is continuous at* ϕ_0 *and* $L_G(\hat{\phi}) \xrightarrow{p} L(\phi_0)$*.*

Proof. Consider a sequence $\{\phi_n\} \to \phi_0$. For the neighborhood $\mathcal{N}$ of ϕ_0 satisfying (d), we have $\|\ell(v, \phi_n)\| \leq \sup_{\phi\in\mathcal{N}}\|\ell(v, \phi)\| =: g(v)$, for all but finite number of n, where $g(v)$ is integrable by (d). Thus, by dominated convergence thoerem, we have $\{E[\ell(\boldsymbol{V}_g, \phi_n)]\} \to E[\ell(\boldsymbol{V}_g, \phi_0)]$, which implies continuity of $L(\phi)$ at ϕ_0. See proof of Lemma 4.3 in Newey and McFadden (1994) for $L_G(\hat{\phi}) \xrightarrow{p} L(\phi_0)$. $\square$

Proof of Proposition 6. Let $\boldsymbol{m} = (\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{M}_1$ be a monotone pair. Recall that

$$\tilde{\boldsymbol{D}}_g = \begin{pmatrix} D_{ig} & D_{jg} & D_{ig}D_{jg} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{jg} & D_{ig} & D_{ig}D_{jg} \end{pmatrix},$$

$$\tilde{\boldsymbol{P}}(\boldsymbol{T}_g, \boldsymbol{\phi}) = \begin{pmatrix} P_i(\boldsymbol{T}_g, \boldsymbol{\phi}) & P_j(\boldsymbol{T}_g, \boldsymbol{\phi}) & P_{ij}(\boldsymbol{T}_g, \boldsymbol{\phi}) & 0 & 0 & 0 \\ 0 & 0 & 0 & P_j(\boldsymbol{T}_g, \boldsymbol{\phi}) & P_i(\boldsymbol{T}_g, \boldsymbol{\phi}) & P_{ij}(\boldsymbol{T}_g, \boldsymbol{\phi}) \end{pmatrix},$$

Since every norm is equivalent for a finite dimensional vector space, let $\|\cdot\|$ be Frobenius norm, i.e., $\|v\| = \mathrm{tr}(vv')^{1/2}$. Then,

$$\left\|\tilde{\boldsymbol{P}}(\boldsymbol{T}_g, \boldsymbol{\phi})\right\| = \sqrt{2(P_i^2(\boldsymbol{T}_g, \boldsymbol{\phi}) + P_j^2(\boldsymbol{T}_g, \boldsymbol{\phi}) + P_{ij}^2(\boldsymbol{T}_g, \boldsymbol{\phi}))} \leq \sqrt{6}, \tag{A.6}$$

$$\left\|\tilde{\boldsymbol{D}}\right\| = \sqrt{2(D_{ig} + D_{jg} + D_{ig}D_{jg})} \leq \sqrt{6}, \tag{A.7}$$

For the neighborhood $\mathcal{N}_\gamma$ satisfying Assumption 8.5,

$$\sup_{\gamma \in \mathcal{N}_\gamma} |\omega_g(\boldsymbol{\gamma})| = \sup_{\gamma \in \mathcal{N}_\gamma} \left| \frac{\mathbb{1}\{\boldsymbol{Z}_g = \boldsymbol{z}\}}{q(\boldsymbol{z}, \boldsymbol{X}_g, \boldsymbol{\gamma})} - \frac{\mathbb{1}\{\boldsymbol{Z}_g = \boldsymbol{z}'\}}{q(\boldsymbol{z}', \boldsymbol{X}_g, \boldsymbol{\gamma})} \right| \leq \frac{2}{q_0}. \tag{A.8}$$

Next, let $\lambda_{\max}(A), \lambda_{\min}(A)$ be maximum, minimum eigenvalue of a square matrix A, respectively. Then, for the compact neighborhood $\mathcal{N}_\phi$ satisfying Assumption 8.6,

$$\begin{aligned} \sup_{\phi \in \mathcal{N}_\phi} \left\|\boldsymbol{S}^{-1}(\boldsymbol{T}_g, \boldsymbol{\phi})\right\| &\leq \sqrt{2} \sup_{\phi \in \mathcal{N}_\phi} \left\|\boldsymbol{S}^{-1}(\boldsymbol{T}_g, \boldsymbol{\phi})\right\|_2 \\ &\leq \sqrt{2} \sup_{\phi \in \mathcal{N}_\phi} \lambda_{\max}(\boldsymbol{S}^{-1}(\boldsymbol{T}_g, \boldsymbol{\phi})) \\ &= \sqrt{2} \sup_{\phi \in \mathcal{N}_\phi} \lambda_{\min}(\boldsymbol{S}(\boldsymbol{T}_g, \boldsymbol{\phi}))^{-1} \\ &= \sqrt{2} \left(\inf_{\phi \in \mathcal{N}_\phi} \lambda_{\min}(\boldsymbol{S}(\boldsymbol{T}_g, \boldsymbol{\phi})) \right)^{-1} \leq \frac{\sqrt{2}}{\lambda_0}, \end{aligned} \tag{A.9}$$

with probability 1, where $\|\cdot\|_2$ is the spectral norm which is the maximum eigenvalue of the matrix. Thus, for a neighborhood $\mathcal{N} := \mathcal{N}_\gamma \times \mathcal{N}_\phi$ of $(\boldsymbol{\phi}_0, \boldsymbol{\gamma}_0)$, we have

