The destinctions of symmetry.

In this section we Investigate the effect of a general QFT action functional. $S[\phi, \partial_\mu \phi]$.

The transformations acts on the position as well as the fields

$$x \to x'$$
$$\phi(x) \to \phi'(x')$$

The post-transform field is then a function of the previous field. *Note that in many QFTs it is common to work in momentum space. This is not the case for us, and we will treat almost every situation in position space.*

This perspective on transformations is often refered to as active transformations, as opposed to passive transformations; where transformations are treated as coordinate transformations.

The action (S) transforms as the following under the general transform.

$$S \to S' = \int d^d x \, \mathcal{L}(\phi'(x), \partial_\mu \phi'(x))$$

change of integration variable.

$$= \int d^d x' \, \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'))$$

$$= \int d^d x' \, \mathcal{L}(\mathcal{F}(\phi(x)), \partial'_\mu \mathcal{F}(\phi(x)))$$

Jacobiane

$$= \int \left|\frac{\partial x'}{\partial x}\right| d^d x \, \mathcal{L}\left(\mathcal{F}(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \mathcal{F}(\phi(x))\right)$$

Ex. 1 Translation

$x' = x + a$

$\phi'(x+a) = \phi(x)$

$$S = \int \left|\frac{\partial x'}{\partial x}\right| d^d x \, \mathcal{L}\left(f(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} f(\phi(x))\right)$$

$$\left|\frac{\partial x'}{\partial x}\right| = 1 \, , \quad \frac{\partial x^\nu}{\partial x'^\mu} = \delta_\mu^\nu$$

$$S = \int d^d x \, \mathcal{L}\left(f(\phi(x)), \partial_\nu f(\phi(x))\right)$$

Remark 1 The action is invariant if not explicitly dependent on pos.

Ex. 2 Lorentz transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \,, \qquad \Phi'(\Lambda x) = L_\Lambda \phi(x)$$

$$S = \int \left|\frac{\partial x'}{\partial x}\right| d^d x \, \mathcal{L}\left(f(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} f(\phi(x))\right)$$

$$\left|\frac{\partial x'}{\partial x}\right| = \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = \det\left(\Lambda^\mu{}_\rho \delta^\rho_\nu\right)$$
$$= \det\left(\Lambda^\mu{}_\nu\right) ?$$

By unitarity

Make net of confusion Lorentz group, boost + rotations. Rotations are obv unitary but are boosts?

Ex. 3 Scale transform

We consider the transformation caused by a set of infinitesimal parameters $\{\omega_a\}$. Specifically, the transformations we are referring to are

$$x^\mu \to x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}$$

$$\phi'(x') = \phi(x) + \omega_a \frac{\delta F}{\delta \omega_a}(x) \qquad *$$

It is common to define Generators (G_a) caused by a symmetry transform as follows

Note definition

$$\frac{\delta \phi(x)}{\delta \omega} = \delta_\omega \phi(x) := \Phi'(x) - \Phi(x) := -i\omega_a G_a \phi(x)$$

We then note that we can expand the field $\Phi(x)$ around the point x' in terms of $\{\omega_a\}$

$$\phi(x) = \phi(x') - \omega_a \left(\frac{\delta x}{\delta \omega_a}\right)\left(\frac{\partial \phi(x')}{\partial x'}\right)$$

combined with $*$ yeilds

$$\phi'(x') = \phi(x') - \omega_a \frac{\delta x}{\delta \omega_a}\frac{\partial \phi(x')}{\partial x'} + \omega_a \frac{\delta F}{\delta \omega_a}(x)$$

$$\phi'(x') = \phi(x') - \omega_a \frac{\delta x}{\delta \omega_a}\frac{\partial \phi(x')}{\partial x'} + \omega_a \frac{\delta F}{\delta \omega_a}\left(x' - \omega_a \frac{\delta x}{\delta \omega_a}\right)$$

$$\phi'(x') = \phi(x') - \omega_a \frac{\delta x}{\delta \omega_a}\frac{\partial \phi(x')}{\partial x'} + \omega_a \frac{\delta}{\delta \omega_a}\left(F(x') - \omega_a \ldots\right)$$

$$\phi'(x') = \phi(x') - \omega_a \frac{\delta x}{\delta \omega_a}\frac{\partial \phi(x')}{\partial x'} + \omega_a \frac{\delta}{\delta \omega_a} F(x')$$

$$\phi'(x') - \phi(x') = -\omega_a \left(\frac{\delta x}{\delta \omega_a}\frac{\partial \phi(x')}{\partial x'} + \frac{\delta F(x')}{\delta \omega_a}\right)$$

Eq. blu

$$= -i\omega_a G_a \phi(x)$$

$$\Rightarrow i G_a \phi(x) = \frac{\delta x}{\delta \omega_a}\frac{\partial \phi(x')}{\partial x'} + \frac{\delta F(x')}{\delta \omega_a}$$

Using eq. blu for said transform yeilds

$$S' = \int d^d x \left(1 + \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right)\right) \times \mathcal{L}\left(\phi + \omega_a \frac{\delta F}{\delta \omega_a},\right.$$
$$\left.\left[\delta^\nu_\mu - \partial_\mu(\omega_a(\delta x^\nu/\delta \omega^a))\right](\partial_\nu \phi + \partial_\nu[\omega_a(\delta F/\delta \omega_a)])\right)$$

Where, in order to calculate the Jacobian we have used the following approximation

$$\det(1+E) \approx 1 + \operatorname{Tr} E \quad (\text{Small } E)$$

To calculate δS we will need to perform an first order expansion S' around x w.r.t. ω_a.

~~For this reason, it is usfull to remind ourselves about the formula for multivariate taylor expansion~~

$$\cancel{f(a+h, b+k) = f(a,b) + \frac{\partial}{\partial x} f(a,b) h + \frac{\partial}{\partial y} f(a,b) k + |(h,k)|\, B(h,k)}$$

~~where $B(h,k)$ is a compact function close to the origin.~~

$$\delta S = \int d^d x\, \partial_\mu j^\mu_a \omega_a$$

$$j^\mu_a = \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right\} \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \mathcal{I}}{\delta \omega_a}$$

The energy - momentum tensor is the conserved current that corresponds with translational invariance.

$$\frac{\delta x^\mu}{\delta \epsilon^\nu} =$$