$$\begin{aligned}
E\left[\sup_{(\boldsymbol{\phi},\boldsymbol{\gamma})\in\mathcal{N}} \left\|\omega_g(\boldsymbol{\gamma})\boldsymbol{R}(\boldsymbol{T}_g,\boldsymbol{\phi})\tilde{\boldsymbol{D}}_g\right\|\right] &= E\left[\sup_{(\boldsymbol{\phi},\boldsymbol{\gamma})\in\mathcal{N}} \left\|\omega_g(\boldsymbol{\gamma})\tilde{\boldsymbol{P}}(\boldsymbol{T}_g,\boldsymbol{\phi})'\boldsymbol{S}^{-1}(\boldsymbol{V}_g,\boldsymbol{\phi})\tilde{\boldsymbol{D}}_g\right\|\right] \\
&\le E\left[\sup_{\boldsymbol{\gamma}\in\mathcal{N}_\gamma} |\omega_g(\boldsymbol{\gamma})| \sup_{\boldsymbol{\phi}\in\mathcal{N}_\phi}\left\|\tilde{\boldsymbol{P}}(\boldsymbol{T}_g,\boldsymbol{\phi})\right\| \sup_{\boldsymbol{\phi}\in\mathcal{N}_\phi}\left\|\boldsymbol{S}^{-1}(\boldsymbol{V}_g,\boldsymbol{\phi})\right\|\left\|\tilde{\boldsymbol{D}}_g\right\|\right] \\
&\le \frac{12\sqrt{2}}{\lambda_0 q_0} < \infty, \\
E\left[\sup_{(\boldsymbol{\phi},\boldsymbol{\gamma})\in\mathcal{N}} \left\|\omega_g(\boldsymbol{\gamma})\boldsymbol{R}(\boldsymbol{T}_g,\boldsymbol{\phi})\boldsymbol{Y}_g\right\|\right] &= E\left[\sup_{(\boldsymbol{\phi},\boldsymbol{\gamma})\in\mathcal{N}} \left\|\omega_g(\boldsymbol{\gamma})\tilde{\boldsymbol{P}}(\boldsymbol{T}_g,\boldsymbol{\phi})'\boldsymbol{S}^{-1}(\boldsymbol{V}_g,\boldsymbol{\phi})\boldsymbol{Y}_g\right\|\right] \\
&\le E\left[\sup_{\boldsymbol{\gamma}\in\mathcal{N}_\gamma} |\omega_g(\boldsymbol{\gamma})| \sup_{\boldsymbol{\phi}\in\mathcal{N}_\phi}\left\|\tilde{\boldsymbol{P}}(\boldsymbol{T}_g,\boldsymbol{\phi})\right\| \sup_{\boldsymbol{\phi}\in\mathcal{N}_\phi}\left\|\boldsymbol{S}^{-1}(\boldsymbol{V}_g,\boldsymbol{\phi})\right\|\left\|\boldsymbol{Y}_g\right\|\right] \\
&\le \frac{4\sqrt{3}}{\lambda_0 q_0} E\left[\|\boldsymbol{Y}_g\|\right] < \infty,
\end{aligned}$$

Thus, by Lemma 6, we have

$$\begin{aligned}
&\frac{1}{G}\sum_{g=1}^{G}\omega_g(\hat{\boldsymbol{\gamma}})\boldsymbol{R}(\boldsymbol{T}_g,\hat{\boldsymbol{\phi}})\tilde{\boldsymbol{D}}_g \xrightarrow{p} E[\omega_g\boldsymbol{R}(\boldsymbol{T}_g)\tilde{\boldsymbol{D}}_{\boldsymbol{g}}] = E[\tilde{\boldsymbol{P}}(\boldsymbol{T}_g)'\boldsymbol{S}^{-1}(\boldsymbol{T}_g)\tilde{\boldsymbol{P}}(\boldsymbol{T}_g)], \\
&\frac{1}{G}\sum_{g=1}^{G}\omega_g(\hat{\boldsymbol{\gamma}})\boldsymbol{R}(\boldsymbol{T}_g,\hat{\boldsymbol{\phi}})\boldsymbol{Y}_g \xrightarrow{p} E[\omega_g\boldsymbol{R}(\boldsymbol{T}_g)\boldsymbol{Y}_g] = E[\tilde{\boldsymbol{P}}(\boldsymbol{T}_g)'\boldsymbol{S}^{-1}(\boldsymbol{T}_g)\tilde{\boldsymbol{P}}(\boldsymbol{T}_g)]\boldsymbol{\beta}_0.
\end{aligned}$$

The consistency follows from the Slutsky's theorem. Let $\boldsymbol{\varepsilon}_g := (\boldsymbol{Y}_g - \tilde{\boldsymbol{D}}_g\boldsymbol{\beta}_{\boldsymbol{0}})$. Since $\boldsymbol{\phi}_0, \boldsymbol{\gamma}_0$ are interior points, by mean value theorem, there exists $\bar{\boldsymbol{\gamma}}$ and $\bar{\boldsymbol{\phi}}$ that satisfy

$$\begin{aligned}
0 &= \frac{1}{\sqrt{G}}\sum_{g=1}^{G}\omega_g(\hat{\boldsymbol{\gamma}})\boldsymbol{R}(\boldsymbol{T}_g,\hat{\boldsymbol{\phi}})(\boldsymbol{Y}_g - \tilde{\boldsymbol{D}}_g\hat{\boldsymbol{\beta}}) \\
&= \frac{1}{\sqrt{G}}\sum_{g=1}^{G}\omega_g\boldsymbol{R}(\boldsymbol{T}_g)\boldsymbol{\varepsilon}_g - \left[\frac{1}{G}\sum_{g=1}^{G}\omega_g(\bar{\boldsymbol{\gamma}})\boldsymbol{R}(\boldsymbol{T}_g,\bar{\boldsymbol{\phi}})\tilde{\boldsymbol{D}}_g\right]\sqrt{G}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \\
&\quad + \sum_{j=1}^{k_\phi}\left[\frac{1}{G}\sum_{g=1}^{G}\omega_g(\bar{\boldsymbol{\gamma}})\boldsymbol{\varepsilon}_g\frac{\partial\boldsymbol{R}(\boldsymbol{T}_g,\bar{\boldsymbol{\phi}})}{\partial\phi_j}\right]\sqrt{G}(\hat{\phi}_j - \phi_{j0}) \\
&\quad + \left[\frac{1}{G}\sum_{g=1}^{G}\boldsymbol{R}(\boldsymbol{T}_g,\bar{\boldsymbol{\phi}})\boldsymbol{\varepsilon}_g\frac{\partial\omega_g(\bar{\boldsymbol{\gamma}})}{\partial\gamma'}\right]\sqrt{G}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0). \qquad \text{(A.10)}
\end{aligned}$$

By (A.6), (A.9), and Assumption 8.6,

$$
\begin{aligned}
& E\left[\sup_{\phi\in\mathcal{N}_\phi}\left\|\frac{\partial \boldsymbol{R}(\boldsymbol{T}_g,\boldsymbol{\phi})}{\partial\phi_j}\right\|\right] \\
&= E\left[\sup_{\phi\in\mathcal{N}_\phi}\left\|\frac{\partial \tilde{\boldsymbol{P}}(\boldsymbol{T}_g,\boldsymbol{\phi})}{\partial\phi_j}\boldsymbol{S}^{-1}(\boldsymbol{T}_g,\boldsymbol{\phi})+\tilde{\boldsymbol{P}}(\boldsymbol{T}_g,\boldsymbol{\phi})\frac{\partial \boldsymbol{S}^{-1}(\boldsymbol{T}_g,\boldsymbol{\phi})}{\partial\phi_j}\right\|\right] \\
&= E\left[\sup_{\phi\in\mathcal{N}_\phi}\left\|\frac{\partial \tilde{\boldsymbol{P}}(\boldsymbol{T}_g,\boldsymbol{\phi})}{\partial\phi_j}\boldsymbol{S}^{-1}(\boldsymbol{T}_g,\phi)-\tilde{\boldsymbol{P}}(\boldsymbol{T}_g,\boldsymbol{\phi})\boldsymbol{S}^{-1}(\boldsymbol{T}_g,\boldsymbol{\phi})\frac{\partial \boldsymbol{S}(\boldsymbol{T}_g,\boldsymbol{\phi})}{\partial\phi_j}\boldsymbol{S}^{-1}(\boldsymbol{T}_g,\boldsymbol{\phi})\right\|\right] \\
&\leq \frac{\sqrt{2}}{\lambda_0}E\left[\sup_{\phi\in\mathcal{N}_\phi}\left\|\frac{\partial \tilde{\boldsymbol{P}}(\boldsymbol{T}_g,\phi)}{\partial\phi_j}\right\|\right]+\frac{2\sqrt{6}}{\lambda_0^2}E\left[\sup_{\phi\in\mathcal{N}_\phi}\left\|\frac{\partial \boldsymbol{S}(\boldsymbol{T}_g,\phi)}{\partial\phi_j}\right\|\right]<\infty \qquad \text{(A.11)}
\end{aligned}
$$

Similarly, for the neighborhood $\mathcal{N}_\gamma$ of $\boldsymbol{\gamma}_0$,

$$
\begin{aligned}
& E\left[\sup_{\gamma\in\mathcal{N}_\gamma}\left\|\frac{\partial\omega_g(\boldsymbol{\gamma})}{\partial\gamma'}\right\|\right] \\
&= E\left[\sup_{\gamma\in\mathcal{N}_\gamma}\left\|-\frac{\mathbb{1}\{\boldsymbol{Z}_g=\boldsymbol{z}\}}{q^2(\boldsymbol{z},\boldsymbol{X}_g,\boldsymbol{\gamma})}\frac{\partial q(\boldsymbol{z},\boldsymbol{X}_g,\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}}+\frac{\mathbb{1}\{\boldsymbol{Z}_g=\boldsymbol{z}'\}}{q^2(\boldsymbol{z}',\boldsymbol{X}_g,\boldsymbol{\gamma})}\frac{\partial q(\boldsymbol{z}',\boldsymbol{X}_g,\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}}\right\|\right] \\
&\leq E\left[\frac{1}{q_0^2}\sup_{\gamma\in\mathcal{N}_\gamma}\left\|\frac{\partial q(\boldsymbol{z},\boldsymbol{X}_g,\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}'}\right\|+\frac{1}{q_0^2}\sup_{\gamma\in\mathcal{N}_\gamma}\left\|\frac{\partial q(\boldsymbol{z}',\boldsymbol{X}_g,\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}'}\right\|\right] \\
&\leq \frac{2}{q_0^2}\max\left\{E\left[\sup_{\gamma\in\mathcal{N}_\gamma}\left\|\frac{\partial q(\boldsymbol{z},\boldsymbol{X}_g,\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}'}\right\|\right],E\left[\sup_{\gamma\in\mathcal{N}_\gamma}\left\|\frac{\partial q(\boldsymbol{z}',\boldsymbol{X}_g,\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}'}\right\|\right]\right\}<\infty \qquad \text{(A.12)}
\end{aligned}
$$

Therefore, we have a neighborhood $\mathcal{M}=\mathcal{N}_\phi\times\mathcal{N}_\gamma$ such that

$$
\begin{aligned}
& E\left[\sup_{(\phi,\gamma)\in\mathcal{M}}\left\|\omega_g(\boldsymbol{\gamma})\boldsymbol{\varepsilon}_g\frac{\partial\boldsymbol{R}(\boldsymbol{T}_g,\phi)}{\partial\phi_j}\right\|\right] \\
&\qquad \leq E\left[\sup_{\phi\in\mathcal{N}_\phi}\left\|\frac{\partial\boldsymbol{R}(\boldsymbol{T}_g,\boldsymbol{\phi})}{\partial\phi_j}\right\|^2\right]^{\frac{1}{2}}E\left[\sup_{\gamma\in\mathcal{N}_\gamma}|\omega_g(\boldsymbol{\gamma})|^4\right]^{\frac{1}{4}}E\left[\|\boldsymbol{\varepsilon}_g\|^4\right]^{\frac{1}{4}}<\infty,
\end{aligned}
$$

because the first term is bounded by (A.11), second term is bounded by (A.8), and the last term is bounded by Assumption 8.2. Furthermore, we have

$$E\left[\sup_{(\boldsymbol{\phi},\gamma)\in\mathcal{M}}\left\|\boldsymbol{R}(\boldsymbol{T}_g,\boldsymbol{\phi})\boldsymbol{\varepsilon}_g\frac{\partial\omega_g(\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}'}\right\|\right]$$
$$\leq E\left[\sup_{\phi\in\mathcal{N}_\phi}\|\boldsymbol{R}(\boldsymbol{T}_g,\phi)\|^4\right]^{\frac{1}{4}}E\left[\|\boldsymbol{\varepsilon}_g\|^4\right]^{\frac{1}{4}}E\left[\sup_{\gamma\in\mathcal{N}_\gamma}\left\|\frac{\partial\omega_g(\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}'}\right\|^2\right]^{\frac{1}{2}}$$

because the first and the second terms are bounded by (A.6), (A.9), and Assumption 8.2, and the last term is

$$E\left[\sup_{\gamma\in\mathcal{N}_\gamma}\left\|\frac{\partial\omega_g(\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}'}\right\|^2\right]^{\frac{1}{2}}\leq\frac{2}{q_0^2}\max_{\boldsymbol{z}\in\{0,1\}^2}E\left[\sup_{\phi\in\mathcal{N}_\phi}\left\|\frac{\partial q(\boldsymbol{z},\boldsymbol{X}_g,\gamma)}{\partial\gamma'}\right\|^2\right]^{\frac{1}{2}}<\infty,$$

by (A.12). Therefore, by applying Lemma 6 for each term in (A.10), we have

$$\begin{aligned}0=&\frac{1}{\sqrt{G}}\sum_{g=1}^{G}\omega_g\boldsymbol{R}(\boldsymbol{T}_g)\boldsymbol{\varepsilon}_g-E\left[\omega_g\boldsymbol{R}(\boldsymbol{T}_g)\tilde{\boldsymbol{D}}_g\right]\sqrt{G}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)\\&+\sum_{j=1}^{k_\phi}E\left[\omega_g\boldsymbol{\varepsilon}_g\frac{\partial\boldsymbol{R}(\boldsymbol{T}_g,\boldsymbol{\phi}_0)}{\partial\phi_j}\right]\sqrt{G}(\hat{\phi}_j-\phi_{j0})+E\left[\boldsymbol{R}(\boldsymbol{T}_g)\boldsymbol{\varepsilon}_g\frac{\partial\omega_g(\boldsymbol{\gamma}_0)}{\partial\gamma'}\right]\sqrt{G}(\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_0)+o_p(1).\end{aligned}\tag{A.13}$$

The third term in (A.13) is zero because

$$E\left[\omega_g\boldsymbol{\varepsilon}_g\frac{\partial\boldsymbol{R}(\boldsymbol{T}_g,\boldsymbol{\phi}_0)}{\partial\phi_j}\right]=E\left[\underbrace{E\left[\omega_g\boldsymbol{\varepsilon}_g|\boldsymbol{T}_g\right]}_{=0}\frac{\partial\boldsymbol{R}(\boldsymbol{T}_g,\boldsymbol{\phi}_0)}{\partial\phi_j}\right],$$

by the given moment condition (10). By Assumption 7, we have

$$\sqrt{G}(\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_0)=\frac{1}{\sqrt{G}}\sum_{g=1}^{G}\psi_\gamma(\boldsymbol{V}_g,\boldsymbol{\gamma}_0)+o_p(1).$$

Therefore, by rearranging (A.13), we have

$$\sqrt{G}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) = \frac{1}{\sqrt{G}} \sum_{g=1}^{G} \underbrace{\boldsymbol{A}^{-1} \left[\omega_g \boldsymbol{R}(\boldsymbol{T}_g)(\boldsymbol{Y}_g - \tilde{\boldsymbol{D}}_g \boldsymbol{\beta}_0) + \boldsymbol{B}\psi_\gamma(\boldsymbol{V}_g, \boldsymbol{\gamma}_0)\right]}_{:=\psi_\beta(\boldsymbol{V}_g, \boldsymbol{\beta}_0)} + o_p(1),$$

where $\boldsymbol{A} = E\left[\omega_g \boldsymbol{R}(\boldsymbol{T}_g)\tilde{\boldsymbol{D}}_g\right] = E\left[\tilde{\boldsymbol{P}}'(\boldsymbol{T}_g)\boldsymbol{S}^{-1}(\boldsymbol{T}_g)\tilde{\boldsymbol{P}}(\boldsymbol{T}_g)\right]$, and $\boldsymbol{B} = E\left[\boldsymbol{R}(\boldsymbol{T}_g)\boldsymbol{\varepsilon}_g \frac{\partial \omega_g(\gamma_0)}{\partial \gamma'}\right]$.

B Additional Tables

B.1 Simulation

Table B.1: Simulation of Design 1

		Use true ω		Estimate ω	
	G	Probit	Linear	Probit	Linear
MSE	200	2710.46	168139.32	3094.09	291167.35
	400	1567.27	2498.3	741.76	1566.72
	500	720.74	765.76	679.73	642.41
	800	325.74	360.43	288.04	319.61
	1,000	259.55	287.12	225.69	250.65
	2,000	121.32	130.04	104.22	111.75
MAE	200	59.84	88.79	58.62	90.81
	400	39.22	43.32	37.05	41.17
	500	34.1	36.58	32.31	34.48
	800	26.01	27.22	24.53	25.7
	1,000	23.41	24.51	21.95	22.98
	2,000	16.26	16.73	15.07	15.54
Coverage	200	0.98	0.95	0.97	0.95
	400	0.97	0.95	0.97	0.94
	500	0.97	0.95	0.96	0.94
	800	0.96	0.95	0.96	0.95
	1,000	0.96	0.94	0.96	0.95
	2,000	0.95	0.94	0.95	0.95

notes. This table presents simulation results for $B = 10{,}000$ replications. Column G denotes the number of groups. MSE is $\sum_{b=1}^{B} \left\| \hat{\beta}_b - \beta_0 \right\|^2 / B$, where $\hat{\beta}_b = (\hat{\delta}_1(0), \hat{\theta}_1(0), \hat{\delta}_2(0), \hat{\theta}_2(0))'$ is the vector of estimates in b^{th} replication, and $\beta_0 = (\delta_1(0), \theta_1(0), \delta_2(0), \theta_2(0))'$ is the true vector of parameters. The actual parameter values are $\delta_1(0) = 20$, $\theta_1(0) = 10$, $\delta_2(0) = 30$, $\theta_2(0) = 15$. MAE is $\sum_{b=1}^{B} \sum_{k=1}^{4} |\hat{\beta}_{kb} - \beta_{0k}|/4B$. Coverage computes the minimum 95% coverage rate among four estimate, i.e., $\min_{1 \leq k \leq 4} \sum_{b=1}^{B} \mathbb{1}\{\hat{\beta}_{kb} - 1.96SE(\hat{\beta}_{kb}) \leq \beta_{0k} \leq \hat{\beta}_{kb} + 1.96SE(\hat{\beta}_{kb})\}/B$. The first two columns ("Use true ω") use the true propensity score $\Pr(\boldsymbol{Z}_i = \boldsymbol{z}|\boldsymbol{X})$ for the weight ω, while the last two columns ("Estimate ω") estimate the propensity score and hence ω. Linear and Probit denote that the optimal instrument $P_i^{\boldsymbol{m}_1}(\boldsymbol{T})$ is estimated by linear probability model, and probit model, respectively.

Table B.2: Simulation of Design 2

	G	Use true ω Probit	Use true ω Linear	Estimate ω Probit	Estimate ω Linear
MSE	200	44.91	42.19	31.01	29.56
	400	22.41	21.38	14.18	13.73
	500	17.75	16.94	11.34	11.06
	800	11.12	10.66	6.97	6.81
	1,000	8.86	8.57	5.52	5.44
	2,000	4.44	4.31	2.72	2.67
MAE	200	7.36	7.06	6.17	5.98
	400	5.25	5.09	4.18	4.1
	500	4.68	4.55	3.74	3.69
	800	3.7	3.62	2.94	2.9
	1,000	3.33	3.27	2.61	2.6
	2,000	2.34	2.31	1.84	1.82
Coverage	200	0.92	0.94	0.93	0.94
	400	0.94	0.94	0.94	0.94
	500	0.94	0.95	0.94	0.94
	800	0.94	0.94	0.94	0.94
	1,000	0.95	0.95	0.94	0.94
	2,000	0.95	0.95	0.94	0.95

notes. This table presents simulation results for $B = 10{,}000$ replications. Column G denotes the number of groups. MSE is $\sum_{b=1}^{B} \left\| \hat{\beta}_b - \beta_0 \right\|^2 / B$, where $\hat{\beta}_b = (\hat{\delta}_1(0), \hat{\theta}_1(0), \hat{\delta}_2(0), \hat{\theta}_2(0))'$ is the vector of estimates in b^{th} replication, and $\beta_0 = (\delta_1(0), \theta_1(0), \delta_2(0), \theta_2(0))'$ is the true vector of parameters. The actual parameter values are $\delta_1(0) = 20$, $\theta_1(0) = 10$, $\delta_2(0) = 30$, $\theta_2(0) = 15$. MAE is $\sum_{b=1}^{B} \sum_{k=1}^{4} |\hat{\beta}_{kb} - \beta_{0k}| / 4B$. Coverage computes the minimum 95% coverage rate among four estimate, i.e., $\min_{1 \leq k \leq 4} \sum_{b=1}^{B} \mathbb{1}\{\hat{\beta}_{kb} - 1.96 SE(\hat{\beta}_{kb}) \leq \beta_{0k} \leq \hat{\beta}_{kb} + 1.96 SE(\hat{\beta}_{kb})\} / B$. The first two columns ("Use true ω") use the true propensity score $\Pr(\boldsymbol{Z}_i = \boldsymbol{z} | \boldsymbol{X})$ for the weight ω, while the last two columns ("Estimate ω") estimate the propensity score and hence ω. Linear and Probit denote that the optimal instrument $P_i^{\boldsymbol{m}_1}(\boldsymbol{T})$ is estimated by linear probability model, and probit model, respectively.

Table B.3: Simulation of Design 3

	G	Use true ω Probit	Use true ω Linear	Estimate ω Probit	Estimate ω Linear
MSE	200	107.9	184.18	105.09	116.68
	400	46.86	47.36	46.45	46.69
	500	38.58	38.79	38.01	38.1
	800	22.78	22.79	22.48	22.48
	1,000	17.76	17.77	17.49	17.5
	2,000	9.3	9.29	9.17	9.16
MAE	200	10.78	11.16	11.02	11.23
	400	7.43	7.46	7.47	7.49
	500	6.79	6.8	6.79	6.8
	800	5.28	5.28	5.26	5.25
	1,000	4.68	4.68	4.65	4.65
	2,000	3.39	3.39	3.38	3.38
Coverage	200	0.96	0.96	0.95	0.95
	400	0.96	0.96	0.95	0.95
	500	0.96	0.96	0.95	0.95
	800	0.96	0.96	0.95	0.95
	1,000	0.96	0.96	0.95	0.95
	2,000	0.95	0.95	0.95	0.95

notes. This table presents simulation results for $B = 10{,}000$ replications. Column G denotes the number of groups. MSE is $\sum_{b=1}^{B} \left\| \hat{\beta}_b - \beta_0 \right\|^2 / B$, where $\hat{\beta}_b = (\hat{\delta}_1(0), \hat{\theta}_1(0), \hat{\delta}_2(0), \hat{\theta}_2(0))'$ is the vector of estimates in b^{th} replication, and $\beta_0 = (\delta_1(0), \theta_1(0), \delta_2(0), \theta_2(0))'$ is the true vector of parameters. The actual parameter values are $\delta_1(0) = 20$, $\theta_1(0) = 10$, $\delta_2(0) = 30$, $\theta_2(0) = 15$. MAE is $\sum_{b=1}^{B} \sum_{k=1}^{4} |\hat{\beta}_{kb} - \beta_{0k}|/4B$. Coverage computes the minimum 95% coverage rate among four estimate, i.e., $\min_{1\le k\le 4} \sum_{b=1}^{B} \mathbb{1}\{\hat{\beta}_{kb} - 1.96SE(\hat{\beta}_{kb}) \le \beta_{0k} \le \hat{\beta}_{kb} + 1.96SE(\hat{\beta}_{kb})\}/B$. The first two columns ("Use true ω") use the true propensity score $\Pr(\boldsymbol{Z}_i = \boldsymbol{z}|\boldsymbol{X})$ for the weight ω, while the last two columns ("Estimate ω") estimate the propensity score and hence ω. Linear and Probit denote that the optimal instrument $P_i^{\boldsymbol{m}_1}(\boldsymbol{T})$ is estimated by linear probability model, and probit model, respectively.

Table B.4: Simulation of Design 1 for each parameter

		Design 1			
	G	$\delta_1(0)$	$\theta_1(0)$	$\delta_2(0)$	$\theta_2(0)$
Mean	200	19.56	10.11	28.44	15.04
	400	19.87	9.9	29.3	14.96
	500	19.83	10.05	29.62	14.93
	800	19.97	9.9	29.71	15.04
	1,000	19.89	9.86	29.71	14.88
	2,000	19.93	10.03	30	14.96
Med	200	19.59	10.91	29.57	15.17
	400	19.8	10.41	30.3	15.01
	500	19.83	10.36	30.28	14.96
	800	19.98	10.24	30.08	15.05
	1,000	19.9	10.1	29.94	14.97
	2,000	19.91	10.13	30.17	14.89
MSE	200	267.41	974.83	1524.68	327.18
	400	56	213.54	394.99	77.24
	500	48.37	252.33	323.38	55.65
	800	22.93	86.1	149.97	29.03
	1,000	18.22	67.54	117.18	22.75
	2,000	8.64	30.9	53.93	10.74
Coverage	200	0.97	0.98	0.97	0.97
	400	0.97	0.97	0.97	0.97
	500	0.96	0.97	0.97	0.97
	800	0.96	0.96	0.97	0.97
	1,000	0.96	0.96	0.96	0.96
	2,000	0.95	0.96	0.96	0.95

notes. This table presents simulation results for $B = 10{,}000$ replications. Column G denotes the number of groups. Mean is $\sum_{b=1}^{B} \hat{\beta}_b / B$, where $\hat{\beta}_b = (\hat{\delta}_1(0), \hat{\theta}_1(0), \hat{\delta}_2(0), \hat{\theta}_2(0))'$ is the vector of estimates in b^{th} replication, and $\beta_0 = (\delta_1(0), \theta_1(0), \delta_2(0), \theta_2(0))' = (20, 10, 30, 15)'$ is the true vector of parameters. Med is the median of $\hat{\beta}_b$ among B simulated estimates. MSE is $\sum_{b=1}^{B} \left\| \hat{\beta}_b - \beta_0 \right\|^2 / B$. Coverage computes the 95% coverage rate, i.e., $\sum_{b=1}^{B} \mathbb{1}\{\hat{\beta}_{kb} - 1.96SE(\hat{\beta}_{kb}) \leq \beta_{0k} \leq \hat{\beta}_{kb} + 1.96SE(\hat{\beta}_{kb})\}/B$.

Table B.5: Simulation of Design 2 and 3 for each parameter

		Design 1			
	G	$\delta_1(0)$	$\theta_1(0)$	$\delta_2(0)$	$\theta_2(0)$
Mean	200	20.24	15.19	19.3	14.47
	400	20.12	15.09	19.75	14.81
	500	20.1	15.08	19.87	14.9
	800	20.07	15.05	19.91	14.93
	1,000	20.04	15.03	19.89	14.92
	2,000	20.04	15.03	19.95	14.96
Med	200	20.67	15.51	18.95	14.22
	400	20.3	15.23	19.64	14.71
	500	20.24	15.18	19.65	14.73
	800	20.14	15.1	19.78	14.84
	1,000	20.13	15.09	19.79	14.84
	2,000	20.07	15.05	19.91	14.92
MSE	200	19.73	11.28	67.15	37.93
	400	9.01	5.16	29.71	16.74
	500	7.22	4.12	24.29	13.72
	800	4.44	2.53	14.37	8.1
	1,000	3.51	2.01	11.19	6.3
	2,000	1.73	0.99	5.86	3.3
Coverage	200	0.93	0.93	0.95	0.95
	400	0.94	0.94	0.95	0.95
	500	0.94	0.94	0.95	0.95
	800	0.94	0.94	0.95	0.95
	1,000	0.94	0.94	0.95	0.95
	2,000	0.95	0.94	0.95	0.95

notes. This table presents simulation results for $B = 10{,}000$ replications. Column G denotes the number of groups. Mean is $\sum_{b=1}^{B} \hat{\beta}_b/B$, where $\hat{\beta}_b = (\hat{\delta}_1(0), \hat{\theta}_1(0), \hat{\delta}_2(0), \hat{\theta}_2(0))'$ is the vector of estimates in b^{th} replication, and $\beta_0 = (\delta_1(0), \theta_1(0), \delta_2(0), \theta_2(0))' = (20, 10, 30, 15)'$ is the true vector of parameters. Med is the median of $\hat{\beta}_b$ among B simulated estimates. MSE is $\sum_{b=1}^{B} \left\| \hat{\beta}_b - \beta_0 \right\|^2 /B$. Coverage computes the 95% coverage rate, i.e., $\sum_{b=1}^{B} \mathbb{1}\{\hat{\beta}_{kb} - 1.96SE(\hat{\beta}_{kb}) \leq \beta_{0k} \leq \hat{\beta}_{kb} + 1.96SE(\hat{\beta}_{kb})\}/B$.

B.2 Non-Additive Separable Outcomes

This section discuss a simulation design under total monotonicity, but the outcome is not additively separable. Following Table 2, there are 5 compliance types for each unit. Let $K_i \in \mathcal{K} := \{AT, SC, C, GC, NT\}$ be a discrete random variable denoting unit i's compliance type. Then, as shown in Appendix C, we can define direct and indirect local average treatment effects for each compliance type in $\mathcal{K}$. Potential outcomes are generated from the following structural equation:

$$Y_i(d_1, d_2) = \alpha_i(d_1) + \phi_i(d_2) + \rho\alpha_i(d_1)\phi_i(d_2) + \beta_i^y Y_j(d_2, d_1) + \beta_{1i}^w W_i + \beta_{2i}^w W_j. \tag{B.1}$$

Here, $\alpha_i(d)$, $\phi_i(d)$ are random components generated by

$$\begin{aligned}
\alpha_i(d) &= \sum_{k\in\mathcal{K}} \mathbb{1}\{K_i = k\}\alpha_i^k(d), & \alpha_i^k(1) &\sim N(\bar{\alpha}_i^k, 1), \quad \alpha_i^k(0) \sim N(0,1),\\
\phi_i(d) &= \sum_{k\in\mathcal{K}} \mathbb{1}\{K_j = k\}\phi_i^k(d), & \phi_i^k(1) &\sim N(\bar{\phi}_i^k, 1), \quad \phi_i^k(0) \sim N(0,1),
\end{aligned}$$

for $i \in \{1,2\}$. This reflects that direct effects depends on the own compliance type, while indirect effects depend on the other's compliance type. The third term in (B.1) is an interaction term, which is a part of last term in (5). This interaction term vanishes when $\rho = 0$. In the fourth term, β_i^y reflects the endogenous peer effects. The above structural equation implies the following reduced form outcome:

$$\begin{aligned}
Y_i(d_1, d_2) = {} & \underbrace{\frac{\alpha_i(d_1) + \beta_i^y \phi_j(d_1)}{1 - \beta_1^y\beta_2^y}}_{:=\tilde{\alpha}_i(d_1)} + \underbrace{\frac{\phi_i(d_2) + \beta_i^y \alpha_j(d_2)}{1 - \beta_1^y\beta_2^y}}_{:=\tilde{\phi}_i(d_2)} + \rho\underbrace{\frac{\alpha_i(d_1)\phi_i(d_2) + \beta_i^y \alpha_j(d_2)\phi_j(d_1)}{1 - \beta_1^y\beta_2^y}}_{:=\tilde{\xi}(d_1,d_2)}\\
& + \underbrace{\frac{(\beta_{1i}^w + \beta_i^y\beta_{2j}^w)W_i + (\beta_{2i}^w + \beta_i^y\beta_{1j}^w)W_j}{1 - \beta_1^y\beta_2^y}}_{:=\boldsymbol{\beta}_i'\boldsymbol{W}_i}.
\end{aligned}$$

Thus (B.1) imposes for potential outcome to satisfy Assumption 1 and 3. The observed outcome is generated as follows:

$$
\begin{aligned}
Y_i &= Y_i(0,0) + D_i(Y_i(1,0) - Y_i(0,0)) + D_j(Y_i(0,1) - Y_i(0,0)) \\
&\quad + D_iD_j(Y_i(1,1) - Y_i(1,0) - Y_i(0,1) + Y_i(0,0)) \\
&= (\tilde{\alpha}_i(0) + \tilde{\phi}_i(0)) + (\tilde{\alpha}_i(1) - \tilde{\alpha}_i(0))D_i + (\tilde{\phi}_i(1) - \tilde{\phi}_i(0))D_j \\
&\quad + (\tilde{\xi}_i(1,1) - \tilde{\xi}_i(1,0) - \tilde{\xi}_i(0,1) - \tilde{\xi}_i(0,0))D_j + \boldsymbol{\beta}_i'\boldsymbol{W}_i.
\end{aligned}
$$

The actual parameter values are given as Table B.6.[11]

Table B.6: Parameter Values

Individual (i)	1	2
$\boldsymbol{m}_1$-Direct Effect ($\delta_i^{\boldsymbol{m}_1}(0)$)	64	49.14
$\boldsymbol{m}_1$-Indirect ($\theta_i^{\boldsymbol{m}_1}(0)$)	36.57	32
$\boldsymbol{m}_1$-Interaction ($\zeta_i^{\boldsymbol{m}_1}$)	10.24	8.96

Table Table B.7 shows the simulation results for this design. Overall, we have the similar result as in Table 5. Considering MSE, all methods show little differences, but using the first-stage estimator for ω, and computing the instrument with the linear probability model appears the most efficient. The inference based on the plug in standard error is asymptotically valid.

[11]The rest of parameters are set as follows. First, the parameters in potential outcome are $(\beta_1^y, \beta_2^y, \beta_{1i}^w, \beta_{2i}^w) = (0.5, 0.25, 1, 0.5)$, $i \in \{1,2\}$. Second, the parameters in the potential treatment generation are $\boldsymbol{\phi}_1^d = (-1.5, 6, 2, 9, 1)', \boldsymbol{\phi}_2^d = (-1.5, 8, 2, 9, 0.5)'$. Third, the true means of random components in potential outcome are $\bar{\boldsymbol{\alpha}}_1(1) = [48, 48, 48, 96, 120]$, $\bar{\boldsymbol{\alpha}}_2(1) = [40, 40, 40, 80, 100]$, $\bar{\boldsymbol{\phi}}_1(1) = [12, 12, 12, 16, 20]$, and $\bar{\boldsymbol{\phi}}_2(1) = [16, 16, 16, 32, 40]$, where the ordering is AT, SC, C, GC, NT. Lastly, $\bar{\boldsymbol{\alpha}}_1(0) = \bar{\boldsymbol{\alpha}}_2(0) = \bar{\boldsymbol{\phi}}_1(0) = \bar{\boldsymbol{\phi}}_2(0) = 0$.

Table B.7: Monte Carlo Simulation

		Use true ω		Estimate ω	
G		Probit	Linear	Probit	Linear
2500	MSE	14.293	12.629	13.781	12.271
	MAE	0.029	0.024	0.031	0.027
	Cov. Rate	0.944	0.961	0.945	0.961
5000	MSE	6.506	6.12	6.406	5.989
	MAE	0.027	0.022	0.021	0.015
	Cov. Rate	0.948	0.959	0.948	0.96
10000	MSE	3.131	3.003	3.037	2.89
	MAE	0.011	0.009	0.007	0.005
	Cov. Rate	0.946	0.951	0.946	0.952
20000	MSE	1.482	1.363	1.461	1.339
	MAE	0.005	0.006	0.003	0.004
	Cov. Rate	0.951	0.961	0.95	0.959

Notes: This table presents simulation results for $B = 10{,}000$ replications. Column G denotes the number of independent groups. The mean squared error (MSE) is calculated by $\sum_{b=1}^{B} \|\hat{\beta}_b - \beta_0\|^2/B$, where $\hat{\beta}_b = (\hat{\delta}_1(0), \hat{\theta}_1(0), \hat{\zeta}_1, \hat{\delta}_2(0), \hat{\theta}_2(0), \hat{\zeta}_2)'$ is the vector of estimates in the b^{th} replication, and $\beta_0 = (\delta_1(0), \theta_1(0), \zeta_1, \delta_2(0), \theta_2(0), \zeta_2)'$ is the true vector of parameters. The actual parameter values are set by Table B.6. The mean absolute error (MAE) is calculated by $\sum_{b=1}^{B} \sum_{k=1}^{6} |\hat{\beta}_{kb} - \beta_{0k}|/(6B)$. Coverage computes the minimum 95% coverage rate among the four estimates, i.e., $\min_{1 \leq k \leq 6} \sum_{b=1}^{B} \mathbb{1}\{\hat{\beta}_{kb} - 1.96 SE(\hat{\beta}_{kb}) \leq \beta_{0k} \leq \hat{\beta}_{kb} + 1.96 SE(\hat{\beta}_{kb})\}/B$. The first two columns ("Use true ω") use the true propensity score $\Pr(\boldsymbol{Z}_i = \boldsymbol{z}|\boldsymbol{X})$ for the weight ω, while the last two columns ("Estimate ω") estimate the propensity score and hence ω. "Linear" and "Probit" denote that the optimal instrument is estimated by the linear probability model and probit model, respectively.

B.3 Empirical Result with Nonlinear First-Stage

Table B.8: Make Deposit

		$\boldsymbol{m}_1$		$\boldsymbol{m}_1$	
Unit	Effects	R	V	R	V
Female	Direct	0.17***	0.08**	-0.03	
		(0.05)	(0.03)	(0.02)	
	Indirect	-1.11**		0.03	-0.002
		(0.47)		(0.02)	(0.03)
Male	Direct	0.22		0.15***	0.1***
		(0.42)		(0.04)	(0.04)
	Indirect	-0.04	0.001	-0.15***	
		(0.04)	(0.03)	(0.04)	

Notes: The dependent variable is 1 if the individual made at least one deposit. "Direct" denotes the local average treatment effects $E[Y_i(1,0) - Y_i(0,0)|K_i^{\boldsymbol{m}} = 1]$, and "Indirect" denotes $E[Y_i(0,1) - Y_i(0,0)|K_j^{\boldsymbol{m}} = 1]$. Plug-in clustered standard errors are reported in parentheses. *,**,*** denote the significance levels at 10%, 5%, and 1%, respectively. The instruments are estimated by using probit model in the first stage.

Table B.9: Make Withdrawal

		$\boldsymbol{m}_1$		$\boldsymbol{m}_1$	
Unit	Effects	R	V	R	V
Female	Direct	0.05*	0.04***	0.27	
		(0.03)	(0.02)	(1.15)	
	Indirect	-0.004		-0.002**	0.01
		(0.25)		(0.0009)	(0.01)
Male	Direct	0.45		0.02	0.05*
		(0.32)		(0.02)	(0.03)
	Indirect	-0.02	0.01	0.52	
		(0.03)	(0.03)	(1.26)	

Notes: The dependent variable is 1 if the individual made at least one withdrawal. "Direct" denotes the local average treatment effects $E[Y_i(1,0) - Y_i(0,0)|K_i^{\boldsymbol{m}} = 1]$, and "Indirect" denotes $E[Y_i(0,1) - Y_i(0,0)|K_j^{\boldsymbol{m}} = 1]$. Plug-in clustered standard errors are reported in parentheses. *,**,*** denote the significance levels at 10%, 5%, and 1%, respectively. The instruments are estimated by using probit model in the first stage.

C Finer Classification of Compliance Types

If there exist multiple monotone pairs, compliance types can be divided into finer and disjoint categories. Consider the monotone pairs $\boldsymbol{m}_1 = ((0,1),(0,0),(1,1))$, $\boldsymbol{m}_2 = ((1,0),(0,0),(1,1))$ and $\boldsymbol{m}_3 = ((1,0),(0,1),(1,-1))$ in Panel (b) of Table 3. For unit 1, the $\boldsymbol{m}_2$-complier is interpreted as either social complier (SC, or $\boldsymbol{m}_1$-complier) or complier (C, or $\boldsymbol{m}_3$-complier). Since the events for unit 1 to be a $\boldsymbol{m}_1$-complier, and $\boldsymbol{m}_3$-complier are disjoint, we have

$$P_1^{\boldsymbol{m}_2}(\boldsymbol{T}) = \Pr(K_1^{\boldsymbol{m}_2} = 1|\boldsymbol{T}) = \Pr(K_1^{\boldsymbol{m}_1} = 1|\boldsymbol{T}) + \Pr(K_1^{\boldsymbol{m}_3} = 1|\boldsymbol{T}) = P_1^{\boldsymbol{m}_1}(\boldsymbol{T}) + P_1^{\boldsymbol{m}_3}(\boldsymbol{T}).$$

It follows that

$$\begin{aligned}
\delta_1^{\boldsymbol{m}_2} P_1^{\boldsymbol{m}_2}(\boldsymbol{T}) &= E[\Delta_1 Y_1(0)|K_1^{\boldsymbol{m}_2} = 1, \boldsymbol{T}] \Pr(K_i^{\boldsymbol{m}_2} = 1|\boldsymbol{T}) \\
&= E[\Delta_1 Y_1(0)|K_1^{\boldsymbol{m}_1} = 1] \Pr(K_1^{\boldsymbol{m}_1} = 1|K_1^{\boldsymbol{m}_2} = 1, \boldsymbol{T}) \Pr(K_i^{\boldsymbol{m}_2} = 1|\boldsymbol{T}) \\
&\qquad + E[\Delta_1 Y_1(0)|K_1^{\boldsymbol{m}_3} = 1] \Pr(K_1^{\boldsymbol{m}_3} = 1|K_1^{\boldsymbol{m}_2} = 1, \boldsymbol{T}) \Pr(K_1^{\boldsymbol{m}_2} = 1|\boldsymbol{T}) \\
&= E[\Delta_1 Y_1(0)|K_1^{\boldsymbol{m}_1} = 1] \Pr(K_1^{\boldsymbol{m}_1} = 1|\boldsymbol{T}) \\
&\qquad + E[\Delta_1 Y_1(0)|K_1^{\boldsymbol{m}_3} = 1] \Pr(K_1^{\boldsymbol{m}_3} = 1|\boldsymbol{T}) \\
&= \delta_1^{\boldsymbol{m}_1} P_1^{\boldsymbol{m}_1}(\boldsymbol{T}) + \delta_1^{\boldsymbol{m}_3} P_1^{\boldsymbol{m}_3}(\boldsymbol{T}).
\end{aligned}$$

Therefore, we have the following relationship between parameters from different monotone pairs:

$$\delta_1^{\boldsymbol{m}_2} = \delta_1^{\boldsymbol{m}_1} \left(\frac{P_1^{\boldsymbol{m}_1}(\boldsymbol{T})}{P_1^{\boldsymbol{m}_1}(\boldsymbol{T}) + P_1^{\boldsymbol{m}_3}(\boldsymbol{T})} \right) + \delta_1^{\boldsymbol{m}_3} \left(\frac{P_1^{\boldsymbol{m}_3}(\boldsymbol{T})}{P_1^{\boldsymbol{m}_1}(\boldsymbol{T}) + P_1^{\boldsymbol{m}_3}(\boldsymbol{T})} \right).$$

The (8) with respect to the monotone pair $\boldsymbol{m}_2$ can be written as

$$\begin{aligned}
E[ITT_1^{\boldsymbol{m}_2}(\boldsymbol{X})|\boldsymbol{T}] &= \delta_1^{\boldsymbol{m}_2} P_1^{\boldsymbol{m}_2}(\boldsymbol{T}) + \theta_1^{\boldsymbol{m}_2} P_2^{\boldsymbol{m}_2}(\boldsymbol{T}) \\
&= \delta_1^{\boldsymbol{m}_1} P_1^{\boldsymbol{m}_1}(\boldsymbol{T}) + \delta_1^{\boldsymbol{m}_3} P_1^{\boldsymbol{m}_3}(\boldsymbol{T}) + \theta_1^{\boldsymbol{m}_2} P_2^{\boldsymbol{m}_2}(\boldsymbol{T}).
\end{aligned}$$